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A Uniform Logic of Information Dynamics*

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Abstract

Unlike standard modal logics, many dynamic epistemic logics are not closed under uniform substitution. A distinction therefore arises between the logic and its *substitution core*, the set of formulas all of whose substitution instances are valid. The classic example of a non-uniform dynamic epistemic logic is Public Announcement Logic (PAL), and a well-known open problem is to axiomatize the substitution core of PAL. In this paper we solve this problem for PAL over the class of all relational models with infinitely many agents, PAL-$\mathbf{K}_\omega$, as well as standard extensions thereof, e.g., PAL-$\mathbf{T}_\omega$, PAL-$\mathbf{S4}_\omega$, and PAL-$\mathbf{S5}_\omega$. We introduce a new Uniform Public Announcement Logic (UPAL), prove completeness of a deductive system with respect to UPAL semantics, and show that this system axiomatizes the substitution core of PAL.

Keywords: dynamic epistemic logic, Public Announcement Logic, schematic validity, substitution core, uniform substitution

1 Introduction

One of the striking features of many of the *dynamic epistemic logics* [28,19,13,9,4] studied in the last twenty years is the failure of closure under *uniform substitution* in these systems. Given a valid principle of information dynamics in such a system, uniformly substituting complex epistemic formulas for atomic sentences in the principle may result in an *invalid* instance. Such failures of closure under uniform substitution turn out to reveal insights into the nature of information change [1,7,11,24,8]. They also raise the question: what are the more robust principles of information dynamics that are valid in all instances, that are *schematically* valid? Even for the simplest system of dynamic epistemic logic, Public Announcement Logic (PAL) [28], the answer has been unknown. In van Benthem's "Open Problems in Logical Dynamics" [3], Question 1 is whether the set of schematic validities of PAL is axiomatizable. [1]

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[1] Dynamic epistemic logics are not the only non-uniform modal logics to have been studied. Other examples include Buss's [16] modal logic of "pure provability," Åqvist's [10] two-

In this paper, we give an axiomatization of the set of schematic validities—or *substitution core*—of PAL over the class of all relational models with infinitely many agents, $\mathsf{PAL}\text{-}\mathbf{K}_\omega$, as well as standard extensions thereof, e.g., $\mathsf{PAL}\text{-}\mathbf{T}_\omega$, $\mathsf{PAL}\text{-}\mathbf{S4}_\omega$, and $\mathsf{PAL}\text{-}\mathbf{S5}_\omega$. After reviewing the basics of PAL in §1.1, we introduce the idea of Uniform Public Announcement Logic (UPAL) in §1.2, prove completeness of a $\mathtt{UPAL}$ deductive system in §3 with respect to alternative semantics introduced in §2, and show that it axiomatizes the substitution core of PAL in §4. In §5, we demonstrate our techniques with examples, and in §6 we conclude by discussing extensions of these techniques to other logics.

Although much could be said about the conceptual significance of UPAL as a uniform logic of information dynamics, here we only present the formal results. For conceptual discussion of PAL, we refer the reader to the textbooks [9,4]. Our work here supports a theme of other recent work in dynamic epistemic logic: despite its apparent simplicity, PAL and its variants prove to be a rich source for mathematical investigation (see, e.g., [3,2,25,24,22,32,26,5,23,33]).

1.1 Review of PAL

We begin our review of PAL with the language we will use throughout.

Definition 1.1 For a set At of atomic sentences and a set Agt of agent symbols with $|\mathsf{Agt}| = \kappa$, the language $\mathcal{L}^{\kappa}_{\mathsf{PAL}}$ is generated by the following grammar:

$$\varphi \ ::= \top \mid \ p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Diamond_a \varphi \mid \langle\varphi\rangle\varphi,$$

where $p \in \mathsf{At}$ and $a \in \mathsf{Agt}$. We define $\Box_a\varphi$ as $\neg\Diamond_a\neg\varphi$ and $[\varphi]\psi$ as $\neg\langle\varphi\rangle\neg\psi$.

- $\mathsf{Sub}(\varphi)$ is the set of subformulas of φ;
- $\mathsf{At}(\varphi) = \mathsf{At} \cap \mathsf{Sub}(\varphi)$;
- $\mathsf{Agt}(\varphi) = \{a \in \mathsf{Agt} \mid \Diamond_a\psi \in \mathsf{Sub}(\varphi) \text{ for some } \psi \in \mathcal{L}^{\kappa}_{\mathsf{PAL}}\}$;
- $\mathsf{An}(\varphi) = \{\chi \in \mathcal{L}^{\kappa}_{\mathsf{PAL}} \mid \langle\chi\rangle\psi \in \mathsf{Sub}(\varphi) \text{ for some } \psi \in \mathcal{L}^{\kappa}_{\mathsf{PAL}}\}$.

We will be primarily concerned with the language $\mathcal{L}^{\omega}_{\mathsf{PAL}}$ with infinitely many agents, which leads to a more elegant treatment than $\mathcal{L}^{n}_{\mathsf{PAL}}$ for some arbitrary finite n. In §6 we will briefly discuss the single-agent and finite-agent cases.

We will consider two interpretations of $\mathcal{L}^{\kappa}_{\mathsf{PAL}}$, one now and one in §2. The standard interpretation uses the following models and truth definition.

Definition 1.2 Models for PAL are tuples of the form $\mathcal{M} = \langle W, \{R_a\}_{a\in\mathsf{Agt}}, V\rangle$, where W is a non-empty set, R_a is a binary relation on W, and $V\colon \mathsf{At} \to \mathcal{P}(W)$.

