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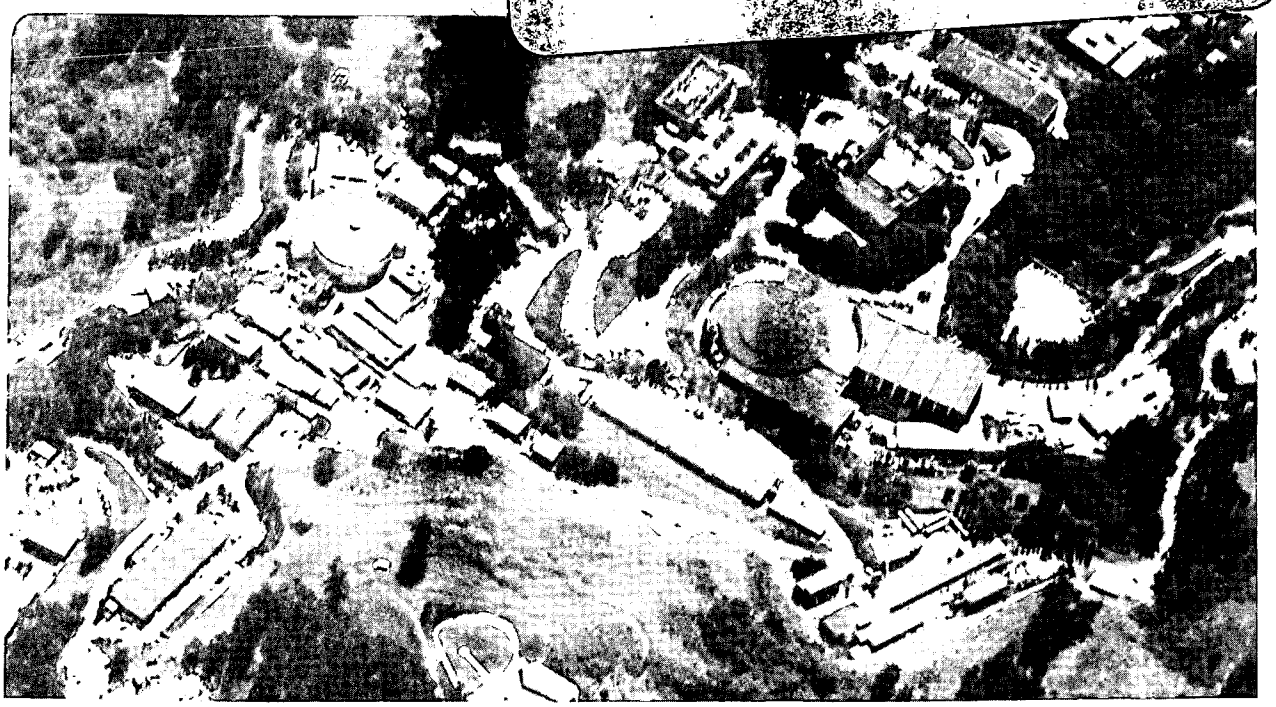

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THE SINE-GORDON AND SINH-GORDON GAUSSIAN EFFECTIVE POTENTIAL IN D + 1 DIMENSIONS *

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ABSTRACT

We compute the Gaussian approximation to the effective potential for the sine-Gordon and the sinh-Gordon model in D + 1 dimensions. Issues such as vacuum energy, renormalization and stability of the vacuum are discussed in detail. Within the Gaussian approximation, we show that 1) for $D \geq 3$, the model exists only as a free-field theory, and 2) for $D < 3$, the vacuum is unstable over a certain range of the coupling constant. We find a particularly elegant expression for the Gaussian effective potential for the special case $D = 1$.

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1 Introduction

The "Gaussian approximation" is a non-perturbative tool for estimating the effective potential in quantum field theories. This approximation has been used for over twenty years, under a variety of names. Its full value was finally seen clearly in the analysis of the $O(N)$ vector model by Bardeen and Moshe [1]. For a recent review, with many interesting examples, see [2].

