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Descriptive Combinatorics on Trees, Grids, and Non-Amenable Graphs

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Descriptive Combinatorics on Trees, Grids, and Non-Amenable Graphs

A dissertation submitted in partial satisfaction
of the requirements for the degree
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by

Clark Richard Lyons

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ABSTRACT OF THE DISSERTATION

Descriptive Combinatorics on Trees, Grids, and Non-Amenable Graphs

by

Clark Richard Lyons
Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2024
Professor Andrew Marks, Co-Chair
Professor Itay Neeman, Co-Chair

This dissertation investigates Baire measurable, measurable, and Borel labeling problems in descriptive combinatorics on Borel graphs that are tree-like or grid-like, and also on graphs that have certain expansion behavior.

Chapter 2 introduces a framework for applying the determinacy method to prove impossibility results in Borel combinatorics to labeling problems on tree-like graphs and hypergraphs. It also establishes a generalized method of round elimination to prove analogous impossibility results in the theory distributed algorithms and shows that these two methods both naturally apply to the same class of sinkless coloring problems.

Chapter 3 provides a proof that the set of games for which a certain player has a winning strategy in a Borel family of games in Baire measurable. This result is used in the previous chapter in the determinacy arguments. This result is proven by classical methods, but it is also shown how the result follows from and fits into the theory of universally Baire sets of reals.

Chapter 4 contains joint work with Felix Weilacher and Anton Bernshteyn to find a locally

checkable labeling problem on two dimensional grids which can always be solved μ-measurably on any Borel grid for any Borel probability measure μ, but cannot always be solved Baire measurably.

Chapter 5 contains joint work with Alexander Kastner to prove that free Borel actions of non-amenable groups admit Baire measurable perfect matchings.

Chapter 6 proves an expander mixing lemma for probability measure preserving graphs and uses this result to obtain a simpler construction of an edge-colored highly mixing graph from the descriptive combinatorics literature.

The dissertation of Clark Richard Lyons is approved.

Igor Pak

Artem Chernikov

Itay Neeman, Committee Co-Chair

Andrew Marks, Committee Co-Chair

University of California, Los Angeles

2024

To my parents.

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VITA

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CHAPTER 1

Introduction

The field of descriptive combinatorics studies definable graphs and definable solutions to labeling problems on these graphs. A Borel graph is a graph G whose vertex set $V(G)$ is a Polish space and whose edge relation $E(G) \subseteq V(G) \times V(G)$ is Borel. Of particular importance are Borel graphs which are locally finite, which means that every vertex has finite degree, or even Borel graphs of bounded degree. A fundamental example of a Borel graph is that of a Schreier graph. If Γ is a finitely generated group with finite generating set S, and $a : \Gamma \times X \to X$ is a Borel action of Γ on a Polish space X, then the resulting Schreier graph $\mathrm{Sh}(a, S)$ is a bounded degree Borel graph. These Schreier graphs for various group actions and Borel graphs derived from them form a good source of examples in descriptive combinatorics.

One simple but illustrative example of such a Schreier graph comes from the irrational rotation action on the circle $T_\theta : \mathbb{Z} \times S^1 \to S^1$ (for a fixed irrational angle θ). The resulting Schreier graph $\mathrm{Sh}(T_\theta, \{\pm 1\})$ is acyclic and every vertex has degree 2. Thus $\mathrm{Sh}(T_\theta, \{\pm 1\})$ consists of continuum many components which are all bi-infinite paths. Therefore $\mathrm{Sh}(T_\theta, \{\pm 1\})$ has a proper 2-coloring (the vertices can be labeled with 2 colors in such a way that adjacent vertices receive different colors). But by a classical ergodicity argument, $\mathrm{Sh}(T_\theta, \{\pm 1\})$ does not have a Borel 2-coloring. That is, the function which labels the vertices with the two colors cannot be Borel. In fact there is no Lebesgue measurable or Baire measurable 2-coloring of $\mathrm{Sh}(T_\theta, \{\pm 1\})$. This example already shows that the descriptive combinatorial properties of a graph can diverge from the classical combinatorial properties of that graph.

A Borel Schreier graph also makes an appearance in the proof of the Banach-Tarski paradox in the form of an action of a free group on the sphere by rotations $a : \mathbb{F}_2 \times S^2 \to S^2$ with $\alpha, \beta \in \mathbb{F}_2$ the generators. The Banach-Tarski paradox amounts to finding a vertex labeling of $\mathrm{Sh}(a, \{\alpha, \beta, \alpha^{-1}, \beta^{-1}\})$, indicating the pieces in the decomposition, such that the labels satisfy certain local conditions in $\mathrm{Sh}(a, \{\alpha, \beta, \alpha^{-1}, \beta^{-1}\})$ which indicate how the pieces can be rotated to double-cover S^2. Since the Banach-Tarski paradox cannot be performed with Borel pieces (or even Lebesgue measurable pieces), we have another example of a Borel graph with interesting descriptive combinatorial properties. But as it turns out, the Banach-Tarski paradox can be performed with Baire measurable pieces by work of Dougherty and Foreman [8].

In general if G is a locally finite graph and Π is a labeling problem whose correctness at a vertex only depends on the labels in a neighborhood of that vertex, then the axiom of choice implies that there is a solution to Π on G if and only if there is a solution to Π on every finite subgraph of G, where the labeling condition is only enforced at the interior vertices of the finite subgraph. But this compactness argument which lifts labelings of finite subgraphs to G does not ensure anything about the definability of the resulting labeling on the whole graph. If for example G is a Borel graph and we want a Borel solution to Π, then we need to take more care than is afforded by this approach using compactness.

In work of Kechris-Solecki-Todorcevic [19], Grebk-Pikhurko [12], and Lyons-Nazarov [23], definable vertex-colorings, edge-colorings, and matchings are constructed algorithmically, in part by analyzing how classical greedy algorithms run in the context of Borel graphs. This algorithmic approach continues to be fruitful in descriptive combinatorics. Analogously in the field of distributed algorithms, labeling problems are studied on large finite graphs where each vertex represents a computer which can only directly communicate with its neighbors. After some number of rounds of communication each vertex must output its label. There have been precise connections made between the field of distributed algorithms and the field of descriptive combinatorics by Bernshteyn [2], and even outside of these proven connections each field employs similar ideas which can sometimes be borrowed into the other field.

1.1 Measure, Category, and Structural Decompositions

What separates descriptive combinatorics from set-theoretic combinatorics on the underlying graph is the regularity properties that definable sets of reals have. For example, Borel subsets of Polish spaces are Baire measurable and μ-measurable for any Borel probability measure μ.

Either Baire measurability or μ-measurability can be used to show that the Borel chromatic number of $\mathrm{Sh}(T_\theta, \{\pm 1\})$ is not 2. And these two methods of proof are analogous to each other in this simple setting. But in general, the kind of regularity imposed by Baire measurability and μ-measurability are very different. As was stated before, the Banach-Tarski paradox can be performed with Baire measurable pieces (but not with Lebesgue measurable pieces). In [25] Marks and Unger recast this as a result about Baire measurable matchings in bipartite graphs with combinatorial expansion. Chapter 5 extends their results to Baire measurable matchings in non-bipartite graphs with combinatorial expansion, with applications to the Schreier graphs of free actions of non-amenable groups.

One main tool in the results of Chapter 5, which goes back to Marks and Unger, is a certain strong structural Borel decomposition, which can be obtained in any Borel graph after discarding an invariant meager set. This kind of decomposition can provide a witness to hyperfiniteness, off of this invariant meager set, providing another way to see the result that every countable Borel equivalence relation is hyperfinite on an invariant comeager set. Hjorth and Kechris [16], and independently Sullivan, Weiss, and Wright [27] and also Woodin in unpublished work, proved that every countable Borel equivalence relation is hyperfinite on an invariant comeager set. Even more, after discarding an invariant meager set, there are other strong strong structural Borel decompositions one can obtain like toast (discussed in chapter 4), Borel asymptotic dimension decompositions, Borel asymptotic separation index decompositions, and others.

It is also known that Schreier graphs of $\mathbb{Z}^d$ admit such strong Borel decompositions without having to discard anything. This connects the study of Baire measurable combinatorics to the Borel combinatorics of certain nicer Borel graphs, especially those generated by nicer groups. The question of which groups are nice is very subtle and connects to Weiss's question, which is open: Are orbit equivalence relations of actions of amenable groups on Polish spaces hyperfinite?

Ornstein and Weiss [26] proved that after discarding an invariant null set, actions of amenable groups are hyperfinite. But non-amenable groups can generate non-measure-hyperfinite equivalence relations. Therefore the Schreier graphs of actions non-amenable groups need not have any of the structural Borel decompositions described above, even after discarding an invariant null set. This provides a major difference between Baire measurable and μ-measurable combinatorics which shows up in the presence of more complicated group actions.

But this still leaves the question of what differences are their between Baire measurable and μ-measurable combinatorics on Schreier graphs of nicer groups. It is asked in [14] whether there are any locally checkable labeling problems on Schreier graphs of $\mathbb{Z}^d$ which can be solved μ-measurably for any action, but cannot always be solved Baire measurably. An example of such a labeling problem is provided in Chapter 4.

1.2 Other Methods

There are also impossibility results in Borel combinatorics which do not come from measure or category. Marks [24] constructed, for every d, an acyclic d-regular Borel graph which has Borel chromatic number $d+1$. In contrast, as shown in [6], for $d \geq 3$ every such graph has a Borel d-coloring on a comeager or conull invariant set. Mark's method uses the Borel determinacy theorem. Chapter 2 provides a framework for applying Borel determinacy to certain labeling problems on graphs and hypergraphs that are tree-like. That chapter also introduces a framework for applying the method of round elimination from distributed algorithms to show that there is no $o(\log(n))$-LOCAL algorithm for solving certain labeling problems on graphs and hypergraphs that are tree-like. These two methods both naturally apply to a general class of problems called sinkless coloring, which are described in the chapter.

In the determinacy arguments in Chapter 2, some measurability questions arise pertaining to sets that have fairly complicated definitions in the sense of descriptive set theory. Chapter 3 is devoted to proving that these sets satisfy the required measurability hypotheses, and to connecting these measurability results to the theory of universally Baire sets of reals.

In addition to the sigma-algebras of μ-measurable and Baire measurable sets of reals, there are other sigma-algebras of interest in descriptive combinatorics. The sigma-algebra of completely Ramsey subsets of $[\omega]^\omega$ is used for the purpose of proving some completeness results in descriptive combinatorics in [15]. They combine results about the Ramsey property with properties of certain edge-labeled measure-preserving graphs that they construct using a local-global limit. Chapter 6 of this thesis provides an alternative construction which avoids local-global limits and instead develops expansion properties of measure-preserving graphs through a spectral analysis.

1.3 Terminology and Notation

We assume standard descriptive set theory terminology and notation as in [17].

A Borel graph $G = (V, E)$ is a graph whose vertex set V is a Polish space and whose edge relation $E \subseteq V \times V$ is Borel. For a Borel graph G and a graph labeling problem P, descriptive combinatorics concerns whether there is a a solution to P on G with a labeling function that is Borel. As mentioned above, if G is locally finite (every vertex has finite degree) and the validity of P at a vertex can be checked by considering the labels in a neighborhood of the vertex, then a consequence of the axiom of choice is that G has a labeling which solves P if and only if every finite subgraph of G has a labeling which solves P (where the labeling condition of P is only enforced at interior vertices of the finite subgraph). But this solution given by the axiom of choice, is not guaranteed to be Borel, or even Baire measurable or μ-measurable for a Borel probability measure μ.

Graphs will be assumed to be irreflexive with no multi-edges. We will use the discrete graph distance $d(x, y)$ between vertices x and y that are in the same component of a graph G. We use $B_r(x)$ to denote set of vertices of distance at most r from x in the graph. And if X is a set of vertices, we use $B_r(X)$ to denote the set of vertices of G of distance at most r from some vertex in X. Denote the set of neighbors of a vertex x by $N(x)$. And similarly let $N(X)$ denote the set of vertices which are adjacent to a vertex in X, when X is a set of vertices. Additional graph-theoretic terminology and notation will be introduced and used as needed, dependent on context.

If $a : \Gamma \curvearrowright X$ is an action of a group Γ on a set X and $S \subseteq \Gamma$ is a set, then we have a Schreier graph $Sh(a, S)$ whose vertex set is X and such that two vertices are related when one can be obtained from the other by an application of a element of S in the action a. In the case that X is a Polish space, Γ is a countable discrete group, and the action a is by Borel transformations, the graph $Sh(a, S)$ is a Borel graph.

