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Characteristic Dependent Linear Rank Inequalities and Applications to Network Coding

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by

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The dissertation of Eric Francis Freiling is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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ABSTRACT OF THE DISSERTATION

Characteristic Dependent Linear Rank Inequalities and Applications to Network Coding

by

Eric Francis Freiling

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Professor Kenneth Zeger, Chair

Let P be a finite or co-finite set of primes. We prove that there exists a linear rank inequality valid for all finite fields with characteristic in P. These linear rank inequalities are then used to prove that there exists a network that is linearly solvable over a field, F, if and only if the characteristic of F is in P.

Chapter 1

Introduction

In 2000, Ahlswede, Cai, Li, and Yeung introduced the field of Network Coding [R. Ahlswede 00]. It has been shown that Network Coding is a useful tool to improve the performance of networks in lieu of routing. However, the field is young, complex, and full of open and deep problems. There are no known algorithms to determine the capacity of a given network, even if you restrict your coding solutions to be linear. In fact, it is not clear if an algorithm exists.

Information inequalities are linear inequalities that hold for all jointly distributed random variables and Shannon inequalities are information inequalities of a certain form [Shannon 48]. Both are properly defined in 1.2. It is known [Yeung 02] that all information inequalities containing three or fewer variables are Shannon inequalities. The first “non-Shannon" information inequality was of four variables and was published in 1998 by Zhang and Yeung [Zhang 98]. Since this publication, many other non-Shannon inequalities have been found. See for example, Lněnička [Lněnička 03], Makarychev, Makarychev, Romashchenko,and Vereshchagin [K. Makarychev 02], Zhang [Zhang 03], Zhang and Yeung [Zhang 97], Dougherty, Freiling, and Zeger [R. Dougherty 06], and Matus [Matus 07]. Additionally, Matus was the first to show that the list of non-Shannon information inequalities is infinite [Matus 07] and provides two lists. A third infinte list was discovered by Xu, Wang, and Sun [W. Xu 08].

There is a close connection between information inequalities and network coding [Chan 07]. Capacities of some networks have been computed by finding matching lower and upper bounds [Dougherty 07]. Lower bounds have been found by deriving coding solutions. Upper bounds have been found by using information inequalities and treating the sources as independent random variables. Information inequalities might also play an important role in one day developing an algorithm to compute the capacity of a given network by looking at the entropy space bounded by the inequalities. So in order to further our understanding of network coding it is vital to analyze these information inequalities.

It has been shown that linear codes are insufficient for network coding in general [R. Dougherty 05]. However, linear codes are very popular to use because they are easier to produce and analyze and most likely what would be used in practice. When restricting the codes to being linear codes, we call the capacity the linear coding capacity. It has been shown that the coding capacity is independent of the alphabet size [J. Cannons 06]. However, the linear coding capacity is dependent on alphabet size, or more specifically the field characteristic. In other words, one can achieve a higher rate of linear communication by choosing one characteristic over another. To provide good upper bounds for the linear coding capacity for a particular field one can look at linear rank inequalities [Dougherty 13]. Linear rank inequalities are linear inequalities that are always satisfied by ranks of subspaces of a vector space. All information inequalities are linear rank inequalities but not all linear rank inequalities are information inequalities. The first example of a linear rank inequality that is not an information inequality was found by Ingleton [Ingleton 71]. Information inequalities can provide an upper bound for the capacity of a network, but this upper bound would hold for all alphabets. Therefore, to determine the linear coding capacity over a certain characteristic one would have to consider linear rank inequalities.

All linear rank inequalities for up to and including five variables are known and are all characteristic independent [R. Dougherty 10]. All the linear rank inequalities for six variables have not yet been determined. The first characteristic dependent linear rank inequalities are of seven variables [Dougherty 13]. One is valid for characteristic two and the other is valid for every characteristic except for two. These inequalities are then used to provide upper bounds for the linear coding capacity of two networks. In Chapter 2, we give two characteristic dependent linear rank inequalities of eight variables. One is valid for characteristic three and the other is valid for every characteristic except for three. These inequalities are then used to provide upper bounds for the linear coding capacity of two networks. In Chapter 3, we give two families of characteristic dependent linear rank inequalities. One is valid for any finite set of primes, and the other is valid for any co-finite set of primes. Again these inequalities are then used to provide upper bounds for the linear coding capacity of two families of networks. In [Dougherty 08], it was shown that every finite or co-finite set of primes, P, there exists a network that is scalar linearly solvable only over primes in P. We generalize this result to linear solvability. In [Ngai 04], an example of a sequence of networks was given where the ratio of coding capacity to routing capacity is arbitrarily large. We give another example of this result with a simpler sequence of networks.

Each of the two Chapters 2-3 in this dissertation are sections of submitted journal papers. These are as follows:

Chapter 2	R. Dougherty, E. Freiling, and K. Zeger, "Characteristic Dependent Linear Rank Inequalities and Applications to Network Coding," submitted to the *IEEE Transactions on Information Theory*, 2013.
Chapter 3	E. Freiling "Characteristic Dependent Linear Rank Inequalities for every Finite and Co-finite Set of Primes with Applications to Network Coding," submitted to the *IEEE Transactions on Information Theory*, 2014.

1.1 Matroids

The book [Oxley 92] is a very useful book on Matroid Theory and we will reference it for the majority of this section. A matroid is an abstract structure that captures a notion of "independence" that is found in matrices and many other topics in mathematics.

Definition 1.1.1. A finite *matroid*, M, is a pair (E, I), where E is a finite set and I is a set of subsets of E that satisfy the following properties:

I1) $\emptyset \in I$.

I2) I is closed under subsets, $\forall \hat{A} \subseteq A \subseteq E$, if $A \in I$ then $\hat{A} \in I$.

I3) I has the augmentation property. If $A, B \subseteq I$ and $|A| > |B|$, then $\exists u \in A$ such that $u \notin B$ and $\{u\} \cup B \in I$.

The sets in I are called independent sets. If a subset of E is not an element in I, then it is called dependent. An example of a matroid is obtained from linear algebra. Suppose A is an $m \times n$ matrix over a field F. If $E = \{1, \ldots, n\}$ and I is the set of all $X \subseteq E$ such that the multiset of columns of A indexed by the elements of X is linearly independent in the vector space $V(m, F)$, then $M = (E, I)$ is a matroid called the *vector matroid* of A. If a matroid is isomorphic to a vector matroid over $V(m, F)$ we say that the matroid is *representable* over the field F. It is clear why $(I1)$ and $(I2)$ hold for this example of a matroid, but the third condition is not obvious at first. To prove that $(I3)$ holds, let $I_1, I_2 \in I$ such that $|I_1| < |I_2|$. Let $W = \langle I_1, I_2 \rangle$, ($W$ is the subspace of $V(m, F)$ spanned by $I_1 \cup I_2$). Then $dim(W) \geq |I_2|$. Now suppose that $\forall e \in I_2 \backslash I_1$, $I_1 \cup \{e\}$ is linearly dependent, then $W \subseteq \langle I_1 \rangle$. Thus $|I_2| \leq dim(W) \leq |I_1| < |I_2|$. We have reached a contradiction, so $(I3)$ must hold.

Consider the matrix

$$A = \begin{array}{c} \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Let a, b, c, d, and e denote the columns of A going from the left to the right. Note, here we are indexing the columns by letters instead of numbers for simplicity. Then there is a vector matroid on A, $M = (E, I)$, where $E = \{a, b, c, d, e\}$ and

$$I = \{\emptyset, \{a\}, \{b\}, \{d\}, \{e\}, \{a, b\}, \{a, e\}, \{b, d\}, \{b, e\}, \{d, e\}\}.$$

A maximal independent set is a set that is not contained in any larger independent set. We will define a *base* to a maximal independent set, and we will denote set of all bases of a matroid M by $B(M)$. Since I is closed under subsets $B(M)$ is sufficient to define a matroid. In our example, $B(M) = \{\{a, b\}, \{a, e\}, \{b, d\}, \{b, e\}, \{d, e\}\}$. It is a well known result that all the bases of a matroid are of the same cardinality. Let $X \subseteq E$ and let $I|X = \{i \subseteq X : i \in I\}$, then it is easy to see that $(X, I|X)$ is a matroid. We will define the *rank* of X, $r(X)$, to be the cardinality of a base in $M|X$. In our example, $r(M) = 2$. We can also define a matroid by its dependent sets. A *circuit* is a minimal dependent set, that is, a circuit is a dependent set in which all of its proper subsets are independent. Unlike bases, circuits can differ in size. The circuits in our example are $\{\{c\}, \{a, d\}, \{a, b, e\}, \{b, d, e\}\}$. The independent sets can be derived from the set of circuits, so the circuits are also sufficient to define a matroid. A *spanning set*, X, of a matroid M is a subset of E such that $r(X) = r(M)$. A *hyperplane* of a matroid is a maximal non-spanning set. In our example, the set $S = \{a, c, d\}$ is a hyperplane because it does not contain a base (or $r(S) = 1$) and for any element $e \in E \backslash S$, $r(S \cup \{e\}) = 2$ (or $S \cup \{e\}$ would contain a base).

1.2 Linear rank Inequalities

Let A, B and C be collections of discrete random variables over an alphabet $\mathcal{X}$, and let p be the probability density function of A. The *entropy* of A is defined by

$$H(A) = -\sum_{u} p(u) \log_{|\mathcal{X}|} p(u)$$

The *conditional entropy* of A given B will be denoted by

$$H(A|B) = H(A, B) - H(B), \tag{1.1}$$

the *mutual information* between A and B will be denoted by

$$I(A; B) = H(A) - H(A|B) = H(A) + H(B) - H(A, B), \tag{1.2}$$

and the *conditional mutual information* between A and B given C will be denoted by

$$I(A; B|C) = H(A, C) - H(A|B, C) = H(A, C) + H(B, C) - H(C) - H(A, B, C). \tag{1.3}$$

We will make use of the following basic information-theoretic facts [Yeung 02]:

$$\begin{aligned} 0 &= H(\emptyset) && (1.4)\\ 0 &\leq H(A) = H(A|\emptyset) && (1.5)\\ 0 &\leq H(A|B) && (1.6)\\ 0 &\leq I(A;B) && (1.7)\\ H(A,B|C) &\leq H(A|C) + H(B|C) && (1.8)\\ H(A|B,C) &\leq H(A|B) \leq H(A,C|B) && (1.9)\\ I(A;B) &= H(A) + H(B) - H(A,B) && (1.10)\\ I(A;B|C) &= H(A,C) + H(B,C) - H(C) - H(A,B,C) && (1.11)\\ I(A;B,C) &= I(B;A|C) + I(A;C) && (1.12) \end{aligned}$$

The equations (1.5)-(1.9) were originally given by Shannon in 1948 [Shannon 48], and can all be summarized by $I(A;B|C) \geq 0$ [Yeung 02].

Definition 1.2.1. Let q be a positive integer, and let $S_1, \ldots, S_k$ be subsets of $\{1, \ldots, q\}$. Let $\alpha_i \in \mathbb{R}$ for $1 \leq i \leq k$. A linear inequality of the form

$$\alpha_1 H(\{A_i : i \in S_1\}) + \cdots + \alpha_k H(\{A_i : i \in S_k\}) \quad \geq \quad 0$$

is called an *information inequality* if it holds for all jointly distributed random variables $A_1, \ldots, A_q$.

As an example, taking $q = 2$, $S_1 = \{1\}$, $S_2 = \{2\}$, $S_3 = \emptyset$, $S_4 = \{1, 2\}$, $\alpha_1 = \alpha_2 = 1$, $\alpha_4 = -1$, and using (1.8) shows that $H(A_1) + H(A_2) - H(A_1, A_2) \geq 0$ is an information inequality.

A *Shannon information inequality* is any information inequality that can be expressed as a finite sum of the form

$$\sum_i \alpha_i I(A_i; B_i|C_i) \geq 0$$

where each α_i is a nonnegative real number. Any information that cannot be expressed in the form above will be called a *non-Shannon information inequality.*

A *linear rank inequality* is a linear inequality that is always satisfied by ranks of subspaces of a vector space. Linear rank inequalities are closely related to information inequalities. For instance, all Shannon inequalities are linear rank inequalities for finite vector spaces, but not all linear rank inequalities are Shannon inequalities. The first known example of a linear rank inequality that is not an information inequality is the *Ingleton inequality* [Ingleton 71]:

$$I(A;B) \quad \leq \quad I(A;B|C) + I(A;B|D) + I(C;D)$$

Let A, B, C, D be binary random variables, and let $X = (A, B, C, D)$ with probabilities:

$$\begin{aligned} P(0000) &= 1/4 \\ P(1111) &= 1/4 \\ P(0101) &= 1/4 \\ P(0110) &= 1/4 \end{aligned}$$

Then the Ingleton inequality fails:

$$\begin{aligned} I(A;B) &\leq I(A;B|C) + I(A;B|D) + I(C;D) \\ 3/2\log_2(4/3) - 1/2 &\leq 0 + 0 + 0 \\ 0.1226 &\leq 0 \end{aligned}$$

When talking about information inequalities, we usually refer to entropies on random variables. However, with linear rank inequalities, we are using the entropies to represent the ranks of subspaces. To see this connection, it is easy to see that the entropy of a uniformly distributed random variable is equal to 1 (here we are taking the base of the logarithm to be the size of the alphabet). If we consider a vector space to be a uniformly distributed random variable, then the entropy of a single dimension of a subspace that consists of finitely many equiprobable field elements would also be 1. So every dimension of a subspace adds 1 to the entropy, or the entropy of a subspace is the rank. So when A, B, and C denote subspaces of a vector space, $H(A)$ denotes rank of A, the notation $H(A, B)$ denotes the rank of $\langle A, B\rangle$, $H(A|B)$ is the excess of the rank of A over that of $A \cap B$, or the codimension of $A \cap B$ in A, and $I(A;B)$ is the rank of $A \cap B$. For background material on this relationship and other topics used here, a useful source is Hammer, Romashchenko, Shen, and Vereshchagin [D. Hammer 00].

1.3 Network Coding

We will first informally discuss some preliminaries of network coding. For more on network coding, see [Yeung 08]. We can think of a network *message* as an arbitrary string of k alphabet symbols and a *packet* as a string of n alphabet symbols, where an alphabet $\mathcal{A}$ is a finite set. More precisely, a message is a variable with domain $\mathcal{A}^k$ and a packet is a variable with domain $\mathcal{A}^n$. A *network* is based on a finite, directed, acyclic multigraph and is assigned a finite set of messages. Each message originates at a particular node called the *source node* for that message and is required by one or more *demand nodes*. In our diagram, whenever the message variable appears above the node, it is a source node that generates that message. If the message appears below the node, it is a receiver node that demands the message. The information about the messages is passed from node to node in the form of packets. There is one packet for each edge of the graph. All edges have the capability of carrying a n dimensional packet. For a given

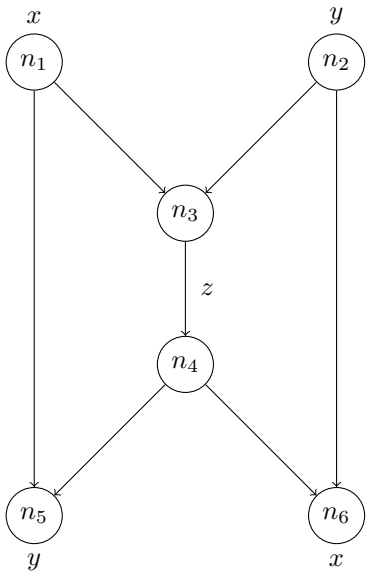

Figure 1.1: The Butterfly Network

network we can consider different values of k and n that remain consistent through out the network.

The inputs to a network node are the packets carried on its in-edges as well as the messages generated at the node. The outputs of a network node are the packets carried on its out-edges as well as the demanded messages. Each output of a node must be a function of its inputs. A *coding solution* for the network is an assignment of such functions. When the values of k and n need to be emphasized, the coding solution will be called a (k, n)-coding solution. We can know define the *capacity* of a network,$\mathcal{C}$:

$$\mathcal{C} \quad = \quad \sup\{k/n : \exists \text{ a (k,n)-coding solution}\}$$

There are also specific types of solutions. In a *linear solution*, we assume the alphabet $\mathcal{A}$ consists of the elements of a finite ring, and usually, it will be a finite field. Hence, all messages are k-long vectors of ring elements while the packets are n-long vectors. The functions in a linear solution must only use the operations of vector addition and multiplication of a vector by a constant matrix (whose components are ring elements). If there exists a (k, n)-coding solution such that $k \geq n$, then we say that the network is *solvable*. If there exists a (k, n)-linear coding solution such that $k \geq n$, then we say that the network is *linearly solvable*. The *linear capacity* would be defined the same as the capacity if we restrict ourselves to only using linear coding solutions. It is also easily verified that if x is a message, then $H(x) = k$, and if x is a packet, then $H(x) \leq n$.

Let's look at a famous example often called the Butterfly Network depicted in figure

1.1. If we assume that the network messages x and y are independent, k-dimensional, random vectors with uniformly distributed components, then in any solution it must be the case that

$$H(y|x,z) \quad = \quad 0 \tag{1.13}$$

We can find the coding capacity of the butterfly network by first finding an upper bound and then finding a coding solution that achieves the upper bound. We can calculate an upper bound by [Dougherty 07]:

$$\begin{aligned}
2k &= H(x) + H(y) \\
&= H(x,y) && \text{[from indep. of x and y]} \\
&\leq H(x,y,z) && \text{[from (1.9)]} \\
&= H(x,z) + H(y|x,z) && \text{[from (1.1)]} \\
&= H(x,z) && \text{[from (1.13)]} \\
&\leq H(x) + H(z) && \text{[from (1.8)]} \\
&\leq k + n
\end{aligned}$$

So we have $2k \leq k + n$ which implies $k/n \leq 1$. Notice if we let $z = x + y$ over any alphabet, then a solution is achieved for $k = n = 1$. Thus the coding capacity for the butterfly network is the same as the linear coding capacity which is 1.

1.4 Preliminaries

In this section, we given some technical lemmas which will be useful for proving the main results of the dissertation.

If A is a subspace of vector space V, and $\overline{A}$ is a subspace of A, then we will use the notation $\text{codim}_A(\overline{A}) = \dim(A) - \dim(\overline{A})$ to represent the codimension of $\overline{A}$ in A. We will omit the subscript when it is obvious from the context which space the codimension is with respect to.

Lemma 1.4.1. *[Dougherty 13] Let V be a finite dimensional vector space with subspaces A and B. Then the subspace $A \cap B$ has codimension at most* $\text{codim}(A) + \text{codim}(B)$.

Proof. We know $H(A) + H(B) - I(A;B) = H(A,B) \leq H(V)$. Then adding $H(V)$ to both sides of the inequality gives $H(V) - I(A;B) \leq H(V) - H(A) + H(V) - H(B)$. Thus $codim(A \cap B) \leq codim(A) + codim(B)$. □

Lemma 1.4.2. *[Dougherty 13] Let A and B be vector spaces with subspaces $\overline{A}$ and $\overline{B}$ respectively. Let $f : A \to B$ be a linear function such that $f(A \backslash \overline{A}) \subseteq B \backslash \overline{B}$. Then the codimension of $\overline{A}$ is at most the codimension of $\overline{B}$.*

Proof. Let $\{a_1, \ldots, a_n\}$ be a set of basis elements that extend $\overline{A}$ to A and let $\{b_1, \ldots, b_k\}$ be a set of basis elements that extend $\overline{B}$ to B. We would like to first show that $\{f(a_1), \ldots, f(a_n)\}$ is a linearly independent set. By way of contradiction assume $\{f(a_1), \ldots, f(a_n)\}$ is a linearly dependent set. Then there exists field elements $\alpha_1, \ldots, \alpha_n$, not all zero, such that $\alpha_1 f(a_1) + \cdots + \alpha_n f(a_n) = 0$. Since f is linear we know that $f(\alpha_1 a_1 + \cdots + \alpha_n a_n) = 0$. Now $\alpha_1 a_1 + \cdots + \alpha_n a_n$ cannot be a non-zero element, because that would contradict the fact that $f(A \backslash \overline{A}) \subseteq B \backslash \overline{B}$. So $\alpha_1 a_1 + \cdots + \alpha_n a_n = 0$. However, $a_1, \ldots, a_n$ are basis elements and thus are linearly independent. We have arrived at a contradiction so $\{f(a_1), \ldots, f(a_n)\}$ must be a linearly independent set. Now since there can be at most k elements in $B \backslash \overline{B}$ that are linearly independent over B, we know that $n \leq k$ or the codimension of $\overline{A}$ is at most the codimension of $\overline{B}$. □

Lemma 1.4.3. *[Dougherty 13] Let Z and A be vector spaces, $\overline{A}$ be a subspace of A, $f : Z \to A$ be a linear function. Then for $t \in Z$, $f(t) \in \overline{A}$ on a subspace of Z of codimension at most the codimension of $\overline{A}$.*

Proof. Let $T = \{t \in Z : f(t) \in \overline{A}\}$. Then $f(Z \backslash T) \subseteq A \backslash \overline{A}$. By Lemma 1.4.2, the codimension of T is at most the codimension of $\overline{A}$. □

Lemma 1.4.4. *[Dougherty 13] Let V be a finite dimensional vector space and let $A_1, \ldots, A_k, B$ be subspaces of V. Then for $i = 1, \ldots, k$, $\exists$ linear functions $f_i : B \to A_i$ such that $f_1 + \cdots + f_k = I$ on a subspace of B of codimension $H(B|A_1, \ldots, A_k)$.*

Proof. Let W be a subspace of B defined by $W = \langle A_1, \ldots, A_k \rangle \cap B$. The subspace on which this lemma holds is W. If $H(W) = 0$, then the lemma would be trivially true. Let's assume that $H(W) > 0$, then let $\{w_1, \ldots, w_n\}$ be a basis for W. For each $j = 1, \ldots, n$, choose $x_{i,j} \in A_i$ for $i = 1, \ldots, k$ such that $w_j = x_{1,j} + \cdots + x_{k,j}$. For each $i = 1, \ldots, k$, define a linear mapping $g_i : W \to A_i$ so that $g_i(w_j) = x_{i,j}$ for all i and j. Then extend g_i arbitrarily to $f_i : B \to A_i$. Now we have linear functions $f_1, \ldots, f_k$ such that $f_1 + \cdots + f_k = I$ on W. The dimension of W is $H(W) = I(A_1, \ldots, A_k; B)$, so the codimension of W is $H(B) - I(A_1, \ldots, A_k; B) = H(B|A_1, \ldots, A_k)$. □

Lemma 1.4.5. *[Dougherty 13] Let V be a finite-dimensional vector space and let A, B, and C be subspaces of V. Let $f : A \to B$ and $g : A \to C$ be linear functions such that $f + g = 0$ on A. Then $f = g = 0$ on a subspace of A of codimension at most $I(B; C)$.*

Proof. Let K be the kernel of f. Clearly, f maps A into $B \cap C$ and since f is linear we have:

$$\begin{aligned} H(K) &\geq H(A) - I(B; C) \\ H(A) - H(K) &\leq I(B; C) \\ codim(K) &\leq I(B; C) \end{aligned}$$

□

Lemma 1.4.6. *[Dougherty 13] Let V be a finite dimensional vector space and let $A, B_1, \ldots, B_k$ be subspaces of V. For each $i = 1, \ldots, k$ let $f_i : A \to B_i$ be a linear function such that $f_1 + \cdots + f_k \equiv 0$ on A. Then $f_1 \equiv \cdots \equiv f_k \equiv 0$ on a subspace of A of codimension at most $H(B_1) + \cdots + H(B_k) - H(B_1, \ldots, B_k)$.*

Proof. First we apply Lemma 1.4.5 to f_1 and $(f_2 + \cdots + f_k)$ to get $f_1 = (f_2 + \cdots + f_k) = 0$ on a subspace A_1 of A of codimension at most $I(B_1; B_2, \ldots, B_k) = H(B_1) + H(B_2, \ldots, B_k) - H(B_1, B_2, \ldots, B_k)$. Then apply Lemma 1.4.5 to f_2 and $(f_3 + \cdots + f_k)$ to get $f_2 = (f_3 + \cdots + f_k) = 0$ on a subspace A_2 of A_1 of codimension at most $I(B_2; B_3, \ldots, B_k) = H(B_2) + H(B_3, \ldots, B_k) - H(B_2, B_3, \ldots, B_k)$. Continue on until we apply Lemma 1.4.5 to f_{k-1} and f_k to get $f_{k-1} = f_k = 0$ on a subspace A_{k-1} of A_{k-2} of codimension at most $I(B_{k-1}; B_k) = H(B_{k-1}) + H(B_k) - H(B_{k-1}, B_k)$. Now A_{k-1} is a subspace of A of codimension at most $H(B_1) + \cdots + H(B_k) - H(B_1, \ldots, B_k)$, on which $f_1 = f_2 = \cdots = f_k = 0$. □

Lemma 1.4.7. *Let A, B, C, D and E be subspaces of a vector space V and let f_R, f_L, g_R, and g_L be functions such that $f_R : A \to C, f_L : C \to A, g_R : B \to D$, and $g_L : D \to E$. If $f_L f_R = I$ on A and $g_L g_R$ is injective on B, then $g_L f_R$ is injective on $f_L(f_R A \cap g_R B)$.*

Proof. Let $x, y \in f_L(f_R A \cap g_R B)$. Since $x \in f_L(f_R A \cap g_R B)$, we know $f_R(x) \in f_R f_L(f_R A \cap g_R B) = f_R A \cap g_R B$, which implies $f_R(x) = g_R(b_x)$ for some $b_x \in B$. Similarly, we know $f_R(y) = g_R(b_y)$ for some $b_y \in B$. So we have $g_L g_R(b_x) = g_L f_R(x)$ and $g_L g_R(b_y) = g_L f_R(y)$. If we assume $g_L f_R(x) = g_L f_R(y)$, then we have $g_L g_R(b_x) = g_L g_R(b_y)$. Since $g_L g_R$ is injective on B, we know $b_x = b_y$. Thus $f_R(x) = g_R(b_x) = g_R(b_y) = f_R(y)$, which implies $f_L f_R(x) = f_L f_R(y)$. Since $f_L f_R = I$ on A, we know $x = y$. Thus $g_L f_R$ is injective on $f_L(f_R A \cap g_R B)$. □

Chapter 2

Characteristic Dependent Linear Rank Inequalities with Applications to Network Coding

2.1 A Linear Rank Inequality for fields of characteristic other than 3

The $T8$ matroid [Oxley 92] is represented by the following matrix, where column dependencies are over characteristic 3:

$$\begin{array}{cccccccc} A & B & C & D & W & X & Y & Z \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

The $T8$ matroid is representable over a field if and only if the the field is of characteristic 3. Figure 3.1 is a network whose dependencies and independencies are consistent with the $T8$ matroid. It was designed by the construction process described in [Dougherty 07], and we will refer to it as the $T8$ network. Theorem 2.1.1 uses the $T8$ network as a guide to derive a linear rank inequality valid for every characteristic except for 3. The new linear rank inequality can then be used to prove the $T8$ network is only linearly solvable if the characteristic is 3.

Theorem 2.1.1. *Let $A, B, C, D, W, X, Y,$ and Z be subspaces of a vector space V. Then the*

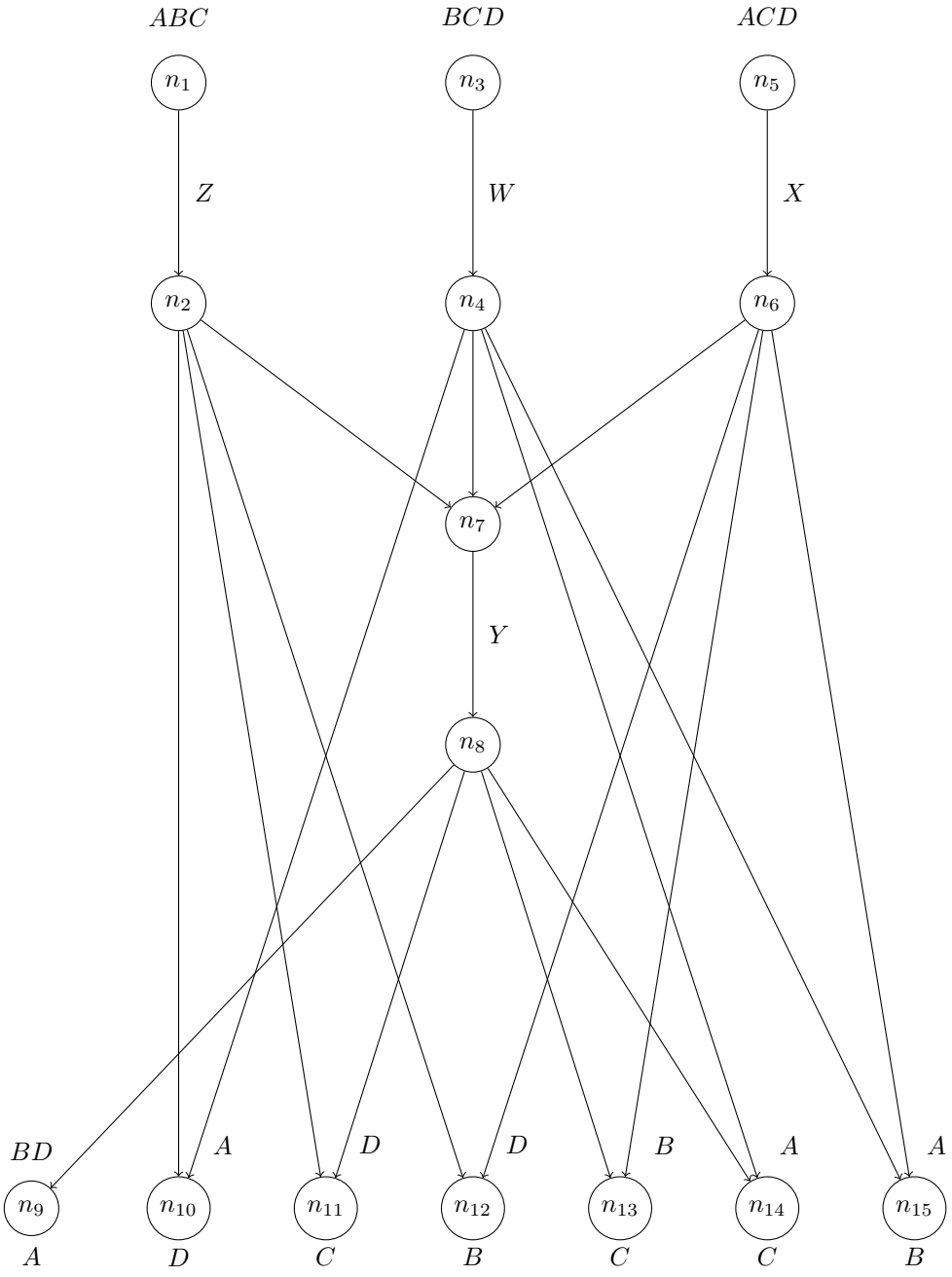

Figure 2.1: The $T8$ network has source messages A, B, C, and D

following is a linear rank inequality for fields of characteristic other than 3,

$$\begin{aligned} H(A) \quad \leq \quad & 8H(Z)+29H(Y)+3H(X)+8H(W)-6H(D)-17H(C)-8H(B)-17H(A) \\ & +55H(Z|A,B,C)+35H(Y|W,X,Z)+50H(X|A,C,D)+45H(W|B,C,D) \\ & +18H(A|B,D,Y)+7H(B|D,X,Z)+H(B|A,W,X)+7H(C|D,Y,Z) \\ & +7H(C|B,X,Y)+3H(C|A,W,Y)+6H(D|A,W,Z) \\ & +49(H(A)+H(B)+H(C)+H(D)-H(A,B,C,D)) \end{aligned}$$

Proof. Let V be a finite dimensional vector space with subspaces A, B, C, D, W, X, Y, Z. By Lemma 1.4.4 we get linear functions:

$$\begin{array}{lll} f_1 : Z \to A, & f_2 : Z \to B, & f_3 : Z \to C, \\ f_4 : W \to B, & f_5 : W \to C, & f_6 : W \to D, \\ f_7 : X \to A, & f_8 : X \to C, & f_9 : X \to D, \\ f_{10} : Y \to Z, & f_{11} : Y \to W, & f_{12} : Y \to X, \\ f_{13} : A \to B, & f_{14} : A \to D, & f_{15} : A \to Y, \\ f_{16} : D \to Z, & f_{17} : D \to W, & f_{18} : D \to A, \\ f_{19} : C \to Z, & f_{20} : C \to Y, & f_{21} : C \to D, \\ f_{22} : B \to Z, & f_{23} : B \to X, & f_{24} : B \to D, \\ f_{25} : C \to Y, & f_{26} : C \to X, & f_{27} : C \to B, \\ f_{28} : C \to Y, & f_{29} : C \to W, & f_{30} : C \to A, \\ f_{31} : B \to W, & f_{32} : B \to X, & f_{33} : B \to A \end{array}$$

such that

$$f_1+f_2+f_3 \equiv I \text{ on a subspace of } Z \text{ of codimension } H(Z|A,B,C) \tag{2.1}$$

$$f_4+f_5+f_6 \equiv I \text{ on a subspace of } W \text{ of codimension } H(W|B,C,D) \tag{2.2}$$

$$f_7+f_8+f_9 \equiv I \text{ on a subspace of } X \text{ of codimension } H(X|A,C,D) \tag{2.3}$$

$$f_{10}+f_{11}+f_{12} \equiv I \text{ on a subspace of } Y \text{ of codimension } H(Y|W,X,Z) \tag{2.4}$$

$$f_{13}+f_{14}+f_{15} \equiv I \text{ on a subspace of } A \text{ of codimension } H(A|B,D,Y) \tag{2.5}$$

$$f_{16}+f_{17}+f_{18} \equiv I \text{ on a subspace of } D \text{ of codimension } H(D|A,W,Z) \tag{2.6}$$

$$f_{19}+f_{20}+f_{21} \equiv I \text{ on a subspace of } C \text{ of codimension } H(C|D,Y,Z) \tag{2.7}$$

$$f_{22}+f_{23}+f_{24} \equiv I \text{ on a subspace of } B \text{ of codimension } H(B|D,Y,Z) \tag{2.8}$$

$$f_{25}+f_{26}+f_{27} \equiv I \text{ on a subspace of } C \text{ of codimension } H(C|B,X,Y) \tag{2.9}$$

$$f_{28}+f_{29}+f_{30} \equiv I \text{ on a subspace of } C \text{ of codimension } H(C|A,W,Y) \tag{2.10}$$

$$f_{31}+f_{32}+f_{33} \equiv I \text{ on a subspace of } B \text{ of codimension } H(B|A,W,X) \tag{2.11}$$

Now let

$$\begin{aligned} f_A &\triangleq f_7 f_{12} + f_1 f_{10} \\ f_B &\triangleq f_4 f_{11} + f_2 f_{10} \\ f_C &\triangleq f_8 f_{12} + f_5 f_{11} + f_3 f_{10} \\ f_D &\triangleq f_9 f_{12} + f_6 f_{11} \end{aligned}$$

Now combining the functions we got from Lemma 1.4.4 we get new functions:

$$\begin{aligned} f_A \circ f_{15} &: A \to A \\ f_B \circ f_{15} + f_{13} &: A \to B \\ f_C \circ f_{15} &: A \to C \\ f_D \circ f_{15} + f_{14} &: A \to D \end{aligned}$$

Using (2.1) - (2.5), Lemma 1.4.1, and Lemma 1.4.3 we know the sum of these functions is equal to I on a subspace of A of codimension at most $H(Z|A,B,C)+H(W|B,C,D)+H(X|A,C,D)+H(Y|W,X,Z)+H(A|B,D,Y)$.

