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# Essays on Welfare in a Disinformation Age

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Doctor of Philosophy
in
Economics

by

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Committee in charge:

Professor Theodore Bergstrom, Chair
Professor Gary Charness
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December 2019

The dissertation of Huy X. Nguyen is approved.

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October 2019

Essays on Welfare in a Disinformation Age

To my students—past, present, and future.

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I thank my parents for their decades raising me amidst endless hardships as a working-class, immigrant family. They accepted sacrifices without hesitation so that I may be the first in our ancestry to attend college. The resilience we forged together helped me traverse an arduous graduate journey.

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**Curriculum Vitæ**

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## Education

2019 **Ph.D. in Economics**
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2014 **M.A. in Economics**
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2014 Referee for ILR Review
2014 Spring School in Behavioral Economics, *UCSD Rady School*
2011 Research Assistant, *Boston U., for Hiroaki Kaido*

***Working Papers***

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# Abstract

Essays on Welfare in a Disinformation Age

by

Huy X. Nguyen

A historical age is often named after its most salient resource. In our current age, that resource is information. Information has impacted economics in transformative ways, from a shift toward services to competition with outsourcing and automation. Technologies like mass and social media have made us more efficient and knowledgeable but also susceptible to untruths.

My dissertation explores the effects of misinformation (inaccurate) and disinformation (intentionally false) on collective choice, namely that they reduce expected welfare. I assume the role of a benevolent social planner and design mechanisms that improve this welfare, given that untruths occur. Then, because information closely relates to perception, I share a fresh approach on understanding and appreciating a well-known principle.

In Chapter 1 (Delayed Information Improves Cascades), I consider how the Internet enables information–be it true, inaccurate, or intentionally false—to cascade instantly and globally. I design a mechanism to mitigate wrong cascades by limiting the first $k$, of $n$, players to observe their own signal but not the signal or even action of previous players. I show that instant information ($k = 0$) performs strictly better than no information ($k = n$) but that delayed information ($0 < k < n$) performs best at an optimal $k^*(n, p)$, where $p$ is signal accuracy. This suggests that polling and review websites can improve welfare by reaching a minimum number of ratings before releasing aggregate results.

In Chapter 2 (Pretending Volunteers), I introduce the ability to pretend in the volunteer's dilemma. Pretending contributes nothing, but it costs less than helping and confers honor if the public good is provided and shame otherwise. The main result is that the ability to pretend weakly reduces provision chance. High values of honor increase provision, especially when coupled with high shame. In the long run, pretenders dilute the honor from helping and discourage actual helpers. Authenticated help at a premium can remedy this. Extensions on sophistication and asymmetry explain why helpers, bystanders, and pretenders coexist.

In Chapter 3 (Visualization of Revenue Equivalence), I construct three-dimensional visualizations of revenue equivalence between first-price, second-price, and all-pay sealed-bid auctions for two bidders with uniformly-distributed private values. The mean height of each solid represents expected revenue and is equal across all three formats. I then present a summation approach using partitioned volumes. As the increments shrink toward zero, the three expected revenues converge to the continuous limit. Lastly, I share a tangible, ham-and-cheese model as example of a pedagogical tool.

# Contents

# Chapter 1

# Delayed Information Improves Cascades

## 1.1 Introduction

Consider a game of $n$ sequential players deciding whether to adopt a new product. This product need not be a commodity or service; it can also represent a new medicine, technology, politician, or fashion. Each player receives a signal on whether the new product will lead to prosperity or ruin. This signal is right with probability $p$. Each player must decide whether to adopt the new product based on the choices (not signals) of previous players plus his own signal. If the count favors one choice over another, he chooses that; if it's a tie, he follows his own signal. A player receives utility 1 if he chooses rightly and 0 otherwise.

The predominant risk among cascade games is that it takes just the first two players to receive wrong signals for everyone to follow suit like lemmings.[1] In the early nineties, independent authors showed that cascades, right or wrong, inevitably occur among sequential players when group size becomes sufficiently large (Welch 1992, Banerjee 1992). This is true even when the signal is non-binary but finitely many (Bikhchandani et al. 1992); however, the signal distribution must be bounded for cascades to occur (Smith and Sorensen 2000).

Though cascades exist only under specific assumptions, they nonetheless provide a useful template in understanding movements and the risks they pose when misinformed. Cascade models quickly found applications in financial markets, particularly asset pricing (Welch 1992) and bank runs (Chen 1999).[2] Theoretical extensions soon covered multidimensional signals (Avery and Zemsky 1998), risk aversion (Decamps and Lovo 2006a), transaction costs (Romano 2007), and reputation concerns (Dasgupta and Prat 2008). Following some empirical and experimental studies, the cascade literature has since waned in popularity.

However, digital advances over the past decade have brought new opportunities and dangers, prompting an urgent reexamination of cascades. Telecommunications now enable ideas—be they true (information), inaccurate (misinformation), or intentionally false (disinformation)—to spread instantly and globally. Social media giants Facebook, YouTube, and Twitter are often criti-

[1]The terms 'information cascade' and 'herd behavior' are often used interchangeably. However, a cascade refers to some players ignoring their own signal while a herd refers to all players choosing the same action in the long run. A cascade implies a herd but not necessarily the reverse.

[2]Pricing models further require exogenous prices (Avery-Zemsky 1998) and discrete actions (Lee 1993) for cascades to form. The intuition is that endogenous, updating prices and continuous actions preserve rather than obscure private information. These are called the price critique and continuous investment critique, respectively.

cized for contributing to fake news, echo chambers, and anti-vaccination campaigns. Meanwhile, avant-garde 'deepfake' artificial intelligences can render lifelike videos of anyone's face and voice.

These challenges exhibit characteristics of cascades because early influencers can disproportionately shape public opinion like never before. People habitually forward trending ideas without investigating further sources. Agencies, too, once took time to collect and publish aggregate data. Now digital polls, ratings, reviews, and other signals update in live time, accessible to all. Their convenience also makes them susceptible to the volatility of first movers. Left unchecked, digital cascades can exacerbate science myths, sociocultural polarizations, and even contagious diseases.

Motivated by these concerns, I propose an improvement by delaying information such that the first $k$ players ($k \leq n$) can see their private signal but not the choices of previous players. This represents websites revealing a statistic only after a minimum number of reviews. Players thereafter can see their own signal as well as previous choices. The tradeoff here is that the initial $k$ players sacrifice information that would otherwise help their own decision in order to provide the remaining $n - k$ players a more accurate signal.

For example, suppose $n = 20$, $p = 0.7$, $k = 4$. The first four players have only their own signal to follow. The fifth player then sees all four of those actions plus his own signal and makes a more informed decision. The aggregate signal is more accurate because, by the law of large numbers, repeated randomization reduces the variance. This intuition leads us to conjecture that delayed information (e.g. $k = 4$ in this example) can outperform instant information (i.e. $k = 0$) in expected welfare.

If $k = 4$ is indeed better, then what $k^*$ is best? In other words, if we are to maximize welfare, for how many $k$ players should we withhold information before the remaining players from $k + 1$ onward can see all previous actions? Equivalently, choose $k^*(n, p)$ to maximize $E(W|n, p, k)$.

Two notable studies exist on delayed cascades. In the first, some players receive a buy option with a fixed exercise price. It is publicly known whenever a player exercises their buy option, which creates a positive (informational) externality about the asset value (Chamley and Gale 1994). The authors find that pessimistic beliefs lead to no investment, optimistic beliefs lead to instant investment, and intermediate beliefs lead to randomizing between instant or delayed investment. In the second, players receive private signals and can endogenously choose both their actions and also the timing of those actions (Zhang 1997). The author shows that a cascade in favor of investment always occurs in equilibrium, and a strategic delay exists prior to cascading. This delay causes a loss of welfare because the model incorporates a discount factor $\delta$ and assumes waiting is costly.

In these preceding papers, players can strategically choose the timing of their action. In my paper, however, the order of player actions is exogenously determined. Furthermore, my model closely follows the original structure of Bikhchandani, Hirshleifer, and Welch in order to focus on the effect of delayed revelation of player actions. My paper thus redirects the agency from simple-strategy players to a benevolent planner who accepts that cascades are inevitable but maximizes welfare via delayed information. This is useful for digital environments where organizations can control when to disclose early polls or ratings. To my best knowledge, my paper is the first to study cascades where player actions are not revealed until a threshold number of actions have been made.

In Section 2 (Instant Information), I present the original cascade model as a baseline I call instant information ($k = 0$) and derive the formula for expected welfare. I provide a table for per-person (i.e. percentage) expected welfare for varying group size $n$ and signal accuracy $p$. I analyze comparative statics to show that this welfare is primarily determined by $p$, whereas $n$ has negligible impact beyond $n > 40$ or so players.

In Section 3 (Delayed Information Mechanism), I introduce a delayed-information mechanism where the first $k$ players see only their private signal and neither the signals nor actions of previous players. In Section 3.1 (Welfare Formula), I derive the formula for expected welfare under delayed information ($0 < k < n$) and compare this among instant ($k = 0$) and no ($k = n$) information. I prove that no information ($k = n$) is welfare-*minimizing* among possible values of $0 \leq k \leq n$. This demonstrates the positive information externality gained from seeing previous actions.

In Section 3.2 (Optimal Delay), I show that delayed information ($0 < k \leq k_T < n$) outperforms instant information ($k = 0$), up to a certain threshold $k_T$.[3] In other words, it is possible to delay information *too much* such that we are better off with no delay at all. For any $n > 2$ and $p > 1/2$, there exists an optimal $k^*(n, p)$ that maximizes expected welfare $E(W|n, p, k)$. While I did

[3]Threshold $k_T$ is different from welfare-maximizing $k^*$; generally, $k^* < k_T$.

not find a closed-form solution, I do derive an algorithm that identifies $k^*$ and use it to compute a table for $k^*$ based on inputs of $n$ and $p$.

In Section 3.3 (Approximation Function), I provide an approximation function to estimate $k^*(n, p)$ by regressing the data generated by the optimal delay algorithm. Two versions are offered: an 'accurate' function with correlation $R^2 = 0.982$ but cumbersome to use, or an 'accessible' function with correlation $R^2 = 0.847$ but convenient to use. I further simplify the accessible version to produce a rule-of-thumb that is best used for group sizes $n \in [10, 50]$.

In Section 4 (Model Extensions), I explore variants as augments or alternatives to the delayed-information mechanism. In Section 4.1 (Early Disclosure), I consider disclosure prior to $k^*$ if the first $2 < m < k^*$ signals match. I show that augmenting this sub-mechanism can further improve welfare. In Section 4.2 (Strict Margin), I propose a mechanism that discloses when one signal outnumbers the other by three. I show that this path-dependent alternative performs comparably to a fixed-time disclosure.

In Section 4.3 (Type I/II Errors), I adjust the delayed-information model to account for Type I and Type II errors by integrating the probabilities a good or bad product appears good ($p_G$, $p_B$), and the benefit and cost of choosing rightly ($b$, $c$). These results greatly extend the model's applicability over a wide range of scenarios.

In Section 5 (Discussion), I compare the delayed information problem to related optimal stopping problems, including confirmation bias, the multi-armed bandit problem, and the secretary problem. Lastly, I suggest a 'Rule of 75' as the most convenient and portable estimator for $k^*(n, p)$.

## 1.2 Instant Information

In the classic cascade model, the first player has only his own signal to follow. We can assume $p > 1/2$ without loss of generality because if a signal is more likely to be wrong than right, a player can always take the complement $1 - p$ as right. For example, a binary signal that is right 3 out of 10 times is, counter-intuitively, more useful than one that is right 6 out of 10 times. This is because we can infer the complement of 3/10, which is 7/10, to be right.

The second player sees the first player's choice and his own signal. If they match, he joins the first player. If they differ, which is a tie, he follows his own signal. Either way, he will follow his own signal.

The third player is where public information can first override private information. If the first two players match, the third ignores his signal and joins them because their two counts already outweigh his one. In this case, the fourth, fifth, and every player thereafter would ignore their signal and join the group. This is called a cascade. On the other hand, if the first two players differ, the third gains no useful information and thus follows his own signal. Effectively, this 'resets' the count because the third player acts *as if* he is the first.

From this we can conclude that cascades always start on even-numbered players. Now let us define a signal 'pair' as $(1^{st}, 2^{nd})$, $(3^{rd}, 4^{th})$, $(5^{th}, 6^{th})$, and so on. If a signal pair ever matches ($AA$ or $BB$), this triggers a cascade. Conversely, so long as signal pairs differ ($AB$ or $BA$), the cascade is postponed. Equivalently, a cascade starts when one signal's count exceeds the other by two. For example, the sequence $ABBA$ resets the count but $ABBB$ triggers a cascade because there are three $B$s to one $A$—*not* because there are two consecutive $B$s! Even if the fifth signal is $A$, $B$s still dominate.

To show that delayed information improves cascades, we must first establish a baseline level of welfare.

**Proposition 1.** *Given group size $n > 2$, signal accuracy $p > 1/2$, and instant information ($k = 0$), the expected welfare is:*[4]

$$E(W|n,p,0) = \sum_{r=0}^{n} r \cdot 2^{min\{r,n-r\}} \cdot p^{min\{r,n-r+2\}} \cdot (1-p)^{min\{r+2,n-r\}} \tag{1.1}$$

*Proof of Proposition 1.* Without loss of generality, let $A$ to be the objectively right choice and $B$ the objectively wrong one; this is unknown to deciding players. For a given $n$, there are finite sequences in which exactly $r$ right signals occur. A forward method is to repeat resets ($AB$ or $BA$) to accumulate $A$s while postponing a cascade. Once $r$ right signals are reached, the sequence cascades-wrong. For example, suppose we want $r = 2$ right out of $n \geq 6$ players. The sequence $(AB)(AB)BB...$ results in exactly $r = 2$ right players. The ellipses indicate that subsequent signals, right or wrong, produce the same result because a cascade has triggered. The parentheses indicate that $A$ and $B$ can be switched inside and produce the same result. That is, $(AB)(BA)BB...$, $(BA)(AB)BB...$, and $(BA)(BA)BB...$ are equivalent variants.

There is a limitation to the forward method: we cannot count $r > n/2$ because each $A$ demands one signal pair. To count past the midpoint, we use a reverse method, which is to instead accumulate $B$s via resets then cascade-

[4] The expression $0^0$ is better understood as equal to 1 rather than 'undefined' in this context. This is relevant in the special case where $p = 1$ and $n = r$.

right. For example, the sequence $(BA)AA...$ results in exactly $r = 4$ right out of $n = 5$ players. Therefore, if:

A)

$0 \leq r \leq n/2$, then the probability of exactly $r$ rights is:

$$P(R = r) = 2^r \cdot p^r \cdot (1 - p)^{min\{r+2,n-r\}} \tag{1.2}$$

There are $2^r$ variants for $r$ reset signal pairs $(AB)$. If $n = 2r + 1$, then the last signal is $B$ for a total of $n - r$ wrong signals. If $n > 2r + 1$, then a cascade-wrong $BB...$ follows the resets. In this case, subsequent signals do not affect the probability; that is, we would be multiplying the event $A \vee B$, which has probability $p + (1 - p) = 1$. In this case, there are $r + 2$ wrong signals that affect the probability. Equivalently, there are $min\{r + 2, n - r\}$ wrong signals that affect the probability. Note that if $n > 2r + 1$, $n - r \geq r + 2$.

B)

$n/2 \leq r \leq n$, then the probability of exactly $r$ rights is:

$$P(R = r) = 2^{n-r} \cdot p^{min\{r,n-r+2\}} \cdot (1 - p)^{n-r} \tag{1.3}$$

This is simply the symmetric case, switching $A$ and $B$. If $r = n/2$, then both forward and reverse methods are equal. In fact, we can merge the two probability equations as a unified formula for $0 \leq r \leq n$:

$$P(R = r) = 2^{min\{r,n-r\}} \cdot p^{min\{r,n-r+2\}} \cdot (1 - p)^{min\{r+2,n-r\}} \tag{1.4}$$

Then, the expected welfare is a sum of the random number of right players ($R = r$) weighted by the probability exactly $r$ rights occur. □

**Examples**

I now walk through a few examples for intuition. For any $n > 2$, the probability of zero rights is $P(R = 0) = (1-p)^2$ because if the first two signals are wrong, everyone will cascade wrong. On the other end, the probability of all rights is $P(R = n) = p^2$. For a given $n > 2$, what is the probability of exactly one right player, $P(R = 1)$? Let us start with $n = 3$. Signals $ABB$ or $BAB$ produce exactly one right; $BBA$ produce zero rights because player 3 chooses $B$. Exactly two right players would be $ABA$ or $BAA$ but not $AAB$. So, the expected welfare is:

$$E(W|n = 3) = \sum_{r=0}^{3} r \cdot P(R = r|n = 3) = \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} (1-p)^2 \\ 2 \cdot p(1-p)^2 \\ 2 \cdot p^2(1-p) \\ p^2 \end{bmatrix} \quad (1.5)$$

For $n = 4$, $ABBB$ or $BABB$ produces one right; $ABAB$, $ABBA$, $BAAB$, or $BABA$, two right; and $ABAA$ or $BAAA$, three. The idea is that resets extend the number of rights or wrongs before cascading toward one direction. The expected welfare is:

$$E(W|n=4)=\sum_{r=0}^{4} r\cdot P(R=r|n=4)=\begin{bmatrix}0 & 1 & 2 & 3 & 4\end{bmatrix}\cdot\begin{bmatrix}(1-p)^2\\ 2\cdot p(1-p)^3\\ 2^2\cdot p^2(1-p)^2\\ 2\cdot p^3(1-p)\\ p^2\end{bmatrix} \tag{1.6}$$

For $n = 5$, $ABBB$... or $BABB$... produces one right; the ellipses indicate that any signals thereafter produce the same result because a cascade has occurred. $ABABB$, $ABBAB$, $BAABB$, $BABAB$, two right; $ABABA$, $ABBAA$, $BAABA$, $BABAA$, three; and $BAAA$... or $ABAA$..., four. So, we can extrapolate:

$$E(W|n=5)=\sum_{r=0}^{5} r\cdot P(R=r|n=5)=$$

$$\begin{bmatrix}0 & 1 & 2 & 3 & 4 & 5\end{bmatrix}\cdot\begin{bmatrix}(1-p)^2\\ 2\cdot p(1-p)^3\\ 2^2\cdot p^2(1-p)^3\\ 2^2\cdot p^3(1-p)^2\\ 2\cdot p^3(1-p)\\ p^2\end{bmatrix}$$

$$E(W|n=6) = \sum_{r=0}^{6} r \cdot P(R=r|n=6) =$$

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} (1-p)^2 \\ 2 \cdot p(1-p)^3 \\ 2^2 \cdot p^2(1-p)^4 \\ 2^3 \cdot p^3(1-p)^3 \\ 2^2 \cdot p^4(1-p)^2 \\ 2 \cdot p^3(1-p) \\ p^2 \end{bmatrix}$$

...

$$E(W|n,p,0) = \sum_{r=0}^{n} r \cdot 2^{min\{r,n-r\}} \cdot p^{min\{r,n-r+2\}} \cdot (1-p)^{min\{r+2,n-r\}} \quad (1.7)$$

Inputting example numbers $(n, p) = (20, 0.7)$ yields $E(W|20, 0.7) \approx 16.40$. By comparison, with no information where everyone follows their own signal, the expected welfare is only $E(20, 0.7) = 20 \cdot 0.7 = 14.00$. Instant information leads about 82% of the group to the right choice compared to no information's 70%. Inputting $p = 0.5$ yields $E(W) = n/2$ because the signal is useless; inputting $p = 1$ yields $E(W) = n$ because the signal is perfect.

| $p =$ | 0.55 | 0.60 | 0.65 | 0.70 | 0.75 | 0.80 | 0.85 | 0.90 | 0.95 |
|---|---|---|---|---|---|---|---|---|---|
| $n =$ | | | | | | | | | |
| 10 | 58% | 66% | 73% | 80% | 85% | 90% | 94% | 97% | 99% |
| 20 | 59% | 67% | 75% | 82% | 88% | 92% | 95% | 98% | 99% |
| 30 | 59% | 68% | 76% | 83% | 88% | 93% | 96% | 98% | 99% |
| 40 | 59% | 68% | 76% | 83% | 89% | 93% | 96% | 98% | 99% |
| 50 | 60% | 69% | 77% | 83% | 89% | 93% | 96% | 98% | 100% |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |
| 100 | 60% | 69% | 77% | 84% | 90% | 94% | 97% | 99% | 100% |

**Fig 1.** Expected % Right By Signal Accuracy & Group Size

More accurate signals lead to a larger percentage of right players. We can see that scaling $n$ from 40 to 50 or even 100 changes $E(W)/n$ by no more than one percent, given rounding. For practical purposes, it is safe to say that signal accuracy $p$ is the primary factor in determining per-person, expected welfare $E(W)/n$ and that group size $n$ has negligible impact beyond $n > 40$ or so players with instant information.