If G is a Borel graph then the connectedness relation, where two vertices are related if they are in the same connected component, is an analytic equivalence relation. If G is locally finite or even just locally countable, then the connectedness relation is Borel and has countable classes. We call such Borel equivalence relations countable Borel equivalence relations. Similarly if a Borel equivalence relation has finite classes, we call it a finite Borel equivalence relation.

If E is a countable Borel equivalence relation on a Polish space X and there exists an increasing sequence of finite Borel equivalence relations E_n on X such that $E = \bigcup E_n$, then we say that E is hyperfinite. The notion of hyperfiniteness, as mentioned above, plays an important role in descriptive combinatorics. This is in part because finite Borel equivalence relations have very simple definable combinatorics.

CHAPTER 2

Round Elimination and The Determinacy Method

This chapter studies an aspect of the connection between descriptive combinatorics and distributed algorithms, especially in the case of graphs that are tree-like, as studied in [3]. The connection of descriptive combinatorics to distributed algorithms is through the LOCAL model of distributed computation, introduced by Linial in [22]. See Distributed Graph Coloring: Fundamentals and Recent Developments [1] by Barenboim and Elkin for a formal introduction.

Consider a graph G with n vertices. Here G represents a network of computers that can communicate with their neighbors in discrete rounds. In each round, the vertices perform a computation locally and then send messages to all their neighbors. After R many rounds, every vertex outputs a label for itself or for its edges/hyperedges, and the this labeling of G is the output of the algorithm. The efficiency of such an algorithm is measured by the number of communication rounds R required, maximized over all n-vertex graphs (or all n-vertex graphs from a specified class). Every vertex of G performs the same algorithm. Symmetry is broken by assuming that each vertex has a unique identifier from $\{1, \ldots, n\}$, and each vertex knows its own identifier. A deterministic LOCAL algorithm solves a labeling problem P if the labeling it outputs on any graph G is a valid solution to P, regardless of the way the identifiers are assigned. There is also a randomized LOCAL model, where the vertices generate their identifiers independently at random from $\{1, \ldots, n\}$ and the algorithm is required to produce a valid solution to P with probability of failure less than $\frac{1}{n}$. We will primarily be concerned with deterministic algorithms.

We will consider classes of graphs of bounded degree, and we study asymptotic bounds (in terms of n) on the number of rounds of communication required to solve a labeling problem, where the implied constants can depend on the degree bound.

Bernshteyn [2] proved the following two theorems which provide a connection between descriptive graph combinatorics and distributed algorithms.

Theorem 1. *If $G = (V, E)$ is a Borel graph of bounded degree and there is a $o(\log n)$-round deterministic LOCAL algorithm for solving a locally checkable labeling problem P on the finite subgraphs of V, then there is a Borel solution to P on G.*

Theorem 2. *If $G = (V, E)$ is a Borel graph of bounded degree and there is a $o(\log n)$-round randomized LOCAL algorithm for solving a locally checkable labeling problem P on the finite subgraphs of V, the for any compatible Polish topology τ on V there is a Baire measurable solution to P on G. And for any Borel probability measure μ on V there is a μ-measurable solution to P on G.*

These theorems turn upper bound results in distributed algorithms directly into positive results in descriptive graph combinatorics. Although the converses to the previous theorems are not true, there is a similarity between the method of round elimination for proving lower bounds in distributed algorithms and the determinacy method for proving impossibility methods in descriptive set theory. Brandt's [5] method of round elimination has been to show that there is no $o(\log n)$-LOCAL deterministic algorithm for finding a Δ vertex coloring in Δ-regular acyclic graphs. The determinacy method of Marks [24] has been used to construct a Borel Δ-regular acyclic graph without a Borel Δ vertex coloring. Both round elimination and the determinacy method have since been applied to a variety of problems. This chapter generalizes both methods to a wider class of problems about labeling hypergraphs and shows that both methods apply to natural problems in this wider class.

Since its introduction, round elimination has been applied to prove lower bounds for a large number of problems in the theory of distributed algorithms. This has happened in part because of the online Round Eliminator tool, which can help test whether the method applies to a particular labeling problem. The generalized notion of round elimination introduced in this thesis may be able to expand the applications of round elimination further, especially if a tool like the Round Eliminator were made available for this new form of round elimination.

2.1 Notation

The technique of round elimination takes place in the setting of a bipartite graph, where the two parts in the partition are the "active" and "passive" vertices. The generalized setting for round elimination proposed in this paper takes place in a k-partite k-uniform hypergraph, which includes the case of a bipartite graph when $k = 2$. A **k-uniform hypergraph** $G = (V, E)$ is a set of vertices V equipped with a collection E of hyperedges, which are cardinality k subsets of V. We will consider labelings of the vertices and hyperedges of G which formally are functions from V or E to a set of labels. We say that a k-uniform hypergraph $G = (V, E)$ is **k-partite** if there is a partition

$$V = V_1 \sqcup \ldots \sqcup V_k$$

of the vertices such that every hyperedge contains exactly one vertex from each V_i. A k-partite hypergraph is **σ-regular** (with respect to a partition $V = V_1 \sqcup \ldots \sqcup V_k$) for a tuple $\sigma = (d_1, \ldots, d_k)$ if for each i each vertex in V_i belongs to exactly d_i many hyperedges. We call such a tuple $\sigma = (d_1, \ldots, d_k)$ for $d_i \geq 2$ a **signature**.

Given a σ-regular k-partite hypergraph $G = (V, E)$ and a set of labels L, a **labeling problem** $\mathscr{P} = (P_1, \ldots, P_k)$ is a tuple where each P_i is a collection of cardinality d_i multisets of elements of L. We can also think of each P_i as a d_i-ary relation on L which is symmetric under all permutations of the d_i coordinates. A labeling $c : E \to L$ is a solution to $\mathscr{P}$ if for every i and every $v \in V_i$,

$$\{c(e) | v \in e\} \in P_i$$

where the left side is treated as a multiset of cardinality d_i. All labeling problems considered here will be of this form.

In the case $k = 2$, so $\sigma = (d_1, d_2)$, we obtain the standard way of encoding labeling problems for the purpose of round elimination. The case that k is larger and $d_i = 2$ for all i is what is considered in the original determinacy method [24]. This encodes vertex-labeling problems on k-regular graphs with a fixed k-edge coloring.

For a fixed signature $\sigma = (d_1, \ldots, d_k)$, if $\mathscr{P} = (P_1, \ldots, P_k)$ is a problem with label set L and $\mathscr{Q} = (Q_1, \ldots, Q_k)$ is a problem with label set M, then a **0-round reduction** of $\mathscr{Q}$ to $\mathscr{P}$ is a function $f : L \to M$ which is a homomorphism in the sense that for each i we have

$$(\ell_1, \ldots, \ell_{d_i}) \in P_i \implies (f(\ell_1), \ldots, f(\ell_{d_i})) \in Q_i.$$

If there is a 0-round reduction of $\mathscr{Q}$ to $\mathscr{P}$ then a solution to $\mathscr{Q}$ can be obtained from a solution to $\mathscr{P}$ by applying f to all labels. Also note that the composition of 0-round reductions is a 0-round reduction.

The graphs we consider to prove lower bounds will be arise from certain σ-regular k-partite hypergraphs associated to the Schreier graphs of free actions of free products of finite groups. Fixing a signature σ we consider the group

$$\Gamma_\sigma = \Delta_1 * \ldots * \Delta_k$$

where Δ_i is a cyclic group of order d_i generated by δ_i for each i. Given an action of Γ_σ on a set X, we can form a k-uniform k-partite hypergraph $G_\sigma(X)$ with vertex set $V = V_1 \sqcup \ldots \sqcup V_k$ where each V_i is the set of cosets $\{x, \delta_i x, \ldots, \delta_i^{d_i - 1} x\}$ of the induced action of Δ_i on X. The hyperedges of $G_\sigma(X)$ are between sets of cosets whose intersection is a singleton $\{x\} \subseteq X$. We call a k-partite σ-regular hypergraph free if it is a subgraph of some $G_\sigma(X)$.

For any signature σ we can consider the sinkless coloring problem $\mathscr{P}_{\text{sc}} = (P_1, \ldots, P_k)$ with label set $\{1, \ldots, k\}$ with relations defined by

$$P_i = \{(\ell_1, \ldots, \ell_{d_i}) | \exists m \leq d_i,\ \ell_m \neq i\}.$$

It follows from the original determinacy method that there is a Borel graph of the form $G_\sigma(X)$ which does not have a Borel solution to $\mathscr{P}_{\mathrm{sc}}$. As demonstrated in [24], the problem $\mathscr{P}_{\mathrm{sc}}$ can be seen as a weakening of the problem of finding a Δ-vertex coloring in Δ-regular graphs equipped with a Δ-edge coloring, by considering the signature $\sigma = (2, \ldots, 2)$. The problem $\mathscr{P}_{\mathrm{sc}}$ also can be seen as a weakening of the sinkless orientation problem or the problem of $2\Delta - 2$ edge coloring.

2.2 Round Elimination

Fix a signature $\sigma = (d_1, \ldots, d_k)$, a problem $\mathscr{P} = (P_1, \ldots, P_k)$ with label set L in this signature, and an index $i \in \{1, \ldots, k\}$ to indicate the **active** vertices. This means that when we consider a distributed algorithm on a free k-partite σ-uniform hypergraph $G = (V, E)$ with k-partition $V = V_1 \sqcup \ldots \sqcup V_k$, only the vertices in V_i will label the hyperedges they belong to. This avoids problems of conflicting outputs of an algorithm since each hyperedge contains exaclty one element of V_i. Also, this does not affect the asymptotic number of rounds needed to solve a problem since each vertex neighbors a vertex from V_i.

In addition to only declaring some of the vertices active, we also insist that a vertex $v \in V_i$ can only directly send a message to vertices in V_{i-1} (or V_k if $i = 1$) with which it shares a hyperedge. This lengthens the number of rounds to communicate a message, but only by a factor of at most $k - 1$ since it now takes at most $k - 1$ rounds to send messages to the other vertices on the same hyperedge.

If P is a d-ary relation on L, then we define two d-ary relations $P^\exists$ and $P^\forall$ on $\mathcal{P}(L)$ by

$$P^\exists = \{(S_1, \ldots, S_d) | \exists \ell_1 \in S_1, \ldots, \exists \ell_d \in S_d,\ P(\ell_1, \ldots, \ell_d)\}$$

$$P^\forall = \{(S_1, \ldots, S_d) | \forall \ell_1 \in S_1, \ldots, \forall \ell_d \in S_d,\ P(\ell_1, \ldots, \ell_d)\}.$$

We define a new problem $\mathrm{re}_i(\mathscr{P}) = (Q_1, \ldots, Q_k)$ with label set $\mathcal{P}(L)$ in the same signature but with active index $i + 1$ (the new active index is 1 if $i = k$). And we let $Q_i = P_i^\exists$ and

$Q_j = P_j^{\forall}$ for all $j \neq i$.

Note that there is a natural 0-round reduction from $\mathrm{re}_i(\mathscr{P})$ to $\mathscr{P}$ which is $f : L \to \mathcal{P}(L)$ with $f(\ell) = \{\ell\}$. We also have the following, which is the fundamental property of round elimination.

Lemma 3. *Let $\mathscr{P} = (P_1, \ldots, P_k)$ be a problem in signature σ and let $1 \leq r \ll \log n$. Then there is an r round solution in the port-labeling model to $\mathscr{P}$ with the V_i vertices active if and only if there is an $r-1$ round solution in the port-labeling model to $\mathrm{re}_i(\mathscr{P})$ with the V_{i+1} vertices active, for free hypergraphs of signature σ with n vertices.*

Proof. Suppose that there is an $r-1$ round solution to $re_i(\mathscr{P})$ with the V_{i+1} vertices active. We show how in one more round of communication, the vertices in V_i can solve $\mathscr{P}$. First, the V_{i+1} vertices send their solution to $\mathrm{re}_i(\mathscr{P})$ to all of the V_i vertices with which they share a hyperedge. Each $v \in V_i$ will recieve d_i many sets $S_1, \ldots, S_{d_i}$. Because $(S_1, \ldots, S_{d_i}) \in P_i^{\exists}$, the vertex v can pick $\ell_m \in S_m$ for all m such that $(\ell_1, \ldots, \ell_{d_i}) \in P_i$. Then v outputs each ℓ_m on the hyperedge where S_m would have been in the solution to $\mathrm{re}_i(\mathscr{P})$. This is a solution to $\mathscr{P}$ because for each $j \neq i$ and $v \in V_j$, the labels on the hyperedges around v were chosen from sets satisfying $P_j^{\forall}$ and hence satisfy P_j.

