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PHASE-SPACE REPRESENTATIONS OF WAVE EQUATIONS WITH APPLICATIONS TO THE EIKONAL APPROXIMATION FOR SHORT-WAVELENGTH WAVES

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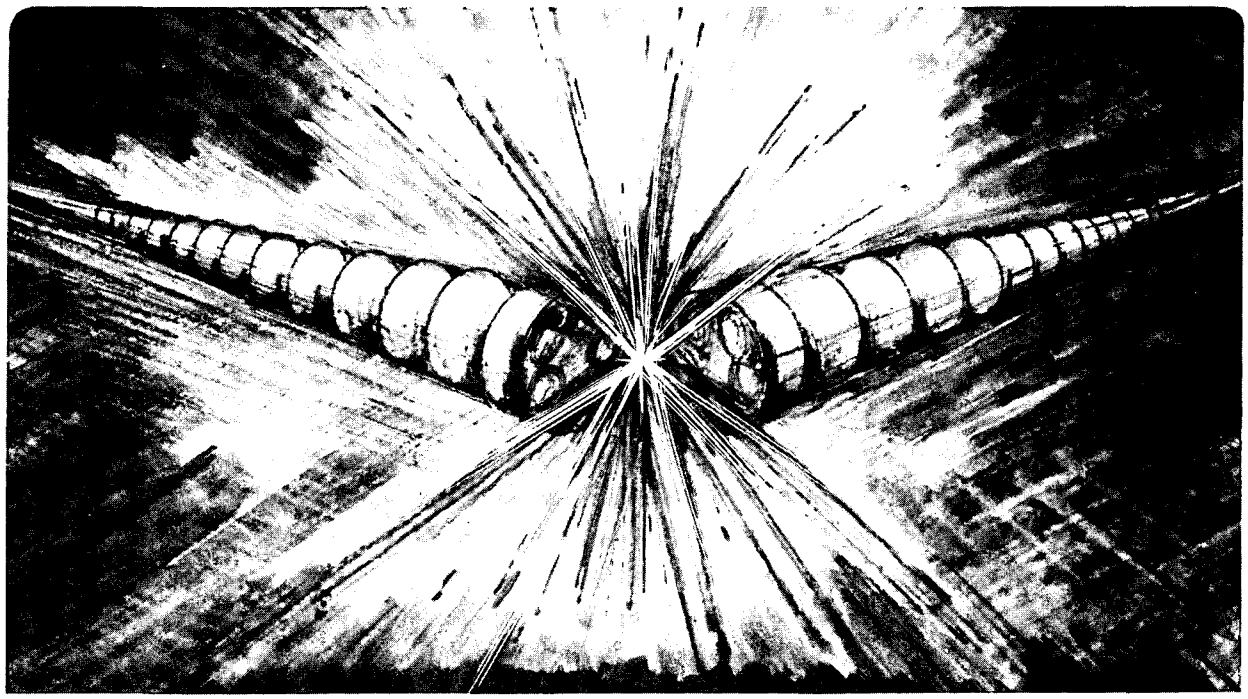

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ABSTRACT

We discuss three formalisms for defining a phase-space representation of a wave field; *i.e.*, a description of the wave as a function of both position x and wavenumber k (the phase space of the associated geometrical optics rays). The motivation for these ideas stems from the recognition that the focusing or caustic singularity problems inherent in the traditional eikonal analysis of short-wavelength wave propagation (responsible for a necessarily piecewise construction of the field in x-space) are due to projection singularities of the ray trajectories in phase space onto configuration space. In this Report, we explore the possibility of treating the equation that governs the phase-space representation of the field in the short-wavelength regime, constructing the wave along the ray trajectories in phase space (where rays do not focus), and only then reconstructing the x-space representation of the wave by projection; our hope is that this procedure avoids caustic singularities and produces a uniformly asymptotic solution which is valid over all x-space.

We consider the mathematical concept of the *symbol* of an operator, the related formalism of the Weyl symbol, and the coherent state representation. Since caustic singularities plague both quantum mechanical and classical wave equations, our treatment addresses a general wave equation and is explicitly applied to both vector and scalar waves, allowing for the presence of weak dissipation and nonlinearities. The emphasis of the presentation is on the potential for each formalism to produce physically interesting results and to be useful as a tool for analyzing short-wavelength wave propagation. Among the results of these theories are a derivation of the kinetic equation governing the wave action density of a vector wave and the construction of a short-wavelength wave in x-space with no caustic singularities.

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1. Introduction

Two of the most fundamental means for the mathematical description of physical phenomena are based on the interpretation of a particular process in terms of either particles or waves. Often the choice of the appropriate description is not at issue, as one's perception and physical intuition may clearly discern the proper formalism through which one can most successfully analyze a given event or effect. In other situations, however, even though the wave description may in principle provide the exact or complete theory of a physical phenomenon, the particle interpretation may permit a simpler method of analysis and to some extent may even furnish a more insightful, advantageous (depending on the application) and satisfying point of view. This was the case in the history of the development of the wave theory of light (*physical optics*) which evolved in competition with the corpuscular theory (*geometrical optics*); the phenomena of reflection, refraction, and focusing of light by optical elements, for example, can be understood in terms of simple ray concepts whereas the description of these effects based on the wave theory of electromagnetic radiation, while more complete, frequently involves a much more complicated analysis. In the same spirit, the Bohr theory of the hydrogen atom provided not only a simple (though to some extent, troubling) interpretation of atomic structure in terms of quantized electron orbits, it successfully predicted the correct frequencies observed in the emission spectrum; the discovery of quantum mechanics, while providing the correct theory, required the solution of a wave equation (and generated debate on the interpretation of the wave function itself). In both of these instances it is implicit that there is a definite correspondence between the dynamics of the particle or *ray* system and the associated wave system.

The nature of an *exact* correspondence between a particular classical mechanical system and a quantum mechanical system has long been the subject of much discussion and can be approached from both sides: how does one "quantize" a given ray system to obtain a wave problem, and the inverse, how is a wave system converted to its associated ray system? A further question could be: is there a direct correspondence between quantities representative of a ray system (*e.g.*, dynamical variables, trajectories and characteristic functions of these, such as particle densities and correlation times) and those typically associated with wave equations (*e.g.*, amplitude, phase, field correlation and spectral functions)?

In this Report we shall explore some of the aspects of the correspondence between rays and waves with a motivation toward the development of techniques by which the solutions to wave problems can be obtained or approximated from the analysis of the properties of the associated ray system. The standard, traditional approach to this exercise is based on the *eikonal* approximation [1,2] of the wave equation; depending on the field of application, this method is also known as the *WKB*, *semiclassical*, or *quasiclassical* approximation. The crux of this formalism lies in the assumption that the wave equation under consideration describes waves propagating in a medium which is slowly-varying in space and time compared with the wavelength and frequency of the wave solutions of interest. In the case of the Schrödinger equation, this can be realized by either considering wave functions which oscillate rapidly compared with the spatial variation of the potential, or by letting Planck's constant $\hbar$ approach zero for fixed particle energy (since the particle momentum is inversely proportional to the wavelength, with $\hbar$ the constant of proportionality). The result of the approximation scheme is a hierarchy of equations, the lowest order of which is an equation for the wave phase $\phi(\mathbf{x}, t)$; an important step

in the investigation of this procedure is the recognition of this *eikonal* equation as the Hamilton-Jacobi equation of classical mechanics, with the wave phase function ϕ being identified as Hamilton's action function. The significance of this relationship is that it immediately implies that the solution for the wave phase is to be obtained by examining the characteristic trajectories of a Hamiltonian system of rays, or classical particles. One of the special features distinguishing a Hamiltonian dynamical system is the existence of canonical variables; here, the Hamilton-Jacobi form naturally specifies that the momentum conjugate to the ray coordinate $\mathbf{x}$ is the *local wave vector* $\mathbf{k}$, the gradient of the phase ϕ. In this way, the concept of the *ray phase space* is introduced as the space of elements $(\mathbf{x}, \mathbf{k})$ through which the rays evolve.

The essential point to be appreciated here is that, while the wave phase and amplitude (governed by the higher-order equations) are properly conceived of as functions evolving in an N-dimensional $\mathbf{x}$-space, they are determined (in the short-wavelength approximation) by properties of the associated Hamiltonian ray system in the $2N$-dimensional $(\mathbf{x}, \mathbf{k})$ phase space. Due to the work of Maslov [3], Keller [4] and Arnold [5], it is now understood how certain aspects of the wave constructed in this fashion arise as a result of the projection of the ray motion in phase space onto the lower-dimensional physical configuration space. A particularly striking example of how wave features can be interpreted in terms of the corresponding ray trajectories projected onto $\mathbf{x}$-space is provided by the *caustic* phenomenon: these are regions of $\mathbf{x}$-space in which the wave intensity is enhanced due to the focusing of the rays in configuration space. In the $(\mathbf{x}, \mathbf{k})$ phase space the rays do not focus since one of the properties of Hamiltonian systems is that phase-space volume is conserved along ray orbits; the focusing and accompanying high wave intensity are the effect of a singularity in the projection of the ray trajectories in phase space onto $\mathbf{x}$-space. Indeed, the eikonal

approximation itself breaks down in these regions, as the manifestation of this projection singularity is the appearance of a singularity in the equation for the amplitude.

The conventional eikonal procedure then can be thought of as follows. The solution to a wave equation is approximated with a formalism which induces a Hamiltonian phase-space structure; the space-time wave problem is thus *lifted* into the higher-dimensional phase space which is the natural setting for the corresponding dynamical ray system. The properties of the ray trajectories are analyzed and then projected back down onto configuration space in order to reconstruct the wave solution. This method sometimes leads to undesirable consequences (such as caustic singularities) which, although modern developments in the theory have been able to overcome, the typical result is a piecewise representation of the solution involving matching conditions. The point of view upon which this Report is based is that since the ray system is most advantageously investigated in the $(\mathbf{x}, \mathbf{k})$ phase space, perhaps the properties of the wave solution could also be more clearly examined if expressed in a *phase-space representation.* That is, can one overcome the difficulties encountered in the conventional eikonal method (and perhaps even construct an approximate wave solution which is uniformly valid, everywhere in $\mathbf{x}$-space) by first casting the wave equation in a joint $(\mathbf{x}, \mathbf{k})$ phase-space representation (rather than the familiar $\mathbf{x}$-space *or* $\mathbf{k}$-space representations), applying approximation techniques (similar to those used in standard eikonal theory) to introduce the ray trajectories from which the approximate solution (in a phase-space representation) is constructed, and *then* project that solution onto configuration space to construct the space-time representation? Viewing the wave as a function on phase space may perhaps provide a more direct association with the ray trajectories. More importantly, perhaps the simplifications achieved by lifting the rays into phase space will have corresponding consequences for the properties of such a

phase-space representation of the wave.

The concept of a phase-space representation of a wave field is not a new one. Since the advent of quantum mechanics, many attempts have been made to establish connections between Hilbert space operators and functions on the classical phase space; the Hamiltonian operator, for example, must somehow be related to the classical Hamiltonian function. Since the wave field itself is not an operator (rather, an element of the Hilbert state space), its phase-space representation is usually considered to be the phase-space representation of the density matrix (or projection operator). It is well-known that there is not a unique correspondence between wave operators and phase-space functions. Due to the non-commutativity of the fundamental position and momentum (or wave vector) operators, the order in which they appear in an operator can change its action on the state vector (although physical observables are generally required to be associated with Hermitian operators).

In the same sense that the coordinate space representation of the Hamiltonian operator of quantum mechanics produces the Schrödinger wave equation, one can extend the notion of a phase-space representation for wave operators to classical wave equations (*e.g.*, Maxwell's equations). This is the theory of *pseudodifferential operators* [6-11] in which the phase-space representation of an operator is known as its *symbol*. Again, there are several different ways in which the symbol of an operator can be defined. In this Report we shall discuss two symbol formalisms and a third technique which, although not strictly a pseudodifferential operator method, does yield a useful phase-space representation related to the other two. An important ingredient in each scheme is the derivation of the equation governing the phase-space representation of the field. This is accomplished in the most direct fashion by viewing the original wave equation for the field in an abstract

(representation-free) operator form. The phase-space representation of this equation is then immediately obtained by introducing the symbols of both the operator and the field and by invoking the corresponding *symbol calculus*. This is nothing more than the rules which translate the abstract operations (*e.g.*, compositions of operators, adjoints, *etc.*) into operations on the corresponding symbols.

The emphasis here will be not so much on the mathematical justification and rigor but rather on the potential for each formalism to produce useful and physically interesting results in the short-wavelength limit. In general, the mathematical foundation for the development presented here can be found in a growing mathematical literature which is concerned to a large extent with the classification of pseudodifferential operators on the basis of the properties of their corresponding symbols. Although this branch of mathematics provides the background for the construction of phase-space representations of operators, its focus as an asymptotic (short-wavelength) theory is primarily on modern geometric interpretations of traditional eikonal theory based on an **x**-space representation of wave equations. The essentially new aspect of the treatment contained in this Report is the application of eikonal-like assumptions to phase-space representations of wave equations, or *Phase-Space Eikonal Techniques*. Furthermore, we shall illustrate our methods in the context of a general vector wave equation when showing how the familiar concept of a partial differential operator is extended to that of a pseudodifferential operator; this is so that the techniques and results can be applied to wave phenomena in fields other than quantum mechanics (such as those which consider electromagnetic wave propagation).

An outline of the Report is as follows. In Section 2, we shall briefly review the concepts, techniques and results of conventional eikonal theory in order to illustrate its relationship to the ray phase space and the inherent reasons for its failure under certain

circumstances. An introduction to phase-space representations, symbols and pseudodifferential operators will follow in Section 3. The symbol formalism discussed here is probably the simplest, in terms of it providing a reasonable extension of partial differential operators, and is the one most often encountered in the mathematical literature; from this discussion it should be clear that pseudodifferential operators indeed arise naturally in many physical wave problems. Furthermore, the calculus for this symbol is easy to derive and will serve as an illustration for obtaining phase-space wave equations. We shall see, however, that for many physical reasons this particular formalism does not produce a suitable representation for exploitation in the short-wavelength regime.

A second, perhaps more advantageous, phase-space representation will be the subject of Section 4. This correspondence between operators and phase-space functions was introduced by Weyl [12] and is therefore often referred to as the *Weyl symbol* of an operator. This formalism is a popular one in that the Weyl symbol of the density matrix representing the wave field is the *Wigner function* [13]. This object has received much attention [14-18] recently in connection with the study of the manifestations of chaos in quantum mechanical systems and has also been used in the asymptotic analysis of classical wave equations [19,20]. In defining this representation, we shall see that it is actually closely related to the symbol considered in Section 3, and this relationship will be exploited to give a shorthand derivation of the Weyl symbol calculus. Using this calculus, it will be seen how the Weyl symbol of the abstract operator equation governing the field directly produces the equation governing the evolution of the Wigner function in phase space. Even though this result is well known when the underlying wave equation is the time-dependent Schrödinger equation for a scalar wave function, the analysis given here for a general vector wave equation (allowing for sources and weak dissipation) should show that

these abstract concepts can have physically meaningful and important consequences for wave problems in a wide variety of disciplines. Here, the Weyl symbol of a vector wave field (a tensor-valued Wigner function) will be related to the more familiar notions of the *local spectral tensor* and the *wave action density*. The Weyl calculus provides an exact equation for the spectral tensor in terms of the local dispersion tensor and sources which account for discreteness and nonlinear effects. When eikonal-like assumptions are made, this exact equation is reduced to the *wave kinetic equation* for the wave action density; this could be thought of as a generalized Liouville equation associated with a general vector wave system in the short-wavelength limit. Thus, the Weyl symbol formalism permits a direct derivation of this important equation in a way which can easily be extended to higher order.

Finally, in Section 5, we shall consider the *coherent state representation*. Although this quantity is perhaps more familiar in the context of quantum field theory, [21-23] it will be seen to provide a useful description of a classical field as well (especially in the short-wavelength regime). The use of this representation for quantum mechanical wave functions has also become very popular [24-28] and is often referred to as a gaussian wave-packet representation. In fact, this representation will be shown to be closely related to the Weyl formalism and again this association is exploited in order to derive the phase-space equation governing it. This equation may be directly treated with a phase-space version of the conventional eikonal method; the result is a procedure for determining both the phase and the leading order amplitude of the short-wavelength wave along rays in phase space. These ideas are applied to the simple example of the Schrödinger equation for the harmonic oscillator. While traditional eikonal methods provide the correct spectrum for this problem, the asymptotic eigenmodes constructed in that theory suffer from caustic

singularities at the turning points. In contrast, the solution of the phase-space eikonal equations for the coherent state representation yields not only the exact spectrum but the exact eigenmodes as well. Thus, the advantage of this description is that it raises the possibility of being a method by which the field could be constructed in the short-wavelength limit as a function on phase space to avoid caustics, and subsequntly projected back down onto configuration space to give a uniform approximation of the field which is everywhere valid.

2. Review of Eikonal Theory

In this Section we shall briefly review the assumptions, methods and results of the conventional eikonal approach to the approximate, asymptotic solution of a wave equation. We do not intend to give a detailed rigorous presentation of the many facets of this well-documented theory, and therefore many of the results will be provided without derivation or mathematical justification. The purpose here, rather, is to discuss the procedure as a foundation upon which the following Sections will be based and against which the phase-space techniques developed there should be contrasted. Thus, the focus will be primarily on the way in which the classical ray phase space is introduced by the application of the eikonal approximation and how the properties of the ray trajectories are related to the asymptotic solution.

2.1 Eikonal Approximation for a Scalar Integral Wave Equation

In order to more clearly draw attention to the important points and to keep the presentation simple, we shall treat here only the case of a scalar wave equation in one spatial dimension and time. In this context, we consider linear wave equations which can be written in the general form

$$\int dx'\,dt'\; \mathcal{D}(x,t;x',t')\,\psi(x',t') = 0 \tag{2.1}$$

Here, the two-point spacetime *kernel* $\mathcal{D}$ is assumed to be Hermitian ($\mathcal{D}(x,t;x',t') = \mathcal{D}^*(x',t';x,t)$) and may contain δ-functions and their derivatives. As examples, both the Schrödinger equation for one-dimensional partical motion in a potential $V(x)$ with

$$\mathcal{D} \equiv i\hbar\delta(x-x')\delta'(t-t') + \left[\frac{\hbar^2}{2m}\delta''(x-x') - V(x)\delta(x-x')\right]\delta(t-t') \tag{2.2}$$

and the equation for waves on a string characterized by a nonuniform wave propagation

speed $u(x)$ with

$$\mathcal{D} \equiv \delta(x-x')\delta''(t-t') + \partial_{x'}[u^2(x')\delta'(x-x')\delta(t-t')] \tag{2.3}$$

can be cast in this form. We say that these equations govern waves propagating in nonuniform media in that the kernel depends on its spacetime arguments independently rather than in the special combination of the separations $s = x - x'$ or $\tau = t - t'$, which would describe a uniform medium (uniform potential $V(x) = V_0$ or homogeneous string $u(x) = u_0$). To emphasize the dependence on the spacetime separation variables, we can rewrite (2.1) as

$$\int ds\, d\tau\ \mathcal{D}(x,t;x-s,t-\tau)\,\psi(x-s,t-\tau) = 0 \tag{2.4}$$

The basic assumption underlying the eikonal approach to the solution of this wave equation is that the waves of interest described by $\psi(x,t)$ are characterized by a typical wavenumber k and frequency ω which are large compared with the spatial and temporal rates of variation of the medium described by $\mathcal{D}$. If this is the case, it is reasonable to assume that at a point (x,t) the wave solution looks roughly like a plane wave, but over a larger scale the amplitude, wavenumber and frequency may vary, as do the properties of the medium. These concepts are embodied in the form of the eikonal assumption for the form of the solution:

$$\psi(x,t) \equiv A(x,t)\, e^{i\,\phi(x,t)} \tag{2.5}$$

In analogy with the plane-wave solutions in a uniform medium where the phase $\phi(x,t)$ is simply $kx - \omega t$, the local wavenumber and frequency are defined to be the measure of the local rate of variation of the phase

$$k(x,t) \equiv \partial_x \phi(x,t) \qquad\qquad \omega(x,t) \equiv -\partial_t \phi(x,t) \tag{2.6}$$

If the length and time scales of variation of the medium are L and T respectively, we define the small dimensionless parameter ε

$$k(x,t)L \sim \omega(x,t)T \equiv \varepsilon^{-1} \gg 1 \tag{2.7}$$

and impose

$$\begin{aligned} A^{-1}(\partial_x A) \sim k^{-1}(\partial_x k) \sim \omega^{-1}(\partial_x \omega) \sim L^{-1} \\ A^{-1}(\partial_t A) \sim k^{-1}(\partial_t k) \sim \omega^{-1}(\partial_t \omega) \sim T^{-1} \end{aligned} \tag{2.8}$$

by which the foregoing assumptions about the variation of the amplitude $A(x,t)$, wavenumber and frequency are made explicit.* All higher derivatives are assumed to be of corresponding higher order in ε.

Substituting (2.5) into (2.4), we expand the amplitude and phase of $\psi(x-s,t-\tau)$ around the point (x,t) by

$$\begin{aligned} A(x-s,t-\tau) &\approx A(x,t) - s\partial_x A(x,t) - \tau\partial_t A(x,t) \\ \phi(x-s,t-\tau) &\approx \phi(x,t) - k(x,t)s + \omega(x,t)\tau + \tfrac{1}{2}s^2\partial_x^2\phi + \tfrac{1}{2}\tau^2\partial_t^2\phi + s\tau\partial_x\partial_t\phi \\ e^{i\,\phi(x-s,t-\tau)} &\approx e^{i\,\phi(x,t)}\, e^{-i\,(ks-\omega\tau)}[1 + \tfrac{i}{2}s^2\partial_x k - \tfrac{i}{2}\tau^2\partial_t\omega + \tfrac{i}{2}s\tau(\partial_t k - \partial_x\omega)] \end{aligned} \tag{2.9}$$

Here we have kept all terms which will contribute to our final result through $\mathcal{O}(\varepsilon)$. Furthermore, we have used the cross derivatives of (2.6) to express $\partial_x\partial_t\phi = \frac{1}{2}(\partial_t k - \partial_x\omega)$. We now have

$$\begin{aligned} A(x,t)e^{i\,\phi(x,t)}\Big\{ &\int ds\,d\tau\ \mathcal{D}(x,t;x-s,t-\tau)\, e^{-i\,(ks-\omega\tau)} \\ &\tfrac{i}{2}\int ds\,d\tau\ \mathcal{D}(x,t;x-s,t-\tau)\,[s^2\partial_x k - \tau^2\partial_t\omega + s\tau(\partial_t k - \partial_x\omega)\, e^{-i\,(ks-\omega\tau)} \\ &- A^{-1}\int ds\,d\tau\ \mathcal{D}(x,t;x-s,t-\tau)\,[s(\partial_x A) + \tau(\partial_t A)]\, e^{-i\,(ks-\omega\tau)}\Big\} = 0 \end{aligned}$$

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* In general, the amplitude should also be expressed as a power series in ε: $A = \sum_n \varepsilon^n A_n$. As we shall consider only the lowest two orders in the approximation treatment, this expansion is not necessary.

If we use the identity $se^{-iks} = i\partial_k e^{-iks}$ and a similar one for $\tau e^{i\omega\tau}$, the integrals with powers of s and τ can be performed in terms of derivatives with respect to k and ω giving

$$A\,e^{i\phi}\Big\{1+\tfrac{i}{2}\left[-(\partial_x k)\partial_k^2+(\partial_t\omega)\partial_\omega^2+(\partial_t k-\partial_x\omega)\partial_k\partial_\omega\right.$$
$$\left.-2A^{-1}[(\partial_x A)\partial_k-(\partial_t A)\partial_\omega]\right]\Big\}\int ds\,d\tau\;\mathcal{D}(x,t;x-s,t-\tau)\,e^{-i(ks-\omega\tau)}=0 \tag{2.10}$$

Given the form of the kernel $\mathcal{D}$ the remaining integral in (2.10) could now be computed; indeed it is, to within a phase factor, just the Fourier transform of $\mathcal{D}$ with respect to the second set (x',t') of its arguments. We define the result of this transform to be the *local dispersion function*

$$d(x,t,k,\omega)\equiv\int ds\,d\tau\;\mathcal{D}(x,t;x-s,t-\tau)\,e^{-i(ks-\omega\tau)} \tag{2.11}$$

As examples of this construction, the kernel for the Schrödinger equation (2.2) inserted into (2.11) gives

$$d(x,t,k,\omega)=\hbar\omega-\frac{\hbar^2k^2}{2m}-V(x) \tag{2.12}$$

while the string wave equation kernel (2.3) yields

$$d(x,t,k,\omega)=-\omega^2+k^2u^2(x)+ik\partial_x u^2(x) \tag{2.13}$$

Evidently, the local dispersion function for the Schrödinger equation is just the difference $E-H(x,p)$, where $E=\hbar\omega$ is the classical particle energy and $H(x,p)$ is the classical Hamiltonian (with particle momentum $p=\hbar k$). While this is perhaps a familiar and expected result, the form of d for the string wave equation may appear somewhat unusual in that it is not real-valued. The first two terms $-\omega^2+k^2u^2(x)$ are real and identifiable as the difference which, when set to zero, gives the expected dispersion relation for these waves. The imaginary part of (2.13) is unexpected because the kernel itself (2.3) is Hermitian.

As we shall see in Section 3, this is a general property of $d(x,t,k,\omega)$ as defined by (2.11) and is related to the fact that the local dispersion relation $\omega^2 = k^2u^2(x)$ for these waves contains a *product of x and k* (note that such a product is not present in the typical classical particle Hamiltonian $H = p^2/2m + V(x)$). For the present development however, let us merely note that this imaginary term, being proportional to the derivative of the local wave speed, is of the same lower order as the quantities in (2.8).

In view of the possibility that in general $d(x,t,k,\omega)$ may be complex, we split it into its real and imaginary parts by writing

$$d \equiv d' + id'' \qquad d' = \tfrac{1}{2}(d + d^*) \qquad d'' = -\tfrac{i}{2}(d - d^*) \tag{2.14}$$

With this definition, we can proceed to order the terms in (2.10) by powers of ϵ. In addition to (2.7,2.8), we need to make the following assumptions on the nth derivatives of d':

$$\partial_k^n d' \sim k^{-n} d' \qquad \partial_\omega^n d' \sim \omega^{-n} d' \tag{2.15}$$

which appear reasonable considering the examples (2.12,2.13). Furthermore, being led by the discussion of the imaginary part of (2.13), we take

$$d'' \sim \epsilon d' \tag{2.16}$$

and its derivatives correspondingly smaller. Now the lowest two orders of (2.10) are

$$\mathcal{O}(1): \qquad d'(x,t,k,\omega)\, A(x,t)\, e^{i\,\phi(x,t)} = 0 \tag{2.17}$$

and

$$\mathcal{O}(\epsilon): \qquad (\partial_\omega d')\partial_t A - (\partial_k d')\partial_x A + d'' A + \tfrac{1}{2}\left[(\partial_\omega^2 d')\partial_t \omega + (\partial_\omega \partial_k d')\partial_t k - (\partial_k^2 d')\partial_x k - (\partial_k \partial_\omega d')\partial_x \omega\right] A(x,t) = 0 \tag{2.18}$$

2.2 The Eikonal Equation

As in the solution of a wave problem for a uniform medium where a condition similar to (2.17) is obtained (in the Fourier-transform k-space), one concludes that either the field ψ or the dispersion function must vanish at every spacetime point. Thus, setting the real part of the local dispersion function to zero generates the *local dispersion relation*

$$d'(x,t,k,\omega) = 0 \qquad \Longrightarrow \qquad \omega = \Omega(x,k,t) \tag{2.19}$$

governing waves propagating in the system. For the Schrödinger equation (2.2,2.12) this is just the relation between the energy and the Hamiltonian function $E = H(x,p)$, while for the string waves (2.3,2.13) it is the relation between frequency, wavenumber and position $\omega^2 = k^2u^2(x)$. For more complicated wave equations (such as those which arise in plasma physics), the condition $d' = 0$ is in general not invertible for the dispersion relation and is therefore only implicitly defined.

Recalling that the local wavenumber and frequency are by (2.6) derivatives of the phase ϕ, the lowest order condition (2.17) is in fact a nonlinear, first order partial differential equation for the phase

$$d'(x,t,\partial_x\phi(x,t),-\partial_t\phi(x,t)) = 0 \tag{2.20}$$

Again considering the dispersion function for the Schrödinger equation (2.12), the *eikonal equation* (2.20) can be recognized as the *Hamilton-Jacobi* equation of classical mechanics, with the phase ϕ identified with the *action function* of the classical system [29,30]. The solution of this equation is constructed in terms of the *characteristic trajectories* defined by

$$\dot{x} = -\frac{\partial_k d'}{\partial_\omega d'} \qquad \dot{k} = \frac{\partial_x d'}{\partial_\omega d'} \qquad \dot{\omega} = -\frac{\partial_t d'}{\partial_\omega d'} \tag{2.21}$$

where the dot notation denotes total derivative with respect to time. Using the standard relations following from (2.19)

$$\begin{aligned}(\partial_x d')_{k,t,\omega} + (\partial_\omega d')_{x,k,t}(\partial_x \Omega)_{k,t} &= 0 \\ (\partial_k d')_{x,t,\omega} + (\partial_\omega d')_{x,k,t}(\partial_k \Omega)_{x,t} &= 0 \\ (\partial_t d')_{x,k,\omega} + (\partial_\omega d')_{x,k,t}(\partial_t \Omega)_{x,k} &= 0\end{aligned} \tag{2.22}$$

(the variables to be held fixed during partial differentiation being denoted as subscripts), these *ray equations* can also be written in the usual Hamiltonian form

$$\dot{x} = \partial_k \Omega(x,k,t) \qquad \dot{k} = -\partial_x \Omega(x,k,t) \qquad \dot{\omega} = \partial_t \Omega(x,k,t) \tag{2.23}$$

A single ray trajectory evolves in the four-dimensional extended phase space (x,t,k,ω) on the three-dimensional *dispersion surface* defined by (2.19). If the medium is stationary so that the local dispersion relation is time-independent (as in our examples), one can view the trajectory as evolving in the familiar two-dimensional (x,k) phase space on the one-dimensional surface of constant ω. In this case, the equations of motion are often written in terms of the *Poisson bracket*

$$\begin{aligned}\dot{x} = \{x,\Omega\} \qquad \dot{k} = \{k,\Omega\} \\ \{f(x,k),g(x,k)\} \equiv (\partial_x f)(\partial_k g) - (\partial_k f)(\partial_x g)\end{aligned} \tag{2.24}$$

For a wave propagation problem in a stationary medium, the initial condition for the eikonal equation is specified by giving the value of the phase at a point x_0 and its spatial derivative $k_0 = \partial_x \phi(x_0)$. This determines an initial condition (x_0,k_0) in the ray phase space, from which a trajectory is generated by (2.21). The solution for the phase is then just the path integral along the trajectory $(x(t;x_0,k_0),k(t;x_0,k_0))$

$$\phi(x,t) = \phi(x_0) - \omega t + \int_{x_0}^{x} k\,dx \tag{2.25}$$

where $\omega = \Omega(x_0,k_0)$.

2.3 The Amplitude Equation: Action Transport and Caustics

We have seen that the lowest order approximation (2.17) to the wave equation (2.1) induces a ray system which evolves in an (x,k) phase space governed by a Hamiltonian (the local dispersion relation) and by which the phase of the wave field can be computed. In many applications, knowledge of these ray trajectories (either in phase space or their projection onto configuration, or x-space) is sufficient to analyze the problem under consideration. After all, the rays move in x-space with the group velocity (2.23) so that one knows that the energy in the wave goes where the rays go (as long as the assumptions for the eikonal approximation hold). In other applications, however, one may be interested in the wave field amplitude structure in x-space in order to evaluate, for example, energy deposition, higher-order wave-particle interactions or nonlinear wave coupling mechanisms. For this purpose, the next higher-order $O(\varepsilon)$ equation (2.18) which governs the wave amplitude is required. We now examine that equation and show how it can be re-expressed in terms of more familiar quantities.

Taking $A(x,t)$ to be real, we multiply (2.18) by $2A$ and rearrange to produce

$$\begin{aligned}(\partial_\omega d')\partial_t A^2 + \left[(\partial_\omega^2 d')\partial_t\omega + (\partial_\omega\partial_k d')\partial_t k\right] A^2 \\ -(\partial_k d')\partial_x A^2 + \left[(\partial_k^2 d')\partial_x k + (\partial_k\partial_\omega d')\partial_x\omega\right] A^2 \\ + 2d'' A^2 = 0\end{aligned} \tag{2.26}$$

recalling that both k and ω are functions of (x,t) (being derivatives of the phase ϕ); these functions can be computed along the ray trajectories discussed above. Now, using this fact in computing the derivatives

$$\begin{aligned}\partial_t\left[A^2(\partial_\omega d')\right] &= (\partial_\omega d')\partial_t A^2 + \left[(\partial_t\partial_\omega d') + (\partial_\omega^2 d')\partial_t\omega + (\partial_k\partial_\omega d')\partial_t k\right]A^2 \\ \partial_x\left[A^2(\partial_k d')\right] &= (\partial_k d')\partial_t A^2 + \left[(\partial_x\partial_k d') + (\partial_k^2 d')\partial_x k + (\partial_\omega\partial_k d')\partial_x\omega\right]A^2\end{aligned} \tag{2.27}$$

we can rewrite (2.26) as

$$\partial_t\left[A^2(\partial_\omega d')\right] - \partial_x\left[A^2(\partial_k d')\right] = A^2\left[(\partial_t\partial_\omega d') - (\partial_x\partial_k d') - 2d''\right] \tag{2.28}$$

The right-hand side of this form of the amplitude transport equation contains derivatives of the real part of the dispersion function d' which vanish if d' does not contain xk or ωt products (*i.e.*, products of *conjugate* variables). We have also seen by example (2.13) that the presence of an imaginary part d'' of the dispersion function is related to the existence of such products in the real part d'. Thus, for the Schrödinger Hamiltonian (2.12), the derivatives and the imaginary part d'' vanish so that the right-hand side is zero. The surprising result, however, is that the right-hand side also vanishes for the string wave dispersion function (2.13), as can be easily verified by computing the appropriate derivatives of d' and comparing this with $2d''$. Actually, we shall show in Section 3 that in general, a dispersion function d defined by the transform (2.11) for a Hermitian kernel D has a real and imaginary part related by

$$(\partial_t\partial_\omega d') - (\partial_x\partial_k d') = 2d'' \tag{2.29}$$

to lowest order (note that the derivatives of d' here are of order $(\omega T)^{-1} \sim (kL)^{-1} \sim \varepsilon$, as is the assumed order of d'').

Focusing on the left-hand side of (2.28) then, we define the quantity

$$J_x(x,t) \equiv A^2(x,t)(\partial_\omega d')\big(x,t,k(x,t),\omega(x,t)\big) \tag{2.30}$$

in terms of which the amplitude transport equation is

$$\partial_t J_x(x,t) + \partial_x\left[\left(-\frac{\partial_k d'}{\partial_\omega d'}\right) J_x(x,t)\right] = \partial_t J_x(x,t) + \partial_x\left[\dot{x}(x,t) J_x(x,t)\right] = 0 \tag{2.31}$$

where we have used the ray equations (2.23). The combination $J_x(x,t)$ is called the *action density* [31] and the equation it obeys (2.31) implies that the value of J_x integrated over space is constant. For the Schrödinger equation (2.12), we have $J_x = \hbar A^2(x,t)$ which

indeed has the units of (action/length), since the units of A are $(\text{length})^{-\frac{1}{2}}$ (for wave function normalization); the conservation of total J_x in this case is of course just the conservation of total particle probability (normalized wave functions). The energy density $U(x,t)$ of the wave is related [31] to the action by $U = \omega J_x$, the same as the relationship between the energy and action of a classical harmonic oscillator. The action density is in a sense a more useful quantity than the energy as the total action is conserved (in this adiabatic approximation) even in a time-varying medium while the energy is not (due to the time-dependence of the frequency (2.23)). We use the subscript x for J to denote that this is the action density on x-space; we shall see in Sections 3 and 4 that one could just as well approach the wave problem and define an action density on k-space, or even on the joint (x,k) phase space.