## 1.3 Delayed Information Mechanism

### 1.3.1 Welfare Formula

Now I introduce a delayed-information mechanism where the first $k$ players see only their private signal and neither the signals nor actions of previous players. The expected net welfare of these initial players is $k \cdot p$. The expected net welfare of remaining players is more complicated. It depends on whether the first players acting on their own signals determine a cascade-right, a cascade-wrong, or a reset.

**Proposition 2.** *For group size $n > 2$, signal accuracy $p > 1/2$, and even delay $k$, the expected welfare is:*

$$E(W|n, p, k) = k \cdot p + P_{CR} \cdot (n - k) + P_{RS} \cdot E(W|n - k, p, 0) \tag{1.8}$$

*where the probability of a cascade-right is:*

$$P_{CR} = \sum_{r=(k/2)+1}^{k} \binom{k}{r} p^r (1 - p)^{k-r}$$

*the probability of a reset is:*

$$P_{RS} = \binom{k}{k/2} p^{k/2} (1 - p)^{k/2}$$

*and the subgame expected welfare of remaining players, should initial players reset, is:*

$$E(W|n - k, p, 0) = \sum_{r=0}^{n-k} r \cdot 2^{min\{r,n-k-r\}} \cdot p^{min\{r,n-k-r+2\}} \cdot (1 - p)^{min\{r+2,n-k-r\}}$$

*Proof of Proposition 2.* If, among the $k$ initial players, there are two or more $A$ (right) signals than $B$ (wrong) signals, then a cascade-right occurs and all $n - k$ remaining players each get utility 1. If instead $B$s outnumber $A$s by two or more, then a cascade-wrong occurs and remaining players get utility 0. If signals are evenly split, then initial players do not affect the decision of remaining players. In this case, the remaining players essentially play an instant-information subgame, so their welfare is identical to Proposition 1 except $n$ players is replaced by $n-k$ players.

Then, the probability that an even $k$ cascades-right is the sum of cases where $A$ signals exceed $B$ by two or more over all $2^k$ possible cases. Using binomial expansion, the probability of a cascade-right is:

$$P_{CR}(R \geq (k/2)+1) = \sum_{r=(k/2)+1}^{k} \binom{k}{r} p^r (1-p)^{k-r} \tag{1.9}$$

The probability of a reset is:

$$P_{RS}(R = k/2) = \binom{k}{k/2} p^{k/2} (1-p)^{k/2} \tag{1.10}$$

And the probability of a cascade-wrong is:

$$P_{CW}(R \leq (k/2)-1) = \sum_{r=0}^{(k/2)-1} \binom{k}{r} p^r (1-p)^{k-r} \tag{1.11}$$

Because $k \geq 2$ is even, $k/2 \in \mathbb{N}$. These three probabilities sum to 1. For even $k$ (or, equivalently, odd $k-1$), the total expected welfare is:

$$E(W|n,p,k) = k \cdot p + P_{CR} \cdot (n-k) + P_{RS} \cdot E(W|n-k,p,0) \tag{1.12}$$

□

For clarity, I interpret each term. The term $k \cdot p$ is the expected welfare of the initial players. The terms $P_{CR}$ and $P_{RS}$ are the respective probabilities of a cascade-right or a reset, given above; notice that $P_{CW}$ does not appear because the welfare of remaining players in a cascade-wrong would be zero. The recursive term $E(W|n-k,p,0)$ is the subgame expected welfare of remaining players, given $k$ fewer players and instant information, should the initial players reset.

**An Odd/Even Technicality**

**Corollary 2.** *For group size $n > 2$, signal accuracy $p > 1/2$, and even delay $k$, the expected welfare can be improved by using the odd delay $k-1$.*

*Proof of Corollary 2.* Among an initial, odd $k-1$ signals, there are four possible cases: $A$ outnumbers $B$ by two or more; $B$ outnumbers $A$ by two or more; $A$ outnumbers $B$ by one; $B$ outnumbers $A$ by one. The first two cases trigger a cascade, but the last two depend on the subsequent signal of the even $k^{th}$ player, which can either reset or cascade toward the leading signal. If the $k^{th}$ signal matches the majority, a cascade occurs; if it differs, then a reset occurs.

How does this compare with even delay $k$? Suppose the even $k^{th}$ player is among the initial group who follow their own signal. If, among the first $k-1$ signals, one signal outnumbers the other by two, then a cascade certainly occurs for players $k+1$ onward regardless of $k$'s signal. This is identical to the case with odd delay $k-1$.

If, among the first $k-1$ signals, one outnumbers the other by one, the fate of players $k+1$ onward depend on $k$'s signal. If it matches the majority, a cascade occurs; if it differs, a reset occurs. This, too, is identical to the case with odd delay $k-1$.

Thus, the expected utility of all but one player is invariant whether we use even delay $k$ or odd delay $k-1$. Only player $k$'s utility is marginally greater under odd delay $k-1$ due to a positive information externality. Specifically, he has a chance to cascade-right, a chance to cascade-wrong, and a chance to follow his own signal. If he follows his own signal, this is no different than what he would have done under even delay $k$. If he cascades, he has a greater than $p$ chance to cascade-right. This is because the signal accuracy is $1/2 < p < 1$ and, by law of large numbers (LLN), more samples are better than one. Specifically, his utility gain from using odd delay $k-1$ instead of even delay $k$ is:

$$\begin{aligned}
&E(U_k|k-1) - E(U_k|k) \\
&= P(R \geq (k/2)+1) + P((k/2)-1 \leq R \leq k/2) \cdot p - p \\
&= \sum_{r=(k/2)+1}^{k-1} \binom{k-1}{r} p^r (1-p)^{k-1-r} \\
&+ \left[ \sum_{r=(k/2)-1}^{k/2} \binom{k-1}{r} p^r (1-p)^{k-1-r} \right] \cdot p - p && (1.13) \\
&\approx \epsilon > 0 && (1.14)
\end{aligned}$$

□

To illustrate, consider the game $(n, p, k) = (20, 0.7, 4)$ with even delay $k = 4$ versus odd delay $k - 1 = 3$. The marginal fourth player gains $E(U_4|k = 4) = 0.7 < 0.78 \approx E(U_4|k-1 = 3)$, and the respective welfares are 16.67 versus 16.75, a very small difference of 0.08. In general, *decreasing* from an even $k > 2$ to its odd, $k-1$ partner in a signal pair is Pareto-improving because the marginal $k^{th}$ player gains a positive information externality from seeing previous actions.[5] This is not to be confused with *increasing* to an odd $k+1$, which is not Pareto-improving because the $k+1^{th}$ player loses information.

That $E(W|k-1) = E(W|k) + \epsilon$ for even $k > 2$ is an important result that simplifies our computation by one-half. The matter of odd versus even $k^*$ is more technical than practical, as the welfare difference is minimal. Moreover, because resets occur only with even $k$, it is magnitudes simpler to compute welfare via the even component in a signal pair. For these two practical reasons, all subsequent equations use even $k$ and treat $E(W|k-1) = E(W|k) + \epsilon$, where $\epsilon$ is negligible.

**Proposition 3.** *Given group size $n > 2$, and signal accuracy $p > 1/2$, no information ($k = n$) is welfare-minimizing among all possible $0 \leq k \leq n$.*

*Proof of Proposition 3.* If $k = n$, no one observes anyone else, so each player has expected payoff $p$. If $k < n$, some players observe others, and those who follow a cascade have expected payoff of at least $p^2/[p^2+(1-p)^2]$, the minimum conditional probability of cascading-right given that a cascade has occurred. This probability is greater than $p$ when:

$$\frac{p^2}{p^2 + (1-p)^2} > p \tag{1.15}$$

$$\frac{p}{p^2 + (1-p)^2} > 1$$

$$p > p^2 + (1-p)^2$$

[5] The exception is $k = 2$ because player two follows his own signal whether or not he sees player one's action. Player two gains no information externality and thus no better chance of choosing rightly.

$$
\begin{aligned}
& p > p^2 + 1 - 2 \cdot p + p^2 \\
& 0 > 2 \cdot p^2 - 3 \cdot p + 1 \\
& 0 > (2 \cdot p - 1)(p - 1) \\
\implies & 1/2 < p < 1 \qquad (1.16)
\end{aligned}
$$

Of course, the signal accuracy is innately $1/2 < p < 1$, so this is always true. If $k < n$, there is a positive probability that at least one player cascades, so any $0 \leq k < n$ is better than $k = n$. □

### 1.3.2 Optimal Delay

Due to the combinatorial nature of the welfare formula, I could not derive a closed-form solution for optimal delay $k^*(n, p)$. However, I do derive an algorithm using a few simplifying deductions. First, I recognized symmetry across the midpoint $n/2$ to derive the expected welfare of the instant-information model, $E(W|n, p, 0)$. Second, I developed the concept of signal pairs in the delayed-information model to show that $E(W|n, p, k-1) = E(W|n, p, k)+\epsilon$ for even $k > 2$, where $\epsilon$ is a few percent higher chance for the marginal $k^{th}$ player to choose rightly. Third, I now substitute the recursive term $E(W|n-k, p, 0)$ in Proposition 2 with a shifted (i.e. using $n-k$ instead of $n$) $E(W|n-k, p, 0)$ from Proposition 1. This transforms $E(W|n, p, k)$ from a recursive formula into an explicit formula.

$$E(W|n, p, k) = k \cdot p + P_{CR} \cdot (n - k) + P_{RS} \cdot \sum_{r=0}^{n-k} r \cdot P(R = r) \qquad (1.17)$$

where

$$P(R = r) = 2^{min\{r,n-k-r\}} \cdot p^{min\{r,n-k-r+2\}}(1-p)^{min\{r+2,n-k-r\}}$$

Let us return to our numerical example $(n, p, k) = (20, 0.7, k)$. When the information delay was $k = 0$, the expected welfare was $E(W|20, 0.7, 0) \approx 16.40$. That is, about 82% of the 20 players are expected to choose rightly. This was an improvement over the other extreme of $k = n$, which yielded $E(W|20, 0.7, 20) = 14$, or 70% of players.

Recall that $k \geq 2$ is even. Because the first two players already follow their own signal, $E(W|n, p, 0) = E(W|n, p, 2)$. So, the minimum effective delay must be $k = 4$. Those initial players gain expected net welfare of $k \cdot p = 4 \cdot 0.7 = 2.80$. If among initial players, right signals outnumber wrong signals by two or more, remaining players will cascade-right. The probability of this is $P_{CR}(R \geq (k/2) + 1) \approx 65\%$, and each of those $n - k = 20 - 4 = 16$ players would gain utility 1. So, the middle term in the formula is $P_{CR}(n-k) \approx 65\% \cdot 16 = 10.43$. If among initial players, right and wrong signals are equally frequent, then remaining players play a 'reset' subgame equivalent to $(n, p, k) = (16, 0.7, 0)$. This occurs with probability $P_{RS}(r = k/2) \approx 26\%$, and the expected welfare from such a subgame would be $E(W|16, 0.7, 0) \approx 13.02$. So, the last term is $P_{RS} \cdot E(W|16, 0.7, 0) \approx 26\% \cdot 13.02 = 3.39$.[6] Combining all three terms yields:

$$E(W|20, 0.7, 4) \approx 2.80 + 10.43 + 3.39 = 16.62 \tag{1.18}$$

[6]A cascade-wrong yields utility 0 for remaining $n - k$ players, so this term does not appear.

This is about 83% of players choosing rightly, an improvement over the 82% obtained with $k = 0$. Computing inputs of delay $k$ yields the following graph for per-person, expected welfare $E(W|20, 0.7, k)/n$:

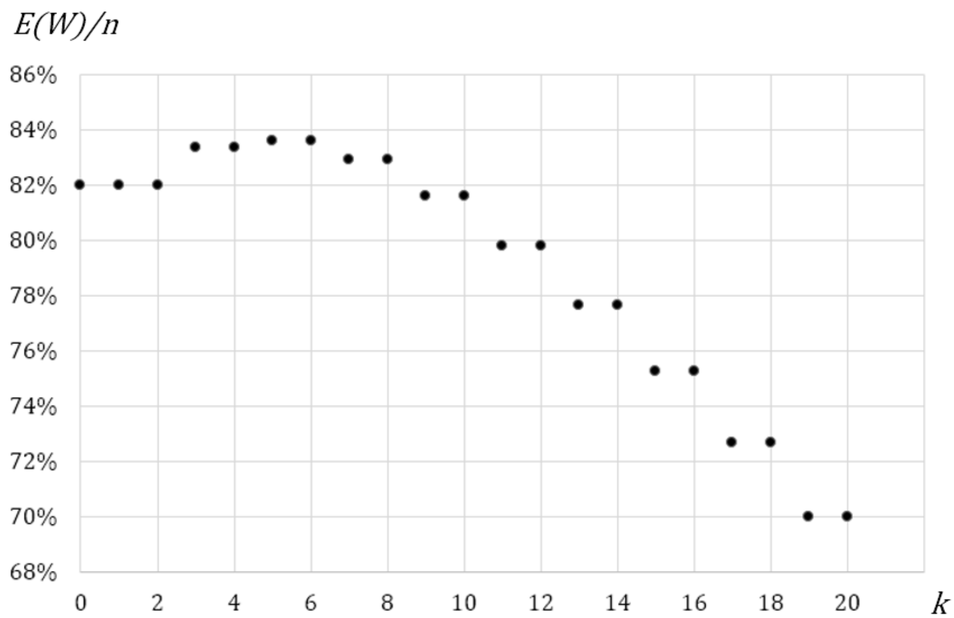



**Fig. 2.** Effect of Information Delay on Expected % of Right Players

The function $E(W)/n$ is discontinuous because its input, $k$, is a whole number. Even $k$ and its odd, $k-1$ partner form a signal pair that produce approximately equal per-person, expected welfare. From left to right, we see that $k = 0$, 1, or 2 has the same effect because the first two players already follow their own signal. Welfare increases with delayed $k > 2$ and peaks at optimal $k^* = 6$. Beyond this, welfare decreases because the marginal loss of initial players exceeds the marginal gain of remaining players; essentially, we are delaying information 'too much'. Next, the threshold $k_T = 8$ is the most we can delay information before no delay at all becomes better. Welfare then continues decreasing until it reaches a global minimum at $k = n$. This value is 70% because without information externalities, every player follows his own signal; in this example, the signal points to the 'right' (utility 1 instead of 0) path with $p = 0.7$ chance. I extend this graph to a wider range of signal accuracies to produce:

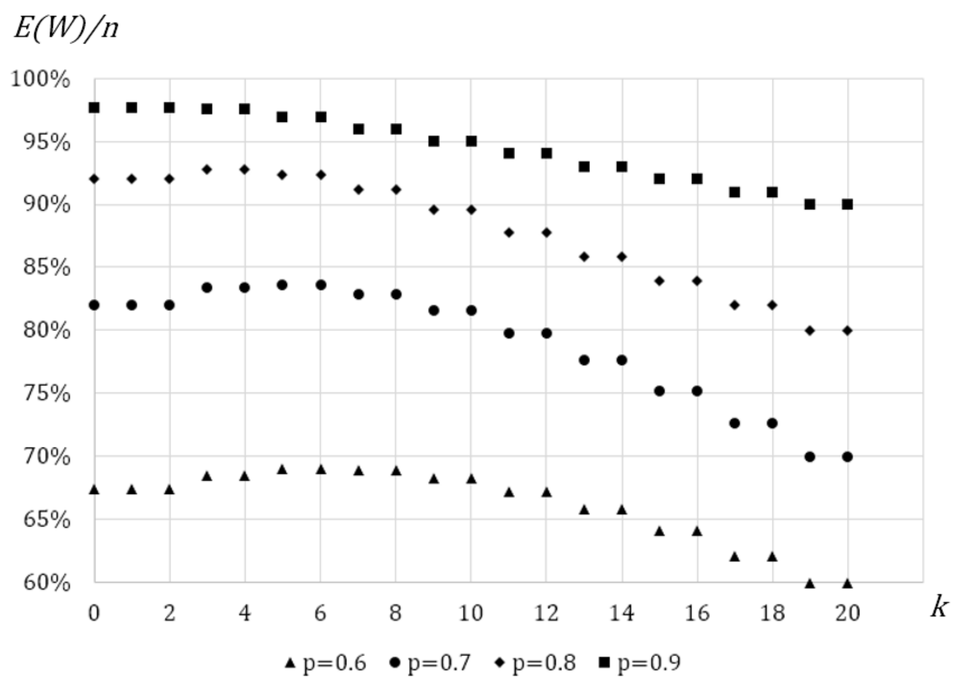



**Fig. 3.** Effect of Information Delay on Expected % of Right Players

Signal accuracy $p$ improves expected welfare $E(W)$, though at a concave rate. Stronger $p$ reduces optimal delay $k^*$ because a few samples from clear signals are enough to ascertain the right choice; a hazy signal instead demands more sampling. The respective $k^*$ are 6, 6, 4, and 2. Extending the data to varying group size $n$ produces:

| $p =$ | 0.55 | 0.60 | 0.65 | 0.70 | 0.75 | 0.80 | 0.85 | 0.90 | 0.95 |
|---|---|---|---|---|---|---|---|---|---|
| $n =$ | | | | | | | | | |
| 10 | 4 | 4 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 20 | 6 | 6 | 6 | 6 | 4 | 4 | 4 | 2 | 2 |
| 30 | 10 | 10 | 8 | 8 | 6 | 6 | 4 | 4 | 2 |
| 40 | 14 | 12 | 12 | 10 | 8 | 6 | 6 | 4 | 2 |
| 50 | 18 | 16 | 14 | 12 | 10 | 8 | 6 | 4 | 4 |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |
| 100 | 34 | 28 | 22 | 16 | 12 | 10 | 8 | 6 | 4 |

**Fig 4.** Optimal Delay By Signal Accuracy & Group Size

Optimal delay $k^*$ naturally scales with group size $n$, though at varying rates. This rate is convex with weak signals, approximately linear with medium signals, and concave with strong signals. Compare, for example, scaling from $n = 10$ to $n = 100$ with $p = 0.55$ versus $p = 0.95$. The weak signal requires 30 more samples while the strong one only 2 more! Then, the expected proportion of players who choose rightly are:

| $p =$ | 0.55 | 0.60 | 0.65 | 0.70 | 0.75 | 0.80 | 0.85 | 0.90 | 0.95 |
|---|---|---|---|---|---|---|---|---|---|
| $n =$ | | | | | | | | | |
| 10 | 58% | 66% | 73% | 80% | 85% | 90% | 94% | 97% | 99% |
| 20 | 60% | 69% | 77% | 84% | 89% | 93% | 96% | 98% | 99% |
| 30 | 61% | 71% | 80% | 86% | 91% | 94% | 97% | 98% | 99% |
| 40 | 62% | 73% | 81% | 88% | 92% | 95% | 97% | 99% | 99% |
| 50 | 63% | 74% | 83% | 89% | 93% | 96% | 98% | 99% | 100% |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |
| 100 | 67% | 80% | 88% | 92% | 95% | 97% | 98% | 99% | 100% |

**Fig 5.** Expected % Right At $k^*$ By Signal Accuracy & Group Size

Large groups perform better than small ones, so a planner may consider group-merging as a sub-mechanism to improve welfare. Notice that two groups of $n = 10$ do not perform as well as one group of $n = 20$, nor do two groups of $n = 20$ as well as one of $n = 40$.

