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Vector potentials in gauge theories in flat spacetime

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A recent suggestion that vector potentials in electrodynamics (ED) are nontensorial objects under four-dimensional (4D) frame rotations is found to be both unnecessary and confusing. As traditionally used in ED, a vector potential A always transforms homogeneously under 4D rotations in spacetime, but if the gauge is changed by the rotation, one can restore the gauge back to the original gauge by adding an inhomogeneous term. It is then not a 4-vector, but two 4-vectors: one for rotation and one for translation. For such a gauge, it is much more important to preserve *explicit* homogeneous Lorentz covariance by simply skipping the troublesome gauge-restoration step. A gauge-independent separation of A into a dynamical term and a nondynamical term in Abelian gauge theories is redefined more generally as the terms caused by the presence and absence, respectively, of the 4-current term in the inhomogeneous Maxwell equations for A. Such a separation *cannot* in general be extended to non-Abelian theories where A satisfies nonlinear differential equations. However, in the linearized iterative solution that is perturbation theory, the usual Abelian quantizations in the usual gauges can be used. Some nonlinear complications are briefly reviewed.

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I. INTRODUCTION

Lorcé [1] has recently given, among many interesting results, a nonstandard description of the vector potentials A_L in Lorenz (L) gauges in classical electrodynamics (CED). His motivation is to be able to claim that the CED vector potential $A(x)$ in *any* gauge is Lorentz covariant under *four-dimensional* (4D) frame rotations. He does this by redefining homogeneous Lorentz covariance for $A(x)$, but not for x itself, as *inhomogeneous* Lorentz covariance where an inhomogeneous term for 4D translation can be added to the result of a 4D rotation. He further claims that the Lorenz gauge is "nothing special" because "one cannot conclude that the only possible Lorentz transformation law for the gauge potential is the four-vector one, unless one removes the residual gauge freedom," and that vector potentials are "nontensorial objects" [1].

One objective of this paper is to explain in Sec. II why Lorcé's revisionist view is both unnecessary and confusing compared to the traditional textbook language (Refs. [2–4], for example) that is already simple, clear, and consistent. In this standard language using special relativity in flat space (and not general relativity in curved space), 4-vectors are *defined* to transform homogeneously under 4D frame rotations, just like x itself. For $A(x)$ in gauge theories, the central issue turns out to be whether its chosen gauge in frame x will remain unchanged after a 4D rotation, i.e., a *homogeneous Lorentz* (hL) transformation. We shall show that the hL transformation of $A(x)$ is gauge preserving in *covariant gauges* (cg) and gauge nonpreserving in *noncovariant* (nc) gauges.

If the gauge is changed for a vector potential $A(x)$ in the rotated frame x', its altered gauge can be restored back to the original gauge used in frame x by adding a change-of-gauge term to give a total gauge-restored but inhomogeneous Lorentz transformation. Such an $A(x)$ is described correctly as "not a 4-vector" in textbooks [2,3] because it is the sum of *two* 4-vectors, one for rotation and one for translation. However, one can easily preserve the important requirement of explicit homogeneous Lorentz covariance by simply skipping this troublesome gauge-restoration step. Some implications of this change of perspective are discussed.

The second objective of this paper is to show in Sec. III that the vector potential A in electrodynamics (ED) is nonunique in general by using the separation $A = A_{\mathrm{dyn}} + A_{\mathrm{ndy}}$ known from the theory of linear differential equations. Here, the *dynamical* (dyn) part A_{dyn} and *nondynamical* (ndy) part A_{ndy} are, respectively, the solutions of the Maxwell equations for A with and without the 4-current density j. This separation further highlights the important role played by the *Coulomb* (C) gauge in clarifying a different known structure of A present in both dynamical and nondynamical parts: The Coulomb gauge transversality condition $\boldsymbol{\nabla} \cdot \mathbf{A}_{\mathrm{C}} = 0$ selects a transverse ($\perp$) or physical (phys) part $\mathbf{A}_{\mathrm{C}} = \mathbf{A}_\perp = \mathbf{A}_{\mathrm{phys}}$ that is known to be gauge invariant (Ref. [5], for example). It excludes the longitudinal ($\parallel$) or "pure-gauge" part $\mathbf{A}_\parallel = \mathbf{A}_{\mathrm{pure}}$ due to gauge transformations known to contribute nothing to the field tensor. The sum $A = A_{\mathrm{phys}} + A_{\mathrm{pure}}$ is thus explicitly gauge invariant, as shown by Chen *et al.* [6].

The dynamical/nondynamical treatment is naturally gauge invariant in ED. The vector potential A is nonunique

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because a nondynamical part A_{ndy} can be added to any particular solution to generate other solutions. A_{ndy} also has a transverse/longitudinal (or physical/gauge) decomposition. Its gauge part is the entire gauge part of A and has no physical consequences. Its physical part generates nonzero field tensors $F^{\mu\nu}$ describing transverse electromagnetic waves in free space and longitudinal electric fields from scalar potentials satisfying the Laplace equation for different boundary conditions. However, in applications such as Ref. [6], where one is only interested in one particular solution A_{dyn} for the gauge field bound inside a distinct atomic state by a unique bound-state boundary condition, there is no need to add a further A_{ndy} term of the physical type. The dynamical/nondynamical treatment can accommodate these special cases, too.