While the form (2.31) is convenient for illustrating the conservation of total action, for purposes of evolving the amplitude along ray trajectories the alternative expression

$$\dot{J}_x \equiv \partial_t J(x,t) + \dot{x}\partial_x J_x(x,t) = -J(x,t)(\partial_x \dot{x}) \tag{2.32}$$

is more useful. This equation implies that the x-space action density is convected at the group velocity along the x-space projection of the ray trajectories. We observe furthermore that although total J_x is conserved, the value of $J_x(x(t),t)$ along a ray is not constant if $\partial_x \dot{x}$ is not everywhere zero. In general, this condition is not satisfied as $\partial_x \dot{x}$ is a measure of the convergence or divergence of rays in x-space. Even though it is a feature of Hamiltonian ray systems that rays do not converge or diverge in the (x,k) phase space (volume in phase space is conserved by Liouville's theorem), this is not true of the projection of these rays onto x-space. Indeed, this result could be disastrous for a numerical scheme attempting to integrate (2.32) along x-space ray paths in regions where the rays tend to focus (or $\partial_x \dot{x}$

becomes large); points (or surfaces in higher dimensions) in x-space where this focusing term approaches infinity are called *caustics*.

A graphical illustration of these concepts is presented in Fig. 1. The Hamiltonian considered is a one-dimensional oscillator $H(x,p) = p^2/2m + V(x)$ so that $\dot{x} = p/m$. The curve in the (x,p) phase space is a trajectory with x-space turning points at $x = x_L, x_R$. The important point is that near the turning points, the momentum p goes to zero and the slope of the curve becomes vertical (or $\partial_x p \to \infty$). This is precisely the region where the projection of the phase space curve (or manifold) onto x-space becomes singular (*i.e.*, the projection mapping ceases to be one-to-one and invertible).

A caustic is an example of a region in x-space where the eikonal approximation as presented here breaks down: not only does the amplitude become singular but often the local wavenumber k vanishes (as in Fig. 1), or the wavelength becomes infinite violating the assumptions (2.7). Methods for treating the wave equation in these regions have been developed so that the ray trajectories can be "continued" through caustics by matching wave solutions on both sides. Modern geometric methods, developed primarily by Maslov [3,1], take advantage of the fact that in the neighborhood of an x-space caustic, the k-space representation is smooth (*i.e.*, the projection of the phase-space trajectory onto k-space is not singular). This can be visualized in Fig. 1 as the slope $\partial_k \dot{k} \propto \partial_k x$ (in the vicinity of the turning point) which is well-behaved. In terms of the asymptotic x-space form (2.5), the asymptotic k-space representation is defined to be

$$\begin{aligned} \hat{\psi}(k,t) &= \int dx' \; A(x',t)\, e^{i\,\phi(x',t)} e^{-ikx'} \\ &\equiv \hat{A}(k,t)\, e^{i\,\hat{\phi}(k,t)} \end{aligned} \tag{2.33}$$

where the integral is to be computed in the *stationary phase approximation*, *i.e.*, by evaluating the integrand at the point x' where the phase $\phi(x') - kx'$ (normally rapidly varying)

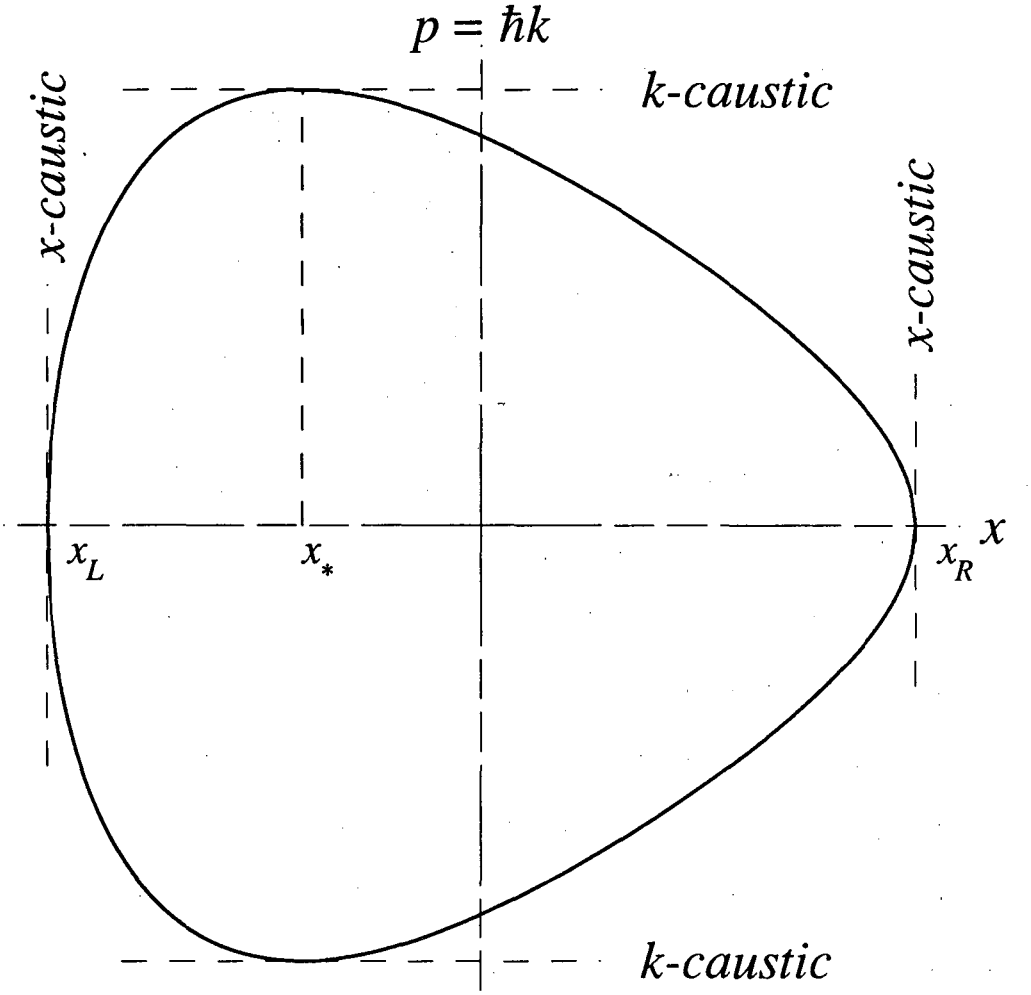

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Fig. 1. Phase-space portrait of a one-dimensional oscillator Hamiltonian $H(x,p) = p^2/2m + V(x)$. The x-space turning points x_L, x_R (where $p = 0$) are also the points where the projection of the phase-space trajectory onto x-space becomes singular ($\partial_x p \to \infty$) producing x-space caustics in the asymptotic wave solution. Similarly, k-caustics are divergences in the k-space representation of the wave where the projection of the ray manifold onto k-space becomes singular.

is stationary:

$$\partial_{x'}[\phi(x',t) - kx'] = k(x',t) - k = 0 \qquad \Longrightarrow \qquad x' = \eta(k,t) \tag{2.34}$$

Expanding the phase around the point $x' = \eta(k,t)$, we find

$$\begin{aligned} \hat{\phi}(k,t) &= \phi(\eta(k,t),t) - k\eta(k,t) \\ \hat{A}(k,t) &\sim A(\eta(k,t),t)\, e^{i\sigma} \end{aligned} \tag{2.35}$$

where the formula for $\hat{A}$ also includes important phase factors σ from the remaining gaussian-like integral centered at the stationary phase point.

Near $x = x_*$ in Fig. 1, the x-space representation is valid while the k-space representation becomes infinite (a *k-caustic*) as the projection of the ray trajectory encounters a singularity (one can show that $\hat{A}$ becomes infinite). Indeed, it has been shown that in N dimensions (N degrees of freedom) one can always find a (possibly mixed) representation for the wave in terms of some set of j x variables and $N - j$ k variables $(x_1, x_2, \ldots, x_j, k_{j+1}, \ldots, k_N)$ in which the eikonal solution is valid. The study of the transformations between the representations and their connection with characteristic functions of catastrophes which describe the field in the neighborhood of the caustic is, however, beyond the scope of the presentation here. As motivation for the development in the following Sections, we only remark that even this beautiful geometric theory does not yield a solution for the wave which is valid everywhere in a single representation, and matching must be performed when a transformation is made between representations.

2.4 Bound State Problems

We conclude this Section with a short review of the application of the eikonal method to *bound state* wave problems, by which we mean boundary-value wave equations which possess a discrete spectrum of eigenvalues and normal mode solutions; such wave equations

correspond to ray systems with trajectories that are localized or bounded in some region of x-space. As we shall also restrict our discussion to wave problems without resonances (*i.e.*, wavelength $\lambda(x)$ everywhere nonzero), the ray systems are bounded in the (x, k) phase space as well. Continuing to work in just one spatial dimension, we shall further restrict our attention to local dispersion functions of the Schrödinger type (2.12) for which these bounded Hamiltonians are simply one-dimensional oscillators, with phase space portraits topologically similar to that in Fig. 1.

The eikonal equation (2.17) for this type of system again induces the ray trajectories in phase space by which the solution for the wave phase ϕ is determined. Now, however, since the rays oscillate between x-space turning points there is interference between left and right propagating waves (*i.e.*, $k(x) \sim \pm\sqrt{E - V(x)}$ is a double-valued function as in Fig. 1). Requiring that the resultant wave function be single-valued in the classically allowed region of x-space leads to the well-known connection formulas [2] and the quantization condition

$$\tfrac{1}{2\pi} \oint k \, dx = n + \tfrac{1}{2} \tag{2.36}$$

In conventional treatments, the factor of $\frac{1}{2}$ arises from a detailed investigation of the wave phase in the neighborhood of the turning points. The modern theory of Maslov [3] has shown how such fractional contributions to the quantization condition emerge from an analysis of the topological structure of the ray manifold in phase space and its projection onto x- and k-space. A good presentation of the theory for these *Maslov indices* can be found in Percival [32].

The amplitude equation written in terms of the action density (2.31) becomes even simpler in the case of normal modes where $\partial_t J_x \equiv 0$. This condition implies

$$J_x(x) = \dot{x} A^2(x) = \text{constant} \qquad \Longrightarrow \qquad A(x) \sim (\dot{x})^{-1/2} \sim [k(x)]^{-1/2} \tag{2.37}$$

For the Schrödinger equation this is the well-known result that the semi-classical probability density A^2 is just the classical probability density, inversely proportional to the x-space speed. As discussed above, the x-space action becomes singular at the classical turning points where $\dot{x} = 0$. The traditional treatment, however, constructs the wave function in the classically disallowed region (on the other side of the turning points) by considering an appropriately-signed imaginary wavenumber so that the wave function decays exponentially as $x \to \pm\infty$. The final result is a piecewise-defined wave function with one formula for $\psi(x)$ asymptotically valid in the classically allowed region (away from the turning points), two more expressions asymptotically valid in the classically disallowed regions ($x \to \pm\infty$) and an additional two forms valid in the turning point regions, obtained by methods discussed in the preceding subsection. In the following Sections we shall explore methods by which one might be able to construct a single solution which is uniformly, asymptotically valid everywhere.

3. Symbols

As discussed in Section 2.3, one of the features of conventional eikonal theory is that while the ray trajectories introduced to determine the wave phase are most naturally analyzed in the (x, k) phase space, the $O(\varepsilon)$ equation for the wave amplitude is cast in terms of the x-space projection of these trajectories. This leads to the amplitude singularities on caustic surfaces in x-space where this projection becomes singular. Even though it was indicated that an alternative nonsingular representation in terms of a projection onto other subspaces can always be constructed in these regions, this procedure invariably requires matching conditions and necessarily implies a piecewise solution. A mark of this concept of projection is that the conventional eikonal assumption tacitly implies that the local wavenumber (or momentum) is really a function of x (being defined as the derivative of the phase). Although k is allowed to be a multiple-valued function of x (as in the oscillator problem of Section 2.4), the traditional method fails [33] if k is *continuously multiple-valued* as found in chaotic ray systems in more than one dimension.

In this Section we will begin to explore methods by which one can generate asymptotic solutions to wave equations in a representation which is a function of both k and x. Such a *phase-space representation* might avoid the caustic projection singularities if a solution is first constructed from the rays in phase space (where volume is conserved and k is truly independent of x) and then this *solution* (not just the ray manifold) is projected back onto x-space.

3.1 The Symbol of a Pseudodifferential Operator

As the development of a phase-space representation is an exact concept (*i.e.*, not

just an asymptotic approximation) we shall start over again with the analysis of the wave equation (2.1) given at the beginning of Section 2. We shall introduce the notion of the *symbol* of an operator and give a brief discussion of its usefulness and its properties. Since there is a considerable body of mathematical literature devoted to this concept [6-11] and its relation to the study of *pseudodifferential operators*, we shall not strive for mathematical precision or completeness. Instead, we shall focus on the physical implications of these ideas and examine them with regard to their potential for providing useful methods (either analytical or numerical) for treating short-wavelength wave propagation problems.

As in the previous Section, we shall continue to work primarily with a scalar wave equation. As the results obtained can be directly extended to more than one dimension, we shall use the notation x to stand for $(\mathbf{x}, t)$ and k to denote $(\mathbf{k}, \omega)$; the standard definition with opposite signs for the spatial and temporal Fourier transform exponents require the product kx to be understood as $\mathbf{k}\cdot\mathbf{x} - \omega t$ and the derivatives ∂_x, ∂_k to be $(\partial_{\mathbf{x}}, -\partial_t)$ and $(\partial_{\mathbf{k}}, \partial_\omega)$ respectively. Furthermore, although we consider a scalar equation, the extension to vector fields and tensor-valued operators is also straightforward; these generalizations will either be apparent at each step or explicitly noted. Finally, in order to include departures from the usual linear treatment of waves (such as sources and higher order nonlinear wave processes*), we allow for a non-zero right-hand side in (2.1) and consider as our basic equation

$$\int dx'\, D(x,x')\psi(x') = F(x) \tag{3.1}$$

* To include nonlinearities, the right-hand side should be a functional of ψ, although our approximation method will treat this term as a perturbation on the linear waves. Furthermore, this is not a general treatment of nonlinearites in that the kernel D is not considered to be a functional of ψ.

The dispersion kernel $\mathcal{D}(x,x')$ is the x-space (or in many dimensions, space-time) representation of the abstract dispersion operator $\mathbf{D}$. Similarly, $\psi(x)$ and $F(x)$ are the x-space representations of the abstract fields ψ and F. It will be important for the following discussion to view (3.1) as just the x-space description of the abstract equation

$$\mathbf{D}\psi = F \tag{3.2}$$

In the case where (3.1) is the Schrödinger equation (with $F = 0$), this abstract operator equation is interpreted as the action of the operator $\mathbf{D}$ on the state vector ψ in the Hilbert space. To recover the x-space representation (3.1) of (3.2), one can use the Dirac notation to project (3.2) onto the basis states $|x\rangle$ of the position operator

$$\langle x|\mathbf{D}\psi\rangle = \langle x|F\rangle \tag{3.3}$$

and with the projection operator (or completeness) identity [34]

$$\int dx' \, |x'\rangle\langle x'| = \mathbf{I} \tag{3.4}$$

one has

$$\int dx' \, \langle x|\mathbf{D}|x'\rangle\langle x'|\psi\rangle = \langle x|F\rangle \tag{3.5}$$

Defining the configuration space representations as

$$\begin{aligned} \langle x|\mathbf{D}|x'\rangle &\equiv \mathcal{D}(x,x') \\ \langle x'|\psi\rangle &\equiv \psi(x') \\ \langle x|F\rangle &\equiv F(x) \end{aligned} \tag{3.6}$$

equation (3.1) is obtained.

Of course other representations of (3.2) are possible; for example, the wavenumber- or k-space description is

$$\int \frac{dk'}{2\pi} \hat{\mathcal{D}}(k,k')\hat{\psi}(k') = \hat{F}(k) \tag{3.7}$$

where the k-space quantities are related to their x-space counterparts by the usual Fourier transform

$$\begin{aligned} \hat{\psi}(k) &\equiv \int dx'\, \psi(x') e^{-ikx'} \\ \hat{D}(k,k') &\equiv \int dx\, dx'\, e^{-ikx} D(x,x') e^{ik'x'} \end{aligned} \tag{3.8}$$

We have stressed the difference between the abstract representation-free expression (3.2) and its x-space representation (3.1) because the abstract form will be the starting point for developing the phase-space equations.

Consider for a moment the case where (3.1) can be expressed as a finite order partial differential equation for $\psi(x)$. That is,

$$\begin{aligned} \int dx'\, D(x,x')\psi(x') &= \mathbf{D}(\overset{2}{x}, \overset{1}{D}_x)\psi(x) \\ &\equiv \sum_{m=0}^{M} d_m(x) D_x^m \psi(x) \end{aligned} \tag{3.9}$$

$$D_x \equiv -i\partial_x \tag{3.10}$$

Here, the superscript notation above the arguments of $\mathbf{D}$ denotes the order of the differentiation and the multiplication by the x-dependent coefficients $d_m(x)$ as shown; here the differentiation D_x^m is to be performed before multiplication by $d_m(x)$. In N dimensions, the obvious generalization for the form of $\mathbf{D}(x, D_x)$ is

$$\mathbf{D}(\overset{2}{x}, \overset{1}{D}_x) = \sum_{\mathbf{m}}^{\mathbf{M}} d_{\mathbf{m}}(\mathbf{x}) D_{\mathbf{x}}^{\mathbf{m}} \tag{3.11}$$

where $\mathbf{m} = (m_1, m_2, \ldots, m_N)$, $m = m_1 + m_2 + \cdots + m_N$, and

$$D_{\mathbf{x}}^{\mathbf{m}} = \frac{\partial^m}{\partial x_1^{m_1} \partial x_2^{m_2} \cdots \partial x_N^{m_N}} \tag{3.12}$$

The sum is over all possible combinations of derivatives with respect to the variables x_i which may include time.

Writing $\psi(x)$ in terms of its Fourier transform, (3.9) becomes

$$\sum_m d_m(x)(-i\partial_x)^m \int \frac{dk}{2\pi} e^{ikx}\hat{\psi}(k) = F(x) \tag{3.13}$$

or

$$\begin{aligned} \int \frac{dk}{2\pi} \left[\sum_m d_m(x)k^m\right] e^{ikx}\hat{\psi}(k) &= F(x) \\ &\equiv \int \frac{dk}{2\pi} d(x,k)e^{ikx}\hat{\psi}(k) \\ d(x,k) &\equiv \sum_m d_m(x)k^m \end{aligned} \tag{3.14}$$

These expressions define the quantity $d(x,k)$ which we refer to as the *symbol* of the partial differential operator $\mathbf{D}$. Simply stated, an operator whose action in the x-representation may be written as in (3.9) is associated with a symbol obtained by replacing the differentiation D_x by k. The differential operator, being a polynomial in D_x with x-dependent coefficients, possesses a symbol which is the same polynomial in k

$$\mathbf{D}(\overset{2}{x}, \overset{1}{D_x}) \qquad \leftrightarrow \qquad d(x,k) = \mathbf{D}(x,k) \tag{3.15}$$

Comparing the action of the operator in terms of its kernel (3.1) and its symbol (3.14) one has

$$\begin{aligned} \int dx'\, \mathcal{D}(x,x')\psi(x') &= \int \frac{dk}{2\pi} d(x,k)e^{ikx}\hat{\psi}(k) \\ &= \int \frac{dk}{2\pi} dx'\, d(x,k)e^{ikx}e^{-ikx'}\psi(x') \end{aligned} \tag{3.16}$$

which implies

$$\mathcal{D}(x,x') = \int \frac{dk}{2\pi} d(x,k)e^{ik(x-x')} \tag{3.17}$$

This relation just involves a Fourier transform which is assumed to be invertible to give

$$d(x,k) = \int ds\, \mathcal{D}(x,x-s)e^{-iks} \tag{3.18}$$

These expressions show that the symbol $d(x,k)$ may be obtained directly from the kernel $\mathcal{D}(x,x')$ by a Fourier transform on just the second argument of the two-point function. For future reference, we also note that this is the same transform used to define the local dispersion function (2.11) in Section 2. Taken together, we shall use these formulas to extend the definition of the symbol for any operator (*i.e.*, not just operators whose x-space representation is of the form (3.9)) in terms of its x-space kernel representation. Since we shall be introducing other definitions of symbols in the following Sections, we shall refer to this particular object as the *common symbol* as it is the symbol most commonly found in the mathematical literature. Being a joint function of (x,k) we shall use the term *symbol* and *phase-space representation* interchangeably.

From (3.9) one concludes that a partial differential operator is represented by a kernel of the form

$$\begin{aligned} \mathbf{D} \quad &\leftrightarrow \quad \mathbf{D}(\overset{2}{x},\overset{1}{D}_x) = \sum_m^M d_m(x) D_x^m \\ &\Rightarrow \quad \mathcal{D}(x,x') = \sum_m^M d_m(x) D_x^m \delta(x-x') \end{aligned} \tag{3.19}$$

When this is inserted into (3.18) the polynomial of (3.14) is again obtained. The extension of the definition (3.18) to operators with kernels that are not of the simple form of (3.19) implies the construction of symbols which are not polynomial in k. Such an operator cannot be written down in the x-representation in the familiar form (3.9) of a differential operator even though its action may be defined in terms of its kernel (or symbol). We call this more general type of operator a *pseudodifferential operator.*

Much of the discussion in the mathematical references concentrates on the meaning of integrals like those in (3.17,3.18), especially in regard to their convergence. The behavior of the differentiability properties of symbols (with respect to both arguments) is examined

and in this way pseudodifferential operators are classified according to the form of their symbols. While such investigations are obviously necessary, they are beyond the scope of this presentation. For the remainder of this Report we shall assume that the requirements of convergence are met (or can be dealt with) when the formalism is applied to wave operators of interest.

As an illustration of the preceding development, consider the tensor kernel $\underset{\sim}{\mathcal{D}}$ associated with wave propagation governed by Maxwell's equations in a plasma with a conductivity tensor $\underset{\sim}{\sigma}$ linearly relating the wave electric field to the wave current density:

$$\begin{aligned}\underset{\sim}{\mathcal{D}}(\mathbf{x},t;\mathbf{x}',t') = &\left[\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\underset{\sim}{I} + \nabla\nabla\right]\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \\ &+ \frac{4\pi}{c^2}\frac{\partial}{\partial t}\underset{\sim}{\sigma}(\mathbf{x},t;\mathbf{x}',t')\end{aligned} \tag{3.20}$$

The first part of this kernel (due to the vacuum Maxwell equations) evidently corresponds to a partial differential operator. In order to see what type of operator the conductivity kernel $\underset{\sim}{\sigma}$ represents, we initially assume a uniform and stationary plasma so that this two-point function depends only on the space-time separation. In this case, the integral (3.18) which produces the symbol from the kernel reduces to the usual Fourier transform; therefore, the symbol associated with (3.20) is the uniform plasma dispersion tensor

$$\begin{aligned}\underset{\sim}{d}(\mathbf{x},t,\mathbf{k},\omega) = \hat{\underset{\sim}{\mathcal{D}}}(\mathbf{k},\omega) &= (k^2\underset{\sim}{I} - \mathbf{k}\mathbf{k}) - \frac{\omega^2}{c^2}\left(\underset{\sim}{I} + \frac{4\pi i}{\omega}\hat{\underset{\sim}{\sigma}}(\mathbf{k},\omega)\right) \\ &\equiv (k^2\underset{\sim}{I} - \mathbf{k}\mathbf{k}) - \frac{\omega^2}{c^2}\hat{\underset{\sim}{\epsilon}}(\mathbf{k},\omega)\end{aligned} \tag{3.21}$$

As expected, the partial differential operator piece of (3.20) is transformed into the $(\mathbf{k},\omega)$-polynomial piece of (3.21). Focusing on just the longitudinal component of this expression for example, one has the scalar dielectric function

$$-\frac{c^2}{\omega^2}\hat{\mathbf{k}}\cdot\underset{\sim}{d}(\mathbf{k},\omega)\cdot\hat{\mathbf{k}} = \hat{\mathbf{k}}\cdot\hat{\underset{\sim}{\epsilon}}(\mathbf{k},\omega)\cdot\hat{\mathbf{k}} \equiv \hat{\epsilon}(\mathbf{k},\omega) \tag{3.22}$$

In the Vlasov model of an unmagnetized plasma in thermal equilibrium the dielectric function $\hat{\epsilon}$ has the form [35]

$$\hat{\epsilon}(\mathbf{k},\omega) = 1 + \sum_{s=species} \frac{1}{k^2\lambda_s^2} Z\left(\frac{\omega}{kv_s}\right) \tag{3.23}$$

where $\lambda_s^2 \equiv T_s/4\pi n_s q_s$ and $v_s^2 \equiv T_s/m_s$ are the Debye length and thermal velocity respectively of species s in terms of species temperature T_s, density n_s, charge q_s and mass m_s. Here the dependence on $(\mathbf{k},\omega)$ through the plasma dispersion function Z is much more complicated than just polynomial; this implies that even in uniform plasma the wave operator $\mathbf{D}$ is generally a pseudodifferential operator.

Naturally, in a nonuniform plasma the integral in the definition of the symbol (3.18) does not reduce to the usual Fourier transform so that without a more specific model for the two-point conductivity kernel the symbol cannot be computed. However, one might expect that if the plasma is only weakly nonuniform then the wave operator $\mathbf{D}$ would be only slightly modified and that the symbol of the longitudinal component, for example, would be similar in form to (3.23). In fact, this is often the method used to obtain the local plasma dispersion tensor; allowing for weak spatial dependence in the density and temperature of a species produces a slowly varying Debye length λ_s and thermal velocity v_s so that in this way the dielectric function becomes a function of both $(\mathbf{k},\omega)$ and $(\mathbf{x},t)$. Evidently, the symbol $\epsilon(\mathbf{x},t,\mathbf{k},\omega)$ remains non-$(\mathbf{k},\omega)$-polynomial in this approximation, supporting the premise that the wave operator $\mathbf{D}$ is in general a pseudodifferential operator in nonuniform plasma.

3.2 The Symbol Calculus: Phase-Space Equations

Having defined the concept of the symbol of an operator, the remainder of this Section

will be devoted to the derivation of an equation governing the phase-space representation of the wave field. This is most directly accomplished by beginning with the representation-free abstract operator expression of the wave equation (3.2). Now whereas $\mathbf{D}$ is an operator with a well-defined symbol, the wave field ψ and source term F are not operators. In order to construct an operator associated with the field, we multiply (3.2) on both sides by the dual element $\psi^\dagger$ of the field

$$\mathbf{D}\psi\psi^\dagger = F\psi^\dagger \tag{3.24}$$

In Dirac notation, this equation would be written

$$\mathbf{D}|\psi\rangle\langle\psi| = |F\rangle\langle\psi| \tag{3.25}$$

Now consider the adjoint of (3.2):

$$\begin{aligned} &\psi^\dagger\mathbf{D}^\dagger = F^\dagger \\ \text{or} \qquad &\langle\psi|\mathbf{D}^\dagger = \langle F| \end{aligned} \tag{3.26}$$

Finally, assuming $\mathbf{D}^\dagger$ is invertible and substituting (3.26) into (3.25) one obtains

$$\begin{aligned} &\mathbf{D}(\psi\psi^\dagger) = (FF^\dagger)(\mathbf{D}^\dagger)^{-1} \\ \text{or} \qquad &\mathbf{D}|\psi\rangle\langle\psi| = |F\rangle\langle F|(\mathbf{D}^\dagger)^{-1} \end{aligned} \tag{3.27}$$

This equation now involves only operators. The operators representing the wave field ψ and the source F are the bilinear quantities

$$\begin{aligned} \mathbf{S} &\equiv (\psi\psi^\dagger) \qquad (\text{ or } |\psi\rangle\langle\psi|) \\ \mathbf{N} &\equiv (FF^\dagger) \qquad (\text{ or } |F\rangle\langle F|) \end{aligned} \tag{3.28}$$

In quantum mechanics, $\mathbf{S}$ is the projection or density operator, but in a more general classical wave context it will be convenient to call $\mathbf{S}$ the *correlation* or *spectral* operator of the field; similarly, $\mathbf{N}$ is the operator associated with the correlation of the source. This terminology can be understood by considering the x-space "kernel" representation of the

operator **S** and its symbol. In Dirac notation and using the definitions (3.6) one has simply

$$S(x,x') \equiv \langle x|\psi\rangle\langle\psi|x'\rangle \equiv \psi(x)\psi^*(x') \tag{3.29}$$

in terms of which the symbol is computed by (3.18) to be

$$\begin{aligned} s(x,k) &\equiv \int ds\, S(x,x-s)e^{-iks} \\ &= \psi(x)e^{-ikx}\int dx'\,\psi^*(x')e^{ikx'} \\ &= \psi(x)\hat{\psi}^*(k)e^{-ikx} \end{aligned} \tag{3.30}$$

The connotation "correlation operator" is due to the result (3.29) for the x-space description of $(\psi\psi^\dagger)$; the autocorrelation function of the field is defined as an average (usually an ensemble average) of this expression. Thus, since the symbol of the field is proportional to the Fourier transform of the "unaveraged" correlation function, it may be interpreted as an "unaveraged" local spectral function. In addition, as the form in (3.30) is the product of the x and k representations, it is almost what one might expect for a phase-space representation. In fact, apart from the multiplicative phase factor, this definition is just the kernel in a *mixed representation* $\langle x|\psi\rangle\langle\psi|k\rangle$. Similar relations are obtained for any field (such as F) and these are easily extended to many dimensions and vector fields.

The basic representation-free equation (3.2) has now been manipulated into equation (3.27) which involves only operators. Each side of (3.27) is an operator (being the product, or composition, of two operators) so that the phase-space representation of this equation is simply

$$[\mathbf{DS}](x,k) = [\mathbf{N}(\mathbf{D}^\dagger)^{-1}](x,k) \tag{3.31}$$

The equality of the two operators implies the equality of their symbols. In order to derive an equation for the field correlation symbol $s(x,k)$, however, one must determine the rule

for expressing the symbol of the product of two operators in terms of the symbols of the individual operators. This rule is an element of the *symbol calculus*; the calculation requires only a short digression and will be instructive of the manipulations involved in many of the other derivations to follow.

Consider the product of two operators in terms of their kernels

$$\begin{aligned} \mathbf{C} &\equiv \mathbf{AB} \\ C(x,y) &= \int dx'\, \mathcal{A}(x,x')\mathcal{B}(x',y) \end{aligned} \tag{3.32}$$

The symbol of **C** is by (3.18)

$$\begin{aligned} c(x,k) &\equiv \int ds\, C(x,x-s)e^{-iks} \\ &= \int ds\, dx'\, \mathcal{A}(x,x')\mathcal{B}(x',x-s)e^{-iks} \end{aligned} \tag{3.33}$$

Now, using (3.17) to express the kernels $\mathcal{A}$ and $\mathcal{B}$ in terms of their symbols, (3.33) becomes

$$\begin{aligned} c(x,k) = \int ds\, dx' \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\, e^{ik_1(x-x')}e^{ik_2(x'-x+s)}e^{-iks} \\ \times a(x,k_1)b(x',k_2) \end{aligned} \tag{3.34}$$

The integrals over s and k_2 yield immediately

$$\begin{aligned} c(x,k) &= [\mathbf{AB}](x,k) \\ &= \int dx' \frac{dk'}{2\pi}\, e^{-i(k'-k)(x'-x)}a(x,k')b(x',k) \end{aligned} \tag{3.35}$$

Thus, the symbol of the product of two operators at the point (x,k) in phase space is not simply given by the product of the symbols at that point. Rather, the symbol of the product is given by this generalization of the convolution rule to a nonuniform medium; in a uniform medium, the symbols a, b and c would be functions of k only, and this integral would reduce to the product $\hat{a}(k)\hat{b}(k)$.

The relation (3.35) may be cast in a more compact and more useful form as follows: change variables in the integral to $x' = x + s$,

$$c(x,k) = \int ds \frac{dk'}{2\pi}\, e^{-i(k'-k)s} a(x,k') b(x+s,k) \tag{3.36}$$

and expand the symbol b in a Taylor series around x

$$c(x,k) = \int ds \frac{dk'}{2\pi}\, e^{-i(k'-k)s} a(x,k') e^{s\partial_x} b(x,k) \tag{3.37}$$

The exponential operator here is a shorthand way of writing the Taylor expansion; it is to be interpreted in terms of its power series and questions of convergence are ignored for simplicity. Now that $b(x,k)$ is independent of the integration variables, it may be taken outside of the integral to the right; the integral is now an operator (of x-differentiation) acting on b. The relation

$$e^{-i(k'-k)s} a(x,k') e^{s\partial_x} = e^{-i(k'-k)s} a(x,k') e^{-i\overleftarrow{\partial_k}\partial_x} \tag{3.38}$$

holds since a is independent of k, and it also may be verified by power series expansion. The left-pointing arrow above the k-derivative indicates that it operates on all functions of k (not k') standing to the left (*i.e.*, not on b). With these steps, (3.37) becomes

$$c(x,k) = \left[\int ds \frac{dk'}{2\pi}\, e^{-i(k'-k)s} a(x,k') \right] e^{-i\overleftarrow{\partial_k}\overrightarrow{\partial_x}} b(x,k) \tag{3.39}$$

The integrations are now trivial and one finally has

$$[\mathbf{AB}](x,k) = a(x,k) e^{-i\overleftarrow{\partial_k}\overrightarrow{\partial_x}} b(x,k) \tag{3.40}$$

This is the desired compact form of (3.35); it is to be interpreted in terms of that integral form or its power series:

$$a(x,k) e^{-i\overleftarrow{\partial_k}\overrightarrow{\partial_x}} b(x,k) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \frac{\partial^n a(x,k)}{\partial k^n} \frac{\partial^n b(x,k)}{\partial x^n} \tag{3.41}$$

From this it is apparent that if a is the symbol of a partial differential operator (*i.e.*, it is an Nth degree polynomial in k), then this is a finite series of x-differentiation on b (of order N). Furthermore, the earlier result for a uniform medium is recovered in that if the

symbols are independent of position (specifically, b not a function of x), then the symbol of the product is the product of the symbols (only the first term of the power series = 1 contributes).

Returning to the wave operator equation (3.27) and its symbol (3.31), one may immediately apply the rule (3.40) to obtain

$$d(x,k)e^{-i\overleftarrow{\partial_k}\overrightarrow{\partial_x}}s(x,k) = n(x,k)e^{-i\overleftarrow{\partial_k}\overrightarrow{\partial_x}}(d^\dagger)^{-1}(x,k) \tag{3.42}$$

This is the desired phase-space equation for the symbol of the field correlation operator $s(x,k)$. The corresponding expression for vector fields and tensor operators in three dimensions and time is

$$\underset{\sim}{d}(\mathbf{x},t,\mathbf{k},\omega)e^{-i(\overleftarrow{\partial}_{\mathbf{k}}\cdot\overrightarrow{\partial}_{\mathbf{x}}-\overleftarrow{\partial}_{\omega}\overrightarrow{\partial}_{t})}\cdot\underset{\sim}{s}(\mathbf{x},t,\mathbf{k},\omega) = \underset{\sim}{n}(\mathbf{x},t,\mathbf{k},\omega)e^{-i(\overleftarrow{\partial}_{\mathbf{k}}\cdot\overrightarrow{\partial}_{\mathbf{x}}-\overleftarrow{\partial}_{\omega}\overrightarrow{\partial}_{t})}\cdot(\underset{\sim}{d}^\dagger)^{-1}(\mathbf{x},t,\mathbf{k},\omega) \tag{3.43}$$

The rather complicated structure of this tensor equation may be somewhat clarified if expressed in explicit component form:

$$\sum_{m=0}^{\infty}\sum_{i=1}^{4}\sum_{\nu=1}^{3}\frac{(-i)^m}{m!}\frac{\partial^m d_{\mu\nu}}{\partial k_i^m}\frac{\partial^m s_{\nu\sigma}}{\partial x_i^m} = \sum_{m=0}^{\infty}\sum_{i=1}^{4}\sum_{\nu=1}^{3}\frac{(-i)^m}{m!}\frac{\partial^m n_{\mu\nu}}{\partial k_i^m}\frac{\partial^m (d^\dagger)^{-1}_{\nu\sigma}}{\partial x_i^m} \tag{3.44}$$

As stated before, the four-dimensional notation requires $(x_4,k_4) = (t,\omega)$ but $(\partial_{x_4},\partial_{k_4}) = (-\partial_t,\partial_\omega)$. In addition, it must be remembered that this is an expression relating two tensors and must be satisfied for all components $(\mu\sigma)$.