Now suppose that there is an r round solution to $\mathrm{re}_i(\mathscr{P})$. Consider a vertex $v \in V_i$. Let $e_1, \ldots, e_{d_i}$ be the hyperedges containing v, and let $\{u_1, \ldots, u_{d_i}\}$ be the corresponding vertices in V_{i+1} which share a hyperedge with v. Using $r-1$ rounds of communication v can simulate the computation and communication to predict what label each u_m would have outputted for e_m. The only information which is inaccessible to v for this simulation are the vertices which can reach u_m in r rounds but cannot reach v in $r-1$ rounds of communication. For each hyperedge e containing v let $X_e(v)$ be the set of vertices which can reach the vertex in $e \cap V_{i+1}$ in r rounds but cannot reach v in $r-1$ rounds of communication. The vertex v lists through all possible identifiers which the vertices of $X_e(v)$ could have and outputs the following label from $\mathcal{P}(L)$ for e

$$\{\ell | u_m \text{ outputs } \ell \text{ for some assignment of identifiers to } X_e(v)\}.$$

Now fix $j \neq i$ and some $w \in V_j$. For each hyperedge e containing w, let v_e be the vertex in $e \cap V_i$. We claim that the different sets $X_e(v_e)$ are disjoint because the hypergraph is free. This is because the vertices which can reach each v_e in r rounds but cannot reach w in $r-1$ rounds must have their information pass into v_e from a vertex other than w and so come from different directions in the free graph. This implies that the output of the algorithm satisfies the relation $P_j^{\forall}$ around each $w \in V_j$. If it were possible for each e around w to get labels that violate P_j for some choice of identifiers on each $X_e(v_e)$, then it is possible for all of the $X_e(v_e)$ to have these identifiers simultaneously which violates the correctness of the algorithm being simulated. Checking the correctness of the algorithm around each $v \in V_i$ is more straightforward. The relation $P_i^{\exists}$ must hold around v since for every e containing v, the vertices from each $e \cap v_{i+1}$ are all simulating the same vertex v to produce their labels. □

This lemma shows that round elimination produces a canonical problem which requires one fewer round of communication, as long as we rotate which vertices are active. As a consequence of this we have the following.

Theorem 4. *If $\mathscr{P}$ is a fixed point of round elimination in the sense that $\mathscr{P}$ is 0-round reducible to*

$$re_k \circ re_{k-1} \circ \ldots \circ re_2 \circ re_1(\mathscr{P})$$

then $\mathscr{P}$ does not have a $o(\log n)$-round deterministic algorithm in the port-labeling model for the class of free hypergraphs of signature σ, unless $\mathscr{P}$ is solvable in $< k$ rounds.

Proof. If $\mathscr{P}$ has a solution in r rounds for $k \leq r \ll \log n$, then $\mathrm{re}_k \circ \ldots \circ \mathrm{re}_1(\mathscr{P})$ has a solution in $r-k$ rounds. And using the zero round reduction from $\mathscr{P}$, we have that $\mathscr{P}$ has a solution in $r-k$ rounds. The theorem follows by induction. □

We can upgrade this result to the deterministic LOCAL model using the same standard technique with typical round elimination [5].

Theorem 5. *If $\mathscr{P}$ is a fixed point of round elimination in the sense that $\mathscr{P}$ is 0-round reducible to*

$$re_k \circ re_{k-1} \circ \ldots \circ re_2 \circ re_1(\mathscr{P})$$

then $\mathscr{P}$ does not have a $o(\log n)$-round LOCAL deterministic algorithm for the class of free hypergraphs of signature σ, unless $\mathscr{P}$ is solvable in $< k$ rounds.

Even though the problem $\mathrm{re}_k \circ \ldots \circ \mathrm{re}_1(\mathscr{P})$ has far more labels than $\mathscr{P}$, it is possible to test whether $\mathscr{P}$ is 0-round reducible to $\mathrm{re}_k \circ \ldots \circ \mathrm{re}_1(\mathscr{P})$ by only considering reductions to problems with one usage of round elimination. But first we prove a simple lemma.

Lemma 6. *If $f : L \to M$ is a 0-round reduction from Q to P, then the direct image function*

$$f" : \mathcal{P}(L) \to \mathcal{P}(M)$$

is a 0-round reduction from $re_i(\mathscr{Q})$ to $re_i(\mathscr{P})$ for any i.

Proof. This follows from the fact that $f"$ is a homomorphism from $\mathcal{P}(L)$ to $\mathcal{P}(M)$ for the relations $P_i^{\exists}$ and $Q_i^{\exists}$ and also for the relations $P_i^{\forall}$ and $Q_i^{\forall}$ whenever f is a homomorphism from L to M for the relations P_i and Q_i. □

Lemma 7. *A problem $\mathscr{P}$ is 0-round reducible to $re_k \circ \ldots \circ re_1(\mathscr{P})$ if and only if for all i, the problem $\mathscr{P}$ is 0-round reducible to $re_i(\mathscr{P})$.*

Proof. First assume that $\mathscr{P}$ is reducible to $\mathrm{re}_k \circ \ldots \circ \mathrm{re}_1(\mathscr{P})$. By a previous remark, applications of round elimination always produce easier problems from the perspective of 0-round reductions. Thus $\mathrm{re}_{i-1} \circ \ldots \circ \mathrm{re}_1(\mathscr{P})$ is reducible to $\mathscr{P}$. Applying the previous lemma gives that $\mathrm{re}_i \circ \ldots \circ \mathrm{re}_1(\mathscr{P})$ is reducible to $\mathrm{re}_i(\mathscr{P})$. We also know that $\mathrm{re}_k \circ \ldots \circ \mathrm{re}_1(\mathscr{P})$ is reducible to $\mathrm{re}_i \circ \ldots \circ \mathrm{re}_1(\mathscr{P})$. Following a sequence of three compositions we have that $\mathscr{P}$ is reducible to $\mathrm{re}_i(\mathscr{P})$.

Conversely assume that $\mathscr{P}$ is reducible to $\mathrm{re}_i(\mathscr{P})$ for all i. Applying the previous lemma $k-i$ times we have for all i that $\mathrm{re}_k \circ \ldots \circ \mathrm{re}_{i+1}(\mathscr{P})$ is reducible to $\mathrm{re}_k \circ \ldots \circ \mathrm{re}_i(\mathscr{P})$. Composiing these reductions together yields that $\mathscr{P}$ is reducible to $\mathrm{re}_k \circ \ldots \circ \mathrm{re}_1(\mathscr{P})$. □

We now show that the sinkless coloring problem $\mathscr{P}_{\mathrm{sc}} = (P_1, \ldots, P_k)$ admits round elimination. Fix $i \in \{1, \ldots, k\}$ and consider $\mathrm{re}_i(\mathscr{P}) = (Q_1, \ldots, Q_k)$. Then

$$Q_i = P_i^{\exists} = \{(S_1, \ldots, S_{d_i}) | \forall m \leq d_i,\ S_{d_i} \not\subseteq \{i\}\}$$

and for $j \neq i$

$$Q_j = P_j^{\forall} = \{(S_1, \ldots, S_{d_j}) | \forall m \leq d_j,\ j \notin S_{d_j}\}.$$

We show that the map $f : \mathcal{P}(L) \to L$ given by

$$f_i(S) = \begin{cases} \ell & \text{if } \ell \text{ is least such that } \ell \neq i \text{ and } \ell \in S \\ i & \text{if } S \subseteq \{i\} \end{cases} \tag{2.1}$$

is a 0-round reduction. To show this suppose that $(S_1, \ldots, S_{d_i}) \in Q_i$. Then $(f_i(S_1), \ldots, f_i(S_{d_i})) \in P_i$ since none of the entries are i. And for $j \neq i$ if $(S_1, \ldots, S_{d_j}) \in Q_j$ then $(f_i(S_1), \ldots, f_i(S_{d_j})) \in P_j$ since none of the entries are j. This can be summarized by the following theorem.

Theorem 8. *The sinkless coloring problem $\mathscr{P}_{sc}$ admits round elimination in the sense that $\mathscr{P}_{sc}$ is 0-round reducible to $re_i(\mathscr{P}_{sc})$ for all i.*

This provides an alternative proof that $\mathscr{P}_{\mathrm{sc}}$ does not have a $o(\log n)$ round determanistic LOCAL algorithm, but this also follows from applying Theorem 1 to [24].

2.3 Determinacy

Again fix a signature $\sigma = (d_1, \ldots, d_k)$. The determinacy method can be used to construct a Borel σ-regular hypergraph of the form $G_\sigma(X)$ without a Borel labeling that solves a problem $\mathscr{P}$ for certain problems $\mathscr{P}$. Let k^L denote the set of functions from L to $\{1, \ldots, k\}$ where L is the label set. We think of such functions as partitions of the labels into k named sets.

Definition 9. A problem $\mathscr{P} = (P_1, \ldots, P_k)$ is **playable** if there is a function $D : k^L \to \{1, \ldots, k\}$ such that for any $i \in \{1, \ldots, k\}$ if $F_1, \ldots, F_{d_i} \in k^L$ are such that $D(F_m) = i$ for all i, then

$$(F_1^{-1}(i), \ldots, F_{d_i}^{-1}(i)) \in P_i^{\exists}.$$

We think of each $F \in k^L$ determining a k-player game where player i tries to avoid the labels in $F^{-1}(i)$. The function D picks out a player which does not have a winning strategy in each game. And if player i loses each of the games F_m, then it must be the case that the label sets $F_m^{-1}(i)$ are compatible by a strategy stealing argument where $d_i - 1$ copies of the winning strategies of the $k-1$ other players are played against each other. The following is proven in [4] for the case that $\sigma = (2, \ldots, 2)$, but the same method yields the corresponding statement for hypergraphs.

Theorem 10. *If $\mathscr{P}$ is not playable, then there is a σ-regular Borel hypergraph of the form $G_\sigma(X)$ which does not have a Borel solution to the problem $\mathscr{P}$.*

Their method goes through the theory of local-global limits of finite graphs. We show that it is possible to avoid this and also that the Borel graph $G_\sigma(X)$ can be taken to be hyperfinite, meaning that the connectedness relation is a hyperfinite Borel equivalence relation. This hyperfiniteness was achieved in [4] when $\mathscr{P}$ is a homomorphism problem, and hyperfiniteness had been achieved earlier for the classical vertex coloring, edge coloring, and matching problems.

Theorem 11. *If $\mathscr{P}$ is not playable, then there is a hyperfinite σ-regular Borel hypergraph of the form $G_\sigma(X)$ which does not have a Borel solution to the problem $\mathscr{P}$.*

This is obtained by replacing the ID graphing of [4] with an alternative Borel graph, and the measure theoretic properties are replaced by Baire measurable properties.

For a fixed signature σ, consider the group

$$\Gamma_\sigma^{*\omega} = (\Delta_{1,1} * \ldots * \Delta_{1,k}) * (\Delta_{2,1} * \ldots * \Delta_{2,k}) * \ldots$$

where $\Delta_{p,i}$ is a cyclic group of order d_i generated by $\delta_{p,i}$. Let $Y \subseteq 2^{\Gamma_\sigma^{*\omega}}$ be a comeager subset on which $\Gamma_\sigma^{*\omega}$ acts freely and the action is hyperfinite, which must exist by [16]. We have the following key property which allows for $\Gamma_\sigma^{*\omega}$ to be used as an ID graph as in [4].

Lemma 12. *If $A_1, \ldots, A_{d_i} \subseteq Y$ are each comeager in a non-empty basic open subset of Y, then there exists $p \in \mathbb{N}$ and $y \in Y$ such that for all $m \leq d_i$ we have $\delta_{p,i}^m \cdot y \in A_m$.*

Proof. For each m let U_m be a non-empty basic open subset of Y such that A_m is comeager in U_m. Then each U_m is determined by a finite set of coordinates $I_m \subseteq \Gamma_\sigma^{*\omega}$. For each distinct $m, m' \leq d_i$, for all but finitely many p we have that $\delta_{p,i}^m I_m$ is disjoint from $\delta_{p,i}^{m'} I_{m'}$. Thus there is a single p such that for all m the sets $\delta_{p,i}^m I_m$ are disjoint. Thus the basic open sets $\delta_{p,i}^{-m} \cdot U_m$ have non-meager intersection. Let $y \in Y$ be in this intersection. Then for all $m \leq d_i$ we have $\delta_{p,i}^m \cdot y \in A_m$. □

Proof. Theorem 10 now follows from the methods of [4] but with Y as the ID graph.