The decomposition of A in ED into parts cannot in general be extended to non-Abelian theories where A satisfies nonlinear differential equations. Even the 4-current and field tensor are themselves gauge dependent. However, a linearized iterative treatment using perturbation theory permits Abelian quantizations in the usual gauges as an approximation. Some nonlinear complications of non-Abelian gauge theories are reviewed, especially from the perspective that second quantization is a linearization process that *unavoidably* leaves most of the native nonlinearity untreated until after quantization.

II. HOMOGENEOUS LORENTZ COVARIANCE

Recall that a (single-valued) vector field $A(x)$ in an inertial frame x of a flat and isotropic spacetime is one where its value at point x is a 4-vector, i.e., a 4D "oriented arrow" of definite length (figuratively speaking) present *in spacetime* itself. A 4-vector as used here is any 4-component object of which the components are defined by the same coordinate axes as the frame x and are compatible in physical units so that its "length" can be calculated. Under a hL transformation (i.e., a 4D rotation) of the coordinate frame defined by $x' = \Lambda x$, such a 4D vector field transforms as

$$A'(x' = \Lambda x) = \Lambda A(x), \qquad \text{because}$$
$$A = e'_{\mu}(x')A'^{\mu}(x') = e_{\nu}(x)A^{\nu}(x):$$
$$A'^{\mu} = e'^{\mu} \cdot A = (e'^{\mu} \cdot e_{\nu})A^{\nu}$$
$$= \Lambda^{\mu}_{\nu}A^{\nu}, \qquad (1)$$

where $e_{\nu}(x)$ is a unit vector. This hL transformation gives the new components A'^{μ} in the rotated frame x' of each spacetime arrow $A(x)$ of unchanged length and orientation. The rotation matrix Λ on the rhs for position $x \neq 0$ uses a local x frame (a copy of the x frame with its origin translated to position x) and not the original x frame centered at $x = 0$. In other words, a real 4-vector field is defined to be based on the 4D real-number system that

transforms as $x' = \Lambda x$ under 4D frame rotations. It can be generalized to a complex 4-vector field by adding an overall phase factor at each position x.

Each 4D vector in $A(x)$ defined by Eq. (1) will be called a hL *covariant 4-vector* in this paper, meaning a 4-vector under 4D frame rotations. This is the same object as Lorcé's "Lorentz 4-vector" or the usual 4-vector of textbooks [2–4]. Tensors built from covariant 4-vectors are covariant 4-tensors, while an invariant scalar has the same single value in all hL frames. The language used here and in textbooks is thus the simplest generalization of 3D spatial vector fields in Euclidean space of signature (3,0) to 4D spacetime vector fields in Euclidean space of signature (3,1), where the spacelike specification comes first, irrespective of the overall sign convention used.

In Lorcé's revised language [1,7,8], 4D rotational covariance of $A(x)$ is redefined as an inhomogenous Lorentz transformation, containing terms for both rotation between two frames and translation in a single frame, even though the most important 4-vector, namely, x itself, satisfies only a hL transformation. (This exception is needed because the associated inhomogeneous Lorentz transformation for x itself defines a full-blown Poincaré transformation that is not the subject under discussion.) This inconsistent treatment is the most serious source of confusion in the proposed revision.

A prime example of the standard usage adopted in this paper appears in the definition of the gauge-covariant derivative in QED involving particles of charge e ($e = -|e|$ for electrons):

$$D^{\mu} \equiv \partial^{\mu} + ieA^{\mu}. \qquad (2)$$

In gauge theories of interaction, it is the requirement of local gauge invariance of the Lagrangian under the local gauge transformation (LGT) of the complex Dirac field $\psi(x) \xrightarrow{\text{LGT}} e^{-ie\omega(x)}\psi(x)$ for a charged particle that forces the introduction of the gauge potential $A(x) = \partial\omega(x)$, where $\omega(x)$ is a real scalar field. Thus, A by design is the same covariant 4-vector object as ∂ in flat space. The fact that this A appears in the invariant scalar product $\bar{\psi}\gamma_{\mu}D^{\mu}\psi$ in QED is what gives the interaction Lagrangian $\bar{\psi}\gamma_{\mu}A^{\mu}\psi = j \cdot A$ (where $j = \bar{\psi}\gamma\psi$) both its interaction and its explicit hL invariance. This A^{μ} is thus intended to be a covariant 4-vector. We shall explain why it sometimes turns out to be not a 4-vector, thus ruining the explicit hL invariance of the interaction Lagrangian, and how it can be chosen to be a covariant 4-vector for any choice of gauge in frame x for gauge-invariant theories in flat spacetime.