Let us recapitulate the meaning of this equation. It is a (possibly infinite order) partial differential equation (or integral equation, *cf.* Eq.(3.35)) for the local spectral tensor $\underset{\sim}{s}$ defined in (3.30) (or its analogous vector form). The left-hand side involves

$(\mathbf{k},\omega)$-derivatives of the local dispersion tensor $\underset{\sim}{d}(\mathbf{x},t,\mathbf{k},\omega)$, defined in the four-dimensional, tensor form of (3.18), and $(\mathbf{x},t)$-derivatives of $\underset{\sim}{s}(\mathbf{x},t,\mathbf{k},\omega)$. It is important to remember that in this formalism $(\mathbf{x},t,\mathbf{k},\omega)$ are all independent variables. If the dispersion tensor is only a polynomial in $(\mathbf{k},\omega)$ (*cf.* Eq.(3.14)) then the left-hand side reduces to a finite order differential action on $\underset{\sim}{s}$.

The right-hand side can be viewed in two ways, depending upon the meaning of the source field $\mathbf{F}$. Taken as a specified source field it simply represents an inhomogeneous term in an otherwise linear equation for $\underset{\sim}{s}$. In this case, the right hand side is a function on phase space composed of $(\mathbf{k},\omega)$ derivatives of the source spectral tensor and space-time derivatives of the inverse adjoint dispersion tensor (the calculation of the symbol $(\underset{\sim}{d}^\dagger)^{-1}$ in terms of $\underset{\sim}{d}$ will be discussed shortly). Thus, the entire right-hand side is a known source for the left-hand side at each point in phase space.

The possibility exists, however, (due to the form of the basic equation (3.1)) that this term may be extended to include nonlinear sources for which $\mathbf{F}$ could be considered a functional of $\boldsymbol{\psi}$. Of course, this might have significant implications for the nature of the abstract vector space in which the operator equation (3.2) is to be viewed, but it seems as if this circumstance could be treated from the standpoint of either (3.1) or (3.43). In this case, then, the right hand side would contain $(\mathbf{k},\omega)$-derivatives of nonlinear terms in $\underset{\sim}{s}$ as well as space-time derivatives of the given tensor $(\underset{\sim}{d}^\dagger)^{-1}$.

Let us now proceed to compute the symbol $d^\dagger$ of the adjoint operator $\mathbf{D}^\dagger$ whose inverse is required for the phase-space equation (3.42). The rule for computing $d^\dagger$ from the symbol d is also an element of the symbol calculus and may be obtained as follows: Beginning with the definition of $d^\dagger$ in terms of the adjoint kernel from (3.18)

$$d^\dagger(x,k) \equiv \int ds\, \mathcal{D}^\dagger(x, x-s) e^{-iks} \tag{3.45}$$

and the definition of the kernel of the adjoint operator

$$\mathcal{D}^\dagger(x,x') \equiv \mathcal{D}^*(x',x) \tag{3.46}$$

one has

$$d^\dagger(x,k) = \int ds\, \mathcal{D}^*(x-s,x) e^{-iks} \tag{3.47}$$

Note from (3.46) that for a partial differential operator which can be written in the form (3.9) with (3.19), the adjoint differential operator is $\mathbf{D}(\overset{1}{x}, \overset{2}{D}_x)$ where the multiplication by the coefficients is performed before the differentiation. Now with the inverse relation (3.17),

$$\mathcal{D}^*(x-s,x) = \int \frac{dk'}{2\pi} d^*(x-s,k') e^{ik's} \tag{3.48}$$

Eq.(3.47) becomes

$$\begin{aligned} d^\dagger(x,k) &= \int ds \frac{dk'}{2\pi} d^*(x-s,k') e^{-iks} e^{ik's} \\ &= \int \frac{dk'}{2\pi} ds\, e^{i(k'-k)s} e^{-s\partial_x} d^*(x,k') \\ &= \int \frac{dk'}{2\pi} ds\, e^{i(k'-k)s} e^{-i\overleftarrow{\partial}_k \overrightarrow{\partial}_x} d^*(x,k') \end{aligned} \tag{3.49}$$

where the intermediate steps are similar to those used in deriving the product rule (3.36–3.39). The integrals are easily performed to give

$$\begin{aligned} d^\dagger(x,k) &= \int \frac{dk'}{2\pi} \delta(k'-k) e^{-i\overleftarrow{\partial}_k \overrightarrow{\partial}_x} d^*(x,k') \\ &= e^{-i\overrightarrow{\partial}_k \overrightarrow{\partial}_x} \int \frac{dk'}{2\pi} \delta(k'-k) d^*(x,k') \end{aligned} \tag{3.50}$$

$$d^\dagger(x,k) = e^{-i\overrightarrow{\partial}_k \overrightarrow{\partial}_x} d^*(x,k)$$

Here the exponential operator has been moved to the left of the integral (and the arrow on the k-differentiation accordingly reversed) since the only k dependence is in the δ-function.

The result (3.50) has several implications. First, it is apparent that the symbol $d^\dagger$ of the adjoint operator $\mathbf{D}^\dagger$ is not simply the adjoint of the symbol d of $\mathbf{D}$. Thus, for scalar symbols one does not have in general $d^\dagger = d^*$; for tensor symbols the relation analogous to (3.50) is

$$d^\dagger_{\mu\nu}(\mathbf{x}, t, \mathbf{k}, \omega) = e^{-i(\overrightarrow{\partial}_{\mathbf{k}} \cdot \overrightarrow{\partial}_{\mathbf{x}} - \overrightarrow{\partial}_{\omega} \overrightarrow{\partial}_{t})} d^*_{\nu\mu}(\mathbf{x}, t, \mathbf{k}, \omega) \tag{3.51}$$

so that $d^\dagger_{\mu\nu} \neq d^*_{\nu\mu}$. These observations are true unless the symbol d contains no products of conjugate variables $x_i k_i$, $(i = 1, 4)$. Furthermore, it follows that if the operator $\mathbf{D}$ is self-adjoint then (3.50,3.51) imply that its symbol is *not* self-adjoint if it contains xk products. In the case of a self-adjoint operator, however, one can derive a relationship between the self-adjoint and anti-self-adjoint parts of the symbol as follows. Considering for simplicity scalar symbols, we use (2.14) to write d in terms of its real and imaginary parts (d' and d''); the self-adjoint condition $\mathbf{D} = \mathbf{D}^\dagger$ then implies the phase-space equation

$$\begin{aligned} d(x,k) = d^\dagger &= e^{-i\overrightarrow{\partial_k}\overrightarrow{\partial_x}} d^*(x,k) \\ d' + id'' &= e^{-i\overrightarrow{\partial_k}\overrightarrow{\partial_x}} (d' - id'') \end{aligned} \tag{3.52}$$

Expanding the exponential operator and equating real and imaginary parts of the resulting expression gives

$$\begin{aligned} d' &= d' - \partial_k \partial_x d'' - \tfrac{1}{2!}\partial_k^2 \partial_x^2 d' + \tfrac{1}{3!}\partial_k^3 \partial_x^3 d'' + \cdots \\ d'' &= -d'' - \partial_k \partial_x d' + \tfrac{1}{2!}\partial_k^2 \partial_x^2 d'' + \tfrac{1}{3!}\partial_k^3 \partial_x^3 d' + \cdots \end{aligned} \tag{3.53}$$

To lowest order in derivatives with respect to x and k, one has the relation $2d'' = -\partial_k \partial_x d'$ which was used in (2.29) to derive the action conservation equation for conventional eikonal theory.

Finally, the symbol d^{-1} of the inverse operator $\mathbf{D}^{-1}$ which satisfies

$$\mathbf{D}\mathbf{D}^{-1} = \mathbf{D}^{-1}\mathbf{D} = \mathbf{I} \tag{3.54}$$

is computed from the phase-space representation of (3.54)

$$d(x,k)e^{-i\overleftarrow{\partial_k}\overrightarrow{\partial_x}}d^{-1}(x,k) = d^{-1}(x,k)e^{-i\overleftarrow{\partial_k}\overrightarrow{\partial_x}}d(x,k) = 1 \tag{3.55}$$

where the identity operator $\mathbf{I}$ has the kernel $\delta(x-x')$ and symbol $i(x,k) = 1$. To lowest order in derivatives of d, both conditions in (3.55) simply imply $d^{-1} = 1/d$; indeed, this result also satisfies the next order equation found by subtracting the two conditions: $(\partial_k d)(\partial_x d^{-1}) - (\partial_k d^{-1})(\partial_x d) = 0$.

3.3 Asymptotic Analysis

The point of introducing the symbol of the wave field (3.29) and thus the phase-space representation of the wave equation (3.27) is to determine whether any useful results can be obtained by analyzing this phase-space equation in the short-wavelength or asymptotic regime. We therefore consider (3.42) and, in order to compare our results with traditional eikonal theory, we set the right-hand side to zero (*i.e.*, we do not consider sources or nonlinearities)

$$d(x,k)e^{-i\overleftarrow{\partial_k}\overrightarrow{\partial_x}}s(x,k) = 0 \tag{3.56}$$

In the conventional theory, one is led by physical intuition to assume a solution of the form (2.5) of a local plane wave in x-space. Here, we can use that assumption to guide us to an appropriate asymptotic form for the field spectral function $s(x,k)$ to use in the solution of (3.56). This is quite easy since (3.30) shows that $s(x,k)$ is just proportional to the product of the x- and k-space representations of ψ, while the asymptotic forms of each of these is given by (2.5) and (2.33-2.35). Recalling that in this phase-space representation x and k are to be treated as independent variables, we take

$$s(x,k) = A(x)\hat{A}(k)e^{i(\phi(x)-\hat{\phi}(k)-ikx)} \tag{3.57}$$

This is a particularly convenient assumption in that (3.56) contains only derivatives of s with respect to x; therefore, the k-dependence of s is not determined by (3.56) and the k-dependent factors of (3.57) divide out. Of course, the k-dependence of s is not arbitrary; in this case, one must also consider the adjoint of (3.56) (corresponding to the adjoint of (3.27) with $F = 0$) which will determine the k-dependence in the form of (3.57) when treated in a manner similar to the following analysis.

Substituting (3.57) into (3.56), expanding the exponentiated operator and splitting d into its real and imaginary parts we have

$$(d' + id'')[1 - i\overleftrightarrow{\partial_k}\overrightarrow{\partial_x} - \tfrac{1}{2!}(\overleftrightarrow{\partial_k}\overrightarrow{\partial_x})^2 + \cdots]A(x)e^{i\phi(x)-ikx} = 0 \tag{3.58}$$

Now as in the conventional theory, we define the local wavenumber $k(x) \equiv \partial_x\phi$ and adopt the ordering assumptions of (2.7,2.8,2.15,2.16) to produce the two lowest-order equations

$$\begin{aligned} O(1): \quad & [d' + (\partial_k d')(k(x) - k) + \tfrac{1}{2}(\partial_k^2 d')(k(x) - k)^2 \\ & + \tfrac{1}{3!}(\partial_k^3 d')(k(x) - k)^3 + \cdots]Ae^{i\phi - ikx} = 0 \end{aligned} \tag{3.59}$$

and

$$\begin{aligned} O(\epsilon): \quad & [(\partial_k d') + (\partial_k^2 d')(k(x) - k) + \tfrac{1}{2}(\partial_k^3 d')(k(x) - k)^2 + \cdots](\partial_x A) \\ & + \tfrac{1}{2}[(\partial_k^2 d') + (\partial_k^3 d')(k(x) - k) + \cdots](\partial_x k)A \\ & - [d'' + (\partial_k d'')(k(x) - k) + \tfrac{1}{2}(\partial_k^2 d'')(k(x) - k)^2 + \cdots]A = 0 \end{aligned} \tag{3.60}$$

The terms in brackets in both of these equations are just those that appear through the third derivative expansion of the exponentiated operator; it is clear, however, that additional terms from higher-order derivatives in the expansion will continue to contribute to the expressions in the brackets. These continuing series have the apparent pattern of a Taylor series about the point k of the various quantities appearing as the first term in each bracket. Assuming that the sums converge (which is the same as assuming that it is indeed justified to expand (3.56) in the first place), the two equations become

$$O(1): \qquad d'(x,k(x)) = d'(x,\partial_x\phi) = 0 \tag{3.61}$$

and

$$O(\varepsilon): \qquad (\partial_k d')(x,k(x))(\partial_x A) + \tfrac{1}{2}(\partial_k^2 d')(x,k(x))(\partial_x k)A = 2d''(x,k(x))A \tag{3.62}$$

With the straightforward extension of these equations to include time we recognize them to be precisely the eikonal and amplitude transport equations derived in conventional eikonal theory (*cf.* Eqs.(2.17,2.18)). Analysis of the adjoint of (3.56) in a similar fashion leads to the the eikonal and amplitude equations for the k-space representation of the wave. Thus, although the present phase-space eikonal method initially considered x and k to be independent variables, the action of the exponentiated operator in (3.56) focuses attention on the ray manifold determined by $k = k(x)$ (or in many dimensions and time, $\mathbf{k} = \mathbf{k}(\mathbf{x},t), \omega = \omega(\mathbf{x},t)$). Since the discussion of the drawbacks (such as caustic singularities) given in Section 2 follows directly from the analysis of these two equations, we conclude that this method offers no improvement over the conventional eikonal method as it reduces to the same theory.

3.4 Discussion

The primary advantage of this particular symbol formalism is the natural way in which it extends the familiar differential operator to the concept of a pseudodifferential operator. Furthermore, the computations required to obtain the rules for translating operations on abstract operators into corresponding symbol operations (the symbol calculus) are perhaps the simplest with regard to alternative representations. Again, possibly for these reasons, this type of symbol is perhaps the one most often studied in the mathematical literature. As we have pointed out, however, there are at least two reasons for not persuing this representation (as defined by (3.18)) for the purpose of developing a useful phase-space

eikonal method:

1) The asymptotic analysis of the exact evolution equation (3.42) for the spectral function $s(x,k)$ (in the absence of sources) has shown that caustic singularities will still plague the construction of the field (by (3.62)). This result could have been anticipated from the expression for $s(x,k)$ (3.30) when the wave field has the local plane wave form (2.5) in x-space and the corresponding form (2.33-2.35) in k-space

$$s(x,k) \sim A(x)A(\eta(k))\, e^{i(\phi(x)-\phi(\eta(k))+k\eta(k)-kx} \tag{3.63}$$

where $\eta(k)$ is the inversion of the stationary phase condition $\partial_x\phi(\eta) = k$. As pointed out by Berry [14], this result is undesirable because it has singularities in phase space not only at all points (x,k) where the asymptotic form of $A(x)$ becomes large (x-space caustics), but also at all points where $A(\eta(k))$ is singular (k-space caustics). This is illustrated in Fig. 2.

For the purposes of a physically meaningful phase-space representation of a wave field, one would intuitively desire a function on phase space which "adheres" to the rays (*i.e.*, one that has large amplitude on and near the ray manifold in phase space and small amplitude off of it). As one can see from Fig. 2 however, the region in phase space where the asymptotic form of the spectral function in this particular symbol formalism has the largest amplitude (indeed singular) is near these linear extensions of the x- and k-space caustics into phase space, while it is much smaller on the remainder of the ray manifold.

2) The expression (3.50) relating the symbol of an operator to the symbol for the adjoint operator shows that the symbol of a scalar Hermitian operator has an unexpected imaginary part (if the symbol contains conjugate products). This could have serious consequences in many applications to classical wave propagation; as the discussion of the

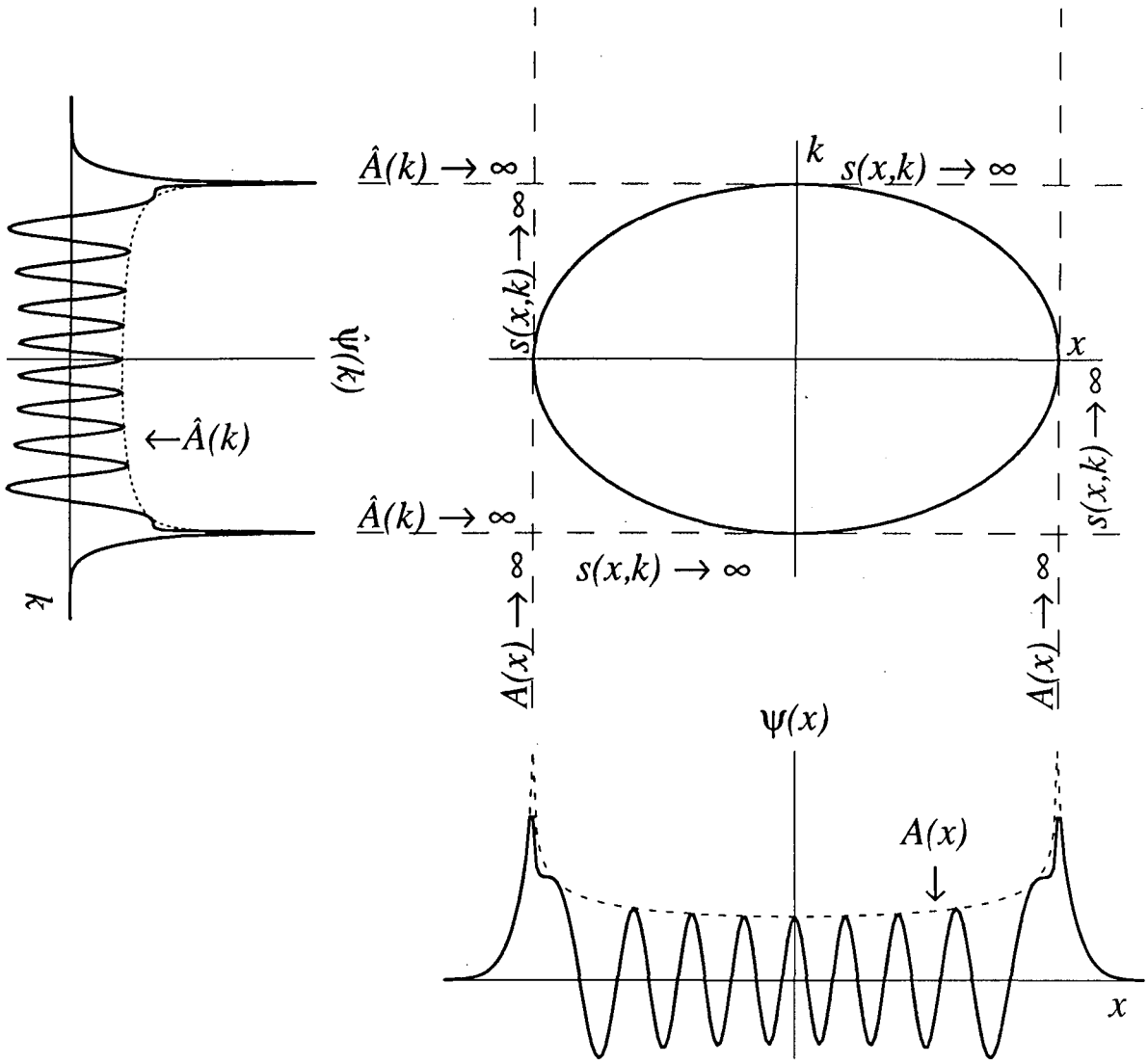

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Fig. 2. Schematic illustration of the singularities in phase space characterizing the common symbol of the spectral tensor $s(x,k)$ when it is evaluated in the geometrical optics approximation. These singularities are simply the linear extension into phase space of the singularities present in both $A(x)$ and $\hat{A}(k)$ due to the caustic singularities in the projection of the ray manifold onto x- or k-space.

example (3.20–3.23) indicates, the local dispersion tensor as well as other phase-space functions of interest in general do involve xk products. This is particularly true in the case of the symbol representing the wave field, where the underlying kernel or correlation function (3.29) is manifestly Hermitian while the spectral function $s(x,k)$ constructed in (3.30) is not purely real. While many other operators are self-adjoint (where relationships like (3.52,3.53) hold), the situation becomes complicated when one considers wave operators which include dissipation mechanisms. In this case one must be careful to distinguish the component of the imaginary part of the dispersion function which actually represents the dissipation from the spurious intrinsic piece that arises from the presence of conjugate products. Intuitively, one would desire a representation in which symbols more faithfully reflect this physically important property of their corresponding operators.

4. Weyl symbols

The main disadvantages of the symbol formalism presented in the previous Section concerned the physical issues of singularities of the spectral function in phase space due to caustics and the somewhat complicated relation between the Hermitian and anti-Hermitian parts of an operator and its corresponding symbol. One also might note from (3.18) that the transformation from the x-space kernel $\mathcal{D}$ to the symbol d involves an integral in which only the second argument of $\mathcal{D}$ varies. Furthermore, the phase-space equation (3.42) appears to treat x and k in an unsymmetric way; that is, x-derivatives act only to the right and k-derivatives act only to the left. In this Section we shall introduce a phase-space representation which not only treats the x-space arguments of the kernel in the defining transformation in a more symmetrical way, but which results in a symbol calculus that treats the phase-space variables x and k on an equal footing. While the extension from partial to pseudodifferential operator is not as clear as in the case of the common symbol of Section 3, we shall see that this new representation corrects the deficiencies of the common symbol formalism and to some extent provides a closer connection in the short-wavelength regime between the evolution of the wave spectral function and the corresponding rays in phase space.

4.1 Properties of the Weyl Symbol

We shall refer to the symmetrized symbol defined by

$$D(x,k) \equiv \int ds\, \mathcal{D}(x+\tfrac{1}{2}s, x-\tfrac{1}{2}s)e^{-iks} \tag{4.1}$$

as the *Weyl symbol* [12] of the operator $\mathbf{D}$. Like the definition of the common symbol (3.18), this expression is a type of Fourier transform on the separation s of the two arguments in

the x-space kernel D; unlike (3.18), however, the transform is "centered" around the point x and involves both arguments. The inverse of (4.1) may be verified to be

$$D(x,x') = \int \frac{dk}{2\pi}\, D(\tfrac{1}{2}(x+x'),k)e^{ik(x-x')} \tag{4.2}$$

As these transformations have the obvious extension to more than one dimension and time, we shall continue to use the one-dimensional notation in the development of the Weyl calculus unless specifically noted. This representation has received somewhat less attention in the mathematical literature, possibly due to its tenuous connection to the theory of pseudodifferential operators [6]. Because of its nice properties, however, it has been used by many authors [13-18] to define a correspondence between phase-space functions and quantum mechanical operators and to study the relationship between classical and quantum chaos. In this regard, the Weyl symbol of the field spectral operator (or density matrix) $\psi\psi^\dagger$

$$S(x,k) = \int ds\, \psi(x+\tfrac{1}{2}s)\psi^*(x-\tfrac{1}{2}s)e^{-iks} \tag{4.3}$$

is also known as the *Wigner function* [13].

Whereas the expression (3.30) for the common symbol of a field ψ was rather transparent for the purpose of examining its structure in phase space, the definition (4.3) is not so intuitive. Therefore, as the first of a few examples of what might be expected from this construction, let us consider the simplest case of the spectral function of a plane wave:

$$\begin{aligned} &\psi(x) \equiv A_0 e^{ik_0 x} \\ \Rightarrow \qquad &S(x,k) = 2\pi\,|A_0|^2\,\delta(k-k_0) \end{aligned} \tag{4.4}$$

This singular distribution is concentrated on the appropriate surface in phase space to describe the uniform wavenumber of the wave and is independent of x.

A generalization of this is the Wigner function associated with a wave of the eikonal form (2.5)

$$\begin{aligned} \psi(x) &\equiv A(x)e^{i\phi(x)} \\ \Rightarrow \qquad S(x,k) &= 2\pi \left|A(x)\right|^2 \delta(k-k(x)) \end{aligned} \tag{4.5}$$

Here we have expanded the phases $\phi(x \pm \frac{1}{2}s)$ around the point x with the definition (2.6) of the local wavenumber $k(x) = \partial_x \phi$ and have assumed the amplitude $A(x)$ as well as the derivatives of the phase to be slowly varying compared with the phase variation itself. This also is a singular distribution, although Balasz [36] has shown that the Wigner function can be *exactly* a δ-function only on linear subspaces of phase space (such as in (4.4)); being only an asymptotic form, however, this distribution is concentrated on the more general manifold $k(x)$ generated by the definition of a local wavenumber inherent in the conventional eikonal approach. While this example appears to indicate that the Weyl symbol associated with a short-wavelength wave will also exhibit the singularities that both the conventional x-space eikonal and common symbol representations suffer from (*i.e.*, near caustics where $A(x) \to \infty$), and indeed appears to be even more singular owing to the presence of the δ-function, we observe the following: The singularities in both the conventional eikonal and common symbol descriptions of short-wavelength wave propagation were inherent in the equations governing the phase and amplitude derived in the asymptotic analysis of the wave equation (in fact, it was seen that the common symbol analysis reduced to the same equations found in the conventional eikonal approach). We shall see, however, that these singularities are *not* intrinsic to the equations which govern the Weyl symbol of the wave in the same limit and that a nonsingular distribution on phase space results from the asymptotic solution of those equations. Therefore, even though the spectral function considered in this Section is singular when constructed from the eikonal form (4.5), this is due to the nature of the conventional eikonal solution and not inherent in the definiton (4.3).

As an example of a Weyl spectral function which is not a δ-function, consider an eigenfunction of the Schrödinger equation for a harmonic oscillator [34] with natural frequency ω_0

$$\begin{aligned} \psi_n(x) &= C_n H_n(\alpha x) e^{-\alpha^2 x^2/2} \\ E_n &= \hbar\omega_n = (n+\tfrac{1}{2})\hbar\omega_0 \end{aligned} \tag{4.6}$$

where C_n is a normalization constant, H_n is the nth Hermite polynomial, $\alpha^{-1} = \sqrt{\hbar/m\omega_0}$ is the natural length scale of the quantum oscillator and the E_n are the energy eigenvalues. Substituting (4.6) into (4.3), the integral may be explicitly evaluated to give

$$\begin{aligned} S_n(x,k) &= 2(-1)^n L_n\big(2(\alpha^2 x^2 + k^2/\alpha^2)\big) e^{-\alpha^2 x^2 - k^2/\alpha^2} \\ &= 2(-1)^n L_n\big(4\Omega(x,k)/\omega_0\big) e^{-2\Omega(x,k)/\omega_0} \\ S_n(r) &= 2(-1)^n L_n(2r^2) e^{-r^2} \end{aligned} \tag{4.7}$$

Here, L_n is the nth Laguerre polynomial, $\Omega(x,k)$ is the classical harmonic oscillator dispersion relation (*i.e.*, found by setting (2.12) to zero), and r is the radius in phase space in dimensionless variables $(\alpha x, k/\alpha)$:

$$\begin{aligned} \Omega(x,k) &= \frac{\hbar k^2}{2m} + \frac{m\omega_0^2 x^2}{2\hbar} \\ &= \tfrac{1}{2}(\alpha^2 x^2 + k^2/\alpha^2)\omega_0 \\ &\equiv \tfrac{1}{2}\omega_0 r^2 \end{aligned} \tag{4.8}$$

Setting the nth eigenvalue ω_n from (4.6) equal to the value of the classical frequency Ω, the radius in phase space r_n of the classical dispersion surface $\omega = \omega_n = \Omega(r_n)$ corresponding to the nth state is $r_n = \sqrt{2n+1}$. In these variables, the form of (4.7) shows that the spectral function is azimuthally symmetric and, considering the behavior of the Laguerre polynomials, oscillates rapidly (with increasing n) out to approximately the radius r_n of the classical dispersion surface, and then rapidly decays to zero (due to the exponential).

This example illustrates two significant points. First, the spectral function is an oscillatory function on phase space (as opposed to the previous two examples); indeed, it

may be shown [37] in general that the phase-space representation defined by (4.3) cannot be non-negative for all functions $\psi(x)$. The second point is that (4.7) is not localized around the expected classical surface as was the case in the previous two examples; instead, the spectral function oscillates with a gradually diminishing amplitude from the phase-space origin out to the classical surface ($r = r_n$) where the last oscillation turns out to be slightly wider (but not of significantly higher amplitude) than the others. It has been shown that these oscillations can be (and, for asymptotic use, should be) removed, however, by subjecting the spectral function to a local x-space or phase-space averaging [38,14], thereby producing a positive-definite density on phase space; an example of this will be seen in the next Section. In the present example, a suitable coarse-graining would conceivably remove all of the oscillations except for perhaps the last one (being slightly broader) which is in the correct location to coincide with the classical radius.

4.2 The Weyl Symbol Calculus

Our primary goal is to derive the phase-space equation which governs the spectral function $S(x,k)$ in terms of the Weyl symbol of the wave operator, or dispersion function $D(x,k)$. As in the case of the common symbol of the previous Section, we could proceed from the abstract operator equation (3.27) and derive a product rule for this representation using the relationship between symbol and kernel, but it will be more efficient and instructive in the calculation of this and other elements of the Weyl symbol calculus to exploit an intimate relationship between Weyl symbols and common symbols. This connection is readily obtained upon substitution of the inverse formula (3.17) into the definition (4.1)

$$\begin{aligned} D(x,k) &= \int ds \frac{dk'}{2\pi}\, d(x+\tfrac{1}{2}s,k') e^{i(k'-k)s} \\ &= 2\int dx' \frac{dk'}{2\pi}\, d(x',k') e^{2i(k'-k)(x'-x)} \end{aligned} \tag{4.9}$$

The factor of 2 arises from the change in variables $x' = x + \frac{1}{2}s$ and becomes 2^N in the N-dimensional form of (4.9). This relation is an integral over all phase space similar to the Fourier-type integrals already encountered in the previous Section (*cf.* Eq.(3.35)), the difference being the factor of 2 in the exponent (which remains 2 in N dimensions). It can easily be verified that (4.9) is invertible (assuming all integrals exist) so that one also has

$$d(x,k) = 2\int dx' \frac{dk'}{2\pi}\, D(x',k') e^{-2i(k'-k)(x'-x)} \tag{4.10}$$

It was also seen in Section 3 that phase-space integrals like this may be cast in a compact differential form which sometimes provides insights into the properties of the relationship. Thus, the manipulations of (4.9) proceed

$$\begin{aligned} D(x,k) &= \int ds \frac{dk'}{2\pi}\, e^{i(k'-k)s} e^{\frac{s}{2}\partial_x} d(x,k') \\ &= \int ds \frac{dk'}{2\pi}\, e^{i(k'-k)s} e^{\frac{i}{2}\overleftarrow{\partial_k}\overrightarrow{\partial_x}} d(x,k') \\ &= e^{\frac{i}{2}\overrightarrow{\partial_k}\overrightarrow{\partial_x}} \int ds \frac{dk'}{2\pi}\, e^{i(k'-k)s} d(x,k') \\ D(x,k) &= e^{\frac{i}{2}\overrightarrow{\partial_k}\overrightarrow{\partial_x}} d(x,k) \end{aligned} \tag{4.11}$$

Similar operations on the inverse integral (4.10) reveal that the naive inverse of relations of the form (4.11) also holds

$$d(x,k) = e^{-\frac{i}{2}\overrightarrow{\partial_k}\overrightarrow{\partial_x}} D(x,k) \tag{4.12}$$

As stated before, we assume that all integrals and infinite series of the type (4.12) converge. The important interpretation of these expressions indicated by the differential form is that if neither type of symbol contains xk products then they are identical.

As an example of the application of these exponential operators, consider the case where $d(x,k)$ is the common symbol of a differential operator (3.14). The Weyl symbol of the differential operator may be computed using (4.11) in its power series expansion:

$$\begin{aligned} D(x,k) &\equiv e^{\frac{i}{2}\overrightarrow{\partial_k}\overrightarrow{\partial_x}} \sum_m d_m(x) k^m \\ &= \sum_m \sum_{n=0}^{\infty} \frac{1}{n!} \left[(\tfrac{i}{2}\partial_x)^n d_m(x) \right] \partial_k^n k^m \end{aligned} \tag{4.13}$$

The k-derivatives reduce the exponents of k^m so that terms in the power series ∂_k^n vanish for $n > m$. Thus, with

$$\partial_k^n k^m = \begin{cases} \frac{m!}{(m-n)!} k^{m-n} & n \le m \\ 0 & n > m \end{cases} \tag{4.14}$$

(4.13) becomes

$$\begin{aligned} D(x,k) &= \sum_m \sum_{n=0}^{m} \frac{m!}{n!(m-n)!} k^{m-n} (\tfrac{i}{2}\partial_x)^n d_m(x) \\ &= \sum_m \left(k + \tfrac{i}{2}\partial_x\right)^m d_m(x) \end{aligned} \tag{4.15}$$

having recognized the form of the binomial expansion. This formula may be verified with the integral relations (4.9). It also suggests that the Weyl symbol, although "symmetrized" to produce nice physical properties, does not provide the natural extension of a differential operator as does the common symbol.

The inverse (4.12) may be applied to (4.15) in order to illustrate the way in which it "undoes" the operation of (4.11,4.13). Begin with

$$e^{-\frac{i}{2}\overrightarrow{\partial_k}\overrightarrow{\partial_x}} \sum_m \left(k + \tfrac{i}{2}\partial_x\right)^m d_m(x) = \sum_m \sum_{n=0}^{\infty} \frac{1}{n!} \partial_k^n \left(k + \tfrac{i}{2}\partial_x\right)^m (-\tfrac{i}{2}\partial_x)^n d_m(x) \tag{4.16}$$

expand the binomial and use (4.14) to find

$$\partial_k^n \left(k + \tfrac{i}{2}\partial_x\right)^m = \partial_k^n \sum_{l=0}^{m} \binom{m}{l} k^l (\tfrac{i}{2}\partial_x)^{m-l}$$
$$= \sum_{l=0}^{m} \binom{m}{l} \frac{l!}{(l-n)!} k^{l-n} (\tfrac{i}{2}\partial_x)^{m-l}, \qquad (n \le l) \tag{4.17}$$

Now (4.16) is

$$\sum_m \sum_{l=0}^{m} \binom{m}{l} (\tfrac{i}{2}\partial_x)^{m-l} \sum_{n=0}^{l} \binom{l}{n} k^{l-n} (-\tfrac{i}{2}\partial_x)^n d_m(x)$$
$$= \sum_m \sum_{l=0}^{m} \binom{m}{l} (\tfrac{i}{2}\partial_x)^{m-l} \left(k - \tfrac{i}{2}\partial_x\right)^l d_m(x)$$

so that the derivatives of d_m cancel when the binomial theorem is used once again

$$\sum_m \left(\tfrac{i}{2}\partial_x + k - \tfrac{i}{2}\partial_x\right)^m d_m(x) = \sum_m k^m d_m(x) \tag{4.18}$$

to recover the form of the common symbol $d(x,k)$.

Having demonstrated that the Weyl symbol can indeed be obtained from the common symbol with this exponential operator method, we may exploit the technique for a quick derivation of the product rule in the Weyl calculus. Recalling that the rule (3.40) which translates the operator composition into the common symbol product is

$$\mathbf{C} = \mathbf{AB}$$
$$\Rightarrow \qquad c(x,k) = a(x,k) e^{-i\overleftarrow{\partial_k}\overrightarrow{\partial_x}} b(x,k) \tag{4.19}$$

we may immediately use (4.11) and (4.12) to introduce the Weyl symbols of the operators into this expression:

$$e^{-\frac{i}{2}\overrightarrow{\partial_k}\overrightarrow{\partial_x}} C(x,k) = \left(e^{-\frac{i}{2}\overrightarrow{\partial_k}\overrightarrow{\partial_x}} A(x,k)\right) e^{-i\overleftarrow{\partial_k}\overrightarrow{\partial_x}} \left(e^{-\frac{i}{2}\overrightarrow{\partial_k}\overrightarrow{\partial_x}} B(x,k)\right) \tag{4.20}$$

The exponential operator on the left-hand side may be inverted to give

$$C(x,k) = e^{\frac{i}{2}\overrightarrow{\partial_k}\overrightarrow{\partial_x}} \left[\left(e^{-\frac{i}{2}\overrightarrow{\partial_k}\overrightarrow{\partial_x}} A(x,k)\right) e^{-i\overleftarrow{\partial_k}\overrightarrow{\partial_x}} \left(e^{-\frac{i}{2}\overrightarrow{\partial_k}\overrightarrow{\partial_x}} B(x,k)\right)\right] \tag{4.21}$$

The term in brackets may be viewed as nothing more than the multiplicative product of two functions of x and k (albeit, not A and B; *cf.* Eq.(3.41)) so that the action of the operator from the left on this product must be determined.