The improvement in welfare is measured by the difference between values in Fig. 1 and Fig. 5. Relative to an instant-information ($k = 0$) model, the delayed-information ($k^*$) mechanism outperforms by:

| $p =$ | 0.55 | 0.60 | 0.65 | 0.70 | 0.75 | 0.80 | 0.85 | 0.90 | 0.95 |
|---|---|---|---|---|---|---|---|---|---|
| $n =$ | | | | | | | | | |
| 10 | 0% | 0% | 0% | 0% | 0% | 0% | 0% | 0% | 0% |
| 20 | 1% | 2% | 2% | 2% | 1% | 1% | 1% | 0% | 0% |
| 30 | 2% | 3% | 4% | 3% | 3% | 1% | 1% | 0% | 0% |
| 40 | 3% | 5% | 5% | 5% | 3% | 2% | 1% | 1% | 0% |
| 50 | 3% | 5% | 6% | 6% | 4% | 3% | 2% | 1% | 0% |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |
| 100 | 7% | 11% | 11% | 8% | 5% | 3% | 1% | 0% | 0% |

**Fig 6.** $k^*$ Outperforms $k = 0$, By Signal Accuracy & Group Size

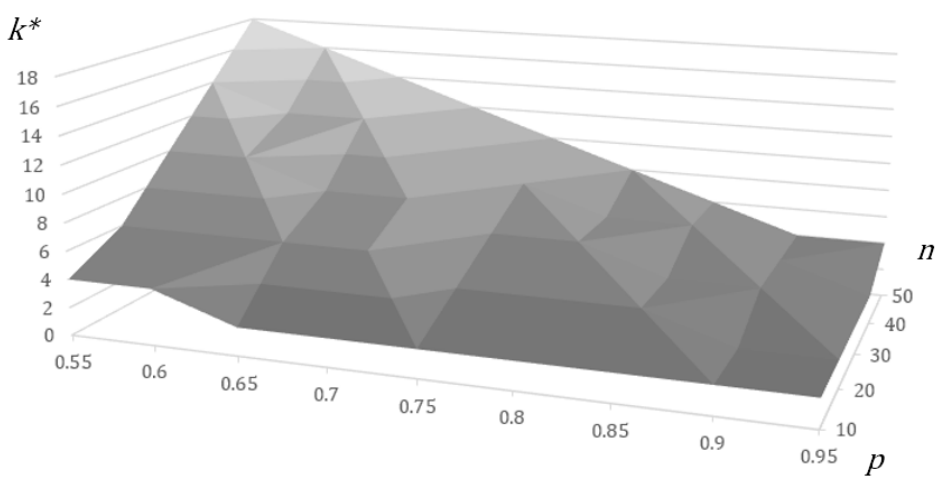
If players are too few or the accuracy is too high, then delayed information has minimal effects. A noticeable improvement begins around $n \approx 15$ players, becoming stronger with increasing $n$ and strongest around accuracy $p = 0.65 \approx 2/3$. While percentages may seem small at first, the expected improvement is 11 more players choosing rightly at $n = 100$. At the scale of Internet connectivity, an improvement of 11% or more translates to thousands or even millions more humans choosing rightly. This is especially important when the great masses forward, share, and retweet signals on public matters like elections, vaccines, climate change, or in rare cases even Earth's geometry.[7]

[7]Science has demonstrated that vaccines work, climate change is real, and Earth is spherical, the proofs of which are beyond the scope of this paper.

### 1.3.3 Approximation Function

I now leave the reader with an approximation function to estimate $k^*(n,p)$.[8] While Fig. 4 conveys much information, readers may wish to estimate intermediate values or extrapolate beyond the data range. Economics already has approximations such as "nominal interest equals real interest plus inflation" or "doubling time equals 70 (or 72) divided by interest rate." These oversimplify the mathematically-precise $i = (1+r)(1+\pi) - 1$ and $t = ln(2)/r \approx 0.69/r$, but are convenient for single-digit percentages.

In this light, I construct a portable rule-of-thumb that maintains a high degree of accuracy. First, I render Fig. 4 as a three-dimensional mesh to observe whether $k^*(n,p)$ is linear, quadratic, or logarithmic in relation to its inputs, $n$ and $p$. It appears linear in both arguments and, because the gradient changes, dependent also on the cross term $n \cdot p$.




[8]Recall that for even $k > 2$, $E(W|k-1) = E(W|k) + \epsilon$, meaning an even $k^*$ and its odd, $k^* - 1$ partner are both considered optimal.

**Fig. 7.** Effect of Information Delay on Expected % of Right Players

Next, I perform a multivariable, linear regression with to estimate:

$$k_i^*(n,p) = \beta_0 + \beta_1(n_i) + \beta_2(p_i) + \beta_3(n_i \cdot p_i) + \epsilon_i \tag{1.19}$$

Because most humans are not numberphiles and cannot easily calculate products mentally, I also perform a regression that omits the cross term. This of course biases the coefficients but exchanges precision for accessibility:

$$k_i^*(n,p) = \beta_0 + \beta_1(n_i) + \beta_2(p_i) + \epsilon_i \tag{1.20}$$

| **Variable** | **Accurate** | **Accessible** |
|---|---|---|
| Constant | -2.794* | 15.956*** |
| | (1.39) | (1.89) |
| Group Size | 0.818*** | 0.193*** |
| | (0.04) | (0.02) |
| Signal Accuracy | 4.467** | -20.533*** |
| | (1.72) | (2.45) |
| Cross Term | -0.833*** | |
| | (0.05) | |
| Observations | 45 | 45 |
| R-squared | 0.982 | 0.847 |

$^{*}p < 0.1$, $^{**}p < 0.05$, $^{***}p < 0.01$

**Fig. 8.** Regression Results for Dependent Variable: Optimal Delay

The accurate model achieves an exceptional correlation at $R^2 = 0.982$ for the range of small ($n = 10$), medium ($n = 30$), and large ($n = 50$) group sizes. The accessible model, though misspecified, still achieves a strong correlation of $R^2 = 0.847$. I then test their projections on an extreme ($n = 100$) group size:

| $p =$ | 0.55 | 0.60 | 0.65 | 0.70 | 0.75 | 0.80 | 0.85 | 0.90 | 0.95 |
|---|---|---|---|---|---|---|---|---|---|
| $n =$ | | | | | | | | | |
| (true) 100 | 34 | 28 | 22 | 16 | 12 | 10 | 8 | 6 | 4 |
| (accurate) 100 | 36 | 32 | 28 | 24 | 20 | 16 | 12 | 8 | 4 |
| (accessible) 100 | 24 | 23 | 22 | 21 | 20 | 19 | 18 | 17 | 16 |

**Fig. 9.** Model Projections At Extreme ($n = 100$) Group Sizes

The accurate model matches the true numbers adequately even at ranges well beyond its data set, overestimating only by single-digit percentages. The accessible model, of course, lacks the cross term $n \cdot p$ and cannot adapt to extreme values of $n$ and $p$. This does not mean the accessible model is less useful. Like the nominal interest and doubling time approximations, the accessible model is convenient for small, common inputs. In fact, we can simplify it further as:

$$k^*(n,p) \approx 15.96 + 0.19 \cdot n - 20.53 \cdot p \tag{1.21}$$

$$\approx \frac{80 + n - P}{5} \tag{1.22}$$

where $P$ is signal accuracy without the % (i.e. $P = 100 \cdot p$), to match interest rate $r$ in both the nominal interest and doubling time approximations. One last time, let us use the example $(n, p) = (20, 0.7)$ to test our rule-of-thumb for $k^*$:

$$\begin{aligned} k^*(n, P) &\approx \frac{80 + n - P}{5} \\ &= \frac{80 + 20 - 70}{5} \\ &= \frac{30}{5} \\ &= 6 \end{aligned} \tag{1.23}$$

This is indeed the true answer.

## 1.4 Model Extensions

### 1.4.1 Early Disclosure

So far, we understand that the delayed-information mechanism improves expected welfare by aggregating signals. This reduces variance and optimizes the risk of a wrong cascade. For example, $k^*(40, 0.7) = 10$ means that given $n = 40$ players and signal accuracy $p = 0.7$, the initial 10 players should see only, and thus follow, their own signal. The remaining 30 players then see all previous actions. This resulted in an expected 35 players choosing rightly compared to only 33 via instant information (i.e. no delay $k = 0$).[9]

What if, while running the above algorithm, we observe the first $m = 3$ signals to all match? How confident are we that these three signals converge on the good product? Should we stop collecting samples and disclose this data

[9] In this section, I round expected welfare (players) to the nearest integer for ease of visualization.

early, prior to $k^* = 10$?

We know the signal indicates the good product with $p$ chance and the bad product with $1 - p$.[10] The conditional probabilities the product is good or bad given a consensus of $m$ signals with accuracy $p$ are:

$$P(G|m, p) = p^m/(p^m + (1 - p)^m) \tag{1.24}$$

$$P(B|m, p) = (1 - p)^m/(p^m + (1 - p)^m) = 1 - P(G|m, p) \tag{1.25}$$

$P(G|3, 0.7) \approx 93\%$ and $P(B|3, 0.7) \approx 7\%$. If we disclose this data early, the remaining 30 players will cascade. All players (not each) choose rightly with 93% chance and wrongly with 7%, so the expected welfare is $E(W) = 0.93 \cdot 40 + 0.07 \cdot 0 = 37.20 \approx 37$. This is greater than the expected 35 from $k^* = 10$, so our instinct may be to disclose early. However, for completeness we must also check the probabilities of a cascade-same $P_{CS}$, reset $P_{RS}$, or cascade-opposite $P_{CO}$ at $k^* = 10$ given that the first three signals matched. These probabilities necessarily differ from the case where we have not observed any signals.

It takes two or more counts of one signal over the other to cascade. With 10 samples, a cascade triggers with six or more matching signals. Given that three already matched, it takes three or more matches (of seven remaining

[10] Earlier, for sake of generality, we assigned $A$ to be the good product. In practice, however, we often do not know which of two choices is '$A$'. We know only that the signal is more likely ($1/2 < p < 1$) to select the good product.

samples) to cascade in the *same* direction:

$$P_{CS}(R \geq 3) = \sum_{r=3}^{7} \binom{7}{r} 0.7^r \cdot 0.3^{7-r} \approx 97\% \tag{1.26}$$

A reset occurs if exactly two signals (of seven remaining) match the existing three:

$$P_{RS}(R = 2) = \binom{7}{2} 0.7^2 \cdot 0.3^5 \approx 3\% \tag{1.27}$$

Lastly, it takes one or zero match (of seven) to cascade in the *opposite* direction:

$$P_{CO}(R \leq 1) = \sum_{r=0}^{1} \binom{7}{r} 0.7^r \cdot 0.3^{7-r} \approx 0\% \tag{1.28}$$

Suppose that upon observing the first $m = 3$ signals matching, we decide to wait until $k^* = 10$ anyway. Then, players 4 through 10 have no information externality and follow their own, random signal. An expected five of those seven will choose rightly ($0.7 \cdot 7 = 4.90 \approx 5$). With 97% chance, the remaining 30 players will cascade-same and face a $P(G|3, 0.7) \approx 93\%$ chance to choose rightly ($0.93 \cdot 30 = 27.90 \approx 28$). With 3% chance, a reset occurs where $E(W|30, 0.7) = 24.85 \approx 25$ (via Proposition 1). So, the overall expected welfare is $0.93 \cdot 3 + 4.90 + 0.97 \cdot 27.90 + 0.03 \cdot 24.85 = 35.50 \approx 36$ players choosing rightly, which is less than the 37 from early disclosure.

Our synthesized algorithm for $(n, p) = (40, 0.7)$ is thus: if the first $m^* = 3$ signals match, disclose the three matching actions immediately; otherwise, wait until $k^* = 10$ to disclose the aggregated ten actions. The first three signals will match $0.7^3 + 0.3^3 = 37\%$ of the time, and expected welfare will be $37.20 \approx 37$. The remaining 63% of the time, expected welfare will be $35.20 \approx 35$.

The expected welfare of the synthesized algorithm, which includes both early disclosure $m^*$ and delayed information $k^*$, is $E(W) = 37\% \cdot 37.20 + 63\% \cdot 35.20 = 35.94 \approx 36$. This is a slight improvement over the delayed-information mechanism alone. It is infeasible to compute optimal $m^*$ for some 50 combinations of $n$ and $p$. Nonetheless, the walkthrough for $(n, p) = (40, 0.7)$ illustrates how a welfare-improving mechanism can be synthesized with—and further improved by—a sub-mechanism.

### 1.4.2 Strict Margin

By default, cascades trigger when one signal outnumbers the other by two. Another idea is to withhold information until one signal 'wins' by a stricter margin, say three. Upon disclosure, all remaining players cascade; that is, there is no post-disclosure reset.

**Conjecture 1.** *Given group size $n > 3$, signal accuracy $p > 1/2$, and win margin of three, the expected welfare is:*

$$E(W|n,p) = \sum_{r=0}^{n} r \cdot 3^{min\{r,n-r\}} \cdot p^{min\{r,n-r+3\}} \cdot (1-p)^{min\{r+3,n-r\}} \cdot (2/3)^{I_{n=2r}} \tag{1.29}$$

$$\textit{where } I_{n=2r} = \begin{cases} 1 & n = 2r \\ 0 & n \neq 2r \end{cases}$$

*Observation of Conjecture 1.* Following the analysis in Proposition 1:

$$E(W|n=4) = \sum_{r=0}^{4} r \cdot P(R=r|n=4) = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} (1-p)^3 \\ 3 \cdot p(1-p)^3 \\ 6 \cdot p^2(1-p)^2 \\ 3 \cdot p^3(1-p) \\ p^3 \end{bmatrix}$$

$$E(W|n=5) = \sum_{r=0}^{5} r \cdot P(R=r|n=5) = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} (1-p)^3 \\ 3 \cdot p(1-p)^4 \\ 9 \cdot p^2(1-p)^3 \\ 9 \cdot p^3(1-p)^2 \\ 3 \cdot p^4(1-p) \\ p^3 \end{bmatrix}$$

$$E(W|n=6) = \sum_{r=0}^{6} r \cdot P(R=r|n=6) =$$

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} (1-p)^3 \\ 3 \cdot p(1-p)^4 \\ 9 \cdot p^2(1-p)^4 \\ 18 \cdot p^3(1-p)^3 \\ 9 \cdot p^4(1-p)^2 \\ 3 \cdot p^4(1-p) \\ p^3 \end{bmatrix}$$

$$E(W|n=7) = \sum_{r=0}^{7} r \cdot P(R=r|n=7) =$$

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{bmatrix} \cdot \begin{bmatrix} (1-p)^3 \\ 3 \cdot p(1-p)^4 \\ 9 \cdot p^2(1-p)^5 \\ 27 \cdot p^3(1-p)^4 \\ 27 \cdot p^4(1-p)^3 \\ 9 \cdot p^5(1-p)^2 \\ 3 \cdot p^4(1-p) \\ p^3 \end{bmatrix}$$

...

$$E(W|n,p) = \sum_{r=0}^{n} r \cdot 3^{min\{r,n-r\}} \cdot p^{min\{r,n-r+3\}} \cdot (1-p)^{min\{r+3,n-r\}} \cdot (2/3)^{I_{n=2r}} \tag{1.30}$$

□

Inputting values for $n$ and $p$ produces the table:

| $p =$ | 0.55 | 0.60 | 0.65 | 0.70 | 0.75 | 0.80 | 0.85 | 0.90 | 0.95 |
|---|---|---|---|---|---|---|---|---|---|
| $n =$ | | | | | | | | | |
| 10 | 58% | 66% | 73% | 80% | 85% | 90% | 93% | 96% | 98% |
| 20 | 61% | 70% | 79% | 86% | 90% | 94% | 96% | 98% | 99% |
| 30 | 62% | 73% | 81% | 88% | 92% | 95% | 97% | 99% | 99% |
| 40 | 63% | 74% | 83% | 89% | 93% | 96% | 98% | 99% | 100% |
| 50 | 63% | 74% | 83% | 90% | 94% | 97% | 98% | 99% | 100% |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |
| 100 | 64% | 76% | 85% | 91% | 95% | 98% | 99% | 99% | 100% |

**Fig 10.** Expected % Right By Signal Accuracy & Group Size

The strict-margin mechanism performs comparably to the optimally-delayed information mechanism even up to large ($n = 50$) group sizes. However, it underperforms for extreme ($n = 100$) group sizes with weak signal accuracy ($0.50 < p < 0.75$); negative differences are parenthesized:

| $p =$ | 0.55 | 0.60 | 0.65 | 0.70 | 0.75 | 0.80 | 0.85 | 0.90 | 0.95 |
|---|---|---|---|---|---|---|---|---|---|
| $n =$ | | | | | | | | | |
| 10 | 0% | 0% | 0% | 0% | 0% | 0% | (1%) | (1%) | (1%) |
| 20 | 1% | 1% | 2% | 2% | 1% | 1% | 0% | 0% | 0% |
| 30 | 1% | 2% | 1% | 2% | 1% | 1% | 0% | 1% | 0% |
| 40 | 1% | 1% | 2% | 1% | 1% | 1% | 1% | 0% | 1% |
| 50 | 0% | 0% | 0% | 1% | 1% | 1% | 0% | 0% | 1% |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |
| 100 | (3%) | (4%) | (3%) | (1%) | 0% | 1% | 1% | 0% | 1% |

**Fig 11.** Strict-Margin Performance Relative to $k^*$

The intuition for this difference is that optimal delay $k^*(n, p)$, being a function of $n$, scales with group size. The strict-margin mechanism would have to increase its win margin as $n \to \infty$. An optimal win margin $m^*(n, p)$ as a function of $n$ and $p$ is beyond the scope of this paper. Still, the strict-margin mechanism shows that path-dependent disclosure exists as a comparable alternative to fixed-time disclosure.