Now applying Lemma 1.4.6 and Lemma 1.4.1 to $f_A \circ f_{15} - I$, $f_B \circ f_{15} + f_{13}$, $f_C \circ f_{15}$, and $f_D \circ f_{15} + f_{14}$ we get a subspace $\overline{A}$ of A of codimension at most

$$\begin{aligned} \Delta_{\overline{A}} = \; & H(Z|A,B,C) + H(W|B,C,D) + H(X|A,C,D) + H(Y|W,X,Z) + H(A|B,D,Y) \\ & + H(A) + H(B) + H(C) + H(D) - H(A,B,C,D) \end{aligned}$$

on which,

$$f_A \circ f_{15} \equiv I \tag{2.12}$$
$$f_B \circ f_{15} + f_{13} \equiv 0 \tag{2.13}$$
$$f_C \circ f_{15} \equiv 0 \tag{2.14}$$
$$f_D \circ f_{15} + f_{14} \equiv 0 \tag{2.15}$$

Similarly, we get a subspace $\overline{B}$ of B of codimension at most

$$\begin{aligned} \Delta_{\overline{B}} = \; & H(Z|A,B,C) + H(X|A,C,D) + H(B|D,X,Z) \\ & + H(A) + H(B) + H(C) + H(D) - H(A,B,C,D) \end{aligned}$$

on which,

$$f_7 \circ f_{23} + f_1 \circ f_{22} \equiv 0 \tag{2.16}$$
$$f_2 \circ f_{22} \equiv I \tag{2.17}$$
$$f_8 \circ f_{23} + f_3 \circ f_{22} \equiv 0 \tag{2.18}$$
$$f_{24} + f_9 \circ f_{23} \equiv 0 \tag{2.19}$$

We get a subspace $\widehat{B}$ of B of codimension at most

$$\begin{aligned} \Delta_{\widehat{B}} = \; & H(W|B,C,D) + H(X|A,C,D) + H(B|A,W,X) \\ & + H(A) + H(B) + H(C) + H(D) - H(A,B,C,D) \end{aligned}$$

on which,

$$f_{33} + f_7 \circ f_{32} \equiv 0 \tag{2.20}$$
$$f_4 \circ f_{31} \equiv I \tag{2.21}$$
$$f_8 \circ f_{32} + f_5 \circ f_{31} \equiv 0 \tag{2.22}$$
$$f_9 \circ f_{32} + f_6 \circ f_{31} \equiv 0 \tag{2.23}$$

We get a subspace $\overline{C}$ of C of codimension at most

$$\begin{aligned}\Delta_{\overline{C}} = {} & 2H(Z|A,B,C) + H(W|B,C,D) + H(X|A,C,D) + H(Y|W,X,Z) + H(C|D,Y,Z) \\ & + H(A) + H(B) + H(C) + H(D) - H(A,B,C,D)\end{aligned}$$

on which,

$$f_A \circ f_{20} + f_1 \circ f_{19} \equiv 0 \tag{2.24}$$
$$f_B \circ f_{20} + f_2 \circ f_{19} \equiv 0 \tag{2.25}$$
$$f_C \circ f_{20} + f_3 \circ f_{19} \equiv I \tag{2.26}$$
$$f_D \circ f_{20} + f_{21} \equiv 0 \tag{2.27}$$

We get a subspace $\widehat{C}$ of C of codimension at most

$$\begin{aligned}\Delta_{\widehat{C}} = {} & H(Z|A,B,C) + H(W|B,C,D) + 2H(X|A,C,D) + H(Y|W,X,Z) + H(C|B,X,Y) \\ & + H(A) + H(B) + H(C) + H(D) - H(A,B,C,D)\end{aligned}$$

on which,

$$f_A \circ f_{25} + f_7 \circ f_{26} \equiv 0 \tag{2.28}$$
$$f_B \circ f_{25} + f_{27} \equiv 0 \tag{2.29}$$
$$f_C \circ f_{25} + f_8 \circ f_{26} \equiv I \tag{2.30}$$
$$f_D \circ f_{25} + f_9 \circ f_{26} \equiv 0 \tag{2.31}$$

We get a subspace $\widetilde{C}$ of C of codimension at most

$$\begin{aligned}\Delta_{\widetilde{C}} = {} & H(Z|A,B,C) + 2H(W|B,C,D) + H(X|A,C,D) + H(Y|W,X,Z) + H(C|A,W,Y) \\ & + H(A) + H(B) + H(C) + H(D) - H(A,B,C,D)\end{aligned}$$

on which,

$$f_A \circ f_{28} + f_{30} \equiv 0 \tag{2.32}$$
$$f_B \circ f_{28} + f_4 \circ f_{29} \equiv 0 \tag{2.33}$$
$$f_C \circ f_{28} + f_5 \circ f_{29} \equiv I \tag{2.34}$$
$$f_D \circ f_{28} + f_6 \circ f_{29} \equiv 0 \tag{2.35}$$

We get a subspace $\overline{D}$ of D of codimension at most

$$\begin{aligned}\Delta_{\overline{D}} = {} & H(Z|A,B,C) + H(W|B,C,D) + H(D|A,W,Z) \\ & + H(A) + H(B) + H(C) + H(D) - H(A,B,C,D)\end{aligned}$$

on which,

$$f_{18} + f_1 \circ f_{16} \equiv 0 \tag{2.36}$$
$$f_4 \circ f_{17} + f_2 \circ f_{16} \equiv 0 \tag{2.37}$$
$$f_5 \circ f_{17} + f_3 \circ f_{16} \equiv 0 \tag{2.38}$$
$$f_6 \circ f_{17} \equiv I \tag{2.39}$$

First notice that (2.12) implies

$$f_{15} \text{ is injective on } \overline{A} \tag{2.40}$$

We need to define a subspace of $\overline{A}$ on which f_{13} and f_{14} are injective. The justifications can be found on (2.44) and (2.45). Let

$$\begin{aligned}\overline{C}^* &\triangleq f_3(f_{19}(\overline{C} \cap f_{20}^{-1} f_{15}\overline{A}) \cap f_{22}\overline{B}) \subseteq \overline{C} \\ \widetilde{C}^* &\triangleq f_5(f_{29}(\widetilde{C} \cap f_{28}^{-1} f_{15}\overline{A}) \cap f_{17}\overline{D}) \subseteq \widetilde{C} \\ \overline{A}^* &\triangleq f_A(f_{15}\overline{A} \cap f_{20}\overline{C}^* \cap f_{28}\widetilde{C}^*) \subseteq \overline{A}\end{aligned}$$

To justify why $\overline{C}^* \subseteq \overline{C}$, by (2.14) we know $f_C f_{15} = 0$ and by (2.26) we know $f_C f_{20} + f_3 f_{19} = I$. Thus $\forall \overline{c} \in \overline{C} \cap f_{20}^{-1} f_{15}\overline{A}$, $f_C f_{20} = 0$ which gives

$$f_3 f_{19} = I \text{ on } \overline{C} \cap f_{20}^{-1} f_{15}\overline{A} \tag{2.41}$$

Using (2.14) and (2.34) we have

$$f_5 f_{29} = I \text{ on } \widetilde{C} \cap f_{28}^{-1} f_{15}\overline{A} \tag{2.42}$$

Using (2.14) and (2.30) we have

$$f_8 f_{26} = I \text{ on } \widehat{C} \cap f_{25}^{-1} f_{15}\overline{A} \tag{2.43}$$

We are now going to show f_{13} is injective on $\overline{A}^*$. First we need to apply Lemma 1.4.7 to show $f_2 f_{19}$ is injective on $\overline{C}^*$ and then again to show $f_B f_{15}$ is injective on $\overline{A}^*$. By (2.17) and (2.41), we know $f_2 f_{22}$ is injective on $\overline{B}$ and $f_3 f_{19} = I$ on $\overline{C} \cap f_{20}^{-1} f_{15} \overline{A}$. So we can apply Lemma 1.4.7 by letting $g_L = f_2$, $g_R = f_{22}$, $f_L = f_3$, and $f_R = f_{19}$ to get that $f_2 f_{19}$ is injective on $\overline{C}^*$. Then using (2.25), we know $f_B f_{20}$ is injective on $\overline{C}^*$. Now we can apply Lemma 1.4.7 again by using the fact that $f_A f_{15} = I$ on $\overline{A}$ and by letting $g_L = f_B$, $g_R = f_{20}$, $f_L = f_A$, and $f_R = f_{15}$ to get $f_B f_{15}$ is injective on $\overline{A}^*$. Thus by (2.13),

$$f_{13} \text{ is injective on } \overline{A}^*. \tag{2.44}$$

Similarly, we are going to show f_{14} is injective on $\overline{A}^*$. We will first apply Lemma 1.4.7 to show $f_6 f_{29}$ is injective on $\widetilde{C}^*$ and then again to show $f_D f_{15}$ is injective on $\overline{A}^*$. By (2.39) and (2.42), we know $f_6 f_{17}$ is injective on $\overline{D}$ and $f_5 f_{29} = I$ on $\widetilde{C} \cap f_{28}^{-1} f_{15} \overline{A}$. So we can apply Lemma 1.4.7 by letting $g_L = f_6$, $g_R = f_{17}$, $f_L = f_5$, and $f_R = f_{29}$ to get that $f_6 f_{29}$ is injective on $\widetilde{C}^*$. Then using (2.35), we know $f_D f_{29}$ is injective on $\widetilde{C}^*$. Now we can apply Lemma 1.4.7 again by using the fact that $f_A f_{15} = I$ on $\overline{A}$ and by letting $g_L = f_D$, $g_R = f_{29}$, $f_L = f_A$, and $f_R = f_{15}$ to get $f_D f_{15}$is injective on $\overline{A}^*$. Thus by (2.15),

$$f_{14} \text{ is injective on } \overline{A}^*. \tag{2.45}$$

Now we are going to find an upper bound for $codim_A(\overline{A}^*)$. First we need to find upper bounds for $codim_C(\overline{C}^*)$ and $codim_C(\widetilde{C}^*)$. Using (2.40) to show $dim(f_{15}\overline{A}) = dim(\overline{A})$, and again using Lemma 1.4.1 and Lemma 1.4.3, we have

$$\begin{aligned}
codim_C(\overline{C}^*) &= H(C) - dim(\overline{C}^*) \\
&= H(C) - dim(f_3[f_{19}(\overline{C} \cap f_{20}^{-1} f_{15}\overline{A}) \cap f_{22}\overline{B}]) \\
&= H(C) - dim(f_{19}(\overline{C} \cap f_{20}^{-1} f_{15}\overline{A}) \cap f_{22}\overline{B}) \\
&= H(C) - H(Z) + codim_Z(f_{19}(\overline{C} \cap f_{20}^{-1} f_{15}\overline{A}) \cap f_{22}\overline{B}) \\
&\leq H(C) - H(Z) + codim_Z(f_{19}(\overline{C} \cap f_{20}^{-1} f_{15}\overline{A})) + codim_Z(f_{22}\overline{B}) \\
&= H(C) - H(Z) + H(Z) - dim(f_{19}(\overline{C} \cap f_{20}^{-1} f_{15}\overline{A})) + H(Z) - dim(f_{22}\overline{B}) \\
&= H(C) + H(Z) - dim(\overline{C} \cap f_{20}^{-1} f_{15}\overline{A}) - dim(\overline{B}) \\
&= H(C) + H(Z) - H(C) + codim_C(\overline{C} \cap f_{20}^{-1} f_{15}\overline{A}) - H(B) + codim_B(\overline{B}) \\
&= H(Z) - H(B) + codim_C(\overline{C} \cap f_{20}^{-1} f_{15}\overline{A}) + codim_B(\overline{B}) \\
&\leq H(Z) - H(B) + \Delta_{\overline{C}} + codim_C(f_{20}^{-1} f_{15}\overline{A}) + \Delta_{\overline{B}} \\
&\leq H(Z) - H(B) + \Delta_{\overline{C}} + codim_Y(f_{15}\overline{A}) + \Delta_{\overline{B}} \\
&\leq H(Z) - H(B) + \Delta_{\overline{C}} + H(Y) - dim(f_{15}\overline{A}) + \Delta_{\overline{B}} \\
&= H(Z) - H(B) + \Delta_{\overline{C}} + H(Y) - dim(\overline{A}) + \Delta_{\overline{B}} \\
&= H(Z) - H(B) + \Delta_{\overline{C}} + H(Y) - H(A) + codim_A(\overline{A}) + \Delta_{\overline{B}} \\
&\leq H(Z) - H(B) + H(Y) - H(A) + \Delta_{\overline{C}} + \Delta_{\overline{A}} + \Delta_{\overline{B}}
\end{aligned} \tag{2.46}$$

$$
\begin{aligned}
codim_C(\widetilde{C}^*) &= H(C) - dim(\widetilde{C}^*) \\
&= H(C) - dim(f_5[f_{29}(\widetilde{C} \cap f_{28}^{-1} f_{15}\overline{A}) \cap f_{17}\overline{D}]) \\
&= H(C) - dim(f_{29}(\widetilde{C} \cap f_{28}^{-1} f_{15}\overline{A}) \cap f_{17}\overline{D}) \\
&= H(C) - H(W) + codim_W(f_{29}(\widetilde{C} \cap f_{28}^{-1} f_{15}\overline{A}) \cap f_{17}\overline{D}) \\
&\leq H(C) - H(W) + codim_W(f_{29}(\widetilde{C} \cap f_{28}^{-1} f_{15}\overline{A})) + codim_W(f_{17}\overline{D}) \\
&= H(C) - H(W) + H(W) - dim(f_{29}(\widetilde{C} \cap f_{28}^{-1} f_{15}\overline{A})) + H(W) - dim(f_{17}\overline{D}) \\
&= H(C) + H(W) - dim(\widetilde{C} \cap f_{28}^{-1} f_{15}\overline{A}) - dim(\overline{D}) \\
&= H(C) + H(W) - H(C) + codim_C(\widetilde{C} \cap f_{28}^{-1} f_{15}\overline{A}) - H(D) + codim_D(\overline{D}) \\
&= H(W) - H(D) + codim_C(\widetilde{C} \cap f_{28}^{-1} f_{15}\overline{A}) + codim_D(\overline{D}) \\
&\leq H(W) - H(D) + \Delta_{\widetilde{C}} + codim_C(f_{28}^{-1} f_{15}\overline{A}) + \Delta_{\overline{D}} \\
&\leq H(W) - H(D) + \Delta_{\widetilde{C}} + codim_Y(f_{15}\overline{A}) + \Delta_{\overline{D}} \\
&= H(W) - H(D) + \Delta_{\widetilde{C}} + H(Y) - dim(f_{15}\overline{A}) + \Delta_{\overline{D}} \\
&= H(W) - H(D) + \Delta_{\widetilde{C}} + H(Y) - dim(\overline{A}) + \Delta_{\overline{D}} \\
&= H(W) - H(D) + \Delta_{\widetilde{C}} + H(Y) - H(A) + codim_A(\overline{A}) + \Delta_{\overline{D}} \\
&\leq H(W) - H(D) + H(Y) - H(A) + \Delta_{\widetilde{C}} + \Delta_{\overline{A}} + \Delta_{\overline{D}} \qquad (2.47)
\end{aligned}
$$

In the justification for (2.44), we concluded that $f_B f_{20}$ is injective on $\overline{C}^*$, which implies f_{20} is injective on $\overline{C}^*$. In the justification for (2.45), we concluded that $f_D f_{28}$ is injective on $\widetilde{C}^*$, which implies f_{28} is injective on $\widetilde{C}^*$. These facts combined with (2.40) will be used to arrive on

line (2.48).

$$\begin{aligned}
codim_A(\overline{A}^*) &= H(A) - dim(f_A[f_{15}\overline{A} \cap f_{20}\overline{C}^* \cap f_{28}\widetilde{C}^*]) \\
&= H(A) - dim(f_{15}\overline{A} \cap f_{20}\overline{C}^* \cap f_{28}\widetilde{C}^*) \\
&= H(A) - H(Y) + codim_Y(f_{15}\overline{A} \cap f_{20}\overline{C}^* \cap f_{28}\widetilde{C}^*) \\
&\leq H(A) - H(Y) + codim_Y(f_{15}\overline{A}) + codim_Y(f_{20}\overline{C}^*) + codim_Y(f_{28}\widetilde{C}^*) \\
&= H(A) - H(Y) + H(Y) - dim(f_{15}\overline{A}) + H(Y) - dim(f_{20}\overline{C}^*) \\
&\quad + H(Y) - dim(f_{28}\widetilde{C}^*) \\
&= H(A) + 2H(Y) - dim(\overline{A}) - dim(\overline{C}^*) - dim(\widetilde{C}^*) \qquad (2.48) \\
&= H(A) + 2H(Y) - H(A) + codim_A(\overline{A}) - H(C) + codim_C(\overline{C}^*) \\
&\quad - H(C) + codim_C(\widetilde{C}^*) \\
&= 2H(Y) - 2H(C) + codim_A(\overline{A}) + codim_C(\overline{C}^*) + codim_C(\widetilde{C}^*) \\
&\leq 2H(Y) - 2H(C) + \Delta_{\overline{A}} \\
&\quad + H(Z) - H(B) + H(Y) - H(A) + \Delta_{\overline{C}} + \Delta_{\overline{A}} + \Delta_{\overline{B}} \\
&\quad + H(W) - H(D) + H(Y) - H(A) + \Delta_{\widetilde{C}} + \Delta_{\overline{A}} + \Delta_{\overline{D}} \\
&= H(W) + 4H(Y) + H(Z) - 2H(A) - H(B) - 2H(C) - H(D) \\
&\quad + 3\Delta_{\overline{A}} + \Delta_{\overline{B}} + \Delta_{\overline{C}} + \Delta_{\widetilde{C}} + \Delta_{\overline{D}} \\
&\triangleq \Delta_{\overline{A}^*} \qquad (2.49)
\end{aligned}$$

Let $t \in A$. Now we will assume t satisfies conditions (2.50) - (2.55). The justification of the conditions can be found below.

We will assume $t \in \overline{A}^*$. This is true on a subspace of A of codimension at most $\Delta_{\overline{A}^*}$ (2.50)

We will assume $f_{10}f_{15}t \in f_{19}(\overline{C} \cap f_{20}^{-1}f_{15}\overline{A}^*)$. This is true on a subspace of A of codimension at most $H(Z) - H(C) + H(Y) - H(A) + \Delta_{\overline{C}} + \Delta_{\overline{A}^*}$ (2.51)

We will assume $f_{11}f_{15}t \in f_{29}(\widetilde{C} \cap f_{28}^{-1}f_{15}\overline{A}^*)$. This is true on a subspace of A of codimension at most $H(W) - H(C) + H(Y) - H(A) + \Delta_{\widetilde{C}} + \Delta_{\overline{A}^*}$ (2.52)

We will assume $f_{12}f_{15}t \in f_{26}(\widehat{C} \cap f_{25}^{-1}f_{15}\overline{A}^*)$. This is true on a subspace of A of codimension at most $H(X) - H(C) + H(Y) - H(A) + \Delta_{\widehat{C}} + \Delta_{\overline{A}^*}$ (2.53)

We will assume $f_{10}f_{15}t \in f_{22}(\overline{B} \cap f_{23}^{-1}f_{26}[\widehat{C} \cap f_{25}^{-1}f_{15}\overline{A}^*])$. This is true on a subspace of A of codimension at most

$$H(Z) - H(B) + H(X) - H(C) + H(Y) - H(A) + \Delta_{\overline{A}^*} + \Delta_{\overline{B}} + \Delta_{\widehat{C}} \qquad (2.54)$$

We will assume $f_{11}f_{15}t \in f_{31}(\widehat{B} \cap f_{32}^{-1}f_{26}[\widehat{C} \cap f_{25}^{-1}f_{15}\overline{A}^*])$. This is true on a subspace of A of codimension at most

$$H(W) - H(B) + H(X) - H(C) + H(Y) - H(A) + \Delta_{\overline{A}^*} + \Delta_{\widehat{B}} + \Delta_{\widehat{C}} \qquad (2.55)$$

To justify (2.51), first we know f_{19} is injective on $\overline{C} \cap f_{20}^{-1} f_{15} \overline{A}^*$ by (2.41). Then by Lemma 1.4.3, we know $f_{10} f_{15} t \in f_{19}(\overline{C} \cap f_{20}^{-1} f_{15} \overline{A}^*)$ on a subspace of A of codimension at most $H(Z) - H(C) + codim_C(\overline{C} \cap f_{20}^{-1} f_{15} \overline{A}^*)$. By Lemma 1.4.1, we know

$$codim_C(\overline{C} \cap f_{20}^{-1} f_{15} \overline{A}^*) \leq \Delta_{\overline{C}} + codim_C(f_{20}^{-1} f_{15} \overline{A}^*)$$

Then using Lemma 1.4.3 and (2.40), we know

$$\begin{aligned} codim_C(\overline{C} \cap f_{20}^{-1} f_{15} \overline{A}^*) &\leq \Delta_{\overline{C}} + codim_Y(f_{15} \overline{A}^*) \\ &= \Delta_{\overline{C}} + H(Y) - dim(f_{15} \overline{A}^*) \\ &= \Delta_{\overline{C}} + H(Y) - dim(\overline{A}^*) \\ &\leq \Delta_{\overline{C}} + H(Y) - H(A) + \Delta_{\overline{A}^*} \end{aligned} \tag{2.56}$$

So we have $f_{10} f_{15} t \in f_{19}(\overline{C} \cap f_{20}^{-1} f_{15} \overline{A}^*)$ on a subspace of A of codimension at most $H(Z) - H(C) + H(Y) - H(A) + \Delta_{\overline{C}} + \Delta_{\overline{A}^*}$.

To justify (2.52), first we know f_{29} is injective on $\widetilde{C} \cap f_{28}^{-1} f_{15} \overline{A}^*$ by (2.42). Then by Lemma 1.4.3, we know $f_{11} f_{15} t \in f_{29}(\widetilde{C} \cap f_{28}^{-1} f_{15} \overline{A}^*)$ on a subspace of A of codimension at most $H(Z) - H(C) + codim_C(\widetilde{C} \cap f_{28}^{-1} f_{15} \overline{A}^*)$. By Lemma 1.4.1, we know

$$codim_C(\widetilde{C} \cap f_{28}^{-1} f_{15} \overline{A}^*) \leq \Delta_{\widetilde{C}} + codim_C(f_{28}^{-1} f_{15} \overline{A}^*)$$

Then using Lemma 1.4.3 and (2.40), we know

$$\begin{aligned} codim_C(\widetilde{C} \cap f_{28}^{-1} f_{15} \overline{A}^*) &\leq \Delta_{\widetilde{C}} + codim_Y(f_{15} \overline{A}^*) \\ &= \Delta_{\widetilde{C}} + H(Y) - dim(f_{15} \overline{A}^*) \\ &= \Delta_{\widetilde{C}} + H(Y) - dim(\overline{A}^*) \\ &\leq \Delta_{\widetilde{C}} + H(Y) - H(A) + \Delta_{\overline{A}^*} \end{aligned} \tag{2.57}$$

So we have $f_{11} f_{15} t \in f_{29}(\widetilde{C} \cap f_{28}^{-1} f_{15} \overline{A}^*)$ on a subspace of A of codimension at most $H(Z) - H(C) + H(Y) - H(A) + \Delta_{\widetilde{C}} + \Delta_{\overline{A}^*}$.

To justify (2.53), first we know f_{26} is injective on $\widehat{C} \cap f_{25}^{-1} f_{15} \overline{A}^*$ by (2.43). Then by Lemma 1.4.3, we know $f_{12} f_{15} t \in f_{26}(\widehat{C} \cap f_{25}^{-1} f_{15} \overline{A}^*)$ on a subspace of A of codimension at most $H(Z) - H(C) + codim_C(\widehat{C} \cap f_{25}^{-1} f_{15} \overline{A}^*)$. By Lemma 1.4.1, we know

$$codim_C(\widehat{C} \cap f_{25}^{-1} f_{15} \overline{A}^*) \leq \Delta_{\widehat{C}} + codim_C(f_{25}^{-1} f_{15} \overline{A}^*)$$

Then using Lemma 1.4.3 and (2.40), we know

$$\begin{aligned} codim_C(\widehat{C} \cap f_{25}^{-1} f_{15} \overline{A}^*) &\leq \Delta_{\widehat{C}} + codim_Y(f_{15} \overline{A}^*) \\ &= \Delta_{\widehat{C}} + H(Y) - dim(f_{15} \overline{A}^*) \\ &= \Delta_{\widehat{C}} + H(Y) - dim(\overline{A}^*) \\ &\leq \Delta_{\widehat{C}} + H(Y) - H(A) + \Delta_{\overline{A}^*} \end{aligned} \tag{2.58}$$

So we have $f_{12}f_{15}t \in f_{26}(\widehat{C} \cap f_{25}^{-1}f_{15}\overline{A}^*)$ on a subspace of A of codimension at most $H(Z) - H(C) + H(Y) - H(A) + \Delta_{\widehat{C}} + \Delta_{\overline{A}^*}$.

To justify (2.54), we first know f_{22} is injective on $\overline{B} \cap f_{23}^{-1}f_{26}[\widehat{C} \cap f_{25}^{-1}f_{15}\overline{A}^*]$ by (2.17). Then by Lemma 1.4.3, we know $f_{10}f_{15}t \in f_{22}(\overline{B} \cap f_{23}^{-1}f_{26}[\widehat{C} \cap f_{25}^{-1}f_{15}\overline{A}^*])$ on a subspace of A of codimension at most $H(Z) - H(B) + codim_B[\overline{B} \cap f_{23}^{-1}f_{26}(\widehat{C} \cap f_{25}^{-1}f_{15}\overline{A}^*)]$. Now again we are going to use Lemma 1.4.1, Lemma 1.4.3, and (2.40). Also on line (2.59) we will use the fact that f_{26} is injective on $\widehat{C} \cap f_{25}^{-1}f_{15}\overline{A}^*$ from (2.43).

$$\begin{aligned}
codim_B[\overline{B} \cap f_{23}^{-1}f_{26}(\widehat{C} \cap f_{25}^{-1}f_{15}\overline{A}^*)] &\leq \Delta_{\overline{B}} + codim_B(f_{23}^{-1}f_{26}[\widehat{C} \cap f_{25}^{-1}f_{15}\overline{A}^*]) \\
&\leq \Delta_{\overline{B}} + codim_X(f_{26}[\widehat{C} \cap f_{25}^{-1}f_{15}\overline{A}^*]) \\
&= \Delta_{\overline{B}} + H(X) - dim(f_{26}[\widehat{C} \cap f_{25}^{-1}f_{15}\overline{A}^*]) \\
&= \Delta_{\overline{B}} + H(X) - dim(\widehat{C} \cap f_{25}^{-1}f_{15}\overline{A}^*) \qquad (2.59) \\
&\leq \Delta_{\overline{B}} + H(X) - H(C) + codim_C(\widehat{C}) + codim_C(f_{25}^{-1}f_{15}\overline{A}^*) \\
&\leq \Delta_{\overline{B}} + H(X) - H(C) + \Delta_{\widehat{C}} + codim_Y(f_{15}\overline{A}^*) \\
&= \Delta_{\overline{B}} + H(X) - H(C) + \Delta_{\widehat{C}} + H(Y) - dim(f_{15}\overline{A}^*) \\
&= \Delta_{\overline{B}} + H(X) - H(C) + H(Y) + \Delta_{\widehat{C}} - dim(\overline{A}^*) \\
&= \Delta_{\overline{B}} + H(X) - H(C) + H(Y) + \Delta_{\widehat{C}} - H(A) + codim_A(\overline{A}^*) \\
&\leq \Delta_{\overline{B}} + H(X) - H(C) + H(Y) - H(A) + \Delta_{\widehat{C}} + \Delta_{\overline{A}^*} \qquad (2.60)
\end{aligned}$$

So we have $f_{10}f_{15}t \in f_{22}(\overline{B} \cap f_{23}^{-1}f_{26}[\widehat{C} \cap f_{25}^{-1}f_{15}\overline{A}^*])$ on a subspace of A of codimension at most $H(Z) - H(C) + H(Y) - H(A) + \Delta_{\overline{A}^*} + \Delta_{\overline{B}} + \Delta_{\widehat{C}}$.

To justify (2.55), we first know f_{31} is injective on $\widehat{B} \cap f_{32}^{-1}f_{26}[\widehat{C} \cap f_{25}^{-1}f_{15}\overline{A}^*]$ by (2.21). Then by Lemma 1.4.3, we know $f_{11}f_{15}t \in f_{31}(\widehat{B} \cap f_{32}^{-1}f_{26}[\widehat{C} \cap f_{25}^{-1}f_{15}\overline{A}^*])$ on a subspace of A of codimension at most $H(W) - H(B) + codim_B[\widehat{B} \cap f_{32}^{-1}f_{26}(\widehat{C} \cap f_{25}^{-1}f_{15}\overline{A}^*)]$. Now again we are going to use Lemma 1.4.1 and Lemma 1.4.3,

$$\begin{aligned}
codim_B(\widehat{B} \cap f_{32}^{-1}f_{26}[\widehat{C} \cap f_{25}^{-1}f_{15}\overline{A}^*]) &\leq \Delta_{\widehat{B}} + codim_B(f_{32}^{-1}f_{26}[\widehat{C} \cap f_{25}^{-1}f_{15}\overline{A}^*]) \\
&\leq \Delta_{\widehat{B}} + codim_X(f_{26}[\widehat{C} \cap f_{25}^{-1}f_{15}\overline{A}^*]) \\
&\leq \Delta_{\widehat{B}} + H(X) - H(C) + H(Y) - H(A) + \Delta_{\widehat{C}} + \Delta_{\overline{A}^*}
\end{aligned}$$

The last line was derived by copying the argument from (2.60). So we have $f_{11}f_{15}t \in f_{31}(\widehat{B} \cap f_{32}^{-1}f_{26}[\widehat{C} \cap f_{25}^{-1}f_{15}\overline{A}^*])$ on a subspace of A of codimension at most $H(W) - H(C) + H(Y) - H(A) + \Delta_{\overline{A}^*} + \Delta_{\widehat{B}} + \Delta_{\widehat{C}}$.