Writing Leibniz's formula for the differentiation of a product in the convenient form

$$\begin{aligned}\partial_x^n f(x)g(x) &= \sum_{m=0}^{n} \binom{n}{m} (\partial_x^{n-m} f)(\partial_x^m g) \\ &= f(x)(\overleftarrow{\partial_x} + \overrightarrow{\partial_x})^n g(x)\end{aligned} \tag{4.22}$$

which generalizes to two variables as

$$\partial_k^n \partial_x^n f(x,k)g(x,k) = f(x,k)(\overleftarrow{\partial_k} + \overrightarrow{\partial_k})^n (\overleftarrow{\partial_x} + \overrightarrow{\partial_x})^n g(x,k) \tag{4.23}$$

and indeed to $2N$ independent commuting derivatives as

$$\partial_{\mathbf{k}}^n \cdot \partial_{\mathbf{x}}^n f(\mathbf{x},\mathbf{k})g(\mathbf{x},\mathbf{k}) = f(\mathbf{x},\mathbf{k})[(\overleftarrow{\partial_{\mathbf{k}}} + \overrightarrow{\partial_{\mathbf{k}}}) \cdot (\overleftarrow{\partial_{\mathbf{x}}} + \overrightarrow{\partial_{\mathbf{x}}})]^n g(\mathbf{x},\mathbf{k}) \tag{4.24}$$

one can easily verify the action of the exponentiated differential operator

$$e^{i\partial_k \partial_x} f(x,k)g(x,k) = f(x,k)e^{i(\overleftarrow{\partial_k}+\overrightarrow{\partial_k})(\overleftarrow{\partial_x}+\overrightarrow{\partial_x})} g(x,k) \tag{4.25}$$

Consequently, with (4.25), the product rule (4.21) becomes

$$C(x,k) = \left(e^{-\frac{i}{2}\overrightarrow{\partial_k}\overrightarrow{\partial_x}} A(x,k)\right) e^{\frac{i}{2}(\overleftarrow{\partial_k}+\overrightarrow{\partial_k})(\overleftarrow{\partial_x}+\overrightarrow{\partial_x})} e^{-i\overleftarrow{\partial_k}\overrightarrow{\partial_x}} \left(e^{-\frac{i}{2}\overrightarrow{\partial_k}\overrightarrow{\partial_x}} B(x,k)\right) \tag{4.26}$$

which may be rewritten as

$$C(x,k) = A(x,k)e^{-\frac{i}{2}\overleftarrow{\partial_k}\overleftarrow{\partial_x}} e^{\frac{i}{2}(\overleftarrow{\partial_k}+\overrightarrow{\partial_k})(\overleftarrow{\partial_x}+\overrightarrow{\partial_x})} e^{-i\overleftarrow{\partial_k}\overrightarrow{\partial_x}} e^{-\frac{i}{2}\overrightarrow{\partial_k}\overrightarrow{\partial_x}} B(x,k) \tag{4.27}$$

where the arrows have kept track of the correct functions to be differentiated by each operator. Now, since the x and k derivatives commute, the exponentials can be combined to give

$$\begin{aligned}C(x,k) &= Ae^{\frac{i}{2}(-\overleftarrow{\partial_k}\overleftarrow{\partial_x}+\overleftarrow{\partial_x}\overleftarrow{\partial_k}+\overleftarrow{\partial_x}\overrightarrow{\partial_k}+\overleftarrow{\partial_k}\overrightarrow{\partial_x}+\overrightarrow{\partial_x}\overrightarrow{\partial_k}-2\overleftarrow{\partial_k}\overrightarrow{\partial_x}-\overrightarrow{\partial_k}\overrightarrow{\partial_x})} B \\ &= A(x,k)e^{\frac{i}{2}(\overleftarrow{\partial_x}\overrightarrow{\partial_k}-\overleftarrow{\partial_k}\overrightarrow{\partial_x})} B(x,k)\end{aligned} \tag{4.28}$$

where again all manipulations may be verified by means of power series expansions. Therefore, the product rule for Weyl symbols is

$$\begin{aligned}C(x,k) &= A(x,k)e^{\frac{i}{2}\overleftrightarrow{\mathcal{L}}} B(x,k) \\ \overleftrightarrow{\mathcal{L}} &\equiv \overleftarrow{\partial_x}\overrightarrow{\partial_k} - \overleftarrow{\partial_k}\overrightarrow{\partial_x}\end{aligned} \tag{4.29}$$

Of course, this formula could have been derived with only the definition (4.1) and its inverse (4.2) in a manner similar to that presented for common symbols although the calculation is much longer [39]. The integral form of (4.29) may also be obtained by combining (4.9), (4.10) and the integral form of the common symbol product (3.35). The resulting expression is

$$C(x,k) = 4\int dx_1\, dx_2\, \frac{dk_1\, dk_2}{(2\pi)^2}\, e^{-2i[(k_1-k)(x_2-x)-(k_2-k)(x_1-x)]} \times A(x_1,k_1)B(x_2,k_2) \tag{4.30}$$

from which (4.29) may be derived using the familiar Taylor series arguments.

Both (4.29) and (4.30) are similar to (3.40) and (3.35) in that they involve exponential bi-directional operators or nonlocal integral relations, but they differ in two significant aspects. In the Weyl symbol product, x and k are treated on an equal basis, ostensibly because of the symmetric x-space definition (4.1). Perhaps more importantly, the symmetrized definition has led to the bi-directional operator $\overleftrightarrow{\mathcal{L}}$ of (4.29) which is precisely the *Poisson bracket* of Hamiltonian ray theory (see Eq.(2.24)). In fact, expanding (4.29) in a power series the first two terms are

$$\begin{aligned} C(x,k) &= A(x,k)B(x,k) + A(x,k)\tfrac{i}{2}\overleftrightarrow{\mathcal{L}}B(x,k) + \ldots \\ &= A(x,k)B(x,k) + \tfrac{i}{2}\left((\partial_x A)(\partial_k B) - (\partial_k A)(\partial_x B)\right) + \ldots \\ &= A(x,k)B(x,k) + \tfrac{i}{2}\{A,B\} + \ldots \end{aligned} \tag{4.31}$$

although the higher terms cannot be expressed so simply by means of the Poisson bracket. Thus, not only has the Weyl symbol rectified the x-space and phase space symmetry deficiencies found in the common symbol formalism, there is also the suggestion of a possibly closer connection to the description of geometric optics in phase space.

The Weyl product rule can be directly extended to many dimensions and tensor operators. Hence, the Weyl symbol or phase-space representation of the basic operator

wave equation (3.2,3.27) is

$$\underset{\sim}{D}(x,k)e^{\frac{i}{2}\overleftrightarrow{\mathcal{L}}}\cdot\underset{\sim}{S}(x,k)=\underset{\sim}{N}(x,k)e^{\frac{i}{2}\overleftrightarrow{\mathcal{L}}}\cdot(\underset{\sim}{D}^{\dagger})^{-1}(x,k) \tag{4.32}$$

where

$$\begin{aligned}(x,k)&=(\mathbf{x},t,\mathbf{k},\omega)\\(\partial_x,\partial_k)&=(\partial_{\mathbf{x}},-\partial_t,\partial_{\mathbf{k}},\partial_\omega)\\\overleftrightarrow{\mathcal{L}}&=\overleftarrow{\partial}_{\mathbf{x}}\cdot\overrightarrow{\partial}_{\mathbf{k}}-\overleftarrow{\partial}_{\mathbf{k}}\cdot\overrightarrow{\partial}_{\mathbf{x}}+\overleftarrow{\partial_\omega}\overrightarrow{\partial_t}-\overleftarrow{\partial_t}\overrightarrow{\partial_\omega}\end{aligned} \tag{4.33}$$

In expanded component form this is

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{j_1,j_2,\dots j_n}^{4}\sum_{\nu=1}^{3}\frac{(i/2)^n}{n!}\binom{n}{m}(-1)^m\frac{\partial^n D_{\mu\nu}}{\partial(x^{n-m})\partial(k^m)}\frac{\partial^n S_{\nu\sigma}}{\partial(k^{n-m})\partial(x^m)}\\&\quad=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{j_1,j_2,\dots j_n}^{4}\sum_{\nu=1}^{3}\frac{(i/2)^n}{n!}\binom{n}{m}(-1)^m\frac{\partial^n N_{\mu\nu}}{\partial(x^{n-m})\partial(k^m)}\frac{\partial^n (D^{\dagger})^{-1}_{\nu\sigma}}{\partial(k^{n-m})\partial(x^m)}\end{aligned} \tag{4.34}$$

with the notation for the derivatives being

$$\begin{aligned}\frac{\partial^n}{\partial(x^{n-m})\partial(k^m)}&\equiv\frac{\partial^n}{\partial x_{j_1}\partial x_{j_2}\cdots\partial x_{j_{n-m}}\partial k_{j_{n-m+1}}\cdots\partial k_{j_n}}\\\frac{\partial}{\partial x_4}&\equiv-\frac{\partial}{\partial t}\end{aligned} \tag{4.35}$$

The elements of this equation are the Weyl symbols of the dispersion tensor $\underset{\sim}{D}$, the spectral tensors $\underset{\sim}{S}$ and $\underset{\sim}{N}$ of the vector wave field $\boldsymbol{\psi}$ and source $\mathbf{F}$ respectively, and the inverse adjoint dispersion tensor $(\underset{\sim}{D}^{\dagger})^{-1}$. The Weyl symbol $\underset{\sim}{D}$ is given by the tensor form of (4.1) in terms of the dispersion kernel $\underset{\sim}{\mathcal{D}}$ while the spectral tensors (or tensor Wigner functions) are constructed from the fields by the tensor form of (4.3) which is explicitly

$$\underset{\sim}{S}(\mathbf{x},t,\mathbf{k},\omega)=\int d^3s\,d\tau\;\boldsymbol{\psi}(\mathbf{x}+\tfrac{1}{2}\mathbf{s},t+\tfrac{1}{2}\tau)\boldsymbol{\psi}^*(\mathbf{x}-\tfrac{1}{2}\mathbf{s},t-\tfrac{1}{2}\tau)e^{-i\mathbf{k}\cdot\mathbf{s}+i\omega\tau} \tag{4.36}$$

and similarly for $\underset{\sim}{N}$. The inverse of this transformation is by analogy with (4.2)

$$\begin{aligned}\boldsymbol{\psi}(x+\tfrac{1}{2}s)\boldsymbol{\psi}^*(x-\tfrac{1}{2}s)&=\int\frac{d^4k}{(2\pi)^4}\;\underset{\sim}{S}(x,k)e^{iks}\\|\boldsymbol{\psi}(x)|^2&=\mathrm{Tr}\int\frac{d^4k}{(2\pi)^4}\;\underset{\sim}{S}(x,k)\end{aligned} \tag{4.37}$$

and similar projection rules may be verified for k-space

$$\begin{aligned} \hat{\psi}(k+\tfrac{1}{2}\kappa)\hat{\psi}^*(k-\tfrac{1}{2}\kappa) &= \int d^4x\ \underset{\sim}{S}(x,k)e^{-i\kappa x} \\ \left|\hat{\psi}(k)\right|^2 &= \mathrm{Tr}\int d^4x\ \underset{\sim}{S}(x,k) \end{aligned} \tag{4.38}$$

As in the common symbol description (for which formulas similar to (4.37,4.38) also hold) these relations indicate that the Weyl symbol of the field correlation operator is an "unaveraged" local spectral tensor and its projections are "unaveraged" (yet "centered") intensities and correlations.

There are two other important properties of Weyl symbols in general that also contribute to its usefulness. Due to the symmetry of the x-space kernel definition (4.1), the symbol of the adjoint operator is simply

$$\begin{aligned} D^\dagger(x,k) &\equiv \int ds\ D^*(x-\tfrac{1}{2}s, x+\tfrac{1}{2}s)e^{-iks} \\ &= \int ds\ D^*(x+\tfrac{1}{2}s, x-\tfrac{1}{2}s)e^{iks} \\ &= \left[\int ds\ D(x+\tfrac{1}{2}s, x-\tfrac{1}{2}s)e^{-iks}\right]^* \\ &= D^*(x,k) \end{aligned} \tag{4.39}$$

The relation for a tensor symbol is also easily derived:

$$(D^\dagger)_{\mu\nu}(x,k) = D^*_{\nu\mu}(x,k) \tag{4.40}$$

In the Weyl representation therefore, we have the fortunate circumstance that the symbol of the adjoint operator is just the adjoint of the symbol. For scalar symbols then, a self-adjoint operator is associated with a real symbol. The manifest Hermiticity of the spectral operator **S** implies that the Wigner tensor is self-adjoint. Most significantly, these assertions are valid irrespective of the presence of xk products which plague the common symbol representation.

Another characteristic of Weyl symbols not only provides further motivation for their use as a suitable phase-space representation, but it will be needed in the next Section as well. This property concerns the *expectation value* of the measurement of some wave attribute A in the field ψ. From the representation-free abstract point of view, this is interpreted as the action of the operator $\mathbf{A}$ (which represents the attribute) on the field ψ with subsequent projection onto ψ. This concept will be seen to be important here although it is perhaps more familiar in the context of quantum mechanics. Therefore, with the help of the Dirac notation, the expectation value is

$$\langle \mathbf{A} \rangle_\psi \equiv \psi^\dagger \mathbf{A} \psi \qquad \text{or} \qquad \langle \psi | \mathbf{A} | \psi \rangle \tag{4.41}$$

In terms of the x-space representation this is

$$\begin{aligned} \langle \mathbf{A} \rangle_\psi &= \int dx\, dy\ \langle \psi | x \rangle \langle x | \mathbf{A} | y \rangle \langle y | \psi \rangle \\ &= \int dx\, dy\ \psi^*(x) \mathcal{A}(x,y) \psi(y) \\ &= \int dx\, dy\ \mathcal{A}(x,y) S(y,x) \\ &\equiv \mathrm{Tr}\ \mathbf{AS} \end{aligned} \tag{4.42}$$

which defines the *trace* of an operator. Thus, the field spectral operator may be interpreted as a density operator against which the observable $\mathbf{A}$ is weighted in the wave field ψ.

The phase-space representation of (4.42) is obtained with the use of (4.2):

$$\begin{aligned} \langle \mathbf{A} \rangle_\psi &= \int dx\, dy \frac{dk_1\, dk_2}{(2\pi)^2}\ e^{ik_1(x-y)} e^{ik_2(y-x)} A(\tfrac{1}{2}(x+y), k_1) S(\tfrac{1}{2}(x+y), k_2) \\ &= \int \frac{dk_1\, dk_2}{(2\pi)^2}\, dr\, ds\ e^{i(k_1-k_2)s} A(r, k_1) S(r, k_2) \\ &= \int \frac{dk}{2\pi}\, dx\ A(x,k) S(x,k) \end{aligned} \tag{4.43}$$

It may be shown by means of the integral form of the Weyl product (4.30) that this expression is equivalent to

$$\langle \mathbf{A} \rangle_\psi = \text{Tr } \mathbf{AS} = \int \frac{dk}{2\pi}\, dx\, [\mathbf{AS}](x,k) \tag{4.44}$$

This result is reminiscent of the classical expression for the expectation value of the phase function $A(x,k)$ with respect to a phase-space density $S(x,k)$. The fact that a similar relation does not hold for the common symbol description marks another advantage for the Weyl representation. Nevertheless, there is a non-classical aspect to (4.43); while $S(x,k)$ is real (for scalar fields), we have seen in Section 4.1 that it is not necessarily non-negative as one would expect a phase-space probability density to be.

4.3 Asymptotic Analysis: The Wave Kinetic Equation

In that the Weyl phase-space representation offers several improvements over the symbol formalism discussed in Section 3, we now proceed to investigate the phase-space representation of the wave equation (4.32) in the short-wavelength regime. We shall show how the corresponding ray system is naturally induced by this equation and how one is led to a natural definition of the wave action density as a function on phase space. The transport equation which results from this analysis will be the analog of the Liouville Theorem of classical mechanics; thus, this formalism enables a formal derivation of the *wave kinetic equation* governing the wave action density. In order to show the direct application of this method to vector wave fields, we retain the tensor notation of (4.32) and work with the explicit space-time and wave-vector-frequency variables $(\mathbf{x},t,\mathbf{k},\omega)$.

In the spirit of the conventional eikonal method for approximating the solution of (3.1), we shall postulate a physically reasonable ordering of the derivatives of the various elements in (4.32). To this end, consider the power series expansion of (4.32)

$$\underset{\sim}{D}(1 + \tfrac{i}{2}\overleftrightarrow{\mathcal{L}} + \cdots) \cdot \underset{\sim}{S} = \underset{\sim}{N}(1 + \tfrac{i}{2}\overleftrightarrow{\mathcal{L}} + \cdots) \cdot (\underset{\sim}{D}^\dagger)^{-1} \tag{4.45}$$

The exponential expression (rather than the integral formula (4.30)) permits the immediate identification of the relative orders of variation involved in the evolution of $\underset{\sim}{S}$. Consistent with the assumption of a weakly inhomogeneous medium (compared to a typical short wavelength λ and high frequency ω of the waves), we require

$$|\partial_{\mathbf{x}} \underset{\sim}{D}| \sim L^{-1} \qquad |\partial_t \underset{\sim}{D}| \sim T^{-1} \tag{4.46}$$

where L and T are again the scalelength and timescale of the variation of the medium. As a quadratic function of the wave field $\boldsymbol{\psi}$, one might expect $\underset{\sim}{S}$ to also exhibit only this slowly varying behavior; therefore, we shall look for solutions which satisfy

$$|\partial_{\mathbf{x}} \underset{\sim}{S}| \sim L^{-1} \qquad |\partial_t \underset{\sim}{S}| \sim T^{-1} \tag{4.47}$$

For order of magnitude estimates, we shall also take the k-derivatives to be

$$|\partial_{\mathbf{k}} \underset{\sim}{D}| \sim |\partial_{\mathbf{k}} \underset{\sim}{S}| \sim \lambda \qquad |\partial_\omega \underset{\sim}{D}| \sim |\partial_\omega \underset{\sim}{S}| \sim \omega^{-1} \tag{4.48}$$

These assumptions imply that terms on the left-hand side of (4.45) have relative scale given by

$$\left| \underset{\sim}{D} (\overleftrightarrow{\mathcal{L}})^n \underset{\sim}{S} \right| \sim \left(\frac{\lambda}{L} \right)^n \sim (\omega T)^{-n} \equiv \varepsilon^n \tag{4.49}$$

which defines the same small expansion parameter ε used in conventional eikonal theory.

In addition to this space-time scale ordering, we shall also restrict attention to that region of phase space for which the wave system is only slightly dissipative and only weakly driven. Thus, the anti-Hermitian part of $\underset{\sim}{D}$ and the sources $\mathbf{F}$ are assumed to be small in the sense that

$$\begin{gathered} \underset{\sim}{D}' \equiv \tfrac{1}{2}(\underset{\sim}{D} + \underset{\sim}{D}^\dagger) \qquad \underset{\sim}{D}'' \equiv -\tfrac{i}{2}(\underset{\sim}{D} - \underset{\sim}{D}^\dagger) \\ |\underset{\sim}{D}''| \sim |\underset{\sim}{N}| \sim \mathcal{O}(\varepsilon) \end{gathered} \tag{4.50}$$

While these ordering arguments are common to traditional eikonal treatments, an actual form for the solution has not been postulated. Naturally, the justification for these assumptions must eventually be based on the properties of the solutions which emerge.

Applying these assumptions (4.46–4.50) to (4.45), the lowest two orders are

$$O(1): \qquad \underset{\sim}{D}'(\mathbf{x},t,\mathbf{k},\omega)\cdot\underset{\sim}{S}(\mathbf{x},t,\mathbf{k},\omega)=0 \tag{4.51}$$

and

$$O(\varepsilon): \qquad \tfrac{i}{2}\underset{\sim}{D}'\overleftrightarrow{\mathcal{L}}\cdot\underset{\sim}{S}+i\underset{\sim}{D}''\cdot\underset{\sim}{S}=\underset{\sim}{N}\cdot(\underset{\sim}{D}^{\dagger})^{-1} \tag{4.52}$$

As we shall see, the $O(1)$ equation (4.51) is a *constraint* on the spectral tensor $\underset{\sim}{S}$ while the $O(\varepsilon)$ equation* (4.52) governs the evolution of $\underset{\sim}{S}$.

We consider first the lowest-order equation (4.51). Unlike the usual dispersion equation in a homogeneous medium (or the corresponding lowest-order result (2.17) of conventional eikonal methods), this is a matrix equation (at each point $(\mathbf{x},t,\mathbf{k},\omega)$) which must be satisfied for each element. Therefore, the standard result $\det\underset{\sim}{D}'=0$ does not immediately follow from (4.51). Fortunately, since the spectral tensor is self-adjoint in the Weyl representation and because only the Hermitian part of the dispersion tensor is involved, the adjoint of (4.51) implies that in this approximation, these two tensors commute and can therefore be simultaneously diagonalized. This property is not only important at lowest order, but it will allow the equation generated in the next order to be decoupled so that the inherent tensor nature of (4.32,4.45) can be treated by scalar equations.

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* In general, one should expand the solution $\underset{\sim}{S}=\sum_n\varepsilon^n\underset{\sim}{S}_n$, in which case (4.52) would contain the term $\underset{\sim}{D}'\cdot\underset{\sim}{S}_1$ on the left-hand side (and all other quantities in that equation would be labeled zeroeth order). Since all of the tensors which appear in (4.52) are Hermitian, however, one finds that the anti-Hermitian part of (4.52) produces the equation shown (due to the presence of i), whereas the Hermitian part gives $\underset{\sim}{D}'\cdot\underset{\sim}{S}_1=0$ (a constraint on $\underset{\sim}{S}_1$ similar to (4.51)). Thus, the first-order quantity $\underset{\sim}{S}_1$ does not appear here, and we have dropped the subscript notation for the zeroeth-order tensors shown.

It is most convenient to express (4.51) in the basis in which both matrices are diagonal. This basis is that of the eigenvectors of $\underset{\sim}{D}'$

$$\underset{\sim}{D}'(\mathbf{x},t,\mathbf{k},\omega)\cdot\hat{\mathbf{e}}^{\alpha}(\mathbf{x},t,\mathbf{k},\omega)=D^{\alpha}(\mathbf{x},t,\mathbf{k},\omega)\hat{\mathbf{e}}^{\alpha}(\mathbf{x},t,\mathbf{k},\omega)$$
$$\text{for}\qquad \alpha=1,2,3 \tag{4.53}$$

The vectors $\hat{\mathbf{e}}^{\alpha}$ are also called the *local polarization vectors* as they give the local direction of the field just as in the uniform medium and traditional eikonal treatments. It must be pointed out, however, that the polarization vectors (4.53) here are functions of the independent phase-space variables $(\mathbf{x},t\mathbf{k},\omega)$, whereas in the conventional eikonal theory one has $\mathbf{k}=\mathbf{k}(\mathbf{x},t)$ and $\omega=\omega(\mathbf{x},t)$ so that

$$\hat{\mathbf{e}}(\mathbf{x},t,\mathbf{k},\omega)\rightarrow\hat{\mathbf{e}}(\mathbf{x},t,\mathbf{k}(\mathbf{x},t),\omega(\mathbf{x},t,))\rightarrow\hat{\mathbf{e}}(\mathbf{x},t) \tag{4.54}$$

As $\underset{\sim}{D}'$ is Hermitian, the polarization vectors are orthonormal and satisfy a completeness relation

$$\hat{\bar{\mathbf{e}}}^{\alpha}\cdot\hat{\mathbf{e}}^{\beta}=\sum_{\mu=1}^{3}\bar{e}_{\mu}^{\alpha}e_{\mu}^{\beta}=\delta^{\alpha\beta}\qquad\text{and}\qquad\sum_{\alpha=1}^{3}\bar{e}_{\mu}^{\alpha}e_{\nu}^{\alpha}=\delta_{\mu\nu} \tag{4.55}$$

The overbar notation here denotes the complex-conjugate transpose vector (the left eigenvectors of $\underset{\sim}{D}'$). The superscripts label the eigenvector while the subscripts denote components with respect to the usual Cartesian basis.

The local eigenvalues D^{α} of $\underset{\sim}{D}'$ are real and are the components of $\underset{\sim}{D}'$ in its diagonal representation $\underset{\sim}{D}$. The unitary transformation which effects this change of basis is, using (4.53,4.55),

$$\tilde{D}^{\alpha\beta}=\sum_{\mu,\nu}\bar{e}_{\mu}^{\alpha}D'_{\mu\nu}e_{\nu}^{\beta}=\sum_{\mu}\bar{e}_{\mu}^{\alpha}D^{\beta}e_{\mu}^{\beta}=D^{\alpha}\delta^{\alpha\beta} \tag{4.56}$$

with the inverse

$$D'_{\mu\nu}=\sum_{\alpha,\beta}e_{\mu}^{\alpha}\tilde{D}^{\alpha\beta}e_{\nu}^{\beta}=\sum_{\alpha}D^{\alpha}e_{\mu}^{\alpha}\bar{e}_{\nu}^{\alpha}\equiv\sum_{\alpha}P_{\mu\nu}^{\alpha}D^{\alpha} \tag{4.57}$$

which defines the projection operators $\underset{\sim}{P}^\alpha$ onto the eigenvector subspaces. Now, as the spectral tensor $\underset{\sim}{S}$ is also diagonal in this basis (in this approximation), we have

$$S_{\mu\nu} \equiv \sum_\alpha P^\alpha_{\mu\nu} W^\alpha \tag{4.58}$$

This defines the real diagonal elements $S^\alpha(\mathbf{x}, t, \mathbf{k}, \omega)$ which will be interpreted as the scalar spectral (or Wigner) functions of the waves with each polarization. The result of multiplying the two diagonal matrices (4.57,4.58) for the lowest order equation (4.51) is also a diagonal matrix, each element of which must satisfy

$$D^\alpha(\mathbf{x}, t, \mathbf{k}, \omega) S^\alpha(\mathbf{x}, t, \mathbf{k}, \omega) = 0 \tag{4.59}$$

at each point $(\mathbf{x}, t, \mathbf{k}, \omega)$ of phase space.

The relationship between this phase-space method and the rays of geometrical optics begins to emerge with this equation. Under the approximation scheme in which (4.59) represents the lowest order term of (4.32), one has the condition that at each point in phase space either D^α or S^α (or both) must vanish. The restriction $D^\alpha(\mathbf{x}, t, \mathbf{k}, \omega) = 0$ is equivalent to the usual eikonal condition $\det \underset{\sim}{D}' = 0$ and in the same way implicitly defines the local dispersion relation

$$D^\alpha(\mathbf{x}, t, \mathbf{k}, \omega) = 0 \qquad \Rightarrow \qquad \omega^\alpha = \Omega^\alpha(\mathbf{x}, \mathbf{k}, t) \tag{4.60}$$

More generally, the vanishing of a single eigenvalue D^α may yield multiple solutions (or branches) for ω^α, all corresponding to the same polarization. In addition, more than one eigenvalue D^α may vanish at a point (or on some manifold) in phase space; this possibility introduces coupling between the linear modes at the next order and requires special treatment [40]. We shall not consider further such degeneracies.

The *dispersion manifold* defined by (4.60) is the surface on which the rays generated in conventional eikonal methods evolve. According to (4.59) it is also the only region of phase

space where $S^\alpha(\mathbf{x},t,\mathbf{k},\omega)$ is allowed to be nonzero. While this apparently establishes a connection between the spectral function S^α and the rays of geometrical optics, two points must remain clear:

1) As explained in Section 2, the dispersion relation (4.60) as it arises in traditional eikonal techniques is in reality a partial differential (Hamilton-Jacobi) equation for the eikonal phase; that is $(\mathbf{k},\omega) \equiv (\partial_{\mathbf{x}}\phi, -\partial_t\phi)$. The rays are introduced as the characteristic trajectories for solution of this equation. In the present treatment, however, $\mathbf{k},\omega$ are independent of $\mathbf{x},t$; therefore, while (4.60) defines the same dispersion manifold, it does not induce the ray trajectories. Indeed, the lowest-order equation (4.51,4.59) in this formalism should be viewed more as a *constraint* on the spectral tensor rather than an equation for it.

2) The condition (4.59) does not require S^α to be nonzero everywhere that D^α vanishes so that it is possible for the spectral function to be concentrated on a submanifold of the dispersion surface.

In view of these considerations, we take as an appropriate solution of (4.59) to be

$$S^\alpha(\mathbf{x},t,\mathbf{k},\omega) = 2\pi\, J^\alpha(\mathbf{x},\mathbf{k};t)\ \delta(D^\alpha(\mathbf{x},t,\mathbf{k},\omega)) \tag{4.61}$$

We shall see that the function $J^\alpha(\mathbf{x},\mathbf{k};t)$ which gives structure to the spectral function on the dispersion manifold can be identified as the *wave action density* on phase space.

The next higher-order terms of (4.45) will determine the evolution of J^α with the use of (4.61). First we express (4.52) in the basis of the polarization vectors $\hat{\mathbf{e}}^\alpha$. In doing so, we make the simplifying assumption that the waves in the system are of only one polarization: that is, we take only S^s to be nonzero ($S^\alpha = 0$ everywhere for $\alpha \neq s$). Furthermore, due to the form of (4.61), the derivatives of S^s also vanish where $D^s \neq 0$.

As previously stated, we assume that there are no degeneracies: in the region where S^s is nonzero, $D^s = 0$ but $D^\alpha \neq 0$ for $\alpha \neq s$. Under these conditions, (4.52) becomes the evolution equation for $S^s(\mathbf{x}, t, \mathbf{k}, \omega)$

$$(\partial_\omega D^s)(\partial_t S^s) - (\partial_t D^s)(\partial_\omega S^s) + \{D^s, S^s\} = -2(D'')^{ss} S^s - \frac{2i\, N^{ss}}{D^s - i(D'')^{ss}} \qquad (4.62)$$

where we have used the notation of the Poisson bracket (2.24,4.31). In this expression, the superscript ss denotes the ss component of a tensor expressed in the polarization-vector basis. This equation is derived by casting all the terms in (4.52) in the polarization-vector basis and then comparing the expressions produced by extracting the ss component and the trace; for the details, see the Appendix.

Equation (4.62) governs the evolution of the spectral function S^s for the polarization s on the dispersion manifold $D^s = 0$. Therefore, the quantities $N^{ss}(\mathbf{x}, t, \mathbf{k}, \omega)$ and $(D'')^{ss}(\mathbf{x}, t, \mathbf{k}, \omega)$ as well as all derivatives of D^s and S^s must be evaluated on this surface by setting $\omega = \Omega^s(\mathbf{x}, \mathbf{k}; t)$. This can be incorporated into (4.62) to some extent by introducing the solution (4.61) of the lowest-order constraint which explicitly exhibits this restriction. Using the antisymmetry of the Poisson bracket we have

$$\{D^s, S^s\} = \{D^s, J^s\, \delta(D^s)\} = \{D^s, J^s\}\, \delta(D^s) \qquad (4.63)$$

so that the substitution of (4.61) into (4.62) yields

$$2\pi \left[(\partial_\omega D^s)\partial_t J^s(\mathbf{x}, \mathbf{k}, t) + \{D^s, J^s\}\right] \delta(D^s) = -4\pi (D'')^{ss}\, J^s\, \delta(D^s) - \frac{2i\, N^{ss}}{D^s - i(D'')^{ss}} \qquad (4.64)$$

We have dropped the term proportional to $\partial_\omega J^s$ since by definition (4.61) the action density $J^s(\mathbf{x}, \mathbf{k}; t)$ is independent of ω. The appearance of δ-functions in (4.64) indicates that this is to be interpreted as a density; integrating with respect to D^s (the local direction transverse to the dispersion manifold) we obtain

$$(\partial_\omega D^s)\partial_t J^s + \{D^s, J^s\} = -2(D'')^{ss}\, J^s + N^{ss} \qquad (4.65)$$

In arriving at this expression, we have replaced the denominator of the source term in (4.64) with its limiting form as $(D'')^{ss}$ tends to zero [41]:

$$[D^s - i(D'')^{ss}]^{-1} \to i\pi + \mathrm{P}[(D^s)^{-1}]$$

where P denotes the Cauchy principal value. The principal-value integral vanishes because of the antisymmetry of the integrand. Using the standard relations from (2.22) that relate derivatives of the dispersion function D^s to derivatives of the local dispersion relation Ω on the dispersion manifold $D^s = 0$ and defining the local growth rate $\gamma^s(\mathbf{x},\mathbf{k};t)$ in the usual manner

$$\gamma^s(\mathbf{x},\mathbf{k};t) \equiv -\left[\frac{(D'')^{ss}}{(\partial_\omega D^s)}\right]_{\omega=\Omega}(\mathbf{x},\mathbf{k};t) \tag{4.66}$$

we have finally

$$\partial_t J^s(\mathbf{x},\mathbf{k};t) + \{J^s,\Omega^s\} = 2\gamma^s(\mathbf{x},\mathbf{k};t)\,J^s + \left[\frac{N^{ss}}{(\partial_\omega D^s)}\right]_{\omega=\Omega}(\mathbf{x},\mathbf{k};t) \tag{4.67}$$

This result is known as the *wave kinetic equation* governing the wave action density $J^s(\mathbf{x},\mathbf{k};t)$ on the dispersion manifold in phase space (in contrast to the x-space action conservation equation derived in Section 2). In that the left-hand side has the form of the Liouville operator of classical mechanics, we observe that this $O(\varepsilon)$ approximation to the exact phase-space wave equation (4.32) introduces the notion of the ray trajectories in phase space. Specifically, defining the Hamiltonian ray system by the usual ray equations

$$\begin{aligned} \dot{\mathbf{x}} &= \partial_{\mathbf{k}}\Omega^s \qquad\qquad \dot{\mathbf{k}} = -\partial_{\mathbf{x}}\Omega^s \\ \text{and} \qquad \frac{dJ^s(\mathbf{x},\mathbf{k};t)}{dt} &\equiv \left[\partial_t + \dot{\mathbf{x}}\cdot\partial_x + \dot{\mathbf{k}}\cdot\partial_k\right] J^s(\mathbf{x},\mathbf{k};t) \end{aligned} \tag{4.68}$$

it is clear that (4.67) gives the total time derivative of J^s along the ray trajectories (4.68) in phase space.

4.4 Discussion

We now turn to a discussion of (4.67) and the nature of our definition (4.61) of $J^s(\mathbf{x},\mathbf{k};t)$ in order to justify its identification as the wave action density on phase space. In the absence of dissipation and sources, the evolution equation (4.67) simply states that J^s is constant along the ray trajectories in phase space (in contrast to the x-space action density which is *not* an invariant along the x-space projection of these trajectories). In analogy with a similar result of classical mechanics, this suggests that J^s should be interpreted as a kind of Liouville phase-space density for the propagation of waves in the short-wavelength regime. Thus, as we have in classical mechanics, this equation implies the conservation of phase-space volume convected with the ray trajectories (rays do not converge or diverge in phase space as they can in x-space). This means that integration of (4.67) along ray trajectories to construct a phase-space representation of the wave in the short-wavelength limit will not encounter the caustic singularities that the corresponding x-space equation (2.32) does. The x-space intensity $|\psi(\mathbf{x},t)|^2$ can then be obtained by using (4.61) in (4.37)

$$\begin{aligned} |\psi(\mathbf{x},t)|^2 &= \sum_\alpha \int \frac{d^3k}{(2\pi)^3}\,\frac{d\omega}{2\pi}\,2\pi J^\alpha(\mathbf{x},\mathbf{k};t)\delta(D^\alpha(\mathbf{x},t,\mathbf{k},\omega)) \\ &= \sum_\alpha \int \frac{d^3k}{(2\pi)^3}\,\frac{J^\alpha(\mathbf{x},\mathbf{k};t)}{(\partial_\omega D^\alpha)_{\omega=\Omega}} \end{aligned} \tag{4.69}$$

where we have expressed the trace indicated in (4.37) as a sum over the diagonal elements S^α (*i.e.*, in the basis where $\underset{\sim}{S}$ is diagonal, in this approximation). As this is just an integral over a well-behaved density on phase space, one does not expect further singularities to arise in this projection procedure of a wave on phase space onto a wave on x-space. As we shall see later, however, phase information of the wave in x-space is lost in general due to the definition of the phase-space representation (the spectral tensor) as a quadratic function of the field.