Recall that in CED alone the electromagnetic field tensor $F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$ in frame x is unchanged by the *gauge transformation* (GT),

$$A^{\mu}(x) \xrightarrow{\text{GT}} A^{\mu}(x) + \partial^{\mu}\Omega(x), \qquad (3)$$

where $\Omega(x)$ is a real scalar field. The vector potential A introduced by Eq. (2) is thus highly nonunique. Part of the resulting redundant gauge degrees of freedom can be eliminated by imposing a constraint called a gauge condition, expressible in the form

$$\mathcal{G}\{A(x)\} \equiv C(x) \cdot A(x) = 0, \qquad (4)$$

where $C(x)$ is a chosen covariant 4-vector operator or field. Under a general hL transformation, the chosen gauge can be either (a) covariant or gauge preserving and denoted cg or (b) noncovariant or gauge nonpreserving, denoted *variable gauge* (vg):

(a)   if $\Lambda C(x) = C'(x')$:

$$C(x) \cdot A_{cg}(x) = C(x) \cdot \Lambda^{-1} \Lambda A_{cg}(x),$$
$$= C'(x') \cdot A'_{cg}(x'),$$
$$A'_{cg}(x') = \Lambda A_{cg}(x).$$

(b)   if $\Lambda C(x) \neq C'(x')$:

$$C(x) \cdot A_{vg}(x) \neq C'(x') \cdot [\Lambda A_{vg}(x)]. \qquad (5)$$

[The identity $C \cdot \Lambda^{-1} = [\Lambda C] \cdot$ has been used in Eq. (5).] Thus, in the standard language, the gauge-preserving covariant 4-vector property of $A_{cg}(x)$ in any covariant gauge cg described in Eq. (5a) is a consequence of the hL invariance of its gauge condition.

The inhomogeneous Maxwell equation for A in ED is

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial^2 A^\nu - \partial^\nu(\partial \cdot A)$$
$$\equiv \mathcal{L}^\nu_\mu A^\mu = j^\nu, \qquad (6)$$

where $\mathcal{L} = \mathcal{L}(x)$ is a *gauge-dependent* linear differential operator and $j = j(x)$ is a gauge-independent covariant 4-vector in spacetime (because it is made up of point charges moving with 4-velocities in frame x [9]). For a covariant gauge cg where $A_{cg}(x)$ is a covariant 4-vector, the full covariant tensor structure of the rank-2 covariant 4-tensor $\mathcal{L}$ is preserved under 4D rotations. So covariant gauges are indeed special.

Among these cg gauges, the Lorenz gauge is the most special and indeed unique because its gauge condition $\partial \cdot A_L(x) = 0$ allows Eq. (6) to be simplified to

$$\partial^2 A_L = j. \qquad (7)$$

Since $\mathcal{L}_L = \partial^2$ is now an invariant scalar operator, the covariant 4-vector nature of A_L is dictated by the covariant 4-vector nature of the 4-current j on the rhs [4], even for the special case $j = 0 = (0, 0, 0, 0)$. So the covariant 4-vector property of *every* solution A_L for both $j \neq 0$ and $j = 0$ is a consequence of the Lorenz condition alone.

Further elaboration of the covariant 4-vector nature of A_L might be helpful. First define $A_L(x)$ as the multivalued set, object, or "function" containing all the multiple solutions of Eq. (7) as its multiple values. Then, display its behavior under 4D rotation of the external local frame x to x',

$$\Lambda \partial^2 A_L(x) = (\Lambda \partial^2 \Lambda^{-1})(\Lambda A_L(x))$$
$$= \partial'^2 A'_L(x') = \Lambda j(x) = j'(x'), \qquad (8)$$

where $A'_L(x') = \Lambda A_L(x)$ from Eq. (5a) has been used. So the wave equations (7) transform covariantly for every solution contained in $A_L(x)$ for the simple reason that the multivalued object $A'_L(x')$ is exactly the same object as the original $A_L(x)$ but with components decomposed relative to the rotated local frame x'. This covariance refers initially to the constancy of the 4-component object $A_L(x)$ under *external* frame rotations.

However, ED as a hL-invariant theory in isotropic spacetime contains an additional symmetry of importance in physics: Spacetime itself shows no 4D orientation preference for ED phenomena so that only the relative orientation between the frame and solution is physically meaningful. There thus exists a solution $A_L(x')$ in frame x where the solution has been rotated in the opposite direction and is numerically indistinguishable from $A'_L(x')$. The covariance of Eq. (8) can then be *interpreted* as referring to such rotated solutions in hL-invariant theories. In theories that are not even implicitly hL invariant, such an interpretation is not admissible.