From (2.51) and (2.54) we know $\exists \overline{c} \in \overline{C}, \overline{b} \in \overline{B}$ such that

$$f_{10}f_{15}t = f_{19}\overline{c} = f_{22}\overline{b} \text{ where } f_{20}\overline{c} \in f_{15}\overline{A}^* \text{ and } f_{23}\overline{b} \in f_{26}(\widehat{C} \cap f_{25}^{-1}f_{15}\overline{A}^*) \qquad (2.61)$$

From (2.52) and (2.55) we know $\exists \widetilde{c} \in \widetilde{C}, \widehat{b} \in \widehat{B}$ such that

$$f_{11}f_{15}t = f_{29}\widetilde{c} = f_{31}\widehat{b} \text{ where } f_{28}\widetilde{c} \in f_{15}\overline{A}^* \text{ and } f_{32}\widehat{b} \in f_{26}(\widehat{C} \cap f_{25}^{-1}f_{15}\overline{A}^*) \qquad (2.62)$$

From (2.53) we know $\exists \widehat{c} \in \widehat{C}$ such that

$$f_{12} f_{15} t = f_{26} \widehat{c} \text{ where } f_{25} \widehat{c} \in f_{15} \overline{A}^* \tag{2.63}$$

From (2.12) and (2.13), we know

$$\begin{aligned} f_B f_{15} &= -f_{13} \\ f_B &= -f_{13} f_A \text{ on } f_{15} \overline{A} \end{aligned} \tag{2.64}$$

From (2.12) and (2.15), we know

$$\begin{aligned} f_D f_{15} &= -f_{14} \\ f_D &= -f_{14} f_A \text{ on } f_{15} \overline{A} \end{aligned} \tag{2.65}$$

From (2.12) we have

$$f_7 f_{12} f_{15} t + f_1 f_{10} f_{15} t = t$$

Then (2.63), (2.61), (2.28), and (2.24) gives

$$\begin{aligned} f_7 f_{12} f_{15} t + f_1 f_{10} f_{15} t &= t \\ f_7 f_{26} \widehat{c} + f_1 f_{19} \overline{c} &= t \\ -f_A f_{25} \widehat{c} - f_A f_{20} \overline{c} &= t \\ f_A f_{25} \widehat{c} + f_A f_{20} \overline{c} &= -t \end{aligned} \tag{2.66}$$

From (2.13) we have

$$f_4 f_{11} f_{15} t + f_2 f_{10} f_{15} t = -f_{13} t$$

Then (2.62), (2.61), (2.33), and (2.25) gives

$$\begin{aligned} f_4 f_{11} f_{15} t + f_2 f_{10} f_{15} t &= -f_{13} t \\ f_4 f_{29} \widetilde{c} + f_2 f_{19} \overline{c} &= -f_{13} t \\ -f_B f_{28} \widetilde{c} - f_B f_{20} \overline{c} &= -f_{13} t \end{aligned}$$

By (2.62) and (2.61), we know $f_{28} \widetilde{c} \in f_{15} \overline{A}^*$ and $f_{20} \overline{c} \in f_{15} \overline{A}^*$. Now by (2.64), we have

$$\begin{aligned} -f_B f_{28} \widetilde{c} - f_B f_{20} \overline{c} &= -f_{13} t \\ f_{13} f_A f_{28} \widetilde{c} + f_{13} f_A f_{20} \overline{c} &= -f_{13} t \end{aligned}$$

Then using (2.12), we know $f_A f_{28} \widetilde{c} \in \overline{A}^*$ and $f_A f_{20} \overline{c} \in \overline{A}^*$. By (2.44), we have

$$\begin{aligned} f_{13} f_A f_{28} \widetilde{c} + f_{13} f_A f_{20} \overline{c} &= -f_{13} t \\ f_A f_{28} \widetilde{c} + f_A f_{20} \overline{c} &= -t \end{aligned} \tag{2.67}$$

From (2.15) we have

$$f_9 f_{12} f_{15} t + f_6 f_{11} f_{15} t = -f_{14} t$$

Then (2.63), (2.62), (2.35), and (2.31) gives

$$\begin{aligned} f_9 f_{12} f_{15} t + f_6 f_{11} f_{15} t &= -f_{14} t \\ f_9 f_{26} \widehat{c} + f_6 f_{29} \widetilde{c} &= -f_{14} t \\ -f_D f_{25} \widehat{c} + -f_D f_{28} \widetilde{c} &= -f_{14} t \end{aligned}$$

By (2.63) and (2.62), we know $f_{25}\widehat{c} \in f_{15}\overline{A}^*$ and $f_{28}\widetilde{c} \in f_{15}\overline{A}^*$. Now by (2.65), we have

$$\begin{aligned} -f_D f_{25} \widehat{c} + -f_D f_{28} \widetilde{c} &= -f_{14} t \\ f_{14} f_A f_{25} \widehat{c} + f_{14} f_A f_{28} \widetilde{c} &= -f_{14} t \end{aligned}$$

Then using (2.12), we know $f_A f_{25}\widehat{c} \in \overline{A}^*$ and $f_A f_{28}\widetilde{c} \in \overline{A}^*$. By (2.45), we have

$$\begin{aligned} f_{14} f_A f_{25} \widehat{c} + f_{14} f_A f_{28} \widetilde{c} &= -f_{14} t \\ f_A f_{25} \widehat{c} + f_A f_{28} \widetilde{c} &= -t \end{aligned} \tag{2.68}$$

From (2.24) and (2.41), we know

$$\begin{aligned} f_1 f_{19} &= -f_A f_{20} \\ f_1 &= -f_A f_{20} f_3 \text{ on } f_{19}(\overline{C} \cap f_{20}^{-1} f_{15} \overline{A}^*) \end{aligned} \tag{2.69}$$

From (2.28) and (2.43), we know

$$\begin{aligned} f_7 f_{26} &= -f_A f_{25} \\ f_7 &= -f_A f_{25} f_8 \text{ on } f_{26}(\widehat{C} \cap f_{25}^{-1} f_{15} \overline{A}^*) \end{aligned} \tag{2.70}$$

From (2.16), we have

$$f_7 f_{23} \overline{b} + f_1 f_{22} \overline{b} = 0$$

By (2.61), we know $f_{23}\overline{b} \in f_{26}(\widehat{C} \cap f_{25}^{-1} f_{15} \overline{A}^*)$. By (2.61), we also know $f_{22}\overline{b} = f_{19}\overline{c}$, which implies $f_{22}\overline{b} \in f_{19}(\overline{C} \cap f_{20}^{-1} f_{15} \overline{A}^*)$. Now we can apply (2.69) and (2.70) to give us

$$\begin{aligned} f_7 f_{23} \overline{b} + f_1 f_{22} \overline{b} &= 0 \\ -f_A f_{25} f_8 f_{23} \overline{b} - f_A f_{20} f_3 f_{22} \overline{b} &= 0 \end{aligned}$$

Now using (2.18), (2.61), and (2.41), we have

$$\begin{aligned} -f_A f_{25} f_8 f_{23} \overline{b} - f_A f_{20} f_3 f_{22} \overline{b} &= 0 \\ f_A f_{25} f_3 f_{22} \overline{b} - f_A f_{20} f_3 f_{22} \overline{b} &= 0 \\ f_A f_{25} f_3 f_{22} \overline{b} &= f_A f_{20} f_3 f_{22} \overline{b} \\ f_A f_{25} f_3 f_{19} \overline{c} &= f_A f_{20} f_3 f_{19} \overline{c} \\ f_A f_{25} \overline{c} &= f_A f_{20} \overline{c} \end{aligned} \tag{2.71}$$

From (2.31) and (2.43), we know

$$\begin{aligned} f_9 f_{26} &= -f_D f_{25} \\ f_9 &= -f_D f_{25} f_8 \text{ on } f_{26}(\widehat{C} \cap f_{25}^{-1} f_{15} \overline{A}^*) \end{aligned} \tag{2.72}$$

From (2.35) and (2.42), we know

$$\begin{aligned} f_6 f_{29} &= -f_D f_{28} \\ f_6 &= -f_D f_{28} f_5 \text{ on } f_{29}(\widetilde{C} \cap f_{28}^{-1} f_{15} \overline{A}^*) \end{aligned} \tag{2.73}$$

From (2.23), we have

$$f_9 f_{32} \widehat{b} + f_6 f_{31} \widehat{b} = 0$$

From (2.62) we know $f_{31}\widehat{b} = f_{29}\widetilde{c}$ so $f_{31}\widehat{b} \in f_{29}(\widetilde{C} \cap f_{28}^{-1} f_{15} \overline{A}^*)$. From (2.62) we also know that $f_{32}\widehat{b} \in f_{26}(\widehat{C} \cap f_{25}^{-1} f_{15} \overline{A}^*)$, so (2.72) and (2.73) give us

$$\begin{aligned} f_9 f_{32} \widehat{b} + f_6 f_{31} \widehat{b} &= 0 \\ -f_D f_{25} f_8 f_{32} \widehat{b} - f_D f_{28} f_5 f_{31} \widehat{b} &= 0 \end{aligned}$$

From (2.62), we know $f_{32}\widehat{b} \in f_{26}(\widehat{C} \cap f_{25}^{-1} f_{15} \overline{A}^*)$. From (2.43), we know $f_8 f_{26} = I$ on $\widehat{C} \cap f_{25}^{-1} f_{15} \overline{A}^*$. So $f_8 f_{32} \widehat{b} \in f_{25}^{-1} f_{15} \overline{A}^*$, which implies $f_{25} f_8 f_{32} \widehat{b} \in f_{15} \overline{A}^*$. By (2.62) and (2.42), we know $f_{28} f_5 f_{31} \widehat{b} = f_{28} f_5 f_{29} \widetilde{c} = f_{28} \widetilde{c} \in f_{15} \overline{A}^*$. Now we can apply (2.65) to give us

$$\begin{aligned} -f_D f_{25} f_8 f_{32} \widehat{b} - f_D f_{28} f_5 f_{31} \widehat{b} &= 0 \\ f_{14} f_A f_{25} f_8 f_{32} \widehat{b} + f_{14} f_A f_{28} f_5 f_{31} \widehat{b} &= 0 \end{aligned}$$

Since we already established that $f_{25} f_8 f_{32} \widehat{b} \in f_{15} \overline{A}^*$ and $f_{28} f_5 f_{31} \widehat{b} \in f_{15} \overline{A}^*$, by (2.12) and (2.45) we know

$$\begin{aligned} f_{14} f_A f_{25} f_8 f_{32} \widehat{b} + f_{14} f_A f_{28} f_5 f_{31} \widehat{b} &= 0 \\ f_A f_{25} f_8 f_{32} \widehat{b} + f_A f_{28} f_5 f_{31} \widehat{b} &= 0 \end{aligned}$$

Now by (2.22)

$$\begin{aligned} f_A f_{25} f_8 f_{32} \widehat{b} + f_A f_{28} f_5 f_{31} \widehat{b} &= 0 \\ -f_A f_{25} f_5 f_{31} \widehat{b} + f_A f_{28} f_5 f_{31} \widehat{b} &= 0 \\ f_A f_{25} f_5 f_{31} \widehat{b} &= f_A f_{28} f_5 f_{31} \widehat{b} \end{aligned}$$

By (2.62) and (2.42), we have

$$\begin{aligned} f_A f_{25} f_5 f_{31} \widehat{b} &= f_A f_{28} f_5 f_{31} \widehat{b} \\ f_A f_{25} f_5 f_{29} \widetilde{c} &= f_A f_{28} f_5 f_{29} \widetilde{c} \\ f_A f_{25} \widetilde{c} &= f_A f_{28} \widetilde{c} \end{aligned} \tag{2.74}$$

Now adding (2.66), (2.67), and (2.68), we have

$$-3t = 2(f_A f_{20}\overline{c} + f_A f_{25}\widehat{c} + f_A f_{28}\widetilde{c})$$

Now using (2.71) and (2.74) we have

$$\begin{aligned} -3t &= 2(f_A f_{25}\overline{c} + f_A f_{25}\widehat{c} + f_A f_{25}\widetilde{c}) \\ -3t &= 2f_A f_{25}(\overline{c} + \widehat{c} + \widetilde{c}) \end{aligned}$$

By (2.41), (2.42), and (2.43) we know

$$-3t = 2f_A f_{25}(f_3 f_{19}\overline{c} + f_8 f_{26}\widehat{c} + f_5 f_{29}\widetilde{c})$$

By (2.61), (2.62), (2.63), and (2.14), we have

$$\begin{aligned} -3t &= 2f_A f_{25}(f_3 f_{10} f_{15} t + f_8 f_{12} f_{15} t + f_5 f_{11} f_{15} t) \\ -3t &= 2f_A f_{25}(0) \\ 3t &= 0 \end{aligned} \tag{2.75}$$

Thus if the field is of characteristic other than 3, then no nonzero t can satisfy conditions (2.50)-(2.55). Therefore the sum of the codimensions given in the assumptions must be at least the dimension of A. So we have a linear rank inequality for fields of characteristic other than 3:

$$\begin{aligned} H(A) \leq\ & \Delta_{\overline{A}^*} + H(Z) - H(C) + H(Y) - H(A) + \Delta_{\overline{C}} + \Delta_{\overline{A}^*} \\ & + H(W) - H(C) + H(Y) - H(A) + \Delta_{\widetilde{C}} + \Delta_{\overline{A}^*} \\ & + H(X) - H(C) + H(Y) - H(A) + \Delta_{\widehat{C}} + \Delta_{\overline{A}^*} \\ & + H(Z) - H(B) + H(X) - H(C) + H(Y) - H(A) + \Delta_{\overline{A}^*} + \Delta_{\overline{B}} + \Delta_{\widehat{C}} \\ & + H(W) - H(B) + H(X) - H(C) + H(Y) - H(A) + \Delta_{\overline{A}^*} + \Delta_{\widehat{B}} + \Delta_{\widehat{C}} \\ =\ & 2H(Z) + 5H(Y) + 3H(X) + 2H(W) - 5H(A) - 2H(B) - 5H(C) \\ & + 6\Delta_{\overline{A}^*} + \Delta_{\overline{B}} + \Delta_{\widehat{B}} + \Delta_{\overline{C}} + \Delta_{\widetilde{C}} + 3\Delta_{\widehat{C}} \\ =\ & 2H(Z) + 5H(Y) + 3H(X) + 2H(W) - 5H(A) - 2H(B) - 5H(C) \\ & + 6(H(W) + 4H(Y) + H(Z) - 2H(A) - H(B) - 2H(C) - H(D)) \\ & + 6(3\Delta_{\overline{A}} + \Delta_{\overline{B}} + \Delta_{\overline{C}} + \Delta_{\widetilde{C}} + \Delta_{\overline{D}}) + \Delta_{\overline{B}} + \Delta_{\widehat{B}} + \Delta_{\overline{C}} + \Delta_{\widetilde{C}} + 3\Delta_{\widehat{C}} \\ =\ & 8H(Z) + 29H(Y) + 3H(X) + 8H(W) - 6H(D) - 17H(C) - 8H(B) - 17H(A) \\ & + 18\Delta_{\overline{A}} + 7\Delta_{\overline{B}} + \Delta_{\widehat{B}} + 7\Delta_{\overline{C}} + 7\Delta_{\widetilde{C}} + 3\Delta_{\widehat{C}} + 6\Delta_{\overline{D}} \\ =\ & 8H(Z) + 29H(Y) + 3H(X) + 8H(W) - 6H(D) - 17H(C) - 8H(B) - 17H(A) \\ & + 55H(Z|A,B,C) + 35H(Y|W,X,Z) + 50H(X|A,C,D) + 45H(W|B,C,D) \\ & + 18H(A|B,D,Y) + 7H(B|D,X,Z) + H(B|A,W,X) + 7H(C|D,Y,Z) \\ & + 7H(C|B,X,Y) + 3H(C|A,W,Y) + 6H(D|A,W,Z) \\ & + 49(H(A) + H(B) + H(C) + H(D) - H(A,B,C,D)) \end{aligned}$$

Now notice that the linear rank inequality does not hold for characteristic 3. A counterexample would be: In $V = GF(3)^4$ where

$$\begin{aligned} A &= \langle(1,0,0,0)\rangle & B &= \langle(0,1,0,0)\rangle \\ C &= \langle(0,0,1,0)\rangle & D &= \langle(0,0,0,1)\rangle \\ W &= \langle(0,1,1,1)\rangle & X &= \langle(1,0,1,1)\rangle \\ Y &= \langle(1,1,0,1)\rangle & Z &= \langle(1,1,1,0)\rangle \end{aligned}$$

Then

$$\begin{aligned} H(Z|A,B,C) &= H(Y|W,X,Z) \\ &= H(X|A,C,D) \\ &= H(W|B,C,D) \\ &= H(A|B,D,Y) \\ &= H(B|D,X,Z) \\ &= H(B|A,W,X) \\ &= H(C|D,Y,Z) \\ &= H(C|B,X,Y) \\ &= H(C|A,W,Y) \\ &= H(D|A,W,Z) \\ &= 0 \end{aligned}$$

Since W, X, Y, Z are independent, we also have $H(W)+H(X)+H(Y)+H(Z) = H(W,X,Y,Z)$. So the inequality becomes

$$\begin{aligned} H(A) &\leq 8H(Z) + 29H(Y) + 3H(X) + 8H(W) - 6H(D) - 17H(C) - 8H(B) - 17H(A) \\ 1 &\leq 8 + 29 + 3 + 8 - 6 - 17 - 8 - 17 \\ 1 &\leq 0 \end{aligned}$$

which is clearly a contradiction. Therefore, the inequality above is a linear rank inequality for fields of characteristic other than 3. □

Corollary 2.1.2. *The linear coding capacity of the $T8$ network is at most 48/49 over any characteristic other than 3. The linear coding capacity over characteristic 3 and the coding capacity is 1.*

Proof. Let us apply the linear rank inequality derived in Theorem 2.1.1 to the $T8$ network. Then we would have:

$$\begin{aligned} H(Z|A,B,C) &= H(Y|W,X,Z) \\ &= H(X|A,C,D) \\ &= H(W|B,C,D) \\ &= H(A|B,D,Y) \\ &= H(B|D,X,Z) \\ &= H(B|A,W,X) \\ &= H(C|D,Y,Z) \\ &= H(C|B,X,Y) \\ &= H(C|A,W,Y) \\ &= H(D|A,W,Z) \\ &= 0. \end{aligned}$$

Since W, X, Y, Z are independent, we also have $H(W)+H(X)+H(Y)+H(Z) = H(W,X,Y,Z)$. So the inequality becomes

$$H(A) \leq 8H(Z) + 29H(Y) + 3H(X) + 8H(W) - 6H(D) - 17H(C) - 8H(B) - 17H(A).$$

Now we know $H(A) = H(B) = H(C) = H(D) = k$ and $H(W) = H(X) = H(Y) = H(Z) = n$, so we have

$$\begin{aligned} k &\leq 8n + 29n + 3n + 8n - 6k - 17k - 8k - 17k \\ 49k &\leq 48n \\ k/n &\leq 48/49 \end{aligned}$$

So the linear coding capacity over every characteristic except for 3 is at most $48/49 < 1$. The $T8$ network is solvable over characteristic 3 by the following coding solution:

$$\begin{aligned} Z &= A + B + C \\ W &= B + C + D \\ X &= A + C + D \\ Y &= W + X + Z \end{aligned}$$

We know the coding capacity is at most 1 because there is a unique path from source A to node n_9 and by the coding solution given above we know the capacity is at least 1, thus the capacity is 1.

□

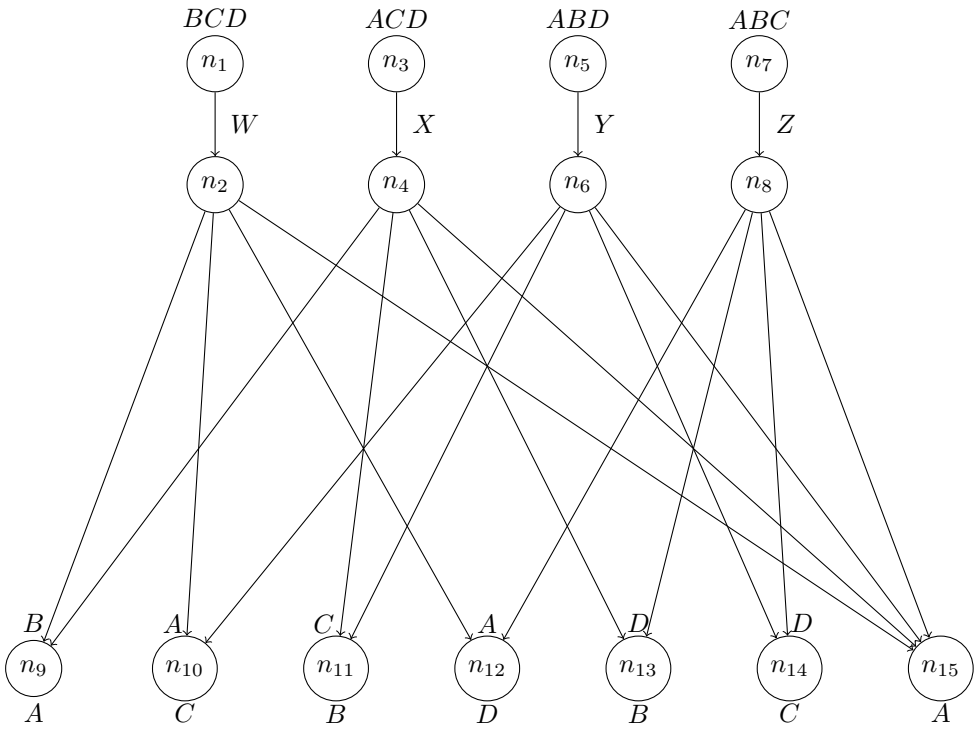

Figure 2.2: Non-$T8$ Network

2.2 A Linear Rank Inequality for Fields of Characteristic 3

We will define the *non-$T8$ matroid* to be the $T8$ matroid except we are going to force the circuit $\{W, X, Y, Z\}$ to be a base. Figure 2.2 is a network whose dependencies and independencies are consistent with the non-$T8$ matroid. It was also designed by the construction process described in [Dougherty 07], and we will refer to it as the non-$T8$ network. Theorem 2.2.1 uses the non-$T8$ network as a guide to derive a linear rank inequality valid for characteristic 3. The new linear rank inequality can then be used to prove the non-$T8$ network is only linearly solvable if the characteristic is not 3.

Theorem 2.2.1. *Let A, B, C, D, W, X, Y, and Z be subspaces of a vector space V. Then the*

following is a linear rank inequality for fields of characteristic 3,

$$\begin{aligned} H(A) \leq\ & 9H(Z)+8H(Y)+5H(X)+6H(W)-4H(D)-12H(C)-11H(B)-H(A) \\ & 19H(Z|A,B,C)+17H(Y|A,B,D)+13H(X|A,C,D)+11H(W|B,C,D) \\ & H(A|W,X,Y,Z)+H(A|B,W,X)+7H(B|D,X,Z)+4H(B|C,X,Y) \\ & 7H(C|D,Y,Z)+5H(C|A,W,Y)+4H(D|A,W,Z) \\ & +29(H(A)+H(B)+H(C)+H(D)-H(A,B,C,D)) \end{aligned}$$

Proof. Let V be a finite dimensional vector space with subspaces A, B, C, D, W, X, Y, Z. By Lemma 1.4.4 we get linear functions:

$$\begin{array}{ccccc} f_1 : W \to B, & & f_2 : W \to C, & & f_3 : W \to D, \\ f_4 : X \to A, & & f_5 : X \to C, & & f_6 : X \to D, \\ f_7 : Y \to A, & & f_8 : Y \to B, & & f_9 : Y \to D, \\ f_{10} : Z \to A, & & f_{11} : Z \to B, & & f_{12} : Z \to C, \\ f_{13} : A \to B, & & f_{14} : A \to W, & & f_{15} : A \to X, \\ f_{16} : C \to A, & & f_{17} : C \to W, & & f_{18} : C \to Y, \\ f_{19} : B \to C, & & f_{20} : B \to X, & & f_{21} : B \to Y, \\ f_{22} : D \to W, & & f_{23} : D \to A, & & f_{24} : D \to Z, \\ f_{25} : B \to X, & & f_{26} : B \to D, & & f_{27} : B \to Z, \\ f_{28} : C \to Y, & & f_{29} : C \to Z, & & f_{30} : C \to D, \\ f_{31} : A \to W, & f_{32} : A \to X, & f_{33} : A \to Y, & f_{34} : A \to Z & \end{array}$$

such that

$$f_1+f_2+f_3 \equiv I \text{ on a subspace of } W \text{ of codimension } H(W|B,C,D) \quad (2.76)$$

$$f_4+f_5+f_6 \equiv I \text{ on a subspace of } X \text{ of codimension } H(X|A,C,D) \quad (2.77)$$

$$f_7+f_8+f_9 \equiv I \text{ on a subspace of } Y \text{ of codimension } H(Y|A,B,D) \quad (2.78)$$

$$f_{10}+f_{11}+f_{12} \equiv I \text{ on a subspace of } Z \text{ of codimension } H(Z|A,B,C) \quad (2.79)$$

$$f_{13}+f_{14}+f_{15} \equiv I \text{ on a subspace of } A \text{ of codimension } H(A|B,W,X) \quad (2.80)$$

$$f_{16}+f_{17}+f_{18} \equiv I \text{ on a subspace of } C \text{ of codimension } H(C|A,W,Y) \quad (2.81)$$

$$f_{19}+f_{20}+f_{21} \equiv I \text{ on a subspace of } B \text{ of codimension } H(B|C,X,Y) \quad (2.82)$$

$$f_{22}+f_{23}+f_{24} \equiv I \text{ on a subspace of } D \text{ of codimension } H(D|A,W,Z) \quad (2.83)$$

$$f_{25}+f_{26}+f_{27} \equiv I \text{ on a subspace of } B \text{ of codimension } H(B|D,X,Z) \quad (2.84)$$

$$f_{28}+f_{29}+f_{30} \equiv I \text{ on a subspace of } C \text{ of codimension } H(C|D,Y,Z) \quad (2.85)$$

$$f_{31}+f_{32}+f_{33}+f_{34} \equiv I \text{ on a subspace of } A \text{ of codimension } H(A|W,X,Y,Z) \quad (2.86)$$

Using (2.76) - (2.79), (2.86), Lemma 1.4.1, and Lemma 1.4.3 we know the sum of these functions is equal to I on a subspace of A of codimension at most $H(W|B,C,D)+H(X|A,C,D)+H(Y|A,B,D)+H(Z|A,B,C)+H(A|W,X,Y,Z)$.

Now applying Lemma 1.4.6 and Lemma 1.4.1 to $f_4 \circ f_{32} + f_7 \circ f_{33} + f_{10} \circ f_{34} - I$, $f_1 \circ f_{31} + f_8 \circ f_{33} + f_{11} \circ f_{34}$, $f_2 \circ f_{31} + f_5 \circ f_{32} + f_{12} \circ f_{34}$, and $f_3 \circ f_{31} + f_6 \circ f_{32} + f_9 \circ f_{33}$ we

get a subspace $\widehat{A}$ of A of codimension at most

$$\begin{aligned}\Delta_{\widehat{A}} &= H(W|B,C,D)+H(X|A,C,D)+H(Y|A,B,D)+H(Z|A,B,C)+H(A|W,X,Y,Z)\\ &\quad +H(A)+H(B)+H(C)+H(D)-H(A,B,C,D)\end{aligned}$$

on which,

$$f_4\circ f_{32}+f_7\circ f_{33}+f_{10}\circ f_{34} \equiv I \tag{2.87}$$
$$f_1\circ f_{31}+f_8\circ f_{33}+f_{11}\circ f_{34} \equiv 0 \tag{2.88}$$
$$f_2\circ f_{31}+f_5\circ f_{32}+f_{12}\circ f_{34} \equiv 0 \tag{2.89}$$
$$f_3\circ f_{31}+f_6\circ f_{32}+f_9\circ f_{33} \equiv 0 \tag{2.90}$$

Similarly, we get a subspace $\overline{A}$ of A of codimension at most

$$\begin{aligned}\Delta_{\overline{A}} &= H(W|B,C,D)+H(X|A,C,D)+H(A|B,W,X)\\ &\quad +H(A)+H(B)+H(C)+H(D)-H(A,B,C,D)\end{aligned}$$

on which,

$$f_4\circ f_{15} \equiv I \tag{2.91}$$
$$f_{13}+f_1\circ f_{14} \equiv 0 \tag{2.92}$$
$$f_2\circ f_{14}+f_5\circ f_{15} \equiv 0 \tag{2.93}$$
$$f_3\circ f_{14}+f_6\circ f_{15} \equiv 0 \tag{2.94}$$

We get a subspace $\overline{B}$ of B of codimension at most

$$\begin{aligned}\Delta_{\overline{B}} &= H(X|A,C,D)+H(Y|A,B,D)+H(B|C,X,Y)\\ &\quad +H(A)+H(B)+H(C)+H(D)-H(A,B,C,D)\end{aligned}$$

on which,

$$f_4\circ f_{20}+f_7\circ f_{21} \equiv 0 \tag{2.95}$$
$$f_8\circ f_{21} \equiv I \tag{2.96}$$
$$f_{19}+f_5\circ f_{20} \equiv 0 \tag{2.97}$$
$$f_6\circ f_{20}+f_9\circ f_{21} \equiv 0 \tag{2.98}$$

We get a subspace $\widehat{B}$ of B of codimension at most

$$\begin{aligned}\Delta_{\widehat{B}} &= H(X|A,C,D)+H(Z|A,B,C)+H(B|D,X,Z)\\ &\quad +H(A)+H(B)+H(C)+H(D)-H(A,B,C,D)\end{aligned}$$

on which,

$$f_4 \circ f_{25} + f_{10} \circ f_{27} \equiv 0 \quad (2.99)$$
$$f_{11} \circ f_{27} \equiv I \quad (2.100)$$
$$f_5 \circ f_{25} + f_{12} \circ f_{27} \equiv 0 \quad (2.101)$$
$$f_6 \circ f_{25} + f_{26} \equiv 0 \quad (2.102)$$

We get a subspace $\overline{C}$ of C of codimension at most

$$\begin{aligned}\Delta_{\overline{C}} &= H(W|B,C,D) + H(Y|A,B,D) + H(C|A,W,Y) \\ &\quad + H(A) + H(B) + H(C) + H(D) - H(A,B,C,D)\end{aligned}$$

on which,

$$f_{16} + f_7 \circ f_{18} \equiv 0 \quad (2.103)$$
$$f_1 \circ f_{17} + f_8 \circ f_{18} \equiv 0 \quad (2.104)$$
$$f_2 \circ f_{17} \equiv I \quad (2.105)$$
$$f_3 \circ f_{17} + f_9 \circ f_{18} \equiv 0 \quad (2.106)$$

We get a subspace $\widehat{C}$ of C of codimension at most

$$\begin{aligned}\Delta_{\widehat{C}} &= H(Y|A,B,D) + H(Z|A,B,C) + H(C|D,Y,Z) \\ &\quad + H(A) + H(B) + H(C) + H(D) - H(A,B,C,D)\end{aligned}$$

on which,

$$f_7 \circ f_{28} + f_{10} \circ f_{29} \equiv 0 \quad (2.107)$$
$$f_8 \circ f_{28} + f_{11} \circ f_{29} \equiv 0 \quad (2.108)$$
$$f_{12} \circ f_{29} \equiv I \quad (2.109)$$
$$f_9 \circ f_{28} + f_{30} \equiv 0 \quad (2.110)$$

We get a subspace $\overline{D}$ of D of codimension at most

$$\begin{aligned}\Delta_{\overline{D}} &= H(W|B,C,D) + H(Z|A,B,C) + H(D|A,W,Z) \\ &\quad + H(A) + H(B) + H(C) + H(D) - H(A,B,C,D)\end{aligned}$$

on which,

$$f_{23} + f_{10} \circ f_{24} \equiv 0 \quad (2.111)$$
$$f_1 \circ f_{22} + f_{11} \circ f_{24} \equiv 0 \quad (2.112)$$
$$f_2 \circ f_{22} + f_{12} \circ f_{24} \equiv 0 \quad (2.113)$$
$$f_3 \circ f_{22} \equiv I \quad (2.114)$$

Let $\widehat{B}^* = f_{11}(f_{27}\widehat{B} \cap f_{29}\widehat{C}) \subseteq \widehat{B}$. Considering (2.100) and (2.109), we can apply Lemma 2.7 1.4.7 to show that $f_{12}f_{27}$ is injective on $\widehat{B}^*$. By (2.101), we know

$$f_5 f_{25} \text{ is injective on } \widehat{B}^*. \tag{2.115}$$

Let $\widehat{C}^* = f_{12}(f_{29}\widehat{C} \cap f_{27}\widehat{B}) \subseteq \widehat{C}$. Considering again (2.100) and (2.109), we can apply Lemma 2.71.4.7 to show that $f_{11}f_{29}$ is injective on $\widehat{C}^*$. By (2.108), we know

$$f_8 f_{28} \text{ is injective on } \widehat{C}^*. \tag{2.116}$$

Let $\overline{A}^* = f_4(f_{15}\overline{A} \cap f_{25}\widehat{B}^*) \subseteq \overline{A}$. Considering (2.91) and (2.115), we can apply Lemma 2.71.4.7 to show that $f_5 f_{15}$ is injective on $\overline{A}^*$. By (2.93), we know $f_2 f_{14}$ is injective on $\overline{A}^*$ which implies

$$f_{14} \text{ is injective on } \overline{A}^*. \tag{2.117}$$

Let $\overline{C}^* = f_2(f_{17}\overline{C} \cap f_{22}\overline{D}) \subseteq \overline{C}$. Considering (2.105) and (2.114),we can apply Lemma 2.71.4.7 to show that $f_3 f_{17}$ is injective on $\overline{C}^*$. Then by (2.106), we know

$$f_9 f_{18} \text{ is injective on } \overline{C}^*. \tag{2.118}$$

Let $\overline{B}^* = f_8(f_{21}\overline{B} \cap f_{18}\overline{C}^*) \subseteq \overline{B}$. Considering (2.96) and the fact that $f_9 f_{18}$ is injective on $\overline{C}^*$, we can apply Lemma 2.71.4.7 to show that

$$f_9 f_{21} \text{ is injective on } \overline{B}^*. \tag{2.119}$$

By (2.98), we know

$$f_6 f_{20} \text{ is injective on } \overline{B}^* \tag{2.120}$$

which implies

$$f_{20} \text{ is injective on } \overline{B}^*. \tag{2.121}$$

Now considering (2.96), (2.100), (2.105), and (2.109) we have

$$\begin{aligned}
(2.104) \qquad & f_1 = -f_8 f_{18} f_2 \text{ on } f_{17}\overline{C} && (2.122)\\
(2.93) \qquad & f_2 = -f_5 f_{15} f_{14}^{-1} \text{ on } f_{14}\overline{A}^* && (2.123)\\
(2.94), (2.106) \qquad & f_3 = -f_6 f_{15} f_{14}^{-1} \text{ on } f_{14}\overline{A}^* \text{ and } f_3 = -f_9 f_{18} f_2 \text{ on } f_{17}\overline{C} && (2.124)\\
(2.95) \qquad & f_4 = -f_7 f_{21} f_{20}^{-1} \text{ on } f_{20}\overline{B}^* && (2.125)\\
(2.98) \qquad & f_6 = -f_9 f_{21} f_{20}^{-1} \text{ on } f_{20}\overline{B}^* && (2.126)\\
(2.95) \qquad & f_7 = -f_4 f_{20} f_8 \text{ on } f_{21}\overline{B} && (2.127)\\
(2.98) \qquad & f_9 = -f_6 f_{20} f_8 \text{ on } f_{21}\overline{B} && (2.128)\\
(2.99), (2.107) \qquad & f_{10} = -f_4 f_{25} f_{11} \text{ on } f_{27}\widehat{B} \text{ and } f_{10} = -f_7 f_{28} f_{12} \text{ on } f_{29}\widehat{C} && (2.129)\\
(2.108) \qquad & f_{11} = -f_8 f_{28} f_{12} \text{ on } f_{29}\widehat{C} && (2.130)\\
(2.101) \qquad & f_{12} = -f_5 f_{25} f_{11} \text{ on } f_{27}\widehat{B} && (2.131)
\end{aligned}$$

Now we need to find upper bounds for the codimensions of $\overline{A}^*, \widehat{B}^*, \overline{B}^*$, and $\overline{C}^*$. From (2.100), we know f_{11} is injective on $f_{27}\widehat{B}$ and f_{27} is injective on $\widehat{B}$. These facts will be used to arrive on lines (2.132) and (2.134). From (2.109), we know f_{29} is injective on $\widehat{C}$, which will also be used to arrive on line (2.134). Lemma 2.11.4.1 will be used to arrive on (2.133).

$$\begin{aligned}
codim_B \widehat{B}^* &= H(B) - dim(\widehat{B}^*) \\
&= H(B) - dim(f_{11}(f_{27}\widehat{B} \cap f_{29}\widehat{C})) \\
&= H(B) - dim(f_{27}\widehat{B} \cap f_{29}\widehat{C}) && (2.132) \\
&= H(B) - H(Z) + codim_Z(f_{27}\widehat{B} \cap f_{29}\widehat{C}) \\
&\leq H(B) - H(Z) + codim_Z(f_{27}\widehat{B}) + codim_Z(f_{29}\widehat{C}) && (2.133) \\
&= H(B) - H(Z) + H(Z) - dim(f_{27}\widehat{B}) + H(Z) - dim(f_{29}\widehat{C}) \\
&= H(B) + H(Z) - dim(\widehat{B}) - dim(\widehat{C}) && (2.134) \\
&\leq H(B) + H(Z) - H(B) + \Delta_{\widehat{B}} - H(C) + \Delta_{\widehat{C}} && (2.135) \\
&\leq H(Z) - H(C) + \Delta_{\widehat{B}} + \Delta_{\widehat{C}} && (2.136) \\
&\triangleq \Delta_{\widehat{B}^*}
\end{aligned}$$

From (2.91), we know f_4 is injective on $f_{15}\overline{A}$ and f_{15} is injective on $\overline{A}$. These facts will be used on lines (2.137) and (2.139). From (2.115), we know f_{25} is injective on $\widehat{B}^*$, which will also be used to arrive on line (2.139). Lemma 2.11.4.1 will be used to arrive on (2.138).

$$\begin{aligned}
codim_A \overline{A}^* &= H(A) - dim(\overline{A}^*) \\
&= H(A) - dim(f_4(f_{25}\widehat{B}^* \cap f_{15}\overline{A})) \\
&= H(A) - dim(f_{25}\widehat{B}^* \cap f_{15}\overline{A}) && (2.137) \\
&= H(A) - H(X) + codim_X(f_{25}\widehat{B}^* \cap f_{15}\overline{A}) \\
&\leq H(A) - H(X) + codim_X(f_{25}\widehat{B}^*) + codim_X(f_{15}\overline{A}) && (2.138) \\
&= H(A) + H(X) - dim(f_{25}\widehat{B}^*) - dim(f_{15}\overline{A}) \\
&= H(A) + H(X) - dim(\widehat{B}^*) - dim(\overline{A}) && (2.139) \\
&\leq H(A) + H(X) - H(B) + \Delta_{\widehat{B}^*} - H(A) + \Delta_{\overline{A}} \\
&= H(X) - H(B) + H(Z) - H(C) + \Delta_{\widehat{B}} + \Delta_{\widehat{C}} + \Delta_{\overline{A}} \\
&\triangleq \Delta_{\overline{A}^*}
\end{aligned}$$

From (2.105), we know f_2 is injective on $f_{17}\overline{C}$ and f_{17} is injective on $\overline{C}$. These facts will be used to arrive on lines (2.140) and (2.142). From (2.114), we know f_{22} is injective on $\overline{D}$,

which will also be used on line (2.142). Lemma 2.11.4.1 will be used to arrive on (2.141).

$$\begin{aligned}
codim_C\overline{C}^* &= H(C) - dim(\overline{C}^*) \\
&= H(C) - dim(f_2(f_{17}\overline{C} \cap f_{22}\overline{D})) \\
&= H(C) - dim(f_{17}\overline{C} \cap f_{22}\overline{D}) && (2.140) \\
&= H(C) - H(W) + codim_W(f_{17}\overline{C} \cap f_{22}\overline{D}) \\
&\leq H(C) - H(W) + codim_W(f_{17}\overline{C}) + codim_W(f_{22}\overline{D}) && (2.141) \\
&= H(C) - H(W) + H(W) - dim(f_{17}\overline{C}) + H(W) - dim(f_{22}\overline{D}) \\
&= H(C) + H(W) - dim(\overline{C}) - dim(\overline{D}) && (2.142) \\
&\leq H(C) + H(W) - H(C) + \Delta_{\overline{C}} - H(D) + \Delta_{\overline{D}} \\
&= H(W) - H(D) + \Delta_{\overline{C}} + \Delta_{\overline{D}} \\
&\triangleq \Delta_{\overline{C}^*}
\end{aligned}$$

From (2.96), we know f_8 is injective on $f_{21}\overline{B}$ and f_{21} is injective on $\overline{B}$. These facts will be used to arrive on lines (2.143) and (2.145). From (2.118), we know f_{18} is injective on $\overline{C}^*$, which will also be used on line (2.145). Lemma 2.11.4.1 will be used to arrive on (2.144).

$$\begin{aligned}
codim_B\overline{B}^* &= H(B) - dim(\overline{B}^*) \\
&= H(B) - dim(f_8(f_{21}\overline{B} \cap f_{18}\overline{C}^*)) \\
&= H(B) - dim(f_{21}\overline{B} \cap f_{18}\overline{C}^*) && (2.143) \\
&= H(B) - H(Y) + codim_Y(f_{21}\overline{B} \cap f_{18}\overline{C}^*) \\
&\leq H(B) - H(Y) + codim_Y(f_{21}\overline{B}) + codim_Y(f_{18}\overline{C}^*) && (2.144) \\
&= H(B) - H(Y) + H(Y) - dim(f_{21}\overline{B}) + H(Y) - dim(f_{18}\overline{C}^*) \\
&= H(B) + H(Y) - dim(\overline{B}) - dim(\overline{C}^*) && (2.145) \\
&\leq H(B) + H(Y) - H(B) + \Delta_{\overline{B}} - H(C) + \Delta_{\overline{C}^*} \\
&= H(Y) - H(C) + \Delta_{\overline{B}} + \Delta_{\overline{C}^*} \\
&= H(Y) - H(C) + H(W) - H(D) + \Delta_{\overline{C}} + \Delta_{\overline{D}} + \Delta_{\overline{B}} \\
&\triangleq \Delta_{\overline{B}^*}
\end{aligned}$$

From (2.109), we know f_{12} is injective on $f_{29}\widehat{C}$ and f_{29} is injective on $\widehat{C}$. These facts will be used to arrive on lines (2.146) and (2.148). From (2.100), we know f_{27} is injective on $\widehat{B}$,

which will also be used on line (2.148). Lemma 2.11.4.1 will be used to arrive on (2.147).

$$\begin{aligned}
codim_C\widehat{C}^* &= H(C) - dim(\widehat{C}^*) \\
&= H(C) - dim(f_{12}(f_{27}\widehat{B} \cap f_{29}\widehat{C})) \\
&= H(C) - dim(f_{27}\widehat{B} \cap f_{29}\widehat{C}) && (2.146) \\
&= H(C) - H(Z) + codim_Z(f_{27}\widehat{B} \cap f_{29}\widehat{C}) \\
&\leq H(C) - H(Z) + codim_Z(f_{27}\widehat{B}) + codim_Z(f_{29}\widehat{C}) && (2.147) \\
&= H(C) - H(Z) + H(Z) - dim(f_{27}\widehat{B}) + H(Z) - dim(f_{29}\widehat{C}) \\
&= H(C) + H(Z) - dim(\widehat{B}) - dim(\widehat{C}) && (2.148) \\
&\leq H(C) + H(Z) - H(B) + \Delta_{\widehat{B}} - H(C) + \Delta_{\widehat{C}} \\
&= H(Z) - H(B) + \Delta_{\widehat{B}} + \Delta_{\widehat{C}} \\
&\triangleq \Delta_{\widehat{C}^*}
\end{aligned}$$

Let $t \in A$. Now we will assume t satisfies conditions (2.149) - (2.154). The justifications can be found below.