Allowing for dissipation, the solution of (4.67) is

$$J^s(\mathbf{x}(t),\mathbf{k}(t),t) = J^s(\mathbf{x}_0,\mathbf{k}_0,0)\exp\left[2\int_0^t \gamma^s(\mathbf{x}(t'),\mathbf{k}(t'),t')\,dt'\right] \tag{4.70}$$

which explicitly conveys the non-Hamiltonian damping (or growth) of this phase-space density from its initial value depending on the local value of γ^s. The factor of two is appropriate as J is quadratic in the field amplitude, yet it arises naturally here from the $\frac{i}{2}\overleftrightarrow{\mathcal{L}}$ in (4.52) and the approximation scheme. As was the case in the common symbol representation of Section 3, the source contribution $\underset{\sim}{N}$ may represent a specified source field or possibly nonlinear effects. If this term is independent of J^s, then (4.67) is a linear inhomogeneous equation for J^s. If, however, the source field $\mathbf{F}(\mathbf{x},t)$ depends nonlinearly on the field $\boldsymbol{\psi}(\mathbf{x},t)$, then using (4.1,4.2) the source $\underset{\sim}{N}$ may be written as a nonlinear functional of J^s; in this case, (4.67) becomes a nonlinear equation.

To see that J is properly identified as the wave action density, we integrate the definition (4.61) with respect to D^s

$$\begin{aligned} J^s(\mathbf{x},\mathbf{k};t) &= \int \frac{dD^s}{2\pi}\, S^s(\mathbf{x},t,\mathbf{k},\omega) \\ &= \int \frac{d\omega}{2\pi}\,(\partial_\omega D^s)(\mathbf{x},t,\mathbf{k},\omega) S^s(\mathbf{x},t,\mathbf{k},\omega) \end{aligned} \tag{4.71}$$

In this integral transverse to the dispersion manifold, the slowly varying term $(\partial_\omega D^s)$ may be approximated by its value at $\omega = \Omega^s$ due to the singular behavior of S^s across this surface. Thus we have

$$\begin{aligned} J^s(\mathbf{x},\mathbf{k};t) &\approx (\partial_\omega D^s)(\mathbf{x},t,\mathbf{k},\Omega^s)\int \frac{d\omega}{2\pi}\, S^s(\mathbf{x},t,\mathbf{k},\omega) \\ &\equiv (\partial_\omega D^s)(\mathbf{x},t,\mathbf{k},\Omega^s)\tilde{S}^s(\mathbf{x},\mathbf{k};t) \end{aligned} \tag{4.72}$$

with $\tilde{S}^s$ defined as

$$\tilde{S}^s(\mathbf{x},\mathbf{k};t) \equiv \hat{\tilde{\mathbf{e}}}^s(\mathbf{x},\mathbf{k},t,\Omega^s)\cdot\int d^3s\;\boldsymbol{\psi}(\mathbf{x}+\tfrac{1}{2}\mathbf{s},t)\boldsymbol{\psi}^*(\mathbf{x}-\tfrac{1}{2}\mathbf{s},t)e^{-i\mathbf{k}\cdot\mathbf{s}}\cdot\hat{\mathbf{e}}^s(\mathbf{x},\mathbf{k},t,\Omega^s) \tag{4.73}$$

where the Weyl projection formula (4.37) has been used. In a stationary uniform medium, (4.72) reduces to the usual expression for the energy of the mode at $(\mathbf{k},\omega)$ divided by the frequency

$$J^s(\mathbf{k}) = |E^s(\mathbf{k})|^2 \left(\frac{\partial D^s(\mathbf{k},\omega)}{\partial\omega}\right)_{\omega=\Omega(\mathbf{k})} = \frac{U^s(\mathbf{k})}{\omega^s(\mathbf{k})} \tag{4.74}$$

Thus, our definition of $J^s(\mathbf{x},\mathbf{k};t)$ reduces to the standard one in the uniform medium case. Furthermore, we have seen that in the absence of dissipation or sources, the action density is invariant under the flow of the rays in phase space. For these reasons, we suggest that $J(\mathbf{x},\mathbf{k};t)$ as defined in (4.61,4.71) is an appropriate extension of the wave action density on the ray phase space to the case of a weakly nonuniform, nonstationary medium.

The most common method [42] used for deriving (4.67) proceeds from the quantum field-theoretic concept of the occupation number $n_{\mathbf{k}}$ of a mode and the changes in that number due to nonlinear interactions with other modes; the classical limit then assumes large occupation numbers with smooth (as opposed to discrete) variation in time. As $n_{\mathbf{k}}$ is proportional to the square of the amplitude of the wave, the connection between the classical relation (4.74) and the usual quantum electrodynamic model of modes as oscillators provides the identification of the occupation number as the wave action in the classical limit. The primary focus of these methods is the form of the source terms $\underset{\sim}{N}$ which are taken to represent nonlinear couplings among modes and the approximations which can be made to simplify them. The classical, irreversible aspect of dissipation is generally just inserted into the kinetic equations which appear, and the extension to a nonuniform nonstationary medium is typically achieved by simply assuming a local spatial dependence of the occupation numbers and postulating [43] the replacement of $(\partial/\partial t)$ by the full convective operator (4.68).

Other non-quantum mechanical derivations have been given based on traditional

eikonal treatments of wave propagation. As we have seen in Section 2, the x-space (or k-space) action conservation equation which results is naturally set in an N-dimensional subspace of the full $2N$-dimensional phase space, and the action densities considered are densities on these lower-dimensional spaces (and therefore suffer from projection singularities near caustics). In order to obtain an equation for an action density $J(\mathbf{x},\mathbf{k};t)$ defined on the ray phase space (as derived in this Section), various techniques have been used to *lift* the x-space action density into a phase-space density. One technique [44] is to label the contribution to the amplitude at a point due to a single ray by the initial value of the wave vector of that ray; in this way the amplitude becomes an implicit function of k. Another method [45] relies on the asymptotic form of the spectral function (4.5) constructed from a wave in the eikonal approximation. Both of these schemes for introducing phase-space representations into fundamentally x-space equations tacitly assume a relationship $\mathbf{k}(\mathbf{x})$ (either through initial conditions or the eikonal phase) and this poses difficulties in each case: either quantities appearing in the resulting equations are tied to initial conditions (requiring the inversion of all trajectories) or the eikonal phase label may be continous (invalidating the assumption used for the spectral function). In addition, neither method incorporates the possibility of local nonlinear sources.

The procedure employed in the present derivation of the wave kinetic equation therefore has several advantages. In contrast to the conventional eikonal approach, this is inherently a phase-space method from the outset: we have treated $\mathbf{x},t$ and $\mathbf{k},\omega$ as independent variables and all functions on phase space are well-defined by the Weyl transform. No assumption has been made on the form of the spectral function(*i.e.*, it is not based on the eikonal description of the wave) yet the exact equation (4.32) which governs its evolution has been solved under an ordering hierarchy compatible with the customary

eikonal approximations. This leads to a natural definition for the wave action density as the "amplitude" of the spectral function on the dispersion manifold, an identification which leads to a reasonable extension of the wave action density in a uniform medium. The form of the exponential operator in (4.32) in conjunction with the approximation scheme used not only produces the Poisson bracket as the time advancement operator (the phase-space total time derivative (4.68)) but it also allows one to proceed to higher order in a straightforward manner.

4.5 Application

As an example of the use of this phase-space eikonal technique to solve a wave propagation problem, we shall consider the extremely simple case of wave packet evolution governed by the Schrödinger equation for the harmonic oscillator potential. In this case, the wave kinetic equation (4.67) can be treated analytically, as can the exact solution [34] for comparison; at appropriate points in the discussion, however, we shall comment on how a numerical scheme could be applied in more complicated situations that cannot be solved analytically. The important result of this example is the way in which the use of this asymptotic method avoids the caustic singularites that would cause the integration of the x-space eikonal transport equation (2.32) to fail when the wave packet approaches the turning points of the oscillator potential well.

The dispersion function (2.12) for the Schrödinger equation in the case of a harmonic oscillator with natural frequency ω_0 is

$$D(x,k;\omega) = \hbar\omega - \frac{\hbar^2 k^2}{2m} - \tfrac{1}{2}m\omega_0^2 x^2 \tag{4.75}$$

The lowest-order equation (4.60) $D(x,k;\omega) = 0$ thus defines the classical harmonic os-

cillator Hamiltonian and the dispersion manifold as the classical energy surface in phase space. We assume an initial wave packet of the form

$$\psi(x,0) = (\sigma\sqrt{\pi})^{-1/2}\, e^{-(x-a)^2/2\sigma^2} \tag{4.76}$$

which is a gaussian state of arbitrary width σ displaced from the center of the well a distance a.

The first step in the use of the wave kinetic equation is to consruct the initial wave action density $J(x,k;0)$ corresponding to the initial wave function (4.76). Thus, neglecting the vector notation in (4.71-4.73) since this is a scalar wave, we obtain

$$J(x,k;0) = 2\hbar\, e^{-(x-a)^2/\sigma^2}\, e^{-k^2\sigma^2} \tag{4.77}$$

With the factor of $\hbar$ from $(\partial_\omega D)$, J has the correct units of an action density on phase space (since phase-space (x,k) volume is unitless). We see that this distribution in phase space is also a gaussian, centered at $(x = a, k = 0)$ with x-width $\sigma/\sqrt{2}$ (appropriate for $|\psi(x)|^2$) and k-width $1/(\sigma\sqrt{2})$. The action density can be thus interpreted as the local wavenumber spectrum of the x-space wave function. Although for this simple initial waveform we could analytically compute the spatial transform (4.73) for the action density, in general one would have to perform this local Fourier integral numerically (where fast Fourier transform methods could be applied).

The solution of the wave kinetic equation in the absence of sources and damping (as in the present case) is simply that the action density J is constant along trajectories in phase space, or $J(x,k;t) = J(x_0,k_0;0)$. This means that if one can invert the trajectories generated by (4.75) and write the ray initial conditions in terms of the location of the ray at time t, $(x_0,k_0) = (x_0(x,k;t), k_0(x,k;t))$, then the solution for all time can be obtained by substituting these expressions into (4.77). In our example this can indeed be accomplished

since the equations of motion can be solved and inverted to give

$$x_0 = x\cos\omega_0 t - \alpha^{-2}k_0\sin\omega_0 t$$
$$k_0 = k\cos\omega_0 t + \alpha^2 x_0\sin\omega_0 t \tag{4.78}$$
$$\alpha^2 \equiv m\omega_0/\hbar$$

Here, we have defined the usual harmonic oscillator parameter α which sets a natural length scale (α^{-1}) for the problem. Introducing dimensionless phase-space variables with the use of this scale

$$\xi \equiv \alpha x \qquad\qquad \kappa \equiv k/\alpha \tag{4.79}$$

and the ratio of the scalelength to the characteristic width σ of the packet

$$\epsilon \equiv \alpha^{-2}\sigma^{-2} \tag{4.80}$$

the evolution of the phase-space density J is now obtained by using (4.78) in (4.77):

$$J(x,k;t) = 2\hbar\, e^{-\epsilon(\xi\cos\omega_0 t - \kappa\sin\omega_0 t - \xi_0)^2}\, e^{(\kappa\cos\omega_0 t + \xi\sin\omega_0 t)^2/\epsilon} \tag{4.81}$$

where $\xi_0 \equiv \alpha a$. This expression describes the motion of a gaussian distribution in phase space with constant maximum height but changing orientation of its axes (directions of minimum and maximum spread).

In the more general case where the equations of motion cannot be solved analytically, one must resort to a numerical integration of the ray trajectories in order to evolve the action density. Thus, after the initial distribution $J(x,k;0)$ is constructed, one could select some number of ray initial conditions in the region where $J(x,k;0)$ has amplitude greater than, for example, 1% of its maximum (these initial conditions could be chosen at random or on a grid in phase space). To each initial condition $(x_i(0), k_i(0))$ would be assigned the corresponding value of the action density $J_i(0) = J(x_i(0), k_i(0); 0)$. The ray equations are then numerically integrated; if damping and sources are present, an additional ordinary

differential equation governing the evolution of $J_i(t)$ for each ray would have to be updated at each timestep. At any later time t then, the action density in phase space is represented by the location of the rays and the values of J_i on each.

To obtain the x-space intensity (or probability density) $|\psi(x,t)|^2$ at a later time t, we can use (4.69) which projects the phase-space action density onto x-space by integrating over k-space at fixed x. In our example, this integral can be computed analytically to give

$$|\psi(x,t)|^2 = \frac{1}{\sqrt{\pi}w(t)} \exp\left(-\frac{(x - a\cos\omega_0 t)^2}{\sigma^2 w^2(t)}\right) \tag{4.82}$$
$$w^2(t) = \cos^2\omega_0 t + \epsilon^2 \sin^2\omega_0 t$$

For the special case of the minimum-uncertainty wave packet considered in Ref. [34] with packet width $\sigma = \alpha^{-1} \Rightarrow \epsilon = 1$, this reduces to the simple form of a wave packet whose center oscillates according to (4.78) (with zero initial momentum) between turning points at $x = \pm a$, and whose width and height remain constant. By direct comparison with the results in Ref. [34] this is indeed the *exact solution* for $|\psi(x,t)|^2$ in this case. If the pulse is not a minimum-uncertainty packet, or $\sigma \neq \alpha^{-1} \Rightarrow \epsilon \neq 1$, the packet height and width change periodically in time; this can be interpreted now in terms of the changing projection of the phase-space gaussian distribution onto x-space as its axes rotate.

In the numerical implementation of this method, the integral (4.69) over k-space could be performed as follows. At the desired time t, the action density is represented by the locations of the rays in phase space and the values of the action density associated with each. To construct $|\psi(x,t)|^2$ on a grid in x-space, one should construct another grid in k-space so that phase space is partitioned into cells. The values of J_i on each ray are then interpolated onto the lattice vertices in phase space (using some appropriate weighting scheme). At each grid-point in x-space, the integral (4.69) is subsequently computed using the discretization along the k-directions. This procedure will undoubtably require

a judgement in each particular case (and perhaps at each evaluation time) as to the most advantageous cell size (and shape) in phase space with regard to ray binning statistics.

Even though one should not expect such good agreement with the exact solution as found here using the approximation of the wave kinetic equation, this example serves to illustrate two important points. First, the caustic singularities that would be encountered by solving this problem using the conventional eikonal transport equation were successfully avoided by constructing the phase-space representation of the initial wave packet, evolving the action density along the ray trajectories in phase space (here, by the simple rule that J is constant) and finally projecting this phase-space distribution back down onto x-space. The second point is perhaps a criticism of this method: Due to the definition of this phase-space representation as a quadratic function of the wave field, phase information has been lost. Even though the initial wave packet considered here is purely real and has no oscillations, the exact analysis in Ref. [34] shows that the wave function does develop complex phase structure as it propagates, even though its intensity $|\psi(x,t)|^2$ continues to be oscillation-free. While the present technique succeeded in producing the exact solution for the wave packet intensity, it is impossible to determine from $J(x,k;t)$ the phase of the wave function $\psi(x,t)$ itself. This phase-space eikonal technique based on the Weyl representation of the wave equation is useful therefore only in cases where the evolution of the wave intensity is of primary interest and the detailed information of the wave phase is unimportant.

5. The Coherent State Representation

Although the Weyl symbol formalism discussed in the previous section possesses many properties which encourage its use in the asymptotic analysis of wave propagation problems and which, for this purpose, improve upon the deficiencies found in the common symbol formalism of Section 3, two major criticisms remain regarding the inherent nature of that definition (4.1) of a phase-space representation: The phase-space density (or Wigner function) constructed to represent the wave field ψ is not necessarily positive, and phase information contained in ψ is lost in the procedure. In this Section we shall investigate yet another phase-space representation for a wave field which corrects both of these problems while retaining many of the nice properties of the Weyl formalism; indeed, we shall again exploit the relationship between the Weyl symbol and this new representation in order to derive the phase-space equation corresponding to (3.1) which governs its evolution. We consider the *coherent state* or Glauber [21] representation which, although originally introduced for the study of the quantum theory of radiation and optics, has been recently applied [23-28] in other attempts to construct uniformly asymptotic quantum mechanical wave functions (where it is often called the *gaussian wave-packet basis*) and the study of chaotic quantum eigenfunctions. As has been our emphasis, however, we shall see that this representation has meaning and potential use in the analysis of short-wavelength solutions of classical wave equations as well.

5.1 Properties of the Coherent State Representation

Motivated by the notion (see Section 4.1) that the oscillatory structure of the spectral function in the Weyl representation might be removed by subjecting it to a local x-space

or phase-space averaging, we define the coherent state representation of a wave ψ by*

$$\Psi(x,k) \equiv (\pi\sigma^2)^{-1/4} \int dx'\ \psi(x') e^{-(x'-x)^2/2\sigma^2} e^{-ik(x'-x)} \tag{5.1}$$

This phase-space representation can be thought of as a "smoothed local Fourier transform" of the field in that $\psi(x)$ is convoluted with a gaussian "window" of width σ centered at the point x and then Fourier-transformed around x. Expressing $\psi(x)$ in terms of its Fourier transform $\hat{\psi}(k)$, the definition of Ψ may also be written

$$\Psi(x,k) = (4\pi\sigma^2)^{1/4} \int \frac{dk'}{2\pi}\ \hat{\psi}(k') e^{-\frac{1}{2}\sigma^2(k'-k)^2} e^{ik'x} \tag{5.2}$$

Despite the smoothing involved in the definition of this representation, both (5.1,5.2) are invertible:

$$\begin{aligned} \psi(x) &= (\pi\sigma^2)^{1/4} \int \frac{dk}{2\pi}\ \Psi(x,k) \\ \hat{\psi}(k) &= (4\pi\sigma^2)^{-1/4} \int dx\ \Psi(x,k) e^{-ikx} \end{aligned} \tag{5.3}$$

The first of these is just a simple projection onto x-space while the second is similar to a Fourier transform; the difference is that the same value of k appears both in the phase and in the argument of Ψ.

The construction of $\Psi(x,k)$ is fundamentally different from that prescribed for the phase-space representations of a wave discussed in the previous two Sections because it is *linear* (not quadratic) in the field ψ. The advantage of such a definition is that phase information about the field $\psi(x)$ is retained in Ψ, as is evident from the inversion formulas (5.3). In addition, even though Ψ will be the primary quantity of interest in this Section, it may itself be viewed as the amplitude of a proper phase-space density; thus, being

* The normalization and phase factors chosen here are those convenient for our application and may differ from those used by others.

inherently complex, it induces a real non-negative phase-space density $P(x,k) \equiv |\Psi(x,k)|^2$. With the choice of normalization factors in (5.1), it is easily verified that

$$\int dx \frac{dk}{2\pi} \, P(x,k) = \int dx \frac{dk}{2\pi} \, |\Psi(x,k)|^2 = \int dx \, |\psi(x)|^2 \tag{5.4}$$

independent of the smoothing length σ. Thus, if $\psi(x)$ is normalized on x-space, Ψ is normalized on phase space.

The definition (5.1) is easily generalized to many dimensions (including time) in an obvious way. Therefore, for simplicity of notation, the discussion of this Section will generally consider only one spatial dimension, although time may be included explicitly in some computations. In addition, only scalar fields will be treated; the techniques of the preceding Section should indicate the extension of the following development to vector fields.

At this point the width σ, which designates the meaning of "local" in this construction, is arbitrary and most of the results of this Section will be independent of its value. In specific cases, however, a judicious choice of the width may greatly simplify the mathematical analysis. For the purpose of future application to short-wavelength fields in weakly inhomogeneous media, it will be convenient to think of σ as intermediate between the wavelength and the scalelength; this provides a method for determining a local wavelength as well as allowing Ψ to reflect the slow amplitude variation of the field in x-space. In this regard, (5.1) has two interesting limiting forms

$$\Psi(x,k) \sim \begin{cases} \sigma^{1/2}\psi(x) & \text{as } \sigma \to 0 \\ \sigma^{-1/2}\hat{\psi}(k) & \text{as } \sigma \to \infty \end{cases} \tag{5.5}$$

which are easily understood: $\sigma \to 0$ signifies an extremely narrow window sampling only the value of ψ at x, whereas $\sigma \to \infty$ is the usual infinite window encompassing the entire field which produces the Fourier transform. In this sense, the finite σ local Fourier

transform can be thought of as an intermediate representation of the field.

The use of a gaussian window function with constant width for the implementation of local averaging can in principle be generalized to a gaussian with a spatially dependent or anisotropic width (in many dimensions), and even to a non-gaussian window. Different window shapes have been studied in application to signal theory and nonstationary spectral analysis [46] and quantum mechanics [47]. The results of this Section, however, are heavily dependent on the choice of a constant-width gaussian; the manipulations involved in many calculations are specific to this selection and may have to be drastically altered for application to other window shapes.

In order to compare the phase-space representation of a wave generated by this smoothed Fourier transform with the Weyl spectral function, we construct $\Psi(x,k)$ for the three examples considered in Section 4.1. Thus, for $\psi(x)$ a plane wave with wavenumber k_0 and amplitude A_0 we have from (5.1)

$$\begin{aligned} & \Psi(x,k) = (4\pi\sigma^2)^{1/4} A_0 e^{ik_0 x} e^{-\frac{1}{2}\sigma^2(k-k_0)^2} \\ \Rightarrow \qquad & P(x,k) = |\Psi(x,k)|^2 = 2\pi^{\frac{1}{2}}\sigma\,|A_0|^2\, e^{-\sigma^2(k-k_0)^2} \end{aligned} \tag{5.6}$$

which exhibits the underlying plane wave structure in the x-direction modulated by a gaussian along k. The form of P should be compared with the corresponding form of the Wigner function (4.4); obviously, this probability density is much smoother (for finite σ) than the singular δ-function behavior found in that representation. This expression also manifestly conforms to the Fourier uncertainty principle: the width in k-space (σ^{-1}) is inversely proportional to the localization length σ of the wave in x-space.

For a wave in the eikonal form (2.5) the local Fourier transform is

$$
\begin{aligned}
\Psi(x,k) &= (\pi\sigma^2)^{-1/4}\int dx'\; A(x')e^{i\phi(x')}e^{-ik(x'-x)}e^{-(x'-x)^2/2\sigma^2} \\
&\approx (\pi\sigma^2)^{-1/4}A(x)e^{i\phi(x)}\int dx'\; e^{-i(k-k(x))(x'-x)}e^{-(x'-x)^2/2\sigma^2} \\
&\approx (4\pi\sigma^2)^{1/4}A(x)e^{i\phi(x)}e^{-\frac{1}{2}\sigma^2(k-k(x))^2} \\
P(x,k) &\approx 2\pi^{\frac{1}{2}}\sigma\,|A(x)|^2\,e^{-\frac{1}{2}\sigma^2(k-k(x))^2}
\end{aligned}
\tag{5.7}
$$

Here, the phase ϕ has been expanded around the point x keeping only the first-derivative term and using the usual eikonal definition $k(x) \equiv \partial_x\phi(x)$; this truncation requires that the quadratic term $\sim (x'-x)^2\partial_x^2\phi(x)$ be much smaller than the linear term. Thus, since the gaussian window effectively cuts off the integral at $|x'-x| \approx \sigma$, this approximation is justified if

$$
\begin{aligned}
(x'-x)k(x) \gg (x'-x)^2\frac{dk}{dx}(x) \qquad &\rightarrow \qquad \frac{\sigma}{\lambda(x)} \gg \frac{\sigma^2}{\lambda(x)L} \\
&\Rightarrow \qquad \sigma \ll L
\end{aligned}
\tag{5.8}
$$

where the second derivative of the phase has been estimated in keeping with the conventional eikonal ordering. In the same spirit, the amplitude A has been evaluated at x and its Taylor expansion neglected. The condition (5.8) reiterates the notion that for short-wavelength fields in a weakly nonuniform medium, the window should be narrow compared to the scalelength. The fact that the result (5.7) is similar to that found for a plane wave implicitly suggests that σ be large enough to include sufficiently many wavelengths in order that the appropriate local wavelength be represented in $\Psi(x,k)$. Again, this result should be compared with that of the Wigner function of (4.5); as discussed following that example, the fact that (5.7) appears to show that the asymptotic form of the coherent state representation of a short-wavelength wave contains singularities where $A \rightarrow \infty$, one should keep in mind that this is based on the asymptotic behavior of the conventional eikonal solution and not the asymptotic analysis of the equation which governs the evolution of Ψ itself.

The final example considered in Section 4.1 and which will be instructive here is the case of the eigenfunctions of the Schrödinger equation for the harmonic oscillator. Inserting (4.6) for $\psi(x)$ into (5.1), one may again explicitly evaluate the integral for any smoothing length σ in terms of Hermite polynomials; unfortunately, the result does not provide immediate insight into the structure of Ψ in phase space. For the special choice of $\sigma = \alpha^{-1}$, however, an extremely transparent expression is obtained:

$$\begin{aligned}
\Psi_n(x,k) &= C_n(\alpha^2/\pi)^{1/4} \int dx'\ H_n(\alpha x') e^{-\frac{1}{2}\alpha^2 x'^2} e^{-\frac{1}{2}\alpha^2(x'-x)^2} e^{-ik(x'-x)} \\
&= C_n(\pi/\alpha^2)^{1/4} e^{-\frac{1}{4}(\alpha^2x^2+k^2/\alpha^2)} e^{\frac{1}{2}ikx} \left(\alpha x - \frac{ik}{\alpha}\right)^n \\
&= C_n(\pi/\alpha^2)^{1/4} e^{-\frac{1}{4}r^2} e^{\frac{1}{4}ir^2\sin 2\theta} r^n e^{-in\theta} \qquad (5.9) \\
P(x,k) &= (2^n n!)^{-1} e^{-\frac{1}{2}(\alpha^2x^2+k^2/\alpha^2)} \left(\alpha^2x^2 + \frac{k^2}{\alpha^2}\right)^n \\
&= (2^n n!)^{-1}\ r^{2n} e^{-\frac{1}{2}r^2}
\end{aligned}$$

where again polar coordinates (r,θ) in phase space have been defined as in (4.8) and the actual value of C_n has been used. Again, this choice of the smoothing length $\sigma = \alpha^{-1}$ is rather special, but it is easy to show that this value represents the geometric mean of the characteristic wavelength and the scalelength of the oscillator in the nth mode. That is, the wavelength at $x = 0$ and the amplitude for the nth state are

$$\begin{aligned}
\frac{\hbar^2 k^2(0)}{2m} = n\hbar\omega \qquad &\Rightarrow \qquad k \sim \sqrt{2n}\,\alpha \\
\tfrac{1}{2}m\omega^2 L^2 = n\hbar\omega \qquad &\Rightarrow \qquad L \sim \sqrt{2n}\,\alpha^{-1} \qquad (5.10) \\
\text{or} \qquad &\lambda L \sim L/k \sim \alpha^{-2} \sim \sigma^2
\end{aligned}$$

Unlike the Wigner function for this example (4.7), the density $P_n(x,k)$ exhibits no oscillations in phase space; instead, as shown in Fig. 3, it rises as a power of the radius to a gaussian peak at $r_n = \sqrt{2n}$ which, for large n, is the radius of the classical dispersion surface. Applying Stirling's formula, one determines that the height of the peak at r_n

scales as $n^{-\frac{1}{2}}$ while its width approaches unity (in dimensionless units); these values are consistent with the fact that $P(x,k)$ is normalized over phase space by (5.4) so that the volume under it is unity. Thus, in the classical limit ($n \to \infty$), the peak shrinks in height although it becomes more localized around the classical orbit in the sense that the ratio of the width to the radius r_n decreases.

5.2 The Operator Basis for $\Psi(x,k)$ and a Connection with Weyl Symbols

Having discussed some of the properties of the coherent state representation, we shall now give a brief outline of the abstract operator structure which, as in the case of the symbol formalisms of previous Sections, provides the foundation for and insights into the development of the theory. The basic idea here is that $\Psi(x,k)$ is a true representation of the field ψ rather than of the spectral operator $\psi\psi^\dagger$ which was used in the previous symbol formalisms to represent the field. In this sense, it is more closely related to $\psi(x)$ or $\hat{\psi}(k)$ because it describes the field by its projection on some set of basis vectors in the abstract function space. Let a member of this basis set be denoted by ϕ so that in Dirac notation one has the transformation

$$\langle\phi|\psi\rangle = \int dx' \, \langle\phi|x'\rangle\langle x'|\psi\rangle \tag{5.11}$$

Comparing this to (5.1) with $\langle x'|\psi\rangle = \psi(x')$ and $\langle\phi|x'\rangle = \phi^*(x')$, it is apparent that the x-space representation of these basis states $\phi(x')$ is given by

$$\phi(x') = (\pi\sigma^2)^{-1/4} e^{ik(x'-x)} e^{-(x'-x)^2/2\sigma^2} \tag{5.12}$$

Evidently, each function ϕ has two labels, the values of x and k. Thus, each $\phi_{x,k}$ can be thought of as representing a point in phase space just as the quantity $|x\rangle$ represents a point in x-space; a difference is that while the x'-representation of $|x\rangle$ is a δ-function,

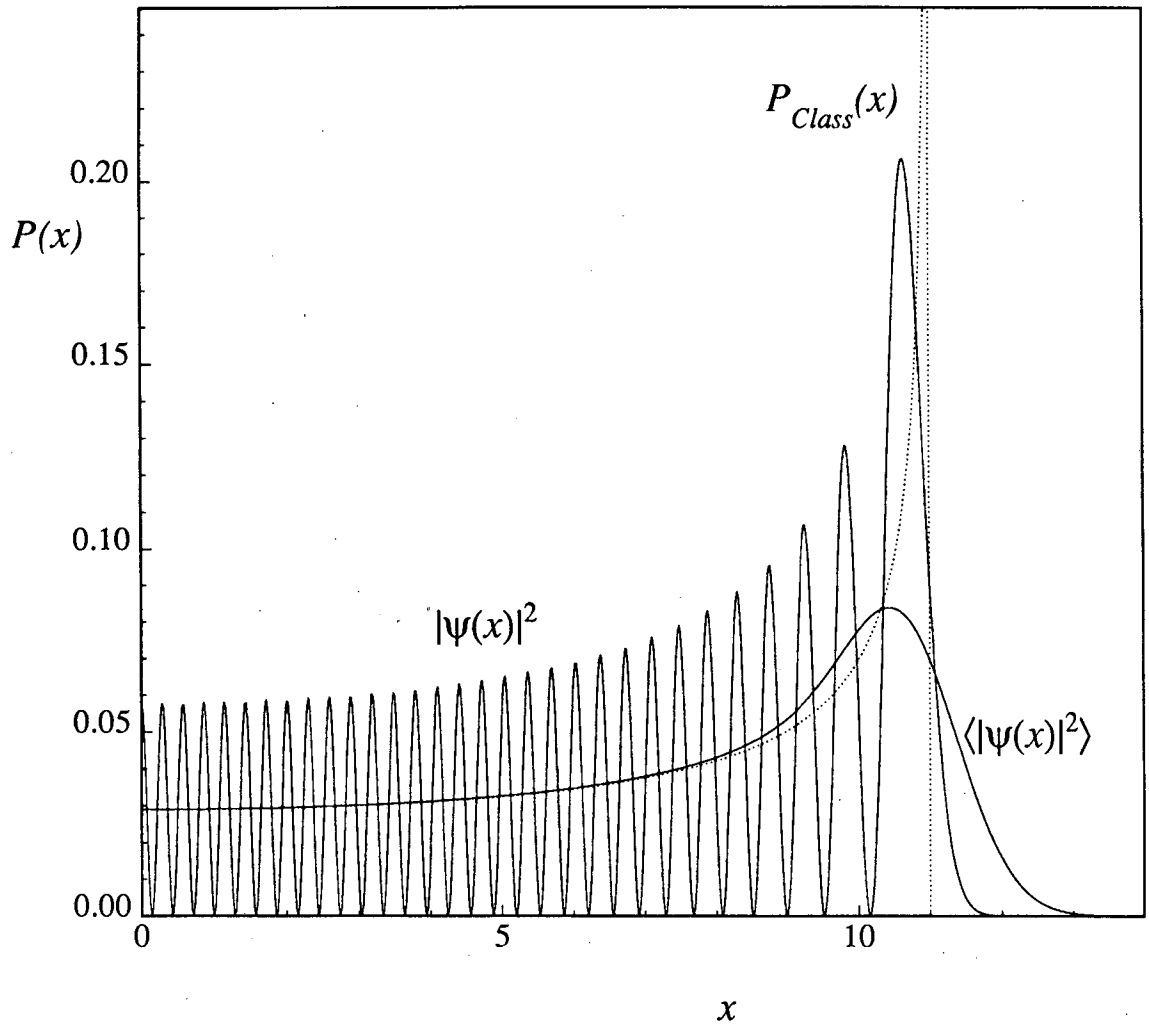

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Fig. 3. Comparison of x-space probability densities (wave intensities) for the harmonic oscillator state $n = 60$. Solid oscillatory curve is exact $|\psi_{60}(x)|^2$, dotted singular curve is exact classical ray probability density (or conventional eikonal intensity), and solid non-oscillatory curve is $\langle|\psi_{60}|^2\rangle$, the gaussian-smoothed wave intensity obtained by projecting $P_{60}(x,k)$ (with $\sigma = \alpha^{-1}$) from phase space onto x-space.

the x'-representation (5.12) of $\phi_{x,k}$ is a gaussian around x (and it is easy to show that its k'-representation is a gaussian around k). Indeed, these elements $\phi_{x,k}$ are often called "gaussian basis states" since their projection onto either x- or k-space is gaussian.

Before discussing other properties of these basis elements, it may be observed that the function $\phi_{x,k}(y)$ satisfies the following differential equation

$$\left[\frac{x}{\sigma} + \sigma\frac{d}{dy}\right]\phi_{x,k}(y) = \left[\frac{x}{\sigma} + i\sigma k\right]\phi_{x,k}(y) \tag{5.13}$$

With the identification of $-i(d/dy)$ as the x-representation of the operator $\mathbf{k}$, it is seen that $\phi_{x,k}$ is an eigenstate of the operator

$$\begin{aligned} \mathbf{a}\,|\phi_{x,k}\rangle &= (\frac{x}{\sigma} + i\sigma k)|\phi_{x,k}\rangle \\ \mathbf{a} &\equiv \frac{\mathbf{x}}{\sigma} + i\sigma\mathbf{k} \end{aligned} \tag{5.14}$$

This operator $\mathbf{a}$ is manifestly non-Hermitian and consequently its eigenvalues are complex with real and imaginary parts given by x/σ and σk respectively. Moreover, the form of $\mathbf{a}$ is familiar from quantum mechanics where it is often introduced (with a factor of $2^{-\frac{1}{2}}$ and $\sigma = \sqrt{\hbar/m\omega}$) as the lowering operator in the analysis of the harmonic oscillator. It may be shown [21] that the basis states $\phi_{x,k}$ are not orthogonal for different labels (x,k); however, the overlap integral of $\phi_{x,k}(y)$ and $\phi_{x',k'}(y)$ does diminish exponentially as $(x'-x)^2/\sigma^2 + \sigma^2(k'-k)^2$ increases. It is also a fact that the basis states $\phi_{x,k}$ form an overcomplete set [21].