The Cayley graphs of

$$\Gamma_\sigma^{*\omega} = (\Delta_{1,1} * \ldots * \Delta_{1,k}) * (\Delta_{2,1} * \ldots * \Delta_{2,k}) * \ldots$$

and

$$\Gamma_\sigma = \Delta_1 * \ldots * \Delta_k$$

both have natural edge-colorings where edges corresponding to the generators $\delta_{i,j}$ or δ_j are colored j. And Y inherits such a coloring since its components are all isomorphic to the Cayley graph of

$$\Gamma_\sigma^{*\omega} = (\Delta_{1,1} * \ldots * \Delta_{1,k}) * (\Delta_{2,1} * \ldots * \Delta_{2,k}) * \ldots.$$

Let H be the set of subgraphs of Y which are isomorphic to $\Gamma_\sigma = \Delta_1 * \ldots * \Delta_k$, with a distinguished vertex. The isomorphism is required to respect the edge-coloring just described. Let $\mathcal{H} = (H, E)$ be the graph with vertex set H where two subgraphs (G, v) and (G', v'), with distinguished vertices, are adjacent if they are the same subgraph $G = G'$ and the distinguished vertices v and v' are adjacent. Then $\mathcal{H}$ is a graph, all of whose connected components are isomorphic to $\Gamma_\sigma = \Delta_1 * \ldots * \Delta_k$. In fact $\mathcal{H}$ can be seen as the Schreier graph of an action of $\Gamma_\sigma = \Delta_1 * \ldots * \Delta_k$ on H. And since Y is hyperfinte, H is as well. Suppose that the σ-regular Borel hypergraph $G_\sigma(H)$ has a Borel solution to $\mathscr{P}$. We show that $\mathscr{P}$ is playable.

For each $y \in Y$ and partition $F \in k^L$ we define a k-player game $\mathbb{G}(y, F)$. In turns the players build an element of H, a subgraph with distinguished vertex y. In the first turn player i chooses among the generators $\delta_{j,i}$ to choose a d_i cycle through y. In subsequent rounds the players choose cycles further and further away from y to build the subgraph (G, v) by finite approximation. Let ℓ be the label that the hyperedge corresponding to (G, v) has in the Borel solution to $\mathscr{P}$. The loser is declared to be the player $F(\ell)$.

In this multi-player Borel game there must be some player i such that the players in $\{1, \ldots, k\} \setminus \{i\}$ have a combined winning strategy to force i to lose. For each y and F, let $L(y, F)$ be the least such losing player. Note also that if player i loses the game $\mathbb{G}(y, F)$, then player i also loses the variant of the game where they make their play first in each turn. The other players can simply use the same combined strategy ignoring the extra information of one extra play each turn. As described in [4], the function $L(y, F)$ is Baire measurable. (In the next chapter we provide an alternative proof of this fact, avoiding the metamathematical theory of provably Δ^1_2 sets.) Thus we can define a function $D : k^L \to \{1, \ldots, k\}$ such that for each $f \in k^L$ we have $L(y, F) = D(F)$ for a non-meager set of y.

Now suppose that $F_1, \ldots, F_{d_i} \in k^L$ are such that $D(F_m) = i$ for all i. Then by the previous lemma there is some $y \in Y$ and some $\delta_{p,i}$ such that for all $m \leq d_i$ we have $L(\delta^m_{p,i} \cdot y, F_m) = i$. We now consider a way to pit the combined winning strategies of the other players against each other. Starting at the cycle $\{y, \ldots, \delta^{d_i - 1}_{p,i} \cdot y\}$, we can build a subgraph of Y by following for each m the winning combined strategy of $\mathbb{G}(\delta^m_{p,i} \cdot y, F_m)$ where $\{y, \ldots, \delta^{d_i-1}_{p,i} \cdot y\}$ is taken to be the opponent's first move. Since these are all winning strategies, it must be the case that the hyperedge corresponding to $\delta_{p,i}^{m} \cdot y$ receives a label in $F^{-1}(\ell)$. Therefore

$$(F_1^{-1}(i), \ldots, F_{d_i}^{-1}(i)) \in P_i^{\exists}.$$

This verifies that $\mathscr{P}$ is playable. $\square$

The following is a rephrasing of the original determinacy argument of [24].

Theorem 13. *The sinkless coloring problem $\mathscr{P}_{sc}$ is not playable.*

Proof. Note that $L = \{1, \ldots, k\}$. Consider the partition $F : L \to \{1, \ldots, k\}$ given by the identity $F(i) = i$. Supposing that $\mathscr{P}_{\text{sc}} = (P_1, \ldots, P_k)$ is playable, we can choose such an assignment $D : k^L \to \{1, \ldots, k\}$. Then we have $D(F) = i$ for some i. Now let $F_1 = \ldots = F_{d_i} = F$. Then

$$(F_1^{-1}(i), \ldots, F_{d_i}^{-1}(i)) \notin P_i^{\exists}$$

since by the definition of $\mathscr{P}_{\text{sc}}$, $(i, \ldots, i) \notin P_i$. This contradicts playability. $\square$

We also mention one simple property of playability and 0-round reductions.

Theorem 14. *If there is a 0-round reduction $f : L \to M$ of $\mathscr{Q}$ to $\mathscr{P}$ and $\mathscr{P}$ is playable, then $\mathscr{Q}$ is playable.*

Proof. If $\mathscr{P}$ is playable, there is a function $D : k^L \to \{1, \ldots, k\}$ such that for any $i \in \{1, \ldots, k\}$ if $F_1, \ldots, F_{d_i} \in k^L$ are such that $D(F_m) = i$ for all i, then

$$(F_1^{-1}(i), \ldots, F_{d_i}^{-1}(i)) \in P_i^{\exists}.$$

We show that $\mathscr{Q}$ is playable by considering the function $D' : k^M \to \{1, \ldots, k\}$ defined by $D'(F) = D(F \circ f)$. Suppose that $F_1, \ldots, F_{d_i} \in k^M$ are such that $D'(F_m) = i$ for all m. Then $D(F_m \circ f) = i$ for all m. Thus

$$((F_1 \circ f)^{-1}(i), \ldots, (F_{d_i} \circ f)^{-1}(i)) \in P_i^{\exists}.$$

But this immediately implies that

$$(F_1^{-1}(i), \dots, F_{d_i}^{-1}(i)) \in Q_i^{\exists}.$$

□

CHAPTER 3

Borel Families of Games and Baire Measurability

This chapter provides a proof that in a Borel family of Borel games, the set of games for which player I has a winning strategy is Baire measurable. This was established by Solovay, in unpublished work. We also connect this fact to the theory of universally Baire sets.

Throughout, games are identified with subsets of $H \subseteq \omega^\omega$. Players I and II alternate playing natural numbers and player II wins if and only if the resulting sequence is in H. The Borel determinacy theorem says that if H is a Borel set, then one of the players has a winning strategy. For more details see [17].

If X is a set and $A \subseteq X \times \omega^\omega$ then for every $x \in X$ there is a game $A_x \subseteq \omega^\omega$ such that

$$A_x = \{\alpha \in \omega^\omega | (x, \alpha) \in A\}.$$

In this way A defines a parameterized family of games. If X is a Polish space and A is Borel, then A defines a Borel family of Borel games, each of which is determined. This is the setting of interest.

In Chapter 2, as part of the construction of a Borel graph without Borel solutions to certain labeling problems, Borel families of games are considered. It is important for those arguments to proceed that the set of games won by a player in a Borel family of games is Baire measurable. Similar arguments can be made to work if this set is μ-measurable for a certain Borel probability measure μ. This measurability problem in the determinacy method first shows up in [4]. They resolve the issue with the metamathematical theory of weakly provably $\mathbf{\Delta}^1_2$ sets. This chapter provides an alternative more "classical" approach, which avoids the metatheory and instead uses an argument that combines a Borel family of games into a single Borel game. We also provide an alternative quick proof that relies on the theory of universally Baire sets.

The set of games for which a certain player has a winning strategy is $\boldsymbol{\Delta}^1_2$. But these sets of games won by a certain player in a Borel family of games have some nicer measurability properties than general $\boldsymbol{\Delta}^1_2$ sets, in particular they are Baire measurable. But if $V = L$, then there is a Δ^1_2 well ordering of ω^ω. This can provide an example of a Δ^1_2 set which is not Baire measurable. Of course, under large cardinals we have Baire measurability of all Δ^1_2 sets, including sets of games won by a player in a Borel family of games. So the main result of this chapter is most relevant in the setting of ZFC without necessarily assuming large cardinal hypotheses.

3.1 Classical Proof

This proof is based off of a proof that the axiom of determinacy implies that all sets of reals are Baire measurable.

Theorem 15. *If X is a Polish space and $A \subseteq X \times \omega^\omega$ is Borel then*

$$W = \{x | \textit{II has a winning strategy for } A_x\}$$

is Baire measurable.

Proof. It suffices to show that for any open set $U \subseteq X$ that either W is comeager in U or $X \setminus W$ is comeager in a nonempty open $V \subseteq U$. But localizing the argument and replacing X with U, it suffices to prove that W is either comeager or $X \setminus W$ is comeager in some nonempty open $V \subseteq X$. This is what we prove.

Consider the following Borel game G_A. Fix a countable basis $\mathcal{U}$ for the topology on X. Player I plays pairs (U_{2k}, n_{2k}) and player II plays pairs (U_{2k+1}, n_{2k+1}) such that for all i,

$n_i \in \omega$, $U_i \in \mathcal{U}$ is a basic open subset of X, $\operatorname{diam}(U_i) \leq 2^{-i}$, and $\overline{U_{i+1}} \subseteq U_i$. By the shrinking conditions on the open sets, we have

$$\bigcap_{i\in\omega} U_i = \{x\}$$

and player II wins if and only if $(x, (n_i)_{i\in\omega}) \in A$.

We will prove that if player II has a winning strategy for G_A then W is comeager, and then that if player I has a winning strategy for G_A then $X \setminus W$ is nonmeager in some nonempty open $V \subseteq U$. Borel determinacy implies that one of the two players has a winning strategy and the result follows.

We first prove that if player II has a winning strategy for G_A, then W is comeager. Let

$$\sigma : \bigcup_{k\in\omega} (\mathcal{U} \times \omega)^{2k} \to \mathcal{U} \times \omega$$

be a winning strategy for II. Let $T \subseteq (\mathcal{U} \times \omega)^{<\omega}$ be a tree on $\mathcal{U} \times \omega$ with the property that any

$$a \in T \cap (\mathcal{U} \times \omega)^{2k}$$

has a unique child

$$a' = a^\frown\sigma(a) \in T \cap (\mathcal{U} \times \omega)^{2k+1}.$$

We also assume that for any $b \in T \cap (\mathcal{U} \times \omega)^{2k-1}$ and any n_{2k} the collection

$$\{U_{2k+1} | \exists U_{2k}\ \exists U_{2k+1}\ \exists n_{2k+1},\ b^\frown(U_{2k}, n_{2k})^\frown(U_{2k+1}, n_{2k+1}) \in T\}$$

is pairwise disjoint with union dense in U_{2k-1}. Such a tree T can be built inductively by choosing a maximal disjoint such collection of U_{2k+1} at each stage.

For each $s = (n_0, n_2, \ldots, n_{2k})$ define

$$D_s = \bigcup\{U_{2k+1} | \exists n_1 \ldots \exists n_{2k+1} \exists U_0, \ldots, \exists U_{2k},\ \big((U_0, n_0), \ldots, (U_{2k+1}, n_{2k+1})\big) \in T\}.$$

Then each D_s is open and dense by construction. We show that $\bigcap_s D_s \subseteq W$ to show that W is comeager. Let $x \in \bigcap_s D_s$. We describe a strategy τ for player II in A_x. If the play of the game so far is $s = (n_0, \dots, n_{2k})$, then player II responds with the unique n_{2k+1} such that there exist $U_0, \dots, U_{2k+1}$ such that $\big((U_0, n_0), \dots, (U_{2k+1}, n_{2k+1})\big) \in T$ and $x \in U_{2k+1}$. We now prove that τ is a winning strategy for player II. Suppose that $(n_i)_{i\in\omega}$ is a result of player II following τ. Then there is a unique sequence $(U_i)_{i\in\omega}$ such that $((U_i, n_i))_{i\in\omega}$ is an infinite path through T and $x \in U_i$ for all i. By the definition of T and because σ is a winning strategy for player II in G_A, we have $(x, (n_i)_{i\in\omega}) \in A$. Thus τ is winning for player II and so $x \in W$.

Now suppose that player I has a winning strategy for G_A with first move (U_0, n_0). Then we can repeat the same arguments as above for the complement of A localized to U_0 to show that $X \setminus W$ is comeager in U_0.

□

The above argument can also be used to prove the Kuratowski-Ulam theorem.