$t \in \widehat{A}$; this is true on a subspace of A of codimension at most $\Delta_{\widehat{A}}$ (2.149)

$f_{32}t \in f_{20}\overline{B}^* \cap f_{25}\widehat{B}^*$; this is true on a subspace of A of codimension at most

$$2H(X) - 2H(B) + \Delta_{\overline{B}^*} + \Delta_{\widehat{B}^*} \quad (2.150)$$

$f_{33}t \in f_{28}\widehat{C}^* \cap f_{21}\overline{B}^*$; this is true on a subspace of A of codimension at most

$$2H(Y) - H(B) - H(C) + \Delta_{\overline{B}^*} + \Delta_{\widehat{C}^*} \quad (2.151)$$

$f_{34}t \in f_{29}\widehat{C}^* \cap f_{27}\widehat{B}^*$; this is true on a subspace of A of codimension at most

$$2H(Z) - H(C) - H(B) + \Delta_{\widehat{C}^*} + \Delta_{\widehat{B}^*} \quad (2.152)$$

$f_{18}f_2f_{31}t \in f_{21}\overline{B}^* \cap f_{28}\widehat{C}^*$; this is true on a subspace of A of codimension at most

$$2H(Y) - H(B) - H(C) + \Delta_{\overline{B}^*} + \Delta_{\widehat{C}^*} \quad (2.153)$$

Now we need to make two assumptions on t simultaneously.

$f_{31}t \in f_{17}\overline{C} \cap f_{14}\overline{A}^*$ and $f_{15}f_{14}^{-1}f_{31}t \in f_{20}\overline{B}^* \cap f_{25}\widehat{B}^*$; this is true on a subspace of A of codimension at most

$$2H(X) - 2H(B) + 2H(W) - H(C) - H(A) + \Delta_{\overline{C}} + \Delta_{\overline{A}^*} + \Delta_{\overline{B}^*} + \Delta_{\widehat{B}^*} \quad (2.154)$$

To justify (2.150), first we know f_{20} is injective on $\overline{B}^*$ by (2.120). Then by Lemma 2.3 1.4.3, we know $f_{32}t \in f_{20}\overline{B}^*$ on a subspace of A of codimension at most $H(X) - H(B) + codim_B(\overline{B}^*) \leq H(X) - H(B) + \Delta_{\overline{B}^*}$. By (2.115), we also know f_{25} is injective on $\widehat{B}^*$. Then by Lemma 2.3 1.4.3, we know $f_{32}t \in f_{25}\widehat{B}^*$ on a subspace of A of codimension at most $H(X) - H(B) + codim_B(\widehat{B}^*) \leq H(X) - H(B) + \Delta_{\widehat{B}^*}$. Then using Lemma 2.1 1.4.1, we have

$f_{32}t \in f_{20}\overline{B}^* \cap f_{25}\widehat{B}^*$ on a subspace of A of codimension at most $2H(X) - 2H(B)\Delta_{\overline{B}^*} + \Delta_{\widehat{B}^*}$. Conditions (2.151) - (2.153) can be justified using similarly.

To justify (2.154), first we know f_{17} is injective on $\overline{C}$ by (2.105). Then by Lemma 2.3 1.4.3, we know $f_{31}t \in f_{17}\overline{C}$ on a subspace of A of codimension at most $H(W) - H(C) + codim_C(\overline{C}) \leq H(W) - H(C) + \Delta_{\overline{C}}$. By (2.117), we also know f_{14} is injective on $\overline{A}^*$. Then by Lemma 2.3 1.4.3, we know $f_{31}t \in f_{14}\overline{A}^*$ on a subspace of A of codimension at most $H(W) - H(A) + codim_A(\overline{A}^*) \leq H(W) - H(A) + \Delta_{\overline{A}^*}$. Then using Lemma 2.1 1.4.1, we have

$$f_{31}t \in f_{17}\overline{C} \cap f_{14}\overline{A}^*$$

on a subspace, S, of A of codimension at most $2H(W) - H(C) - H(A) + \Delta_{\overline{C}} + \Delta_{\overline{A}^*}$. Since f_{14} is injective on $\overline{A}^*$, the function $f_{15}f_{14}^{-1}f_{31}$ is defined on S. Using the same technique as before we can show that

$$f_{15}f_{14}^{-1}f_{31}t \in f_{20}\overline{B}^* \cap f_{25}\widehat{B}^*$$

on a subspace, $\overline{S}$, of codimension with respect to S at most $2H(X) - 2H(B) + \Delta_{\overline{B}^*} + \Delta_{\widehat{B}^*}$. Thus both conditions are true on $\overline{S}$, which has codimension with respect to A at most $codim_S\overline{S} + codim_A S \leq 2H(X) - 2H(B) + 2H(W) - H(C) - H(A) + \Delta_{\overline{C}} + \Delta_{\overline{A}^*} + \Delta_{\overline{B}^*} + \Delta_{\widehat{B}^*}$.

Our final goal is to show that $t = 3x$ for some x so that we may conclude that $t = 0$ if the characteristic is 3. We will accomplish this by using (2.87) and by proving that $f_4f_{32}t = f_7f_{33}t = f_{10}f_{34}t$.

Claim 2.2.2. $f_4f_{32}t = f_{10}f_{34}t$

Proof. First we must show that $f_{28}f_{12}f_{34}t = f_{21}f_{20}^{-1}f_{32}t$. By (2.88), we know

$$f_8f_{33}t \quad = \quad -f_{11}f_{34}t - f_1f_{31}t$$

Then by using (2.130) and condition (2.152), we have

$$f_8f_{33}t \quad = \quad f_8f_{28}f_{12}f_{34}t - f_1f_{31}t$$

Now by using (2.122) and condition (2.154), we have

$$f_8f_{33}t \quad = \quad f_8f_{28}f_{12}f_{34}t + f_8f_{18}f_2f_{31}t$$

By (2.116), we know f_8 is injective on $f_{28}\widehat{C}^*$. By condition (2.151), we know $f_{33}t \in f_{28}\widehat{C}^*$. By condition (2.153), we know $f_{18}f_2f_{31}t \in f_{28}\widehat{C}^*$. By condition (2.152), we know $f_{34}t \in f_{29}\widehat{C}^*$. Using (2.109), we know $f_{12}f_{34}t \in \widehat{C}^*$. Thus, we have

$$f_{33}t \quad = \quad f_{28}f_{12}f_{34}t + f_{18}f_2f_{31}t \tag{2.155}$$

By (2.90), we have

$$f_9f_{33}t \quad = \quad -f_6f_{32}t - f_3f_{31}t$$

Then by using (2.126) and condition (2.150), we have

$$f_9 f_{33} t \quad = \quad f_9 f_{21} f_{20}^{-1} f_{32} t - f_3 f_{31} t$$

Now by using (2.124) and condition (2.154), we have

$$f_9 f_{33} t \quad = \quad f_9 f_{21} f_{20}^{-1} f_{32} t + f_9 f_{18} f_2 f_{31} t$$

By (2.119), we know f_9 is injective on $f_{21}\overline{B}^*$. By condition (2.151), we know $f_{33}t \in f_{21}\overline{B}^*$. By condition (2.150), we know $f_{32}t \in f_{20}\overline{B}^*$ so $f_{21}f_{20}^{-1}f_{32}t \in f_{21}\overline{B}^*$. By condition (2.153), we know $f_{18}f_2f_{31}t \in f_{21}\overline{B}^*$. Thus, we have

$$f_{33} t \quad = \quad f_{21} f_{20}^{-1} f_{32} t + f_{18} f_2 f_{31} t \tag{2.156}$$

Now setting (2.155) and (2.156) equal to each other, we have

$$f_{21} f_{20}^{-1} f_{32} t \quad = \quad f_{28} f_{12} f_{34} t \tag{2.157}$$

By (2.125) and condition (2.150), we know

$$f_4 f_{32} t \quad = \quad -f_7 f_{21} f_{20}^{-1} f_{32} t$$

Using (2.157), we have

$$f_4 f_{32} t \quad = \quad -f_7 f_{28} f_{12} f_{34} t$$

Then using (2.129) and condition (2.152), we know

$$f_4 f_{32} t \quad = \quad f_{10} f_{34} t$$

□

Claim 2.2.3. $f_7 f_{33} t = f_{10} f_{34} t$

Proof. First we must show that $f_{25}f_{11}f_{34}t = f_{20}f_8f_{33}t$. By (2.89), we know

$$f_5 f_{32} t \quad = \quad -f_{12} f_{34} t - f_2 f_{31} t$$

Then by using (2.131) and condition (2.152), we have

$$f_5 f_{32} t \quad = \quad f_5 f_{25} f_{11} f_{34} t - f_2 f_{31} t$$

Now by using (2.123) and condition (2.154), we have

$$f_5 f_{32} t \quad = \quad f_5 f_{25} f_{11} f_{34} t + f_5 f_{15} f_{14}^{-1} f_{31} t$$

By (2.115), we know f_5 is injective on $f_{25}\widehat{B}^*$. By condition (2.150), we know $f_{32}t \in f_{25}\widehat{B}^*$. By condition (2.152), we know $f_{34}t \in f_{27}\widehat{B}^*$. Now using (2.100), we know $f_{11}f_{34}t \in \widehat{B}^*$. By condition (2.154), we know $f_{15}f_{14}^{-1}f_{31}t \in f_{25}\widehat{B}^*$. Thus, we have

$$f_{32} t \quad = \quad f_{25} f_{11} f_{34} t + f_{15} f_{14}^{-1} f_{31} t \tag{2.158}$$

By (2.90), we have

$$f_6 f_{32} t \quad = \quad -f_9 f_{33} t - f_3 f_{31} t$$

Then using (2.128) and condition (2.151), we have

$$f_6 f_{32} t \quad = \quad f_6 f_{20} f_8 f_{33} t - f_3 f_{31} t$$

Now by using (2.124) and condition (2.154), we have

$$f_6 f_{32} t \quad = \quad f_6 f_{20} f_8 f_{33} t + f_6 f_{15} f_{14}^{-1} f_{31} t$$

By (2.120), we know that f_6 is injective on $f_{20}\overline{B}^*$. By condition (2.150), we know $f_{32}t \in f_{20}\overline{B}^*$. By condition (2.151), we know $f_{33}t \in f_{21}\overline{B}^*$. Now using (2.96), we know $f_8 f_{33} t \in \overline{B}^*$. By condition (2.154), we know $f_{15} f_{14}^{-1} f_{31} t \in f_{20}\overline{B}^*$. Thus, we have

$$f_{32} t \quad = \quad f_{20} f_8 f_{33} t + f_{15} f_{14}^{-1} f_{31} t \tag{2.159}$$

Now setting (2.158) and (2.159) equal to each other, we have

$$f_{25} f_{11} f_{34} t \quad = \quad f_{20} f_8 f_{33} t \tag{2.160}$$

By (2.127) and condition (2.151), we know

$$f_7 f_{33} t \quad = \quad -f_4 f_{20} f_8 f_{33} t$$

Using (2.160), we have

$$f_7 f_{33} t \quad = \quad -f_4 f_{25} f_{11} f_{34} t$$

Then using (2.129) and condition (2.152), we know

$$f_7 f_{33} t \quad = \quad f_{10} f_{34} t$$

□

Now by (2.87), Claim 2.2.2 and Claim 2.2.3, we have

$$\begin{aligned} t &= f_4 f_{32} t + f_7 f_{33} t + f_{10} f_{34} t \\ &= f_{10} f_{34} t + f_{10} f_{34} t + f_{10} f_{34} t \\ &= 3 f_{10} f_{34} t \end{aligned}$$

Thus if the field has characteristic 3, then

$$t \quad = \quad 0 \tag{2.161}$$

So no nonzero t can satisfy all of the conditions (2.149) - (2.152), so we must have

$$\begin{aligned}
H(A) \leq\ & \Delta_{\widehat{A}} + 2H(W) - H(C) - H(A) + \Delta_{\overline{C}} + \Delta_{\overline{A}^*} \\
& +2H(X) - 2H(B) + \Delta_{\overline{B}^*} + \Delta_{\widehat{B}^*} \\
& +2H(Y) - H(B) - H(C) + \Delta_{\overline{B}^*} + \Delta_{\widehat{C}^*} \\
& +2H(Z) - H(C) - H(B) + \Delta_{\widehat{C}^*} + \Delta_{\widehat{B}^*} \\
& +2H(Y) - H(B) - H(C) + \Delta_{\overline{B}^*} + \Delta_{\widehat{C}^*} \\
& +2H(X) - 2H(B) + \Delta_{\overline{B}^*} + \Delta_{\widehat{B}^*} \\
=\ & 2H(Z) + 4H(Y) + 4H(X) + 2H(W) - 4H(C) - 7H(B) - H(A) \\
& +\Delta_{\overline{A}^*} + 4\Delta_{\overline{B}^*} + 3\Delta_{\widehat{B}^*} + 3\Delta_{\widehat{C}^*} + \Delta_{\widehat{A}} + \Delta_{\overline{C}} \\
=\ & 2H(Z) + 4H(Y) + 4H(X) + 2H(W) - 4H(C) - 7H(B) - H(A) \\
& +H(X) - H(B) + H(Z) - H(C) + \Delta_{\widehat{B}} + \Delta_{\widehat{C}} + \Delta_{\overline{A}} \\
& +4(H(Y) - H(C) + H(W) - H(D) + \Delta_{\overline{C}} + \Delta_{\overline{D}} + \Delta_{\overline{B}}) \\
& +3(H(Z) - H(C) + \Delta_{\widehat{B}} + \Delta_{\widehat{C}}) \\
& +3(H(Z) - H(B) + \Delta_{\widehat{B}} + \Delta_{\widehat{C}}) \\
& +\Delta_{\widehat{A}} + \Delta_{\overline{C}} \\
=\ & 9H(Z) + 8H(Y) + 5H(X) + 6H(W) - 4H(D) - 12H(C) - 11H(B) - H(A) \\
& +\Delta_{\widehat{A}} + \Delta_{\overline{A}} + 7\Delta_{\widehat{B}} + 4\Delta_{\overline{B}} + 7\Delta_{\widehat{C}} + 5\Delta_{\overline{C}} + 4\Delta_{\overline{D}} \\
=\ & 9H(Z) + 8H(Y) + 5H(X) + 6H(W) - 4H(D) - 12H(C) - 11H(B) - H(A) \\
& +H(W|B,C,D) + H(X|A,C,D) + H(Y|A,B,D) + H(Z|A,B,C) \\
& +H(A|W,X,Y,Z) + H(W|B,C,D) + H(X|A,C,D) + H(A|B,W,X) \\
& +7(H(X|A,C,D) + H(Z|A,B,C) + H(B|D,X,Z)) \\
& +4(H(X|A,C,D) + H(Y|A,B,D) + H(B|C,X,Y)) \\
& +7(H(Y|A,B,D) + H(Z|A,B,C) + H(C|D,Y,Z)) \\
& +5(H(W|B,C,D) + H(Y|A,B,D) + H(C|A,W,Y)) \\
& +4(H(W|B,C,D) + H(Z|A,B,C) + H(D|A,W,Z)) \\
& +29(H(A) + H(B) + H(C) + H(D) - H(A,B,C,D)) \\
=\ & 9H(Z) + 8H(Y) + 5H(X) + 6H(W) - 4H(D) - 12H(C) - 11H(B) - H(A) \\
& 19H(Z|A,B,C) + 17H(Y|A,B,D) + 13H(X|A,C,D) + 11H(W|B,C,D) \\
& H(A|W,X,Y,Z) + H(A|B,W,X) + 7H(B|D,X,Z) + 4H(B|C,X,Y) \\
& 7H(C|D,Y,Z) + 5H(C|A,W,Y) + 4H(D|A,W,Z) \\
& +29(H(A) + H(B) + H(C) + H(D) - H(A,B,C,D))
\end{aligned}$$

Now notice that the linear rank inequality does not hold for characteristic other than 3.

A counterexample would be: In $V = GF(3)^4$ where

$$\begin{aligned} A &= \langle(1,0,0,0)\rangle & B &= \langle(0,1,0,0)\rangle \\ C &= \langle(0,0,1,0)\rangle & D &= \langle(0,0,0,1)\rangle \\ W &= \langle(0,1,1,1)\rangle & X &= \langle(1,0,1,1)\rangle \\ Y &= \langle(1,1,0,1)\rangle & Z &= \langle(1,1,1,0)\rangle \end{aligned}$$

Then

$$\begin{aligned} H(Z|A,B,C) &= H(Y|A,B,D) \\ &= H(X|A,C,D) \\ &= H(W|B,C,D) \\ &= H(A|B,W,X) \\ &= H(A|W,X,Y,Z) \\ &= H(B|C,X,Y) \\ &= H(B|D,X,Z) \\ &= H(C|A,W,Y) \\ &= H(C|D,Y,Z) \\ &= H(D|A,W,Z) \\ &= 0 \end{aligned}$$

Since W, X, Y, Z are independent, we also have $H(W)+H(X)+H(Y)+H(Z) = H(W,X,Y,Z)$. So the inequality becomes

$$\begin{aligned} H(A) &\leq 9H(Z)+8H(Y)+5H(X)+6H(W)-4H(D)-12H(C)-11H(B)-H(A) \\ 1 &\leq 9+8+5+6-4-12-11-1 \\ 1 &\leq 0 \end{aligned}$$

which is clearly a contradiction. Therefore, the inequality above is a linear rank inequality for fields of characteristic 3.

□

Corollary 2.2.4. *The linear coding capacity of the non-$T8$ network is at most 28/29 over any characteristic 3. The linear coding capacity over any characteristic other than 3 and the coding capacity is 1.*

Proof. Let us apply the linear rank inequality derived in Theorem 2.2.1 to the non-$T8$ network.

Then we would have:

$$\begin{aligned} H(Z|A,B,C) &= H(Y|A,B,D) \\ &= H(X|A,C,D) \\ &= H(W|B,C,D) \\ &= H(A|B,W,X) \\ &= H(A|W,X,Y,Z) \\ &= H(B|C,X,Y) \\ &= H(B|D,X,Z) \\ &= H(C|A,W,Y) \\ &= H(C|D,Y,Z) \\ &= H(D|A,W,Z) \\ &= 0 \end{aligned}$$

Since W, X, Y, Z are independent, we also have $H(W)+H(X)+H(Y)+H(Z) = H(W,X,Y,Z)$. So the inequality becomes

$$H(A) \leq 9H(Z)+8H(Y)+5H(X)+6H(W)-4H(D)-12H(C)-11H(B)-H(A).$$

Now we know $H(A) = H(B) = H(C) = H(D) = k$ and $H(W) = H(X) = H(Y) = H(Z) = n$, so we have

$$\begin{aligned} k &\leq 9n+8n+5n+6n-4k-12k-11k-k \\ 29k &\leq 28n \\ k/n &\leq 28/29 \end{aligned}$$

So the linear coding capacity over characteristic 3 is at most $28/29 < 1$. The non-$T8$ network is solvable over every characteristic except for 3 by the following coding solution:

$$\begin{aligned} W &= B+C+D \\ X &= A+C+D \\ Y &= A+B+D \\ Z &= A+B+C \end{aligned}$$

We know the coding capacity is at most 1 because there is a unique path from source A to node n_9 and by the coding solution given above we know the capacity is at least 1, thus the capacity is 1.

□

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Chapter 3

Characteristic Dependent Linear Rank Inequalities for every Finite and Co-finite Set of Primes with Applications to Network Coding

3.1 A Linear Rank Inequality for any Finite Set of Primes

In [Dougherty 07], an algorithm is given for constructing networks from matroids, or matroidal networks. Figure 3.1 was first constructed using the given algorithm from the $T8$ matroid [Oxley 92]. The $T8$ matroid is represented by the following matrix, where column dependencies are over characteristic 3.

$$\begin{array}{c} \begin{array}{cccccccc} S_1 & S_2 & S_3 & S_4 & C_1 & C_2 & C_3 & Z \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Theorem 3.1.1. *For every finite set of primes, P, there exists a linear rank inequality for fields with characteristic in P.*

Proof. For convenience we will use the MATLAB notation $[a : b]$ to denote $\{z \in \mathbb{Z} : a \leq z \leq b\}$. Let n be the product of all the primes in P. We will assume $n \geq 3$. For the case where $P = \{2\}$, we can let $n = 4$ to get the desired result. Recent work, [Dougherty 13], has also handled the case for $n = 2$ and arrives at a simpler inequality than the following.

We will construct a network as a guide to construct the inequality, but the network is not necessary. Construct $n+1$ independent sources and label them $S_1, \ldots, S_{n+1}$. We will define a *channel* to be an input node, an output node, and a single directed edge connecting the input and output nodes with the direction going towards the output node. Then construct n channels and label them $C_1, \ldots, C_n$. Then for each $i \in [1:n]$, add a directed edge connecting the input node of C_i and S_j for every $j \in [1:n+1]\backslash i$ with the direction going towards the input node of C_i. Now create one more channel, label it Z. For each $i \in [1:n]$, and add a directed edge connecting the input node of Z and the output node of C_i with the direction going towards the input node of Z.

For every $i \in [1:n]$, create a receiver node, R_i, that demands source S_{n+1}. Add a directed edge that connects R_i and the output node of C_i with the direction going towards of R_i. Add an edge that connects R_i and the output node of Z with the direction going towards R_i. Add a directed edge that connects R_i and source S_i with the direction going towards R_i.

For every $i \in [2:n]$, create a receiver node, T_i, that demands source S_i. Add a directed edge that connects T_i and the output node of C_1 with the direction going towards T_i. Add a directed edge that connects T_i and the output node of C_i with the direction going towards T_i. Add a directed edge that connects T_i and source S_1 with the direction going towards T_i.

Create a receiver node, A, that demands source S_1. For every $i \in [2:n]$, add a directed edge that connects A and source S_i with the direction going towards A. Add a directed edge that connects A to the output node of Z with the direction going towards A. We will denote the network constructed above by $\mathcal{N}_2$. Figure 3.1 depicts the resulting network with $n = 3$ obtained by the above procedure.

For notational purposes, let $X_i \triangleq S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_{n+1}$. Let V be a finite dimensional vector space with subspaces $S_1, \ldots, S_{n+1}, C_1, \ldots, C_n, Z$. By Lemma 1.4.4, for every $i \in [1:n]$ and $j \in [1:n+1]\backslash i$, we get linear functions:

$$f_i^j : C_i \to S_j,$$

such that

$$\sum_{j \in [1:n+1]\backslash i} f_i^j \quad = \quad I \tag{3.1}$$

on a subspace of C_i, G_i, of codimension $H(C_i|X_i)$. By Lemma 1.4.4, for $i \in [1:n]$, we get linear functions:

$$f_Z^i : Z \to C_i,$$

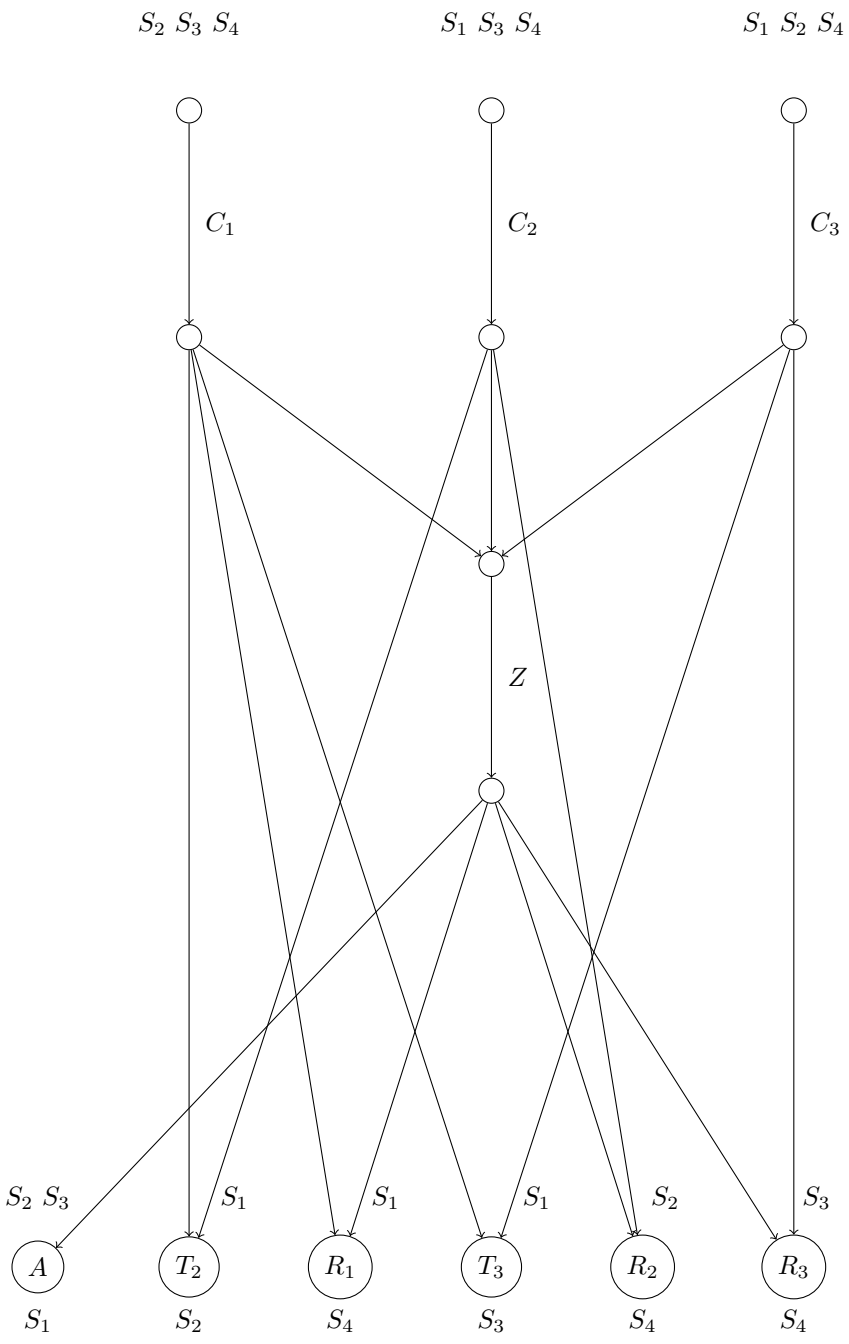

Figure 3.1: The resulting network for $n = 3$. When an source S_i appears above a node, it implies that there is an edge connecting the node to the source.

such that

$$\sum_{i\in[1:n]} f_Z^i \quad = \quad I \tag{3.2}$$

on a subspace of Z of codimension $H(Z|C_1, C_2, \ldots, C_n)$. By Lemma 1.4.4, we get linear functions:

$$A^Z : S_1 \to Z,$$
$$A^j : S_1 \to S_j \text{ for } j \in [2:n],$$

such that

$$A^Z + \sum_{j\in[2:n]} A^j \quad = \quad I \tag{3.3}$$

on a subspace of S_1 of codimension $H(S_1|Z, S_2, \ldots, S_n)$. By Lemma 1.4.4, for $j \in [2:n]$, we get linear functions:

$$T_j^j : S_j \to C_j,$$
$$T_j^1 : S_j \to C_1,$$
$$T_j^S : S_j \to S_1$$

such that

$$T_j^j + T_j^1 + T_j^S \quad = \quad I \tag{3.4}$$

on a subspace of S_j of codimension $H(S_j|S_1, C_1, C_j)$. By Lemma 1.4.4, for $j \in [1:n]$, we get linear functions:

$$R_j^j : S_{n+1} \to C_j,$$
$$R_j^Z : S_{n+1} \to Z,$$
$$R_j^S : S_{n+1} \to S_j$$

such that

$$R_j^j + R_j^Z + R_j^S \quad = \quad I \tag{3.5}$$

on a subspace of S_{n+1} of codimension $H(S_{n+1}|S_j, C_j, Z)$. For $j \in [1:n]$, let

$$f_{S_j} \quad \triangleq \quad \sum_{i\in[1:n]\setminus j} f_i^j f_Z^i$$
$$f_{S_{n+1}} \quad \triangleq \quad \sum_{i=1}^{n} f_i^{n+1} f_Z^j.$$

Claim 3.1.2. There is a subspace of Z, H, of codimension at most $H(Z|C_1, C_2, \ldots, C_n) + \sum_{i=1}^n H(C_i|X_i)$, such that $\sum_{i=1}^{n+1} f_{S_i} = I$ on H.