In Eq. (5b), on the other hand, $\Lambda A_{vg}(x)$ satisfies the gauge $\Lambda C(x)$ that differs from $C'(x')$; it is thus a gauge nonpreserving covariant 4-vector. *If* one insists on restoring the gauge in x' from gauge ΛC back to the original nc gauge, it will be necessary in Abelian theories to *add* an extra inhomogeneous 4-vector term $R'_{\Lambda C \to nc}(x') \equiv \partial' \Omega_{\Lambda C \to nc}(x')$ for gauge restoration to give the inhomogeneous transformation

$$A'_{nc}(x' = \Lambda x) = \Lambda A_{nc}(x) + R'_{\Lambda C \to nc}(x'),$$
$$\text{where } \mathcal{G}_{nc}\{A'_{nc}(x')\} = 0 \qquad (9)$$

is the restored gauge condition, and $A_{nc}(x) = A_{vg}(x)$ has been renamed "nc" for greater clarity. Note that the extra term $R'_{\Lambda C \to nc}$ is *not* concerned with the *residual* gauge degree of freedom describing the nonuniqueness of A'_g itself in a single gauge g in frame x' for the same field tensor F.

The new term $R'_{\Lambda C \to nc}$ in Eq. (9) ruins the hL covariance property of $\Lambda A_{nc}(x)$, however, because the four spacetime components are treated asymmetrically in noncovariant gauges. An asymmetry then appears in the differential operator $\mathcal{L}$ on the lhs of Eq. (6) for a noncovariant gauge. So after gauge restoration, $A'_{nc}(x')$ of Eq. (9) is no longer a covariant 4-vector but an hL-noncovariant 4-vector. This is the standard picture described in textbooks [2,3,5]. See also

Ref. [10]. In particular, the interaction Lagrangian $j \cdot A_{\mathrm{nc}} = (\Lambda j) \cdot (\Lambda A_{\mathrm{nc}}) \neq j' \cdot A'_{\mathrm{nc}}$, where the 4-current j remains a covariant 4-vector, is no longer explicitly hL invariant. The differential equation (DE) (6), too, is no longer covariant even though all tensor indices correctly describe matrix multiplications because $\Lambda \mathcal{L}_{\mathrm{nc}}(x) \Lambda^{-1} \neq \mathcal{L}'_{\mathrm{nc}}(x')$; the rotated solutions in frame x also do not solve the same Eq. (6) with $\mathcal{L}_{\mathrm{nc}}$.

If a gauge-restored but noncovariant 4-vector potential A_{nc} is used in a *gauge-invariant* formulation of ED, it must contain hidden hL covariance because one can always gauge transform to the Lorenz gauge (or any other covariant gauge cg) where the hL covariance of A_{cg} can be explicitly displayed.

How can this hidden hL covariance be made explicit? This question can be answered by using the sequential (gauge → Lorentz → gauge) transformations [11], $A_{\mathrm{nc}}(x) \to^{\mathrm{GT}} A_{\mathrm{cg}}(x) \to^{\mathrm{LT}} A'_{\mathrm{cg}}(x') \to^{\mathrm{GT}} A'_{\mathrm{nc}}(x')$, where x defines any initial inertial frame and $x' = \Lambda x$ is the new hL (4D rotated) frame. The last three A s have the interesting structure

(a) $A_{\mathrm{cg}}(x) = A_{\mathrm{nc}}(x) + R_{\mathrm{nc}\to\mathrm{cg}}(x),$ where

$$R_{\mathrm{nc}\to\mathrm{cg}}(x) = \partial \Omega_{\mathrm{nc}\to\mathrm{cg}}(x);$$

(b) $A'_{\mathrm{cg}}(x' = \Lambda x) = \Lambda A_{\mathrm{cg}}(x)$

$$= \Lambda [A_{\mathrm{nc}}(x) + R_{\mathrm{nc}\to\mathrm{cg}}(x)];$$

(c) $A'_{\mathrm{nc}}(x') = A'_{\mathrm{cg}}(x') + R'_{\mathrm{cg}\to\mathrm{nc}}(x').$ (10)

Here, $\Lambda \equiv \Lambda(x, x')$. These expressions can be combined to give the gauge-restored Lorentz transformation relation for the noncovariant but fixed-gauge 4-vector A_{nc},

$$A'_{\mathrm{nc}}(x') = \Lambda A_{\mathrm{nc}}(x) + R'_{\Lambda C\to\mathrm{nc}}(x, x'), \quad (11)$$

$$R'_{\Lambda C\to\mathrm{nc}}(x, x') = R'_{\mathrm{cg}\to\mathrm{nc}}(x') + \Lambda R_{\mathrm{nc}\to\mathrm{cg}}(x). \quad (12)$$

Equation (11) is a refinement of Eq. (9) for the same end result, namely, an inhomogeneous transformation in the 4D functional space of A involving two hL-related local frames x and x' in 4D spacetime with no translation between them.

All the gauge transformations involved in Eqs. (9), (10) are change-of-gauge ones; none is concerned only with a residual gauge term causing no gauge change. However, a necessarily longitudinal residual gauge term $R_{\mathrm{g}}(x) \equiv \partial \Omega_{\mathrm{g}}(x)$ or $R'_{\mathrm{g}}(x')$ can in general be added to a vector potential in any *nontransverse* gauge g and in the frame x or x' in these equations without changing their validity. This R_{g} is by definition gauge preserving. It is often called a gauge transformation of the second kind [12]. In Lorenz gauges, for example, the term satisfies the wave equation $\partial^2 \Omega_{\mathrm{L}} = 0$, or $\partial^2 R_{\mathrm{L}} = 0$.