Theorem 16 (Kuratowski-Ulam)**.** *If X and Y are Polish spaces and $Q \subseteq X \times Y$ is comeager, then for comeager many $x \in X$, the fiber $Q_x \subseteq Y$ is comeager.*

Proof. For each $x \in X$ we can consider the Banach-Mazur game where players I and II play a decreasing sequence of non-empty basic open subsets of Y and II wins if the intersection is contained in Q_x. Player II has a winning strategy if and only if $Q_x \subseteq Y$ is comeager. Since Y is Polish, this game can be encoded into a standard game on ω^ω.

This defines a Borel family of games $A \subseteq X \times \omega^\omega$. Consider the combined game G_A. In this game the players alternate to play a decreasing sequence of non-empty basic open subsets of both X and Y simultaneously. This can be viewed as playing non-empty basic open subsets of the product $X \times Y$, and in fact the game G_A is exactly the Banach Mazur game for A as a subset of $X \times Y$.

If $Q \subseteq X \times Y$ is comeager, then player II wins G_A. So by the above arguments, player II wins the Banach-Mazur game for Q_x for comeager many fibers x. Thus Q_x is comeager in Y for a comeager set of $x \in X$. □

3.2 Universally Baire Sets

Note that if $S \subseteq \omega^\omega$ is the set of games that one player wins in a Borel family of Borel games, then S is $\mathbf{\Delta}^1_2$. In fact S is absolutely $\mathbf{\Delta}^1_2$ in in the following sense. There are Σ^1_2 formulas Φ and Ψ defining S and its complement from a parameter $t \in \omega^\omega$ such that for any forcing notion $\mathbb{P}$ we have

$$\Vdash_{\mathbb{P}} \forall x \subseteq \check{\omega}\big(\Phi(x, \check{t}) \leftrightarrow \neg\Psi(x, \check{t})\big).$$

The formulas Φ and Ψ state that a player has a winning strategy in the game above x.

This works because the forcing extensions still satisfy ZFC and therefore also the Borel determinacy theorem. In general if Φ and Ψ are Σ^1_2 formulas which define complementary sets of reals, then they will define disjoint sets in any forcing extension by Shoenfield absoluteness, but they need not remain complementary.

Here we follow Feng, Magidor, and Woodin who proved:

Theorem 17. *If $S \subseteq \omega^\omega$, the following are equivalent:*

- *For any topological space X and any continuous function $f : X \to \omega^\omega$, the set $f^{-1}(S)$ is Baire measurable.*
- *For any forcing notion $\mathbb{P}$, there exists a cardinal κ and trees T_1 and T_2 on $\omega \times \kappa$ such that*

  $$S = \pi[T_1] \quad \text{and} \quad \omega^\omega \setminus S = \pi[T_2]$$

  and we have

  $$\Vdash_{\mathbb{P}} \omega^\omega = \pi[T_1] \cup \pi[T_2].$$

If the above conditions hold we say that S is univerally Baire. Feng, Magidor, and Woodin [9] proved that universally Baire sets are Baire measurable, Lebesgue measurable, Ramsey, and Bernstein measurable.

Theorem 18. *If $S \subseteq \omega^\omega$ is absolutely $\mathbf{\Delta}^1_2$ then then S is universally Baire.*

Proof. We will let $\kappa = \omega_1$ independent of $\mathbb{P}$. Because every $\mathbf{\Sigma}^1_2$ set is ω_1-Suslin, there are trees T_1 and T_2 on $\omega \times \omega_1$ obtained from Φ and Ψ such that $S = \pi[T_1]$ and $\omega^\omega \setminus S = \pi[T_2]$, and for any forcing notion $\mathbb{P}$ we have

$$\Vdash_{\mathbb{P}} \omega^\omega = \pi[T_1] \cup \pi[T_2].$$

This verifies that S is universally Baire. □

3.3 Extensions

Because the set of games won by a particular player is universally Baire, it is also Baire measurable in the density topology and the Ellentuck topology. Therefore such a set is Lebesgue measurable and completely Ramsey. The "classical" proof can also be used to show Baire measurability in these topologies.

CHAPTER 4

Toast and LCLs on Grids

The results of this chapter are part of join work with Felix Weilacher and Anton Bernshteyn.

This chapter is concerned with measurable solutions to locally checkable labeling problems on grids. A grid is the Schreier graph $G(X, S_d)$ of a free Borel action $\mathbb{Z}^d \curvearrowright X$ of the group $\mathbb{Z}^d$ for some d with respect to the standard symmetric set S_d of $2d$ generators on a Polish space X. So the set of vertices of the grid $G(X, S_d)$ is X, and two vertices are connected by an edge if and only if one can be obtained from the other by an application of one of the generators in S_d.

After fixing a dimension d, a locally checkable labeling (LCL) problem Π on d-dimensional grids is specified by a finite set of labels L and a set of allowed configurations of the labels in a radius 1 ball around a vertex in the grid. A coloring for the LCL is a function from X to L, such that the configuration of labels in the 1-ball around each vertex is allowed. Because we will only work with LCLs on grids in this chapter, we will use some specific conventions and notation for describing them which differ from those introduced in Chapter 2.

Definition 19. Fix a dimension d. Let

$$S_d = \left\{(x_1, \ldots, x_d) \in \mathbb{Z}^d \middle| |x_1| + \ldots + |x_d| = 1\right\}$$

be the standard set of $2d$ many generators. An LCL Π is a finite set of labels L and set

$$\Pi \subseteq L^{S_d \cup \{(0,\ldots,0)\}}$$

of allowed configurations. Let $a : \mathbb{Z}^d \curvearrowright X$ be a free Borel action (a grid) and $F : X \to L$ a labeling. For any $x \in X$, there is a unique function $f_x \in L^{S_d \cup \{(0,\ldots,0)\}}$ such that for all

$g \in S_d \cup \{(0, \ldots, 0)\}$ we have $F(g \cdot x) = f_x(g)$. We say that F is a Π-coloring if for all $x \in X$ we have $f_x \in \Pi$.

In the above definition, the functions f_x encode the configuration of labels in the radius 1 ball around x. Note that each vertex "knows" which edges in the graph come from which generators of $\mathbb{Z}^d$ for the purposes of checking the correctness of an LCL.

It was asked in [14] whether for any d and any LCL Π the following are equivalent.

(1) For any free Borel action $\mathbb{Z}^d \curvearrowright X$ on a standard Borel space X, $G(X, S_d)$ admits a Borel Π-coloring.

(2) For any free Borel action $\mathbb{Z}^d \curvearrowright X$ on a Polish space X, $G(X, S_d)$ admits a Baire measurable Π-coloring.

(3) For any free Borel action $\mathbb{Z}^d \curvearrowright X$ on a standard probability space (X, μ), $G(X, S_d)$ admits a μ-measurable Π-coloring.

It is proven in [13] that the answer is positive for the case $d = 1$. However the main result of this chapter is the following.

Theorem 20 (Bernshteyn-L.-Weilacher)**.** *For $d = 2$ there is a locally checkable labeling problem Π such that (3) holds but (2) and (1) fail.*

The main tool for constructing Borel Π-colorings is a Borel structural decomposition, known to exist in the grid setting by [10], called toast. The construction of the μ-measurable Π-coloring in the above counterexample comes from a restricted form of toast called rectangular toast, which is proven to exist for grids after discarding an invariant null set. And the construction of the free Borel action without a Baire measurable Π-coloring comes from an inverse limit argument, in particular a twisted action on a p-adic space.

4.1 Rectangular Toast

We recall the following fundamental definition from Borel combinatorics.

Definition 21. Let (X, E) be a locally finite Borel graph and $r > 0$. An r-toast structure on (X, E) is a Borel set $\mathcal{T} \subseteq [X]^{<\omega}$ such that $\bigcup \mathcal{T} = X$ and for any $S, T \in \mathcal{T}$ one of the following hold.

(1) $B_r(S) \subseteq T$

(2) $B_r(T) \subseteq S$

(3) $B_r(S) \cap T = \emptyset$ [or equivalently $S \cap B_r(T) = \emptyset$].

Elements $S \in \mathcal{T}$ are called tiles. In the case where the Borel graph is a grid $G(X, S_d)$, a tile S is a rectangle if it is of the form

$$S = [a_0, b_0) \times \ldots \times [a_{d-1}, b_{d-1}) \cdot x$$

for some $[a_0, b_0) \times \ldots \times [a_{d-1}, b_{d-1}) \subseteq \mathbb{Z}^d$ and some $x \in X$. A toast on a grid is called rectangular if all tiles are rectangles.

It is proven in [11] that not every grid (even for $d = 2$) admits a rectangular toast. We next prove that any grid on a standard probability space admits a toast off of an invariant null set. First we need the following lemma which is related to the well known Kakutani-Rokhlin lemma as in [26].

Lemma 22. *Fix d. For any $r > 0$ and any $\varepsilon > 0$ there exists $n = n(r, \varepsilon)$ such that the following holds for any d-dimensional grid (X, E) on a standard probability space (X, μ). There exists a Borel set $\mathcal{R} \subseteq [X]^{<\omega}$ of rectangles with all side lengths n with $\mu(\bigcup \mathcal{R}) > 1 - \varepsilon$ and for any distinct $S, T \in \mathcal{R}$, $B_r(S) \cap B_r(T) = \emptyset$.*

Proof. Fix r and ε. For any value of n, consider the set of rectangles $\mathcal{D}_n \subseteq [\mathbb{Z}^d]^{<\omega}$ in $\mathbb{Z}^d$ with all side lengths n whose lexicographically least entries are in the lattice $(n + 2r + 3) \cdot \mathbb{Z}^d$.

These rectangles are pairwise separated by distances greater than $2r$. When n is large enough the union $U_n = \bigcup \mathcal{D}_n \subseteq \mathbb{Z}^d$ of this collection of rectangles has natural density greater than $1 - \frac{\varepsilon}{2}$ as a subset of $\mathbb{Z}^d$. Fix $n = n(r, \varepsilon)$ to be this large.

By [26] actions of $\mathbb{Z}^d$ are measure-hyperfinite. Let F be the connectedness relation on the grid and let F_k be an increasing sequence of finite Borel equivalence relations on X such that $F = \bigcup F_k$. For each k consider

$$A_k = \{x \in X | B_{d^2(n+2r+3)}(x) \subseteq [x]_{F_k}\}.$$

We have $\bigcup A_k = X$. Fix k large enough that $\mu(A_k) > 1 - \frac{\varepsilon}{2}$. Since F_k is a finite Borel equivalence relation, it has a selector. We can choose a Borel set $Y \subseteq A_k$ such that Y contains exactly one vertex from each F_k class than meets A_k. We can also choose a Borel selector $f : A_k/F_k \to Y$ such that for all $x \in A_k$ we have $(x, f([x])) \in F_k$.

We choose our rectangles to be from a dense grid like $U_n = \bigcup \mathcal{D}_n \subseteq \mathbb{Z}^d$ as considered at the start of the proof. The offset of the grid is determined by the vertices in Y and also a choice of parameter

$$p \in [0, n + 2r + 3)^d.$$

For each possible choice of p, consider the set of rectangles $\mathcal{R}_p$ which have all side lengths n, meet A_k in the F_k-class C, and have lexicographically least entry in the lattice

$$[p + (n + 2r + 3) \cdot \mathbb{Z}^d] \cdot f(C).$$

By construction the rectangles in $\mathcal{R}_p$ are pairwise of distance greater than $2r$ from each other. Also, for each $x \in A_k$, we have that $x \in \bigcup \mathcal{R}_p$ for greater than $1 - \frac{\varepsilon}{2}$ proportion of the choices for p. Therefore for some choice of p we must have

$$\mu\Big(\bigcup \mathcal{R}_p\Big) \geq \Big(1 - \frac{\varepsilon}{2}\Big)\mu(A_k) \geq \Big(1 - \frac{\varepsilon}{2}\Big)^2 > 1 - \varepsilon.$$

Therefore $\mathcal{R} = \mathcal{R}_p$ satisfies the conclusion of the lemma for this choice of p. $\square$

Note that by making $n = n(r, \varepsilon)$ larger we may assume that

$$\mu\Big(\{x \in X | B_r(x) \subseteq \bigcup \mathcal{R}\}\Big) > 1 - \varepsilon.$$

We can now construct μ-measurable rectangular toast layer-by-layer through repeated applications of the proceeding lemma.

Theorem 23. *If (X, E) is a d-dimensional grid on a standard probability space (X, μ), then for any $r > 0$ there exists a rectangular toast on a co-null invariant subset of X.*

Proof. First, we may assume that μ is quasi-invariant, since there is a measure μ' whose null sets are μ-null which is quasi-invariant as shown in [18].