Proof.

$$\sum_{i=1}^{n+1} f_{S_i} = f_{S_{n+1}} + \sum_{i=1}^{n} f_{S_i}$$
$$= \sum_{j=1}^{n} f_j^{n+1} f_Z^j + \sum_{j=1}^{n} \sum_{i\in[1:n]\setminus j} f_i^j f_Z^i$$

Using (3.1), for each $i \in [1:n]$, since $\sum_{j\in[1:n+1]\setminus i} f_i^j = I$ on $G_i \subseteq C_i$, we know

$$\sum_{j\in[1:n+1]\setminus i} f_i^j f_Z^i(t) = f_Z^i(t)$$

for every t such that $t \in Z$ and $f_Z^i(t) \in G_i$. By Lemma 1.4.3, $f_Z^i(t) \in G_i$ on a subspace of Z of codimension at most $codim_{C_i}(G_i)$. So

$$\sum_{j\in[1:n+1]\setminus i} f_i^j f_Z^i = f_Z^i$$

on a subspace of Z of codimension at most $H(C_i|X_i)$. Using Lemma 1.4.1, the sum above becomes

$$\sum_{i=1}^{n} f_Z^i$$

on a subspace of Z of codimension at most $\sum_{i=1}^{n} H(C_i|X_i)$. Then using (3.2) and Lemma 1.4.1, we know there is a subspace of Z, H, of codimension at most $H(Z|C_1, C_2, \ldots, C_n) + \sum_{i=1}^{n} H(C_i|X_i)$, such that $\sum_{i=1}^{n+1} f_{S_i} = I$ on H. □

Combining the functions above, we get new functions:

$$\begin{aligned} f_{S_1}A^Z &: S_1 \to S_1 \\ A^j + f_{S_j}A^Z &: S_1 \to S_j \text{ for } j \in [2:n] \\ f_{S_{n+1}}A^Z &: S_1 \to S_{n+1} \end{aligned}$$

Using Claim 3.1.2, (3.3), Lemma 1.4.1, and Lemma 1.4.3 we know the sum of these functions is equal to I on a subspace of S_1 of codimension at most

$$H(S_1|Z, S_2, \ldots, S_n) + H(Z|C_1, C_2, \ldots, C_n) + \sum_{i=1}^{n} H(C_i|X_i)$$

Now applying Lemma 1.4.6 and Lemma 1.4.1 to $f_{S_1}A^Z - I$, $f_{S_{n+1}}A^Z$, and $A^j + f_{S_j}A^Z$, for $j \in [2:n]$, we get a subspace $\overline{A}$ of S_1 of codimension at most

$$\begin{aligned} \Delta_A &= H(S_1|Z, S_2, \ldots, S_n) + H(Z|C_1, C_2, \ldots, C_n) + \sum_{i=1}^{n} H(C_i|X_i) \\ &\quad -H(S_1, \ldots, S_{n+1}) + \sum_{i=1}^{n+1} H(S_i) \end{aligned}$$

on which

$$\begin{aligned} f_{S_1}A^Z &= I && (3.6)\\ A^j + f_{S_j}A^Z &= 0 \text{ for } j \in [2:n] && (3.7)\\ f_{S_{n+1}}A^Z &= 0. && (3.8) \end{aligned}$$

Combining the functions above, for $j \in [1:n]$, we get new functions:

$$\begin{aligned} f_j^{n+1}R_j^j + f_{S_{n+1}}R_j^Z &: S_{n+1} \to S_{n+1}\\ R_j^S + f_{S_j}R_j^Z &: S_{n+1} \to S_j\\ f_j^iR_j^j + f_{S_i}R_j^Z &: S_{n+1} \to S_i \text{ for } i \in [1:n]\backslash j. \end{aligned}$$

Using (3.1), Claim 3.1.2, (3.5), Lemma 1.4.1, and Lemma 1.4.3 we know the sum of these functions is equal to I on a subspace of S_{n+1} of codimension at most

$$H(S_{n+1}|S_j, C_j, Z) + H(Z|C_1, C_2, \ldots, C_n) + H(C_j|X_j) + \sum_{i=1}^{n} H(C_i|X_i)$$

Now applying Lemma 1.4.6 and Lemma 1.4.1 to $f_j^{n+1}R_j^j + f_{S_{n+1}}R_j^Z - I$, $R_j^S + f_{S_j}R_j^Z$, and $\sum_{i\in[1:n]\backslash j} f_j^iR_j^j + f_{S_i}R_j^Z$ we get a subspace $\overline{S}_{j,n+1}$ of S_{n+1} of codimension at most

$$\begin{aligned} \Delta_{j,n+1} &= H(S_{n+1}|S_j, C_j, Z) + H(Z|C_1, C_2, \ldots, C_n) + H(C_j|X_j) + \sum_{i=1}^{n} H(C_i|X_i)\\ &\quad -H(S_1, \ldots, S_{n+1}) + \sum_{i=1}^{n+1} H(S_i) \end{aligned}$$

on which

$$\begin{aligned} f_j^{n+1}R_j^j + f_{S_{n+1}}R_j^Z &= I && (3.9)\\ R_j^S + f_{S_j}R_j^Z &= 0 && (3.10)\\ f_j^iR_j^j + f_{S_i}R_j^Z &= 0 \text{ for } i \in [1:n]\backslash j. && (3.11) \end{aligned}$$

Combining the functions above, for $j \in [2:n]$, we get new functions:

$$\begin{aligned} f_1^jT_j^1 &: S_j \to S_j && (3.12)\\ T_j^S + f_j^1T_j^j &: S_j \to S_1 && (3.13)\\ f_j^iT_j^j + f_1^iT_j^1 &: S_j \to S_i \text{ for } i \in [2:n+1]\backslash j. && (3.14) \end{aligned}$$

Using (3.1), (3.4), Lemma 1.4.1, and Lemma 1.4.3 we know the sum of these functions is equal to I on a subspace of S_j of codimension at most

$$H(S_j|S_1, C_1, C_j) + H(C_1|X_1) + H(C_j|X_j)$$

Now applying Lemma 1.4.6 and Lemma 1.4.1 to $f_1^j T_j^1 - I$, $T_j^S + f_j^1 T_j^j$, and $\sum_{i\in[2:n+1]\setminus j} f_j^i T_j^j + f_1^i T_j^1$ we get a subspace $\overline{S}_j$ of S_j of codimension at most

$$\begin{aligned}\Delta_j &= H(S_j|S_1,C_1,C_j) + H(C_1|X_1) + H(C_j|X_j)\\ &\quad -H(S_1,\ldots,S_{n+1}) + \sum_{i=1}^{n+1} H(S_i)\end{aligned}$$

on which

$$f_1^j T_j^1 = I \tag{3.15}$$
$$T_j^S + f_j^1 T_j^j = 0 \tag{3.16}$$
$$f_j^i T_j^j + f_1^i T_j^1 = 0 \text{ for } i \in [2:n+1]\setminus j. \tag{3.17}$$

By (3.8) we know $f_{S_{n+1}} A^Z = 0$ on $\overline{A}$. By (3.9) we know $f_j^{n+1} R_j^j + f_{S_{n+1}} R_j^Z = I$ on $\overline{S}_{j,n+1}$. So we know $f_{S_{n+1}} R_j^Z = 0$ on $\overline{S}_{j,n+1} \cap (R_j^Z)^{-1} A^Z(\overline{A})$. Thus

$$f_j^{n+1} R_j^j = I \text{ on } \overline{S}_{j,n+1} \cap (R_j^Z)^{-1} A^Z(\overline{A}) \text{ for } j \in [1:n]. \tag{3.18}$$

Notice here we do not know if R_j^Z is injective, so consider $(R_j^Z)^{-1} A^Z(\overline{A})$ to be a set.

Claim 3.1.3. There exists a subspace, $\overline{A}^* \subseteq \overline{A}$, such that, for $j \in [2:n]$, A^j is injective on $\overline{A}^*$.

Proof. By (3.15) and (3.18), for $j \in [2:n]$, we know

$$\begin{aligned} f_1^j T_j^1 &= I \text{ on } \overline{S}_j, \text{ and}\\ f_1^{n+1} R_1^1 &= I \text{ on } \overline{S}_{1,n+1} \cap (R_1^Z)^{-1} A^Z(\overline{A}).\end{aligned}$$

Then by Lemma 1.4.7, we know $f_1^j R_1^1$ is injective on

$$\widetilde{S}_{j,n+1} \triangleq f_1^{n+1}\left[R_1^1\left(\overline{S}_{1,n+1} \cap (R_1^Z)^{-1} A^Z(\overline{A})\right) \cap T_j^1(\overline{S}_j)\right].$$

Then by (3.11), we know

$$f_{S_j} R_1^Z \text{ is injective on } \widetilde{S}_{j,n+1}. \tag{3.19}$$

Then applying Lemma 1.4.7 to (3.19) and (3.6), we know

$$f_{S_j} A^Z \text{ is injective on } f_{S_1}\left[A^Z(\overline{A}) \cap R_1^Z(\widetilde{S}_{j,n+1})\right].$$

Thus for each $j \in [2:n]$,

$$f_{S_j} A^Z \text{ is injective on } \overline{A}^* \triangleq f_{S_1}\left[A^Z(\overline{A}) \cap \left(\textstyle\bigcap_{j=2}^n R_1^Z(\widetilde{S}_{j,n+1})\right)\right].$$

Then using (3.7), we know for $j \in [2:n]$, A^j is injective on $\overline{A}^*$. □

Now we would like to find an upper bound on the codimension of $\overline{A}^*$. We will use (3.6) to justify (3.20). For $j \in [1:n]$, by Lemma 1.4.3, we know

$$\begin{aligned} codim_{S_{n+1}}\left((R_j^Z)^{-1}A^Z(\overline{A})\right) &\leq codim_Z\left(A^Z(\overline{A})\right) \\ &= H(Z) - dim\left(A^Z(\overline{A})\right) \\ &= H(Z) - dim(\overline{A}) \qquad (3.20) \\ &= H(Z) - H(S_1) + codim_{S_1}(\overline{A}) \\ &= H(Z) - H(S_1) + \Delta_A \qquad (3.21) \end{aligned}$$

We will use (3.18) to justify (3.22). We will use Lemma 1.4.1 to justify (3.23) and (3.25). We will use (3.21) to justify (3.26).

$$\begin{aligned} codim_{S_{n+1}}(\widetilde{S}_{j,n+1}) &= H(S_{n+1}) - dim\left(\widetilde{S}_{j,n+1}\right) \\ &= H(S_{n+1}) - dim\left(f_1^{n+1}\left[R_1^1\left(\overline{S}_{1,n+1} \cap (R_1^Z)^{-1}A^Z(\overline{A})\right) \cap T_j^1(\overline{S}_j)\right]\right) \\ &= H(S_{n+1}) - dim\left(R_1^1\left(\overline{S}_{1,n+1} \cap (R_1^Z)^{-1}A^Z(\overline{A})\right) \cap T_j^1(\overline{S}_j)\right) \qquad (3.22) \\ &= H(S_{n+1}) - H(C_1) + codim_{C_1}\left(R_1^1\left(\overline{S}_{1,n+1} \cap (R_1^Z)^{-1}A^Z(\overline{A})\right) \cap T_j^1(\overline{S}_j)\right) \\ &\leq H(S_{n+1}) - H(C_1) + codim_{C_1}\left(R_1^1\left(\overline{S}_{1,n+1} \cap (R_1^Z)^{-1}A^Z(\overline{A})\right)\right) \\ &\quad + codim_{C_1}\left(T_j^1(\overline{S}_j)\right) \qquad (3.23) \\ &= H(S_{n+1}) - H(C_1) + H(C_1) - dim\left(R_1^1\left(\overline{S}_{1,n+1} \cap (R_1^Z)^{-1}A^Z(\overline{A})\right)\right) \\ &\quad + H(C_1) - dim\left(T_j^1(\overline{S}_j)\right) \\ &= H(S_{n+1}) + H(C_1) - dim\left(\overline{S}_{1,n+1} \cap (R_1^Z)^{-1}A^Z(\overline{A})\right) - dim(\overline{S}_j) \qquad (3.24) \\ &= H(S_{n+1}) + H(C_1) - H(S_{n+1}) + codim_{S_{n+1}}\left(\overline{S}_{1,n+1} \cap (R_1^Z)^{-1}A^Z(\overline{A})\right) \\ &\quad - H(S_j) + codim_{S_j}(\overline{S}_j) \\ &\leq H(C_1) - H(S_j) + codim_{S_{n+1}}\left(\overline{S}_{1,n+1}\right) \\ &\quad + codim_{S_{n+1}}\left((R_1^Z)^{-1}A^Z(\overline{A})\right) + codim_{S_j}(\overline{S}_j) \qquad (3.25) \\ &\leq H(C_1) - H(S_j) + H(Z) - H(S_1) + \Delta_{1,n+1} + \Delta_A + \Delta_j \qquad (3.26) \end{aligned}$$

We will use (3.6) to justify (3.27). We will use Lemma 1.4.1 to justify (3.28). We will use (3.6)

and (3.19) to justify (3.29). We will use (3.26) to justify (3.30).

$$
\begin{aligned}
codim_{S_1}(\overline{A}^*) &= H(S_1) - dim(\overline{A}^*) \\
&= H(S_1) - dim\left(f_{S_1}\left[A^Z(\overline{A}) \cap \left(\bigcap_{j=2}^{n} R_1^Z(\widetilde{S}_{j,n+1})\right)\right]\right) \\
&= H(S_1) - dim\left(A^Z(\overline{A}) \cap \left(\bigcap_{j=2}^{n} R_1^Z(\widetilde{S}_{j,n+1})\right)\right) && (3.27) \\
&= H(S_1) - H(Z) + codim_Z\left(A^Z(\overline{A}) \cap \left(\bigcap_{j=2}^{n} R_1^Z(\widetilde{S}_{j,n+1})\right)\right) \\
&\leq H(S_1) - H(Z) + codim_Z\left(A^Z(\overline{A})\right) + \sum_{j=2}^{n} codim_Z\left(R_1^Z(\widetilde{S}_{j,n+1})\right) && (3.28) \\
&= H(S_1) - H(Z) + H(Z) - dim\left(A^Z(\overline{A})\right) + (n-1)H(Z) - \sum_{j=2}^{n} dim\left(R_1^Z(\widetilde{S}_{j,n+1})\right) \\
&= H(S_1) + (n-1)H(Z) - dim(\overline{A}) - \sum_{j=2}^{n} dim(\widetilde{S}_{j,n+1}) && (3.29) \\
&= H(S_1) + (n-1)H(Z) - H(S_1) + codim_{S_1}(\overline{A}) - (n-1)H(S_{n+1}) \\
&\quad + \sum_{j=2}^{n} codim_{S_{n+1}}(\widetilde{S}_{j,n+1}) \\
&\leq (n-1)H(Z) - (n-1)H(S_{n+1}) + \Delta_A \\
&\quad + \sum_{j=2}^{n} H(C_1) - H(S_j) + H(Z) - H(S_1) + \Delta_{1,n+1} + \Delta_A + \Delta_j && (3.30) \\
&= (n-1)H(Z) - (n-1)H(S_{n+1}) + (n-1)H(C_1) - (n-1)H(S_1) \\
&\quad + (n-1)H(Z) + n\Delta_A + (n-1)\Delta_{1,n+1} + \sum_{j=2}^{n} \Delta_j - H(S_j) \\
&\triangleq \Delta_A^*
\end{aligned}
$$

Claim 3.1.4. For $j \in [2:n]$, there exists a subspace, $\overline{S}_j^* \subseteq \overline{S}_j$, such that T_j^j is injective on $\overline{S}_j^*$.

Proof. For $j \in [2:n-1]$, by (3.15), we know

$$
\begin{aligned}
f_1^j T_j^1 &= I \text{ on } \overline{S}_j, \text{ and} \\
f_1^{j+1} T_{j+1}^1 &= I \text{ on } \overline{S}_{j+1}.
\end{aligned}
$$

Then by Lemma 1.4.7, we know $f_1^{j+1} T_j^1$ is injective on $\overline{S}_j^* \triangleq f_1^j\left[T_j^1(\overline{S}_j) \cap T_{j+1}^1(\overline{S}_{j+1})\right]$. Then by (3.17), we know $f_j^{j+1} T_j^j$ is injective on $\overline{S}_j^*$, thus T_j^j is injective on $\overline{S}_j^*$. For $j = n$, by (3.15),

we know

$$f_1^{n-1}T_{n-1}^1 = I \text{ on } \overline{S}_{n-1}, \text{ and}$$
$$f_1^n T_n^1 = I \text{ on } \overline{S}_n.$$

Then by Lemma 1.4.7, we know $f_1^{n-1}T_n^1$ is injective on $\overline{S}_n^* \triangleq f_1^n\left[T_n^1(\overline{S}_n) \cap T_{n-1}^1(\overline{S}_{n-1})\right]$. Then by (3.17), we know $f_n^{n-1}T_n^n$ is injective on $\overline{S}_n^*$, thus T_n^n is injective on $\overline{S}_n^*$. □

Now we would like to find an upper bound for the codimension of $\overline{S}_j^*$. We will use (3.15) to justify (3.31) and (3.33). We will use Lemma 1.4.1 to justify (3.32). For $j \in [2 : n-1]$, we have

$$\begin{aligned}
codim_{S_j}\left(\overline{S}_j^*\right) &= H(S_j) - dim(\overline{S}_j^*) \\
&= H(S_j) - dim\left(f_1^j\left[T_j^1(\overline{S}_j) \cap T_{j+1}^1(\overline{S}_{j+1})\right]\right) \\
&= H(S_j) - dim\left(T_j^1(\overline{S}_j) \cap T_{j+1}^1(\overline{S}_{j+1})\right) && (3.31) \\
&= H(S_j) - H(C_1) + codim_{C_1}\left(T_j^1(\overline{S}_j) \cap T_{j+1}^1(\overline{S}_{j+1})\right) \\
&\leq H(S_j) - H(C_1) + codim_{C_1}\left(T_j^1(\overline{S}_j)\right) + codim_{C_1}\left(T_{j+1}^1(\overline{S}_{j+1})\right) && (3.32) \\
&= H(S_j) + H(C_1) - dim(\overline{S}_j) - dim(\overline{S}_{j+1}) && (3.33) \\
&= H(S_j) + H(C_1) - H(S_j) + codim_{S_j}(\overline{S}_j) - H(S_{j+1}) + codim_{S_{j+1}}(\overline{S}_{j+1}) \\
&\leq H(C_1) - H(S_{j+1}) + \Delta_j + \Delta_{j+1} \\
&\triangleq \Delta_j^*
\end{aligned}$$

We will use (3.15) to justify (3.34) and (3.36). We will use Lemma 1.4.1 to justify (3.35).

$$\begin{aligned}
codim_{S_n}\left(\overline{S}_n^*\right) &= H(S_n) - dim(\overline{S}_n^*) \\
&= H(S_n) - dim\left(f_1^n\left[T_n^1(\overline{S}_n) \cap T_{n-1}^1(\overline{S}_{n-1})\right]\right) \\
&= H(S_n) - dim\left(T_n^1(\overline{S}_n) \cap T_{n-1}^1(\overline{S}_{n-1})\right) && (3.34) \\
&= H(S_n) - H(C_1) + codim_{C_1}\left(T_n^1(\overline{S}_n) \cap T_{n-1}^1(\overline{S}_{n-1})\right) \\
&\leq H(S_n) - H(C_1) + codim_{C_1}\left(T_n^1(\overline{S}_n)\right) + codim_{C_1}\left(T_{n-1}^1(\overline{S}_{n-1})\right) && (3.35) \\
&= H(S_n) + H(C_1) - dim(\overline{S}_n) - dim(\overline{S}_{n-1}) && (3.36) \\
&= H(S_n) + H(C_1) - H(S_n) + codim_{S_n}(\overline{S}_n) - H(S_{n-1}) + codim_{S_{n-1}}(\overline{S}_{n-1}) \\
&\leq H(C_1) - H(S_{n-1}) + \Delta_n + \Delta_{n-1} \\
&\triangleq \Delta_n^*
\end{aligned}$$

Let $t \in S_1$. Now we will assume t satisfies conditions (D1) - (D3). The justifications can be found below.

(D1) We will assume $t \in \overline{A}^*$. This is true on a subspace of S_1 of codimension at most Δ_A^*.

(D2) We will assume, for $j \in [1:n]$,

$$f_Z^j A^Z t \in R_j^j \left(\overline{S}_{j,n+1} \cap (R_j^Z)^{-1} A^Z(\overline{A}^*) \right).$$

This is true on a subspace of S_1 of codimension at most

$$H(C_j) - H(S_{n+1}) + H(Z) - H(S_1) + \Delta_A^* + \Delta_{j,n+1}.$$

(D3) We will assume, for $j \in [2:n]$,

$$f_Z^j A^Z t \in T_j^j \left(\overline{S}_j^* \cap (T_j^1)^{-1} R_1^1 \left[\overline{S}_{1,n+1} \cap (R_1^Z)^{-1} A^Z(\overline{A}^*) \right] \right).$$

This is true on a subspace of S_1 of codimension at most

$$H(C_j) - H(S_j) + H(C_1) - H(S_{n+1}) + H(Z) - H(S_1) + \Delta_{1,n+1} + \Delta_A^* + \Delta_j^*.$$

To justify (D2), by (3.18) we know R_j^j is injective on $\overline{S}_{j,n+1} \cap (R_j^Z)^{-1} A^Z(\overline{A}^*)$. Then by Lemma 1.4.3, we know

$$f_Z^j A^Z t \in R_j^j \left(\overline{S}_{j,n+1} \cap (R_j^Z)^{-1} A^Z(\overline{A}^*) \right)$$

on a subspace of S_1 of codimension at most

$$H(C_j) - H(S_{n+1}) + codim_{S_{n+1}} \left(\overline{S}_{j,n+1} \cap (R_j^Z)^{-1} A^Z(\overline{A}^*) \right).$$

By Lemma 1.4.1 and (3.21), we know

$$\begin{aligned} codim_{S_{n+1}} \left(\overline{S}_{j,n+1} \cap (R_j^Z)^{-1} A^Z(\overline{A}^*) \right) &\leq codim_{S_{n+1}} \left(\overline{S}_{j,n+1} \right) + codim_{S_{n+1}} \left((R_j^Z)^{-1} A^Z(\overline{A}^*) \right) \\ &\leq H(Z) - H(S_1) + \Delta_A^* + \Delta_{j,n+1} \end{aligned}$$

So we know

$$f_Z^j A^Z t \in R_j^j \left(\overline{S}_{j,n+1} \cap (R_j^Z)^{-1} A^Z(\overline{A}^*) \right)$$

on a subspace of S_1 of codimension at most

$$H(C_j) - H(S_{n+1}) + H(Z) - H(S_1) + \Delta_A^* + \Delta_{j,n+1}.$$

To justify (D3), by Claim 3.1.4, we know T_j^j is injective on

$$\overline{S}_j^* \cap (T_j^1)^{-1} R_1^1 \left[\overline{S}_{1,n+1} \cap (R_1^Z)^{-1} A^Z(\overline{A}^*) \right].$$

Then by Lemma 1.4.3, we know

$$f_Z^j A^Z t \in T_j^j \left(\overline{S}_j^* \cap (T_j^1)^{-1} R_1^1 \left[\overline{S}_{1,n+1} \cap (R_1^Z)^{-1} A^Z(\overline{A}^*) \right] \right)$$

on a subspace of codimension at most

$$H(C_j) - H(S_j) + codim_{S_j}\left(\overline{S}_j^* \cap (T_j^1)^{-1} R_1^1 \left[\overline{S}_{1,n+1} \cap (R_1^Z)^{-1} A^Z(\overline{A}^*)\right]\right).$$

We will use Lemma 1.4.3 to justify (3.38). We will use Lemma 1.4.1 to justify (3.37) and (3.39). We will use (3.21) to justify (3.40)

$$\begin{aligned}
&codim_{S_j}\left(\overline{S}_j^* \cap (T_j^1)^{-1} R_1^1 \left[\overline{S}_{1,n+1} \cap (R_1^Z)^{-1} A^Z(\overline{A}^*)\right]\right) \\
&\quad \leq codim_{S_j}\left(\overline{S}_j^*\right) + codim_{S_j}\left((T_j^1)^{-1} R_1^1 \left[\overline{S}_{1,n+1} \cap (R_1^Z)^{-1} A^Z(\overline{A}^*)\right]\right) && (3.37)\\
&\quad \leq \Delta_j^* + codim_{C_1}\left(R_1^1 \left[\overline{S}_{1,n+1} \cap (R_1^Z)^{-1} A^Z(\overline{A}^*)\right]\right) && (3.38)\\
&\quad = H(C_1) - dim\left(R_1^1 \left[\overline{S}_{1,n+1} \cap (R_1^Z)^{-1} A^Z(\overline{A}^*)\right]\right) + \Delta_j^* \\
&\quad = H(C_1) - dim\left(\overline{S}_{1,n+1} \cap (R_1^Z)^{-1} A^Z(\overline{A}^*)\right) + \Delta_j^* \\
&\quad = H(C_1) - H(S_{n+1}) + codim_{S_{n+1}}\left(\overline{S}_{1,n+1} \cap (R_1^Z)^{-1} A^Z(\overline{A}^*)\right) + \Delta_j^* \\
&\quad \leq H(C_1) - H(S_{n+1}) + codim_{S_{n+1}}(\overline{S}_{1,n+1}) \\
&\qquad + codim_{S_{n+1}}\left((R_1^Z)^{-1} A^Z(\overline{A}^*)\right) + \Delta_j^* && (3.39)\\
&\quad \leq H(C_1) - H(S_{n+1}) + H(Z) - H(S_1) + \Delta_{1,n+1} + \Delta_A^* + \Delta_j^* && (3.40)
\end{aligned}$$

So we know

$$f_Z^j A^Z t \in T_j^j\left(\overline{S}_j^* \cap (T_j^1)^{-1} R_1^1 \left[\overline{S}_{1,n+1} \cap (R_1^Z)^{-1} A^Z(\overline{A}^*)\right]\right)$$

on a subspace of codimension at most

$$H(C_j) - H(S_j) + H(C_1) - H(S_{n+1}) + H(Z) - H(S_1) + \Delta_{1,n+1} + \Delta_A^* + \Delta_j^*.$$

By (D2), we know $\exists c_1 \in \overline{S}_{1,n+1}$ such that

$$f_Z^1 A^Z t = R_1^1 c_1 \text{ where } R_1^Z c_1 \in A^Z(\overline{A}^*). \tag{3.41}$$

By (D2) and (D3), we know for $j \in [2:n]$, $\exists c_j \in \overline{S}_{j,n+1}, b_j \in \overline{S}_j^*$, such that

$$f_Z^j A^Z t = R_j^j c_j = T_j^j b_j \text{ where } R_j^Z c_j \in A^Z(\overline{A}^*) \text{ and } T_j^1 b_j \in R_1^1\left(\overline{S}_{1,n+1} \cap (R_1^Z)^{-1} A^Z(\overline{A}^*)\right). \tag{3.42}$$

Claim 3.1.5.

$$(n-1)\sum_{i=1}^{n} f_{S_1} R_i^Z f_i^{n+1} f_Z^i A^Z t \quad = \quad -nt$$

Proof. Using (3.6), (C1), and the definition of f_{S_1}, we have

$$\sum_{j=2}^{n} f_j^1 f_Z^j A^Z t \quad = \quad t \tag{3.43}$$

Using (3.11) and (3.18), for $j \in [1:n]$ we have

$$\begin{aligned} f_{S_i} R_j^Z + R_j^j f_j^i &= 0 \text{ on } \overline{S}_{j,n+1} \text{ for } i \in [1:n] \backslash j \\ f_j^i R_j^j &= -f_{S_i} R_j^Z \\ f_j^i &= -f_{S_i} R_j^Z f_j^{n+1} \text{ on } R_j^j \left(\overline{S}_{j,n+1} \cap (R_j^Z)^{-1} A^Z(\overline{A}) \right). \end{aligned} \tag{3.44}$$

So we have,

$$f_j^1 = -f_{S_1} R_j^Z f_j^{n+1} \text{ on } R_j^j \left(\overline{S}_{j,n+1} \cap (R_j^Z)^{-1} A^Z(\overline{A}) \right) \text{ for } j \in [2:n] \tag{3.45}$$

Then applying (D2) and (3.45) to (3.43), we have

$$\sum_{j=2}^{n} f_{S_1} R_j^Z f_j^{n+1} f_Z^j A^Z t = -t. \tag{3.46}$$

By (3.7) and the definition of f_{S_i}, for $i \in [2:n]$, we have

$$\begin{aligned} A^i t + f_{S_i} A^Z t &= 0 \\ A^i t + \sum_{j \in [1:n] \backslash i} f_j^i f_Z^j A^Z t &= 0. \end{aligned}$$

By (D2) and (3.44), we have

$$A^i t + \sum_{j \in [1:n] \backslash i} -f_{S_i} R_j^Z f_j^{n+1} f_Z^j A^Z t = 0.$$

By (3.42) and (3.41), we know

$$A^i t + \sum_{j \in [1:n] \backslash i} -f_{S_i} R_j^Z f_j^{n+1} R_j^j c_j = 0.$$

Then by (3.18), we have

$$A^i t + \sum_{j \in [1:n] \backslash i} -f_{S_i} R_j^Z c_j = 0. \tag{3.47}$$

From (3.6) and (3.7), for $i \in [2:n]$ we have

$$f_{S_i} = -A^i f_{S_1} \text{ on } A^Z(\overline{A}). \tag{3.48}$$

We know $R_j^Z c_j \in A^Z(\overline{A}^*)$, so (3.47) becomes

$$A^i t + \sum_{j \in [1:n] \backslash i} A^i f_{S_1} R_j^Z c_j = 0.$$

By Claim 3.1.3, we know that for $i \in [2:n]$, A^i is injective on $\overline{A}^*$. Then by (C1) and (3.6), we have

$$\sum_{j \in [1:n] \backslash i} f_{S_1} R_j^Z c_j = -t.$$

Then using (3.18), (3.42), and (3.41), we have

$$\sum_{j\in[1:n]\backslash i} f_{S_1} R_j^Z f_j^{n+1} f_Z^j A^Z t \quad = \quad -t. \text{ for } i \in [2:n]. \tag{3.49}$$

Now adding (3.46) to the sum over i in (3.49), we have

$$(n-1)\sum_{j=1}^{n} f_{S_1} R_j^Z f_j^{n+1} f_Z^j A^Z t \quad = \quad -nt.$$

□

Claim 3.1.6. For $j \in [2:n]$

$$f_{S_1} R_1^Z f_j^{n+1} f_Z^j A^Z t \quad = \quad f_{S_1} R_j^Z f_j^{n+1} f_Z^j A^Z t.$$

Proof. By (3.17), for $j \in [2:n]$ and $i \in [2:n]\backslash j$, we know

$$f_1^i T_j^1 b_j + f_j^i T_j^j b_j \quad = \quad 0.$$

By (3.42), we know

$$T_j^1 b_j \in R_1^1 \left(\overline{S}_{1,n+1} \cap (R_1^Z)^{-1} A^Z(\overline{A}^*) \right)$$

and

$$T_j^j b_j = R_j^j c_j \in R_j^j \left(\overline{S}_{j,n+1} \cap (R_j^Z)^{-1} A^Z(\overline{A}^*) \right).$$

Then by (3.44), we have

$$-f_{S_i} R_1^Z f_1^{n+1} T_j^1 b_j - f_{S_i} R_j^Z f_j^{n+1} T_j^j b_j \quad = \quad 0. \tag{3.50}$$

From (3.18), we know $f_1^{n+1} R_1^1 = I$ on $\overline{S}_{1,n+1} \cap (R_1^Z)^{-1} A^Z(\overline{A})$. So

$$f_1^{n+1} T_j^1 b_j \in (R_1^Z)^{-1} A^Z(\overline{A}^*).$$

Then applying R_1^Z, we have

$$R_1^Z f_1^{n+1} T_j^1 b_j \in A^Z(\overline{A}^*). \tag{3.51}$$

From (3.42), we also know $T_j^j b_j = R_j^j c_j$. So using (3.18), we know

$$f_j^{n+1} T_j^j b_j = c_j.$$

Then by (3.42), we know

$$R_j^Z f_j^{n+1} T_j^j b_j = R_j^Z c_j \in A^Z(\overline{A}^*). \tag{3.52}$$

Then by applying (3.48), (3.51), and (3.52) to (3.50), we have

$$A^i f_{S_1} R_1^Z f_1^{n+1} T_j^1 b_j + A^i f_{S_1} R_j^Z f_j^{n+1} T_j^j b_j \quad = \quad 0.$$

Since $R_1^Z f_1^{n+1} T_j^1 b_j \in A^Z(\overline{A}^*)$ and $R_j^Z f_j^{n+1} T_j^j b_j \in A^Z(\overline{A}^*)$, by (3.6), we know

$$f_{S_1} R_1^Z f_1^{n+1} T_j^1 b_j \in \overline{A}^*, \text{ and}$$
$$f_{S_1} R_j^Z f_j^{n+1} T_j^j b_j \in \overline{A}^*.$$

So we can apply Claim 3.1.3, to get

$$f_{S_1} R_1^Z f_1^{n+1} T_j^1 b_j + f_{S_1} R_j^Z f_j^{n+1} T_j^j b_j = 0. \tag{3.53}$$

Now letting $i = n+1$ in (3.17), we have

$$f_1^{n+1} T_j^1 b_j = -f_j^{n+1} T_j^j b_j.$$

So (3.53) becomes

$$f_{S_1} R_1^Z f_j^{n+1} T_j^j b_j = f_{S_1} R_j^Z f_j^{n+1} T_j^j b_j.$$

By (3.42), we know

$$f_{S_1} R_1^Z f_j^{n+1} f_Z^j A^Z t = f_{S_1} R_j^Z f_j^{n+1} f_Z^j A^Z t.$$

□

Now combining Claim 3.1.5 and Claim 3.1.6, we have

$$(n-1) \sum_{i=1}^{n} f_{S_1} R_i^Z f_i^{n+1} f_Z^i A^Z t = -nt$$
$$(n-1) f_{S_1} R_1^Z \sum_{i=1}^{n} f_i^{n+1} f_Z^i A^Z t = -nt$$

By the definition of $f_{S_{n+1}}$ and (3.8), we have

$$(n-1) f_{S_1} R_1^Z f_{S_{n+1}} A^Z t = -nt$$
$$0 = nt$$

If the field characteristic is in P, then the characteristic will divide n. So if the field characteristic is not in P, then no nonzero t can satisfy (D1)-(D3). Therefore, the sum of the codimensions in (D1)-(D3) must be at least $H(S_1)$. So we have

$$H(S_1) \leq \Delta_A^* + \sum_{j=1}^{n} H(C_j) - H(S_{n+1}) + H(Z) - H(S_1) + \Delta_A + \Delta_{j,n+1} \tag{3.54}$$
$$+ \sum_{j=2}^{n} H(C_j) - H(S_j) + H(C_1) - H(S_{n+1}) + H(Z) - H(S_1) + \Delta_{1,n+1} + \Delta_A + \Delta_j^* \tag{3.55}$$

Notice that the inequality does not hold for fields of a characteristic that divide n (characteristics in P). Let $p \in P$. Then a counterexample would be: In $V = GF(p)^{n+1}$, let

$$\begin{aligned} S_1 &= \langle(1,0,0,\ldots,0,0)\rangle & C_1 &= \langle(0,1,1,\ldots,1,1)\rangle \\ S_2 &= \langle(0,1,0,\ldots,0,0)\rangle & C_2 &= \langle(1,0,1,\ldots,1,1)\rangle \\ S_3 &= \langle(0,0,1,\ldots,0,0)\rangle & C_3 &= \langle(1,1,0,\ldots,1,1)\rangle \\ &\vdots & &\vdots \\ S_n &= \langle(0,0,0,\ldots,1,0)\rangle & C_n &= \langle(1,1,1,\ldots,0,1)\rangle \\ S_{n+1} &= \langle(0,0,0,\ldots,0,1)\rangle & Z &= \langle(1,1,1,\ldots,1,0)\rangle. \end{aligned}$$

For $i \in [1:n]$, we would have $H(S_i) = H(C_i) = 1$ and $H(S_{n+1}) = H(Z) = 1$. We would also have $\Delta_{j,n+1} = \Delta_j = \Delta_A = \Delta_A^* = 0$. Thus, the inequality would reduce to $1 \leq 0$, which is clearly a contradiction. Therefore, the above inequality is a linear rank inequality for fields of characteristics not in P. □

3.2 A Linear Rank Inequality for any Co-finite Set of Primes

The matroidal network in figure 3.2 was first constructed using the algorithm from [Dougherty 07]. The matroid used in the construction we will call the non-$T8$ matroid. The non-$T8$ matroid is identical to the $T8$ matroid except $\{C_1, C_2, C_3, Z\}$ is a base, where in the $T8$ matroid it is a circuit.