Since ED is a linear theory in A, an ED gauge condition $\mathcal{G}_{\mathrm{g}}\{A\} = 0$ is almost always one satisfying the linearity property

$$\mathcal{G}_{\mathrm{g}}\{A_{\mathrm{g}}(x) + R_{\mathrm{g}}(x)\} = \mathcal{G}_{\mathrm{g}}\{A_{\mathrm{g}}(x)\} + \mathcal{G}_{\mathrm{g}}\{R_{\mathrm{g}}(x)\}$$
$$= 0, \quad (13)$$

where $R_{\mathrm{g}}(x)$ is a residual gauge term. This linearity has the consequence that A_{g} and R_{g} separately or together satisfy the gauge condition. Since the gauge condition also dictates the hL covariance property of vector potentials, A_{g} and R_{g} separately or together must be only covariant 4-vectors or only noncovariant 4-vectors. This means that all residual gauge terms can simply be *absorbed* into their parent terms [such as A_{g} in Eq. (13)] and not shown explicitly, if each A_{g} denotes a multivalued object containing *all possible values* allowed by the residual gauge degree of freedom.

For the covariant Lorenz gauge, for example, the hL transform $A'_{\mathrm{L}}(x') = \Lambda A_{\mathrm{L}}(x)$ is multivalued if the original $A_{\mathrm{L}}(x)$ is multivalued, both containing residual gauge terms. On the other hand, for the noncovariant Coulomb gauge where the residual gauge degree of freedom is absent, the hL transform $\Lambda A_{\mathrm{C}}(x)$ requires an appropriate multivalued gauge restoration $A'_{\mathrm{C}}(x') = \Lambda A_{\mathrm{C}}(x) + R'_{\Lambda C\to\mathrm{C}}$ to remove all unwanted residual gauge terms from $\Lambda A_{\mathrm{C}}(x)$. There is thus also no need for any final gauge transformation or gauge rotation of the type discussed in Ref. [1]. The fact that any allowed residual gauge degree of freedom has been included in the multivalued object A_{g} will be expressed mathematically as Eq. (17) in Sec. III from a more general perspective.

Why should one remain in the same gauge g in frame x' in a noncovariant gauge? It is good to know how to do it, but since the gauge degree of freedom under consideration causes no change in $F^{\mu\nu}$ and the classical properties it describes, this gauge degree of freedom can be used to enforce not the noncovariant gauge condition but the hL covariance of the gauge nonpreserving covariant 4-vector A_{vg}. That is, one can simply use the hL transform $\Lambda A_{\mathrm{vg}}(x)$ of Eq. (5b) alone without adding the troublesome gauge-restoring term $R'_{\Lambda C\to\mathrm{nc}}(x')$, thus allowing the tensor structure of the inhomogeneous Maxwell Eq. (6) to retain its usual meaning in flat spacetime.

For the Coulomb gauge, for example, the hL transform $\Lambda \mathcal{L}_{\mathrm{C}}(x) \Lambda^{-1}$ in frame x' in the variable-gauge approach differs from the operator $\mathcal{L}'_{\mathrm{C}}(x')$ in the gauge-restored approach. In either case, the DE has to be solved with the same operator $\mathcal{L}_{\mathrm{C}}(x)$ in frame x only; the solution is then transformed differently to frame x' in different treatments. In the variable-gauge treatment, all residual gauge terms $R'_{\mathrm{vg}}(x')$ that appear now should also be included. So Eq. (5b) can be rewritten as the covariant but variable-gauge gauge condition $C \cdot A_{\mathrm{vg}} = (\Lambda C) \cdot (\Lambda A_{\mathrm{vg}}) = 0$ to define a special kind of hL invariance/covariance for the original vg = nc gauge in frame x. For the Coulomb gauge in any frame x, the result is a *covariant Coulomb* (cC) gauge. It is also a subset of A_{L} of the Lorenz gauge of which the element for frame x also satisfies the Coulomb gauge

condition. This special element can be located from any element of $A_L(x)$ by the residual gauge transformation:

$$A_{cC}^{\parallel}(x) = A_L^{\parallel}(x) + \partial^{\parallel}\Omega_L(x)$$
$$= 0. \tag{14}$$

The vector potentials for covariant axial and temporal gauges can be similarly located.

For an hL-invariant theory such as ED in flat and isotropic spacetime, physics is independent of the choice of frame x; all Coulomb-gauge results obtained in any one frame describe the same physics as Lorenz gauges. This completes our demonstration that the vector potential A introduced by the gauge-covariant derivative (2) of Abelian gauge theories can always be chosen to be a covariant 4-vector in flat space satisfying the hL transformation law $A'(x' = \Lambda x) = \Lambda A(x)$ for any gauge in frame x including a noncovariant gauge, thus preserving the explicit hL of the theory.