### 1.4.3 Type I/II Errors

Suppose it is known that among a population, 1/2 of products are good and 1/2 are bad. A good product appears good with probability $p_G > 1/2$, and a bad product appears good with probability $p_B < 1/2$. As before, we can assume these inequalities without loss of generality due to complementarity; in essence, it means the signal is useful. A product that turns out to be good benefits $b > 0$, while a product that turns out to be bad costs $c \geq 0$. Players privately receive a good or bad signal and publicly choose, sequentially, to buy or not buy the product. Not buying gives utility 0.

Each player receives a good signal with independent probability:

$$p = (p_G + p_B)/2 \tag{1.31}$$

Let $\gamma$ denote the number of good signals observed among $k$ signals. The conditional probability the product is good given $\gamma = 1, k = 1)$ is:

$$p(G|1,1) = \frac{p_G}{p_G + p_B} \tag{1.32}$$

The conditional probability the product is good given $\gamma = 0, k = 1$ is:

$$p(G|0,1) = \frac{1 - p_G}{2 - p_G - p_B} \tag{1.33}$$

By complementarity, $p(B|\gamma, k) = 1 - p(G|\gamma, k)$. A rational player who sees $\gamma = 1, k = 1$ buys if:

$$\begin{aligned} p(G|1,1) \cdot b - (1 - p(G|1,1)) \cdot c > 0 \\ \implies \frac{p_G}{p_B} > \frac{c}{b} \end{aligned} \tag{1.34}$$

A rational player who sees $\gamma = 0, k = 1$ buys if:

$$\begin{aligned} p(G|0,1) \cdot b - (1 - p(G|0,1)) \cdot c > 0 \\ \implies \frac{1 - p_G}{1 - p_B} > \frac{c}{b} \end{aligned} \tag{1.35}$$

Because players are identical and because $p_G$, $p_B$, $c$, $b$ are all constants, the reaction of all players who see only one signal is identical and predetermined. Thus, there are three cases. First, if players would always buy regardless, then the expected welfare (prior to any observations) is simply $E(W) = n \cdot (b - c)/2$. Second, if they would always not buy, then welfare is $W = 0$. These two cases exist for a variety of parameter combinations, but perhaps it is easiest to imagine the consequences of buying a bad product as minor (low $c$) or major (high $c$).

In both (pooling) cases, player one's action conveys no information about his private signal. Even if player two sees player one's action, he really sees only one signal, his own. Because player two faces the same situation as player one, he also performs the same action. The same is true for players three, four, and onward.

Third, players who see only one bad signal do not buy and receive utility 0. Players who see only one good signal buy and receive $E(u_B) = p(G|1,1) \cdot b - (1 - p(G|1,1)) \cdot c$. This is the expected utility of an unknown product, given one good signal. In this (separating) case, a player's public action conveys information about his private signal.

What of the second or third player who sees multiple signals? They would use a more complex conditional probability. A non-cascading player who sees multiple signals infers the probability the product is good given his sample signals. Suppose player two sees that player one buys, which implies he received a good signal.[11] Then, player two himself receives a bad signal. What should

[11] As a reminder, we are in the third (separating) case where actions imply signals; in the other two (pooling) cases, players cannot infer any signals.

he do?

In the original model, players follow their own signal in case of ties. In the Bayesian model, however, the best action depends on $p_G$, $p_B$, $b$, and $c$. Player two sees $\gamma = 1, k = 2$ and buys if:

$$\begin{aligned}
p(G|1,2) \cdot b - [1 - p(G|1,2)] \cdot c &> 0 \\
\frac{\binom{2}{1}}{\binom{2}{1}} \cdot \frac{p_G \cdot (1-p_G) \cdot b - p_B \cdot (1-p_B) \cdot c}{p_G \cdot (1-p_G) + p_B \cdot (1-p_B)} &> 0 \\
\implies \frac{p_G \cdot (1-p_G)}{p_B \cdot (1-p_B)} &> \frac{c}{b} \qquad (1.36)
\end{aligned}$$

It is easy to see that sufficiently large $b$ makes a player buy while sufficiently large $c$ makes him not buy. It is harder to see the effects of $p_G$ and $p_B$, so let us fix $b = 2$, $c = 1$ and consider the following examples:

**Example 1.** $(p_G, p_B) = (0.7, 0.3)$. The left side is 1, which is greater than 1/2 on the right side, so player two buys. Let this be a basis for comparison.

**Example 2.** $(p_G, p_B) = (0.9, 0.3)$. $0.09/0.27 = 1/3 < 1/2$, so player two does not buy. The intuition is that if good products usually appear good, any bad signal is a major red flag. Inversely, if $(p_G, p_B) = (0.7, 0.1)$, then $0.27/0.09 = 3 > 1/2$, so he buys.

**Example 3.** $(p_G, p_B) = (0.6, 0.4)$. $1 > 1/2$, so player two buys. This is true also for $(p_G, p_B) = (0.9, 0.1)$. In fact, whenever $p_G + p_B = 1$, the left side is 1, so a player buys if and only if $b > c$. The intuition is that if $p_G, p_B$ are symmetric from 1/2, it does not matter whether they are weak or strong signals. A player simply receives an *accurate* signal $1/2 < p < 1$ whether the product is good or bad.

How about players three, four, and onward? In general, a player who sees $\gamma$ good out of $k$ signals buys if:

$$
\begin{aligned}
p(G|\gamma,k)\cdot b-(1-p(G|\gamma,k)\cdot c &> 0\\
\frac{\binom{k}{\gamma}}{\binom{k}{\gamma}}\cdot\frac{p_G^{\gamma}\cdot(1-p_G)^{k-\gamma}\cdot b-p_B^{\gamma}\cdot(1-p_B)^{k-\gamma}\cdot c}{p_G^{\gamma}\cdot(1-p_G)^{k-\gamma}+p_B^{\gamma}\cdot(1-p_B)^{k-\gamma}} &> 0\\
\implies \frac{p_G^{\gamma}\cdot(1-p_G)^{k-\gamma}}{p_B^{\gamma}\cdot(1-p_B)^{k-\gamma}} &> \frac{c}{b}
\end{aligned}
\tag{1.37}
$$

Given $k$ initial players who see their own signal but not the signal or even action of previous players, if:[12]

A)

$$
min\left\{\frac{p_G^{\gamma+1}\cdot(1-p_G)^{k-\gamma}}{p_B^{\gamma+1}\cdot(1-p_B)^{k-\gamma}},\frac{p_G^{\gamma}\cdot(1-p_G)^{k-\gamma+1}}{p_B^{\gamma}\cdot(1-p_B)^{k-\gamma+1}}\right\} > \frac{c}{b} \tag{1.38}
$$

then a buy cascade occurs where every player from $k+1$ onward buys. This expression means the $k+1^{th}$ player buys unconditionally, which adds 1 to the existing count of $\gamma$ good or $k-\gamma$ bad signals.

B)

$$
max\left\{\frac{p_G^{\gamma+1}\cdot(1-p_G)^{k-\gamma}}{p_B^{\gamma+1}\cdot(1-p_B)^{k-\gamma}},\frac{p_G^{\gamma}\cdot(1-p_G)^{k-\gamma+1}}{p_B^{\gamma}\cdot(1-p_B)^{k-\gamma+1}}\right\} \leq \frac{c}{b} \tag{1.39}
$$

then a not-buy cascade occurs where every player from $k+1$ onward does not buy.

[12] The expression $min\{a,b\} > c$ simply means $a > c$ and also $b > c$.

C)

$$\frac{p_G^{\gamma}\cdot(1-p_G)^{k-\gamma+1}}{p_B^{\gamma}\cdot(1-p_B)^{k-\gamma+1}} \leq \frac{c}{b} < \frac{p_G^{\gamma+1}\cdot(1-p_G)^{k-\gamma}}{p_B^{\gamma+1}\cdot(1-p_B)^{k-\gamma}} \tag{1.40}$$

then a 'reset' occurs where player $k+1$ follows his own signal. That is, he buys on a good signal and does not buy on a bad signal.

I now construct a welfare formula for the conditional-model, separating case.[13] Recall that a social planner understands the stochastic process but chooses an optimal $k^*$ without having observed any signals. A reasonable objective function thus takes the social planner's perspective prior to any realized signals. From this perspective, every buyer gains expected utility $E(u_B) = (b-c)/2$.

Among the initial $k$ players, $p = (p_G + p_B)/2$ fraction receive a good signal and buy. The $1-p$ fraction who receive a bad signal do not buy and receive utility 0. The initial $k$ players therefore have expected welfare:

$$\begin{aligned} E(W_{IP}|p_G, p_B, b, c, k) &= k \cdot p \cdot E(u_B) \\ &= k \cdot (p_G + p_B)/2 \cdot (b-c)/2 \end{aligned} \tag{1.41}$$

The remaining $n-k$ players randomly face a cascade-buy, cascade-not-buy, or reset. These events are exhaustive and mutually-exclusive; that is, exactly one must occur.[14] If a cascade-buy occurs, remaining players gain expected utility $E(u_B) = (b-c)/2$. If a cascade-not-buy occurs, remaining players

[13]That is, where good signals suggest buy and bad signals suggest not-buy.
[14]Technically, at *least* one and at *most* one must occur, but this is equivalent.

receive utility 0. If a reset occurs, the $k+1^{th}$ player follows his own signal, which says 'good' with probability $p$, and gains expected utility $E(u_{k+1}) = p\cdot(b-c)/2$. Then, player $k+2$ faces $k+1$ signals that may cascade-buy, cascade-not-buy, or reset, and so forth recursively. The remaining $n-k$ players therefore have expected welfare:

$$\begin{aligned} E(W_{RP}|n, p_G, p_B, b, c, k) = P_{CB}(k) \cdot (n-k) \cdot E(u_B) \\ +P_{RS}(k) \cdot [p \cdot (b-c)/2 + E(W_{RP}|n, p_G, p_B, b, c, k+1)] \end{aligned} \tag{1.42}$$

where the probabilities of cascade-buy and reset are functions of: $k$ observed signals, which increase by one each reset; $\gamma$ good signals, which is stochastic; and parameters $n, p_G, p_B, b, c$, which are fixed:

$$P_{CB}(k) = P\left( min\left\{ \frac{p_G^{\gamma+1} \cdot (1-p_G)^{k-\gamma}}{p_B^{\gamma+1} \cdot (1-p_B)^{k-\gamma}}, \frac{p_G^{\gamma} \cdot (1-p_G)^{k-\gamma+1}}{p_B^{\gamma} \cdot (1-p_B)^{k-\gamma+1}} \right\} > \frac{c}{b} \right) \tag{1.43}$$

$$P_{RS}(k) = P\left( \frac{p_G^{\gamma} \cdot (1-p_G)^{k-\gamma+1}}{p_B^{\gamma} \cdot (1-p_B)^{k-\gamma+1}} \leq \frac{c}{b} < \frac{p_G^{\gamma+1} \cdot (1-p_G)^{k-\gamma}}{p_B^{\gamma+1} \cdot (1-p_B)^{k-\gamma}} \right) \tag{1.44}$$

Net welfare combines the welfare of initial and remaining players, $E(W) = E(W_{IP}) + E(W_{RP})$. An optimal delay $k^*$ is the argument $k$ that maximizes this $E(W)$.

Due to five-dimensional generalizations, I cannot compute tables for $k^*$ as before. However, the framework here demonstrates the robustness of the delayed-information mechanism, which can account for the probabilities a good or bad product appears good ($p_G$, $p_B$) and the benefit and cost of choosing rightly or wrongly ($b$, $c$). Because the 'product' may be a new medicine, the conditional model accounts for Type I and Type II errors.

## 1.5 Discussion

The irony of instant information is that its convenience also facilitates dangerous cascades. Humans routinely take information at face value rather than seek out additional, especially contrary, information.

In a related study on confirmation bias, experimenters told subjects the sequence 2, 4, 6 follows a hidden rule. Subjects could then write their own sequence, and the experimenters confirmed whether it follows the rule. This continued until the subject felt confident to guess the rule. The rule was simply that numbers strictly increase, yet the vast majority was convinced it was evens, multiples, or other patterns after little to no checking. In fact, only one-fifth deduced the rule correctly (Wason 1960).[15] A minimum delay of observations would likely improve performance.

The delayed information problem also bears some similarity to a multi-armed bandit problem or the secretary problem. The multi-armed bandit problem, whose name refers to an array of slot machines, is a dilemma between exploiting an asset known to be profitable versus exploring an asset whose profitability is unknown (Robbins 1952). Therein lies the similar question of how many signals a player should take before committing the remainder

[15] An interactive version of this game can be found online at: https://www.nytimes.com/interactive/2015/07/03/upshot/a-quick-puzzle-to-test-your-problem-solving.html

of actions to the expected-best product.

The secretary problem, sometimes called the marriage or best-choice problem, essentially features $n$ random numbers from the same, unknown distribution. A single player draws observations one at a time and must either permanently reject or accept the draw. Only a single acceptance can be made, which ends the game. The objective is to maximize the expected value of the accepted draw. For large $n$, the solution is to record the greatest number in the first $1/e \approx 37\%$ of observations, then accept the first draw that meets or exceeds this value (Bruss 1984). This is an optimal stopping rule similar to $k^*$ of my delayed-information cascade model.

One nice feature of the optimal delay $k^*(n, p)$ in the non-conditional model is independence from the utilities gained from choosing the objectively better or worse product. That is, we can always normalize the utilities gained to one and zero. This results from the fact that utility is not discounted, in terms of time or money, from waiting for another draw from the distribution. Regarding the rule-of-thumb, recall that if even $k^*$ is optimal, then its odd partner $k^* - 1$ is also optimal. As such, we can exchange 80 for 75 to shift $k^*$ downward by one. In fact, a Rule of 75 may be easier to remember because 75 coincides with the midpoint between $1/2 < p < 1$. An easy benchmark to remember is for moderately-accurate $p = 0.75$, the optimal stopping rule is about $1/5 = 20\%$ of group size.

To focus on the characteristics and benefits of a fixed-time disclosure mechanism, this paper only briefly discusses two variable-time sub-mechanisms that could serve as augments. In practice, the accessibility of variable-time disclosure is limited relative to fixed-time due to greater monitoring and computation costs. A forward path could be to explore the integration of fixed- and variable-time mechanisms for further optimization.

# Chapter 2

# Pretending Volunteers

## 2.1 Introduction

Digital media has blurred the reality between what a person promises and what they deliver. Meanwhile, developments in economic theory have emphasized the importance of image motivation (Bénabou and Tirole 2006; Andreoni and Bernheim 2009).[1] A positive image confers real benefits like influence and access to resources.[2] One overlooked aspect of image motivation, however, is that it incentivizes pretense. If we value what others think, then appearing to be helpful can be just as good as actually helping. In fact, while pretense has been studied in some contexts like quality uncertainty or entry deterrence, it has not received much attention in public economics.[3]

[1]In the dictator game, one player of two decides how to split a sum of money. A 'rational' player takes all, yet many split 50-50 not only out of fairness but also because they wish to be *perceived* as fair by the observers.

[2]Maslow's (1943) hierarchy of needs lists self-esteem and self-respect as major motivations after physiological needs.

[3]A person may disclose or mimic strength to deter conflict. In an I.Q. contest, players signaled over-confidence to deter entry and under-confidence provoke entry (Charness et al.

Pretense in public economics occurs everywhere from a society's base to its very peak. For example, Twitter users often bandwagon on trending hashtags of social movements. Facebook users can, without cost, display photo frames to support popular causes. For a minimal $1 on fundraising platforms, a person can boast contribution toward public goods. At best, bandwagoners free ride the merit generated by real activists and benefactors; at worst, their digital gesturing substitutes for and even discourages material aid.

Digital media provides both the audience for image-crafting and the anonymity for pretense-masking. Indeed, anyone in modern society has almost certainly witnessed a neighbor, coworker, executive, or politician extol a virtue but act contrarily. The year 2018 alone saw many examples of corporate and political media stunts that resulted in no real provision. At the highest levels, United Nations member states laud development goals in the media but take little to no real action. When these pretenses impact climate change or humanitarian crises, the consequences demand critical examination.

Motivated by these considerations, my paper explores the effects of image concerns in a fundamental economic context—public good provision—when pretense is possible. A classic game called the volunteer's dilemma traditionally features players deciding whether to help or bystand. I introduce a third action called pretend. Pretending does not contribute, but it costs less than helping and confers honor if the pretense is believed and shame otherwise.

A formal model can help us predict equilibrium levels of pretending and identify conditions conducive to pretense. Consequently, we might design

2013).

mechanisms that increase the provision of public goods. Thus I ask: How does the option to pretend affect the expected level of provision? How do the equilibrium proportions of helpers, bystanders, and pretenders vary with group size or the values ascribed to honor and shame? Given that pretense does occur, what countermeasures might influence it to increase provision?

In Section 2, I formalize pretense in the context of a volunteer's dilemma. In addition to help or bystand actions, players can also pretend in order to free ride an *honor* benefit if someone else provides the public good. Pretending is costly but cheaper than helping.[4] In the short run, players are naïve and confer honor equally to all claimants—both helpers and pretenders—if provision succeeds. If provision fails, all claimants are exposed as pretenders and suffer a *shame* cost. Helpers also experience a psychological *warm glow* benefit, which bystanders and pretenders do not experience.[5]

In Section 2.1 (Characterizing Equilibria), I establish key assumptions and fully characterize pure- and mixed-strategy equilibria. One pure-strategy equilibrium exists, where one player helps while all others pretend. Two possible mixed-strategy equilibria exist, one mixing on only help/bystand and one mixing on only help/pretend. Only one equilibrium exists at a time, and which depends on every parameter except group size. Group size still increases the probability of provision, converging to a constant much like in the original game. Importantly, I prove that the ability to pretend weakly decreases this

[4]Mimicry costs some effort to convince others. Further, most people are averse to lying because it exacts some cognitive or emotional toll. This is true even when lying is undetectable, absent strategic motives, or leads to improved monetary outcomes for everyone (Gneezy et al. 2013; Abeler et al. 2014; Erat and Gneezy 2011).

[5]Egoistic utility from giving, as opposed to *pure* altruism for the recipient's welfare. The colloquial term was first coined by Andreoni in 1989.

probability relative to when players cannot pretend.