Theorem 3.2.1. *For every co-finite set of primes, P, there exists a linear rank inequality for fields with characteristics in P.*

Proof. For convenience we will use the MATLAB notation $[a:b]$ to denote $\{z \in \mathbb{Z} : a \leq z \leq b\}$. Let n be the product of all the primes not in P. We will assume $n \geq 3$. For the case where P is the set of all primes except 2, we can let $n = 4$ to get the desired result. Recent work, [Dougherty 13], has also handled the case for $n = 2$ and arrives at a simpler inequality than the following.

We will construct a network as a guide to construct the inequality, but the network is not necessary. Construct $n+1$ independent sources and label them $S_1, \ldots, S_{n+1}$. We will define a *channel* to be an input node, an output node, and a single directed edge connecting the input and output nodes with the direction going towards the output node. Then construct $n+1$ channels and label them $C_1, \ldots, C_{n+1}$. Then for each $i \in [1:n+1]$, add a directed edge connecting the input node of C_i and S_j for every $j \in [1:n+1]\backslash i$ with the direction going towards the input node of C_i.

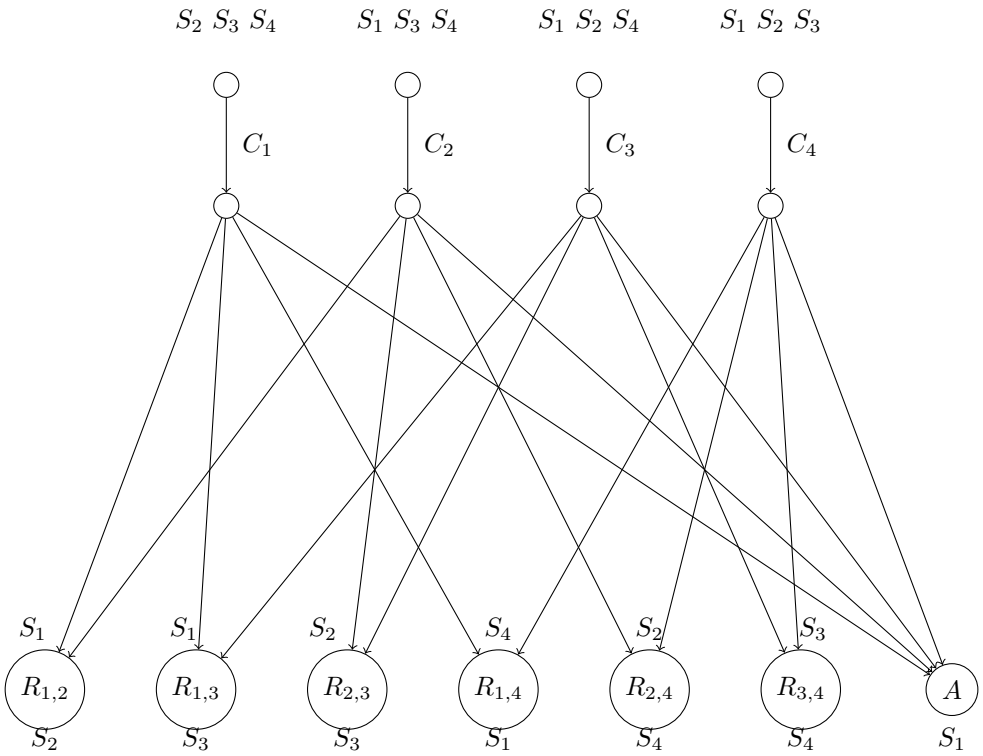

Figure 3.2: The resulting network for $n = 3$. When an source S_i appears above a node, it implies that there is an edge connecting the node to the source.

For short hand purposes, let the term "for every pair (i, j)" denote "for every $i \in [1 : n]$ and $j \in [i + 1 : n + 1]$." Now for every pair (i, j) except $(1, n + 1)$, create a receiver node, $R_{i,j}$, that demands source S_j. Add a directed edge that connects $R_{i,j}$ and the output node of C_i with the direction going towards $R_{i,j}$. Add a directed edge that connects $R_{i,j}$ and the output node of C_j with the direction going towards $R_{i,j}$. Add a directed edge that connects $R_{i,j}$ and source S_i with the direction going towards $R_{i,j}$. Create a receiver node, $R_{1,n+1}$, that demands source S_1. Add a directed edge that connects $R_{1,n+1}$ and the output node of C_1 with the direction going towards $R_{1,n+1}$. Add a directed edge that connects $R_{1,n+1}$ and the output node of C_{n+1} with the direction going towards $R_{1,n+1}$. Add a directed edge that connects $R_{1,n+1}$ and source S_{n+1} with the direction going towards $R_{1,n+1}$.

Create a receiver node, A, that demands S_1. For every $i \in [1 : n+1]$ add a directed edge that connects A and the output node of C_i with the direction going towards A. We will denote the network constructed above by $\mathcal{N}$. Figure 3.2 depicts the resulting network with $n = 3$ obtained by the above procedure. For notational purposes, let $X_i \triangleq S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_{n+1}$. Let V be a finite dimensional vector space with subspaces $S_1, \ldots, S_{n+1}, C_1, \ldots, C_{n+1}$. By Lemma 1.4.4,

for every $i \in [1 : n+1]$ and $j \in [1 : n+1] \backslash i$, we get linear functions:

$$f_i^j : C_i \to S_j,$$

such that

$$\sum_{j \in [1:n+1] \backslash i} f_i^j \quad = \quad I \tag{3.56}$$

on a subspace of C_i, T_i, of codimension $H(C_i|X_i)$. By Lemma 1.4.4, for every pair (i, j) except $(1, n+1)$, we get linear functions:

$$R_{i,j}^i : S_j \to C_i,$$
$$R_{i,j}^j : S_j \to C_j,$$
$$R_{i,j}^S : S_j \to S_i,$$

such that

$$R_{i,j}^S + R_{i,j}^i + R_{i,j}^j \quad = \quad I \tag{3.57}$$

on a subspace of S_j of codimension $H(S_j|S_i, C_i, C_j)$. By Lemma 1.4.4, we get linear functions:

$$R_{1,n+1}^1 : S_1 \to C_1,$$
$$R_{1,n+1}^{n+1} : S_1 \to C_{n+1},$$
$$R_{1,n+1}^S : S_1 \to S_{n+1},$$

such that

$$R_{1,n+1}^S + R_{1,n+1}^1 + R_{1,n+1}^{n+1} \quad = \quad I \tag{3.58}$$

on a subspace of S_1 of codimension $H(S_1|S_{n+1}, C_1, C_{n+1})$. By Lemma 1.4.4, for every $i \in [1 : n+1]$, we get linear functions:

$$A^i : S_1 \to C_i$$

such that

$$\sum_{i=1}^{n+1} A^i \quad = \quad I \tag{3.59}$$

on a subspace of S_1 of codimension $H(S_1|C_1, \ldots, C_{n+1})$. Combining these functions we get new functions:

$$\sum_{i=2}^{n+1} f_i^1 A^i \quad : \quad S_1 \to S_1$$
$$\sum_{i \in [1:n+1] \backslash j} f_i^j A^i \quad : \quad S_1 \to S_j \text{ for } j \in [2 : n+1].$$

Using (3.56), for each $i \in [1 : n+1]$, since $\sum_{j \in [1:n+1]\setminus i} f_i^j = I$ on $T_i \subseteq C_i$, we know

$$\sum_{j \in [1:n+1]} f_i^j A^i(t) = A^i(t)$$

for every t such that $t \in S_1$ and $A^i(t) \in T_i$. By Lemma 1.4.3, $A^i(t) \in T_i$ on a subspace of S_1 of codimension at most $codim_{C_i}(T_i)$. So

$$\sum_{j \in [1:n+1]} f_i^j A^i = A^i$$

on a subspace of S_1 of codimension at most $H(C_i|X_i)$. Summing over i and using Lemma 1.4.1, we get

$$\sum_{i=1}^{n+1} \sum_{j \in [1:n+1]} f_i^j A^i = \sum_{i=1}^{n+1} A^i$$

on a subspace of S_1 of codimension at most $\sum_{i=1}^{n+1} H(C_i|X_i)$. Using (3.59) and Lemma 1.4.1, we get that

$$\sum_{i=1}^{n+1} \sum_{j \in [1:n+1]} f_i^j A^i = I$$

on a subspace of S_1 of codimension at most $H(S_1|C_1, \ldots, C_{n+1}) + \sum_{i=1}^{n+1} H(C_i|X_i)$. Now applying Lemma 1.4.6 and Lemma 1.4.1 to $\sum_{i=2}^{n+1} f_i^1 A^i - I$ and $\sum_{i \in [1:n+1]\setminus j} f_i^j A^i$ (for $j \in [2 : n+1]$) we get a subspace $\overline{S}_1$ of S_1 of codimension at most

$$\begin{aligned} \Delta_A &= H(S_1|C_1, \ldots, C_{n+1}) + \sum_{i=1}^{n+1} H(C_i|X_i) \\ &\quad -H(S_1, \ldots, S_{n+1}) + \sum_{i=1}^{n+1} H(S_i) \end{aligned}$$

on which,

$$\sum_{i=2}^{n+1} f_i^1 A^i = I \tag{3.60}$$

$$\sum_{i \in [1:n+1]\setminus j} f_i^j A^i = 0 \text{ for } j \in [2 : n+1]. \tag{3.61}$$

Combining functions again, for every pair (i, j) except $(1, n+1)$, we get new functions :

$$\begin{aligned} f_i^j R_{i,j}^i &: S_j \to S_j \\ R_{i,j}^S + f_j^i R_{i,j}^j &: S_j \to S_i \\ f_i^k R_{i,j}^i + f_j^k R_{i,j}^j &: S_j \to S_k \text{ for } k \in [1 : n+1]\setminus\{i, j\}. \end{aligned}$$

Using (3.56), (3.57), Lemma 1.4.1, and Lemma 1.4.3 we know the sum of these functions, fixing i and j and summing over k, is equal to I on a subspace of S_j of codimension at most

$H(C_i|X_i) + H(C_j|X_j) + H(S_j|S_i, C_i, C_j)$. Now applying Lemma 1.4.6 and Lemma 1.4.1 to $f_i^j R_{i,j}^i - I$, $R_{i,j}^S + f_j^i R_{i,j}^j$, and $f_i^k R_{i,j}^i + f_j^k R_{i,j}^j$ (for $k \in [1:n+1]\backslash\{i,j\}$) we get a subspace $\overline{S}_{i,j}$ of S_j of codimension at most

$$\begin{aligned}\Delta_{i,j} &= H(C_i|X_i) + H(C_j|X_j) + H(S_j|S_i, C_i, C_j) \\ &\quad -H(S_1, \ldots, S_{n+1}) + \sum_{i=1}^{n+1} H(S_i)\end{aligned}$$

on which,

$$f_i^j R_{i,j}^i = I \tag{3.62}$$

$$R_{i,j}^S + f_j^i R_{i,j}^j = 0 \tag{3.63}$$

$$f_i^k R_{i,j}^i + f_j^k R_{i,j}^j = 0 \text{ for } k \in [1:n+1]\backslash\{i,j\}. \tag{3.64}$$

Combining functions again, we get new functions :

$$\begin{aligned} f_{n+1}^1 R_{1,n+1}^{n+1} &: S_1 \to S_1 \\ R_{1,n+1}^S + f_1^{n+1} R_{1,n+1}^1 &: S_1 \to S_{n+1} \\ f_1^k R_{1,n+1}^1 + f_{n+1}^k R_{1,n+1}^{n+1} &: S_1 \to S_k \text{ for } k \in [2:n]. \end{aligned}$$

Using (3.56), (3.58), Lemma 1.4.1, and Lemma 1.4.3 we know the sum of these functions, summing over k, is equal to I on a subspace of S_1 of codimension at most $H(C_1|X_1) + H(C_{n+1}|X_{n+1}) + H(S_1|S_{n+1}, C_1, C_{n+1})$. Now applying Lemma 1.4.6 and Lemma 1.4.1 to $f_{n+1}^1 R_{1,n+1}^{n+1} - I$, $R_{1,n+1}^S + f_1^{n+1} R_{1,n+1}^1$, and $f_1^k R_{1,n+1}^1 + f_{n+1}^k R_{1,n+1}^{n+1}$ (for $k \in [2:n]$) we get a subspace $\overline{S}_{1,n+1}$ of S_1 of codimension at most

$$\begin{aligned}\Delta_{1,n+1} &= H(C_1|X_1) + H(C_{n+1}|X_{n+1}) + H(S_1|S_{n+1}, C_1, C_{n+1}) \\ &\quad -H(S_1, \ldots, S_{n+1}) + \sum_{i=1}^{n+1} H(S_i)\end{aligned}$$

on which,

$$f_{n+1}^1 R_{1,n+1}^{n+1} = I \tag{3.65}$$

$$R_{1,n+1}^S + f_1^{n+1} R_{1,n+1}^1 = 0 \tag{3.66}$$

$$f_1^k R_{1,n+1}^1 + f_{n+1}^k R_{1,n+1}^{n+1} = 0 \text{ for } k \in [2:n]. \tag{3.67}$$

Claim 3.2.2. For every pair (i,j), there is a subspace, $\overline{S}_{i,j}^* \subseteq \overline{S}_{i,j}$, such that $R_{i,j}^j$ and $R_{i,j}^i$ are both injective on $\overline{S}_{i,j}^*$.

Proof. For every pair (i,j), except $(i,n+1)$ and $(1,n)$, by (3.62), we have

$$\begin{aligned} f_i^j R_{i,j}^i &= I \text{ on } \overline{S}_{i,j} \text{ and} \\ f_i^{j+1} R_{i,j+1}^i &= I \text{ on } \overline{S}_{i,j+1}. \end{aligned}$$

By Lemma 1.4.7, we know $f_i^{j+1}R_{i,j}^i$ is injective on

$$\overline{S}_{i,j}^* \triangleq f_i^j\left(R_{i,j}^i(\overline{S}_{i,j}) \cap R_{i,j+1}^i(\overline{S}_{i,j+1})\right) \subseteq \overline{S}_{i,j},$$

so we know $R_{i,j}^i$ is injective on $\overline{S}_{i,j}^*$. Now by (3.64), we know $f_j^{j+1}R_{i,j}^j$ is injective on $\overline{S}_{i,j}^*$ and thus $R_{i,j}^j$ is injective on $\overline{S}_{i,j}^*$.

For $(1,n)$, by (3.62), we have

$$\begin{aligned} f_1^n R_{1,n}^1 &= I \text{ on } \overline{S}_{1,n} \text{ and} \\ f_1^2 R_{1,2}^1 &= I \text{ on } \overline{S}_{1,2}. \end{aligned}$$

By Lemma 1.4.7, we know $f_1^2R_{1,n}^1$ is injective on

$$\overline{S}_{1,n}^* \triangleq f_1^n\left(R_{1,n}^1(\overline{S}_{1,n}) \cap R_{1,2}^1(\overline{S}_{1,2})\right) \subseteq \overline{S}_{1,n},$$

so we know $R_{1,n}^1$ is injective on $\overline{S}_{1,n}^*$. Now by (3.64), we know $f_n^2R_{1,n}^n$ is injective on $\overline{S}_{1,n}^*$ and thus $R_{1,n}^n$ is injective on $\overline{S}_{1,n}^*$.

For $(i,n+1)$ with $i \in [2:n-1]$, by (3.62), we have

$$\begin{aligned} f_i^{n+1} R_{i,n+1}^i &= I \text{ on } \overline{S}_{i,n+1} \text{ and} \\ f_i^{i+1} R_{i,i+1}^i &= I \text{ on } \overline{S}_{i,i+1}. \end{aligned}$$

By Lemma 1.4.7, we know $f_i^{i+1}R_{i,n+1}^i$ is injective on

$$\overline{S}_{i,n+1}^* \triangleq f_i^{n+1}\left(R_{i,n+1}^i(\overline{S}_{i,n+1}) \cap R_{i,i+1}^i(\overline{S}_{i,i+1})\right) \subseteq \overline{S}_{i,n+1},$$

so we know $R_{i,n+1}^i$ is injective on $\overline{S}_{i,n+1}^*$. Now by (3.64), we know $f_{n+1}^{i+1}R_{i,n+1}^{n+1}$ is injective on $\overline{S}_{i,n+1}^*$ and thus $R_{i,n+1}^{n+1}$ is injective on $\overline{S}_{i,n+1}^*$.

For $(1,n+1)$, by (3.62), we have

$$\begin{aligned} f_2^3 R_{2,3}^2 &= I \text{ on } \overline{S}_{2,3} \text{ and} \\ f_2^{n+1} R_{2,n+1}^2 &= I \text{ on } \overline{S}_{2,n+1}. \end{aligned}$$

By Lemma 1.4.7, we know

$$f_2^3R_{2,n+1}^2 \text{ is injective on } \overline{S}_{2,n+1}^{**} \triangleq f_2^{n+1}\left(R_{2,n+1}^2(\overline{S}_{2,n+1}) \cap R_{2,3}^2(\overline{S}_{2,3})\right). \tag{3.68}$$

Now by (3.64), we know

$$f_{n+1}^3R_{2,n+1}^{n+1} \text{ is injective on } \overline{S}_{2,n+1}^{**}. \tag{3.69}$$

Applying (3.69) and (3.65) to Lemma 1.4.7, we know

$$f_{n+1}^3R_{1,n+1}^{n+1} \text{ is injective on } \overline{S}_{1,n+1}^* \triangleq f_{n+1}^1\left(R_{1,n+1}^{n+1}(\overline{S}_{1,n+1}) \cap R_{2,n+1}^{n+1}(\overline{S}_{2,n+1}^{**})\right) \subseteq \overline{S}_{1,n+1},$$

so we know $R^{n+1}_{1,n+1}$ is injective on $\overline{S}^*_{1,n+1}$. Then by (3.67), we know $f_1^3 R^1_{1,n+1}$ is injective on $\overline{S}^*_{1,n+1}$ and thus $R^1_{1,n+1}$ is injective on $\overline{S}^*_{1,n+1}$.

For $(n, n+1)$, by (3.62), we have

$$\begin{aligned} f_1^2 R^1_{1,2} &= I \text{ on } \overline{S}_{1,2} \text{ and} \\ f_1^n R^1_{1,n} &= I \text{ on } \overline{S}_{1,n}. \end{aligned}$$

By Lemma 1.4.7, we know

$$f_1^2 R^1_{1,n} \text{ is injective on } \overline{S}^{**}_{1,n} \triangleq f_1^n \left(R^1_{1,n}(\overline{S}_{1,n}) \cap R^1_{1,2}(\overline{S}_{1,2}) \right) \subseteq \overline{S}_{1,n}. \tag{3.70}$$

Now by (3.64), we know

$$f_n^2 R^n_{1,n} \text{ is injective on } \overline{S}^{**}_{1,n}. \tag{3.71}$$

By (3.62), we know

$$f_n^{n+1} R^n_{n,n+1} = I \text{ on } \overline{S}_{n,n+1}. \tag{3.72}$$

Applying (3.71) and (3.72) to Lemma 1.4.7, we know

$$f_n^2 R^n_{n,n+1} \text{ is injective on } \overline{S}^*_{n,n+1} \triangleq f_n^{n+1} \left(R^n_{n,n+1}(\overline{S}_{n,n+1}) \cap R^n_{1,n}(\overline{S}^{**}_{1,n}) \right) \subseteq \overline{S}_{n,n+1},$$

so $R^n_{n,n+1}$ is injective on $\overline{S}^*_{n,n+1}$. Then by (3.64), we know $f_{n+1}^2 R^{n+1}_{n,n+1}$ is injective on $\overline{S}^*_{n,n+1}$ and thus $R^{n+1}_{n,n+1}$ is injective on $\overline{S}^*_{n,n+1}$. □

Now we would like to find upper bounds on the codimensions of the subspaces found in Claim 3.2.2. We will use Lemma 1.4.1 to justify lines (3.74), (3.77), (3.80), (3.83), (3.86), (3.89), and (3.92). We will use (3.62) to justify lines (3.73), (3.76), (3.78), (3.75), (3.79), (3.81), (3.82), (3.84), (3.88), (3.90), and (3.91). We will use (3.65) to justify (3.85). We will use (3.65) and (3.69) to justify (3.87). We will use (3.62) and (3.71) to justify (3.93). For every pair (i, j), except $(i, n+1)$ and $(1, n)$, we have

$$\begin{aligned} codim_{S_j} \left(\overline{S}^*_{i,j} \right) &= H(S_j) - dim(\overline{S}^*_{i,j}) \\ &= H(S_j) - dim \left[f_i^j \left(R^i_{i,j}(\overline{S}_{i,j}) \cap R^i_{i,j+1}(\overline{S}_{i,j+1}) \right) \right] \\ &= H(S_j) - dim \left(R^i_{i,j}(\overline{S}_{i,j}) \cap R^i_{i,j+1}(\overline{S}_{i,j+1}) \right) \qquad (3.73) \\ &= H(S_j) - H(C_i) + codim_{C_i} \left(R^i_{i,j}(\overline{S}_{i,j}) \cap R^i_{i,j+1}(\overline{S}_{i,j+1}) \right) \\ &\leq H(S_j) - H(C_i) + codim_{C_i} \left(R^i_{i,j}(\overline{S}_{i,j}) \right) + codim_{C_i} \left(R^i_{i,j+1}(\overline{S}_{i,j+1}) \right) \qquad (3.74) \\ &= H(S_j) - H(C_i) + H(C_i) - dim \left(R^i_{i,j}(\overline{S}_{i,j}) \right) + H(C_i) - dim \left(R^i_{i,j+1}(\overline{S}_{i,j+1}) \right) \\ &= H(S_j) + H(C_i) - dim \left(R^i_{i,j}(\overline{S}_{i,j}) \right) - dim \left(R^i_{i,j+1}(\overline{S}_{i,j+1}) \right) \\ &= H(S_j) + H(C_i) - dim(\overline{S}_{i,j}) - dim(\overline{S}_{i,j+1}) \qquad (3.75) \\ &= H(S_j) + H(C_i) - H(S_j) + codim_{S_j}(\overline{S}_{i,j}) - H(S_{j+1}) + codim_{S_{j+1}}(\overline{S}_{i,j+1}) \\ &\leq H(C_i) - H(S_{j+1}) + \Delta_{i,j} + \Delta_{i,j+1} \\ &\triangleq \Delta^*_{i,j}. \end{aligned}$$

For $(1,n)$, we have

$$
\begin{aligned}
codim_{S_n}\left(\overline{S}^*_{1,n}\right) &= H(S_n) - dim(\overline{S}^*_{1,n}) \\
&= H(S_n) - dim\left[f_1^n\left(R^1_{1,n}(\overline{S}_{1,n}) \cap R^1_{1,2}(\overline{S}_{1,2})\right)\right] \\
&= H(S_n) - dim\left(R^1_{1,n}(\overline{S}_{1,n}) \cap R^1_{1,2}(\overline{S}_{1,2})\right) && (3.76)\\
&= H(S_n) - H(C_1) + codim_{C_1}\left(R^1_{1,n}(\overline{S}_{1,n}) \cap R^1_{1,2}(\overline{S}_{1,2})\right) \\
&\leq H(S_n) - H(C_1) + codim_{C_1}\left(R^1_{1,n}(\overline{S}_{1,n})\right) + codim_{C_1}\left(R^1_{1,2}(\overline{S}_{1,2})\right) && (3.77)\\
&= H(S_n) - H(C_1) + H(C_1) - dim\left(R^1_{1,n}(\overline{S}_{1,n})\right) + H(C_1) - dim\left(R^1_{1,2}(\overline{S}_{1,2})\right) \\
&= H(S_n) + H(C_1) - dim\left(R^1_{1,n}(\overline{S}_{1,n})\right) - dim\left(R^1_{1,2}(\overline{S}_{1,2})\right) \\
&= H(S_n) + H(C_1) - dim(\overline{S}_{1,n}) - dim(\overline{S}_{1,2}) && (3.78)\\
&= H(S_n) + H(C_1) - H(S_n) + codim_{S_n}(\overline{S}_{1,n}) - H(S_2) + codim_{S_2}(\overline{S}_{1,2}) \\
&\leq H(C_1) - H(S_2) + \Delta_{1,n} + \Delta_{1,2} \\
&\triangleq \Delta^*_{1,n}.
\end{aligned}
$$

For every $i \in [2 : n-1]$, we have

$$
\begin{aligned}
codim_{S_{n+1}}\left(\overline{S}^*_{i,n+1}\right) &= H(S_{n+1}) - dim(\overline{S}^*_{i,n+1}) \\
&= H(S_{n+1}) - dim\left[f_i^{n+1}\left(R^i_{i,n+1}(\overline{S}_{i,n+1}) \cap R^i_{i,i+1}(\overline{S}_{i,i+1})\right)\right] \\
&= H(S_{n+1}) - dim\left(R^i_{i,n+1}(\overline{S}_{i,n+1}) \cap R^i_{i,i+1}(\overline{S}_{i,i+1})\right) && (3.79)\\
&= H(S_{n+1}) - H(C_i) + codim_{C_i}\left(R^i_{i,n+1}(\overline{S}_{i,n+1}) \cap R^i_{i,i+1}(\overline{S}_{i,i+1})\right) \\
&\leq H(S_{n+1}) - H(C_i) + codim_{C_i}\left(R^i_{i,n+1}(\overline{S}_{i,n+1})\right) + codim_{C_i}\left(R^i_{i,i+1}(\overline{S}_{i,i+1})\right) && (3.80)\\
&= H(S_{n+1}) - H(C_i) + H(C_i) - dim\left(R^i_{i,n+1}(\overline{S}_{i,n+1})\right) \\
&\qquad + H(C_i) - dim\left(R^i_{i,i+1}(\overline{S}_{i,i+1})\right) \\
&= H(S_{n+1}) - H(C_i) + H(C_i) - dim(\overline{S}_{i,n+1}) + H(C_i) - dim(\overline{S}_{i,i+1}) && (3.81)\\
&= H(S_{n+1}) + H(C_i) - dim(\overline{S}_{i,n+1}) - dim(\overline{S}_{i,i+1}) \\
&= H(S_{n+1}) + H(C_i) - H(S_{n+1}) + codim_{S_{n+1}}(\overline{S}_{i,n+1}) \\
&\qquad - H(S_{i+1}) + codim_{S_{i+1}}(\overline{S}_{i,i+1}) \\
&\leq H(C_i) - H(S_{i+1}) + \Delta_{i,n+1} + \Delta_{i,i+1} \\
&\triangleq \Delta^*_{i,n+1}.
\end{aligned}
$$

The following will be used for the case $(1, n+1)$,

$$
\begin{aligned}
codim_{S_{n+1}}\left(\overline{S}^{**}_{2,n+1}\right) &= H(S_{n+1}) - dim(\overline{S}^{**}_{2,n+1}) \\
&= H(S_{n+1}) - dim\left[f_2^{n+1}\left(R^2_{2,n+1}(\overline{S}_{2,n+1}) \cap R^2_{2,3}(\overline{S}_{2,3})\right)\right] \\
&= H(S_{n+1}) - dim\left(R^2_{2,n+1}(\overline{S}_{2,n+1}) \cap R^2_{2,3}(\overline{S}_{2,3})\right) \qquad (3.82) \\
&= H(S_{n+1}) - H(C_2) + codim_{C_2}\left(R^2_{2,n+1}(\overline{S}_{2,n+1}) \cap R^2_{2,3}(\overline{S}_{2,3})\right) \\
&\leq H(S_{n+1}) - H(C_2) + codim_{C_2}\left(R^2_{2,n+1}(\overline{S}_{2,n+1})\right) + codim_{C_2}\left(R^2_{2,3}(\overline{S}_{2,3})\right) \qquad (3.83) \\
&= H(S_{n+1}) - H(C_2) + H(C_2) - dim\left(R^2_{2,n+1}(\overline{S}_{2,n+1})\right) \\
&\quad + H(C_2) - dim\left(R^2_{2,3}(\overline{S}_{2,3})\right) \\
&= H(S_{n+1}) + H(C_2) - dim(\overline{S}_{2,n+1}) - dim(\overline{S}_{2,3}) \qquad (3.84) \\
&= H(S_{n+1}) + H(C_2) - H(S_{n+1}) + codim_{S_{n+1}}(\overline{S}_{2,n+1}) \\
&\quad - H(S_3) + codim_{S_3}(\overline{S}_{2,3}) \\
&\leq H(C_2) - H(S_3) + \Delta_{2,n+1} + \Delta_{2,3} \\
&\triangleq \Delta^{**}_{2,n+1}.
\end{aligned}
$$

For $(1, n+1)$, we have

$$
\begin{aligned}
codim_{S_1}\left(\overline{S}^{*}_{1,n+1}\right) &= H(S_1) - dim(\overline{S}^{*}_{1,n+1}) \\
&= H(S_1) - dim\left[f^1_{n+1}\left(R^{n+1}_{1,n+1}(\overline{S}_{1,n+1}) \cap R^{n+1}_{2,n+1}(\overline{S}^{**}_{2,n+1})\right)\right] \\
&= H(S_1) - dim\left(R^{n+1}_{1,n+1}(\overline{S}_{1,n+1}) \cap R^{n+1}_{2,n+1}(\overline{S}^{**}_{2,n+1})\right) \qquad (3.85) \\
&= H(S_1) - H(C_{n+1}) + codim_{C_{n+1}}\left(R^{n+1}_{1,n+1}(\overline{S}_{1,n+1}) \cap R^{n+1}_{2,n+1}(\overline{S}^{**}_{2,n+1})\right) \\
&\leq H(S_1) - H(C_{n+1}) + codim_{C_{n+1}}\left(R^{n+1}_{1,n+1}(\overline{S}_{1,n+1})\right) \\
&\quad + codim_{C_{n+1}}\left(R^{n+1}_{2,n+1}(\overline{S}^{**}_{2,n+1})\right) \qquad (3.86) \\
&= H(S_1) - H(C_{n+1}) + H(C_{n+1}) - dim\left(R^{n+1}_{1,n+1}(\overline{S}_{1,n+1})\right) \\
&\quad + H(C_{n+1}) - dim\left(R^{n+1}_{2,n+1}(\overline{S}^{**}_{2,n+1})\right) \\
&= H(S_1) + H(C_{n+1}) - dim(\overline{S}_{1,n+1}) - dim(\overline{S}^{**}_{2,n+1}) \qquad (3.87) \\
&= H(S_1) + H(C_{n+1}) - H(S_1) + codim_{S_1}(\overline{S}_{1,n+1}) - H(S_{n+1}) \\
&\quad + codim_{S_{n+1}}(\overline{S}^{**}_{2,n+1}) \\
&\leq H(C_{n+1}) - H(S_{n+1}) + \Delta_{1,n+1} + \Delta^{**}_{2,n+1} \\
&\triangleq \Delta^{*}_{1,n+1}
\end{aligned}
$$

The following will be used for the case $(n, n+1)$,

$$
\begin{aligned}
codim_{S_n}\left(\overline{S}^{**}_{1,n}\right) &= H(S_n) - dim\left(\overline{S}^{**}_{1,n}\right) \\
&= H(S_n) - dim\left[f_1^n\left(R^1_{1,n}(\overline{S}_{1,n}) \cap R^1_{1,2}(\overline{S}_{1,2})\right)\right] \\
&= H(S_n) - dim\left(R^1_{1,n}(\overline{S}_{1,n}) \cap R^1_{1,2}(\overline{S}_{1,2})\right) && (3.88) \\
&= H(S_n) - H(C_1) + codim_{C_1}\left(R^1_{1,n}(\overline{S}_{1,n}) \cap R^1_{1,2}(\overline{S}_{1,2})\right) \\
&\leq H(S_n) - H(C_1) + codim_{C_1}\left(R^1_{1,n}(\overline{S}_{1,n})\right) + codim_{C_1}\left(R^1_{1,2}(\overline{S}_{1,2})\right) && (3.89) \\
&= H(S_n) - H(C_1) + H(C_1) - dim\left(R^1_{1,n}(\overline{S}_{1,n})\right) + H(C_1) - dim\left(R^1_{1,2}(\overline{S}_{1,2})\right) \\
&= H(S_n) + H(C_1) - dim(\overline{S}_{1,n}) - dim(\overline{S}_{1,2}) && (3.90) \\
&= H(S_n) + H(C_1) - H(S_n) + codim_{S_n}(\overline{S}_{1,n}) - H(S_2) + codim_{S_2}(\overline{S}_{1,2}) \\
&\leq H(C_1) - H(S_2) + \Delta_{1,n} + \Delta_{1,2} \\
&\triangleq \Delta^{**}_{n,n+1}
\end{aligned}
$$

For $(n, n+1)$, we have

$$
\begin{aligned}
codim_{S_{n+1}}\left(\overline{S}^{*}_{n,n+1}\right) &= H(S_{n+1}) - dim(\overline{S}^{*}_{n,n+1}) \\
&= H(S_{n+1}) - dim\left[f_n^{n+1}\left(R^n_{n,n+1}(\overline{S}_{n,n+1}) \cap R^n_{1,n}(\overline{S}^{**}_{1,n})\right)\right] \\
&= H(S_{n+1}) - dim\left(R^n_{n,n+1}(\overline{S}_{n,n+1}) \cap R^n_{1,n}(\overline{S}^{**}_{1,n})\right) && (3.91) \\
&= H(S_{n+1}) - H(C_n) + codim_{C_n}\left(R^n_{n,n+1}(\overline{S}_{n,n+1}) \cap R^n_{1,n}(\overline{S}^{**}_{1,n})\right) \\
&\leq H(S_{n+1}) - H(C_n) + codim_{C_n}\left(R^n_{n,n+1}(\overline{S}_{n,n+1})\right) + codim_{C_n}\left(R^n_{1,n}(\overline{S}^{**}_{1,n})\right) && (3.92) \\
&= H(S_{n+1}) - H(C_n) + H(C_n) - dim\left(R^n_{n,n+1}(\overline{S}_{n,n+1})\right) \\
&\qquad + H(C_n) - dim\left(R^n_{1,n}(\overline{S}^{**}_{1,n})\right) \\
&= H(S_{n+1}) + H(C_n) - dim(\overline{S}_{n,n+1}) - dim(\overline{S}^{**}_{1,n}) && (3.93) \\
&= H(S_{n+1}) + H(C_n) - H(S_{n+1}) + codim_{S_{n+1}}(\overline{S}_{n,n+1}) \\
&\qquad - H(S_n) + codim_{S_n}(\overline{S}^{**}_{1,n}) \\
&\leq H(C_n) - H(S_n) + \Delta_{n,n+1} + \Delta^{**}_{1,n+1} \\
&\triangleq \Delta^{*}_{n,n+1}
\end{aligned}
$$

Claim 3.2.3.

a) There exists a subspace $\widehat{S}_{1,n+1} \subseteq \overline{S}_{1,n+1}$, such that $f^2_{n+1}R^{n+1}_{1,n+1}$ is injective on $\widehat{S}_{1,n+1}$.

b) There exists a subspace $\widehat{S}_{2,n+1} \subseteq \overline{S}_{2,n+1}$, such that $f^3_{n+1}R^{n+1}_{2,n+1}$ is injective on $\widehat{S}_{2,n+1}$.