For $k \in \omega$ let $\varepsilon_k = 2^{-k}$. Let $r_0 = r$ and for $k \in \omega$ let $n_k = n(r_k, \varepsilon_k)$ and $r_{k+1} = 2d^2 n_k + 2r + 3$. Then for each k let $\mathcal{R}_k$ be a Borel collection of rectangles separated by distances greater than $2r_k$ such that

$$\mu\Big(\{x \in X | B_{r_k}(x) \subseteq \bigcup \mathcal{R}_k\}\Big) > 1 - \varepsilon_k.$$

Inductively add all rectangles from $\mathcal{R}_k$ to $\mathcal{T}$ and remove any rectangles in $\bigcup_{i<k} \mathcal{R}_i$ whose radius r-ball is not contained in a rectangle in $\mathcal{R}_k$.

Let $Z = \{x \in X | B_r(x) \subseteq \bigcup \mathcal{R}_k\}$ for infinitely many k. By the Borel-Cantelli lemma, the set Z is co-null. And in the construction of $\mathcal{T}$ rectangles are only removed when they fail to be in the r_k-interior of a rectangle at a later stage k. Therefore $\mathcal{T}$ is a toast on the co-null set Z. Since we can assume μ is quasi-invariant, we may assume that Z is invariant by replacing it with its saturation. □

4.2 Description of the Counterexample

In this section we describe the counterexample that is used to establish Threorem 20. Fix $d = 2$. The problem Π has a set of 10 labels. The first 8 labels indicate "top", "bottom", "left", "right", and the four corner positions "top-right", "top-left", "bottom-left", and "bottom-right". And the other 2 labels are 0 and 1. The intension of Π is to encode the boundaries of rectangles in a rectangular toast, with the 0 and 1 labels forming a checkerboard pattern inside and outside of each rectangular boundary.

If a vertex x has label 0, then the vertex above it must have label 1 or “top”. The label to the right of x must have label 1 or “right”, and similarly for the other two directions. The same is true reversing the roles of 0 and 1.

If a vertex x has label “top” then the vertices above and below it must have labels that are either 0 or 1. The vertex to the right of x must either have the label “top” or “top-right”. And the vertex to the left of x must either have the label “top” or “top-right”. The analogous conditions hold for a vertex with the “bottom”, “left”, “right” labels.

If a vertex x has the label “top-right”, then the vertices above and to the right of x must both have label 0 or both label 1. The vertex to the left of x must have label “top”, and the vertex below x must have label “right”. The analogous conditions hold for a vertex with the other corner labels.

This defines a labeling problem Π. By construction, Π-colorings can be obtained from rectangular toast and can therefore can be constructed μ-measurably.

Lemma 24. *If (X, E) is a 2-dimensional grid which admits a rectangular 1-toast, then (X, E) admits a Borel Π-coloring.*

Proof. Let $\mathcal{T}$ be a rectangular 1-toast on (X, E). Then along the boundaries of tiles in $\mathcal{T}$ we can use the labels for “top”, “bottom”, “left”, “right”, and the four corner positions. The remainder of the graph has finite connected components, and therefore there is a Borel 2-coloring of this remainder with the labels 0 and 1. This defines a Borel Π-coloring. $\square$

But note that Borel Π-colorings do not necessarily encode toast. In particular, an infinite checkerboard pattern of 0 and 1 can be part of a valid Π-coloring. It remains to define a free action of $\mathbb{Z}^2$ by homeomorphisms on a Polish space, such that the resulting grid does not have a Baire measurable solution to Π.

Let $X = \mathbb{Z}_3 \times \mathbb{Z}_2$ where $\mathbb{Z}_3$ is the Polish group of 3-adic integers and $\mathbb{Z}_2$ is the Polish group of 2-adic integers. We define an action $a : \mathbb{Z}^2 \times X \to X$ by

$$a\Big((s,t),(\alpha,\beta)\Big) = (s+\alpha, s+t+\beta).$$

The action is free by homeomorphisms and the Schreier graph defines a 2-dimensional grid (X,E). We first prove a lemma about the regularity of Baire measurable subsets of X under this action.

Lemma 25. *If $Y \subseteq X$ is Baire measurable and non-meager then there exists $y \in Y$, $m \in \omega$, and $k \in \omega$ such that $A \cdot y \subseteq Y$ where A is the subgroup of $\mathbb{Z}^2$ generated by $(3^m, 0)$ and $(1,k)$.*

Proof. Since Y is non-meager and Baire measurable, there is a basic open set U in which Y is comeager. The set U can be taken to be of the form

$$U = \Big\{(\alpha,\beta)\Big|\alpha \equiv u \bmod 3^m \text{ and } \beta \equiv v \bmod 2^n\Big\}$$

for some $u, v \in \mathbb{Z}$ and some $m, n \in \omega$. We see that $(3^m, 0) \in \mathbb{Z}^2$ preserves U under the action a of $\mathbb{Z}^2$. Choose $k \in \omega$ such that $k \equiv 0 \bmod 2^n$ and $k \equiv -1 \bmod 3^m$. Then U is also preserved by $(1,k)$ under the action a of $\mathbb{Z}^2$. Thus U is preserved by A, the subgroup generated by $(3^m, 0)$ and $(1,k)$.

Since the action is by homeomorphisms and Y is comeager in U, the result follows. □

This allows us to prove the following.

Lemma 26. *There is no Baire measurable Π-coloring of (X,E).*

Proof. Suppose there is such a Baire measurable coloring, and consider the set B of vertices which have boundary labels. By the definition of Π, the set B is 2-regular.

In fact, each component of B can only have at most 4 corner labels, and there are a limited number of configurations for such a finite boundary component. Each boundary component with 4 corner labels is the boundary of a standard rectangle. The other components have 0, 1, or 2 corner labels and all have infinite rays of boundary labels.

We first prove that the union of components which contain an infinite ray of boundary labels is meager. This follows from an application of Lemma 25. For example, if there is a nonmeager set of "top" labels T such that $(1,0)\cdot T\subseteq T$, then we may assume without loss of generality that there is a nonmeager set Z of vertices labeled 0 which are immediately above elements of T. But then Z is preserved under some $(3^m,0)\in\mathbb{Z}^2$, which is a contradiction because 3^m is odd. And if there is a nonmeager set of "right" labels R such that $(0,1)\cdot R\subseteq R$, then R is preserved by some $(1,k)\in\mathbb{Z}$, which is a contradiction because a right label cannot be above another right label. The other cases for infinite rays of boundary labels are similar.

Therefore, after modifying the labeling on a meager set, we may assume that all boundary labels are parts of standard rectangles. Now we prove that the set of vertices which are not inside of boundary rectangles is meager. This then shows that the boundary labels do define a rectangular toast structure.

If the set of vertices which are not inside of boundary rectangles is non-meager then we can assume without loss of generality that the set Y of such vertices labeled 0 is non-meager. Thus there is some $y\in Y$ such that also $(3^m,0)\cdot y\in Y$ for some m. But since y and $(3^m,0)\cdot y$ are both not inside of the boundary of any rectangles and are in the same component of the grid, they must both be from the same checkerboard pattern. But they are at an odd distance, which contradicts the fact that they are both labeled 0.

Therefore we have reduced to the case that the boundary labels define a rectangular 1-toast. Because every rectangular boundary component is contained in a larger one, in every component there are arbitrarily long vertical line segments with the label "right". But we know that the set Y of vertices which are labeled 0 or 1 is non-meager. And so Z must contain a lattice of the form $A\cdot y$ for $A\subseteq\mathbb{Z}^2$ generated by some $(3^m,0)$ and $(1,k)$. But such a lattice Z meets every vertical line segment of sufficiently large length. This is a contradiction.

□

The preceding lemma completes the construction required in the proof of Theorem 20.

CHAPTER 5

Baire Measurable Matching in Non-Amenable Graphs

The results of this chapter are part of joint work with Alexander Kastner.

We say an infinite connected graph G of bounded degree is **non-amenable** if there exists $\delta > 0$ such that whenever $F \subseteq V(G)$ is finite, the set of edges $E(F, V(G) \setminus F)$ between F and $V(G) \setminus F$ satisfies $|E(F, V(G) \setminus F)| \geq \delta |F|$. For example, the Cayley graphs of finitely generated non-amenable groups with respect to any finite symmetric generating set (not containing the identity) are non-amenable graphs.

In this chapter, we consider non-amenable Borel graphs on Polish spaces, and prove theorems about the existence of perfect matchings:

Theorem 27 (Kastner-L.)**.** *Let G be a Borel graph such that each component is an infinite, bounded degree, non-amenable vertex transitive graph. Then G admits a Borel perfect matching on a Borel comeager invariant set.*

Corollary 28. *Every Schreier graph of a free Borel action of a finitely generated nonamenable group admits a Borel perfect matching on a Borel comeager invariant set.*

In [25], Marks and Unger studied Baire measurable matchings in the context of *bipartite* Borel graphs, with a view towards applications for Baire measurable equidecompositions. Though not explicitly stated in their paper, their Theorem 1.3 implies that every bipartite Borel graph whose components are bounded degree, regular, and non-amenable has a Baire measurable perfect matching. Thus, our theorem can be viewed as an extension of their result to the non-bipartite setting. The existence of regular, quasi-transitive, non-amenable graphs without perfect matchings, leads us to assume vertex transitivity, not just regularity.

5.1 Perfect matchings

The classical theorem of Tutte, repeated below, characterizes when a locally finite graph admits a perfect matching.

Theorem 29 (Tutte's theorem)**.** *A locally finite graph G admits a perfect matching if and only if whenever $X \subseteq V(G)$ is finite, the graph $G - X$ has at most $|X|$ many finite components of odd size.*

By **Tutte's condition** we will mean the condition that "$G - X$ has at most $|X|$ many odd components for each finite $X \subseteq V(G)$".

The proof of Theorem 27 consists in two steps. First, we establish a Baire measurable variant of Tutte's theorem which gives a sufficient condition for a locally finite Borel graph to admit a perfect matching on a Borel comeager invariant set (Theorem 31). Second, we show that nonamenable vertex transitive graphs satisfy this sufficient condition (Lemma 33).

Definition 30. If G is a locally finite graph and $X \subseteq G$ is finite, define

$$\mathcal{C}_{\text{fin}}(X) := \{\text{finite components of } G - X\}$$

and

$$\mathcal{C}_{\text{odd}}(X) := \{\text{odd components of } G - X\}.$$

Also let

$$\text{hull}_{\text{fin}}(X) := X \cup \bigcup \mathcal{C}_{\text{fin}}(X),$$

and

$$\text{hull}_{\text{odd}}(X) := X \cup \bigcup \mathcal{C}_{\text{odd}}(X).$$

We sometimes add superscripts to indicate the ambient graph when there is ambiguity.

Theorem 31 (Kastner-L.)**.** *Let G be a locally finite Borel graph on a Polish space $V(G)$, and suppose there exists $\varepsilon > 0$ such that for every finite set $X \subseteq V(G)$, we have*

$$|X| \geq |\mathcal{C}_{\text{odd}}(X)| + \varepsilon|\text{hull}_{\text{odd}}(X)|.$$

Then G admits a Borel perfect matching on a Borel comeager invariant set.

For the proof, we say a locally finite graph G satisfies $\text{Tutte}_{\varepsilon,k}$ if (i) Tutte's condition holds, and (ii) whenever $X \subseteq V(G)$ is finite such that $\text{hull}_{\text{odd}}(X)$ is connected and has size at least k,

$$|X| \geq |\mathcal{C}_{\text{odd}}(X)| + \varepsilon|\text{hull}_{\text{odd}}(X)|.$$

Observe that the condition in Theorem 31 is equivalent to $\text{Tutte}_{\varepsilon,1}$. This is an analogue of $\text{Hall}_{\varepsilon,k}$ in the proof of Theorem 1.3 in [25]. Our proof of Theorem 31 follows the same general strategy as the proof in [25]. In particular, we will need the following lemma from that paper.

Lemma 32. *Let G be a locally finite Borel graph on a Polish space $V(G)$, and let $f : \mathbb{N} \to \mathbb{N}$. Then there exist Borel sets $A_n \subseteq V(G)$, $n \in \mathbb{N}$, such that $\bigcup_n A_n$ is a Borel comeager invariant set and $d_G(x, y) > f(n)$ whenever x, y are distinct vertices in A_n.*

Proof of Theorem 31. Let $f : \mathbb{N} \to \mathbb{N}$ be a sufficiently fast-growing increasing function so that

1. $\sum_n \frac{4}{f(n)} < \varepsilon$;
2. letting $\varepsilon_n = \varepsilon - \sum_{m \leq n} \frac{4}{f(n)}$, we have $\varepsilon_{n-1} f(n) > 4$ for each n.