With this brief account of the basic framework of the coherent state representation, we can now derive an important relationship between the coherent state phase-space density $P(x,k)$ and the Weyl spectral function $S(x,k)$. Consider the Weyl symbol $W_{x,k}$ of the spectral operator $|\phi_{x,k}\rangle\langle\phi_{x,k}|$, or in other words, the Wigner function associated with the eigenstate $\phi_{x,k}$:

$$
\begin{aligned}
W_{x,k}(x',k') &= \int ds\ \phi_{x,k}(x'+\tfrac{1}{2}s)\phi^*_{x,k}(x'-\tfrac{1}{2}s)e^{-ik's} \\
&= (\pi\sigma^2)^{-1/2}\int ds\ e^{-(x'-x+\frac{1}{2}s)^2/2\sigma^2}e^{-(x'-x-\frac{1}{2}s)^2/2\sigma^2} \\
&\qquad \times e^{ik(x'-x+\frac{1}{2}s)}e^{-ik(x'-x-\frac{1}{2}s)}e^{-ik's} \\
&= 2e^{-(x'-x)^2/\sigma^2}e^{-\sigma^2(k'-k)^2}
\end{aligned}
\tag{5.15}
$$

This formula explicitly illustrates the notion that the state $\phi_{x,k}$ represents the point (x,k) in phase space. Now the results of (4.41–4.43) of Section 4 may be directly applied by forming the trace of the operator product

$$
\begin{aligned}
\mathrm{Tr}(|\phi_{x,k}\rangle\langle\phi_{x,k}||\psi\rangle\langle\psi|) &= |\langle\phi_{x,k}|\psi\rangle|^2 = P(x,k) \\
P(x,k) &= \int dx'\frac{dk'}{2\pi}\ W_{x,k}(x',k')\ S(x',k') \\
&= 2\int dx'\frac{dk'}{2\pi}\ S(x',k')e^{-(x'-x)^2/\sigma^2}e^{-\sigma^2(k'-k)^2}
\end{aligned}
\tag{5.16}
$$

This relation shows that $P(x,k)$ is a local gaussian-weighted phase-space average of the Wigner function. It is interesting that the result of such a coarse-graining is non-negative (as $P \equiv |\Psi|^2$) independent of the value chosen for the smoothing length σ. Unlike the phase-space integrals (4.9,4.10) encountered in previous Sections which transform one symbol representation into another, and in contrast to the defining integrals (5.1,5.2) which also involve smoothing, this connection between $P(x,k)$ and $S(x,k)$ is noninvertible.

In addition to being a smooth phase-space density, $P(x,k)$ induces locally averaged statistical quantities as well. Projecting (5.16) onto x-space by integrating over k, one obtains

$$\begin{aligned}\int \frac{dk}{2\pi}\, P(x,k) &= 2\int dx' \frac{dk'}{2\pi}\, S(x',k') e^{-(x'-x)^2/\sigma^2} \int \frac{dk}{2\pi}\, e^{-\sigma^2(k'-k)^2} \\ &= \frac{1}{\sigma\sqrt{\pi}} \int dx'\, e^{-(x'-x)^2/\sigma^2} \int \frac{dk'}{2\pi}\, S(x',k') \\ &= \frac{1}{\sigma\sqrt{\pi}} \int dx'\, |\psi(x')|^2\, e^{-(x'-x)^2/\sigma^2} \\ &\equiv \langle |\psi(x)|^2 \rangle_\sigma \end{aligned} \tag{5.17}$$

where (4.37) has been used for the projection of S. The result is the locally-averaged (over length σ) field intensity in x-space. The Fourier transform of (5.16) gives

$$\begin{aligned}\int \frac{dk}{2\pi}\, P(x,k) e^{iks} &= 2\int dx' \frac{dk'}{2\pi}\, S(x',k') e^{-(x'-x)^2/\sigma^2} e^{ik's} \\ &\qquad \times \int \frac{dk}{2\pi}\, e^{-i(k'-k)s} e^{-\sigma^2(k'-k)^2} \\ &= \frac{e^{-s^2/4\sigma^2}}{\sigma\sqrt{\pi}} \int dx'\, e^{-(x'-x)^2/\sigma^2} \int \frac{dk'}{2\pi}\, S(x',k') e^{ik's} \\ &= \frac{e^{-s^2/4\sigma^2}}{\sigma\sqrt{\pi}} \int dx'\, \psi(x'+\tfrac{1}{2}s)\psi^*(x'-\tfrac{1}{2}s) e^{-(x'-x)^2/\sigma^2} \\ &\equiv e^{-s^2/4\sigma^2} \langle \psi(x+\tfrac{1}{2}s)\psi^*(x-\tfrac{1}{2}s) \rangle_\sigma \end{aligned} \tag{5.18}$$

again with the use of (4.37). Thus, the local σ-averaged spatial autocorrelation function is

$$\begin{aligned} C_\psi(x,s) &= \frac{\langle \psi(x+\frac{1}{2}s)\psi^*(x-\frac{1}{2}s) \rangle_\sigma}{\langle |\psi(x)|^2 \rangle_\sigma} \\ &= e^{s^2/4\sigma^2} \frac{\int dk\; P(x,k) e^{iks}}{\int dk\; P(x,k)} \end{aligned} \tag{5.19}$$

Evidently, the Fourier transform of $P(x,k)$ in (5.18) has an inherent exponential decay in s which must be corrected for in (5.19). Similar expressions for the local k-space average intensity and correlation function also exist. These relations explicitly illustrate the result of Wigner [37] that any non-negative phase-space density cannot yield the exact x- or k-space intensities upon projection; here, locally-averaged intensities are obtained although (5.3) may be used to compute the exact intensities from $\Psi(x,k)$ itself. Because

of these properties, the coherent state representation appears to offer closer contact with experimentally measured quantities (such as the local field intensity, the field correlation and spectral functions) since these measurements can always be considered to be coarse-grained in some sense, and usually refer to a single realization of the field.

5.3 The Phase-Space Equation for $\Psi(x,k)$

Having discussed some of the important general properties of the coherent state representation $\Psi(x,k)$ and its associated phase-space density $P(x,k)$, we shall now turn to the derivation of the equation (corresponding to (3.1)) which governs the evolution of Ψ. As in preceding Sections, this is most easily achieved in a rather indirect fashion, although unlike the methods used previously, we shall not introduce a "coherent state symbol calculus". Instead, we shall first derive yet another relationship between this representation and the Weyl formalism so that the Weyl symbol calculus may be immediately applied.

To this end, consider the form of the definition (5.1) when Ψ is evaluated at $(2x,2k)$:

$$\Psi(2x,2k) = (\pi\sigma^2)^{-1/4} \int dx'\ \psi(x') e^{-(x'-2x)^2/2\sigma^2} e^{-2ik(x'-2x)} \tag{5.20}$$

which, although unmotivated, can be manipulated with the change of variables $x' = x + \frac{1}{2}s$ to become

$$\Psi(2x,2k) = \tfrac{1}{2}(\pi\sigma^2)^{-1/4} e^{2ikx} \int ds\ \psi(x+\tfrac{1}{2}s) e^{-(x-\frac{1}{2}s)^2/2\sigma^2} e^{-iks} \tag{5.21}$$

The form of this integral is much like that used in the definition of Weyl symbols (4.1,4.36) except that two different functions appear in the integrand. Indeed, recalling the definition (5.12) of the coherent state basis functions, this is the Weyl symbol or *mixed Wigner function* associated with the operator and kernel

$$|\psi\rangle\langle\phi_{0,0}| \qquad \leftrightarrow \qquad W_{\psi\phi_{0,0}}(x,y) = (\pi\sigma^2)^{-1/4} \psi(x) e^{-y^2/2\sigma^2} \tag{5.22}$$

which is composed of the field ψ and the "ground state" of the coherent basis representing the origin in phase space. Therefore, (5.21) may be written

$$2\Psi(2x,2k)e^{-2ikx} = W_{\psi\phi_{0,0}}(x,k) \tag{5.23}$$

Now that the local Fourier transform (5.1) which defines the coherent state representation has also been identified as the Weyl symbol of some operator (albeit, with a nonlocal correspondence), the Weyl product rule can be invoked to translate the operator equation which governs $(\psi\phi^{\dagger}_{0,0})$ into a phase-space equation for $\Psi(x,k)$. The abstract representation-free equation for the field is again, from (3.2)

$$\mathbf{D}|\psi\rangle = |F\rangle \tag{5.24}$$

where $\mathbf{D}$ is the wave dispersion operator and $|F\rangle$ is the source field. Multiplying this equation from the right by $\langle\phi_{0,0}|$, one has

$$\mathbf{D}|\psi\rangle\langle\phi_{0,0}| = |F\rangle\langle\phi_{0,0}| \tag{5.25}$$

A similar step was taken in the derivation of the equation (3.27) for the field spectral operator $|\psi\rangle\langle\psi|$, followed by the introduction of the adjoint of (5.24). That step was crucial because it specified that the adjoint or dual element introduced in the multiplication was indeed the dual element of the field $\langle\psi|$. Here, in order to complete the specification of the representation, one must supply an operator equation for $|\phi_{0,0}\rangle$. Since $\phi_{0,0}$ is the ground state in the coherent basis, (5.14) holds with eigenvalue $x = k = 0$

$$\mathbf{a}|\phi_{0,0}\rangle = 0 \tag{5.26}$$

with the adjoint

$$\langle\phi_{0,0}|\mathbf{a}^{\dagger} = \langle\phi_{0,0}|\left(\frac{\mathbf{x}}{\sigma} - i\sigma\mathbf{k}\right) = 0 \tag{5.27}$$

Now, multiplying (5.27) by $|\psi\rangle$ from the left, one finds that the mixed density operator must satisfy

$$|\psi\rangle\langle\phi_{0,0}|\left(\frac{\mathbf{x}}{\sigma}-i\sigma\mathbf{k}\right)=0 \tag{5.28}$$

with a similar expression for the source $|F\rangle\langle\phi_{0,0}|$. The entire set of operator equations which correspond to (5.24) for this representation are therefore

$$\begin{aligned}\mathbf{D}|\psi\rangle\langle\phi_{0,0}| &= |F\rangle\langle\phi_{0,0}| \\ |\psi\rangle\langle\phi_{0,0}|\left(\frac{\mathbf{x}}{\sigma}-i\sigma\mathbf{k}\right) &= 0 \\ |F\rangle\langle\phi_{0,0}|\left(\frac{\mathbf{x}}{\sigma}-i\sigma\mathbf{k}\right) &= 0\,.\end{aligned} \tag{5.29}$$

These operator relations can be immediately written as phase-space equations for the corresponding Weyl symbols using the product or composition rule (4.29) from the previous Section. The result is

$$\begin{aligned}D(x,k)e^{\frac{i}{2}\overleftrightarrow{\mathcal{L}}}W_{\psi\phi_{0,0}}(x,k) &= W_{F\phi_{0,0}}(x,k) \\ W_{\psi\phi_{0,0}}(x,k)e^{\frac{i}{2}\overleftrightarrow{\mathcal{L}}}\left(\frac{x}{\sigma}-i\sigma k\right) &= 0 \\ W_{F\phi_{0,0}}(x,k)e^{\frac{i}{2}\overleftrightarrow{\mathcal{L}}}\left(\frac{x}{\sigma}-i\sigma k\right) &= 0\end{aligned} \tag{5.30}$$

where the Weyl symbol of the operator a is simply given by its scalar form since it involves no xk products. Finally, the equations for the mixed Wigner functions $W_{\psi\phi_{0,0}}$ and $W_{F\phi_{0,0}}$ become equations for the corresponding coherent state representations Ψ and Ψ_F with the use of (5.23). Thus, the first of (5.30) may be written

$$D(x,k)e^{\frac{i}{2}\overleftrightarrow{\mathcal{L}}}\Psi(2x,2k)e^{-2ikx}=\Psi_F(2x,2k)e^{-2ikx} \tag{5.31}$$

which is subject to the conditions provided by the last two of (5.30)

$$\begin{aligned}\Psi(2x,2k)e^{-2ikx}e^{\frac{i}{2}\overleftrightarrow{\mathcal{L}}}\left(\frac{x}{\sigma}-i\sigma k\right) &= 0 \\ \Psi_F(2x,2k)e^{-2ikx}e^{\frac{i}{2}\overleftrightarrow{\mathcal{L}}}\left(\frac{x}{\sigma}-i\sigma k\right) &= 0\end{aligned} \tag{5.32}$$

These are the equations which govern the coherent state representation of the field $\Psi(x,k)$ defined in (5.1). The main equation (5.31) is driven by the local Fourier transform Ψ_F of the source field which is also defined by (5.1) in terms of $F(x)$. The local dispersion function $D(x,k)$ which appears here is not the coherent state representation of the operator **D** (indeed, such a quantity has not even been defined in this presentation); instead, it is the usual Weyl symbol of **D** constructed from the two-point dispersion kernel under the "centered" Weyl transform (4.1). This is a (possibly infinite order) partial differential equation for $\Psi(x,k)$ with the operator $e^{\frac{i}{2}\overleftrightarrow{\mathcal{L}}}$ defined by (4.29).

While (5.31) is the basic equation of interest since it contains the information about the medium in $D(x,k)$, it must be solved subject to the conditions given in (5.32). Naturally, from the procedure which was used to develop these equations, these conditions should be automatically satisfied; that is, if Ψ or Ψ_F is constructed from ψ and F by (5.1), then direct computation verifies that (5.32) are identically satisfied. However, if the field ψ is unknown and (5.31) is used to determine Ψ (and hence ψ, by (5.3)) these conditions must be solved in conjunction with (5.31). More generally, if time is included so that (5.31) describes the propagation of Ψ from initial conditions, the subsidiary equations might have to be applied in order that $\Psi(x,k;t)$ remains of the form of a local Fourier transform (5.1).

5.4 Asymptotic Analysis: The Eikonal Theory in Phase Space

The discussion of the properties of $\Psi(x,k)$ and its density $P(x,k)$ encourage the investigation of the set of equations (5.31,5.32); the basic motivation lies in the observation that this representation holds the promise of producing a much smoother phase-space density than found in previous formalisms. The hope is that these equations can be

treated in the asymptotic short-wavelength regime in such a way that the construction of the solution involves the phase-space ray trajectories yet does not generate singularities at caustics. Again, as this formulation is inherently a phase-space technique with x and k treated as independent variables, the reliance on the eikonal description (and thus, the identification of $k(x)$) is absent.

It is convenient to cast Ψ in a form which automatically satisfies (5.32) (so that these conditions can be discarded) by introducing the complex dimensionless variables

$$\begin{aligned} z &= \frac{1}{\sqrt{2}}\left(\frac{x}{\sigma} + i\sigma k\right) \qquad & \overline{z} &= \frac{1}{\sqrt{2}}\left(\frac{x}{\sigma} - i\sigma k\right) \\ x &= \frac{\sigma}{\sqrt{2}}(z+\overline{z}) \qquad & k &= \frac{-i}{\sigma\sqrt{2}}(z-\overline{z}) \end{aligned} \tag{5.33}$$

where $\overline{z}$ denotes the complex conjugate of z. Substituting $(z,\overline{z})$ for (x,k) in the definition (5.1) of $\Psi(x,k)$, one has

$$\begin{aligned} \Psi(x,k) = \tilde{\Psi}(z,\overline{z}) &= (\pi\sigma^2)^{-1/4}\int dx'\ \psi(x')e^{-i(\frac{-i}{\sigma\sqrt{2}})(z-\overline{z})(x'-\frac{\sigma}{\sqrt{2}}(z+\overline{z}))} \\ &\qquad \times e^{-(x'-\frac{\sigma}{\sqrt{2}}(z+\overline{z}))^2/2\sigma^2} \\ &= (\pi\sigma^2)^{-1/4}e^{\frac{1}{4}(z-\overline{z})^2}\int dx'\ \psi(x')e^{-(x'-\sqrt{2}\,\sigma\overline{z})^2/2\sigma^2} \end{aligned} \tag{5.34}$$

Interestingly, the integral involves only the combination $\overline{z}$ of x and k. Furthermore, the basic equation (5.31) and the conditions (5.32) include the multiplicative phase factor e^{-2ikx} with $\Psi(2x,2k)$ so that one should consider the form of [21]

$$\begin{aligned} \tilde{\Psi}(z,\overline{z})e^{-\frac{i}{2}kx} &= (\pi\sigma^2)^{-1/4}e^{-\frac{1}{4}(z^2-\overline{z}^2)}e^{\frac{1}{4}(z-\overline{z})^2}\int dx'\ \psi(x')e^{-(x'-\sqrt{2}\,\sigma\overline{z})^2/2\sigma^2} \\ &= (\pi\sigma^2)^{-1/4}e^{-\frac{1}{2}z\overline{z}}e^{\frac{1}{2}\overline{z}^2}\int dx'\psi(x')e^{-(x'-\sqrt{2}\,\sigma\overline{z})^2/2\sigma^2} \\ &\equiv e^{-\frac{1}{2}z\overline{z}}\Phi(\overline{z}) \end{aligned} \tag{5.35}$$

Thus, the only dependence on the combination of x and k given by z is in the gaussian factor $\exp(-\frac{1}{2}|z|^2)$ and the remainder of $\Psi(x,k)e^{-\frac{i}{2}kx}$ has been assembled in the definition of $\Phi(\overline{z})$.

In order to see how these complex coordinates simplify the conditions (5.32), the bi-directional operator $\overleftrightarrow{\mathcal{L}}$ should be expressed in terms of $(z,\bar{z})$. The form of the derivatives with respect to $(z,\bar{z})$ follow from (5.33)

$$\begin{aligned} \partial_x &= \frac{1}{\sigma\sqrt{2}}(\partial_z + \partial_{\bar{z}}) & \partial_k &= \frac{i\sigma}{\sqrt{2}}(\partial_z - \partial_{\bar{z}}) \\ \partial_z &= \frac{1}{\sqrt{2}}(\sigma\partial_x - i\sigma^{-1}\partial_k) & \partial_{\bar{z}} &= \frac{1}{\sqrt{2}}(\sigma\partial_x + i\sigma^{-1}\partial_k) \end{aligned} \tag{5.36}$$

and it is easily verified that $\overleftrightarrow{\mathcal{L}}$ becomes

$$\overleftrightarrow{\mathcal{L}} = \overleftarrow{\partial_x}\overrightarrow{\partial_k} - \overleftarrow{\partial_k}\overrightarrow{\partial_x} = i(\overleftarrow{\partial_{\bar{z}}}\overrightarrow{\partial_z} - \overleftarrow{\partial_z}\overrightarrow{\partial_{\bar{z}}}) \tag{5.37}$$

As $\overleftrightarrow{\mathcal{L}}$ has been previously identified as the classical Poisson bracket operator, this result implies that the change of variables $(x,k) \rightarrow (\bar{z}, -iz)$ is a (complex) canonical transformation on phase space.

Recognizing that the function on the right of the exponentiated operator in (5.32) is just $\bar{z}$, the substitution of (5.35) and (5.37) into this equation produces

$$e^{-2z\bar{z}}\Phi(2\bar{z})\; e^{-\frac{1}{2}(\overleftarrow{\partial_{\bar{z}}}\overrightarrow{\partial_z} - \overleftarrow{\partial_z}\overrightarrow{\partial_{\bar{z}}})}\; \bar{z} = 0 \tag{5.38}$$

Expanding the exponential operator in power series, only the first two terms survive since the right operand is linear in $\bar{z}$; (5.38) becomes

$$e^{-2z\bar{z}}\Phi(2\bar{z})[1 + \tfrac{1}{2}\overleftarrow{\partial_z}\overrightarrow{\partial_{\bar{z}}}]\; \bar{z} = e^{-2z\bar{z}}\Phi(2\bar{z})(\bar{z} - \bar{z}) = 0 \tag{5.39}$$

Hence, expressing Ψ in terms of $(z,\bar{z})$ in the form (5.35) identically satisfies the condition (5.32) on Ψ and the same treatment of the source field would of course relieve the condition on Ψ_F.

Having dispensed with these subsidiary conditions, attention may now be focused on the basic equation (5.31) which governs Ψ. In complex coordinates $(z,\bar{z})$ and with the

definition (5.35) this is

$$\tilde{D}(z,\overline{z})\; e^{\frac{1}{2}(\overleftarrow{\partial_z}\overrightarrow{\partial_{\overline{z}}}-\overleftarrow{\partial_{\overline{z}}}\overrightarrow{\partial_z})}\; e^{-2z\overline{z}}\Phi(2\overline{z}) = e^{-2z\overline{z}}\Phi_F(2\overline{z}) \tag{5.40}$$

where the functions of complex variables are defined by

$$\begin{aligned} \tilde{D}(z,\overline{z}) &\equiv D(x,k) \\ e^{-\frac{i}{2}kx}\Psi_F(x,k) &\equiv e^{-\frac{1}{2}z\overline{z}}\Phi_F(\overline{z}) \end{aligned} \tag{5.41}$$

For simplicity in the following analysis of this equation, we shall continue to consider only one-dimensional scalar fields, and we shall now also make the following assumptions: (1) $D(x,k)$ is real (no damping), and (2) $\Psi_F(x,k) = 0$ (no sources). Both of these assumptions as well as the stipulation of scalar fields in one dimension could be relaxed in the ensuing discussion, although somewhat tedious calculations similar to those detailed in the Appendix for the development of Section 4 would be required. As our procedure will again involve ordering arguments, the inclusion of weak damping and sources should be straightforward at the appropriate order.

From the outset, the method we shall employ to solve the phase-space equation (5.40) differs from that used in Section 4 for the Weyl symbol S in that while the assumption that the Wigner function is slowly varying (being quadratic in the field ψ) was substantiated by the result (4.67), inspection of the examples given in (5.6,5.7,5.9) of the form of $\Psi(x,k)$ indicate that it displays the same rapid phase behavior as the field in x-space from which it is constructed. It must be remembered that the equation under consideration (5.31,5.40) governs Ψ itself (linear in the field ψ) rather than the phase-space density $P(x,k)$ which exhibits a much smoother, positive and non-oscillatory behavior in phase space.

Again considering short-wavelength solutions to (5.40) in a weakly inhomogeneous medium, the conditions on the variation of D are

$$|\partial_x D(x,k)| \sim L^{-1} \qquad\qquad |\partial_k D(x,k)| \sim \lambda \tag{5.42}$$

where the scalelength L is to be much larger than a typical local wavelength λ. For use in (5.40), these estimates should be written in terms of the complex variables $(z,\overline{z})$; with (5.36) these become

$$\begin{aligned}\left|\partial_z \tilde{D}(z,\overline{z})\right| &\sim \left|(\sigma\partial_x + i\sigma^{-1}\partial_k)D(x,k)\right| \sim \left|\frac{\sigma}{L} + i\frac{\lambda}{\sigma}\right| \\ \left|\partial_{\overline{z}} \tilde{D}(z,\overline{z})\right| &\sim \left|(\sigma\partial_x - i\sigma^{-1}\partial_k)D(x,k)\right| \sim \left|\frac{\sigma}{L} - i\frac{\lambda}{\sigma}\right|\end{aligned} \tag{5.43}$$

Thus, the variation of $\tilde{D}$ with respect to either of the dimensionless complex variables is the same. We shall now take the smoothing length σ to be intermediate between the two length scales

$$\lambda \ll \sigma \ll L \tag{5.44}$$

and, for concreteness, we shall choose it to be approximately the geometric mean

$$\sigma \approx \sqrt{L\lambda} \qquad \Rightarrow \qquad \frac{\sigma}{L} \approx \frac{\lambda}{\sigma} \ll 1 \tag{5.45}$$

As a consequence, the terms in the magnitude of $(\partial_z \tilde{D})$ and $(\partial_{\overline{z}} \tilde{D})$ in (5.43) are roughly equal and both quantities are much less than unity.

Since it has been pointed out that $\Psi(x,k)$ exhibits a rapidly oscillating wave structure on phase space, we propose to solve (5.40) in a manner very similar to conventional eikonal methods. That is, the (x,k) plane will be treated as a two-dimensional configuration space (since in this formalism x and k are independent) and the usual eikonal techniques will be applied. In this spirit, we assume a solution of the form

$$\Phi(\overline{z}) = \Gamma(\overline{z})e^{i\Theta(\overline{z})} \tag{5.46}$$

and define the "local wavenumber" in phase space to be

$$\kappa(\overline{z}) \equiv \frac{d\Theta(\overline{z})}{d\overline{z}} \equiv \Theta'(\overline{z}) \tag{5.47}$$

In terms of the ansatz (5.46), it should be recalled from (5.35) that the desired solution $\Psi(x,k)$ is

$$\begin{aligned}\Psi(x,k) &= \tilde{\Psi}(z,\bar{z}) = e^{\frac{i}{2}kx}e^{-\frac{1}{2}z\bar{z}}\Phi(\bar{z}) \\ &= e^{\frac{1}{4}z^2-\frac{1}{4}\bar{z}^2-\frac{1}{2}z\bar{z}}e^{i\Theta(\bar{z})}\Gamma(\bar{z}) \\ &= e^{\frac{i}{2}kx}e^{-\frac{1}{4}(x^2/\sigma^2+\sigma^2k^2)}e^{i\Theta(\frac{1}{\sqrt{2}}(\frac{x}{\sigma}-i\sigma k))}\Gamma(\tfrac{1}{\sqrt{2}}(\tfrac{x}{\sigma}-i\sigma k))\end{aligned} \tag{5.48}$$

As in the traditional eikonal procedure, one should first postulate the expected relative ordering of the derivatives of the wavenumber $\kappa(\bar{z})$ and the amplitude $\Gamma(\bar{z})$. For this, we are guided by the forms of κ and Γ found in the exact solutions (5.6) and (5.9). For the plane wave example (5.6), it is easily shown that $\Psi(x,k)$ expressed in terms of $(z,\bar{z})$ is

$$\tilde{\Psi}(z,\bar{z}) \sim e^{i\sqrt{2}\sigma k_0\bar{z}+\frac{1}{4}z^2+\frac{1}{4}\bar{z}^2-\frac{1}{2}z\bar{z}-\frac{1}{2}\sigma^2k_0^2} \tag{5.49}$$

Setting this equal to the second line of (5.48), the phase $\Theta(\bar{z})$ is determined to be

$$\Theta(\bar{z}) = \sqrt{2}\sigma k_0\bar{z} - \tfrac{i}{2}\bar{z}^2 - \tfrac{i}{2}\sigma^2k_0^2 \tag{5.50}$$

so that by (5.47) $\kappa(\bar{z})$ is

$$\kappa(\bar{z}) = \sqrt{2}\,\sigma k_0 - i\bar{z} \tag{5.51}$$

These formulas indicate that unlike the usual application of the eikonal method, the eikonal phase and its derivative will in general be complex in this formalism. That these imaginary contributions should be included in the phase (as opposed to being in the amplitude Γ) can be justified by noting the relative sharpness of the gaussian peaks in the examples compared to their location in phase space. In the plane wave case, inspection of (5.6) reveals a gaussian peak at $k = k_0$ of width $1/\sigma$ so that this modulation has relative scale $\sigma/\lambda_0 \gg 1$; this rapid variation therefore is appropriately included in the phase rather than in the slowly varying amplitude. Indeed, in this case the amplitude $\Gamma = \psi_0$ is constant.

The magnitude of κ in (5.51) is $\sim \sigma/\lambda_0$ which, again, is to be much greater than unity by choice of σ. In order for both terms which constitute κ to be of similar order, one must have that $|\overline{z}| \sim |x/\sigma - i\sigma k| \sim \sigma/\lambda_0$; or, by (5.45), both coordinates on phase space should be taken to be large, $(x,k) \sim (L,k_0)$. Thus, this plane wave example suggests that both $\Theta' = \kappa(\overline{z})$ and $(z,\overline{z})$ be taken as $O(\sigma/\lambda \sim L/\sigma)$.

These conclusions are supported by a similar analysis of the harmonic oscillator result (5.9). Expressing Ψ again in terms of $(z,\overline{z})$ one has

$$\tilde{\Psi}(z,\overline{z}) \sim e^{\frac{1}{4}z^2 - \frac{1}{4}\overline{z}^2 - \frac{1}{2}z\overline{z}}\,\overline{z}^n \tag{5.52}$$

and setting this equal to the second line of (5.48), the phase Θ and the local phase-space wavenumber κ are found to be simply

$$\Theta(\overline{z}) = -in\ln\overline{z} \qquad \Rightarrow \qquad \kappa(\overline{z}) = \frac{-in}{\overline{z}} \tag{5.53}$$

If κ and $\overline{z}$ are to be of the same order as suggested by the previous example, this relation implies that both have magnitude $\sim \sqrt{n}$. Indeed, this estimate is consistent with the estimates given in (5.10), which may be translated into conditions on $\overline{z}$, as well as the fact that the variation scale of (5.52) in the radial direction is $\sim n^{-1/2} \sim \kappa^{-1}$ (as given by the ratio of the gaussian width to the radius of the classical orbit). Further inspection of the solution (5.9) reveals that the oscillations of Ψ in the angular direction have wavelength $r^{-1}\partial_\theta \Psi(r,\theta) \sim n/r \sim n^{-1/2}$ which is also consistent with the assumed magnitude of κ.

These arguments therefore suggest that in the following eikonal treatment of (5.46) in (5.40), one should take

$$\Theta'(\overline{z}) = \kappa(\overline{z}) \sim \overline{z} \sim \frac{\sigma}{\lambda} \sim \frac{L}{\sigma} \equiv \frac{1}{\epsilon} \tag{5.54}$$

Once this scale has been set, the derivatives of κ and Γ may be asserted to be of lower order (in the usual eikonal spirit)

$$\left|\partial_{\overline{z}}^{n+1}\kappa\right| \sim \left|\Gamma^{-1}\partial_{\overline{z}}^{n}\Gamma\right| \sim \epsilon^n \qquad n \geq 0 \tag{5.55}$$

Observe that $\kappa' \sim O(1)$, as substantiated by (5.51) and (5.53). These estimates are accompanied by the weak inhomogeneity assumptions on the medium which follow from (5.43)

$$\left|\partial_z^n \tilde{D}(z,\overline{z})\right| \sim \left|\partial_{\overline{z}}^n \tilde{D}(z,\overline{z})\right| \sim \epsilon^n \tag{5.56}$$

Now (5.46) can be substituted into (5.40) to obtain

$$\tilde{D}(z,\overline{z})e^{\frac{1}{2}(\overleftarrow{\partial_z}\overrightarrow{\partial_{\overline{z}}}-\overleftarrow{\partial_{\overline{z}}}\overrightarrow{\partial_z})}\Gamma(2\overline{z})e^{i\Theta(2\overline{z})-2z\overline{z}} = 0 \tag{5.57}$$

where Ψ_F has been discarded as previously discussed. As before, the exponential operator is to be expanded in power series, the derivatives applied and the relative order of the resulting terms to be assessed. The first few terms in the expansion are

$$\begin{aligned}
&\tilde{D}(z,\overline{z})\Gamma(2\overline{z}) + \left\{\tilde{D}_z(z,\overline{z})[(i\kappa(2\overline{z})-z)\Gamma(2\overline{z})+\Gamma'(2\overline{z})] + \tilde{D}_{\overline{z}}(z,\overline{z})\overline{z}\Gamma(2\overline{z})\right\} \\
&\quad + \tfrac{1}{2}\left\{\tilde{D}_{zz}[(i\kappa-z)^2\Gamma + 2(i\kappa-z)\Gamma' + i\kappa'\Gamma + \Gamma''] \right. \\
&\qquad \left. + 2\tilde{D}_{z\overline{z}}[(i\kappa-z)\overline{z}\Gamma + \overline{z}\Gamma' + \tfrac{1}{2}\Gamma] + \tilde{D}_{\overline{z}\overline{z}}\overline{z}^2\Gamma\right\} \\
&\quad + \tfrac{1}{6}\left\{\tilde{D}_{zzz}[(i\kappa-z)^3\Gamma + 3(i\kappa-z)^2\Gamma' + 3(i\kappa-z)i\kappa'\Gamma \right. \\
&\qquad + 3(i\kappa-z)\Gamma'' + 3i\kappa'\Gamma' + i\kappa''\Gamma + \Gamma'''] \\
&\qquad + 3\tilde{D}_{zz\overline{z}}[(i\kappa-z)^2\overline{z}\Gamma + (i\kappa-z)\Gamma + 2(i\kappa-z)\overline{z}\Gamma' + \overline{z}i\kappa'\Gamma \\
&\qquad\quad + \Gamma' + \overline{z}\Gamma''] \\
&\qquad + 3\tilde{D}_{z\overline{z}\overline{z}}[(i\kappa-z)\overline{z}^2\Gamma + \overline{z}\Gamma + \Gamma'] + \tilde{D}_{\overline{z}\overline{z}\overline{z}}\overline{z}^3\Gamma\Big\} + \cdots
\end{aligned} \tag{5.58}$$

It should be noted that since the right operand of (5.57) is evaluated at the double argument $2\overline{z}$, κ and Γ as well as their derivatives should be evaluated at $2\overline{z}$ everywhere they appear in this expansion, just as in the first line. The derivatives denoted by primes therefore indicate differentiation with respect to the proper argument (here, $2\overline{z}$) and the factors of 2 from the chain rule have been incorporated in the numerical coefficients as

they appear. The arguments of $\tilde{D}$ are, of course, still $(z,\overline{z})$ and the subscripts denote partial derivatives. In addition, the multiplicative phase factors $\exp(i\Theta - 2z\overline{z})$ have been divided out.

Applying the ordering assumptions (5.54–5.56) to the terms in the expansion (5.58), the lowest-order collection is $O(1)$:

$$\begin{aligned}\Big\{\tilde{D}(z,\overline{z}) &+ [(i\kappa - z)\tilde{D}_z + \overline{z}\tilde{D}_{\overline{z}}] \\ &+ \tfrac{1}{2}[(i\kappa - z)^2\tilde{D}_{zz} + 2(i\kappa - z)\overline{z}\tilde{D}_{z\overline{z}} + \overline{z}^2\tilde{D}_{\overline{z}\overline{z}}] \\ &+ \tfrac{1}{6}[(i\kappa - z)^3\tilde{D}_{zzz} + 3(i\kappa - z)^2\overline{z}\tilde{D}_{zz\overline{z}} + 3(i\kappa - z)\overline{z}^2\tilde{D}_{z\overline{z}\overline{z}} \\ &+ \overline{z}^3\tilde{D}_{\overline{z}\overline{z}\overline{z}}] + \cdots\Big\}\Gamma(2\overline{z}) = 0\end{aligned} \tag{5.59}$$

where again, κ is meant to be $\kappa(2\overline{z})$. The dots indicate that although terms only through the third derivatives have been retained, the familiar Taylor series pattern is suggested (as in the similar treatment (3.59) of the common symbol). Thus, the terms in the braces of (5.59) is the Taylor expansion of the dispersion function about the point $(z,\overline{z})$ which can be resummed to give

$$\tilde{D}(z + (i\kappa - z), \overline{z} + \overline{z}) = \tilde{D}(i\kappa(2\overline{z}), 2\overline{z}) \tag{5.60}$$

The lowest order equation may now be written

$$\begin{aligned}\tilde{D}(i\kappa(2\overline{z}), 2\overline{z})\Gamma(2\overline{z}) &= 0 \\ \text{or} \qquad \tilde{D}(i\kappa(\overline{z}), \overline{z}) &= 0\end{aligned} \tag{5.61}$$

Here, since only $2\overline{z}$ appears as an argument, it has been replaced by just $\overline{z}$.