Proof.

a) By (3.62), we know

$$\begin{aligned} f_1^2 R_{1,2}^1 &= I \text{ on } \overline{S}_{1,2} \text{ and} \\ f_1^3 R_{1,3}^1 &= I \text{ on } \overline{S}_{1,3}. \end{aligned}$$

By Lemma 1.4.7, we know $f_1^2 R_{1,3}^1$ is injective on

$$\widehat{S}_{1,3}^* \triangleq f_1^3 \left[R_{1,3}^1(\overline{S}_{1,3}) \cap R_{1,2}^1(\overline{S}_{1,2}) \right] \subseteq \overline{S}_{1,3}$$

Now by (3.64) and (3.62), we know

$$f_3^2 R_{1,3}^3 \text{ is injective on } \widehat{S}_{1,3}^* \text{ and} \tag{3.94}$$

$$f_3^{n+1} R_{3,n+1}^3 = I \text{ on } \overline{S}_{3,n+1}. \tag{3.95}$$

By Lemma 1.4.7, we know $f_3^2 R_{3,n+1}^3$ is injective on

$$\widehat{S}_{3,n+1}^* \triangleq f_3^{n+1} \left[R_{3,n+1}^3(\overline{S}_{3,n+1}) \cap R_{1,3}^3(\widehat{S}_{1,3}^*) \right] \subseteq \overline{S}_{3,n+1}.$$

Now by (3.64) and (3.65), we know

$$\begin{aligned} f_{n+1}^2 R_{3,n+1}^{n+1} &\text{ is injective on } \widehat{S}_{3,n+1}^* \text{ and} \\ f_{n+1}^1 R_{1,n+1}^{n+1} &= I \text{ on } \overline{S}_{1,n+1}. \end{aligned}$$

By Lemma 1.4.7, we know $f_{n+1}^2 R_{1,n+1}^{n+1}$ is injective on

$$\widehat{S}_{1,n+1} \triangleq f_{n+1}^1 \left[R_{1,n+1}^{n+1}(\overline{S}_{1,n+1}) \cap R_{3,n+1}^{n+1}(\widehat{S}_{3,n+1}^*) \right] \subseteq \overline{S}_{1,n+1}.$$

b) By (3.62), we know

$$\begin{aligned} f_2^{n+1} R_{2,n+1}^2 &= I \text{ on } \overline{S}_{2,n+1} \text{ and} \\ f_2^3 R_{2,3}^2 &= I \text{ on } \overline{S}_{2,3}. \end{aligned}$$

By Lemma 1.4.7, we know $f_2^3 R_{2,n+1}^2$ is injective on

$$\widehat{S}_{2,n+1} \triangleq f_2^{n+1} \left[R_{2,n+1}^2(\overline{S}_{2,n+1}) \cap R_{2,3}^2(\overline{S}_{2,3}) \right] \subseteq \overline{S}_{2,n+1}.$$

Now by (3.64), $f_{n+1}^3 R_{2,n+1}^{n+1}$ is injective on $\widehat{S}_{2,n+1}$.

□

Now we would like to find upper bounds for the codimensions found Claim 3.2.3. We will use Lemma 1.4.1 to justify lines (3.97), (3.100), (3.103) and (3.107). We will use (3.62) to justify lines (3.96), (3.98), (3.99), (3.106), and (3.108). We will use (3.62) and (3.94) to justify

(3.101). We will use (3.65) to justify (3.102). We will use (3.65) and (3.96) to justify (3.104).

$$
\begin{aligned}
codim_{S_3}\widehat{S}^*_{1,3} &= H(S_3) - dim(\widehat{S}^*_{1,3}) \\
&= H(S_3) - dim\left[f_1^3\left(R^1_{1,3}(\overline{S}_{1,3}) \cap R^1_{1,2}(\overline{S}_{1,2})\right)\right] \\
&= H(S_3) - dim\left(R^1_{1,3}(\overline{S}_{1,3}) \cap R^1_{1,2}(\overline{S}_{1,2})\right) && (3.96) \\
&= H(S_3) - H(C_1) + codim_{C_1}\left(R^1_{1,3}(\overline{S}_{1,3}) \cap R^1_{1,2}(\overline{S}_{1,2})\right) \\
&\leq H(S_3) - H(C_1) + codim_{C_1}\left(R^1_{1,3}(\overline{S}_{1,3})\right) + codim_{C_1}\left(R^1_{1,2}(\overline{S}_{1,2})\right) && (3.97) \\
&= H(S_3) - H(C_1) + H(C_1) - dim\left(R^1_{1,3}(\overline{S}_{1,3})\right) + H(C_1) - dim\left(R^1_{1,2}(\overline{S}_{1,2})\right) \\
&= H(S_3) + H(C_1) - dim(\overline{S}_{1,3}) - dim(\overline{S}_{1,2}) && (3.98) \\
&= H(S_3) + H(C_1) - H(S_3) + codim_{S_3}(\overline{S}_{1,3}) - H(S_2) + codim_{S_2}(\overline{S}_{1,2}) \\
&\leq H(C_1) - H(S_2) + \Delta_{1,3} + \Delta_{1,2}
\end{aligned}
$$

$$
\begin{aligned}
codim_{S_{n+1}}(\widehat{S}^*_{3,n+1}) &= H(S_{n+1}) - dim(\widehat{S}^*_{3,n+1}) \\
&= H(S_{n+1}) - dim\left[f_3^{n+1}\left(R^3_{3,n+1}(\overline{S}_{3,n+1}) \cap R^3_{1,3}(\widehat{S}^*_{1,3})\right)\right] \\
&= H(S_{n+1}) - dim\left(R^3_{3,n+1}(\overline{S}_{3,n+1}) \cap R^3_{1,3}(\widehat{S}^*_{1,3})\right) && (3.99) \\
&= H(S_{n+1}) - H(C_3) + codim_{C_3}\left(R^3_{3,n+1}(\overline{S}_{3,n+1}) \cap R^3_{1,3}(\widehat{S}^*_{1,3})\right) \\
&\leq H(S_{n+1}) - H(C_3) + codim_{C_3}\left(R^3_{3,n+1}(\overline{S}_{3,n+1})\right) + codim_{C_3}\left(R^3_{1,3}(\widehat{S}^*_{1,3})\right) && (3.100) \\
&= H(S_{n+1}) - H(C_3) + H(C_3) - dim\left(R^3_{3,n+1}(\overline{S}_{3,n+1})\right) \\
&\qquad + H(C_3) - dim\left(R^3_{1,3}(\widehat{S}^*_{1,3})\right) \\
&= H(S_{n+1}) + H(C_3) - dim(\overline{S}_{3,n+1}) - dim(\widehat{S}^*_{1,3}) && (3.101) \\
&= H(S_{n+1}) + H(C_3) - H(S_{n+1}) + codim_{S_{n+1}}(\overline{S}_{3,n+1}) - H(S_3) + codim_{S_3}(\widehat{S}^*_{1,3}) \\
&\leq H(C_3) - H(S_3) + H(C_1) - H(S_2) + \Delta_{3,n+1}) + \Delta_{1,3} + \Delta_{1,2}
\end{aligned}
$$

$$
\begin{aligned}
codim_{S_{n+1}}(\widehat{S}_{1,n+1}) &= H(S_1) - dim(\widehat{S}_{1,n+1}) \\
&= H(S_1) - dim\left[f^1_{n+1}\left(R^{n+1}_{1,n+1}(\overline{S}_{1,n+1}) \cap R^{n+1}_{3,n+1}(\widehat{S}^*_{3,n+1})\right)\right] \\
&= H(S_1) - dim\left(R^{n+1}_{1,n+1}(\overline{S}_{1,n+1}) \cap R^{n+1}_{3,n+1}(\widehat{S}^*_{3,n+1})\right) && (3.102) \\
&= H(S_1) - H(C_{n+1}) + codim_{C_{n+1}}\left(R^{n+1}_{1,n+1}(\overline{S}_{1,n+1}) \cap R^{n+1}_{3,n+1}(\widehat{S}^*_{3,n+1})\right) \\
&\leq H(S_1) - H(C_{n+1}) + \\
&\quad codim_{C_{n+1}}\left(R^{n+1}_{1,n+1}(\overline{S}_{1,n+1})\right) + codim_{C_{n+1}}\left(R^{n+1}_{3,n+1}(\widehat{S}^*_{3,n+1})\right) && (3.103) \\
&= H(S_1) - H(C_{n+1}) + H(C_{n+1}) - dim\left(R^{n+1}_{1,n+1}(\overline{S}_{1,n+1})\right) \\
&\quad + H(C_{n+1}) - dim\left(R^{n+1}_{3,n+1}(\widehat{S}^*_{3,n+1})\right) \\
&= H(S_1) + H(C_{n+1}) - dim(\overline{S}_{1,n+1}) - dim(\widehat{S}^*_{3,n+1}) && (3.104) \\
&= H(S_1) + H(C_1) - H(S_1) + codim_{S_1}(\overline{S}_{1,n+1}) - H(S_{n+1}) + codim_{S_{n+1}}(\widehat{S}^*_{3,n+1}) \\
&\leq 2H(C_1) - H(S_{n+1}) - H(S_2) + H(C_3) - H(S_3) \\
&\quad + \Delta_{1,n+1} + \Delta_{3,n+1} + \Delta_{1,3} + \Delta_{1,2} \\
&\triangleq \widehat{\Delta}_{1,n+1}, && (3.105)
\end{aligned}
$$

$$
\begin{aligned}
codim_{S_{n+1}}(\widehat{S}_{2,n+1}) &= H(S_{n+1}) - dim(\widehat{S}_{2,n+1}) \\
&= H(S_{n+1}) - dim\left[f^{n+1}_2\left(R^2_{2,n+1}(\overline{S}_{2,n+1}) \cap R^2_{2,3}(\overline{S}_{2,3})\right)\right] \\
&= H(S_{n+1}) - dim\left(R^2_{2,n+1}(\overline{S}_{2,n+1}) \cap R^2_{2,3}(\overline{S}_{2,3})\right) && (3.106) \\
&= H(S_{n+1}) - H(C_2) + codim_{C_2}\left(R^2_{2,n+1}(\overline{S}_{2,n+1}) \cap R^2_{2,3}(\overline{S}_{2,3})\right) \\
&\leq H(S_{n+1}) - H(C_2) + codim_{C_2}\left(R^2_{2,n+1}(\overline{S}_{2,n+1})\right) + codim_{C_2}\left(R^2_{2,3}(\overline{S}_{2,3})\right)) && (3.107) \\
&= H(S_{n+1}) - H(C_2) + H(C_2) - dim\left(R^2_{2,n+1}(\overline{S}_{2,n+1})\right) \\
&\quad + H(C_2) - dim\left(R^2_{2,3}(\overline{S}_{2,3})\right) \\
&= H(S_{n+1}) + H(C_2) - dim(\overline{S}_{2,n+1}) - dim(\overline{S}_{2,3}) && (3.108) \\
&= H(S_{n+1}) + H(C_2) - H(S_{n+1}) + codim_{S_{n+1}}(\overline{S}_{2,n+1}) - H(S_3) + codim_{S_3}(\overline{S}_{2,3}) \\
&\leq H(C_2) - H(S_3) + \Delta_{2,n+1} + \Delta_{2,3} \\
&\triangleq \widehat{\Delta}_{2,n+1}. && (3.109)
\end{aligned}
$$

Now using Claim 3.2.2, (3.64), and (3.67), for every pair (i,j) and $k \in [1:n+1]\backslash\{i,j\}$, we have

$$f^k_i \quad = \quad -f^k_j R^j_{i,j}(R^i_{i,j})^{-1} \text{ on } R^i_{i,j}(\overline{S}^*_{i,j}). \tag{3.110}$$

Using Claim 3.2.2, (3.64), and (3.67), for every pair (i,j) and $k \in [1:n+1]\backslash\{i,j\}$, we have

$$f^k_j \quad = \quad -f^k_i R^i_{i,j}(R^j_{i,j})^{-1} \text{ on } R^j_{i,j}(\overline{S}^*_{i,j}). \tag{3.111}$$

Let $t \in S_1$. Now we will assume t satisfies conditions (C1) - (C7). The justifications can be found below.

C1) We will assume $t \in \overline{S}_1$. This is true on a subspace of S_1 of codimension at most Δ_A.

C2) We will assume for $k \in [3:n]$, $A^{k-1}t \in R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k})$. This is true on a subspace of S_1 of codimension at most $H(C_{k-1}) - H(S_k) + \Delta^*_{k-1,k}$.

C2a) For $i \in [1:k-2]$, we will assume

$$A^i t \quad \in \quad R^i_{i,k-1}(R^{k-1}_{i,k-1})^{-1}\left[R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k}) \cap R^{k-1}_{i,k-1}(\overline{S}^*_{i,k-1})\right].$$

This is true on a subspace of S_1 of codimension at most

$$H(C_i) - H(S_k) + H(C_{k-1}) - H(S_{k-1}) + \Delta^*_{k-1,k} + \Delta^*_{i,k-1}.$$

C2b) For $i \in [k+1:n+1]$, we will assume

$$A^i t \quad \in \quad R^i_{k-1,i}(R^{k-1}_{k-1,i})^{-1}\left[R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k}) \cap R^{k-1}_{k-1,i}(\overline{S}^*_{k-1,i})\right].$$

This is true on a subspace of S_1 of codimension at most

$$H(C_i) - H(S_k) + H(C_{k-1}) - H(S_i) + \Delta^*_{k-1,k} + \Delta^*_{k-1,i}.$$

C3) We will assume for $k \in [3:n]$, $A^{k-1}t \in R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1})$. This is true on a subspace of A of codimension at most $H(C_{k-1}) - H(S_{k+1}) + \Delta^*_{k-1,k+1}$.

C3a) For $i \in [1:k-2]$, we will assume

$$A^i t \quad \in \quad R^i_{i,k-1}(R^{k-1}_{i,k-1})^{-1}\left[R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1}) \cap R^{k-1}_{i,k-1}(\overline{S}^*_{i,k-1})\right].$$

This is true on a subspace of S_1 of codimension at most

$$H(C_i) - H(S_{k+1}) + H(C_{k-1}) - H(S_{k-1}) + \Delta^*_{k-1,k+1} + \Delta^*_{i,k-1}.$$

C3b) For $i \in [k:n+1]\backslash k+1$, we will assume

$$A^i t \quad \in \quad R^i_{k-1,i}(R^{k-1}_{k-1,i})^{-1}\left[R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1}) \cap R^{k-1}_{k-1,i}(\overline{S}^*_{k-1,i})\right].$$

This is true on a subspace of S_1 of codimension at most

$$H(C_i) - H(S_{k+1}) + H(C_{k-1}) - H(S_i) + \Delta^*_{k-1,k+1} + \Delta^*_{k-1,i}.$$

C4) We will assume $A^{n+1}t \in R^{n+1}_{1,n+1}(\widehat{S}_{1,n+1}) \cap R^{n+1}_{2,n+1}(\widehat{S}_{2,n+1})$. This is true on a subspace of S_1 of codimension at most $2H(C_{n+1}) - H(S_1) - H(S_{n+1}) + \widehat{\Delta}_{1,n+1} + \widehat{\Delta}_{2,n+1}$.

C5) For $i \in [3:n]$, we will assume

$$A^i t \in R^i_{i,n+1}(R^{n+1}_{i,n+1})^{-1}\left[R^{n+1}_{1,n+1}(\widehat{S}_{1,n+1}) \cap R^{n+1}_{i,n+1}(\overline{S}^*_{i,n+1})\right].$$

This is true on a subspace of S_1 of codimension at most

$$H(C_i) - H(S_1) + H(C_{n+1}) - H(S_{n+1}) + \widehat{\Delta}_{1,n+1} + \Delta^*_{i,n+1}.$$

C5a) We will assume

$$A^1 t \in R^1_{1,n+1}\left(\widehat{S}_{1,n+1} \cap \overline{S}^*_{1,n+1}\right).$$

This is true on a subspace of S_1 of codimension at most

$$H(C_1) - H(S_1) + \widehat{\Delta}_{1,n+1} + \Delta^*_{1,n+1}.$$

C6) For $i \in [2:n]\backslash 3$, we will assume

$$A^i t \in R^i_{i,n+1}(R^{n+1}_{i,n+1})^{-1}\left[R^{n+1}_{2,n+1}(\widehat{S}_{2,n+1}) \cap R^{n+1}_{i,n+1}(\overline{S}^*_{i,n+1})\right].$$

This is true on a subspace of S_1 of codimension at most

$$H(C_i) + H(C_{n+1}) - 2H(S_{n+1}) + \widehat{\Delta}_{2,n+1} + \Delta^*_{i,n+1}.$$

C6a) We will assume

$$A^1 t \in R^1_{1,n+1}(R^{n+1}_{1,n+1})^{-1}\left[R^{n+1}_{2,n+1}(\widehat{S}_{2,n+1}) \cap R^{n+1}_{1,n+1}(\overline{S}^*_{1,n+1})\right].$$

This is true on a subspace of S_1 of codimension at most

$$H(C_1) + H(C_{n+1}) - H(S_{n+1}) - H(S_1) + \widehat{\Delta}_{2,n+1} + \Delta^*_{1,n+1}.$$

C7) For $k \in [4:n+1]$, we will assume

$$A^k t \in R^k_{k-1,k}(\overline{S}^*_{k-1,k}).$$

This is true on a subspace of S_1 of codimension at most

$$H(C_k) - H(S_k) + \Delta^*_{k-1,k}.$$

To justify (C2), by Claim 3.2.2 we know $R^{k-1}_{k-1,k}$ is injective on $\overline{S}^*_{k-1,k}$. Then by Lemma 1.4.3, we know $A^{k-1}t \in R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k})$ on a subspace of S_1 of codimension at most

$$H(C_{k-1}) - H(S_k) + codim_{S_k}(\overline{S}^*_{k-1,k}) \quad \leq \quad H(C_{k-1}) - H(S_k) + \Delta^*_{k-1,k}.$$

To justify (C2a), by Claim 3.2.2 we know

$$R^i_{i,k-1}(R^{k-1}_{i,k-1})^{-1} \text{ is injective on } R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k}) \cap R^{k-1}_{i,k-1}(\overline{S}^*_{i,k-1}) \subseteq R^{k-1}_{i,k-1}(\overline{S}^*_{i,k-1}).$$

Then by Lemma 1.4.3, we know

$$A^i t \in R^i_{i,k-1}(R^{k-1}_{i,k-1})^{-1}\left[R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k}) \cap R^{k-1}_{i,k-1}(\overline{S}^*_{i,k-1})\right]$$

on a subspace of S_1 of codimension at most

$$H(C_i) - H(C_{k-1}) + codim_{C_{k-1}}\left(R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k}) \cap R^{k-1}_{i,k-1}(\overline{S}^*_{i,k-1})\right).$$

We will use Lemma 1.4.1 to justify (3.112). We will use Claim 3.2.2 to justify (3.113). So we know

$$\begin{aligned}
&codim_{C_{k-1}}\left(R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k})\cap R^{k-1}_{i,k-1}(\overline{S}^*_{i,k-1})\right)\\
&\quad\le codim_{C_{k-1}}\left(R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k})\right)+codim_{C_{k-1}}\left(R^{k-1}_{i,k-1}(\overline{S}^*_{i,k-1})\right) &(3.112)\\
&\quad= H(C_{k-1})-dim\left(R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k})\right)+H(C_{k-1})-dim\left(R^{k-1}_{i,k-1}(\overline{S}^*_{i,k-1})\right)\\
&\quad= 2H(C_{k-1})-dim(\overline{S}^*_{k-1,k})-dim(\overline{S}^*_{i,k-1}) &(3.113)\\
&\quad= 2H(C_{k-1})-H(S_k)+codim_{S_k}(\overline{S}^*_{k-1,k})-H(S_{k-1})+codim_{S_{k-1}}(\overline{S}^*_{i,k-1})\\
&\quad\le 2H(C_{k-1})-H(S_k)-H(S_{k-1})+\Delta^*_{k-1,k}+\Delta^*_{i,k-1}.
\end{aligned}$$

So we have,

$$A^it\in R^i_{i,k-1}(R^{k-1}_{i,k-1})^{-1}\left[R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k})\cap R^{k-1}_{i,k-1}(\overline{S}^*_{i,k-1})\right]$$

on a subspace of S_1 of codimension at most

$$H(C_i)-H(S_k)+H(C_{k-1})-H(S_{k-1})+\Delta^*_{k-1,k}+\Delta^*_{i,k-1}.$$

To justify (C2b), by Claim 3.2.2, we know

$$R^i_{k-1,i}(R^{k-1}_{k-1,i})^{-1}\text{ is injective on }R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k})\cap R^{k-1}_{k-1,i}(\overline{S}^*_{k-1,i})\subseteq R^{k-1}_{k-1,i}(\overline{S}^*_{k-1,i}).$$

Then by Lemma 1.4.3, we know

$$A^it\in R^i_{k-1,i}(R^{k-1}_{k-1,i})^{-1}\left[R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k})\cap R^{k-1}_{k-1,i}(\overline{S}^*_{k-1,i})\right]$$

on a subspace of S_1 of codimension at most

$$H(C_i)-H(C_{k-1})+codim_{C_{k-1}}\left(R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k})\cap R^{k-1}_{k-1,i}(\overline{S}^*_{k-1,i})\right).$$

We will use Lemma 1.4.1 to justify (3.114). We will use Claim 3.2.2 to justify (3.115). So we know

$$\begin{aligned}
&codim_{C_{k-1}}\left(R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k})\cap R^{k-1}_{k-1,i}(\overline{S}^*_{k-1,i})\right)\\
&\quad\le codim_{C_{k-1}}\left(R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k})\right)+codim_{C_i}\left(R^{k-1}_{k-1,i}(\overline{S}^*_{k-1,i})\right) &(3.114)\\
&\quad= H(C_{k-1})-dim\left(R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k})\right)+H(C_{k-1})-dim\left(R^{k-1}_{k-1,i}(\overline{S}^*_{k-1,i})\right)\\
&\quad= 2H(C_{k-1})-dim(\overline{S}^*_{k-1,k})-dim(\overline{S}^*_{k-1,i}) &(3.115)\\
&\quad= 2H(C_{k-1})-H(S_k)+codim_{S_k}(\overline{S}^*_{k-1,k})-H(S_i)+codim_{S_i}(\overline{S}^*_{k-1,i})\\
&\quad\le 2H(C_{k-1})-H(S_k)-H(S_i)+\Delta^*_{k-1,k}+\Delta^*_{k-1,i}
\end{aligned}$$

(3.116)

So we have,

$$A^i t \in R^i_{k-1,i}(R^{k-1}_{k-1,i})^{-1}\left[R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k}) \cap R^{k-1}_{k-1,i}(\overline{S}^*_{k-1,i})\right]$$

on a subspace of S_1 of codimension at most

$$H(C_i) - H(S_k) + H(C_{k-1}) - H(S_i) + \Delta^*_{k-1,k} + \Delta^*_{k-1,i}.$$

To justify (C3), by Claim 3.2.2 we know $R^{k-1}_{k-1,k+1}$ is injective on $\overline{S}^*_{k-1,k+1}$. Then by Lemma 1.4.3, we know $A^{k-1}t \in R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1})$ on a subspace of S_1 of codimension at most

$$H(C_{k-1}) - H(S_{k+1}) + codim_{S_{k+1}}(\overline{S}^*_{k-1,k+1}) \quad \leq \quad H(C_{k-1}) - H(S_{k+1}) + \Delta^*_{k-1,k+1}.$$

To justify (C3a), by Claim 3.2.2, we know

$$R^i_{i,k-1}(R^{k-1}_{i,k-1})^{-1} \text{ is injective on } R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1}) \cap R^{k-1}_{i,k-1}(\overline{S}^*_{i,k-1}) \subseteq R^{k-1}_{i,k-1}(\overline{S}^*_{i,k-1}).$$

Then by Lemma 1.4.3, we know

$$A^i t \in R^i_{i,k-1}(R^{k-1}_{i,k-1})^{-1}\left[R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1}) \cap R^{k-1}_{i,k-1}(\overline{S}^*_{i,k-1})\right]$$

on a subspace of S_1 of codimension at most

$$H(C_i) - H(C_{k-1}) + codim_{C_{k-1}}\left(R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1}) \cap R^{k-1}_{i,k-1}(\overline{S}^*_{i,k-1})\right).$$

We will use Lemma 1.4.1 to justify (3.117). We will use Claim 3.2.2 to justify (3.118). So we know

$$\begin{aligned}
&codim_{C_{k-1}}\left(R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1}) \cap R^{k-1}_{i,k-1}(\overline{S}^*_{i,k-1})\right) \\
&\quad \leq codim_{C_{k-1}}\left(R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1})\right) + codim_{C_{k-1}}\left(R^{k-1}_{i,k-1}(\overline{S}^*_{i,k-1})\right) \qquad (3.117)\\
&\quad = H(C_{k-1}) - dim\left(R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1})\right) + H(C_{k-1}) - dim\left(R^{k-1}_{i,k-1}(\overline{S}^*_{i,k-1})\right) \\
&\quad = 2H(C_{k-1}) - dim(\overline{S}^*_{k-1,k+1}) - dim(\overline{S}^*_{i,k-1}) \qquad (3.118)\\
&\quad = 2H(C_{k-1}) - dim(\overline{S}^*_{k-1,k+1}) - dim(\overline{S}^*_{i,k-1}) \\
&\quad = 2H(C_{k-1}) - H(S_{k+1}) + codim_{S_{k+1}}(\overline{S}^*_{k-1,k+1}) - H(S_{k-1}) + codim_{S_{k-1}}(\overline{S}^*_{i,k-1}) \\
&\quad \leq 2H(C_{k-1}) - H(S_{k+1}) - H(S_{k-1}) + \Delta^*_{k-1,k+1} + \Delta^*_{i,k-1}.
\end{aligned}$$

So we have,

$$A^i t \in R^i_{i,k-1}(R^{k-1}_{i,k-1})^{-1}\left[R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1}) \cap R^{k-1}_{i,k-1}(\overline{S}^*_{i,k-1})\right]$$

on a subspace of S_1 of codimension at most

$$H(C_i) - H(S_{k+1}) + H(C_{k-1}) - H(S_{k-1}) + \Delta^*_{k-1,k+1} + \Delta^*_{i,k-1}.$$

To justify (C3b), by Claim 3.2.2, we know

$$R^i_{k-1,i}(R^{k-1}_{k-1,i})^{-1} \text{ is injective on } R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1}) \cap R^{k-1}_{k-1,i}(\overline{S}^*_{k-1,i}) \subseteq R^{k-1}_{k-1,i}(\overline{S}^*_{k-1,i}).$$

Then by Lemma 1.4.3, we know

$$A^i t \in R^i_{k-1,i}(R^{k-1}_{k-1,i})^{-1}\left[R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1}) \cap R^{k-1}_{k-1,i}(\overline{S}^*_{k-1,i})\right]$$

on a subspace of S_1 of codimension at most

$$H(C_i) - H(C_{k-1}) + codim_{C_{k-1}}\left(R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1}) \cap R^{k-1}_{k-1,i}(\overline{S}^*_{k-1,i})\right).$$

We will use Lemma 1.4.1 to justify (3.119). We will use Claim 3.2.2 to justify (3.120). So we know

$$\begin{aligned}
&codim_{C_{k-1}}\left(R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1}) \cap R^{k-1}_{k-1,i}(\overline{S}^*_{k-1,i})\right) \\
&\quad \leq codim_{C_{k-1}}\left(R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1})\right) + codim_{C_{k-1}}\left(R^{k-1}_{k-1,i}(\overline{S}^*_{k-1,i})\right) \qquad (3.119)\\
&\quad = H(C_{k-1}) - dim\left(R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1})\right) + H(C_{k-1}) - dim\left(R^{k-1}_{k-1,i}(\overline{S}^*_{k-1,i})\right)\\
&\quad = 2H(C_{k-1}) - dim(\overline{S}^*_{k-1,k+1}) - dim(\overline{S}^*_{k-1,i}) \qquad (3.120)\\
&\quad = 2H(C_{k-1}) - H(S_{k+1}) + codim_{S_{k+1}}(\overline{S}^*_{k-1,k+1}) - H(S_i) + codim_{S_i}(\overline{S}^*_{k-1,i})\\
&\quad \leq 2H(C_{k-1}) - H(S_{k+1}) - H(S_i) + \Delta^*_{k-1,k+1} + \Delta^*_{k-1,i}.
\end{aligned}$$

So we have,

$$A^i t \in R^i_{k-1,i}(R^{k-1}_{k-1,i})^{-1}\left[R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1}) \cap R^{k-1}_{k-1,i}(\overline{S}^*_{k-1,i})\right]$$

on a subspace of S_1 of codimension at most

$$H(C_i) - H(S_{k+1}) + H(C_{k-1}) - H(S_i) + \Delta^*_{k-1,k+1} + \Delta^*_{k-1,i}.$$

To justify (C4), by (3.2.3), we know $R^{n+1}_{1,n+1}$ is injective on $\widehat{S}_{1,n+1} \subseteq \overline{S}_{1,n+1}$. Then by Lemma 1.4.3, we know $A^{n+1}t \in R^{n+1}_{1,n+1}(\widehat{S}_{1,n+1})$ on a subspace of S_1 of codimension at most

$$H(C_{n+1}) - H(S_1) + codim_{S_1}(\widehat{S}_{1,n+1}) \quad \leq \quad H(C_{n+1}) - H(S_1) + \widehat{\Delta}_{1,n+1}.$$

Similarly by Claim 3.2.2 and Lemma 1.4.3,

$$A^{n+1}t \in R^{n+1}_{2,n+1}(\widehat{S}_{2,n+1})$$

on a subspace S_1 of codimension at most $H(C_{n+1}) - H(S_{n+1}) + \widehat{\Delta}_{2,n+1}$. Then by Lemma 1.4.1, we know

$$A^{n+1}t \in R^{n+1}_{1,n+1}(\widehat{S}_{1,n+1}) \cap R^{n+1}_{2,n+1}(\widehat{S}_{2,n+1})$$

on a subspace S_1 of codimension at most $2H(C_{n+1}) - H(S_1) - H(S_{n+1}) + \widehat{\Delta}_{1,n+1} + \widehat{\Delta}_{2,n+1}$.