In Section 2.2 (Three Cultural Archetypes), I identify the forces that enable pretense. Honor encourages helping but also pretending, while shame discourages pretending. Broadly, three cultural archetypes exist. Low-honor cultures develop a norm to bystand as the alternative to not helping. High-honor, low-shame cultures develop a norm to appear helpful, even if that means pretending. High-honor, high-shame cultures revert to bystanding as the alternative to not helping and do not pretend. This three-culture model helps explain why pretense is more prevalent in some environments than others.

In Section 2.3 (Honor as Subsidy, Shame as Tax), I analyze the effect of key variables, particularly honor and shame, on the group's welfare. Because welfare as aggregate utility can be trivially maximized via infinite honor and zero shame, I instead measure welfare as the probability of provision.[6] Honor always increases provision, while shame increases provision only when pretending exists. I generate a three-dimensional graph of provision as a function of honor and shame, $P^*(h, s)$, and show that provision is best maximized by first increasing honor, then by increasing shame.

In Section 2.4 (Long-Run Sophistication), I model a variant where players in the long run become aware that those who appear helpful may actually be pretenders. That is, honor becomes endogenous and discounted by the conditional probability a claimant is a helper. The major consequence is that the presence of pretenders discourages players from helping because their honor is diluted by fakes, much like a lemons market. Discouraged helpers in theory

[6]Indeed, players among some cabals are celebrated for empty gestures and face no repercussion when caught. We can only imagine the magnificent 'welfare' of such groups.

would be willing to pay a premium to distinguish their help as authentic, provided the premium is not too costly. Authentication improves the probability of provision in the long run.

In Section 2.5 (Asymmetry and Coexistence), I treat players as having independent and identically distributed (i.i.d.) private values of honor and warm glow.[7] These represent a player's utility from looking versus feeling good. In equilibrium, players select one of three actions based on their individual preferences. Generally, players who do not value looking good bystand; players who value feeling good help; and players who value looking good but not feeling good pretend. An equilibrium consisting of all three actions can thus be sustained this way. An illustrated partition helps explain why we often see helpers, bystanders, and pretenders coexisting in the real world.

In Section 3, I discuss my findings in the context of prominent literature and suggest subsequent avenues of exploration. To my best knowledge, my paper is the first to model pretense in public goods provision.

## 2.2 Volunteer's Trilemma

The volunteer's dilemma (Diekmann 1985) is an $n$-player game in which each player gains a benefit $b$ from the provision of a public good if at least one player volunteers to pay a cost $c$, where $0 < c < b$. Each player gains from the good's provision but prefers to let someone else pay for the good, a phenomenon known in psychology as *diffusion of responsibility* or the *bystander*

[7] One might argue that players should vary in the help cost, but warm glow serves the same purpose.

*effect* (Darley and Latane 1968). Examples include taking out the trash, helping an injured victim, or enforcing a social norm.[8] Various authors have since covered extensions like asymmetric costs (Diekmann 1993), incomplete information (Weesie 1994), or cost sharing (Weesie and Franzen 1998). The studies closest to my paper incorporate *warm glow* (Andreoni 1990; Bergstrom et al. 2015) and *prestige* (Harbaugh 1998; Andreoni and Petrie 2004) but do not model the pretense of helping.

A problem occurs, and each of $n$ players simultaneously decides to help ($H$), bystand ($B$), or pretend ($P$).

Helping costs $c$ and provides the public good. If at least one player helps, three benefits occur. First, all players enjoy a material benefit $b > 0$. Second, only helpers experience a warm glow $w > 0$. Third, claimants—both helpers and pretenders—gain honor $h > 0$. So long as the public good is provided, everyone is happy and honors claimants fully.[9] Thus, I decompose the helper's benefit into material, warm glow, and honor components.

Bystanding costs zero and contributes nothing. A bystander receives the material benefit $b$ only if someone else helped.

Pretending costs more than zero but less than helping, $0 < k < c$, and confers honor $h$ only if someone else helped.[10] If no one helped, the public good is not provided so all claimants must be pretenders. Everyone is unhappy, and

[8] Animals like penguins and marmots are also known to volunteer serving as a lookout for predators (Dawkins 1976).

[9] In the short run, players are naïve. In a later section, I compare the long run where players become sophisticated and discount honor by the conditional probability a claimant is a helper.

[10] Honor is non-rival in this setting. One player's honor gain or loss does not affect another's.

exposed pretenders suffer a shame cost $-s < 0$. A player $i$'s utility $u_i$ thus depends on their own action and on whether someone else helps. I summarize this below:

| | $u_i$ (if someone else helps) | $u_i$ (if no one else helps) |
|---|---|---|
| help ($H$) | $b+w+h-c$ | $b+w+h-c$ |
| bystand ($B$) | $b$ | $0$ |
| pretend ($P$) | $b+h-k$ | $-s-k$ |

where

| | | |
|---|---|---|
| $b$ = material benefit | $w$ = warm glow | $h$ = honor |
| $c$ = help cost | $k$ = pretend cost | $s$ = shame |

**Table 1.** Actions and Payoffs

**Assumption 1.** $0 < k < h < w + h < c < b$.

All parameters have positive value. The condition $c < b$ is from the original volunteer's dilemma. That $w + h < c$ means warm glow and honor alone do not incentivize a bystander to help; else, $H$ weakly dominates $B$. That $k < h$ means pretending is profitable if someone else helps; else, $B$ weakly dominates $P$. $H$ is the best response if no one else helps, so it cannot be weakly dominated. Together, Assumption 1 ensures that no action is dominated and thus every action is salient.

### 2.2.1 Characterizing Equilibria

**Proposition 1.** *Given Assumption 1, there exist $n$ pure-strategy Nash equilibria where exactly one player helps and all others pretend.*

*Proof of Proposition 1.* If no one helps, then $H$ is the best response. The helper gains $u_i|H = b+w+h-c > 0$. (Recall that $b > c$.) Given that someone else helps, $P$ is the best response. The pretender gains $u_i|P = b+h-k$, which is preferred to either $u_i|B = b$ or $u_i|H = b + w + h - c$. (Recall that $h > k$ and $w + h < c$.) □

Pretenders not only free ride the material benefit but also share in the honor. Whether this is socially wasteful is a matter of perspective, as the resources expended to pretend are compensated by the utility gained from shared honor. A materialist view would measure welfare solely by the expected level of provision.

**Proposition 2.** *Given Assumption 1, in any symmetric, mixed-strategy Nash equilibrium, each player helps with probability* $p^*_H > 0$ *and gains expected utility* $b + h + w - c > 0$.

*Proof of Proposition 2.* Suppose $p^*_H = 0$ such that no one helps and thus no good exists. Then, a bystander receives $u_i|B = 0$ while a pretender suffers $u_i|P = -s - k < 0$. If an action is excluded from a mixed strategy, then its payoff must be weakly less than any included action. However, a helper would gain $u_i|H = b + h + w - c > 0$, a contradiction. Therefore, $p^*_H > 0$. In equilibrium, the expected payoff from all actions are equal, so each player gains unconditional $E(u_i) = b + h + w - c > 0$. □

**Proposition 3.** *Let* $p^*_H$, $p^*_B$, *and* $p^*_P$ *be the respective equilibrium probabilities that a player helps, bystands, or pretends. Given Assumption 1, if:*

*A)*

$$\frac{c-h-w}{b} > \frac{c-k-w}{b+h+s}$$

*then there exists a unique, symmetric Nash equilibrium where* $p_H^* + p_B^* = 1$ *and* $p_P^* = 0$.

*B)*

$$\frac{c-h-w}{b} < \frac{c-k-w}{b+h+s}$$

*then there exists a unique, symmetric Nash equilibrium where* $p_H^* + p_P^* = 1$ *and* $p_B^* = 0$.

*C)*

$$\frac{c-h-w}{b} = \frac{c-k-w}{b+h+s}$$

*then the set of symmetric, mixed-strategy Nash equilibria consists of all* $(p_H^*, p_B^*, p_P^*) \geq 0$ *such that* $p_B^* + p_P^* = \left(\frac{c-h-w}{b}\right)^{1/(n-1)}$ *and* $p_H^* + p_B^* + p_P^* = 1$.

*Proof of Proposition 3.* A game with three actions has potentially up to four symmetric, mixed-strategy equilibria: mixing on only actions $B/P$ (i.e. $p_B^* + p_P^* = 1, p_H^* = 0$); on only $H/B$; on only $H/P$; or on all three $H/B/P$.[11] A helper always gains $b + h + w - c$. A bystander gains $b$ only if at least one other player helps, which occurs with probability $1 - (1 - p_H)^{n-1}$. A pretender gains $b + h - k$ if the public good is provided and $-s - k$ if it is not. In terms of expected utilities:

$$E(u_i|H) = b + h + w - c \tag{2.1}$$

$$E(u_i|B) = b[1 - (1 - p_H)^{n-1}] \tag{2.2}$$

[11]In general, a game with $k$ actions has up to $2^k - k - 1$ symmetric, (non-pure) mixed-strategy equilibria.

$$E(u_i|P) = (b+h)[1-(1-p_H)^{n-1}] - s(1-p_H)^{n-1} - k \tag{2.3}$$

Proposition 2 states that $p_H^* > 0$, so $B/P$ is not an equilibrium. Mixing on only $H/B$ is an equilibrium if there exists an equilibrium probability $p_B^*$ (and implicitly $p_H^* = 1 - p_B^*$) that makes a player indifferent between $H$ and $B$ but still weakly prefer either to $P$:

$$E(u_i|H) = E(u_i|B) \geq E(u_i|P) \tag{2.4}$$

$$b+h+w-c = b[1-(p_B)^{n-1}] \geq (b+h)[1-(p_B)^{n-1}] - s(p_B)^{n-1} - k \tag{2.5}$$

$$\implies p_B^* = \left(\frac{c-h-w}{b}\right)^{1/(n-1)} \geq \left(\frac{c-k-w}{b+h+s}\right)^{1/(n-1)} \tag{2.6}$$

If instead mixing on only $H/P$ were an equilibrium, then:

$$E(u_i|H) = E(u_i|P) \geq E(u_i|B) \tag{2.7}$$

$$b+h+w-c = (b+h)[1-(p_P)^{n-1}] - s(p_B)^{n-1} - k \geq b[1-(p_P)^{n-1}] \tag{2.8}$$

$$\implies p_P^* = \left(\frac{c-k-w}{b+h+s}\right)^{1/(n-1)} \geq \left(\frac{c-h-w}{b}\right)^{1/(n-1)} \tag{2.9}$$

By Assumption 1 ($0 < k < h < w+h < c < b$), the fractions have positive numerators and greater denominators. A fraction between zero and one (exclusive) raised to a positive power $1/(n-1)$ remains between zero and one, so $p_B^*, p_P^* \in (0,1)$. Mixing on $H/B/P$ is an equilibrium if a player is indifferent between all three actions. In this case, the probability at least one player helps, and thus provides the public good, is updated to $1-(p_B+p_P)^{n-1}$.

$$E(u_i|H) = E(u_i|B) = E(u_i|P) \tag{2.10}$$

$$b + h + w - c = b[1-(p_B+p_P)^{n-1}] = (b+h)[1-(p_B+p_P)^{n-1}] - s(p_B+p_P)^{n-1} - k \tag{2.11}$$

$$\implies p_B^* + p_P^* = \left(\frac{c-h-w}{b}\right)^{1/(n-1)} = \left(\frac{c-k-w}{b+h+s}\right)^{1/(n-1)} \tag{2.12}$$

$$\implies p_H^* = 1 - p_B^* - p_P^* \in (0,1) \tag{2.13}$$

The set of all $(p_H^*, p_B^*, p_P^*) \geq 0$ that satisfy these two conditions include the cases where either $p_B^* = 0$ or $p_P^* = 0$. However, the parenthesized expressions are equal only on a set of measure zero in parameter space. In other words, a three-way, symmetric, mixed strategy generally does not exist except in a knife-edge case. □

**Corollary 3.** *Group size n does not determine which mixed-strategy equilibrium exists.*

This is not to be confused with $n$ affecting equilibrium probabilities, which it does as seen below where $n$ appears in the exponent:

**Proposition 4.** *Given Assumption 1, for any mixed-strategy equilibrium, the individual's probability of helping is:*

$$p_H^* = 1 - \left(max\left\{\frac{c-h-w}{b}, \frac{c-k-w}{b+h+s}\right\}\right)^{1/(n-1)}$$

*and the group's probability of provision is:*

$$P^* = 1 - \left(max\left\{\frac{c-h-w}{b}, \frac{c-k-w}{b+h+s}\right\}\right)^{n/(n-1)}$$

*Proof of Proposition 4.* Refer to Inequalities 6 and 9. When the left element (fraction) in the maximum function is greater, only the $H/B$ equilibrium exists and $p_H^* = 1 - p_B^*$. When the right element is greater, only the $H/P$ equilibrium exists and $p_H^* = 1 - p_P^*$. When both elements are equal, the $H/B/P$ equilibrium exists (which includes only $H/B$ and only $H/P$) and $p_H^* = 1 - p_B^* - p_P^*$. The probability that at least one player helps is the complement of no one helping, $P^* = 1 - (1 - p_H^*)^n$. □

**Corollary 4.** *As* $n \longrightarrow \infty$, $p_H^* \longrightarrow 0$ *and* $P^* \longrightarrow 1 - max\left\{\dfrac{c-h-w}{b}, \dfrac{c-k-w}{b+h+s}\right\}$.

*Proof of Corollary 4.* The exponents containing $n$ convergence to 0 for $p_H^*$ and 1 for $P^*$ as $n \longrightarrow \infty$. This is the single intermediary step. □

How does this probability of provision compare to a world where pretending is not possible, as assumed in previous models? In those models, only the help/bystand equilibrium exists, so $\hat{P}^* = 1 - \left(\dfrac{c-h-w}{b}\right)^{n/(n-1)}$. In fact, if we disable also honor ($h = 0$) and warm glow ($w = 0$), we get the original probability of provision in Diekmann's volunteer's dilemma, $\hat{P}^* = 1 - \left(\dfrac{c}{b}\right)^{n/(n-1)}$. Certainly, $P^* \leq \hat{P}^*$ because the maximum function in $P^*$ selects from two elements as opposed to only the left one.

This leads to the first major result:

**Main Result 1.** *The ability to pretend in a volunteer's dilemma weakly decreases the probability that the public good is provided.*

### 2.2.2 Three Cultural Archetypes

We now understand that pretense, a hereto 'unconsidered', sometimes reduces the provision of public goods. Specifically, it occurs when the help/pretend equilibrium exists instead of the help/bystand equilibrium. What factors incentivize or disincentivize this equilibrium?

**Proposition 5.** *Given Assumption 1, the unique help/pretend equilibrium exists when—ceteris paribus—the help cost parameter $c$ is sufficiently small, when material benefit $b$ is sufficiently large, when warm glow $w$ is sufficiently large, or when pretend cost $k$ is sufficiently small.*

*Proof of Proposition 5.* Proposition 3 states that when $\frac{c-h-w}{b} < \frac{c-k-w}{b+h+s}$, only the $H/P$ equilibrium exists. Rearranging in terms of $c$, $b$, $k$, or $w$ shows that the $H/P$ equilibrium exists when $c < h + w + b(h-k)/(h+s)$, $b > (c-h-w)(h+s)/(h-k)$, $k < [h^2 - (c-h-w)(h+s)]/b$, or $w > c - h + (b \cdot k - h^2)/(h+s)$. To be clear, these are not four conditions but the same condition rearranged four ways. If any one condition is true, they are all true together. □

Intuitively, when helping costs little and benefits much, the public good is more likely to be provided. Knowing this, players are more willing to pretend in order to free ride both the material benefit and the shared honor. When players feel good about helping, warm glow serves equivalently as a reduction in help cost. Lastly and most naturally, pretending happens when it is easier to pretend.

Costs and benefits are long- and well-understood in economics whereas the new variables I introduce, honor and shame, traditionally belong to anthropology. In an effort to bridge the two fields, I first quantitatively analyze the effects of honor and shame on equilibrium selection then qualitatively interpret their roles in shaping a cultural norm.

**Proposition 6.** *Given Assumption 1, the unique help/pretend equilibrium exists when—ceteris paribus—the honor parameter $h$ is sufficiently large and also the shame parameter $s$ is sufficiently small. When $h$ is sufficiently small or $s$ is sufficiently large, the unique help/bystand equilibrium exists.*

*Proof of Proposition 6.* Proposition 3 states that when $\dfrac{c-h-w}{b} < \dfrac{c-k-w}{b+h+s}$, only the $H/P$ equilibrium exists. Rearranging in terms of honor $h$ yields:

$$f(h) = h^2 + (b + w - c + s) \cdot h + [s(w - c) - b \cdot k] > 0 \tag{2.14}$$

This is a convex parabola with vertical intercept at $f(0) = s(w-c)-b\cdot k < 0$. The intercept is negative because Assumption 1 implies parameters are positive and $w < c$. This means somewhere in the domain $h > 0$, there exists a unique threshold $h_T$ where $f(h_T) = 0$ crosses the horizontal axis. Specifically, this crossing point occurs at $h_T = (-B+\sqrt{B^2-4C})/2$ where $B = b+w-c+s$ and $C = s(w-c)-b\cdot k$. Values above this threshold $h > h_T$ determine a $H/P$ equilibrium while $h < h_T$ determines a $H/B$ equilibrium.

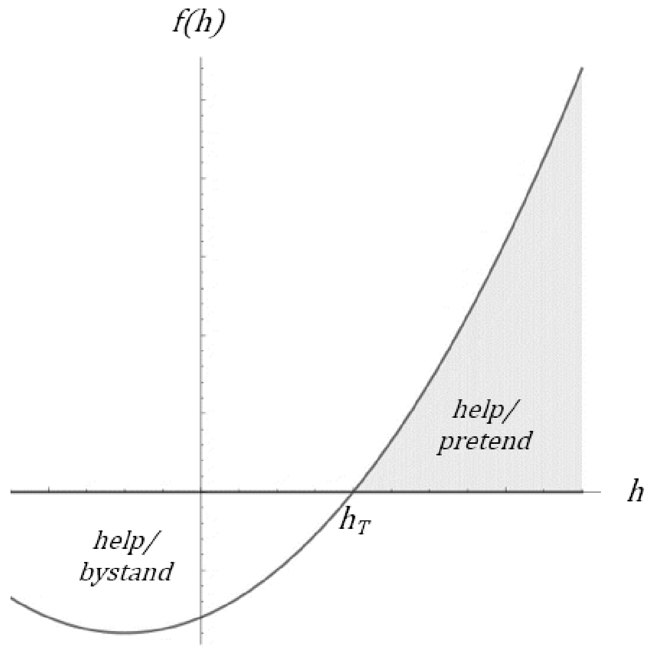



**Fig. 1.** Large Values of Honor Induce Pretense (always)

Similarly, rearranging in terms of shame $s$ shows that the $H/P$ equilibrium exists when:

$$f(s) = s + \left[h - \frac{b(h-k)}{c-h-w}\right] < 0 \tag{2.15}$$

This is a linear function of $s$ with a slope of 1 and an intercept of ambiguous sign. Assumption 1 implies only that the fraction term is positive because $h > k$ and $c > h + w$. This matters because if the intercept is zero or positive, then the unique $H/B$ equilibrium exists and $s$ has no effect. If the intercept is negative, then values below this threshold $s < s_T = b(h-k)/(c-h-w) - h$ determine a $H/P$ equilibrium while $s > s_T$ determines a $H/B$ equilibrium.