The purpose of this paper is to provide a concrete example of a model where the Gaussian approximation works out very cleanly. The model we will examine is the generalized sinh-Gordon model in $D+1$ dimensions. The Lagrangian for this model is

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{\gamma^2}[\cosh(\gamma\phi) - 1]. \tag{1.1}$$

Here, m and γ are the bare mass and bare coupling constant, respectively. If $\gamma^2 > 0$, the potential is a cosh curve, with a single minimum at the origin; if $\gamma^2 < 0$, the potential is the familiar cosine curve of the sine-Gordon model, with an infinite number of degenerate minima. The limiting case $\gamma^2 \to 0$ will be understood to be a free theory of mass m. In this paper, we will assume that $m \neq 0$ and we will not consider the possible generalizations of the model in which γ^2 is not real.

When $D = 1$, the sine-Gordon model is particularly interesting, partly because it is equivalent to a number of other models and partly because it has been solved exactly.

The model is equivalent to the massive Thirring model [3, 4], to the two-dimensional coulomb gas [5], to the continuum limit of the lattice $x - y - z$ spin-$\frac{1}{2}$ model [6] and to the massive $O(2)$ non-linear σ-model [5]. (More generally, we might add that the sine-Gordon model in $D+1$ spacetime dimensions is equivalent to the coulomb gas model in $D+1$ spatial dimensions [5].)

To discuss the exact solution of the $D = 1$ sine-Gordon model, it is convenient to define $\beta^2 \equiv -\gamma^2$. For $\beta^2 > 0$, the classical model has static soliton solutions, as well as time-dependent bound states of soliton-antisoliton pairs. The exact quantum spectrum can be computed, using the WKB approximation [7]. Although it is not obvious that WKB should give exact results, this seems to be the case; confirmation comes via S-matrix methods [8] or by the quantum inverse scattering method [9]. One can also map exact

results from equivalent models to the sine-Gordon model [6, 10].

A major surprise is that the model does not exist for $\beta^2 > 8\pi$. Coleman was the first to discover this, using a variational argument [3]. He found that the energy of trial vacuum states is unbounded below when $\beta^2 > 8\pi$.

This result can be translated to the equivalent models. The Thirring model does not exist, if the coupling is positive and is too large. In the coulomb gas, the pathology shows up as a phase transition from a plasma to a dipole gas, as the temperature falls below some critical temperature. In the $x-y-z$ model, the continuum limit of the lattice theory does not exist when a certain parameter becomes too large.

Much more can be said about the $D = 1$ sine-Gordon model and its cousins [3–10]. However, the above comments are all that is directly relevant to this paper, in which we are interested in deriving an approximate expression for the effective potential. As a by-product, we will also explicitly derive the constraint $\beta^2 < 8\pi$.

It is natural to ask whether an analogous constraint occurs in arbitrary dimensions. Here, a technical point intrudes. γ carries a mass dimension of $\frac{1}{2}(1-D)$; for $D > 1$, the model is perturbatively non-renormalizable, per the usual power-counting rules. Curiously, this problem is swept under the rug in the Gaussian approximation, a fact which will be discussed in the conclusion to this paper. In our calculation, γ will be renormalized by only a finite amount, so we are free to use the bare value or the renormalized value, whichever is more convenient. Looking ahead, it turns out that the bare value is more convenient. On the other hand, the bare mass m undergoes an infinite multiplicative renormalization, to be replaced by the finite parameter m_R. When this is done, we can then consider the question posed above. One of the main results of this paper is that the vacuum is unstable if the bare coupling satisfies the inequality

$$\beta^2 = -\gamma^2 > \frac{2^{D+2}\pi^{(D+1)/2}}{\Gamma\left(\frac{3-D}{2}\right)} m_R^{1-D} \qquad (\text{when } D < 3). \tag{1.2}$$

For $D = 1$, this reduces to Coleman's inequality.

The caveat in (1.2) alludes to another problem that should be expected to arise when $D \geq 3$. It is widely believed that the ϕ^4 theory in 4 or more dimensions is consistent only as a free-field theory, due to quantum effects (see, *e.g.*, [11]). We shall see that this is also true of our model, at least within the Gaussian approximation.