To justify (C5), by Claim 3.2.2, we know

$$R^i_{i,n+1}(R^{n+1}_{i,n+1})^{-1} \text{ is injective on } R^{n+1}_{1,n+1}(\widehat{S}_{1,n+1}) \cap R^{n+1}_{i,n+1}(\overline{S}^*_{i,n+1}) \subseteq R^{n+1}_{i,n+1}(\overline{S}^*_{i,n+1}).$$

By Lemma 1.4.3, we know

$$A^i t \in R^i_{i,n+1}(R^{n+1}_{i,n+1})^{-1}\left[R^{n+1}_{1,n+1}(\widehat{S}_{1,n+1}) \cap R^{n+1}_{i,n+1}(\overline{S}^*_{i,n+1})\right]$$

on a subspace of S_1 of codimension at most

$$H(C_i) - H(C_{n+1}) + codim_{C_{n+1}}\left(R^{n+1}_{1,n+1}(\widehat{S}_{1,n+1}) \cap R^{n+1}_{i,n+1}(\overline{S}^*_{i,n+1})\right).$$

We will use Lemma 1.4.1 to justify (3.121). We will use Claim 3.2.3 and Claim 3.2.2 to justify (3.122). So we know

$$\begin{aligned}
&codim_{C_{n+1}}\left(R^{n+1}_{1,n+1}(\widehat{S}_{1,n+1}) \cap R^{n+1}_{i,n+1}(\overline{S}^*_{i,n+1})\right) \\
&\quad \leq codim_{C_{n+1}}\left(R^{n+1}_{1,n+1}(\widehat{S}_{1,n+1})\right) + codim_{C_{n+1}}\left(R^{n+1}_{i,n+1}(\overline{S}^*_{i,n+1})\right) \qquad (3.121)\\
&\quad = H(C_{n+1}) - dim\left(R^{n+1}_{1,n+1}(\widehat{S}_{1,n+1})\right) + H(C_{n+1}) - dim\left(R^{n+1}_{i,n+1}(\overline{S}^*_{i,n+1})\right) \\
&\quad = 2H(C_{n+1}) - dim(\widehat{S}_{1,n+1}) - dim(\overline{S}^*_{i,n+1}) \qquad (3.122)\\
&\quad = 2H(C_{n+1}) - H(S_1) + codim_{S_1}(\widehat{S}_{1,n+1}) - H(S_{n+1}) + codim_{S_{n+1}}(\overline{S}^*_{i,n+1}) \\
&\quad \leq 2H(C_{n+1}) - H(S_1) - H(S_{n+1}) + \widehat{\Delta}_{1,n+1} + \Delta^*_{i,n+1}.
\end{aligned}$$

So we have,

$$A^i t \in R^i_{i,n+1}(R^{n+1}_{i,n+1})^{-1}\left[R^{n+1}_{1,n+1}(\widehat{S}_{1,n+1}) \cap R^{n+1}_{i,n+1}(\overline{S}^*_{i,n+1})\right]$$

on a subspace of S_1 of codimension at most

$$H(C_i) - H(S_1) + H(C_{n+1}) - H(S_{n+1}) + \widehat{\Delta}_{1,n+1} + \Delta^*_{i,n+1}.$$

To justify (C5a), by Claim 3.2.2, we know

$$R^1_{1,n+1} \text{ is injective on } \widehat{S}_{1,n+1} \cap \overline{S}^*_{1,n+1} \subseteq \overline{S}^*_{1,n+1}.$$

By Lemma 1.4.3, we know

$$A^1 t \in R^1_{1,n+1}\left(\widehat{S}_{1,n+1} \cap \overline{S}^*_{1,n+1}\right)$$

on a subspace of S_1 of codimension at most

$$H(C_1) - H(S_1) + codim_{S_1}\left(\widehat{S}_{1,n+1} \cap \overline{S}^*_{i,n+1}\right),$$

which by Lemma 1.4.1 is at most

$$H(C_1) - H(S_1) + \widehat{\Delta}_{1,n+1} + \Delta^*_{i,n+1}.$$

To justify (C6), by Claim 3.2.2, we know

$$R^i_{i,n+1}(R^{n+1}_{i,n+1})^{-1} \text{ is injective on } R^{n+1}_{2,n+1}(\widehat{S}_{2,n+1}) \cap R^{n+1}_{i,n+1}(\overline{S}^*_{i,n+1}) \subseteq R^{n+1}_{i,n+1}(\overline{S}^*_{i,n+1}).$$

By Lemma 1.4.3, we know

$$A^i t \in R^i_{i,n+1}(R^{n+1}_{i,n+1})^{-1}\left[R^{n+1}_{2,n+1}(\widehat{S}_{2,n+1}) \cap R^{n+1}_{i,n+1}(\overline{S}^*_{i,n+1})\right]$$

on a subspace of S_1 of codimension at most

$$H(C_i) - H(C_{n+1}) + codim_{C_{n+1}}\left(R^{n+1}_{2,n+1}(\widehat{S}_{2,n+1}) \cap R^{n+1}_{i,n+1}(\overline{S}^*_{i,n+1})\right).$$

We will use Lemma 1.4.1 to justify (3.123). We will use Claim 3.2.3 and Claim 3.2.2 to justify (3.124). So we know

$$\begin{aligned}
&codim_{C_{n+1}}\left(R^{n+1}_{2,n+1}(\widehat{S}_{2,n+1}) \cap R^{n+1}_{i,n+1}(\overline{S}^*_{i,n+1})\right) \\
&\quad\leq codim_{C_{n+1}}\left(R^{n+1}_{2,n+1}(\widehat{S}_{2,n+1})\right) + codim_{C_{n+1}}\left(R^{n+1}_{i,n+1}(\overline{S}^*_{i,n+1})\right) \qquad (3.123)\\
&\quad= H(C_{n+1}) - dim\left(R^{n+1}_{2,n+1}(\widehat{S}_{2,n+1})\right) + H(C_{n+1}) - dim\left(R^{n+1}_{i,n+1}(\overline{S}^*_{i,n+1})\right) \\
&\quad= 2H(C_{n+1}) - dim(\widehat{S}_{2,n+1}) - dim(\overline{S}^*_{i,n+1}) \qquad (3.124)\\
&\quad= 2H(C_{n+1}) - H(S_{n+1}) + codim_{S_{n+1}}(\widehat{S}_{2,n+1}) - H(S_{n+1}) + codim_{S_{n+1}}(\overline{S}^*_{i,n+1}) \\
&\quad\leq 2H(C_{n+1}) - 2H(S_{n+1}) + \widehat{\Delta}_{2,n+1} + \Delta^*_{i,n+1}.
\end{aligned}$$

So we have,

$$A^i t \in R^i_{i,n+1}(R^{n+1}_{i,n+1})^{-1}\left[R^{n+1}_{2,n+1}(\widehat{S}_{2,n+1}) \cap R^{n+1}_{i,n+1}(\overline{S}^*_{i,n+1})\right]$$

on a subspace of S_1 of codimension at most

$$H(C_i) + H(C_{n+1}) - 2H(S_{n+1}) + \widehat{\Delta}_{2,n+1} + \Delta^*_{i,n+1}.$$

To justify (C6a), by Claim 3.2.2, we know

$$R^1_{1,n+1}(R^{n+1}_{1,n+1})^{-1} \text{ is injective on } R^{n+1}_{2,n+1}(\widehat{S}_{2,n+1}) \cap R^{n+1}_{1,n+1}(\overline{S}^*_{1,n+1}) \subseteq R^{n+1}_{1,n+1}(\overline{S}^*_{1,n+1}).$$

By Lemma 1.4.3, we know

$$A^1 t \in R^1_{1,n+1}(R^{n+1}_{1,n+1})^{-1}\left[R^{n+1}_{2,n+1}(\widehat{S}_{2,n+1}) \cap R^{n+1}_{1,n+1}(\overline{S}^*_{1,n+1})\right]$$

on a subspace of S_1 of codimension at most

$$H(C_1) - H(C_{n+1}) + codim_{C_{n+1}}\left(R^{n+1}_{2,n+1}(\widehat{S}_{2,n+1}) \cap R^{n+1}_{1,n+1}(\overline{S}^*_{1,n+1})\right).$$

We will use Lemma 1.4.1 to justify (3.125). We will use Claim 3.2.3 and Claim 3.2.2 to justify (3.126). So we know

$$\begin{aligned}
&codim_{C_{n+1}}\left(R^{n+1}_{2,n+1}(\widehat{S}_{2,n+1})\cap R^{n+1}_{1,n+1}(\overline{S}^*_{1,n+1})\right)\\
&\quad\leq codim_{C_{n+1}}\left(R^{n+1}_{2,n+1}(\widehat{S}_{2,n+1})\right)+codim_{C_{n+1}}\left(R^{n+1}_{1,n+1}(\overline{S}^*_{1,n+1})\right) \qquad (3.125)\\
&\quad= H(C_{n+1})-dim\left(R^{n+1}_{2,n+1}(\widehat{S}_{2,n+1})\right)+H(C_{n+1})-dim\left(R^{n+1}_{1,n+1}(\overline{S}^*_{1,n+1})\right)\\
&\quad= 2H(C_{n+1})-dim(\widehat{S}_{2,n+1})-dim(\overline{S}^*_{1,n+1}) \qquad (3.126)\\
&\quad= 2H(C_{n+1})-H(S_{n+1})+codim_{S_{n+1}}(\widehat{S}_{2,n+1})-H(S_1)+codim_{S_1}(\overline{S}^*_{1,n+1})\\
&\quad\leq 2H(C_{n+1})-H(S_{n+1})-H(S_1)+\widehat{\Delta}_{2,n+1}+\Delta^*_{1,n+1}.
\end{aligned}$$

So we have,

$$A^1t\in R^1_{1,n+1}(R^{n+1}_{1,n+1})^{-1}\left[R^{n+1}_{2,n+1}(\widehat{S}_{2,n+1})\cap R^{n+1}_{1,n+1}(\overline{S}^*_{1,n+1})\right]$$

on a subspace of S_1 of codimension at most

$$H(C_1)+H(C_{n+1})-H(S_{n+1})-H(S_1)+\widehat{\Delta}_{2,n+1}+\Delta^*_{1,n+1}.$$

To justify (C7), by Claim 3.2.2, we know

$$R^k_{k-1,k}\text{ is injective on }\overline{S}^*_{k-1,k}.$$

By Lemma 1.4.3, we know

$$A^kt\in R^k_{k-1,k}(\overline{S}^*_{k-1,k})$$

on a subspace of S_1 of codimension at most

$$H(C_k)-H(S_k)+codim_{S_k}(\overline{S}^*_{k-1,k})\leq H(C_k)-H(S_k)+\Delta^*_{k-1,k}.$$

Claim 3.2.4. $f^1_2A^2t=f^1_3A^3t$

Proof. By (3.61) and (C1), we know

$$\begin{aligned}
\sum_{i\in[1:n+1]\backslash 2}f^2_iA^it &= 0,\text{ so}\\
f^2_{n+1}A^{n+1}t &= \sum_{i\in[1:n]\backslash 2}-f^2_iA^it.
\end{aligned}$$

By (3.110), for $i\in[1:n]\backslash 2$, we know

$$f^2_i = -f^2_{n+1}R^{n+1}_{i,n+1}(R^i_{i,n+1})^{-1}\text{ on }R^i_{i,n+1}(\overline{S}^*_{i,n+1}).$$

By (C5) and (C5a), for $i\in[1:n]\backslash 2$, we know $A^it\in R^i_{i,n+1}(\overline{S}^*_{i,n+1})$, so we have

$$f^2_{n+1}A^{n+1}t = \sum_{i\in[1:n]\backslash 2}f^2_{n+1}R^{n+1}_{i,n+1}(R^i_{i,n+1})^{-1}A^it. \qquad (3.127)$$

By Claim 3.2.3, we know

$$f^2_{n+1} \text{ is injective on } R^{n+1}_{1,n+1}(\widehat{S}_{1,n+1}). \tag{3.128}$$

By (C5) and (C5a), for $i \in [1:n]\backslash 2$, we know

$$A^i t \quad \in \quad R^i_{i,n+1}(R^{n+1}_{i,n+1})^{-1}\left[R^{n+1}_{1,n+1}(\widehat{S}_{1,n+1}) \cap R^{n+1}_{i,n+1}(\overline{S}^*_{i,n+1})\right].$$

By Claim 3.2.2, we know $R^i_{i,n+1}$ is injective on $(R^{n+1}_{i,n+1})^{-1}\left[R^{n+1}_{1,n+1}(\widehat{S}_{1,n+1}) \cap R^{n+1}_{i,n+1}(\overline{S}^*_{i,n+1})\right] \subseteq \overline{S}^*_{i,n+1}$. So we know

$$R^{n+1}_{i,n+1}(R^i_{i,n+1})^{-1}A^i t \quad \in \quad R^{n+1}_{1,n+1}(\widehat{S}_{1,n+1}) \cap R^{n+1}_{i,n+1}(\overline{S}^*_{i,n+1}),$$

or more specifically

$$R^{n+1}_{i,n+1}(R^i_{i,n+1})^{-1}A^i t \quad \in \quad R^{n+1}_{1,n+1}(\widehat{S}_{1,n+1}). \tag{3.129}$$

Then applying (C4), (3.128), and (3.129), to (3.127), we have

$$A^{n+1}t \quad = \quad \sum_{i\in[1:n]\backslash 2} R^{n+1}_{i,n+1}(R^i_{i,n+1})^{-1}A^i t. \tag{3.130}$$

By (3.61) and (C1), we know

$$\begin{aligned} \sum_{i\in[1:n+1]\backslash 3} f^3_i A^i \quad &= \quad 0, \text{ so} \\ f^3_{n+1}A^{n+1} \quad &= \quad \sum_{i\in[1:n]\backslash 3} -f^3_i A^i \end{aligned}$$

By (3.110), for $i \in [1:n]\backslash 3$, we know

$$f^3_i \quad = \quad -f^3_{n+1}R^{n+1}_{i,n+1}(R^i_{i,n+1})^{-1} \text{ on } R^i_{i,n+1}(\overline{S}^*_{i,n+1}).$$

By (C6) and (C6a), for $i \in [1:n]\backslash 3$, we know $A^i t \in R^i_{i,n+1}(\overline{S}^*_{i,n+1})$, so we have

$$f^3_{n+1}A^{n+1}t \quad = \quad \sum_{i\in[1:n]\backslash 3} f^3_{n+1}R^{n+1}_{i,n+1}(R^i_{i,n+1})^{-1}A^i t. \tag{3.131}$$

By Claim 3.2.3, we know

$$f^3_{n+1} \text{ is injective on } R^{n+1}_{2,n+1}(\widehat{S}_{2,n+1}). \tag{3.132}$$

By (C6) and (C6a), for $i \in [1:n]\backslash 3$, we know

$$A^i t \quad \in \quad R^i_{i,n+1}(R^{n+1}_{i,n+1})^{-1}\left[R^{n+1}_{2,n+1}(\widehat{S}_{2,n+1}) \cap R^{n+1}_{i,n+1}(\overline{S}^*_{i,n+1})\right]$$

By Claim 3.2.2, we know $R^i_{i,n+1}$ is injective on $(R^{n+1}_{i,n+1})^{-1}\left[R^{n+1}_{2,n+1}(\widehat{S}_{2,n+1}) \cap R^{n+1}_{i,n+1}(\overline{S}^*_{i,n+1})\right] \subseteq \overline{S}^*_{i,n+1}$. So we have

$$R^{n+1}_{i,n+1}(R^i_{i,n+1})^{-1}A^i t \quad \in \quad R^{n+1}_{2,n+1}(\widehat{S}_{2,n+1}) \cap R^{n+1}_{i,n+1}(\overline{S}^*_{i,n+1}),$$

or more specifically

$$R^{n+1}_{i,n+1}(R^i_{i,n+1})^{-1}A^i t \quad \in \quad R^{n+1}_{2,n+1}(\widehat{S}_{2,n+1}). \tag{3.133}$$

Then applying (C4), (3.132), and (3.133) to (3.131), we have

$$A^{n+1}t \quad = \quad \sum_{i\in[1:n]\setminus 3} R^{n+1}_{i,n+1}(R^i_{i,n+1})^{-1}A^i t. \tag{3.134}$$

Now setting (3.130) and (3.134) equal to each other, we can derive

$$R^{n+1}_{2,n+1}(R^2_{2,n+1})^{-1}A^2 t \quad = \quad R^{n+1}_{3,n+1}(R^3_{3,n+1})^{-1}A^3 t \tag{3.135}$$

By (3.110), we know

$$f^1_2 \quad = \quad -f^1_{n+1}R^{n+1}_{2,n+1}(R^2_{2,n+1})^{-1} \text{ on } R^2_{2,n+1}(\overline{S}^*_{2,n+1}).$$

By (C6), we know $A^2 t \in R^2_{2,n+1}(\overline{S}^*_{2,n+1})$, so we have

$$f^1_2 A^2 t \quad = \quad -f^1_{n+1}R^{n+1}_{2,n+1}(R^2_{2,n+1})^{-1}A^2 t.$$

Then using (3.135), we have

$$f^1_2 A^2 t \quad = \quad -f^1_{n+1}R^{n+1}_{3,n+1}(R^3_{3,n+1})^{-1}A^3 t. \tag{3.136}$$

By (3.110), we know

$$f^1_3 \quad = \quad -f^1_{n+1}R^{n+1}_{3,n+1}(R^3_{3,n+1})^{-1} \text{ on } R^3_{3,n+1}(\overline{S}^*_{3,n+1}).$$

By (C5), we know $A^3 t \in R^3_{3,n+1}(\overline{S}^*_{3,n+1})$. Now (3.136) becomes

$$f^1_2 A^2 t \quad = \quad f^1_3 A^3 t.$$

□

Claim 3.2.5. For $k \in [3:n]$, $f^1_k A^k t = f^1_{k+1}A^{k+1}t$

Proof. By (3.61) and (C1), we know

$$\begin{aligned}
\sum_{i\in[1:n+1]\setminus k} f^k_i A^i t \quad &= \quad 0 \\
f^k_{k-1}A^{k-1}t \quad &= \quad \sum_{i\in[1:n+1]\setminus\{k-1,k\}} -f^k_i A^i t \\
&= \quad \sum_{i\in[1:k-2]} -f^k_i A^i t + \sum_{i\in[k+1:n+1]} -f^k_i A^i t.
\end{aligned}$$

By condition (C2a), for $i \in [1:k-2]$, we know $A^i t \in R^i_{i,k-1}(\overline{S}^*_{i,k-1})$. Then by (3.110), we have

$$f^k_{k-1}A^{k-1}t \quad = \quad \sum_{i\in[1:k-2]} f^k_{k-1}R^{k-1}_{i,k-1}(R^i_{i,k-1})^{-1}A^i t + \sum_{i\in[k+1:n+1]} -f^k_i A^i t.$$

By condition (C2b), for $i \in [k+1:n+1]$, we know $A^i t \in R^i_{k-1,i}(\overline{S}^*_{k-1,i})$. Then by (3.111), we have

$$\begin{aligned} f^k_{k-1}A^{k-1}t &= \sum_{i\in[1:k-2]} f^k_{k-1}R^{k-1}_{i,k-1}(R^i_{i,k-1})^{-1}A^i t \\ &\quad + \sum_{i\in[k+1:n+1]} f^k_{k-1}R^{k-1}_{k-1,i}(R^i_{k-1,i})^{-1}A^i t. \end{aligned} \tag{3.137}$$

By (3.62), we know

$$f^k_{k-1} \text{ is injective on } R^{k-1}_{k-1,k}(\overline{S}_{k-1,k}) \supseteq R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k}). \tag{3.138}$$

By (C2a), for $i \in [1:k-2]$, we know

$$A^i t \in R^i_{i,k-1}(R^{k-1}_{i,k-1})^{-1}\left[R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k}) \cap R^{k-1}_{i,k-1}(\overline{S}^*_{i,k-1})\right].$$

By Claim 3.2.2, we know $R^i_{i,k-1}$ is injective on $(R^{k-1}_{i,k-1})^{-1}\left[R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k}) \cap R^{k-1}_{i,k-1}(\overline{S}^*_{i,k-1})\right] \subseteq \overline{S}^*_{i,k-1}$. So we have

$$R^{k-1}_{i,k-1}(R^i_{i,k-1})^{-1}A^i t \in R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k}) \cap R^{k-1}_{i,k-1}(\overline{S}^*_{i,k-1}),$$

or more specifically,

$$R^{k-1}_{i,k-1}(R^i_{i,k-1})^{-1}A^i t \in R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k}). \tag{3.139}$$

By (C2b), for $i \in [k+1:n+1]$, we know

$$A^i t \in R^i_{k-1,i}(R^{k-1}_{k-1,i})^{-1}\left[R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k}) \cap R^{k-1}_{k-1,i}(\overline{S}^*_{k-1,i})\right].$$

By Claim 3.2.2, we know $R^i_{k-1,i}$ is injective on $(R^{k-1}_{k-1,i})^{-1}\left[R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k}) \cap R^{k-1}_{k-1,i}(\overline{S}^*_{k-1,i})\right] \subseteq \overline{S}^*_{k-1,i}$. So we have

$$R^{k-1}_{k-1,i}(R^i_{k-1,i})^{-1}A^i t \in R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k}) \cap R^{k-1}_{k-1,i}(\overline{S}^*_{k-1,i}).$$

or more specifically,

$$R^{k-1}_{k-1,i}(R^i_{k-1,i})^{-1}A^i t \in R^{k-1}_{k-1,k}(\overline{S}^*_{k-1,k}). \tag{3.140}$$

Then applying (C2), (3.138), (3.139), and (3.140) to (3.137), we have

$$A^{k-1}t = \sum_{i\in[1:k-2]} R^{k-1}_{i,k-1}(R^i_{i,k-1})^{-1}A^i t + \sum_{i\in[k+1:n+1]} R^{k-1}_{k-1,i}(R^i_{k-1,i})^{-1}A^i t. \tag{3.141}$$

By (3.61) and (C1), we know

$$\begin{aligned} \sum_{i\in[1:n+1]\setminus k+1} f^{k+1}_i A^i t &= 0 \\ f^{k+1}_{k-1}A^{k-1}t &= \sum_{i\in[1:n+1]\setminus\{k-1,k+1\}} -f^{k+1}_i A^i t \\ f^{k+1}_{k-1}A^{k-1}t &= \sum_{i\in[1:k-2]} -f^{k+1}_i A^i t + \sum_{i\in[k:n+1]\setminus k+1} -f^{k+1}_i A^i t. \end{aligned}$$

By condition (C3a), for $i \in [1:k-2]$, we know $A^i t \in R^i_{i,k-1}(\overline{S}^*_{i,k-1})$. Then by (3.110), we have

$$\begin{aligned} f^{k+1}_{k-1}A^{k-1}t &= \sum_{i\in[1:k-2]} f^{k+1}_{k-1}R^{k-1}_{i,k-1}(R^i_{i,k-1})^{-1}A^i t \\ &\quad + \sum_{i\in[k:n+1]\backslash k+1} -f^{k+1}_i A^i t. \end{aligned}$$

By condition (C3b), for $i \in [k:n+1]\backslash k+1$, we know $A^i t \in R^i_{k-1,i}(\overline{S}^*_{k-1,i})$. Then by (3.111), we have

$$\begin{aligned} f^{k+1}_{k-1}A^{k-1}t &= \sum_{i\in[1:k-2]} f^{k+1}_{k-1}R^{k-1}_{i,k-1}(R^i_{i,k-1})^{-1}A^i t \\ &\quad + \sum_{i\in[k:n+1]\backslash k+1} f^{k+1}_{k-1}R^{k-1}_{k-1,i}(R^i_{k-1,i})^{-1}A^i t \end{aligned} \tag{3.142}$$

By (3.62), we know

$$f^{k+1}_{k-1} \text{ is injective on } R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1}). \tag{3.143}$$

By (C3a), for $i \in [1:k-2]$, we know

$$A^i t \in R^i_{i,k-1}(R^{k-1}_{i,k-1})^{-1}\left[R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1}) \cap R^{k-1}_{i,k-1}(\overline{S}^*_{i,k-1})\right].$$

By Claim 3.2.2, we know $R^i_{i,k-1}$ is injective on $(R^{k-1}_{i,k-1})^{-1}\left[R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1}) \cap R^{k-1}_{i,k-1}(\overline{S}^*_{i,k-1})\right] \subseteq \overline{S}^*_{i,k-1}$. So we have

$$R^{k-1}_{i,k-1}(R^i_{i,k-1})^{-1}A^i t \in R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1}) \cap R^{k-1}_{i,k-1}(\overline{S}^*_{i,k-1}),$$

or more specifically,

$$R^{k-1}_{i,k-1}(R^i_{i,k-1})^{-1}A^i t \in R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1}). \tag{3.144}$$

By (C3b), for $i \in [k:n+1]\backslash k+1$, we know

$$A^i t \in R^i_{k-1,i}(R^{k-1}_{k-1,i})^{-1}\left[R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1}) \cap R^{k-1}_{k-1,i}(\overline{S}^*_{k-1,i})\right].$$

By Claim 3.2.2, we know $R^i_{k-1,i}$ is injective on

$$(R^{k-1}_{k-1,i})^{-1}\left[R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1}) \cap R^{k-1}_{k-1,i}(\overline{S}^*_{k-1,i})\right].$$

So we have

$$R^{k-1}_{k-1,i}(R^i_{k-1,i})^{-1}A^i t \in R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1}) \cap R^{k-1}_{k-1,i}(\overline{S}^*_{k-1,i}),$$

or more specifically,

$$R^{k-1}_{k-1,i}(R^i_{k-1,i})^{-1}A^i t \in R^{k-1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1}). \tag{3.145}$$

Then applying (C3), (3.143), (3.144), and (3.145) to (3.142), we have

$$A^{k-1}t = \sum_{i\in[1:k-2]} R^{k-1}_{i,k-1}(R^i_{i,k-1})^{-1}A^i t + \sum_{i\in[k:n+1]\setminus k+1} R^{k-1}_{k-1,i}(R^i_{k-1,i})^{-1}A^i t. \quad (3.146)$$

Now setting (3.141) and (3.146) equal to each other, we can derive

$$R^{k-1}_{k-1,k}(R^k_{k-1,k})^{-1}A^k t = R^{k-1}_{k-1,k+1}(R^{k+1}_{k-1,k+1})^{-1}A^{k+1}t \quad (3.147)$$

By (3.111), on $R^k_{k-1,k}(\overline{S}^*_{k-1,k})$ we have

$$f^1_k = -f^1_{k-1}R^{k-1}_{k-1,k}(R^k_{k-1,k})^{-1}. \quad (3.148)$$

Similarly, on $R^{k+1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1})$

$$f^1_{k+1} = -f^1_{k-1}R^{k-1}_{k-1,k+1}(R^{k+1}_{k-1,k+1})^{-1}. \quad (3.149)$$

By (C7), we know that for $k \in [4 : n+1]$, $A^k t \in R^k_{k-1,k}(\overline{S}^*_{k-1,k})$. By (C3b), we know that $A^3 t \in R^3_{2,3}(\overline{S}^*_{2,3})$. So for $k \in [3 : n+1]$, we know $A^k t \in R^k_{k-1,k}(\overline{S}^*_{k-1,k})$. Now using (3.148), we have

$$f^1_k A^k t = -f^1_{k-1}R^{k-1}_{k-1,k}(R^k_{k-1,k})^{-1}A^k t.$$

Now using (3.147), we have

$$f^1_k A^k t = -f^1_{k-1}R^{k-1}_{k-1,k+1}(R^{k+1}_{k-1,k+1})^{-1}A^{k+1}t.$$

By (C2b), we know that $A^{k+1}t \in R^{k+1}_{k-1,k+1}(\overline{S}^*_{k-1,k+1})$. Then using (3.149), we have

$$f^1_k A^k t = f^1_{k+1}A^{k+1}t$$

□

Using Claim 3.2.4 and Claim 3.2.5, for $k \in [2 : n]$, we have $f^1_k A^k t = f^1_{k+1}A^{k+1}t$. By (3.60) and (C1), we have

$$\begin{aligned} \sum_{i=2}^{n+1} f^1_i A^i t &= t \\ n f^1_2 A^2 t &= t \end{aligned}$$

If the field characteristic is not in P, then the characteristic will divide n. So if the field characteristic is not in P, then no nonzero t can satisfy (C1)-(C7). Therefore, the sum of the

codimensions in (C1)-(C7) must be at least $H(S_1)$. So we have

$$\begin{aligned}
H(S_1) \leq\ & \Delta_A + \sum_{k=3}^{n} H(C_{k-1}) - H(S_k) + \Delta^*_{k-1,k} \\
& + \sum_{k=3}^{n} \sum_{i=1}^{k-2} H(C_i) - H(S_k) + H(C_{k-1}) - H(S_{k-1}) + \Delta^*_{k-1,k} + \Delta^*_{i,k-1} \\
& + \sum_{k=3}^{n} \sum_{i=k+1}^{n+1} H(C_i) - H(S_k) + H(C_{k-1}) - H(S_i) + \Delta^*_{k-1,k} + \Delta^*_{k-1,i} \\
& + \sum_{k=3}^{n} H(C_{k-1}) - H(S_{k+1}) + \Delta^*_{k-1,k+1} \\
& + \sum_{k=3}^{n} \sum_{i=1}^{k-2} H(C_i) - H(S_{k+1}) + H(C_{k-1}) - H(S_{k-1}) + \Delta^*_{k-1,k+1} + \Delta^*_{i,k-1} \\
& + \sum_{k=3}^{n} H(C_k) - H(S_{k+1}) + H(C_{k-1}) - H(S_k) + \Delta^*_{k-1,k+1} + \Delta^*_{k-1,k} \\
& + \sum_{k=3}^{n} \sum_{i=k+2}^{n+1} H(C_i) - H(S_{k+1}) + H(C_{k-1}) - H(S_i) + \Delta^*_{k-1,k+1} + \Delta^*_{k-1,i} \\
& + 2H(C_{n+1}) - H(S_1) - H(S_{n+1}) + \widehat{\Delta}_{1,n+1} + \widehat{\Delta}_{2,n+1} \\
& + \sum_{k=3}^{n} H(C_k) - H(S_1) + H(C_{n+1}) - H(S_{n+1}) + \widehat{\Delta}_{1,n+1} + \Delta^*_{k,n+1} \\
& + H(C_1) - H(S_1) + \widehat{\Delta}_{1,n+1} + \Delta^*_{1,n+1} \\
& + H(C_2) + H(C_{n+1}) - 2H(S_{n+1}) + \widehat{\Delta}_{2,n+1} + \Delta^*_{2,n+1} \\
& + \sum_{k=4}^{n} H(C_k) + H(C_{n+1}) - 2H(S_{n+1}) + \widehat{\Delta}_{2,n+1} + \Delta^*_{k,n+1} \\
& + H(C_1) + H(C_{n+1}) - H(S_{n+1}) - H(S_1) + \widehat{\Delta}_{2,n+1} + \Delta^*_{1,n+1} \\
& + \sum_{k=4}^{n+1} H(C_k) - H(S_k) + \Delta^*_{k-1,k}
\end{aligned} \tag{3.150}$$

Notice that the inequality does not hold for fields of a characteristic that do not divide n (characteristics in P). Let $p \in P$. Then a counterexample would be: In $V = GF(p)^{n+1}$, let

$$\begin{aligned}
S_1 &= \langle (1,0,0,\ldots,0,0) \rangle & C_1 &= \langle (0,1,1,\ldots,1,1) \rangle \\
S_2 &= \langle (0,1,0,\ldots,0,0) \rangle & C_2 &= \langle (1,0,1,\ldots,1,1) \rangle \\
S_3 &= \langle (0,0,1,\ldots,0,0) \rangle & C_3 &= \langle (1,1,0,\ldots,1,1) \rangle \\
&\vdots & &\vdots \\
S_n &= \langle (0,0,0,\ldots,1,0) \rangle & C_n &= \langle (1,1,1,\ldots,0,1) \rangle \\
S_{n+1} &= \langle (0,0,0,\ldots,0,1) \rangle & C_{n+1} &= \langle (1,1,1,\ldots,1,0) \rangle.
\end{aligned}$$

For $i \in [1 : n+1]$, we would have $H(S_i) = H(C_i) = 1$ and for every pair (i, j), $\Delta_{i,j} = \Delta^*_{i,j} = \widehat{\Delta}_{1,n+1} = \widehat{\Delta}_{2,n+1} = 0$. Thus, the inequality would reduce to $1 \leq 0$, which is clearly a contradiction. Therefore, the above inequality is a linear rank inequality for fields of characteristics not in P.

□

3.3 Applications

It was recently shown in [Dougherty 08] that for every finite or co-finite set of primes, P, there exists a network that is scalar linearly solvable over a field, F, if and only if the characteristic of F is in P. Here we generalize this result to linear solvability.

Theorem 3.3.1. *Let P be a co-finite set of primes. There exists a network that is linearly solvable over a field, F, if and only if the characteristic of F is in P.*

Proof. Let P be a co-finite set of primes and let n be the product of all the primes not in P. Then the network $\mathcal{N}$ is linearly solvable over a field F, if and only if the characteristic of F is in P. We can show $\mathcal{N}$ is not solvable over characteristics not in P by applying the linear rank inequality in (3.150) to $\mathcal{N}$. Let $k' = H(S_1)$ and $n' = H(C_1)$. Then for every pair (i, j), $\Delta_{i,j} = 0$. We would also know that $\widehat{\Delta}_{1,n+1}$, $\widehat{\Delta}_{2,n+1}$, and every $\Delta^*_{i,j}$ would be equal to a multiple of $(n' - k')$. So the inequality would reduce to $k' \leq m(n' - k')$, for some $m \in \mathbb{N}$. Thus

$$k'/n' \leq m/(m+1) < 1,$$

so the linear coding capacity of $\mathcal{N}$ is less than 1 over characteristics not in P. The network $\mathcal{N}$ is solvable over characteristics in P by the coding solution:

$$\begin{aligned}
S_1 &= \langle (1,0,0,\ldots,0,0) \rangle & C_1 &= \langle (0,1,1,\ldots,1,1) \rangle \\
S_2 &= \langle (0,1,0,\ldots,0,0) \rangle & C_2 &= \langle (1,0,1,\ldots,1,1) \rangle \\
S_3 &= \langle (0,0,1,\ldots,0,0) \rangle & C_3 &= \langle (1,1,0,\ldots,1,1) \rangle \\
&\vdots & &\vdots \\
S_n &= \langle (0,0,0,\ldots,1,0) \rangle & C_n &= \langle (1,1,1,\ldots,0,1) \rangle \\
S_{n+1} &= \langle (0,0,0,\ldots,0,1) \rangle & C_{n+1} &= \langle (1,1,1,\ldots,1,0) \rangle.
\end{aligned}$$

□

Theorem 3.3.2. *Let P be a finite set of primes. There exists a network that is linearly solvable over a field, F, if and only if the characteristic of F is in P.*

Proof. Let P be a finite set of primes and let n be the product of all the primes in P. Then the network $\mathcal{N}_2$ is linearly solvable over a field F, if and only if the characteristic of F is in P. We can show $\mathcal{N}_2$ is not solvable over characteristics not in P by applying the linear rank inequality in (3.55) to $\mathcal{N}_2$. Let $k' = H(S_1)$ and $n' = H(C_1)$. Then $\Delta_j = \Delta_{j,n+1} = \Delta_A = 0$. We would also know that Δ_A^* and every Δ_j^* would be equal to a multiple of $(n' - k')$. So the inequality would reduce to $k' \leq m(n' - k')$, for some $m \in \mathbb{N}$. Thus

$$k'/n' \leq m/(m+1) < 1,$$

so the linear coding capacity of $\mathcal{N}_2$ is less than 1 over characteristics not in P. The network $\mathcal{N}_2$ is solvable over characteristics in P by the coding solution:

$$\begin{aligned}
S_1 &= \langle (1,0,0,\ldots,0,0) \rangle & C_1 &= \langle (0,1,1,\ldots,1,1) \rangle \\
S_2 &= \langle (0,1,0,\ldots,0,0) \rangle & C_2 &= \langle (1,0,1,\ldots,1,1) \rangle \\
S_3 &= \langle (0,0,1,\ldots,0,0) \rangle & C_3 &= \langle (1,1,0,\ldots,1,1) \rangle \\
&\vdots & &\vdots \\
S_n &= \langle (0,0,0,\ldots,1,0) \rangle & C_n &= \langle (1,1,1,\ldots,0,1) \rangle \\
S_{n+1} &= \langle (0,0,0,\ldots,0,1) \rangle & Z &= \langle (1,1,1,\ldots,1,0) \rangle.
\end{aligned}$$

□

Corollary 3.3.3. *Let P be a finite or co-finite set of primes. There exists a network that is linearly solvable over a field, F, if and only if the characteristic of F is in P.*

We will define the *network coding gain* to be the coding capacity divided by the routing capacity. In [Ngai 04], it has been shown there exist a sequence of networks $\mathcal{N}(n)$, such that the network coding gain $\to \infty$ as $n \to \infty$. Here we show the same result, but with a simpler sequence of networks.

Theorem 3.3.4. *The network coding gain of $\mathcal{N}_2$ is $n - 1$.*

Proof. Consider a (k', n') routing solution to $\mathcal{N}_2$ Then for $j \in [2 : n]$, the receiver T_j demands S_j, which must pass through C_1. So C_1 must contain all k' components of $S_2, S_3, \ldots, S_n$. So $n' \geq (n-1)k'$, or

$$\frac{k'}{n'} \leq \frac{1}{n-1}.$$

A $(1, n-1)$ routing solution exists by:

$$\begin{aligned} C_1 &= [S_2, S_3 \ldots, S_n], \\ C_2 &= [S_1, S_{n+1}, 0, \ldots, 0], \\ C_3 &= [0, 0, \ldots, 0], \\ &\vdots \\ C_n &= [0, 0, \ldots, 0] \\ Z &= [S_1, S_{n+1}, 0, \ldots, 0]. \end{aligned}$$

Thus, the routing capacity is $1/(n-1)$. The network $\mathcal{N}_2$ is solvable over fields whose characteristic divides n. Since there is a unique path from S_2 to T_2, the coding capacity is at most 1. Thus the coding capacity is 1. Then we know that the network coding gain is $n-1$. □

3.4 Open Questions

Let P be a finite or co-finite set of primes. It was shown in [Dougherty 08], that there exists a network that is scalar linearly solvable over a field, F, if and only if the characteristic of F is in P. It was also shown that this result is not possible for any other set of primes that is not finite nor co-finite. So the following is a question that remains to be answered:

1) Is there a set of primes, P, that is not finite nor co-finite such that there exists a network that is linearly solvable over a field, F, if and only if the characteristic of F is in P?

2) Are there other techniques to tighten these inequalities?

3) Given an upper bound on the number of edges or nodes in a network, is there an upper bound on the possible characteristics that the network could be solvable over?

4) Let $n \in \mathbb{N}$. Is there a bound on the number of characteristic dependent linear rank inequalities of n variables ?

This chapter, in full, is a reprint of the material as it appears in: E. Freiling, "Characteristic Dependent Linear Rank Inequalities and Applications to Network Coding," submitted to the *IEEE Transactions on Information Theory*, May 2013. The dissertation author was the primary investigator of this paper.

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