Inspection of (5.58) shows that under the ordering (5.54–5.56) only even powers of ϵ are represented in the expansion. It is interesting to note that this fact implies that the true expansion parameter is the physical quantity $\epsilon^2 \sim (\sigma/L)(\lambda/\sigma) \sim (\lambda/L)$ which does not involve the value of the arbitrary smoothing length σ (although the inequality (5.44)

must still apply). Assembling the next higher order $O(\epsilon^2)$ terms of (5.58), one has

$$\begin{aligned}
&\Big\{\tilde{D}_z(z,\overline{z}) + [(i\kappa - z)\tilde{D}_{zz} + \overline{z}\tilde{D}_{z\overline{z}}] \\
&\qquad + \tfrac{1}{2}[(i\kappa - z)^2\tilde{D}_{zzz} + 2(i\kappa - z)\overline{z}\tilde{D}_{zz\overline{z}} + \overline{z}^2\tilde{D}_{z\overline{z}\overline{z}}] + \cdots\Big\}\,\Gamma'(2\overline{z}) \\
&\qquad + \tfrac{1}{2}\Big\{\tilde{D}_{zz} + [(i\kappa - z)\tilde{D}_{zzz} + \overline{z}\tilde{D}_{zz\overline{z}}] + \cdots\Big\}\,i\kappa'(2\overline{z})\Gamma(2\overline{z}) \\
&\qquad + \tfrac{1}{2}\Big\{\tilde{D}_{z\overline{z}} + [(i\kappa - z)\tilde{D}_{zz\overline{z}} + \overline{z}\tilde{D}_{z\overline{z}\overline{z}}] + \cdots\Big\}\,\Gamma(2\overline{z}) = 0
\end{aligned} \tag{5.62}$$

As in the lowest order equation, the terms have been arranged to suggest the appearance of the Taylor series for $\tilde{D}_z$, $\tilde{D}_{zz}$ and $\tilde{D}_{z\overline{z}}$ around the point $(z,\overline{z})$. Thus, these can be re-summed as before to give

$$\begin{aligned}
&\tilde{D}_z(i\kappa(2\overline{z}),2\overline{z})\Gamma'(2\overline{z}) + \tfrac{1}{2}[i\kappa'(2\overline{z})\tilde{D}_{zz}(i\kappa(2\overline{z}),2\overline{z}) \\
&\qquad\qquad + \tilde{D}_{z\overline{z}}(i\kappa(2\overline{z}),2\overline{z})]\Gamma(2\overline{z}) = 0
\end{aligned} \tag{5.63}$$

which, since all quantities are evaluated at $2\overline{z}$, becomes

$$\begin{aligned}
\tilde{D}_z(i\kappa,\overline{z})\frac{d\Gamma(\overline{z})}{d\overline{z}} &= -\tfrac{1}{2}\left[\left(\frac{\partial\tilde{D}_z}{\partial\overline{z}}\right)_{\kappa} + \left(\frac{\partial\tilde{D}_z}{\partial z}\right)_{\overline{z}}\frac{di\kappa(\overline{z})}{d\overline{z}}\right]\Gamma(\overline{z}) \\
&= -\tfrac{1}{2}\left(\frac{d\tilde{D}_z}{d\overline{z}}\right)\Gamma(\overline{z})
\end{aligned} \tag{5.64}$$

The results at the lowest two orders of this perturbation method seem to imply that the nonlocal nature of the exact equation (5.57) might just be an artifact of the differential representation. That is, although the left and right operands of (5.57) are evaluated at different points in phase space, the technique of expanding the exponential operator, assuming a solution of the form (5.46), differentiating and re-summing terms at each order (assuming convergence) finally produces the local equations (5.61,5.64). In this regard, it is crucial that all terms in the expansion of the exponential operator be retained because the ordering (5.54) generates low-order contributions at each power of $\overleftrightarrow{\mathcal{L}}$.

The application of "conventional" eikonal concepts to the phase-space equation (5.57) results in the two lowest-order equations (5.61) and (5.64) which bear a striking resem-

blence to the corresponding eikonal and amplitude transport equations (2.17,2.18) that emerge in the traditional analysis on x-space. The ordering assumptions (5.54) are slightly different than in the customary eikonal formulation although it was shown that the true expansion parameter turns out to be the same (λ/L)). Most significant however, is that the final equations involve only *one complex dimension* describing a wave in the two-dimensional phase space.

The one-dimensional analysis given here has neglected the time dependence of the wave and the dispersion kernel; thus, in this model of a stationary medium, the field ψ (and consequently, Ψ) has an overall $\exp(-i\omega t)$ time dependence (which factors out of the equations) and $D(x,k)$ should be parameterized by the frequency ω. Accounting for this in (5.61), the eikonal equation is

$$\tilde{D}(i\kappa,\overline{z};\omega)=0 \qquad \Rightarrow \qquad \omega=\tilde{\Omega}(\overline{z},\kappa) \tag{5.65}$$

which, as in the conventional eikonal method, is the Hamilton-Jacobi differential equation for the phase, with $\kappa(\overline{z}) \equiv \partial_{\overline{z}}\Theta(\overline{z})$. By its construction, this is just the expected local dispersion relation

$$D(x,k;\omega)=\tilde{D}(z,\overline{z};\omega)=0 \qquad \Rightarrow \qquad \omega=\Omega(x,k)=\tilde{\Omega}(\overline{z},-iz) \tag{5.66}$$

except with z replaced by $i\kappa$. The reason for expressing $\tilde{\Omega}$ in terms of $-iz$ is so that the dispersion relation is defined in terms of canonical variables. That is, using the definitions (5.33), the usual Poisson bracket on (x,k) phase space of these variables is

$$\{\overline{z},-iz\}=(\partial_x\overline{z})(-i\partial_k z)-(-i\partial_x z)(\partial_k\overline{z})=1 \tag{5.67}$$

which identifies $-iz$ as the "momentum" conjugate to the coordinate $\overline{z}$.

Following the formalism of Hamilton-Jacobi theory and conventional eikonal methods, the differential equation (5.65) may be solved by introducing characteristic trajectories.

In view of the analogy between (5.65) and (5.66), these trajectories are generated by Hamilton's equations in canonical form

$$\dot{\overline{z}} = \frac{\partial \tilde{\Omega}(\overline{z}, \kappa)}{\partial \kappa} \qquad \dot{\kappa} = -\frac{\partial \tilde{\Omega}(\overline{z}, \kappa)}{\partial \overline{z}} \tag{5.68}$$

With the introduction of these "rays", it is essential to recognize the implications of applying eikonal methods to wave equations in phase space. The traditional eikonal analysis of x-space wave equations generates ray trajectories in (x, k) phase space and the properties of this flow and its projection onto x-space is responsible for the form of the short-wavelength field in x-space. That phase space in which the rays evolve is now the "configuration space" for the present wave equation (5.31,5.57) and, to avoid confusion with the rays generated by (5.68), we shall refer to them as the "underlying" or "physical" trajectories. The concept of a "local phase-space wavenumber", induced by the eikonal phase Θ and its derivative κ, therefore necessitates the consideration of a $(\overline{z}, \kappa)$ phase space above the physical (x, k) or $\overline{z}$ phase space. It is in this "doubled" phase space that the trajectories governed by (5.68) evolve. For clarity, we denote the (x, k) phase space by $\Re$ and the "doubled" phase space $(\overline{z}, \kappa)$ by $\wp$; a schematic illustration of this structure is given in Fig. 4.

Other authors have introduced the concept of a "doubled" phase space [48,49] in connection with equation (4.32) for the Wigner function $S(x, k)$. The space $\wp$ considered here, in relation to the coherent state representation, has the property that although it is four-dimensional (for an underlying one-dimensional wave problem with a two-dimensional physical phase space $\Re$), a complex structure has arisen naturally so that it may be treated as a two complex-dimensional space; this is an advantage in both analytical and numerical investigations. A logical question may be raised, however, as to the relationship between

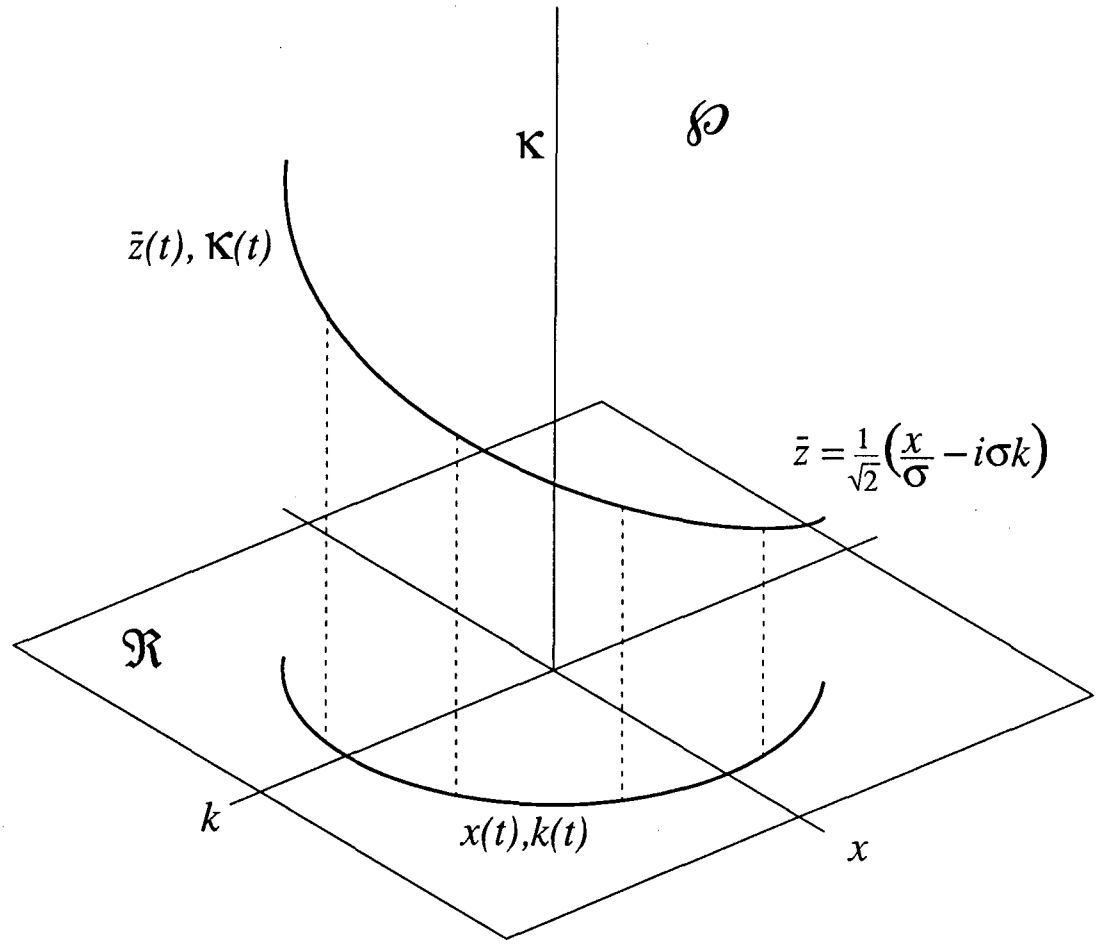

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Fig. 4. Schematic illustration of the "doubled phase space" $\wp$ induced by the eikonal solution of the phase-space equation (5.40). The familiar "underlying" phase space $\Re$ is coordinatized by either real variables (x, k) or the complex variable $\bar{z}$; this is to be viewed as the "configuration space" of $\wp$ with the complex-valued variable κ being the momentum conjugate to $\bar{z}$ ("local phase-space wavenumber"). Orbits in $\wp$ are projected onto $\Re$ in order to determine their relationship to "physical" ray trajectories.

the trajectories generated by (5.68) in $\wp$ and the physical rays which evolve in $\Re$. More precisely, one should examine the projection of the trajectories in $\wp$ onto $\Re$ in order to discover the correspondence with the physical rays.

Consider, for example, points in $\wp$ given by $(\overline{z}, \kappa) = (\overline{z}, -iz)$. Observe that these points are indeed elements of $\wp$ (because both $\overline{z}$ and κ are dimensionless complex variables) but it is the rather special set above the $\overline{z}$ plane ($\Re$) for which $\kappa = \overline{(i\overline{z})}$. Now it is easily shown that the Hamiltonian flow (5.68) preserves this relationship, *i.e.*, that $\kappa(t) = -iz(t)$ satisfies these equations. Thus, one has for $\kappa = -iz$

$$\dot{\kappa}(\overline{z}, \kappa = -iz) = -\frac{\partial \tilde{\Omega}(\overline{z}, \kappa)}{\partial \overline{z}}\bigg|_{\kappa=-iz} = -\frac{\partial \tilde{\Omega}(\overline{z}, -iz)}{\partial \overline{z}} \tag{5.69}$$

which, when (5.36) is applied, becomes

$$\begin{aligned} -\frac{\partial \tilde{\Omega}(\overline{z}, -iz)}{\partial \overline{z}} &= -\frac{1}{\sqrt{2}}\left(\sigma \frac{\partial \Omega(x,k)}{\partial x} + \frac{i}{\sigma}\frac{\partial \Omega(x,k)}{\partial k}\right) \\ &= -\frac{1}{\sqrt{2}}\left(-\sigma \dot{k}(x,k) + \frac{i}{\sigma}\dot{x}(x,k)\right) \\ \dot{\kappa}(\overline{z}, \kappa = -iz) &= -\frac{i}{\sqrt{2}}\left(\frac{\dot{x}(x,k)}{\sigma} + i\sigma \dot{k}(x,k)\right) = -i\dot{z}(\overline{z}, -iz) \end{aligned} \tag{5.70}$$

Furthermore, at these points $\dot{\overline{z}}$ is given by

$$\begin{aligned} \dot{\overline{z}}(\overline{z}, \kappa = -iz) &= \frac{\partial \tilde{\Omega}(\overline{z}, \kappa)}{\partial \kappa}\bigg|_{\kappa=-iz} = \frac{\partial \tilde{\Omega}(\overline{z}, -iz)}{\partial(-iz)} \\ &= i\frac{\partial \tilde{\Omega}(\overline{z}, -iz)}{\partial z} \\ \frac{\dot{x}}{\sigma} - i\sigma \dot{k} &= i\left(\sigma \frac{\partial \Omega(x,k)}{\partial x} - \frac{i}{\sigma}\frac{\partial \Omega(x,k)}{\partial k}\right) \end{aligned} \tag{5.71}$$

The logical interpretation of this calculation is then as follows: at all points in $\wp$ where $\kappa = -iz$ the flow is given by $\dot{\kappa} = -i\dot{z}(\overline{z}, -iz)$ and $\dot{\overline{z}} = \dot{\overline{z}}(\overline{z}, -iz)$ so that by the uniqueness of solutions of Hamiltons equations, $\kappa(t) = -iz(t)$ along trajectories which pass through these points. Moreover, identifying the real and imaginary parts in the last

line of (5.71) shows that the projection $\overline{z}(t)$ of these rays onto the $\overline{z}$ plane $\mathfrak{R}$ reproduces the physical trajectories.

The conclusion to be drawn here is that all of the "physical" orbits in the underlying phase space $\mathfrak{R}$ (generated by the usual dispersion relation $\Omega(x,k)$) are contained in the projection onto $\mathfrak{R}$ of all trajectories in the "doubled" phase space $\wp$ generated by $\tilde{\Omega}(\overline{z},\kappa)$. The special class of trajectories in $\wp$ which participates in this correspondence are those which lie on the two-dimensional manifold given by $\kappa(\overline{z}) = \overline{(i\overline{z})}$ or

$$\mathrm{Re}\,\overline{z} = \frac{x}{\sigma\sqrt{2}} \qquad \mathrm{Im}\,\overline{z} = \frac{-\sigma k}{\sqrt{2}} \qquad \mathrm{Re}\,\kappa = \frac{\sigma k}{\sqrt{2}} \qquad \mathrm{Im}\,\kappa = \frac{-x}{\sigma\sqrt{2}} \tag{5.72}$$

In other words, the projection onto $\mathfrak{R}$ of the flow in $\wp$ restricted to this manifold produces a flow in $\mathfrak{R}$; this should be compared with the projection onto x-space of the flow in phase space $\mathfrak{R}$, which does not result in a flow. Therefore, one would expect that the construction of Ψ along the "physical" trajectories in $\mathfrak{R}$ should be free of singularities because these "configuration-space" orbits are divergence-free.

The projection of all other trajectories in $\wp$ (*i.e.*, those for which $\kappa \neq -iz$) onto $\mathfrak{R}$ are "nonphysical" in the sense that they do not correspond to possible orbits in $\mathfrak{R}$ generated by $\Omega(x,k)$. One may conclude from the foregoing discussion of the "physical" trajectories that these "nonphysical" paths in $\mathfrak{R}$ are important for the construction of Ψ in the "non-classical" regions of (x,k) phase space off the classical orbit. That is, at a fixed value of the frequency ω, the projection of the dispersion surface $\omega = \tilde{\Omega}(\overline{z},\kappa)$ in $\wp$ onto $\mathfrak{R}$ will produce the "physical" orbits, which lie on the dispersion manifold $\omega = \Omega(x,k)$ in $\mathfrak{R}$, as well as "nonphysical" trajectories which are not on this manifold. Naturally, all "physical" and "nonphysical" orbits are important since Ψ should be constructed everywhere in phase space $\mathfrak{R}$ in order to reconstruct the field ψ in x- or k- space by yet another projection.

However, the examples given indicate that in the short-wavelength limit, Ψ decays rapidly away from the classical orbit so that in practice, perhaps only the "physical" rays and nearby "nonphysical" trajectories (for broadening) need be considered.

As previously remarked, the equation derived at next order for the amplitude $\Gamma(\overline{z})$ is very similar to the corresponding equation typically found in conventional x-space eikonal treatments. Indeed, (5.64) may be rearranged and more compactly expressed as

$$\frac{d}{d\overline{z}}\left[\Gamma^2(\overline{z})\left(\frac{\partial\tilde{D}}{\partial z}\right)(i\kappa(\overline{z}),\overline{z})\right]=0 \tag{5.73}$$

The notation here for the partial derivative $\tilde{D}_z$ means the derivative of $\tilde{D}(z,\overline{z})$ with respect to z evaluated at $z=i\kappa(\overline{z})$; thus, this is simply

$$\frac{d}{d\overline{z}}\left[\Gamma^2(\overline{z})\frac{\partial\tilde{D}}{\partial\kappa}(i\kappa(\overline{z}),\overline{z})\right]=0 \tag{5.74}$$

Being subject to the lower order solution, the usual relation

$$\left(\frac{\partial\tilde{D}(i\kappa,\overline{z};\omega)}{\partial\kappa}\right)_{\overline{z},\omega}=-\left(\frac{\partial\tilde{\Omega}(\overline{z},\kappa)}{\partial\kappa}\right)_{\overline{z},\omega}\left(\frac{\partial\tilde{D}(i\kappa,\overline{z};\omega)}{\partial\omega}\right)_{\overline{z},\kappa} \tag{5.75}$$

applies so that (5.74) may be written

$$\frac{d}{d\overline{z}}\left[\Gamma^2(\overline{z})\ \dot{\overline{z}}(\overline{z},\kappa(\overline{z}))\ \tilde{D}_\omega(i\kappa(\overline{z}),\overline{z};\tilde{\Omega}(\overline{z},\kappa(\overline{z})))\right]=0 \tag{5.76}$$

where (5.68) has been used to introduce $\dot{\overline{z}}$.

Evidently, the solution of the amplitude equation is

$$\Gamma(\overline{z})\sim\left[\dot{\overline{z}}(\overline{z},\kappa(\overline{z}))\ \tilde{D}_\omega(i\kappa(\overline{z}),\overline{z};\tilde{\Omega}(\overline{z},\kappa(\overline{z})))\right]^{-\frac{1}{2}} \tag{5.77}$$

This has an intuitive physical interpretation when compared to the solution (2.37) for the traditional x-space eikonal amplitude in a one-dimensional stationary medium, where the x-space amplitude $\propto[\dot{x}(x,k(x))]^{-1/2}$ diverges near x-space turning points ($\dot{x}\to 0$). Similarly, the expression in (5.77) tends to infinity near *fixed points* in phase space where

the phase-space velocity vanishes:

$$\dot{\bar{z}}(\bar{z}, \kappa(\bar{z})) \sim (\sigma^{-1}\dot{x} - i\sigma\dot{k}) = 0 \tag{5.78}$$

In view of the foregoing discussion of the "physical" and "nonphysical" trajectories in the (x, k) phase space $\Re$ which result from the projection of the rays in the $(\bar{z}, \kappa)$ space $\wp$, there are two categories of points $\bar{z}$ in $\Re$ for which (5.78) is satisfied. The "physical" fixed points are where $\dot{x} = \dot{k} = 0$ as determined by the underlying dispersion relation $\Omega(x, k)$ (the same points given by $\dot{\bar{z}}(\bar{z}, \kappa) = 0$ with $\kappa = -iz$) and are unique, separate phase-space trajectories (*i.e.*, not points visited by other orbits). This is quite different than in the case of traditional eikonal solution, where turning points (responsible for divergent amplitudes) are visited by almost all orbits in x-space. Therefore, if Ψ is to be constructed along classical ray orbits in phase space, the fixed point trajectories could be easily neglected and the worst case would be that the amplitude Γ would grow (remaining bounded) as an orbit passes near a fixed point. A further consideration is that, since the phase Θ is complex, the amplitude singularity at and near a fixed point may in practice be eliminated automatically by the behavior of $\exp(i\Theta)$ in that region of phase space. An example of this is provided by the harmonic oscillator $\Psi(x, k)$ which is finite (even for $n = 0$) at the fixed point at the origin.

The existence of fixed points among the "nonphysical" trajectories could provide trouble for numerical applications of this method. Again, however, the examples given indicate that Ψ is quite small in magnitude away from the classical orbits so that one should not expect singularity difficulties in the non-classical regions of phase space. In addition, it was previously pointed out that in the short-wavelength regime, it may suffice to construct Ψ only in the immediate neighborhood of the "physical" trajectories in order

that the projection onto x-space be reasonable.

5.5 Examples

To illustrate the technique developed in this Section, we shall apply them to two examples. The first is the simple case of a plane wave in a uniform medium with a local dispersion relation given by

$$\begin{aligned} & D(x,k;\omega) = D(k,\omega) = 1 - (k^2c^2/\omega^2) = 0 \\ \Rightarrow \qquad & \omega \equiv \pm k_0 c = \Omega(k) = kc \end{aligned} \tag{5.79}$$

where c is the phase velocity of the wave. In order to begin, the dispersion relation Ω must be expressed in terms of the complex variables $(z,\overline{z})$; using (5.33) and choosing the positive frequency wave, this is

$$k_0 = \tilde{\Omega}(\overline{z}, -iz) = \frac{-ic}{\sigma\sqrt{2}}(z - \overline{z}) \tag{5.80}$$

Now the Hamiltonian in the "doubled" phase space $(\overline{z},\kappa)$ is obtained by just replacing $-iz$ by κ in (5.80):

$$\omega = k_0 c = \tilde{\Omega}(\overline{z},\kappa) = \tilde{\Omega}(\overline{z}, -iz \to \kappa) = \frac{-ic}{\sigma\sqrt{2}}(i\kappa - \overline{z}) \tag{5.81}$$

which generates the trajectories

$$\dot{\overline{z}} = \frac{\partial\tilde{\Omega}}{\partial\kappa} = \frac{c}{\sigma\sqrt{2}} \qquad\qquad \dot{\kappa} = -\frac{\partial\tilde{\Omega}}{\partial\overline{z}} = \frac{-ic}{\sigma\sqrt{2}} \tag{5.82}$$

Translating the first of these into terms of (x,k) and identifying real and imaginary parts, the usual trajectories in phase space are recovered

$$\begin{aligned} \dot{\overline{z}} = \frac{1}{\sqrt{2}}\left(\frac{\dot{x}}{\sigma} - i\sigma\dot{k}\right) = \frac{c}{\sigma\sqrt{2}} \qquad & \Rightarrow \qquad \dot{x} = c, \quad \dot{k} = 0 \\ \Rightarrow \qquad x(t) = x(0) + ct \qquad & k(t) = k(0) \end{aligned} \tag{5.83}$$

Since $\overline{z}$ evolves independently of κ, the projection of all trajectories in $\wp$ onto $\Re$ appears to give just the "physical" trajectories in $\Re$. We note, however, that *all* of these

phase-space trajectories $(x(t), k(t))$ are allowed for *any fixed value of* ω; this is because for fixed ω, the initial conditions $(x(0), k(0))$ (or, equivalently, $\overline{z}(0)$) can be chosen arbitrarily (since the initial condition for $\kappa(0)$ can still be determined to satisfy (5.81)). The "physical" trajectories for this system at fixed ω are those with $k(0) = \omega/c$ ($x(0)$ arbitrary), or $\overline{z}(0) = 2^{-1/2}[(x(0)/\sigma) - (i\sigma\omega/c)]$ and, from (5.81), $\kappa(0) = \sigma\sqrt{2}(\omega/c) - i\overline{z}(0) = -i[(x(0)/\sigma) + (i\sigma\omega/c)] = -i\overline{z(0)}$ (as stated earlier); the projection of these trajectories in $\wp$ onto $\Re$ lie on the "physical" ray manifold $k = \omega/c$ for this value of ω. Changing the initial value $k(0)$ away from this "physical" dispersion surface is therefore allowed at the same value of ω by simply changing the initial value $\kappa(0)$; the projection of these trajectories in $\wp$ onto $\Re$ are the "unphysical" trajectories (straight lines parallel to the "physical" trajectories with $k = \omega/c$).

In order to construct Ψ, first use (5.81) to determine $\kappa(\overline{z})$ and then integrate to compute the phase Θ:

$$\begin{aligned} \kappa(\overline{z}) &= \sigma\sqrt{2}\,k_0 - i\overline{z} \equiv \frac{d\Theta(\overline{z})}{d\overline{z}} \\ \Theta(\overline{z}) &= \int^{\overline{z}} d\overline{z}'\kappa(\overline{z}') = \sigma\sqrt{2}\,k_0\overline{z} - \tfrac{i}{2}\overline{z}^2 \end{aligned} \tag{5.84}$$

Comparison of these expressions with (5.50,5.51) reveals that these are the exact forms of κ and Θ which are obtained by constructing Ψ as a local Fourier transform of the plane wave (note that since $\dot{\overline{z}}$ is a constant (5.83), the amplitude Γ is also constant according to (5.77)). Therefore, apart from a multiplicative constant, it is apparent that this eikonal method of computing Ψ will produce the exact expression (5.6).

The success of the phase-space eikonal method in this simplest of examples is perhaps not surprising, although it does demonstrate an important point: the asymptotic formalism is capable of constructing a smooth waveform in two dimensions (complexified phase

space) which exhibits oscillations along the phase-space ray with gaussian modulation transverse to the ray. Of course, conventional eikonal methods experience no difficulty when applied to this problem and also produce the exact solution; there are no caustic singularities since all x-space trajectories are straight lines which do not focus. For this reason, the application of the phase-space analysis to this particular problem seems inordinately cumbersome and unnecessary.

An example in which this technique is able to display its possible relevance and application is the Schrödinger equation for the harmonic oscillator. The dispersion function (4.75) for this problem written in terms of the natural length scale $\alpha^{-1} = \sqrt{\hbar/m\omega_0}$ in (4.8) can be immediately transformed using (5.33) to complex coordinates $(\overline{z}, -iz)$ as

$$\tilde{\Omega}(\overline{z}, -iz) = \tfrac{1}{4}\omega_0[(z^2 + \overline{z}^2)(\alpha^2\sigma^2 - \alpha^{-2}\sigma^{-2}) + 2z\overline{z}(\alpha^2\sigma^2 + \alpha^{-2}\sigma^{-2})] \tag{5.85}$$

The next step in the procedure is to replace z by $i\kappa$ in this expression to obtain the Hamiltonian on the $(\overline{z}, \kappa)$ phase space $\wp$. However, since the smoothing length σ is arbitrary (within the limits (5.44) for the theory to be valid), a suitable choice here will extremely simplify the algebra. Thus, specifying

$$\sigma^2 = \alpha^{-2} = \frac{c}{\omega_0} L \sim \lambda_0 L \tag{5.86}$$

not only provides a smoothing length which is the geometric mean of the wave and density scalelengths (see (5.10)), but also allows the Hamiltonian to be written simply

$$\tilde{\Omega}(\overline{z}, -iz) = \omega_0 z\overline{z} \tag{5.87}$$

Now in $\wp$, the ray Hamiltonian is

$$\omega = \tilde{\Omega}(\overline{z}, \kappa) = i\omega_0 \overline{z}\kappa \tag{5.88}$$

which may be promptly solved for the local phase-space wave number $\kappa(\overline{z})$

$$\kappa(\bar{z}) = -\frac{i\omega}{\bar{z}\omega_0} \tag{5.89}$$

and hence, the phase $\Theta(\bar{z})$ is

$$\Theta(\bar{z}) = \int^{\bar{z}} d\bar{z}'\kappa(\bar{z}') = -\frac{i\omega}{\omega_0}\ln\bar{z} \tag{5.90}$$

The amplitude $\Gamma(\bar{z})$ requires

$$\begin{aligned} \dot{\bar{z}}(\bar{z},\kappa(\bar{z})) &= \frac{\partial\tilde{\Omega}}{\partial\kappa} = i\omega_0\bar{z} \\ (\partial_\omega D) &= \hbar \end{aligned} \tag{5.91}$$

and is therefore, by (5.77),

$$\Gamma(\bar{z}) \sim [\dot{\bar{z}}D_\omega]^{-\frac{1}{2}} \sim \bar{z}^{-\frac{1}{2}} \tag{5.92}$$

Finally, collecting (5.90) and (5.92), the solution for $\Phi(\bar{z})$ is

$$\begin{aligned} \Phi(\bar{z}) &= \Gamma(\bar{z})e^{i\Theta(\bar{z})} \\ &\sim \bar{z}^{-\frac{1}{2}}\exp[(\omega/\omega_0)\ln\bar{z}] \\ &\sim \bar{z}^{p}\,, \qquad p = (\omega/\omega_0) - \tfrac{1}{2} \end{aligned} \tag{5.93}$$

The form of this result is interesting from the standpoint of the theory of analytic functions of a complex variable: in order that $\Phi(\bar{z})$ be single-valued on the complex phase space, the exponent p in (5.93) must be an integer. In effect, this requirement plays the role of a phase-space quantization condition for this system and yields

$$\begin{aligned} p &= (\omega/\omega_0) - \tfrac{1}{2} = n \\ \Rightarrow \qquad \Phi_n(\bar{z}) &\sim \bar{z}^n \end{aligned} \tag{5.94}$$

From the definition (5.48), the phase-space representation $\Psi(x,k)$ is therefore

$$\begin{aligned} \Psi_n(x,k) = \tilde{\Psi}_n(z,\bar{z}) &\sim e^{\frac{1}{4}z^2 - \frac{1}{4}\bar{z}^2 - \frac{1}{2}z\bar{z}}\bar{z}^n \\ &\sim e^{-\frac{1}{4}(\alpha^2x^2 + k^2/\alpha^2)}e^{\frac{1}{2}ikx}(\alpha x - ik/\alpha)^n \end{aligned} \tag{5.95}$$

This is a striking result in that it is the *exact solution* (5.9) (within normalization factors) for Ψ (the normalization constant in (5.95) could be determined from the condition

(5.4) if desired). This in turn implies that the exact eigenfunctions are obtained when (5.95) is projected by (5.3) onto x-space; indeed, the integral which is involved serves as an integral definition of the Hermite polynomials [50].

5.6 Discussion

Evidently, this phase-space eikonal technique has demonstrated that it is capable of treating a two-dimensional (phase-space) wave equation with a one complex-dimensional formalism. In fact, it provides the solution for Ψ not only in the "classical" region of phase space explored by the rays, but in the "nonclassical" region off the trajectories as well. Moreover, these two regions are treated on the same footing (as opposed to the piecewise formulation of traditional eikonal) so that, at least in the examples, no matching analysis is required; in part, this feature is due to the use of a complex phase Θ defined on the complexification of phase space.

The most remarkable feature of the solution (5.95) is the fact that it is exact. This should be compared with the results of conventional eikonal techniques applied to the same problem, where the spectrum is obtained exactly (see (2.36)) but the eigenfuctions are determined in their asymptotic form, and then only piecewise due to the caustic singularities at the turning points. The construction of Ψ in phase space, as might be expected of a phase-space method, encountered no singularites except the existence of a branch point at the origin, which is also a fixed point for the ray system. Thus, since the exact phase-space representation of the wave function is computed in this procedure, its projection onto x-space produces the exact x-space representation over all x, complete with decaying amplitude outside the turning points, oscillations between them and, significantly, maximum amplitude (but no singularity) in the neighborhood of the turning points. In

this respect, the phase-space eikonal technique is more accurate and more complete than conventional eikonal methods.

Naturally, one should not place too much emphasis on a result obtained in the example of the harmonic oscillator. This is especially true with regard to the present formalism as it is based on the introduction of complex variables which (for suitable choice of σ) are intimately related to the harmonic oscillator Hamiltonian. Indeed, the coherent state representation is generated by the eigenstates of the lowering operator which, together with its Hermitian conjugate, are a natural operator basis for this problem. However, it may be significant to recall that the phase-space technique also provided the exact solution to the plane wave (free particle) problem for which the Hamiltonian is not so nicely expressed in terms of these fundamental operators.

As illustrated in the examples, the coherent state representation of a wave is a distribution on phase space which has a form that depends on the choice of the smoothing length σ. For this theory to be accepted as a reliable method for solving short-wavelength problems, however, it should produce certain physical results independent of the value of σ. In the sense that the projection rule (5.3) is independent of σ, this may not be a physically significant defect if the desired result is the form of the wave field in x-space. Thus, different values of σ can be expected to produce different forms of Ψ when the phase-space eikonal technique is applied, and these will be in general just approximations to the exact form for each value of σ. The projection of these σ-dependent approximations to Ψ can be expected to provide σ-dependent approximations to $\psi(x)$ and in practice one would like to develop some rule for choosing a smoothing length so that this approximation is optimized. A further question is whether the projection of an approximately constructed phase-space function Ψ with arbitrary σ will produce a uniform approximation to the

x-space wave, or will the projection produce singularities at the turning points (or perhaps elsewhere). In the examples, the phase-space eikonal technique demonstrated the capability for determining a phase-space density which is smooth and which incorporates the wave-like broadening off the ray manifold; it is this "nonclassical" wave broadening in phase space which mitigates projection singularities at caustics and thereby produces the uniform approximation of the x-space field.

The relationship between the quantization condition for the harmonic oscillator and the analytic single-valuedness criterion is another remarkable result of the application of this method to the harmonic oscillator. In this regard, we note that a similar result has been obtained by Weissman [24] (also for the harmonic oscillator) from the study of canonical transformations of the coherent states. It would be interesting to explore its generality, since while the form of Ψ is to an extent understandably dependent on the smoothing length (even for an eigenfunction), the quantization condition is an example of a physical result which should be independent of σ.

Another curious aspect of this derivation of the quantization rule is the way in which the ground state correction factor of $\frac{1}{2}$ arises. In the usual eikonal analysis this factor appears because of phase matching conditions at the two turning points, while in the modern theory of the Maslov index it arises from the matching of alternate x and k representations of the wave around the classical ray orbit in phase space. Here, however, the factor of $\frac{1}{2}$ is the contribution of the amplitude $\Gamma(\overline{z}) \sim (\overline{z})^{-\frac{1}{2}}$ to the exponent of $\overline{z}$ in the final form (5.93) of $\Phi(\overline{z})$. It may be that there is a connection between the standard interpretation of this factor in terms of the influence of caustics and the effect here of the phase-space amplitude Γ which has a square root singularity at the fixed point encircled by the classical orbit.

Since the contribution of the amplitude Γ to the quantization condition is negligible as $(n \to \infty)$, a possible practical implication for this method is that the amplitude might be ignorable altogether. In the case of the harmonic oscillator, the single-valuedness condition would still apply (resulting in the quantization condition (5.94) without the $\frac{1}{2}$) while the form of Ψ would remain the same $\sim \overline{z}^n$. That both the oscillatory and modulated features of the phase-space representation Ψ are present, even without including the amplitude Γ, is of course due to the complex-valued phase Θ. Now, however, no singularities will be encountered in the construction of Ψ since these arise as the effect of fixed points on Γ.

Finally, it should be noted that the method of solution employed here did not in either example rely on the construction of the "physical" or "nonphysical" trajectories generated by the Hamiltonian $\tilde{\Omega}(\overline{z}, \kappa)$. This is because in the simple one-dimensional models examined, the dispersion relation was easily inverted to determine $\kappa(\overline{z})$ and subsequently the phase $\Theta(\overline{z})$ by integration. However, due to the complications introduced by the complex canonical transformation $(x, k) \to (\overline{z}, -iz)$, this inversion cannot be performed explicitly even for other simple one-dimensional systems (consider the pendulum Hamiltonian, for which a transcendental equation for $\kappa(\overline{z})$ results). For most systems then, these trajectories will have to be introduced in order to determine the phase Θ. Now, important issues are raised in regard to appropriate initial conditions in the $(\overline{z}, \kappa)$ phase space (for unbound systems) or the determination of quantization rules (for bound systems) in terms of the orbits either in the "doubled" phase space or in the physical one. Furthermore, since any practical application of this method to the investigation of wave propagation (or normal modes) in nonuniform media would entail the use of much more complicated dispersion relations (ray Hamiltonians) than considered here, the translation of these ideas into a tractable numerical procedure will be necessary.