The intercept is negative when:

$$\frac{b(h-k)}{c-h-w} - h > 0 \implies h^2 - (c-w-b) \cdot h - b \cdot k > 0 \qquad (2.16)$$

This, too, is a convex parabola with negative vertical intercept. So, the function is positive when $h$ is sufficiently large.

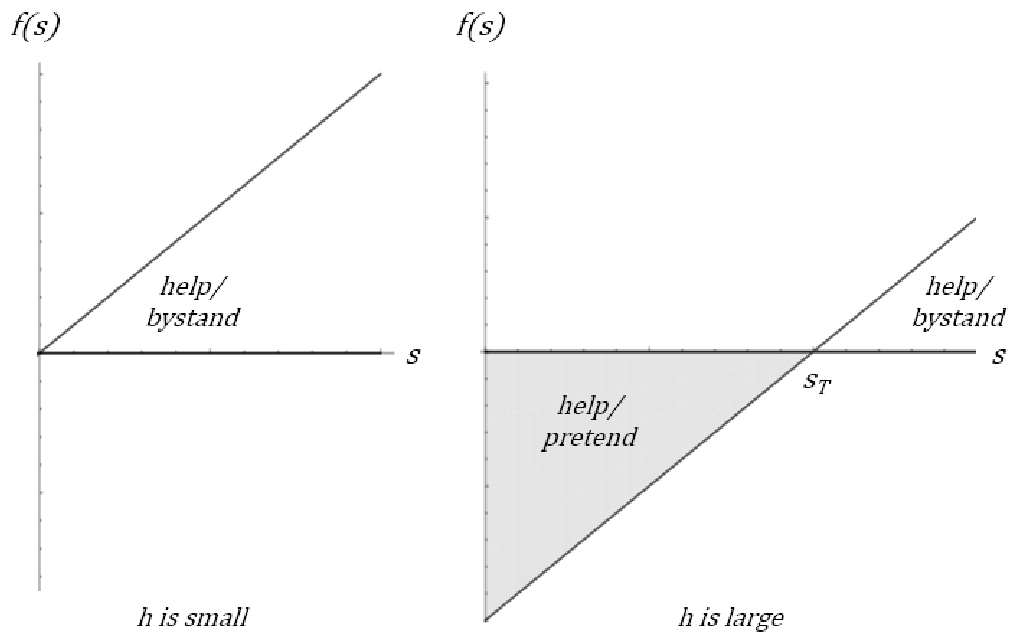



**Fig. 2.** Large Values of Shame Dissuade Pretense (only when $h$ is large)

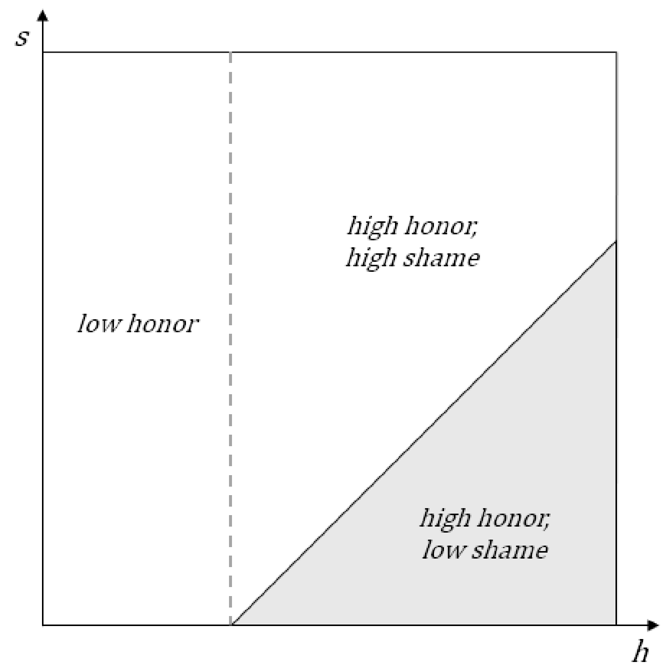



**Fig. 3.** High-Honor, Low-Shame Cultures Pretend (topology of merged $f(h, s)$)

□

How should we interpret these equations? First, honor encourages helping but also pretending. Second, cultures can be categorized broadly as one of three archetypes. Cultures that weakly value honor develop a norm to bystand as the alternative to not helping. Cultures that strongly value honor develop a norm to appear helpful, even if that means pretending. Cultures with high values of both honor *and* shame revert back to bystanding as the alternative to helping. A motto for such cultures might be, "Help, or help not; there is no pretend." This three-culture model helps explain why pretense is more prevalent in some cultures than others.

This leads to the second major result:

**Main Result 2.** *Pretense exists among cultures which strongly honor claimants but weakly shame pretenders.*

### 2.2.3 Honor as Subsidy, Shame as Tax

From an anthropological lens, honor and shame function like subsidies and taxes to shape outcomes desired by one's tribe. In the previous subsection, we discovered how these parameters affect which equilibrium exists, and thus whether people pretend. But is pretending good or bad for the tribe, and how much honor or shame is optimal?

To answer this, we must first define the group's welfare. One proposal might be to aggregate individual utilities. However, this leads to the trivial solution of 'infinite honor'. That is, welfare is maximized by deifying claimants. Not only does this violate Assumption 1 ($h < c$), it is also devoid of insight. Instead, let us favor a materialist view and take the probability of provision as the welfare measure.

**Proposition 7.** *Given Assumption 1, for any mixed-strategy equilibrium,* $\partial P^*/\partial b > 0$, $\partial P^*/\partial h > 0$, $\partial P^*/\partial w > 0$, $\partial P^*/\partial c < 0$, $\partial P^*/\partial k \geq 0$, *and* $\partial P^*/\partial s \geq 0$.

*Proof of Proposition 7.* Proposition 4 is the closed-form probability of provision. The positive or negative sign and numerator or denominator position determine each parameter's marginal effect. Assumption 1 implies the numerators are positive, $c - h - w > 0$ and $c - k - w > 0$, and the denominators are greater because $c < b$. This ensures $P^* \in (0, 1)$. □

The partial derivatives for all parameters except $c$ are non-negative, which means increasing anything except help cost weakly increases provision. Pretend cost $k$ and shame $s$ matter only in the help/pretend equilibrium and have zero effect in the help/bystand one.

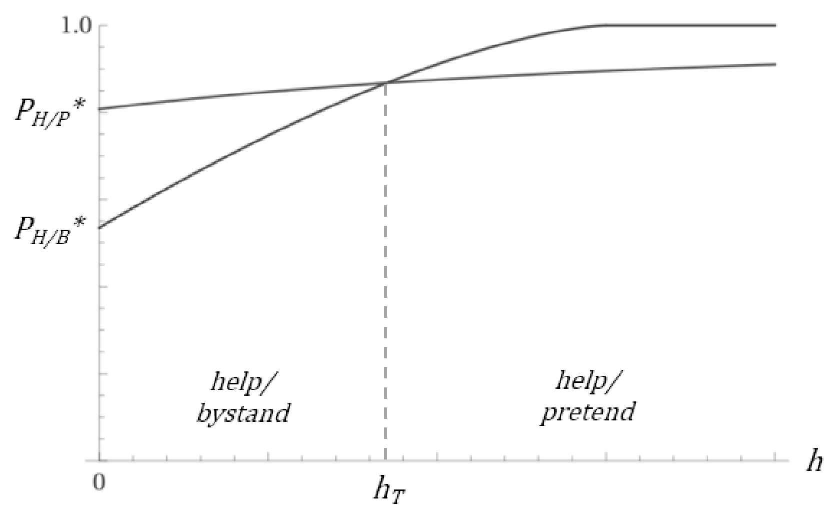



**Fig. 4.** Honor Increases Provision (always, in both equilibria)

$$P^* = min\{P^*_{H/B}, P^*_{H/P}\}$$

The graph above shows the probability of provision as a function of honor, $P^*(h)$. Because $P^*$ has a nested maximum function, it assumes one of two forms depending on which equilibrium exists. Prior to the threshold value of honor $h_T$, $P^* = P^*_{H/B}$. At $h_T$, the function is continuous but kinked. After $h_T$, $P^* = P^*_{H/P}$.

Increasing honor always increases provision albeit with diminishing marginal returns. Noticeably, honor's effect is stronger at low levels of $h$ when the help/bystand equilibrium exists. This is because honor motivates helping, but too much honor makes pretending more attractive.

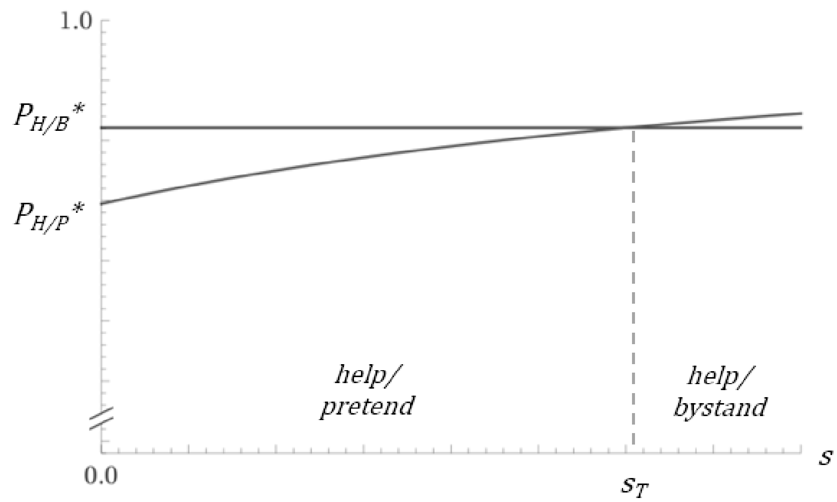



**Fig. 5.** Shame Increases Provision (only in the pretend equilibrium)

Merging both two-dimensional figures produces a three-dimensional graph of the provision probability as a function of honor and shame, $P^*(h, s)$. I provide an accompanying contour graph. These help us better observe key results. Firstly, $P^*(h, s)$ is non-decreasing in both arguments. Secondly, shame ($s$) matters only if honor ($h$) is sufficiently high. It follows that provision chance is maximized first by increasing honor then by increasing shame.

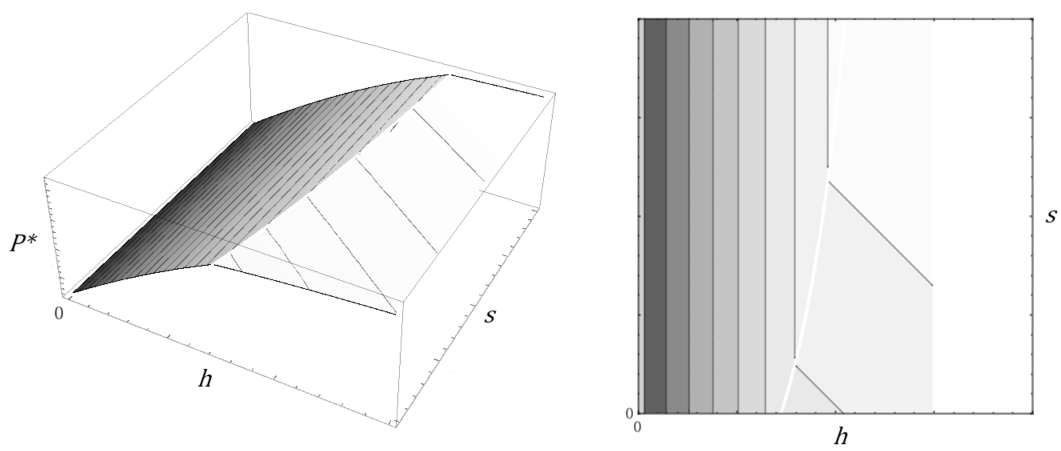



**Fig. 6.** High Honor and High Shame Maximize Provision Chance

This leads to the third major result:

**Main Result 3.** *Cultures which best provide a public good feature high social reward (honor) and high social punishment (shame). Second-best cultures tolerate pretense via high honor but low shame. Cultures which least provide exhibit low honor and low shame.*

### 2.2.4 Long-Run Sophistication

We recurringly say that players 'receive' honor or shame, but from whom? Let us call the people who confer these sanctions 'the audience.' The audience need not be limited to the players themselves. Consider, for instance, a cabal of politicians acting before the public. It is not so much the judgment of their peers as it is the judgment of their constituents that matters. The audience varies by context, but always their belief matters to the players.

In the short run, the audience is naïve. So long as the public good is provided, the audience is happy and honors all claimants fully. In the long run, the audience becomes sophisticated and discounts honor $h$ by the conditional probability a claimant is a helper. This discount factor is $p_H^*/(p_H^*+p_P^*)$, where $p_H^*$ and $p_P^*$ are the equilibrium probabilities of helping and pretending. This makes honor endogenous, affected by the composition of helpers and pretenders in equilibrium. Let $h_D$ represent this discounted honor.

**Assumption 2.** $h_D = h \cdot p_H^*/(p_H^* + p_P^*)$.

**Proposition 8.** *Given Assumptions 1 & 2, exactly one of three symmetric, mixed-strategy Nash equilibria exists: 1) players mix on help/bystand and no one pretends; 2) players mix on help/pretend and no one bystands; or 3) players mix on all three actions—help, bystand, and pretend–with positive probability.*

*Proof of Proposition 8.* An equilibrium that mixes only on $H/B$ exists when:

$$E(u_i|H) = E(u_i|B) \geq E(u_i|P) \tag{2.17}$$

$$b + h + w - c = b[1 - (p_B)^{n-1}] \geq (b+h)[1 - (p_B)^{n-1}] - s(p_B)^{n-1} - k \tag{2.18}$$

$$\implies p_B^* = \left(\frac{c-h-w}{b}\right)^{1/(n-1)} \geq \left(\frac{c-k-w}{b+h+s}\right)^{1/(n-1)} \tag{2.19}$$

An equilibrium that mixes only on $H/P$ exists when:

$$E(u_i|H) = E(u_i|P) \geq E(u_i|B) \tag{2.20}$$

$$b + h \cdot p_H + w - c = (b + h \cdot p_H)[1 - (p_P)^{n-1}] - s(p_P)^{n-1} - k \geq b[1 - (p_P)^{n-1}] \tag{2.21}$$

$$\implies p_P^* = \left(\frac{c-k-w}{b+h\cdot p_H+s}\right)^{1/(n-1)} \geq \left(\frac{c-h\cdot p_H-w}{b}\right)^{1/(n-1)} \tag{2.22}$$

When Inequalities 19 and 22 are both false, then the three-way mixed strategy equilibrium exists. Given that each other player helps, pretends, or bystands with respective probabilities $p_H$, $p_P$, $p_B$, the expected utility from each action is:

$$E(u_i|H) = b + h\frac{p_H}{p_H+p_P} + w - c \tag{2.23}$$

$$E(u_i|P) = (b + h\frac{p_H}{p_H+p_P})[1-(1-p_H)^{n-1}] - s(1-p_H)^{n-1} - k \tag{2.24}$$

$$E(u_i|B) = b[1-(1-p_H)^{n-1}] \tag{2.25}$$

Subtracting $E(u_i|B)$ from each and equating $E(u_i|H) = E(u_i|P) = E(u_i|B)$ yields:

$$0 = b(1-p_H)^{n-1} + h\frac{p_H}{p_H+p_P} + w - c \tag{2.26}$$

$$0 = h\frac{p_H}{p_H+p_P}[1-(1-p_H)^{n-1}] - s(1-p_H)^{n-1} - k \tag{2.27}$$

Equilibrium $(p_H^*, p_P^*)$ must satisfy this system of two equations. There is no general, closed-form solution due to the interaction between exponents and fractions. However, by the Nash Existence Theorem, every $n$-player game with finite actions has at least one equilibrium. If there are no pure-strategy equilibria, then there must be a unique mixed-strategy equilibrium. If the mixed-strategy equilibrium does not mix on only $B/P$, $H/B$, or $H/P$, then it must mix on $H/B/P$. □

**Proposition 9.** *Given Assumptions 1 & 2, a player's probability of helping and the group's probability of provision in the help/pretend equilibrium are strictly less than under Assumption 1 only.*

*Proof of Proposition 9.* When the equilibrium mixes only on actions $H/B$, any claimants are unambiguously helpers. When the equilibrium mixes only on $H/P$, honor $h$ is discounted. In this case, $p_P^* = 1 - p_H^*$ so the discount factor simplifies to $p_H^*$. Equating $E(u_i|H) = E(u_i|P)$ reduces to:

$$\frac{c-k-w}{b+h \cdot p_H + s} = (1-p_H)^{n-1} \in (0,1) \tag{2.28}$$

A discounted $h_D = h \cdot p_H < h$ makes the denominator smaller, the fraction larger, and thus $p_H$ smaller on the right side. □

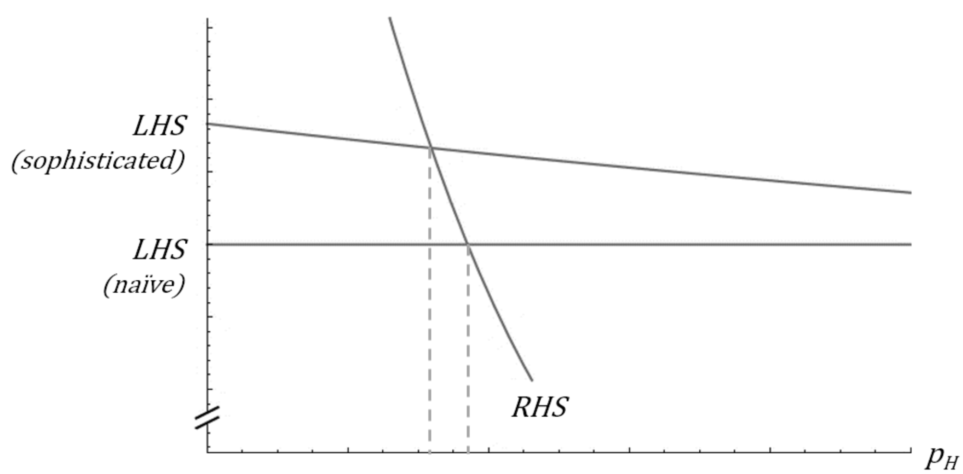



**Fig. 7.** Long-Run Sophistication Reduces Provision

Fig. 6 illustrates how Equation 28 determines equilibrium $p_H^*$ depending on whether the left side uses endogenous honor $h_D$ (long-run, sophisticated audience) or exogenous honor $h$ (short-run, naïve audience). The presence of pretenders has a double-negative effect on provision. First, pretenders do not contribute. Second, pretenders dilute honor, which discourages potential helpers who realize their help may be doubted as fake. This discouragement shares a structural similarity to the market for lemons (Akerlof et al. 1970).[12]

In an environment where audiences are sophisticated and some players pretend, helpers suffer from being pooled with pretenders. Authentication is relevant only in a help/pretend equilibrium. In a help/bystand equilibrium, every helper is already believed. Suppose now those helpers have the option to authenticate ($A$) their action. Authenticating costs more than helping ($c_A > c$), but it appears distinct from any other action and guarantees that the helper earns an undiluted honor $h > h_D = h \cdot p_H$.