2 Computations

We begin by rewriting the Lagrangian (1.1) as

$$\mathcal{L} = \tfrac{1}{2}\dot{\phi}^2 - V(\phi)$$

where

$$V(\phi) \equiv \tfrac{1}{2}(\partial_i\phi)^2 + \frac{m^2}{\gamma^2}[\cosh(\gamma\phi) - 1].$$

From here on, $\phi(\vec{x})$ will be denoted by ϕ_x. The canonical momentum conjugate to ϕ_x is just

$$\Pi_x = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_x} = \dot{\phi}_x$$

and the Hamiltonian is

$$H = \int d^D\vec{x}\left[\tfrac{1}{2}\Pi_x^2 + V(\phi_x)\right].$$

To quantize the model, we implement the usual commutation relation

$$[\phi_x, \Pi_y] = i\delta^D_{xy}$$

via a functional derivative

$$\Pi_x = -i\frac{\delta}{\delta\phi_x}.$$

So much for the Hamiltonian. A more difficult question is: what state is the system in? The full solution to this problem would require solving the functional Schrödinger equation

$$H\Psi = E\Psi \tag{2.1}$$

where Ψ is a wave-functional of the variable ϕ_x, representing one of the energy eigenstates of the system. This is far beyond our present abilities. However, in the limit $\gamma \to 0$, (*i.e.*, the free-field theory), it is standard lore that the solution to (2.1) is given by

$$\Psi[\phi] = N\exp\left[-\tfrac{1}{2}\int_x \phi_x\sqrt{-\nabla_x^2 + m^2}\,\phi_x\right], \tag{2.2}$$

where N is a normalization constant.

Given the obvious difficulty of solving (2.1), it is reasonable to try a variational approach. Thus, we should choose some general ansatz for Ψ, compute the expectation value of the Hamiltonian, and minimize the energy by varying parameters in the ansatz. The problem is that the computation of

the expectation value requires us to perform a path integral over $d\phi_x$. Since the only path integrals we can do are Gaussian, this puts a severe constraint on the form of our ansatz for Ψ. Note, however, that (2.2) *is* Gaussian. It is tempting to take as an ansatz the most general Gaussian wave-functional

$$\Psi[\phi;\Phi,\mathcal{P},f] = N_f \exp\left[i\int_x \mathcal{P}_x\phi_x - \tfrac{1}{2}\int_{x,y}(\phi_x-\Phi_x)f_{xy}(\phi_y-\Phi_y)\right]. \qquad (2.3)$$

Note the following:

1. N_f is f_{xy}-dependent. It is chosen such that

$$\langle\Psi|\Psi\rangle = 1.$$

2. It is trivial to show that

$$\langle\Psi|\,\Pi_x\,|\Psi\rangle = \mathcal{P}_x.$$

3. Regardless of the particular function f_{xy},

$$\langle\Psi|\,\phi_x\,|\Psi\rangle = \Phi_x.$$

The variational "parameters" are therefore the functions $\mathcal{P}_x$, Φ_x and f_{xy}.

Some general comments are appropriate here. The state described by the wave-functional Ψ turns out to be a generalization of the "coherent states", *i.e.*, eigenstates of the usual destruction operators of the second quantized theory. Coherent states are simply the special case of (2.3) in which f_{xy} is replaced by $\sqrt{-\nabla_x^2+\mathcal{M}^2}\delta^D_{xy}$, where $\mathcal{M}$ is some mass parameter. It was in terms of coherent trial states that Coleman performed a variational analysis of the soliton sector of the $D=1$ sine-Gordon model [3].

Actually, Coleman's calculation was done in a different notation, employing creation and destruction operators. For the purposes of this paper (computation of the effective potential), the choice of notation is irrelevant; calculations are just as easy in one scheme as in another. Our choice of notation is guided by the following consideration: when one combines the Gaussian method with the collective coordinate method, so as to adequately approximate the physics of solitons (with their incumbent zero modes), computations seem to be simpler in the wave-functional notation given above. We will deal with solitons in a forthcoming paper.