For convenience, we write $\varepsilon_{-1} = \varepsilon$. Let A_n be the Borel sets given by Lemma 32 for this f. Given a matching M, we write $G - M$ for the graph obtained from G by removing all the *vertices* covered by M (that is, $G - M$ is the induced subgraph on the set of vertices not covered by M). We define increasing Borel matchings M_n such that their union will be a perfect matching of the Borel comeager invariant set $\bigcup_n A_n$. We will ensure that M_n

covers the vertices in A_n and $G - M_n$ satisfies $\text{Tutte}_{\varepsilon_n, f(n)}$. We can take M_{-1} to be the empty matching, and the hypothesis of the theorem implies that $G - M_{-1}$ satisfies $\text{Tutte}_{\varepsilon_{-1},1}$.

Assume M_{n-1} has been defined. For each vertex $x \in A_n \cap V(G - M_{n-1})$, let e_x be the least edge not in M_{n-1} such that $(G - M_{n-1}) - e_x$ satisfies Tutte's condition, equivalently such that $(G - M_{n-1}) - e_x$ admits a perfect matching. We know such an edge exists as the hypothesis that $\text{Tutte}_{\varepsilon_{n-1}, f(n-1)}$ holds for $G - M_{n-1}$ implies in particular that Tutte's condition holds for $G - M_{n-1}$, hence $G - M_{n-1}$ has a perfect matching. If we pick an edge e_x that belongs to a perfect matching of $G - M_{n-1}$, then $(G - M_{n-1}) - e_x$ will still satisfy Tutte's condition. Since Tutte's condition quantifies over finite sets, the matching

$$M_n := M_{n-1} \cup \{e_x : x \in A_n \cap V(G - M_{n-1})\}$$

is Borel.

We verify that $G - M_n$ satisfies $\text{Tutte}_{\varepsilon_n, f(n)}$. As a first step, we show that $G - M_n$ has no odd component (this is verifying Tutte's condition for $X = \emptyset$). Assume for contradiction that C is an odd component of $G - M_n$, and let X' denote the set of endpoints of edges $e_x \in M_n - M_{n-1}$ such that e_x is adjacent to C. Since $G - M_{n-1}$ had no odd component, $X' \neq \emptyset$ and $\text{hull}_{\text{odd}}^{G-M_{n-1}}(X')$ must be connected.

<u>Case 1</u>: Suppose $|X'| \geq 4$, so that there are at least two distinct edges $e_x \in M_n - M_{n-1}$ that are adjacent to C. Since $C \cup X'$ is connected and the vertices in X' corresponding to distinct edges are a distance of at least $f(n)$ from one another, we have

$$|\text{hull}_{\text{odd}}^{G-M_{n-1}}(X')| = |C \cup X'| \geq \frac{|X'|}{2} \cdot \frac{f(n)}{2} \geq \frac{f(n)}{4}|X'|.$$

In particular, $|\text{hull}_{\text{odd}}^{G-M_{n-1}}(X')| \geq f(n) \geq f(n-1)$. So, applying the inductive assumption of $\text{Tutte}_{\varepsilon_{n-1}, f(n-1)}$ to $G - M_{n-1}$ and X', we obtain

$$|X'| \geq \varepsilon_{n-1}|\text{hull}_{\text{odd}}^{G-M_{n-1}}(X')| + |\mathcal{C}_{\text{odd}}^{G-M_{n-1}}(X')| \geq \varepsilon_{n-1}\frac{f(n)}{4}|X'|$$

Since f was chosen so that $\varepsilon_{n-1} f(n) > 4$, this is impossible.

Case 2: Suppose $|X'| = 2$, so that there is a single edge $e_x \in M_n - M_{n-1}$ that is adjacent to C. But this case is impossible as we chose e_x specifically so that $M_{n-1} \cup \{e_x\}$ extends to a perfect matching, so the appearance of the odd component C in $G - M_n$ cannot only be due to e_x.

So far we have proved that $G - M_n$ has no odd component. Let $X \subseteq V(G - M_n)$ be a finite set such that $\mathrm{hull}_{\mathrm{odd}}(X)$ is connected. Let E_X be the set of edges $e_x \in M_n - M_{n-1}$ such that at least one of the endpoints of e_x is adjacent to $\mathrm{hull}_{\mathrm{odd}}^{G-M_n}(X)$ in G.

Case 1: Suppose that $|E_X| \geq 2$. Since $\mathrm{hull}_{\mathrm{odd}}^{G-M_n}(X)$ is connected and distinct edges in E_x are a distance of at least $f(n)$ from one another, we have

$$|\mathrm{hull}_{\mathrm{odd}}^{G-M_n}(X)| \geq |E_x| \frac{f(n)}{2}.$$

In particular, $|\mathrm{hull}_{\mathrm{odd}}^{G-M_{n-1}}(X')| \geq f(n) \geq f(n-1)$. So, applying the inductive assumption of $\mathrm{Tutte}_{\varepsilon_{n-1}, f(n-1)}$ to $G - M_{n-1}$ and

$$X' = X \cup \{v \in V(G) : v \text{ is an endpoint of some } e \text{ in } E_X\},$$

yields

$$|X'| \geq |\mathcal{C}_{\mathrm{odd}}^{G-M_{n-1}}(X')| + \varepsilon_{n-1} |\mathrm{hull}_{\mathrm{odd}}^{G-M_{n-1}}(X')|.$$

Therefore

$$\begin{aligned} |\mathcal{C}_{\mathrm{odd}}^{G-M_n}(X)| &= |\mathcal{C}_{\mathrm{odd}}^{G-M_{n-1}}(X')| \\ &\leq |X'| - \varepsilon_{n-1} |\mathrm{hull}_{\mathrm{odd}}^{G-M_{n-1}}(X')| \\ &\leq |X'| - \varepsilon_{n-1} |\mathrm{hull}_{\mathrm{odd}}^{G-M_n}(X)| \\ &= |X| + 2|E_X| - \varepsilon_{n-1} |\mathrm{hull}_{\mathrm{odd}}^{G-M_n}(X)| \\ &\leq |X| + \frac{4}{f(n)} |\mathrm{hull}_{\mathrm{odd}}^{G-M_n}(X)| - \varepsilon_{n-1} |\mathrm{hull}_{\mathrm{odd}}^{G-M_n}(X)| \\ &= |X| - \varepsilon_n |\mathrm{hull}_{\mathrm{odd}}^{G-M_n}(X)|. \end{aligned}$$

Case 2: Suppose that $|E_X| \leq 1$. If E_X is empty, then the fact that X does not violate $\mathrm{Tutte}_{\varepsilon_n, f(n)}$ simply follows from the fact that $G - M_{n-1}$ satisfies the (stronger) $\mathrm{Tutte}_{\varepsilon_{n-1}, f(n-1)}$.

So suppose that E_X consists of a single edge e_x, for some $x \in A_n \cap V(G - M_{n-1})$. We chose e_x so that Tutte's condition holds for $(G - M_{n-1}) - e_x$, so in particular

$$|\mathcal{C}_{\text{odd}}^{G-M_{n-1}-e_x}(X)| \leq |X|.$$

But e_x is the only edge adjacent to $\text{hull}_{\text{odd}}^{G-M_n}(X)$ in $G - M_n$, so the odd components of $(G - M_{n-1} - e_x) - X$ are precisely the same as the odd components of $(G - M_n) - X$. Hence, X does not violate Tutte's condition in $G - M_n$. Suppose now that $|\text{hull}_{\text{odd}}^{G-M_n}(X)| \geq f(n) \geq f(n-1)$, and as in Case 1 let

$$X' = X \cup \{v \in V(G) : v \text{ is an endpoint of some } e \text{ in } E_X\}.$$

Applying $\text{Tutte}_{\varepsilon_{n-1}, f(n-1)}$ to $G - M_{n-1}$ and X' yields

$$\begin{aligned}
|\mathcal{C}_{\text{odd}}^{G-M_n}(X)| &= |\mathcal{C}_{\text{odd}}^{G-M_{n-1}}(X')| \\
&\leq |X'| - \varepsilon_{n-1}|\text{hull}_{\text{odd}}^{G-M_{n-1}}(X')| \\
&\leq |X| + 2 - \varepsilon_{n-1}|\text{hull}_{\text{odd}}^{G-M_n}(X)| \\
&\leq |X| + \frac{2}{f(n)}|\text{hull}_{\text{odd}}^{G-M_n}(X)| - \varepsilon_{n-1}|\text{hull}_{\text{odd}}^{G-M_n}(X)| \\
&\leq |X| + \varepsilon_n|\text{hull}_{\text{odd}}^{G-M_n}(X)|.
\end{aligned}$$

So X does not violate $\text{Tutte}_{\varepsilon_n, f(n)}$ in Case 2 either. □

Next, we show that non-amenable vertex transitive graphs satisfy the condition in Theorem 31.

Lemma 33. *Let G be an infinite, connected, locally finite, non-amenable, vertex transitive graph. Then there exists $\varepsilon > 0$ such that for all finite $X \subseteq V(G)$,*

$$|X| \geq |\mathcal{C}_{\text{fin}}(X)| + \varepsilon|\text{hull}_{\text{fin}}(X)|.$$

In particular, there exists $\varepsilon > 0$ such that for all finite $X \subseteq V(G)$,

$$|X| \geq |\mathcal{C}_{\text{odd}}(X)| + \varepsilon|\text{hull}_{\text{odd}}(X)|.$$

Proof. Fix a finite set $X \subseteq V(G)$. By Lemma 2.3 of [7], the assumption that G is a (connected, infinite) d-regular, vertex transitive graph implies that each element of $\mathcal{C}_{\text{fin}}(X)$ has at least d many edges in its boundary. And so

$$\left|E\left(X, \bigcup \mathcal{C}_{\text{fin}}(X)\right)\right| = \sum_{F \in \mathcal{C}_{\text{fin}}(X)} |E(X,F)| \geq d|\mathcal{C}_{\text{fin}}(X)|.$$

Also by the expansion property

$$\Big|E(X, V(G) \setminus \text{hull}_{\text{fin}}(X))\Big| \geq \delta|\text{hull}_{\text{fin}}(X)|,$$

where δ is the expansion constant of the graph. Therefore

$$d|X| \geq \left|E\left(X, \bigcup \mathcal{C}_{\text{fin}}(X)\right)\right| + \Big|E\Big(X, V(G) \setminus \text{hull}_{\text{fin}}(X)\Big)\Big| \geq d|\mathcal{C}_{\text{fin}}(X)| + \delta|\text{hull}_{\text{fin}}(X)|.$$

And so

$$|X| \geq |\mathcal{C}_{\text{fin}}(X)| + \varepsilon|\text{hull}_{\text{fin}}(X)|,$$

where $\varepsilon = \frac{\delta}{d}$. □

As discussed earlier, combining Theorem 31 and Lemma 33 immediately yields Theorem 27.

5.2 Comparison with Bipartite Matching Results

It is worth comparing Theorem 31 with the corresponding result in [25].

Theorem 34 (Marks-Unger)**.** *Let G be a locally finite bipartite Borel graph on a Polish space $V(G)$, and suppose there exists $\varepsilon > 0$ such that for every finite independent set $X \subseteq V(G)$, we have*

$$|N(X)| \geq (1+\varepsilon)|X|.$$

Then G admits a perfect matching on a Borel comeager invariant set.

Theorem 31 does not immediately imply Theorem 34. But there is still a connection that can be drawn between the two results. Marks and Unger remark that their method also proves the following one-sided matching theorem.

Theorem 35 (Marks-Unger)**.** *Let G be a locally finite Borel graph on a Polish space $V(G)$ with Borel bipartition $V(G) = P_0 \sqcup P_1$, and suppose there exists $\varepsilon > 0$ such that for every finite independent set $X \subseteq P_0$ we have*

$$|N(X)| \geq (1 + \varepsilon)|X|.$$

Then G admits a matching covering P_0.

Note that a Cantor-Schroder-Bernstein argument shows that Theorem 35 implies Theorem 34 in the case where the graph has a Borel bipartition.

We remark that the same method to prove Theorem 31 can be used to prove the following generalization.

Theorem 36. *Let G be a locally finite Borel graph on a Polish space $V(G)$, let $P \subseteq V(G)$ be Borel, and suppose there exists $\varepsilon > 0$ such that for every finite set $X \subseteq V(G)$, we have*

$$|X| \geq |\mathcal{C}^P_{\text{odd}}(X)| + \varepsilon|\text{hull}^P_{\text{odd}}(X)|.$$

Then G admits a Baire measurable matching that covers P.