To summarize, Lorcé's proposed revision of the 4-vector language for flat spacetime is unnecessary because the traditional textbook usage in ED is simpler. The proposal is confusing because the parent hL transformation of the spacetime 4-coordinate x under 4D rotations of the inertial frame centered at $x = 0$ is not duplicated by the proposed inhomogenous Lorentz transformation of the vector potential under 4D rotations of the local frames centered at x. Finally, the added inhomogeneous term has nothing to do with a hL transformation to another frame but is instead a change-of-gauge transformation in one inertial frame needed to repair or anticipate the gauge damage caused by the hL transformation.

The last comment applies even to non-Abelian gauge theories in flat spacetime where $A(x) = A_a(x)t_a$ is a sum over *internal* components $A_a(x)$ associated with the ath generators t_a of the gauge group. What has changed in non-Abelian (nAb) theories is not the 4D rotation of the external frame *but* their GT at point x in frame x [13],

$$A_a^{\mu} \overset{\text{nAbGT}}{\to} A_a^{\mu} + \partial^{\mu}\Omega_a + gf_{abc}A_b^{\mu}\Omega_c + \cdots, \tag{15}$$

showing only the leading term of an infinite series in powers of gf, f_{abc} being a structure constant of the gauge group.

For $SU(N)$ theories with $N \geq 2$ where $f_{abc} \neq 0$, the presence of non-Abelian terms dependent on gf causes serious complications. First, just the first non-Abelian gauge term shown in Eq. (15) depends on both Ω and A, allowing it to "twist" the internal structure of A itself in different ways depending on the exact circumstances in every gauge transformation. Lorcé's revision misses the real culprit that is this troublesome non-Abelian gauge term and wrongly blames the external frame rotation that is working properly.

Second, non-Abelian vector potentials satisfy *nonlinear differential equations* (nLDE) that cannot accommodate the non-Abelian gauge transformation (15) easily or even allow an easy solution of a chosen gauge definition. We shall return to describe this basic nonlinear obstacle more fully after first setting the stage by discussing the stated second objective of this paper.

III. DYNAMICAL AND NONDYNAMICAL PARTS OF THE VECTOR POTENTIAL

The *linear differential equation* (LDE) (6) also plays a central role in determining the origin of the dynamical part of A in ED:

$$A = A_{\text{dyn}} + A_{\text{ndy}}:$$
$$\mathcal{L}A_{\text{dyn}} = j, \qquad \mathcal{L}A_{\text{ndy}} = 0. \tag{16}$$

That is, A_{dyn} is a particular solution of the inhomogeneous LDE with a nonzero 4-current j that is gauge independent. (Note that in QED $j \equiv \bar{\psi}(x)\gamma\psi(x)$ is a local density where any arbitrary phases in the fermion fields always cancel in pairs. The γ^{μ}s are hL-invariant numerical 4×4 matrices acting on 4-component, hL *variant* Dirac spinors $\psi(x)$. The spacetime (μ index) structure of the expression is designed to guarantee the covariant 4-vector property of j under hL transformation by the hL covariance of the Dirac equation, or vice versa [12].) An additional nondynamical homogeneous solution A_{ndy} (for the equation with $j = 0$ that is also gauge independent) can be added to A_{dyn} to change the 4D boundary condition satisfied by their sum to some desirable value without changing the dynamics induced by the source 4-current j that is already contained in a particular A_{dyn}. This separation into dynamical and non-dynamical parts is a *gauge-independent* process; it can and should be made before a choice of gauge.

Note that A_{dyn} contains the same dynamics as the original gauge-invariant but higher-ranked field tensor F. The nondynamical part $A_{\text{ndy}} = A - A_{\text{dyn}}$ that is left must include the gauge term $\partial^{\mu}\Omega(x)$ of Eq. (3) because with $F^{\mu\nu} = 0$ the gauge term satisfies the homogeneous LDE. However, this LDE also has other solutions with nonzero $F^{\mu\nu}$ that contain real physics, as we shall now discuss.

The dynamical/nondynamical treatment does exact a price: A gauge degree of freedom now appears explicitly in the inhomogeneous LDE (6) so that a choice of gauge is now required before Eq. (6) can be solved in practice. The LDE itself takes different forms in different gauges, thus showing explicitly the variety of vector potentials that can be used to describe the same physics. In Lorenz gauges, both $A_{\text{ndy,L}}$ and the residual gauge term R_L satisfy the same homogeneous wave equation. So homogeneous solutions of the gauge type exist in the Lorenz gauge. There are additional *physical* solutions with nonzero $F^{\mu\nu}$ in any gauge. In Lorenz gauges, they describe free electromagnetic waves in all

spatial directions in 3D space. In the Coulomb gauge, the transversality condition excludes all pure-gauge solutions because they are longitudinal. One is then left with only "physical" solutions that satisfy wave equations in the 2D transverse momentum space and a timelike scalar potential A^0 that satisfies a Laplace equation [9].