We can now begin the calculation. Using standard path integral methods, we find the energy of the variational state to be

$$E[\Phi,\mathcal{P},f] \equiv \langle\Psi|\,H\,|\Psi\rangle$$
$$= \int_x\left[\tfrac{1}{2}\mathcal{P}_x^2 + \tfrac{1}{2}(\partial_i\Phi_x)^2 + \frac{m^2}{\gamma^2}\left[Z_x\cosh(\gamma\Phi_x)-1\right] + \tfrac{1}{4}\left[f_{xx} - \int_y \delta_{xy}\nabla_x^2 f^{-1}_{xy}\right]\right]$$

where

$$Z_x \equiv \exp\left[\tfrac{1}{4}\gamma^2 f^{-1}_{xx}\right].$$

We are interested in finding the effective potential, the energy of the state with constant classical field Φ, so we set $\partial_i\Phi_x = 0$. The minimum energy configuration clearly satisfies the constraint $\mathcal{P}_x = 0$. All that remains is to determine the function f_{xy} which minimizes E; we expect that f_{xy} will have some Φ-dependence.

Because of the symmetric form of (2.3), we can assume that $f_{xy} = f_{yx}$. It is not hard to verify that this fixes the solution to the variational equation

$$\frac{\delta E}{\delta f_{xy}} = 0 \qquad (2.4)$$

to be of the general form

$$f_{xy} = \int \frac{d^D\vec{p}}{(2\pi)^D}\cos[\vec{p}\cdot(\vec{x}-\vec{y})]\sqrt{\vec{p}^2+\mathcal{M}^2}$$

which is trivial to invert

$$f^{-1}_{xy} = \int \frac{d^D\vec{p}}{(2\pi)^D}\frac{\cos[\vec{p}\cdot(\vec{x}-\vec{y})]}{\sqrt{\vec{p}^2+\mathcal{M}^2}}.$$

$\mathcal{M}$ is some constant which is uniquely determined by (2.4). Computationally, the easiest way to determine $\mathcal{M}$ is to substitute the above general form for f_{xy} into the energy functional. $\mathcal{M}$ is then obtained by solving a one-dimensional variational equation, rather than by solving (2.4).

To simplify the notation, we will define the integrals

$$I_n^D(\mathcal{M}^2) \equiv \int \frac{d^D\vec{p}}{(2\pi)^D}\frac{\sqrt{\vec{p}^2+\mathcal{M}^2}}{(\vec{p}^2+\mathcal{M}^2)^n}.$$

Note that

$$f_{xx} = I_0(\mathcal{M}^2)$$
$$f^{-1}_{xx} = I_1(\mathcal{M}^2).$$

(We will suppress the superscript D where it is not needed.) The I_n^D's may be divergent or finite, depending on n and D. We will regard them as

formal quantities which can be differentiated with respect to M^2. One can imagine regulating them with a suitable cutoff procedure, but this is not really necessary [2]. The only property needed to renormalize the model is the formal differentiability property:

$$\frac{\partial I_0}{\partial(M^2)} = \tfrac{1}{2} I_1(M^2)$$
$$\frac{\partial I_1}{\partial(M^2)} = -\tfrac{1}{2} I_2(M^2), \qquad \text{etc.}$$

With this notation, we can write the energy density $\mathcal{E}$ as

$$\mathcal{E}(\Phi, M^2) = \tfrac{1}{2} I_0(M^2) - \tfrac{1}{4} M^2 I_1(M^2) + \frac{m^2}{\gamma^2}\left[Z_m(M^2)\cosh(\gamma\Phi) - 1\right]$$

where

$$Z_m(M^2) \equiv \exp\left[\tfrac{1}{4}\gamma^2 I_1(M^2)\right].$$

At this stage, we can find an implicit expression for the unrenormalized effective potential by the following two-step procedure:

1. Define $\mu(\Phi)$ to be the value of M which minimizes $\mathcal{E}$:

$$\left.\frac{\partial \mathcal{E}}{\partial(M^2)}\right|_{M^2=\mu^2(\Phi)} = 0. \tag{2.5}$$