**Proposition 10.** *Given Assumptions 1 & 2, when authenticating is an option, exactly one of two symmetric, mixed-strategy Nash equilibria exists: 1) players mix on authenticate/bystand and no one pretends or helps; or 2) players mix on help/pretend and no one bystands or authenticates. If the premium is less than the honor loss, the authenticate/bystand equilibrium exists. If the premium is greater than the honor loss, the help/pretend equilibrium exists.*

*Proof of Proposition 10.* Helpers who are pooled with pretenders authenticate

[12] Akerlof, Spence, and Stiglitz shared the 2001 Nobel Prize in Economics for their work in asymmetric information, of which the market for lemons was a central idea.

if $E(u_i|A) > E(u_i|H)$. This occurs when:

$$(c_A - c) < h(1 - p_H^*) \tag{2.29}$$

The left side is the extra cost to authenticate while the right side is the extra benefit from restoring full honor. If marginal benefit outweighs marginal cost, players pay this premium and mix on actions $A/B$ (i.e. they play $H$ and $P$ with probability zero). If the premium is too costly, no one plays $A$ and the equilibrium reverts to mixing on $H/P$. Mixing on $A/P$ cannot be an equilibrium because the audience can deduce that any claimant who does not authenticate is a pretender. □

### 2.2.5 Incomplete Information

The base model assumes every player is identical. A more realistic assumption would be that players are similar in preferences, but each varies in some individual way. To continue a trend in recent literature on extrinsic versus intrinsic motivation, I introduce heterogeneity via honor $h$ and warm glow $w$. These represent how much a person cares about looking good to others versus feeling good about oneself.

Let each player have fixed values of $h_i \sim U[0, \bar{h}]$ and $w_i \sim U[0, \bar{w}]$ that are identically, independently, and uniformly distributed. Individuals know their own types but not those of others; they know only the distribution from which these types are drawn. To derive an equilibrium, I use the Bayesian approach for games of incomplete information (Harsanyi 1967-1968). A strategy here maps a player $i$'s type $(h_i, w_i)$ to an optimal action.[13] In a Bayesian equilibrium, each player best-responds to maximize expected utility given the randomized types ($h_j, w_j$, where $j \neq i$) of other players.

**Proposition 11.** *Given Assumption 1, under heterogeneity and incomplete information, a unique equilibrium exists where players choose exactly one of three actions—help, pretend, bystand—based on their type.*

*Proof of Proposition 11.* Let the binary function $p_{Hi}(h_i, w_i) \in \{0, 1\}$ input a player $i$'s types and output the probability $i$ helps, either zero or one. From $i$'s perspective, the public good is provided if at least one other player helps, meaning $p_{Hj} = 1$ for some $j \neq i$. The good is not provided if no one helps, meaning $p_{Hj} = 0 \; \forall \; j \neq i$. Thus, the probability at least one other player helps is $E(1-\prod_{j\neq i}(1-p_{Hj}))$, where $p_{Hj}(h_j, w_j)$ is a function of uniformly-distributed random variables $h_j, w_j$. Then, $i$'s expected utility from each action depends on their own type $(h_i, w_i)$ and the vector $p_H = (p_{H1}, p_{H2}, ..., p_{Hn-1})$ that maps types to actions:

$$E(u_i|H) = b + h_i + w_i - c \tag{2.30}$$

$$\begin{aligned} E(u_i|B) &= b \cdot E(1 - \prod_{j\neq i}(1 - p_{Hj})) \\ &= b \cdot (1 - \prod_{j\neq i} E(1 - p_{Hj})) \end{aligned} \tag{2.31}$$

$$E(u_i|P) = (b + h_i)(1 - \prod_{j\neq i} E(1 - p_{Hj})) - s \prod_{j\neq i} E(1 - p_{Hj}) - k \tag{2.32}$$

[13] As shown in Weesie 1994, randomization among alternatives is unnecessary for an equilibrium to exist and is thus discarded.

where

$$E(1 - p_{Hj}) = \int_0^{\bar{w}} \int_0^{\bar{h}} (1 - p_{Hj}(h_j, w_j)) \cdot dh_j \cdot dw_j \tag{2.33}$$

Player $i$ chooses the action with the greatest expected utility. If this action is help, then $p_{Hi} = 1$. In equilibrium, the function $p_{Hi}$ must hold true given all other $p_{Hj}$ where $j \neq i$. □

While the strategy functions are infeasibly complex, we can still visualize a mapping of types to optimal actions. A rectangle of dimensions $\bar{h} \times \bar{w}$ represents the type space for $h_i \sim U[0, \bar{h}] \times w_i \sim U[0, \bar{w}]$. This type space is divided by three concentric rays into distinct regions corresponding to the three actions—help, bystand, or pretend. Every player uses this same mapping to select a pure strategy corresponding to their given type.[14]

[14] In fact, I conjecture that a valid partition always exists for any ratio of $H : B : P$. That is, given three distinct angles of rays originating from a central point $P$ and given a desired ratio of $H : B : P$, I claim that there always exists at least one placement of $P$, either inside or outside the rectangle, that partitions the type space into the desired shares. However, the geometric proof for this is beyond the scope of this paper. I simply mention this for interested mathematicians.

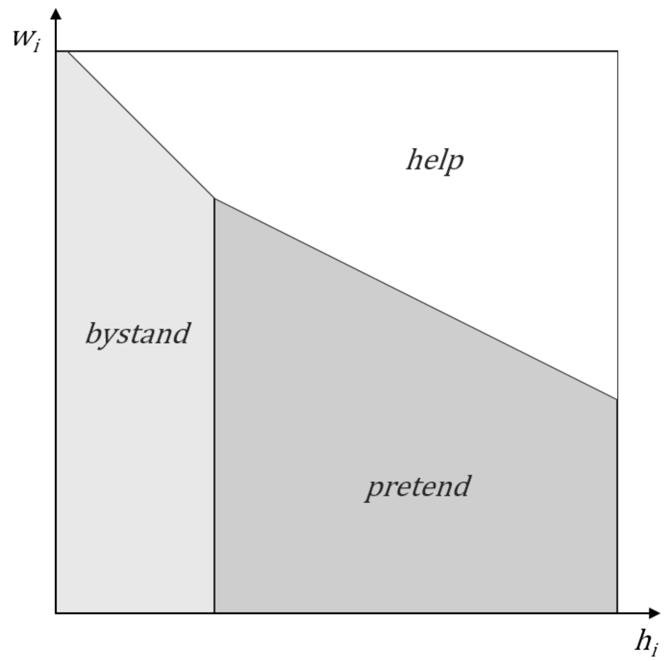



**Fig. 8.** Mapping of Types to Optimal Action

The threshold between bystand and pretend depends only on honor $h_i$. This is because warm glow $w_i$ does not apply to bystanders and pretenders, but honor incentivizes pretending. At the threshold between help and bystand, any increase in either $h_i$ or $w_i$ tips a marginal player toward helping. This is also true for the threshold between help and pretend, albeit with a flatter slope. This is because a marginal increase in honor $h_i$ certainly increases a helper's utility, whereas it has only a chance to increase a pretender's utility (i.e. if the good is provided). An alternative intuition is that pretenders already gain $h_i$ if the good is provided, so it takes greater $h_i$ than $w_i$ to incentivize a marginal pretender into helping.[15]

[15]Mathematically, the help/bystand threshold slope is exactly 45° because $\partial[E(u_i|H) - E(u_i|B)]/\partial h_i = \partial[E(u_i|H) - E(u_i|B)]/\partial w_i$. The help/pretend threshold slope is flatter

In this hypothetical illustration, the areas of help, bystand, and pretend are respectively 35%, 25%, and 40%. These percentages approximate the mean populations of altruistic, selfish, and 'reluctant' types across previous studies: Lazear et al. (2012), Dana et al. (2006), Dana et al. (2007). For emphasis, in an incomplete information equilibrium, a player's chance to help is either zero or one. However, from an observer's perspective, it appears as if $p^* = 35\%$ per individual.

## 2.3 Discussion

The key insight of this paper is that in many public goods contexts—economic, political, social—a 'help' signal may not correspond to a 'help' action. That is, people sometimes pretend to contribute, potentially free riding a positive image without paying the full help cost. How does the ability to pretend affect good provision? How do group size, honor and shame, sophistication, or asymmetry affect outcomes? What are consequences of pretense and how might we mitigate them? Until now, this has been a gap in the public goods literature.

I fill this gap by modeling the volunteer's dilemma with a third alternative: pretending to help. Pretending appears identical to helping, contributes zero toward the public good, and costs less than helping ($0 < k < c$). Secondly, I decompose the benefits from helping (or pretending) into material ($b$), honor ($h$), and warm glow ($w$). These three factors comprise the primary motivations for contributing toward a public good. Helping confers a material benefit, honor, and warm glow. Pretending confers honor if the public good is

than 45° because $\partial[E(u_i|H) - E(u_i|P)]/\partial h_i < \partial[E(u_i|H) - E(u_i|P)]/\partial w_i$.

provided and shame otherwise. Bystanding costs nothing and confers nothing.

In Diekmann's (1985) original, $n$-player volunteer's dilemma, there are $n$ pure-strategy Nash equilibria where exactly one player helps and everyone else bystands. When I introduce the pretend option, these same equilibria instead become exactly one player helping while everyone else pretends. The intuition is that if the good is provided, pretenders can also share in honor (Proposition 1). Whether this outcome is socially wasteful is a matter of perspective, as the utility gained from honor is countered by the cost of pretending. Like the original model, my pretense model has a unique, symmetric, mixed-strategy Nash equilibrium where everyone mixes on help/bystand. In addition, it has one where everyone mixes on help/pretend. Only one of these two equilibria exists at a time (Proposition 3).

Regarding the effect of group size, in both equilibria as $n$ approaches infinity, the probability an individual helps approaches zero and the likelihood the good is provided approaches a constant between zero and one. This is consistent with Diekmann's model (Corollary 4). Whether the mixed-strategy equilibrium is help/bystand or help/pretend depends on costs and benefits, but surprisingly not on group size. Specifically, the equilibrium with the smaller probability of provision is the one that exists. A somber logic implies that when people can pretend, the public good is *less* likely to be provided than when people cannot pretend (Main Result 1).

Pretense exists for sufficiently large values of honor and sufficiently small values of shame. From an anthropological view, cultures which value image heavily develop a norm to appear helpful, which motivates pretending (Main

Result 2). The rise of social media, especially, has created audience effects and pro-social norms. At the same time, unverifiability behind technological barriers can and often do incentivize pretense. Fortunately, cultures can improve their probability of public good provision by first raising social reward (honor), followed by social punishment (shame) (Main Result 3).

The pretending volunteer's dilemma shares some structural features to Akerlof's (1970) famous market for lemons. Akerlof argued that when buyers cannot distinguish between a high-quality car ('peach') and a low-quality car ('lemon'), they will in expectation pay only the average price of a peach and lemon. Sellers, on the other hand, know the quality of their car. Given the lower, average price at which buyers would buy, 'lemons' sell while 'peaches' leave the market. Similarly, when the audience becomes sophisticated in the long run, they discount honor by the probability a claimant is really a helper. Thus, the presence of pretenders has a double-negative effect. First, pretenders do not help. Second, pretenders dilute honor and discourage potential helpers from helping (Proposition 9).

Interestingly, discouraged helpers facing diluted honor are willing to pay extra costs, up to the honor loss, to authenticate their help. When an authentication option is introduced, either everyone mixes on authenticate/bystand or everyone mixes on help/pretend (Proposition 10). While it may seem strange, people sometimes do pay a premium to authenticate their help. A worker might work in an inconvenient but public space for visibility. A philanthropist might increase donation to a higher bracket to avoid being listed among lower-tier donors who may be donating the bare minimum (e.g. $100+ group instead of \$1 to \$99 group). This result is supported by data from Harbaugh's (1998a) work on philanthropy and prestige, which shows that when donations are both tiered and publicized, most donations bunch at the minimum required for inclusion in each tier.

In an incomplete information equilibrium, a two-dimensional space exists that maps types to actions (Proposition 11). Recent work on 'reluctant' helpers by Lazear et al. (2012), Dana et al. (2006, 2007) suggest that people can be grouped into altruistic, selfish, or 'reluctant' types. Reluctant helpers are motivated by image but would rather not help if unobserved. These types correspond closely to helpers, bystanders, and pretenders in my heterogeneity model. The work of these authors estimate that about 35% of the population are altruistic, 25% are selfish, and 40% are reluctant. In crafting Fig. 8, I took special effort to equate the areas of the three regions to these percentages.

To focus on the impact of pretense on good provision, I modeled pretense in the volunteer's dilemma. A future extension could cover a theoretical model of a (simplified) public goods game. Experimentally, when people can pretend, should we expect an increase in claimed volunteers but a decrease in the provision rate? If so, this would imply that some helpers were relucant and perhaps also that some bystanders would 'buy' honor if only honor were cheaper.

# Chapter 3

# Visualization of Revenue Equivalence

## 3.1 Introduction

Rarely are the words ‘geometry’ and ‘economics’ heard together, but in the instances where they do overlap, it is worthwhile to illuminate their intersection. Geometric representations are important in economics because they enhance our mathematical understanding via visual intuition. A classic example is identifying surplus, tax, and deadweight areas in a Marshallian supply-and-demand graph. In two-good consumption, what are indifference curves but contour lines of a three-dimensional utility mesh? Geometry helps us perceive the Slutsky decomposition of income versus substitution effect, shift allocations in the Edgeworth box, and pin-point equilibria from best response intersections.

Similarly, a visual representation of the revenue equivalence principle from auction theory can help us better understand and appreciate a central idea. Revenue equivalence states that, given certain conditions[1], any auction mechanism that produces the same allocation for buyers also yields the same expected revenue for the seller (Myerson 1981, Riley and Samuelson 1981). Several auction formats have been proven algebraically to be revenue equivalent, and I focus only on the geometric aspects.

To be clear, this paper is of an expository and pedagogical nature. It is true that revenue equivalence holds for any number of bidders and any distribution. However, figures are difficult enough to visualize beyond three dimensions, much less amorphous figures. Thus, I limit the geometry to two symmetric bidders with uniform distribution. It is my intent that these two approaches, geometric and summation, offer visual intuition to an algebraically-established principle.

In Section 2 (Geometric), I provide a geometric interpretation of revenue equivalence between first-price, second-price, and all-pay sealed-bid auctions for two bidders with uniformly-distributed private values. Importantly, the mean height of each solid represents expected revenue and is equal across all three solids. I also overlay the solids to identify regions in the value space where each auction format yields the greatest revenue.

In Section 3 (Summation), I present a summation approach of revenue equivalence using partitioned volumes. As these increments shrink toward

[1] Across auction formats, players must bid according to their type, a specific type must have the same probability to win, and the lowest type must have the same expected utility, most commonly zero.

zero, the expected revenues of all three converge to the same continuous limit. I also use summations to calculate the variance of each auction format for a risk-averse seller. In order of ascending variance are the first-price (least), second-price, and all-pay (greatest) auction formats.

In section 4 (Discussion), I share a tangible, ham-and-cheese model as a pedagogical tool. I then suggest materials and techniques to construct such tangible models.

## 3.2 Geometric

### First-Price, Sealed-Bid

Two bidders, $i = 1, 2$, have private values $v_i \sim U[0, 1]$. In a first-price auction, the player with the highest bid wins and pays his bid amount; all others exchange nothing. In linear bidding strategies with $n = 2$ players, each player optimally bids $b_i = [(n-1)/n] \cdot v_i = v_i/2$. The seller receives as revenue the highest bid, or $R_{1st} = max\{v_1/2, v_2/2\}$, and expected revenue $E(R_{1st}) = (n-1)/(n+1) = 1/3$ (Vickrey 1961).[2] The contour curves for a max function are L-shaped, and the solid is quasi-convex.

[2]Vickrey originally solved the descending (Dutch) auction, but it is strategically equivalent to a first-price, sealed-bid auction.

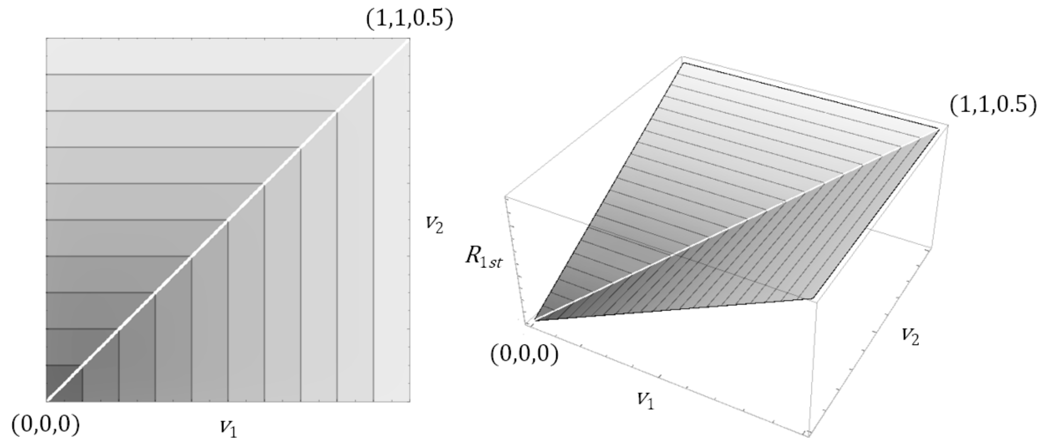



**Fig. 1.** First-Price Geometry

The axes represent values $v_1$ and $v_2$ while the height is revenue $R_{1st}$. For example, the coordinate $(v_1, v_2, R_{1st}) = (0.8, 0.4, 0.4)$ means the seller receives $max\{v_1/2, v_2/2\} = 0.4$. Expected revenue $E(R_{1st})$ is thus the mean height of the solid.

$$E(R_{1st}) = E(height) = volume/base = (prism - pyramid)/square \quad (3.1)$$

$$= (1^2 \cdot 0.5 - 1^2 \cdot 0.5/3)/1^2 = 1/3 \quad (3.2)$$

### Second-Price, Sealed-Bid

Now compare the second-price auction, where the winner pays the *second* highest bid. Here, each player optimally bids his true private value, so $b_i = v_i$. The seller receives as revenue the second highest bid, which in this case is also the minimum $R_{2nd} = min(v_1, v_2)$, and expected revenue is also $E(R_{2nd}) = (n-1)/(n+1) = 1/3$ (Vickrey 1961). The contours for a min function are inverted Ls, and the solid is quasi-concave.