There are infinitely many solutions for both types of A_{ndy} because of the infinite variety of gauge functions Ω that appear in gauge solutions and of boundary conditions for physical solutions. Since a dynamical solution remains a dynamical solution after the addition of any homogeneous solution, there are infinitely many A_{dyn} too. Furthermore, the multivalued $A_{\mathrm{ndy,g}}$ for any gauge g is already contained in the multivalued $A_{\mathrm{dyn,g}}$ and is not independent of it; if we enumerate the different values in these multivalued functions as $A_{\mathrm{dyn,g}}(x, n)$ and $A_{\mathrm{ndy,g}}(x, m, n)$ by adding counting numbers m, n as additional arguments, then

$$A_{\mathrm{ndy,g}}(x, m, n) = A_{\mathrm{dyn,g}}(x, m) - A_{\mathrm{dyn,g}}(x, n). \quad (17)$$

There is thus no additional final gauge transformation, gauge rotation, or even boundary condition of the types described in Refs. [1,7] for the simple reason that all solutions can be included in the multivalued object $A_{\mathrm{dyn,g}}$, previously called A_g. Its different values have the same symmetry properties, especially that under hL transformations, differing only in numerical values.

The situation is similar to that in a much simpler problem: The one-dimensional (1D) Newton equation $\ddot{x}(t) = j(t)$ has the multivalued solution $x(t) \equiv x_{\mathrm{dyn}}(t)$ containing the entire 2D infinity of numerical solutions for all possible choices of initial conditions (ICs) from points of the 2D IC space. A 2D infinity of force-free homogeneous solutions denoted collectively as $x_{\mathrm{ndy}}(t)$ also exist, but they can all be extracted from $x_{\mathrm{dyn}}(t)$ using a 1D version of Eq. (17). The gauge terms are absent, just like the Maxwell A_C in the Coulomb gauge.

Of course, none of the other 2D infinity of multiple solutions is needed if the particular solution on hand is already the physical solution satisfying the desired ICs. So the word "multivalued" is used in this paper in a generic or familial sense and not in the literal sense that a physical state shows multiple realities. On the contrary, each completely specified physical state is described by only one of these multiple solutions.

The multiple solutions of the vector potential A play a more complex role, as we have already discussed. For example, the free electromagnetic waves contained in the physical part of the nondynamical vector potential A_{ndy} quantize to the infinite variety of free photon states with a nonzero number of photons of different energies propagating in all directions in space from spatial infinity to spatial infinity. The multiple solutions of the dynamical vector potential A_{dyn} contain the additional dynamics

associated with distinct choices of the 4-current j. For problems involving only the unique gauge field bound in a specific atomic state, an appropriate "particular" A_{dyn} solution can be defined without using any A_{ndy} term of the physical type, but the gauge part of A_{ndy} can still be kept to display explicit gauge invariance [6]. More generally, however, the dynamical/nondynamical treatment provides a more complete description of the physical contents of A.

The gauge-invariant dynamical/nondynamical separation of A is also hL covariant, since it depends on the presence or absence of a covariant 4-vector current j. Equation (6) then retains its standard hL-covariant tensor structure for all hL-covariant 4-vectors A, as intended in the original covariant formalism; this result holds both for the gauge-preserving covariant 4-vector A_{cg} in a covariant gauge cg and for a gauge nonpreserving but covariant 4-vector A_{vg} in the vg treatment of any other gauge choice. For the gauge-restored but noncovariant (nc) 4-vector construct A_{nc} for a noncovariant gauge, however, the additional gauge-restoring term $R'_{\Lambda C \to \mathrm{nc}}(x')$ destroys the explicit hL invariance of the interaction Lagrangian. This defect does not prevent QED quantization in the Coulomb gauge in a *single frame* [2,3].

So the answer to the objection [14] that a noncovariant gauge like the Coulomb gauge is not hL covariant is that in a gauge-independent and hL-invariant theory the Coulomb gauge used in frame x alone gives correct results because of either a hidden hL covariance or an actual hL covariance when used in a variable-gauge context. It is not possible to preserve explicit hL covariance when the gauge is *fixed* at the Coulomb gauge in all frames.

Of course, covariant gauges in general, and Lorenz gauges in particular, are *special* because they preserve both their gauge and the full hL-covariant structure of all expressions in all hL frames. Nevertheless, it is the Coulomb (radiation or Landau) gauge that provides the simplest and physically most intuitive description of electromagnetic radiation or photons. The 3D space rotation invariance in its gauge condition allows the radiation/photon to travel in the same way along any spatial direction $\mathbf{e_k}$, while the explicit spacetime asymmetry in its gauge condition is designed to confine $\mathbf{A_C}$ entirely to the 2D subspace of transverse polarizations perpendicular to $\mathbf{e_k}$. An electromagnetic wave traveling in a definite direction $\mathbf{e_k}$ described so nicely by the Coulomb gauge is an example of a commonplace phenomenon that the spacetime symmetry of a physical state can differ from the hL invariance of the underlying Lagrangian. Finally, the exclusion of the entire gauge degree of freedom actually means that $\mathbf{A_C} = \mathbf{A_\perp}$ is gauge independent or invariant (Ref. [5], for example).