2. Define the effective potential

$$\mathcal{V}(\Phi) \equiv \mathcal{E}(\Phi, \mu^2(\Phi)).$$

This is problematic; it can fail in two ways:

1. If $\mu^2(\Phi) < 0$, then the various integrals $I_n(\mu^2)$ in $\mathcal{E}$ will be complex.

2. If

$$\left.\frac{\partial^2 \mathcal{E}}{\partial(M^2)^2}\right|_{M^2=\mu^2(\Phi)} < 0,$$

then $\mu^2(\Phi)$ would be a local *maximum* of $\mathcal{E}$, not a local minimum.

It is simple to do the variation (2.5). The result is

$$\frac{\partial \mathcal{E}}{\partial(M^2)} = \tfrac{1}{8} I_2(M^2)\left[M^2 - m^2 Z_m(M^2)\cosh(\gamma\Phi)\right]$$

so that

$$\mu^2(\Phi) = m^2 Z_m(\mu^2)\cosh(\gamma\Phi). \tag{2.6}$$

Clearly, $\mu^2 > 0$ if and only if $\cosh(\gamma\Phi) > 0$. If $\gamma^2 > 0$, then $\mu^2(\Phi)$ is a positive function.

But consider the case $\gamma^2 = -\beta^2 < 0$. Then the classical potential is periodic in ϕ, so with no loss of generality, we can restrict $\beta\Phi$ to be in the range $[-\pi, \pi]$. Note that μ^2 is positive only when $\cos(\beta\Phi) > 0$. Thus, we are able to define the Gaussian effective potential only for $\beta\Phi$ in the range $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$. This range of definition is good enough for our purposes. Primarily, we want to renormalize the model, which requires knowledge of the effective potential only in some neighborhood of $\Phi = 0$. (Presumably, the effective potential exists outside this range, but the true wave-functional is far from being a Gaussian.)

Note that $\mu(\Phi)$ is defined implicitly in terms of itself in (2.6), so it is conceivable that (2.6) has no solution. We will defer this question until after renormalization. Assuming a solution *does* exist, we compute

$$\left.\frac{\partial^2 \mathcal{E}}{\partial(M^2)^2}\right|_{M^2=\mu^2(\Phi)} = \tfrac{1}{8} I_2(\mu^2)\left[1 + \frac{\mu^2\gamma^2 I_2(\mu^2)}{8}\right]. \tag{2.7}$$

This equation will prove useful later on, when we are ready to discuss the stability of the vacuum.

Now we are ready to renormalize the theory. We have a function $\mathcal{V}(\Phi)$ which represents the energy of the state in which $\langle\phi\rangle = \Phi$. Let $\overline{\Phi}$ be the value of Φ which minimizes $\mathcal{V}$:

$$\left.\frac{\partial \mathcal{V}}{\partial \Phi}\right|_{\Phi=\overline{\Phi}} = 0. \tag{2.8}$$

Using the notation $\overline{f(\Phi)} \equiv f(\overline{\Phi})$, we can define our renormalization prescription as

$$m_R^2 \equiv \overline{\frac{\partial^2 \mathcal{V}}{\partial \Phi^2}} \qquad \text{and} \qquad m_R^2\gamma_R^2 \equiv \overline{\frac{\partial^4 \mathcal{V}}{\partial \Phi^4}}$$

in analogy with the corresponding classical definitions.