6. Summary

In this Report we have discussed the concept of a phase-space representation of a wave field and its application to the study of short-wavelength wave propagation in nonuniform media. The motivation for introducing this formulation of wave propagation is the recognition that certain features of a short-wavelength field $\psi(x)$ in x-space (or $\hat{\psi}(k)$ in k-space) can be understood in terms of the properties of the associated Hamiltonian ray system which evolves in the ray phase space. In conventional eikonal theory it is tacitly assumed that the local wavenumber k is the derivative of the phase and hence a function of x; this implies that it is the projection of these rays onto x-space which gives the spatial variation of field intensity and, in particular produces divergent amplitudes in the vicinity of caustics where this projection becomes singular. Since the natural setting for the ray trajectories is in the (x, k) phase space (where the Hamiltonian induces a flow and the rays do not converge or diverge), we have speculated that a joint (x, k) representation of the field might lead to a more natural, simplified description of the asymptotic wave problem (with x and k treated as independent variables) that would avoid the projection singularities inherent in the conventional eikonal approach. We have shown that a phase-space representation of a wave can be defined, the equation which governs its evolution can be derived and analyzed in the short-wavelength limit and the asymptotic form of the solution constructed by means of the ray trajectories in phase space; the subsequent projection of this "phase-space wave" back down onto x-space has succeeded to a varying degree (depending upon the particular representation) in avoiding these caustic singularities. In this sense, these Phase-Space Eikonal techniques may provide useful tools for investigating both classical and quantum-mechanical short-wavelength waves (vector or scalar) in nonuniform media.

Since the phase-space representation of a wave is not unique, we have explored three different (although related) formalisms. Along with the development of each formalism has evolved a set of criteria which an appropriate representation should be expected to satisfy in order that it be useful in the investigation of short-wavelength fields:

A) A suitable representation should be "supported" by the rays in phase space. In other words, the magnitude of the phase-space function describing the wave should be large in the region of phase space visited by the rays associated with the wave (*i.e.*, in the neighborhood of the ray manifold) and small in other regions. This is to ensure the faithfulness of the representation in accurately describing the local spectrum, *i.e.*, the values of k present in the wave at the position x. Furthermore, the solution of the asymptotic phase-space equation should provide a prescription for constructing the phase-space wave function along the rays in phase space.

B) The construction of the phase-space wave function along the ray trajectories should not encounter singularities in phase space. Since the rays do not focus due to Liouville's Theorem, there should naturally be no analog of caustic singularities as found in the x-space asymptotic analysis. Nevertheless, the existence of singularities arising from other unforseen effects would be undesirable.

C) Of course, an important requirement on any phase space representation is that it may be given a physical interpretation. The most closely related quantity of physical interest is the spectral function (or tensor) commonly defined for waves in a homogeneous, stationary medium, and it is the extension of this concept to a nonuniform medium for which the various phase-space wave representations are intended. Indeed, it is just the definition of what is meant by "local spectral tensor" which is ambiguous and which leads to the introduction of different phase-space representations of the wave field. Certain

other physically important features of the underlying x-space wave equation (or, more precisely, the abstract representation-free operator form of the wave problem) should also be retained. In particular, the Hermiticity properties of an operator should be reflected in its phase-space representation.

D) Finally, after the phase-space representation of the wave is constructed, it must be possible to reconstruct the field in x-space by some projection or inversion relation (an integral over k-space). Even though it is hoped that this procedure will give the asymptotic x-space wave intensity (without singularities), for some applications it may be important that the wave phase is also faithfully produced.

The phase-space formalisms which we have discussed in this Report are based on the theory of pseudodifferential operators and their symbols. An integral part of each formalism is the derivation of the equation which governs the symbol of the field spectral operator based on the underlying wave equation; this involves the translation of operator relations into a calculus of the corresponding symbols. It is this equation which is analyzed in the short-wavelength limit in order to investigate the asymptotic form of the spectral function and its relation to the phase-space ray trajectories.

In Section 3, we first considered the symbol formalism or phase-space representation which is most commonly found in the mathematical literature on pseudodifferential operators. The advantage of this description is that it provides a natural extension of the partial differential operator to the more general case of a pseudodifferential operator; by constructing the symbols associated with typical wave operators found in physically relevant propagation problems in nonuniform media we observed that in these operators indeed often fall into this more general class. Despite the simplicity of this formalism, however, by the criteria set above the *common* symbol theory has two fundamental defects which

detract from its usefulness as an asumptotic method. First, by exact construction the Hermiticity properties of an operator are not shared by its symbol. The second problem is that the asymptotic analysis of the phase-space wave equation was shown to reduce to the same equations for the phase and amplitude generated by conventional x-space eikonal methods and therefore intrinsically contain the singularities which a phase-space method attempts to avoid.

In Section 4, we introduced the "symmetrized" or Weyl symbol formalism in which the phase-space representation for the wave field is the Wigner function (or wave spectral function) $S(\mathbf{x},t,\mathbf{k},\omega)$. It was immediately seen that the Hermiticity problem found in the common symbol was corrected (owing to the definition in terms of a "centered" transform). A relationship between the common symbol formalism and that of the Weyl symbol was established and then exploited in order to derive the Weyl symbol calculus and thus the phase-space wave equation. Here we extended the usual one-dimensional treatment of scalar waves to multiple dimensions and vector waves (and, therefore, tensor-valued Wigner functions). An asymptotic analysis of this equation produces at lowest order the definition of the proper wave dispersion surface $D(\mathbf{x},t,\mathbf{k},\omega) = 0$ in phase space but, unlike traditional eikonal methods, does not induce the ray trajectories. The next higher-order equation was shown to reduce to the wave kinetic equation for the wave action density $J(\mathbf{x},\mathbf{k};t)$, here reasonably extended to the case of a nonuniform medium as the amplitude of the Wigner function on the dispersion surface. This Liouville equation naturally generates the proper ray trajectories as those paths along which the action density is convected (and hence, may be constructed) in phase space. By example, we illustrated how this phase-space technique could be implemented to propagate an x-space wave packet via the motion of the action density in phase space, and then to reconstruct the x-space field at later times avoiding

the caustic singularities which would certainly have resulted if conventional methods were used. In spite of this apparent success, a fundamental disadvantage of this formalism is that the phase of the wave is lost. Furthermore, the Wigner function was seen to be in general an oscillatory function on phase space; thus, not being necessarily everywhere non-negative detracts from its interpretation as a probability density on phase space.

The third and final phase-space representation discussed was the coherent state or Glauber representation of the field $\Psi(x,k)$. Unlike the previous symbol formalisms, this object is a true representation of the field itself rather than of the field spectral operator. As such, it retains wave phase information and, being defined as a smoothed local Fourier transform, induces a real non-negative phase-space density $P(x,k) = |\Psi(x,k)|^2$. It was shown that $P(x,k)$ is also the local phase-space average of the Wigner function; this averaging also implies a non-classical broadening of the phase-space density in this formalism off of the dispersion surface on which the ray trajectories lie. A further connection with the Weyl symbol formalism permitted the use of that symbol calculus for the derivation of the equation which governs $\Psi(x,k)$. In this case, a generalization of the conventional eikonal method applied to this phase-space equation supplied a prescription for solution: the crucial ingredients of the technique are the use of a complex eikonal phase and the imposition of a complex structure on phase space. The result of these measures is a procedure for constructing $\Psi(x,k)$ which treats the "classical" and "non-classical" regions of phase space on the same footing and at the same order; $\Psi(x,k)$ is computed in terms of the rays on the ray manifold and in terms of "unphysical" trajectories off the manifold. In special examples, the approximate technique produces exact results with no singularities in phase space and therefore exact results with no singularities when projected onto x-space. As encouraging as the indications of Section 5 are, the coherent state representa-

tion also presents several problems with interpretation and implementation. The apparent arbitrariness of the spatial smoothing scale raises questions in regard to the robustness of the results. The full implications of the "doubled phase space" and the "unphysical" trajectories in regard to actual computation of $\Psi(x,k)$ have not been investigated here; they certainly require a deeper understanding to be useful in a numerical scheme. Nevertheless, this technique (or a numerical adaptation of it) suggests the possibility that it might be capable of producing a nonsingular uniform approximation to a short-wavelength x-space wave field in a typical problem.

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Appendix : Tensor Analysis of the Wave Kinetic Equation

In this Appendix we supply the details for the reduction of the tensor form of the $O(\lambda/L)$ equation (4.52) to the scalar equation (4.62). As was the case at lowest order, this equation simplifies when expressed in the basis of the polarization vectors. The introduction of this basis, however, initially increases the complexity of the term involving the bi-directional operator $\overleftrightarrow{\mathcal{L}}$; therefore, in order to clarify the ensuing discussion we shall treat this term first and then return to the source and dissipation terms later. In component form and using (4.57), this term becomes

$$\sum_{\nu}[D'_{\mu\nu}, S_{\nu\sigma}] = \sum_{\nu}\sum_{\alpha\beta}[D^{\alpha}P^{\alpha}_{\mu\nu}, S^{\beta}P^{\beta}_{\nu\sigma}] \tag{A.1}$$

where we have used the square brackets $[\cdot,\cdot]$ (instead of the operator $\overleftrightarrow{\mathcal{L}}$) to designate the usual Poisson bracket $\{\cdot,\cdot\}$ extended to the eight-dimensional $(\mathbf{x}, t, \mathbf{k}, \omega)$ phase space

$$[A, B] \equiv A\overleftrightarrow{\mathcal{L}}B = (\partial_{\omega}A)(\partial_t B) - (\partial_t A)(\partial_{\omega}B) + \{A, B\} \tag{A.2}$$

Since the projection operators $\underset{\sim}{P}^{\alpha}$ are functions on phase space by their definition (4.57) in terms of the polarization vectors, the bracket on the right contains many more terms than the one on the left.

As stated in Section 4, the first simplifying assumption we shall make is that only one polarization is in the system; that is, only S^s is nonzero ($S^{\beta} = 0$ everywhere for $\beta \neq s$). Also, because of (4.61), not only does S^s vanish where $D^s \neq 0$, but so do its derivatives; this is in contrast to the derivatives of D^s itself which may not vanish even where D^s does. As previously remarked, we assume that where S^s is nonzero ($D^s = 0$) there is no degeneracy of eigenvalues ($D^{\alpha} \neq 0$ for $\alpha \neq s$). Under these conditions, (A.1) is

$$\sum_{\nu\alpha}[D^\alpha P^\alpha_{\mu\nu}, S^s P^s_{\nu\sigma}] = \sum_{\nu\alpha}\{D^\alpha S^s[P^\alpha_{\mu\nu}, P^s_{\nu\sigma}] + P^\alpha_{\mu\nu}P^s_{\nu\sigma}[D^\alpha, S^s] + D^\alpha[P^\alpha_{\mu\nu}, S^s]P^s_{\nu\sigma} + P^\alpha_{\mu\nu}[D^\alpha, P^s_{\nu\sigma}]S^s\} \tag{A.3}$$

where the rule for differentiating a product has been used to expand the bracket in the usual fashion.

The first and third terms on the right-hand side of (A.3) reduce to sums over only $\alpha \neq s$ since either D^s or S^s vanishes. Contracting the projection operators in the second term and using their orthogonality (4.55), only the $\alpha = s$ contribution survives. Thus, the $\mu\sigma$ component of the bracket is

$$P^s_{\mu\sigma}[D^s, S^s] + \sum_{\nu,\alpha\neq s} D^\alpha \{S^s[P^\alpha_{\mu\nu}, P^s_{\nu\sigma}] + [P^\alpha_{\mu\nu}, S^s]P^s_{\nu\sigma}\} + \sum_{\nu\alpha} P^\alpha_{\mu\nu}[D^\alpha, P^s_{\nu\sigma}]S^s \tag{A.4}$$

Although this expression is written in terms of quantities in the polarization basis, the $\mu\sigma$ index refers to the component with respect to the Cartesian basis; one must yet perform the transformation (4.56) in order to obtain matrix elements of (A.4) in the polarization basis. Fortunately, however, not every component of (A.4) in the new basis will be required.

The evolution of S^s will be obtained by extracting the ss component of (A.4). According to (4.56), this is accomplished by the premultiplication of $\bar{e}^s_\mu$, postmultiplication by e^s_σ and contracting. This operation yields

$$\begin{aligned}([\underset{\sim}{D}', \cdot\underset{\sim}{S}])^{ss} &= \sum_{\mu\nu\sigma}\bar{e}^s_\mu[D'_{\mu\nu}, (E_\nu E^\dagger_\sigma)_S]e^s_\sigma \\ &= [D^s, S^s] + S^s\sum_{\mu\nu\sigma,\alpha\neq s} D^\alpha \bar{e}^s_\mu[P^\alpha_{\mu\nu}, P^s_{\nu\sigma}]e^s_\sigma \\ &\quad + \sum_{\mu\nu,\alpha\neq s} D^\alpha \bar{e}^s_\mu[P^\alpha_{\mu\nu}, S^s]e^s_\nu + S^s\sum_{\nu\sigma}\bar{e}^s_\nu[D^s, P^s_{\nu\sigma}]e^s_\sigma\end{aligned} \tag{A.5}$$

The third and fourth terms of the result vanish identically on inserting polarization vectors for the projection operators, expanding the brackets and using the orthogonality properties (4.55) as follows

$$\begin{aligned}
\sum_{\mu\nu,\alpha\neq s} D^{\alpha}\bar{e}^{s}_{\mu}[P^{\alpha}_{\mu\nu},S^{s}]e^{s}_{\nu} &= \sum_{\mu\nu,\alpha\neq s} D^{\alpha}\{\bar{e}^{s}_{\mu}e^{\alpha}_{\mu}[\bar{e}^{\alpha}_{\nu},S^{s}]e^{s}_{\nu}+\bar{e}^{s}_{\mu}[e^{\alpha}_{\mu},S^{s}]\bar{e}^{\alpha}_{\nu}e^{s}_{\nu}\} \\
&= \sum_{\nu,\alpha\neq s} D^{\alpha}\delta^{\alpha s}\{\bar{e}^{s}_{\nu}[e^{\alpha}_{\nu},S^{s}]+\text{c.c.}\}=0 \\
\sum_{\nu\sigma}\bar{e}^{s}_{\nu}[D^{s},P^{s}_{\nu\sigma}]e^{s}_{\sigma} &= \sum_{\nu\sigma}\bar{e}^{s}_{\nu}e^{s}_{\nu}[D^{s},\bar{e}^{s}_{\sigma}]e^{s}_{\sigma}+\bar{e}^{s}_{\nu}[D^{s},e^{s}_{\nu}]\bar{e}^{s}\sigma e^{s}_{\sigma} \\
&= \sum_{\nu}[D^{s},\bar{e}^{s}_{\nu}e^{s}_{\nu}]=0
\end{aligned} \tag{A.6}$$

In the last step we have used the fact that the polarization vectors are normalized to a constant everywhere; thus, derivatives of their norm vanish. Finally, the *ss* diagonal component of the bracket in the polarization basis is

$$([\underset{\sim}{D}',\cdot\underset{\sim}{S}])^{ss}=[D^{s},S^{s}]+S^{s}\sum_{\mu\nu\sigma,\alpha\neq s} D^{\alpha}\bar{e}^{s}_{\mu}[P^{\alpha}_{\mu\nu},P^{s}_{\nu\sigma}]e^{s}_{\sigma} \tag{A.7}$$

It will also be necessary to consider the trace of the matrix (A.4). Setting $\mu=\sigma$ and summing one has

$$\begin{aligned}
\text{Tr}\,([\underset{\sim}{D}',\cdot\underset{\sim}{S}]) = [D^{s},S^{s}]&+S^{s}\sum_{\mu\nu,\alpha\neq s} D^{\alpha}[P^{\alpha}_{\mu\nu},P^{s}_{\nu\mu}] \\
&+\sum_{\mu\nu,\alpha\neq s} D^{\alpha}[P^{\alpha}_{\mu\nu},S^{s}]P^{s}_{\nu\mu}+S^{s}\sum_{\mu\nu\alpha}[D^{\alpha},P^{s}_{\nu\mu}]P^{\alpha}_{\mu\nu}
\end{aligned} \tag{A.8}$$

With expansion of the brackets in terms of polarization vectors similar to that used in (A.6), it is easily shown that the last two terms of this expression are also identically zero. The trace is then given by

$$\text{Tr}\,([\underset{\sim}{D}',\cdot\underset{\sim}{S}])=[D^{s},S^{s}]+S^{s}\sum_{\mu\nu,\alpha\neq s} D^{\alpha}[P^{\alpha}_{\mu\nu},P^{s}_{\nu\mu}] \tag{A.9}$$

It is interesting to compare this expression with the result (A.7) for the single diagonal *ss* component of the bracket matrix. Again resorting to the substitution of eigenvectors $\hat{e}$ for projection operators $\underset{\sim}{P}$, the only piece of the second term in (A.7) which survives is

$$\sum_{\mu\nu\sigma,\alpha\neq s} D^{\alpha}\bar{e}^{s}_{\mu}[P^{\alpha}_{\mu\nu},P^{s}_{\nu\sigma}]e^{s}_{\sigma}=\sum_{\mu\nu,\alpha\neq s} D^{\alpha}\bar{e}^{s}_{\mu}\bar{e}^{\alpha}_{\nu}[e^{\alpha}_{\mu},e^{s}_{\nu}] \tag{A.10}$$

The similar summation in (A.9) becomes

$$\sum_{\mu\nu,\alpha\neq s} D^\alpha [P^\alpha_{\mu\nu}, P^s_{\nu\mu}] = \sum_{\mu\nu,\alpha\neq s} D^\alpha \left\{ \overline{e}^s_\mu \overline{e}^\alpha_\nu [e^\alpha_\mu, e^s_\nu] + e^\alpha_\mu e^s_\nu [\overline{e}^\alpha_\nu, \overline{e}^s_\mu] \right\} \tag{A.11}$$

Evidently, the trace (A.9) exceeds the ss component by

$$\sum_{\mu\nu,\alpha\neq s} D^\alpha e^\alpha_\mu e^s_\nu [\overline{e}^\alpha_\nu, \overline{e}^s_\mu] \tag{A.12}$$

This is the complex conjugate of (A.10) and, as expected, is easily shown to be the sum of the other two diagonal ($\alpha \neq s$) components of the bracket matrix (A.4). This observation will be of use when we return to the bracket; for now, however, we wish to focus attention on the other terms in (4.52).

The damping and source contributions to the $O(\lambda/L)$ evolution equation for S^s are much easier to transform into the polarization basis than was the Poisson bracket because they contain no derivatives. The ss component of the dissipation term is simply

$$(\underset{\sim}{D}'' \cdot \underset{\sim}{S})^{ss} = (D'')^{ss} S^s \tag{A.13}$$

since $\underset{\sim}{S}$ has only one nonvanishing element in this basis. For the same reason, this expression is also the value of the trace

$$(\underset{\sim}{D}'' \cdot \underset{\sim}{S})^{ss} = \mathrm{Tr}\,(\underset{\sim}{D}'' \cdot \underset{\sim}{S}) \tag{A.14}$$

The source term of (4.52) requires a little more care since we have not yet computed the symbol of the inverse operator $(\underset{\sim}{\mathbf{D}}^\dagger)^{-1}$. As in Section 3.2, the inverse operator and its symbol are defined by

$$\begin{aligned} \underset{\sim}{\mathbf{D}}^\dagger \cdot (\underset{\sim}{\mathbf{D}}^\dagger)^{-1} &= \underset{\sim}{\mathbf{I}} \\ \Rightarrow \qquad \underset{\sim}{D}^\dagger(\mathbf{x},t,\mathbf{k},\omega) e^{\frac{i}{2}\overleftrightarrow{\mathcal{L}}} \cdot (\underset{\sim}{D}^\dagger)^{-1}(\mathbf{x},t,\mathbf{k},\omega) &= \underset{\sim}{I} \end{aligned} \tag{A.15}$$

Fortunately, the source contribution enters at first order (due to the assumed magnitude of $\underset{\sim}{N}$) so that considering the estimates (4.46–4.50), the derivatives in (A.15) may be ignored.

Therefore, consistent with these approximations one may take

$$\underset{\sim}{D}^{\dagger}(\mathbf{x},t,\mathbf{k},\omega)\cdot(\underset{\sim}{D}^{\dagger})^{-1}(\mathbf{x},t,\mathbf{k},\omega)=\underset{\sim}{I} \tag{A.16}$$

The dispersion tensor has been separated into its Hermitian and anti-Hermitian parts with relative magnitudes assumed to be as in (4.50). It is easily verified that the standard formula

$$(\underset{\sim}{D}^{\dagger})^{-1}=(\underset{\sim}{D}'-i\underset{\sim}{D}'')^{-1}\approx(\underset{\sim}{D}')^{-1}+i(\underset{\sim}{D}')^{-1}\cdot\underset{\sim}{D}''\cdot(\underset{\sim}{D}')^{-1}+\cdots \tag{A.17}$$

satisfies (A.16) to $\mathcal{O}(\lambda/L)$. In the polarization basis this expression is just

$$\begin{aligned}\left(D^{\dagger\,-1}\right)^{\alpha\beta}&\approx\left(D'^{\,-1}\right)^{\alpha\beta}+i\sum_{\gamma\lambda}\left(D'^{\,-1}\right)^{\alpha\gamma}\left(D''\right)^{\gamma\lambda}\left(D'^{\,-1}\right)^{\lambda\beta}\\&\approx\frac{\delta^{\alpha\beta}}{D^{\alpha}}+i\frac{(D'')^{\alpha\beta}}{D^{\alpha}D^{\beta}}\end{aligned} \tag{A.18}$$

Here we have used the fact that the Hermitian part $\underset{\sim}{D}'$ (and hence $(\underset{\sim}{D}')^{-1}$) is diagonal in this basis. From (A.18), both the ss component of the source contribution in (4.52)

$$\begin{aligned}[\underset{\sim}{N}\cdot(\underset{\sim}{D}^{\dagger})^{-1}]^{ss}&=\sum_{\alpha}N^{s\alpha}(D^{\dagger\,-1})^{\alpha s}\\&\approx\frac{N^{ss}}{D^{s}}+i\sum_{\alpha}\frac{N^{s\alpha}(D'')^{\alpha s}}{D^{\alpha}D^{s}}\end{aligned} \tag{A.19}$$

and the trace

$$\begin{aligned}\mathrm{Tr}[\underset{\sim}{N}\cdot(\underset{\sim}{D}')^{-1}]&=\sum_{\alpha\beta}N^{\beta\alpha}(D^{\dagger\,-1})^{\alpha\beta}\\&\approx\sum_{\alpha}\frac{N^{\alpha\alpha}}{D^{\alpha}}+i\sum_{\alpha\beta}\frac{N^{\beta\alpha}(D'')^{\alpha\beta}}{D^{\alpha}D^{\beta}}\end{aligned} \tag{A.20}$$

may be obtained.

The discussion of the lowest order equation and the Poisson bracket at this order has emphasized that the manifold on which the Wigner function S^{s} (or, more precisely, the action density J^{s}) evolves is the surface on which $D^{s}=0$. Taking note of this, it would seem that a few terms in both (A.19) and (A.20) are much larger than the others as D^{s}

appears in the denominator in these expressions. To demonstrate that these terms are in fact not singular, consider the ss component (A.19); the largest term in the sum is produced when $\alpha = s$:

$$\begin{aligned}[\underset{\sim}{N}\cdot(\underset{\sim}{D}^{\dagger})^{-1}]^{ss} &\approx \frac{N^{ss}}{D^s} + i\frac{N^{ss}(D'')^{ss}}{(D^s)^2} \\ &\approx \frac{N^{ss}}{D^s}\left(1 + i\frac{(D'')^{ss}}{D^s}\right) \\ &\approx \frac{N^{ss}}{D^s - i(D'')^{ss}}\end{aligned} \tag{A.21}$$

In view of the vanishing of D^s, the approximation here that $(D'')^{ss} \ll D^s$ which permits the final step of (A.21)) is questionable. It should be borne in mind, however, that the same approximation was invoked in the derivation of the inverse formula (A.17,A.18) so that the result of (A.21) is justified. The fact that the denominator now has a small imaginary part removes the apparent singularity as $D^s \to 0$ and is the appropriate expression.

A similar argument can be made for the trace formula (A.20) where the largest terms are obviously those for which $\alpha = \beta = s$

$$\begin{aligned}\mathrm{Tr}[\underset{\sim}{N}\cdot(\underset{\sim}{D}^{\dagger})^{-1}] &\approx \frac{N^{ss}}{D^s} + i\frac{N^{ss}(D'')^{ss}}{(D^s)^2} \\ &\approx \frac{N^{ss}}{D^s - i(D'')^{ss}}\end{aligned} \tag{A.22}$$

As to be expected from (A.18), the nondegenerate vanishing of D^s ($D^{\alpha\neq s} \neq 0$) makes the ss element of the diagonal much larger than the others so that it is also equal to the trace in this approximation.

Inserting the ss component results for the Poisson bracket (A.7,A.10), the dissipation (A.13) and the sources (A.21) into the first order equation (4.52), it is found that the evolution of the Wigner function S^s expressed in the polarization basis must obey

$$[D^s, S^s] + S^s \sum_{\mu\nu\alpha\neq s} D^{\alpha}\bar{e}^s_{\mu}\bar{e}^{\alpha}_{\nu}[e^{\alpha}_{\mu}, e^s_{\nu}] = -2(D'')^{ss}S^s - \frac{2i\,N^{ss}}{D^s - i(D'')^{ss}} \tag{A.23}$$

The corresponding trace expressions (A.9,A.11,A.14,A.22) require

$$[D^s, S^s] + S^s \sum_{\mu\nu\alpha\neq s} D^\alpha \left\{ \bar{e}^s_\mu \bar{e}^\alpha_\nu [e^\alpha_\mu, e^s_\nu] + \text{c.c.} \right\} = -2(D'')^{ss} S^s - \frac{2i\, N^{ss}}{D^s - i(D'')^{ss}} \qquad \text{(A.24)}$$

Comparison of these two equations implies that if S^s is to satisfy both, one must have that the polarization vector coupling in the Poisson bracket must at least be of higher order. That is, it cannot be shown that these terms vanish identically due to their orthogonality; instead, this is a dynamical result of the evolution equation to this order and in this approximation. In view of this, both (A.23) and (A.24) agree and the evolution equation is finally

$$(\partial_\omega D^s)(\partial_t S^s) - (\partial_t D^s)(\partial_\omega S^s) + \{D^s, S^s\} = -2(D'')^{ss} S^s - \frac{2i\, N^{ss}}{D^s - i(D'')^{ss}} \qquad \text{(A.25)}$$

where we have now used (A.2) to explicitly write the bracket $[\cdot,\cdot]$ in terms of the usual Poisson bracket $\{\cdot,\cdot\}$.

References

[1] V. P. Maslov and M. V. Fedoriuk, *Semi-Classical Approximation in Quantum Mechanics* (Reidel, 1981).

[2] J. Heading, *An Introduction to Phase-integral Methods* (Methuen, London, 1962); M. V. Berry and K. E. Mount, *Reps. Prog. Phys* **35**, 315 (1972).

[3] V. P. Maslov, *Théorie des Perturbations et Méthods Asymptotique* (Dunod, Paris, 1972).

[4] J. B. Keller, *Ann. Phys.* **4**, 180 (1958); J. B. Keller and S. I. Rubinow, *Ann. Phys.* **9**, 24 (1960).

[5] V. I. Arnold, *Functional Analysis and its Applications* **1**, 1 (1966); *Russ. Math. Surv.* **30**, 1 (1975)(English Trans.).

[6] L. Hörmander, *Am. Math. Soc. Symp. Pure Math.* **10**, 138 (1966); *Acta Math.* **127**, 79 (1971); *Comm. Pure Appl. Math* **32**, 359 (1979).

[7] J. J. Kohn and L. Nirenberg, *Comm. Pure Appl. Math.* **18**, 269.

[8] V. Guillemin and S. Sternberg, *Geometric Asymptotics*, Amer. Math. Survey (1976).

[9] J. J. Duistermaat, *Comm. Pure Appl. Math.* **27**, 207 (1974).

[10] C. L. Mehta, *J. Math. Phys.* **5**, 677 (1964); F. A. Berezin and M. A. Šubin, in *Colloquia Mathematica Societatis János Bolyai, 5. Hilbert Space Operators*, Hungary (1970) (North-Holland, Amsterdam, 1972); G. S. Agarwal and E. Wolf, *Phys. Rev.* **D2**, 2161, 2187, 2206 (1970).

[11] V. P. Maslov, *Operational Methods* (Mir Publ., Moscow, 1976); M. Taylor, *Pseudodifferential Operators*, (Princeton University Press, Princeton, 1981).

[12] H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover, New York, 1931), p274.

[13] E. P. Wigner, *Phys. Rev.* **40**, 749 (1932).

[14] M. V. Berry, *Phil. Trans. Roy. Soc. A* **287**, 237 (1977); *J. Phys. A* **10**, 2083 (1977).

[15] E. J. Heller, *J. Chem. Phys.* **65**, 1289 (1976); J. S. Hutchinson and R. E. Wyatt, *Chem. Phys. Lett.* **72**, 378; .

[16] J. E. Moyal, *Proc. Camb. Phil. Soc. Math. Phys. Sci.* **45**, 99 (1947); W. Siegel, *Acta Phys. Pol.* **B7**, 29 (1976).

[17] S. W. McDonald, Ph. D. Thesis, University of California, Lawrence Berkeley Laboratory, Report No. LBL-14837 (1983).

[18] R. F. O'Connell and E. P. Wigner, *Phys. Lett.* **83A**, 145 (1981); R. F. O'Connell, L. Wang and H. A. Williams, *Phys. Rev.* **A 30**, 2187 (1984).

[19] H. Bremmer, *Radio Science* **8**, 511 (1973); M. J. Bastiaans, *J. Opt. Soc. Am.* **69**, 1710 (1979).

[20] S. W. McDonald and A. N. Kaufman, *Phys. Rev.* **A 32**, 1708 (1985).

[21] R. J. Glauber, *Phys. Rev.* **131**, 2766 (1963).

[22] W. H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973).

[23] J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (W. A. Benjamin, New York, 1968); J. R. Klauder, *Phys. Rev. Lett.* **56**, 897 (1986).

[24] Y. Weissman, *J. Chem. Phys.* **76**, 4067 (1982); *J. Phys. A* **16**, 2693 (1983).

[25] E. J. Heller, *J. Chem. Phys.* **66**, 5777 (1977), **67**, 3339 (1977); M. J. Davis and E. J. Heller, *J. Chem. Phys.* **71**, 3383 (1979).

[26] A. Jannussis, N. Patargias, A. Leodaris, P. Filippakis, T. Filippakis, A. Streclas and V. Papatheou, *Lett. Nuovo Cim.* **34**, 553 (1982); A. Jannussis, A. Leodaris, N. Patargias, T. Filippakis, P. Filippakis and K. Vlachos, *Lett. Nuovo Cim.* **34**, 433 (1982); F. Soto and P. Claverie, *Physica* **109A**, 193 (1981); N. D. Cartwright, *Physica* **83A**, 210 (1976); R. F. O'Connell and E. P. Wigner, *Phys. Lett.* **85A**, 121 (1981); K. Takahashi and N. Saito, *Phys. Rev. Lett.* **55**, 645 (1985).

[27] S. W. McDonald, *Phys. Rev. Lett.* **54**, 1211 (1985).

[28] R. G. Littlejohn, *Phys. Rev. Lett.* **54**, 1742 (1985); *Phys. Rep.* **138**, 193 (1986).

[29] H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, Mass., 1950).

[30] V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, New York, 1978).

[31] G. B. Whitham, *Linear and Nonlinear Waves* (John Wiley & Sons, New York, 1974).

[32] I. C. Percival, *Adv. Chem. Phys.* **36**, 1 (1977).

[33] A. Einstein, *Verhand. Deut. Phys. Ges.* **19**, 82 (1917).

[34] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1968).

[35] S. Ichimaru, *Basic Principles of Plasma Physics, A Statistical Approach* (W. A. Benjamin, Reading, Mass., 1973).

[36] N. L. Balasz, *Physica* **102A**, 236 (1980).

[37] E. P. Wigner, in *Perspectives in Quantum Theory*, Eds. W. Yourgrau and A. van der Merwe (Dover, New York, 1979), p25.

[38] H. Mori, I. Oppenheim and J. Ross, in *Studies in Statistical Mechanics*, Vol. I, Eds. J. De Boer and G. Uhlenbeck (North-Holland, Amsterdam, 1962).

[39] B. Leaf, *J. Math. Phys.* **9**, 65, 769 (1968).

[40] L. Friedland and I. Bernstein, *IEEE Trans. Plasma Sci.* **PS-8**, 90 (1980); *Phys. Rev. A* **22**, 1680 (1980).

[41] J. Mathews and R. L. Walker, *Mathematical Methods of Physics*, 2nd Ed. (W. A. Benjamin, New York, 1970), p481.

[42] R. Z. Sagdeev and A. A. Galeev, *Nonlinear Plasma Theory* (W. A. Benjamin, New York, 1969).

[43] B. B. Kadomtsev and V. I. Petviashvili, *Sov. Phys. JETP* **16**, 1578 (1963); M. Camac, A. R. Kantrowitz, M. M. Litvak, R. M. Patrick and H. E. Petschek, *Nucl. Fus.*, 1962 Suppl., 423.

[44] I. Bernstein, *Phys. Fluids* **18**, 320 (1975).

[45] K. Katou, *J. Phys. Soc. Jap.* **50**, 642 (1981).

[46] M. B. Priestly, *Spectral Analysis and Time Series*, Vol. 2 (Academic Press, London, New York, 1981); W. D. Mark, *J. Sound Vib.* **11**, 19 (1970).

[47] P. Bertrand, J. P. Doremus, B. Izrar, Nguyen V. T. and M. R. Feix, *Phys. Lett.* **94A**, 415 (1983).

[48] T. B. Smith, *J. Phys. A* **11**, 2179 (1978).

[49] M. S. Marinov, *J. Phys A* **12**, 31 (1979).

[50] I. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, New York, 1980), p1033.

Figure Captions

Fig. 1. Phase-space of a one-dimensional oscillator Hamiltonian $H(x,p) = p^2/2m + V(x)$. The x-space turning points x_L, x_R (where $p = 0$) are also the points where the projection of the phase-space trajectory onto x-space becomes singular ($\partial_x p \to \infty$) producing x-space caustics in the asymptotic wave solution. Similarly, k-caustics are divergences in the k-space representation of the wave where the projection of the ray manifold onto k-space becomes singular.

Fig. 2. Schematic illustration of the singularities in phase space characterizing the common symbol of the spectral tensor $s(x,k)$ when it is evaluated in the geometrical optics approximation. These singularities are simply the linear extension into phase space of the singularities present in both $A(x)$ and $\hat{A}(k)$ due to the caustic singularities in the projection of the ray manifold onto x- or k-space.

Fig. 3. Comparison of x-space probability densities (wave intensities) for the harmonic oscillator state $n = 60$. Solid oscillatory curve is exact $|\psi_{60}(x)|^2$, dotted singular curve is exact classical ray probability density (or conventional eikonal intensity), and solid non-oscillatory curve is $\langle|\psi_{60}|^2\rangle$, the gaussian-smoothed wave intensity obtained by projecting $P_{60}(x,k)$ (with $\sigma = \alpha^{-1}$) from phase space onto x-space.

Fig. 4. Schematic illustration of the "doubled phase space" $\wp$ induced by the eikonal solution of the phase-space equation (5.40). The familiar underlying phase space $\Re$ is coordinatized by either real variables (x,k) or the complex variable $\overline{z}$; this is to be viewed as the "configuration space" of $\wp$ with the complex-valued variable κ being the momentum conjugate to $\overline{z}$ ("local phase-space wavenumber"). Orbits in $\wp$ are projected onto $\Re$ in order to determine their relationship to "physical" ray trajectories.

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