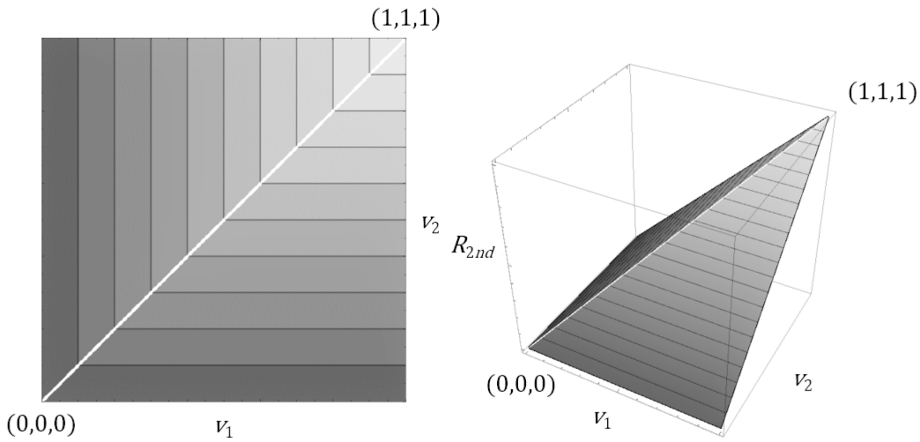



**Fig. 2.** Second-Price Geometry

The height now represents $R_{2nd}$, and so the expected revenue $E(R_{2nd})$ is the mean height of the pyramid, which is equivalent to $E(R_{1st}) = 1/3$.

$$E(R_{2nd}) = E(height) = volume/base = pyramid/square \tag{3.3}$$

$$= (1^3/3)/1^2 = 1/3 \tag{3.4}$$

**All-Pay, Sealed-Bid**

Lastly compare a third auction, the all-pay, where the highest bidder wins, but all players pay their bid. Here, players optimally bid according to $b_i = v_i^2/2$. The seller collects $R_{all} = v_1^2/2 + v_2^2/2$, which is a quarter-paraboloid with arc contours.

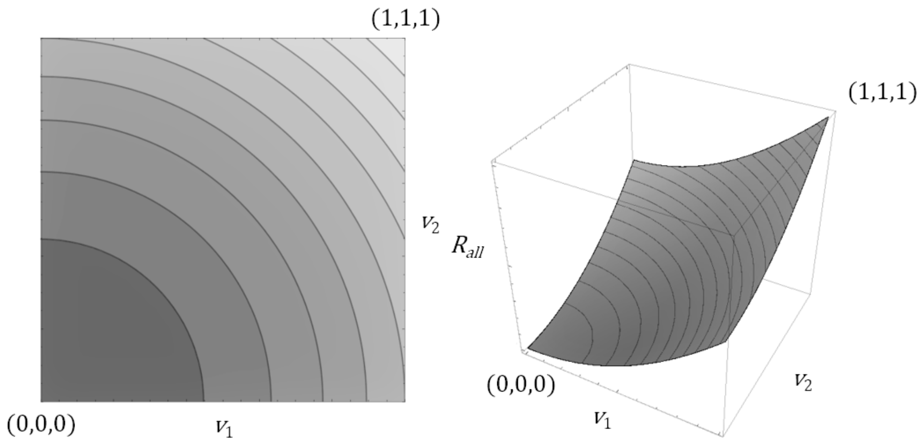


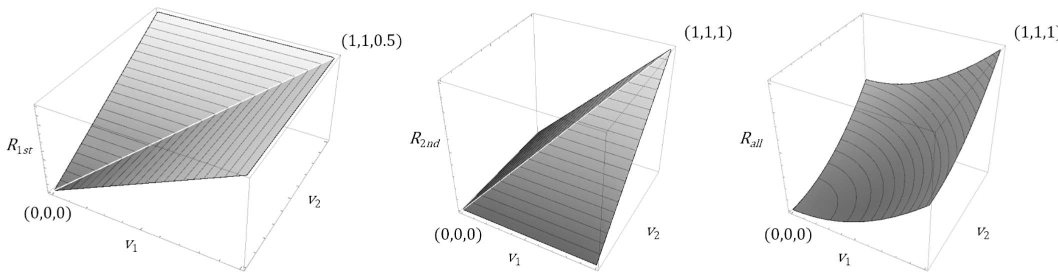

**Fig. 3.** All-Pay Geometry

The expected revenue $E(R_{all})$ is the mean height of this paraboloid, which is also equivalent to $R_{1st} = R_{2nd} = 1/3$. A simplifying step can be made by noting that $v_1$ and $v_2$ are independently and identically distributed, so a single (as opposed to double) integral is sufficient.

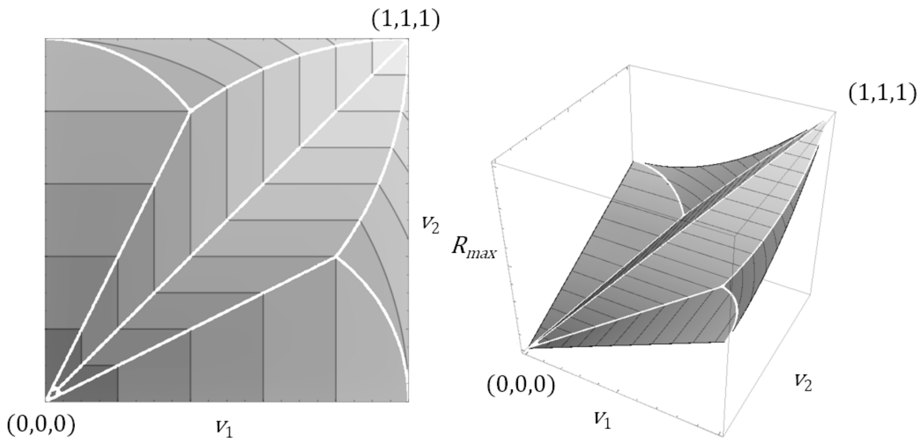
$$E(R_{all}) = E(height) = volume/base = (paraboloid)/(square) \tag{3.5}$$

$$= \Big[\int_0^1 (v^2/2 + v^2/2) \cdot dv\Big]/1^2 = \Big(\int_0^1 v^2 \cdot dv\Big)/1^2 = 1/3 \tag{3.6}$$

Here are the three auction formats again, side-by-side and overlaid for comparison.




$$E(R_{1st}) = E(R_{2nd}) = E(R_{all}) = 1/3$$




$$\begin{aligned} R_{max} &= max\{R_{1st}, R_{2nd}, R_{all}\} \\ &= max\{max\{\frac{v_1}{2}, \frac{v_2}{2}\}, min\{v_1, v_2\}, \frac{v_1^2}{2} + \frac{v_2^2}{2}\} \end{aligned}$$

**Fig. 4.** Geometry Comparison

Out of context, there is little intuition that these separate solids would share the same volume, much less significance regarding their shape. Yet in auction theory each solid represents a distinct mechanism, an optimal response, and an expected revenue. We have thus transformed expected revenue from an algebraic object to a geometric one that is mean height. Further, because the base areas are normalized to one, mean height is equal to volume. Another way geometry helps our understanding is that overlaying the three solids quickly shows which format returns the greatest revenue for a given $(v_1, v_2)$. For example, when bidder values are equal, along the diagonal, second-price is best.

## 3.3 Summation

In this section, I present a summation approach using partitioned volumes for each of the three auction formats: first-price, second-price, and all-pay. I then increase the number of partitions toward infinity and show that the asymptotic limit is consistent with the continuous case.

It may seem as though summations are but a precursor to integrals, but there are several advantages. First, we can observe the behavior of revenue functions near the limit. Second, two of the three solids are not 'smooth' (i.e. twice-differentiable) due to a kink along the $v_1 = v_2$ axis. This makes summations a valuable approach to calculating mean and variance. Third, the material can be presented to advanced youth who have not had exposure to calculus.

**First-Price, Sealed-Bid**

Let us partition each player's continuous value set $v_i \sim U[0,1]$ into $n+1$ discrete elements, such that $v_i \in \{\frac{0}{n}, \frac{1}{n}, \frac{2}{n}, ..., \frac{n}{n}\}$ with equal probability $1/(n+1)$ for each element.

| | | | | | |
|---|---|---|---|---|---|
| $1$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $\dots$ | $\dots$ | $\dots$ | $\dots$ | $\dots$ | $\frac{1}{2}$ |
| $\frac{2}{n}$ | $\frac{1}{n}$ | $\frac{1}{n}$ | $\frac{1}{n}$ | $\dots$ | $\frac{1}{2}$ |
| $\frac{1}{n}$ | $\frac{1}{2n}$ | $\frac{1}{2n}$ | $\frac{1}{n}$ | $\dots$ | $\frac{1}{2}$ |
| $0$ | $0$ | $\frac{1}{2n}$ | $\frac{1}{n}$ | $\dots$ | $\frac{1}{2}$ |
| $v_i$ | $0$ | $\frac{1}{n}$ | $\frac{2}{n}$ | $\dots$ | $1$ |

**Fig. 5.** First-Price Summation

The axes represent $v_1$ and $v_2$, normalized to $[0, 1]$, and each intersection on the lattice gives the revenue at $(v_1, v_2)$. Recall that players are quasi-rational and bid as if continuous, so the seller receives $R_{1st} = max[v_1/2, v_2/2]$. The expected revenue $E(R_{1st})$ is then the sum of revenues weighted by their probabilities. Observe that the elements in each L-shape are increasing odd numbers. Hence,

$$E(R_{1st}) = \sum_{k=0}^{n} (revenue)(probability)$$

$$= \sum_{k=0}^{n} (k/2n)[(2k+1)/(n+1)^2] = \frac{1}{2n(n+1)^2} \sum_{k=0}^{n} 2k^2 + k \tag{3.7}$$

$$= \frac{1}{2n(n+1)^2} \Big[ 2 \cdot \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \Big] \tag{3.8}$$

$$= \frac{1}{2(n+1)} \Big[ \frac{2n+1}{3} + \frac{1}{2} \Big] = \frac{1}{2n+2} \Big( \frac{4n+5}{6} \Big) = \frac{4n+5}{12n+12} \tag{3.9}$$

$$\Rightarrow \lim_{n\to\infty} E(R_{1st}) = \lim_{n\to\infty} \frac{4n+5}{12n+12} = 1/3 \tag{3.10}$$

**Second-Price, Sealed-Bid**

Partitioning the second-price auction the same way yields:

| | | | | | |
|---|---|---|---|---|---|
| 1 | 0 | $\frac{1}{n}$ | $\frac{2}{n}$ | ... | 1 |
| ... | ... | ... | ... | ... | ... |
| $\frac{2}{n}$ | 0 | $\frac{1}{n}$ | $\frac{2}{n}$ | ... | $\frac{2}{n}$ |
| $\frac{1}{n}$ | 0 | $\frac{1}{n}$ | $\frac{1}{n}$ | ... | $\frac{1}{n}$ |
| 0 | 0 | 0 | 0 | ... | 0 |
| $v_i$ | 0 | $\frac{1}{n}$ | $\frac{2}{n}$ | ... | 1 |

**Fig. 6.** Second-Price Summation

$$E(R_{2nd}) = \sum_{k=0}^{n} (revenue)(probability)$$

$$= \sum_{k=0}^{n} (k/n)\{[2(n-k)+1]/(n+1)^2\} = \frac{1}{n(n+1)^2} \sum_{k=0}^{n} 2nk - 2k^2 + k \tag{3.11}$$

$$= \frac{1}{n(n+1)^2}\Big[(2n+1)\cdot\frac{n(n+1)}{2} - 2\cdot\frac{n(n+1)(2n+1)}{6}\Big] \tag{3.12}$$

$$= \frac{1}{n+1}(\frac{2n+1}{2} - \frac{2n+1}{3}) = \frac{1}{n+1}(\frac{2n+1}{6}) = \frac{2n+1}{6n+6} \tag{3.13}$$

$$\Rightarrow \lim_{n\to\infty} E(R_{2nd}) = \lim_{n\to\infty} \frac{2n+1}{6n+6} = 1/3 \tag{3.14}$$

That $\lim_{n\to\infty} E(R_{2nd}) = \lim_{n\to\infty} E(R_{1st}) = 1/3$ means the two expected revenues converge, and that this limit is consistent with the continuous result. However, note that $\forall\ n \in \mathbb{N}$,

$$E(R_{2nd}) = \frac{2n+1}{6n+6} = \frac{4n+2}{12n+12} < \frac{4n+5}{12n+12} = E(R_{1st}) \tag{3.15}$$

**All-Pay, Sealed-Bid**

Deriving the summation formula for the all-pay lattice may seem complex, but $E(R_{all}) = v_1^2/2 + v_2^2/2$, and $v_1$ and $v_2$ are independent and identically distributed. So, the sum is simply double a single buyer's expected payment.

| | | | | | |
|---|---|---|---|---|---|
| $1$ | $\frac{1}{2}$ | $\frac{n^2+1}{2n^2}$ | $\frac{n^2+4}{2n^2}$ | ... | $1$ |
| ... | ... | ... | ... | ... | ... |
| $\frac{2}{n}$ | $\frac{4}{2n^2}$ | $\frac{5}{2n^2}$ | $\frac{8}{2n^2}$ | ... | $\frac{n^2+4}{2n^2}$ |
| $\frac{1}{n}$ | $\frac{1}{2n^2}$ | $\frac{2}{2n^2}$ | $\frac{5}{2n^2}$ | ... | $\frac{n^2+1}{2n^2}$ |
| $0$ | $0$ | $\frac{1}{2n^2}$ | $\frac{4}{2n^2}$ | ... | $\frac{1}{2}$ |
| $v_i$ | $0$ | $\frac{1}{n}$ | $\frac{2}{n}$ | ... | $1$ |

**Fig. 7.** All-Pay Summation

$$
\begin{aligned}
E(R_{all}) &= \sum_{k=0}^{n} (revenue)(probability) \\
&= 2 \cdot \sum_{k=0}^{n} (k^2/2n^2)[1/(n+1)] = \frac{1}{n^2(n+1)} \sum_{k=0}^{n} k^2 && (3.16) \\
&= \frac{1}{n^2(n+1)} [\frac{n(n+1)(2n+1)}{6}] = \frac{1}{n} (\frac{2n+1}{6}) = \frac{2n+1}{6n} && (3.17)
\end{aligned}
$$

$$
\Rightarrow \lim_{n \to \infty} E(R_{all}) = \lim_{n \to \infty} \frac{2n+1}{6n} = 1/3 \quad (3.18)
$$

The asymptotic limit is equal to the other auctions, and to the continuous limit. How does $E(R_{all})$ compare to $E(R_{2nd})$ or $E(R_{1st})$ for finite $n$? $E(R_{all}) = (2n+1)/6n$ has a smaller denominator than $E(R_{2nd}) = (2n+1)/(6n+6)$ and therefore must be larger. Taking common denominators via cross-multiply shows that $\forall \; n \in \mathbb{N}$,

$$
E(R_{all}) = \frac{2n+1}{6n} = \frac{24n^2 + 36n + 12}{72n^2 + 72n} > \frac{24n^2 + 30n}{72n^2 + 72n} = \frac{4n+5}{12n+12} = E(R_{1st}) \quad (3.19)
$$

$$
\Rightarrow E(R_{all}) > E(R_{1st}) \quad (3.20)
$$

Now consider a risk-averse seller who, expected revenues being equal, prefers low variance. Which format should he use? Because two of the three solids are not 'smooth' (i.e. twice-differentiable) due to a kink along the $v_1, v_2$ axis, summations offer an alternate approach to calculating variance. The variances are:

$$Var(R_{1st}) = \lim_{n\to\infty} -[E(revenue)]^2 + \sum_{k=0}^{n} (revenue)^2 (probability) \tag{3.21}$$

$$= -(1/3)^2 + \lim_{n\to\infty} \sum_{k=0}^{n} (k/2n)^2 [(2k+1)/(n+1)^2] \tag{3.22}$$

$$= -(1/3)^2 + \lim_{n\to\infty} \frac{1}{4n^2(n+1)^2} \sum_{k=0}^{n} 2k^3 + k^2 \tag{3.23}$$

$$= -(1/3)^2 + \lim_{n\to\infty} \frac{1}{4n^2(n+1)^2} [2 \cdot \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6}] \tag{3.24}$$

$$= -(1/3)^2 + \lim_{n\to\infty} \frac{1}{8} + \frac{2n+1}{24n^2+24n} = -1/9 + 1/8 + 0 = 1/72 \tag{3.25}$$

$$Var(R_{2nd}) = \lim_{n\to\infty} -[E(revenue)]^2 + \sum_{k=0}^{n} (revenue)^2 (probability) \tag{3.26}$$

$$= -(1/3)^2 + \lim_{n\to\infty} \sum_{k=0}^{n} (k/n)^2 \{[2(n-k)+1]/(n+1)^2\} \tag{3.27}$$

$$= -(1/3)^2 + \lim_{n\to\infty} \frac{1}{n^2(n+1)^2} \sum_{k=0}^{n} 2nk^2 - 2k^3 + k^2 \tag{3.28}$$

$$= -(1/3)^2 + \lim_{n\to\infty} \frac{1}{n^2(n+1)^2}$$

$$\cdot [(2n+1) \cdot \frac{n(n+1)(2n+1)}{6} - 2 \cdot \frac{n^2(n+1)^2}{4}] \tag{3.29}$$

$$= -(1/3)^2 + \lim_{n\to\infty} \frac{4n^2+4n+1}{6n^2+6n} - \frac{1}{2} = -1/9 + 2/3 - 1/2 = 1/18 \tag{3.30}$$

$$Var(R_{all}) = -[E(revenue)]^2 + \int_0^1 (v^2)^2 \cdot dv = -1/9 + 1/5 = 4/45 \tag{3.31}$$

Here are the three auction formats again, side-by-side for comparison.

| | 1st-Price | 2nd-Price | All-Pay |
|---|---|---|---|
| $E[R(n)]$ | $\frac{4n+5}{12n+12}$ | $\frac{2n+1}{6n+6}$ | $\frac{2n+1}{6n}$ |
| $\lim_{n\to\infty} Var[R(n)]$ | $\frac{1}{72}$ | $\frac{1}{18}$ | $\frac{4}{45}$ |

$$\forall n \in \mathbb{N}, E(R_{2nd}) < E(R_{1st}) < E(R_{all})$$
$$\lim_{n\to\infty} E(R_{1st}) = \lim_{n\to\infty} E(R_{2nd}) = \lim_{n\to\infty} E(R_{all}) = 1/3$$
$$Var(R_{1st}) < Var(R_{2nd}) < Var(R_{all})$$

**Fig. 8.** Summation Comparison

## 3.4 Discussion

To conclude, I share a tangible, ham-and-cheese model as an example of a pedagogical tool.

**Fig. 9.** Tangible Example

The ham (left) represents first-price while the cheese (right) represents second-price; their square bases are normalized to one. Revenue equivalence states that the ham and cheese have equal volumes and therefore equal mean heights.

Individuals learn best in different ways: sight, sound, touch, interaction, perhaps all the above. Tangible models reinforce learning because they are interactive and audiences can freely rotate the solids. Materials are accessible at local grocers and cost about 5-10 USD. Carving time takes about 10-15 minutes. Vegans may substitute styrofoam, three-dimensional printing, or other durable materials.

There is perhaps an innate fascination with food or play that keeps us engaged. Trials presenting to graduate, undergraduate, and advanced high school audiences yielded positive feedback about the tangible model. Educators may use these visualizations to enhance a reader's understanding and appreciation of auction theory. Future work can identify other areas of economics that could benefit from visual or interactive learning.

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