First, note that (2.6) allows us to write $\mathcal{V}(\Phi)$ with no explicit Φ-dependence

$$\mathcal{V} = \tfrac{1}{2} I_0(\mu^2) - \tfrac{1}{4}\mu^2 I_1(\mu^2) + \frac{\mu^2 - m^2}{\gamma^2}.$$

The Φ-dependence enters only through the dependence on the function $\mu^2(\Phi)$. It is straightforward to check that

$$\begin{aligned} \frac{\partial \mathcal{V}}{\partial(\mu^2)} &= \frac{1}{\gamma^2}\left(1+\frac{\mu^2\gamma^2 I_2(\mu^2)}{8}\right) \\ \frac{\partial \mu^2}{\partial \Phi} &= \gamma\mu^2\tanh(\gamma\Phi)\left(1+\frac{\mu^2\gamma^2 I_2(\mu^2)}{8}\right)^{-1}, \end{aligned} \tag{2.9}$$

and so

$$\frac{\partial \mathcal{V}}{\partial \Phi} = \frac{\mu^2}{\gamma}\tanh(\gamma\Phi).$$

Comparing this with (2.8), we see that $\overline{\Phi}=0$, in agreement with our classical expectations.

The mass renormalization turns out to be simple:

$$m_R^2 = \overline{\frac{\partial^2 \mathcal{V}}{\partial \Phi^2}} = \overline{\mu^2} = m^2\overline{Z_m(\mu^2)} = m^2\exp\left[\tfrac{1}{4}\gamma^2 I_1(m_R^2)\right]. \tag{2.10}$$

This equation formally determines the bare mass m in terms of m_R, γ and the divergent integral $I_1(m_R^2)$.

[This is the best place to insert a long technical note. For $D \geq 3$, observe that $I_2^D(\mu^2)$ is divergent. From (2.9), we see that $\frac{\partial \mu^2}{\partial \Phi}$ then vanishes. So $\mu^2(\Phi)$ is a constant function,

$$\mu^2(\Phi) = \mu^2(\overline{\Phi}) = m_R^2.$$

But it is easy to combine (2.6) and (2.10) to find

$$\mu^2(\Phi) = m_R^2\cosh(\gamma\Phi)\frac{Z_m(\mu^2)}{Z_m(m_R^2)}.$$

The above two equations can be simultaneously true only if $\gamma = 0$, the free-field case. This proves the claim made in the Introduction that the interacting theory is inconsistent for $D \geq 3$.]

Restricting attention now to $D < 3$, we can rescale all the quantities in the problem so that m_R is the only dimensionful constant:

$$\begin{aligned} \tilde{\mu} &\equiv \frac{\mu}{m_R} \\ \tilde{\gamma} &\equiv \gamma m_R^{(D-1)/2} \\ \tilde{\Phi} &\equiv \Phi m_R^{-(D-1)/2} \\ \tilde{I}_2^D &\equiv \frac{I_2^D(\mu^2)}{\mu^{D-3}}. \end{aligned}$$

Note that $\tilde{I}_2^D$ is a finite parameter depending only on D. It is very handy to define the dimensionless parameter

$$\theta \equiv \tfrac{1}{8}\tilde{\gamma}^2\tilde{I}_2.$$

After a fair amount of algebra, we can compute the renormalized coupling

$$\gamma_R^2 = \gamma^2\left(\frac{1-2\theta}{1+\theta}\right). \tag{2.11}$$

Except for the degenerate case in which $\theta = -1$, γ_R is related to γ by a finite relation. This proves the other claim made in the Introduction, that the coupling constant renormalization is finite. We could easily solve the quadratic equation (2.11) to obtain γ^2 explicitly in terms of γ_R^2, but this would serve no purpose. Both quantities are finite and our equation for the effective potential is simpler when expressed in terms of the bare coupling γ.

There is very little analysis left to be done. In terms of the rescaled quantities, we can rewrite (2.9) as

$$\begin{aligned} \frac{\partial \mathcal{V}}{\partial(\tilde{\mu}^2)} &= \frac{m_R^{D+1}}{\tilde{\gamma}^2}(1+\theta\tilde{\mu}^{D-1}) \\ \frac{\partial \tilde{\mu}^2}{\partial \tilde{\Phi}} &= \frac{\tilde{\gamma}\tilde{\mu}^2\tanh(\tilde{\gamma}\tilde{\Phi})}{1+\theta\tilde{\mu}^{D-1}}. \end{aligned}$$