For the inhomogeneous nLDE satisfied by non-Abelian vector potentials, the separation of A into parts faces a serious obstacle: There are dynamical solutions $A_{\mathrm{dyn},a}$ for a nonzero Dirac 4-current $j_a = \bar{\psi}\gamma t_a \psi$ in the inhomogeneous nLDE and nontrivial nondynamical solutions $A_{\mathrm{ndy},a}$ for the associated homogeneous nLDE with $j_a = 0$. However,

linear superpositions of $A_{\mathrm{dyn},a}$ and $A_{\mathrm{ndy},a}$, or decomposed parts of both, do not in general satisfy nLDEs simply related to the original equations.

One can use systematic perturbation theory [13], however, when there are no nonlinear instabilities or complications. An iterative perturbation theory can be set up by first writing all expressions with the dimensionless non-Abelian coupling constant g shown explicitly in formulas. Then, the non-Abelian nLDE for A can be rearranged so that all terms dependent on g are moved to the rhs,

$$\partial_\mu(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu) = -g(j_a^\nu + j_{2,a}^\nu) - g^2 j_{3,a}^\nu, \quad (18)$$

where the non-Abelian nonlinear terms $j_{2,a}, j_{3,a}$ containing two and three vector potentials, respectively, may be treated as gauge 4-currents in an iterative solution; the calculation starts from a chosen unperturbed A_{ndy} (with $g = 0$ on the rhs). The calculated terms to order n can then be used in the nonlinear 4-currents on the rhs to drive the solution to order $n + 1$ in the resulting linearized DE. In this linearized theory, Abelian quantizations in the usual gauges used in Abelian theories and Feynman diagrams can be used. This perturbative method is different from the procedure suggested in Ref. [6].

Such perturbation methods may not always work well because physical states may contain very significant components where one or more linearized gauge bosons appear when g is large. Certain collective phenomena may require a great deal of effort to describe.

The nonlinearity of non-Abelian gauge theories causes further complications [13,15–17]. The non-Abelian fermion 4-current $j_a \equiv \bar{\psi}(x)\gamma t_a \psi(x)$ and field tensor F_a are both gauge dependent and more difficult to handle. Unlike the simple mathematical structure allowed by the linear superposition property of Abelian theories, non-Abelian nonlinearity admits a multitude of structures at the classical level: A gauge condition may have multiple solutions called Gribov copies or no solution at all. The non-Abelian nLDE (18) with 4-current $j = 0$ satisfied by $A_{\mathrm{ndy}}(x)$ of pure gauge can be used to define a nonlinear (with $g \neq 0$) classical vacuum that has infinitely many distinct solutions of different internal structures characterized by different topological winding numbers or kinks $-\infty \leq n \leq \infty$. This topological structure for all simple Lie groups, including $SU(N \geq 2)$, turns out to be the same as that for their $SU(2)$ subgroup alone, according to Bott's theorem.

The perturbation theory sketched in Eq. (18) can be used to start from any nonlinear solution $A_{\mathrm{ndy},n}(x)$ to give an approximate iterative-perturbative solution with the same winding number n (or in the same homotopy class) as $A_{\mathrm{ndy},n}(x)$. The resulting linearized perturbation theory can accommodate second quantization into gauge bosons based on the topological vacuum state $|n\rangle$ centered around the classical n vacuum. Quantum tunnelings between neighboring $|n\rangle$ vacua give rise to transient events lasting only instances in time called instantons and anti-instantons, instantons' time-reversed twins.

Quantization confers the non-native ability of linear superposition; the true vacuum that includes quantum couplings between $|n\rangle$ vacua is one of the θ vacua $|\theta\rangle = \sum_n e^{in\theta}|n\rangle$, where the arbitrary Bloch phase $n\theta$ appearing with a quantum $|n \neq 0\rangle$ vacuum is n dependent, as required by the periodic appearance of the degenerate $|n\rangle$ vacua in the 1D winding number space. For θ-independent Lagrangians and for certain gauges, these θ vacua can be considered disjoint and duplicate mathematical realizations of the same physical vacuum.

The native nonlinearity persists even among the quantized gauge bosons, however, for they clump together with or without interacting fermions into clusters of zero total non-Abelian charge g. At ultrashort distances, the particles inside each cluster have been found unexpectedly to be free and noninteracting, thus leading to the unfamiliar situation that the physical picture gets progressively simpler as the distance scale of observation decreases. Finally, gluons of nonzero charge g cannot propagate freely in free space. Hence, some of the nondynamical but physical solutions of A_{ndy} of the current-free nonlinear Yang–Mills equations for A giving nonzero $F^{\mu\nu}$ quantize to physical states of glueballs of total $g = 0$ propagating freely in free space.

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