The superscript P indicates that we only consider odd components that are entirely contained in P. Now Theorem 36 directly implies Theorem 35. To see this, let G be a locally finite Borel graph on a Polish space $V(G)$ with Borel bipartition $V(G) = P_0 \sqcup P_1$. And let ε be as in the hypothesis of Theorem 35. Let $\varepsilon' > 0$ be small enough that $\frac{1+\varepsilon'}{1-\varepsilon'} \leq 1 + \varepsilon$. We show that G satisfies the hypothesis of Theorem 36 for $P = P_0$ and for this value ε'. Let $X \subseteq V(G)$. For the purpose of verifying the inequality we may assume that $X \subseteq P_1$ since removing elements of P_0 only decreases the left side and increases the right side. Each odd component of the complement of X entirely contained in P_0 is a singleton. Let Y be the union of these singletons. Then $Y \subseteq P_0$ with $|\mathcal{C}^P_{\text{odd}}(X)| = |Y|$ and $|\text{hull}^P_{\text{odd}}(X)| = |X| + |Y|$. We therefore have by assumption $|X| = |N(Y)| \geq (1 + \varepsilon)|Y|$. Putting this all together we see that

$$
\begin{aligned}
|\mathcal{C}^P_{\text{odd}}(X)| + \varepsilon'|\text{hull}^P_{\text{odd}}(X)| &= |Y| + \varepsilon'(|X| + |Y|) \\
&= |Y|(1+\varepsilon') + \varepsilon'|X| \\
&\leq |X|\Big(\frac{1+\varepsilon'}{1+\varepsilon} + \varepsilon'\Big) \\
&\leq |X|
\end{aligned}
$$

which verifies the hypothesis of Theorem 36. And so G has a Baire measurable matching covering P_0. This completes the argument.

Therefore in some sense, the Baire measurable expansive Tutte results do generalize the Baire measurable expansive Hall results.

CHAPTER 6

Expander Mixing in PMP Graphs

In this chapter, we provide a proof of the expander mixing lemma for probability measure preserving (pmp) graphs. Then we use this along with a spectral gap computation to provide an example of an acyclic pmp graph equipped with an edge labeling which has nice expansion properties. The spectral gap computation is due to Kechris and Tsankov [20], using a result of Kesten [21]. The construction provides a proof of the following theorem of Grebík and Vidnyánszky [15], avoiding the theory of local-global limits.

Theorem 37. *Let $k \geq 1$ and $n \geq 3$. There exist disjoint Borel graphs $\mathcal{G}_j$ for $j < k$ on a probability measure space (Y, μ) such that*

1. *$\bigcup_{j<k} \mathcal{G}_j$ is acyclic and has bounded degree.*
2. *For every $j < k$ if $B, B' \subseteq Y$ are measurable and $\mu(B), \mu(B') \geq \frac{1}{n}$ then there exist $z \in B$ and $z' \in B'$ that are adjacent in $\mathcal{G}_j$.*

6.1 Expander Mixing Lemma

In order to define the spectral gap of a d-regular pmp graph, we introduce a bounded operator on the Hilbert space of square-integrable functions on the graph.

Definition 38. If $(X, \mathcal{G})$ is a probability measure preserving d-regular graph, then we define the adjacency operator $T : L^2(X) \to L^2(X)$ by

$$(Tf)(x) = \sum_{(x,y)\in\mathcal{G}} f(y).$$

We have $T \in \mathcal{B}(L^2(X))$ is a self adjoint bounded operator with $\|T\| = d$. Let

$$L_0^2(X) = \Big\{ f \in L^2(X) \Big| \int_X f d\mu = 0 \Big\}.$$

We say that $(X, \mathcal{G})$ has spectral gap ε if

$$\Big\| T \restriction_{L_0^2(X)} \Big\| = d - \varepsilon.$$

The following is the expander mixing lemma for d-regular pmp graphs. The proof is nearly identical to the proof for finite graphs but with vertex and edge counting replaced by integration.

Theorem 39. *Let $(X, \mathcal{G})$ be a d-regular pmp graph which has spectral gap ε. For any measurable subsets $B, B' \subseteq X$ we have*

$$\Big| |\mathcal{G}(B, B')| - d \cdot \mu(B)\mu(B') \Big| \leq (d - \varepsilon)\sqrt{\mu(B)(1 - \mu(B))} \cdot \sqrt{\mu(B')(1 - \mu(B'))}$$

where we are using the edge measure

$$|\mathcal{G}(B, B')| = \int_B \Big| \{ y \in X | (x, y) \in \mathcal{G} \} \Big| d\mu(x).$$

Proof. Let $B, B' \subseteq X$. First we observe that

$$|\mathcal{G}(B, B')| = \langle 1_B, T 1_{B'} \rangle$$

is the inner product on $L^2(X)$. We can decompose the indicator functions 1_B and $1_{B'}$ as a constant part and an orthogonal part in $L_0^2(X)$. Then

$$\begin{aligned}
\langle 1_B, T 1_{B'} \rangle &= \langle \mu(B) 1_X + 1_B - \mu(B) 1_X, T(\mu(B') 1_X + 1_{B'} - \mu(B') 1_X) \rangle \\
&= \langle \mu(B) 1_X, T(\mu(B') 1_X) \rangle + \langle 1_B - \mu(B) 1_X, T(1_{B'} - \mu(B') 1_X) \rangle \\
&= d \cdot \mu(B)\mu(B') + \langle 1_B - \mu(B) 1_X, T(1_{B'} - \mu(B') 1_X) \rangle.
\end{aligned}$$

Also by Cauchy-Schwarz

$$|\langle 1_B - \mu(B) 1_X, T(1_{B'} - \mu(B') 1_X) \rangle| \leq (d - \varepsilon) \|1_B - \mu(B) 1_X\|_2 \|1_{B'} - \mu(B') 1_X\|_2$$

and we can compute

$$\|1_B - \mu(B)1_X\|_2^2 = \mu(B) - \mu(B)^2 \quad \text{and} \quad \|1_{B'} - \mu(B')1_X\|_2^2 = \mu(B') - \mu(B')^2$$

which gives the result. □

We can also define the expansion constant of a d-regular pmp graph and connect this with the spectral gap.

Definition 40. Define the expansion constant of $(X, \mathcal{G})$ to be

$$\Phi_X = \inf\left\{\frac{|\mathcal{G}(B, X \setminus B)|}{\mu(B)(1-\mu(B))}\middle| B \subseteq X, 0 < \mu(B) < 1\right\}.$$

Theorem 41. *If $(X, \mathcal{G})$ has spectral gap ε then $\Phi_X \geq \varepsilon$.*

Proof. For $B \subseteq X$ we apply the expander mixing lemma to B and $X \setminus B$ and have

$$\begin{aligned}\Big||\mathcal{G}(B, X \setminus B)| - d \cdot \mu(B)(1-\mu(B))\Big| &\leq (d-\varepsilon)\sqrt{\mu(B)(1-\mu(B))}\sqrt{(1-\mu(B))\mu(B)} \\ &= (d-\varepsilon)\mu(B)(1-\mu(B))\end{aligned}$$

and so

$$\frac{|\mathcal{G}(B, X \setminus B)|}{\mu(B)(1-\mu(B))} \geq d - (d-\varepsilon) = \varepsilon.$$

□

6.2 Spectral Decomposition

In order to prove Theorem 37 we will analyze spectral properties of the left shift action of a countable group Γ on $[0,1]^\Gamma$ given by $(\gamma f)(x) = f(\gamma^{-1}x)$. This induces a unitary representation of Γ on $L^2([0,1]^\Gamma)$ given by $(\gamma F)(f) = F(\gamma^{-1}f)$.

Kechris and Tsankov [20] analyze this representation of Γ to compute the spectral gap of the Schreier graph. We repeat their argument for the sake of completeness. Let

$$S = \{f : \Gamma \to \mathbb{Z} \mid f(\gamma) = 0 \text{ for all but finitely many } \gamma\}$$

and for $r \in S$ we consider $H_r \in L^2([0,1]^\Gamma)$

$$H_r(f) = \prod_{\gamma \in \Gamma} e^{2\pi i r(\gamma) f(\gamma)}.$$

Then H_r are orthonormal and closed under products and complex conjugation. The Stone-Weierstrass theorem implies that $\mathcal{B} = \{H_r | r \in S\}$ forms a Hilbert basis for $L^2([0,1])$.

Also note that Γ preserves $\mathcal{B}$. There is one vector H_0 in $\mathcal{B}$ which is fixed by all elements of Γ, and every other vector in $\mathcal{B}$ has finite stabilizer (with some elements having trivial stabilizer). Together this shows an isomorphism of Γ-representations

$$L^2([0,1]^\Gamma) \cong \ell^2(\mathcal{B}) \cong \mathbb{C} \oplus \bigoplus_{i \in I} \ell^2(\Gamma/\Delta_i)$$

where I is a countable set, each Δ_i is a finite subgroup of Γ, and H_i is trivial for at least one i. The one dimensional space $\mathbb{C}$ corresponds to the constant functions in $L^2_0(X)$. In the orthogonal part we have an isomorphism

$$L^2_0(X) \cong \bigoplus_{i \in I} \ell^2(\Gamma/\Delta_i).$$

It follows that for any $a \in \mathbb{C}[\Gamma]$ we have

$$\|a\|_{L^2_0([0,1]^\Gamma)} = \sup_{i \in I} \|a\|_{\ell^2(\Gamma/H_i)} = \|a\|_{\ell^2(\Gamma)}.$$

6.3 Spectral Gap in Free Group Actions

Now consider the free group $\mathbb{F}_{kd}$ with standard generators $\{\gamma_0, \ldots, \gamma_{kd-1}\}$. Let

$$T = \sum_{i=0}^{d-1} \gamma_i + \gamma_i^{-1} \in \mathbb{C}[\mathbb{F}_{kd}].$$

We compute $\|T\|_{\ell^2(\mathbb{F}_{kd})}$. Let $\mathbb{F}_d \leq \mathbb{F}_{kd}$ be the subgroup generated by $\{\gamma_0, \ldots, \gamma_{d-1}\}$. Then $T \in \mathbb{C}[\mathbb{F}_d]$ and $\|T\|_{\ell^2(\mathbb{F}_{kd})} = \|T\|_{\ell^2(\mathbb{F}_d)}$ because

$$\ell^2(\mathbb{F}_{kd}) \cong \ell^2(\mathbb{F}_d)^{\oplus k}$$

as $\mathbb{F}_d$-representations. Kesten [21] computed

$$\|T\|_{\ell^2(\mathbb{F}_d)} = 2\sqrt{2d-1}.$$

We can now prove Theorem 37.

Proof of Theorem 37. Fix $k \geq 1$ and $n \geq 3$. Choose $d \in \mathbb{N}$ large enough that $\frac{d}{\sqrt{2d-1}} > n$. Let $Y = [0,1]^{\mathbb{F}_{dk}}$ with the product measure and left shift action of $\mathbb{F}_{kd}$. For $j < k$ define

$$\mathcal{G}_j = \{(z, z') \in Y \times Y \mid \text{for some } 0 \leq i < d \text{ either } y' = \gamma_{jd+i} y \text{ or } y' = \gamma_{jd+i}^{-1} y\}.$$

Then $\bigcup_{j<k} \mathcal{G}_j$ is the Schreier graph of the action of $\mathbb{F}_{dk}$ as so is acyclic of degree $2dk$ (mod null).

For $j < k$ the adjacency operator for $\mathcal{G}_j$ is

$$T_j = \sum_{i=0}^{d-1} \gamma_{jd+i} + \gamma_{jd+i}^{-1}.$$

And we have computed

$$\|T_j\|_{L^2_0(Y)} = 2\sqrt{2d-1}$$

so $\mathcal{G}_j$ is regular of degree $2d$ with spectral gap $2d - 2\sqrt{2d-1}$. Fix j and let $B, B' \subseteq Y$ be measurable with $\mu(B), \mu(B') \geq \frac{1}{n}$. The expander mixing lemma for $\mathcal{G}_j$ implies

$$\begin{aligned}
|\mathcal{G}_j(B, B')| &\geq 2d \cdot \mu(B)\mu(B') - 2\sqrt{2d-1}\sqrt{\mu(B)(1-\mu(B))}\sqrt{\mu(B')(1-\mu(B'))} \\
&= 2\sqrt{\mu(B)\mu(B')}\Big(d\sqrt{\mu(B)\mu(B')} - \sqrt{2d-1}\sqrt{(1-\mu(B))(1-\mu(B'))}\Big) \\
&\geq 2\sqrt{\mu(B)\mu(B')}\Big(\frac{d}{n} - \sqrt{2d-1}\Big) > 0.
\end{aligned}$$

This shows that there exist $z \in B$ and $z' \in B'$ which are adjacent in $\mathcal{G}_j$. $\square$

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