These are easily integrated:

$$\mathcal{V}(\tilde{\mu}^2) = \mathcal{V}(1) + \frac{m_R^{D+1}}{\tilde{\gamma}^2}\left[(\tilde{\mu}^2-1)+\frac{2\theta(\tilde{\mu}^{D+1}-1)}{D+1}\right] \tag{2.12a}$$

$$\cosh(\tilde{\gamma}\tilde{\Phi}) = \begin{cases} \tilde{\mu}^{2(1+\theta)} & \text{if } D=1 \\ \tilde{\mu}^2\exp\left[2\theta(\tilde{\mu}^{D-1}-1)/(D-1)\right] & \text{if } D\neq 1\,. \end{cases} \tag{2.12b}$$

(2.12b) defines implicitly the function $\tilde{\mu}(\tilde{\Phi})$. Substitution of this function into (2.12a) gives the effective potential $\mathcal{V}(\tilde{\Phi})$, a procedure that generally must be done numerically. (Note that $\mathcal{V}(1)$ is just the vacuum energy, an infinite constant which would disappear if we took the trouble to normal-order the Hamiltonian.)

For the special case $D = 1$, (2.12) can be combined into a single simple formula

$$\mathcal{V}(\Phi) = \mathcal{V}_{vac} + \frac{m_R^2}{\gamma^2}(1+\theta)\left[\cosh^{1/(1+\theta)}(\gamma\Phi)-1\right]. \tag{2.13}$$

The nature of this effective potential changes quite drastically as θ decreases below -1. The meaning of this can be guessed when we compute θ explicitly

for this case:

$$\theta = \frac{\gamma^2}{8\pi} \qquad (\text{ when } D = 1) .$$

Setting $\gamma = i\beta$, the transition point $\theta = -1$ becomes $\beta^2 = 8\pi$. Apparently, we have rediscovered Coleman's phase transition in a different guise.

The above paragraph reminds us that we have left a pair of potential difficulties unresolved.

One of these is the stability problem. The "vacuum" will turn out to be at a maximum in the variational calculation if

$$\left.\frac{\partial^2 \mathcal{E}}{\partial (M^2)^2}\right|_{M^2=\mu^2(\Phi)} < 0 \quad .$$

But (2.7) tells us that this is equivalent to

$$1 + \theta < 0.$$

Thus, we find a phase transition from a stable to an unstable vacuum at $\theta = -1$. This is the underlying cause of the transition that we observed above for the special case $D = 1$.

The other problem we have to face is the issue of whether (2.6) actually *has* a solution. It is not hard to see that, for fixed Φ, (2.6) has a solution if and only if (2.12b) is soluble. This problem must be considered separately for each of the cases $D < 1$, $D = 1$ and $1 < D < 3$. Each of these must be further subdivided into the cases $\theta > 0$ and $-1 < \theta < 0$. (2.12b) can be solved graphically. The results are summarized as follows:

1. For $\theta > 0$ and for all $D < 3$, (2.12b) has a unique solution for all Φ.

2. For $-1 < \theta < 0$ and $D \neq 1$, (2.12b) has two solutions for all values of Φ sufficiently close to the point $\Phi = 0$. One of these solutions can be discarded because it gives a higher energy in (2.12a) than the other solution does.

3. For $-1 < \theta < 0$ and $D = 1$, (2.12b) has a unique solution for all Φ.

Thus, all difficulties in defining the vacuum are resolved provided that $\theta > -1$. θ depends on the dimension D through the family of integrals

$$\begin{aligned} \tilde{I}_2^D &= \int \frac{d^D\vec{p}}{(2\pi)^D} \frac{1}{(\vec{p}^2+1)^{3/2}} \\ &= \frac{\Gamma(\frac{3-D}{2})}{2^{D-1}\pi^{(D+1)/2}}. \end{aligned}$$

We can therefore rewrite the stability requirement $\theta > -1$ as

$$\tilde{\gamma}^2 > \frac{-2^{D+2}\pi^{(D+1)/2}}{\Gamma(\frac{3-D}{2})}, \tag{2.14}$$

which is the rescaled version of the result we quoted in (1.2).

Some special cases of (2.14) are of interest. When $D = 1$, the right hand side is -8π. When $D = 2$, it is -16π. When $D \to 3$, it approaches 0.

3 Comments

The advertised result of this paper is the Gaussian effective potential given in (2.12) for arbitrary D, which simplifies to (2.13) for $D = 1$. (2.12a) is valid for all Φ such that (2.12b) has a solution; (2.13) is valid for all Φ such that $\cosh(\gamma\Phi) > 0$. Although the spectrum and S-matrix of the $D = 1$ sine-Gordon model have been computed exactly [7–10] , the exact effective potential has not been calculated. This would provide a useful check on the Gaussian approximation.

Some technical comments are in order.

First, we have managed to renormalize the model for all $D < 3$. This is curious, since the model is perturbatively non-renormalizable for $D > 1$. The Gaussian approximation somehow evades this difficulty; how does this happen? The answer rests on the fact that the Gaussian approximation sums a certain class of diagrams to all orders in perturbation theory. As it turns out, these are the same diagrams that are relevant in large-N models [12]. But it can be shown that a large class of non-renormalizable large-N models can be renormalized to leading order in N [13, 14]; the large-N version of the generalized sinh-Gordon model is in this class. It follows that, within the Gaussian approximation, our model is renormalizable. (It is not obvious that the Gaussian approximation is *good* in a non-renormalizable model. We should accordingly take our results with a grain of salt for $1 < D < 3$. Note, however, that there is a good reason to suspect that the sine-Gordon model is physically sensible for $D > 1$, despite its non-renormalizability: the model is equivalent to the classical coulomb gas, which has no obvious difficulties of physical interpretation.)

Second, we have generalized Coleman's phase transition to models in arbitrary dimension. This has physical consequences in a familiar model: a coulomb gas in three dimensions should go from plasma to dipole phase as the

temperature goes below some critical temperature. (See [5] for a complete translation dictionary between the sine-Gordon model and the coulomb gas, in arbitrary dimension.)

Third, we can say a little more about this phase transition. Recall that the transition arises because the solution to the variational equation (2.5) switches from a minimum to a maximum at the critical value of β^2. Thus, to solve the variational problem beyond the critical point, we must investigate the endpoints $M \to 0$ and $M \to \infty$. This is straightforward for the two cases of greatest interest, $D = 1$ and $D = 2$; however, the outcome depends on the order of operation. We summarize the results as follows.

1. If we use a finite cutoff Λ, then the variational energy is minimized by $M \to 0$. In the limit $\Lambda \to \infty$, the resulting effective potential approaches some constant and the model is apparently a free-field theory.

2. On the other hand, we can send the cutoff Λ to infinity before solving the variational problem. The energy is then unbounded below as M increases. This was Coleman's approach [3]; no vacuum exists in this continuum theory. This is reminiscent of Luther's result in the $x - y - z$ model [6], in which the lattice theory has no continuum limit for a certain range of the parameters (corresponding precisely to the forbidden range of θ in the sine-Gordon model).

There still remains some interesting work to be done in our model.

Recall that, for the $D = 1$ sine-Gordon model, the limit $\beta^2 \to 4\pi$ is almost as important as the limit $\beta^2 \to 8\pi$ [3, 7–10]. The quantized soliton-antisoliton doublet ceases to exist as β^2 rises above 4π. The reason is easily understood by using the correspondence to the Thirring model [3]. The sine-Gordon soliton is the Thirring elementary fermion. The point $\beta^2 = 4\pi$ corresponds to the free fermion theory and the region $\beta^2 > 4\pi$ corresponds to a *repulsive* coupling constant in the Thirring model. Thus, no fermion-antifermion bound state is expected to exist in this region. (We also note that the point $\beta^2 = 4\pi$ corresponds to a temperature in the coulomb gas at which loose dipoles just begin to form [5].)

Our analysis, being just a generalization of Coleman's variational calculation, has missed this important feature altogether. We can conjecture that there is some analogous critical coupling in arbitrary dimensions, but

we have no way to test this idea with our variational method. Some new scheme would have to be devised to tackle this question.

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