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dimensional modal logic (see [31]), Carnap's [17] modal system for logical necessity (see [12,30]), an epistemic-doxastic logic proposed by Halpern [21], and the full computation tree logic CTL* (see [29]). Among propositional logics, inquisitive logic [27,18] is a non-uniform example. In some of these cases, the schematically valid fragment—or *substitution core*—turns out to be another known system. For example, the substitution core of Carnap's system **C** is **S5** [30], and the substitution core of inquisitive logic is Medvedev Logic [18, §3.4].

Definition 1.3 Given a PAL model $\mathcal{M} = \langle W, \{R_a\}_{a\in\mathsf{Agt}}, V\rangle$ with $w \in W$, $\varphi, \psi \in \mathcal{L}^{\kappa}_{\mathsf{PAL}}$, and $p \in \mathsf{At}$, we define $\mathcal{M}, w \vDash \varphi$ as follows:

$$\begin{array}{lll}
\mathcal{M}, w \vDash \top; & & \\
\mathcal{M}, w \vDash p & \text{iff} & w \in V(p); \\
\mathcal{M}, w \vDash \neg\varphi & \text{iff} & \mathcal{M}, w \nvDash \varphi; \\
\mathcal{M}, w \vDash \varphi \wedge \psi & \text{iff} & \mathcal{M}, w \vDash \varphi \text{ and } \mathcal{M}, w \vDash \psi; \\
\mathcal{M}, w \vDash \Diamond_a\varphi & \text{iff} & \exists v \in W\colon wR_a v \text{ and } \mathcal{M}, v \vDash \varphi; \\
\mathcal{M}, w \vDash \langle\varphi\rangle\psi & \text{iff} & \mathcal{M}, w \vDash \varphi \text{ and } \mathcal{M}_{|\varphi}, w \vDash \psi,
\end{array}$$

where $\mathcal{M}_{|\varphi} = \langle W_{|\varphi}, \{R_{a_{|\varphi}}\}_{a\in\mathsf{Agt}}, V_{|\varphi}\rangle$ is the model such that

$$\begin{array}{l}
W_{|\varphi} = \{v \in W \mid \mathcal{M}, v \vDash \varphi\}; \\
\forall a \in \mathsf{Agt}\colon R_{a_{|\varphi}} = R_a \cap (W_{|\varphi} \times W_{|\varphi}); \\
\forall p \in \mathsf{At}\colon V_{|\varphi}(p) = V(p) \cap W_{|\varphi}.
\end{array}$$

We use the notation $[\![\varphi]\!]^{\mathcal{M}} = \{v \in W \mid \mathcal{M}, v \vDash \varphi\}$. For a class of models C, $\mathrm{Th}_{\mathcal{L}^{\kappa}_{\mathsf{PAL}}}(\mathsf{C})$ is the set of formulas of $\mathcal{L}^{\kappa}_{\mathsf{PAL}}$ that are valid over C.

For the following statements, we use the standard nomenclature for normal modal logics, e.g., $\mathbf{K}$, $\mathbf{T}$, $\mathbf{S4}$, and $\mathbf{S5}$ for the unimodal logics and $\mathbf{K}_\kappa$, $\mathbf{T}_\kappa$, $\mathbf{S4}_\kappa$, and $\mathbf{S5}_\kappa$ for their multimodal versions with $|\mathsf{Agt}| = \kappa$ (assume κ countable). Let $\mathrm{Mod}(\mathbf{L}_\kappa)$ be the class of all models of the logic $\mathbf{L}_\kappa$, so $\mathrm{Mod}(\mathbf{K}_\kappa)$ is the class of all models, $\mathrm{Mod}(\mathbf{T}_\kappa)$ is the class of models with reflexive R_a relations, etc. We write $\mathtt{L}_\kappa$ for the Hilbert-style deductive system whose set of theorems is $\mathbf{L}_\kappa$, and for any deductive system $\mathtt{S}$, we write $\vdash_{\mathtt{S}} \varphi$ when φ is a theorem of $\mathtt{S}$.

Theorem 1.4 (PAL Axiomatization [28]) Let $\mathtt{PAL}\text{-}\mathtt{L}_\kappa$ be the system extending $\mathtt{L}_\kappa$ with the following rule and axioms: [2]

i.	(replacement)	$\dfrac{\psi \leftrightarrow \chi}{\varphi(\psi/p) \leftrightarrow \varphi(\chi/p)}$
ii.	(atomic reduction)	$\langle\varphi\rangle p \leftrightarrow (\varphi \wedge p)$
iii.	(negation reduction)	$\langle\varphi\rangle\neg\psi \leftrightarrow (\varphi \wedge \neg\langle\varphi\rangle\psi)$
iv.	(conjunction reduction)	$\langle\varphi\rangle(\psi \wedge \chi) \leftrightarrow (\langle\varphi\rangle\psi \wedge \langle\varphi\rangle\chi)$
v.	(diamond reduction)	$\langle\varphi\rangle\Diamond_a\psi \leftrightarrow (\varphi \wedge \Diamond_a\langle\varphi\rangle\psi)$.

For all $\varphi \in \mathcal{L}^{\kappa}_{\mathsf{PAL}}$,

$$\vdash_{\mathtt{PAL}\text{-}\mathtt{K}_\kappa} \varphi \ \text{ iff } \ \varphi \in \mathrm{Th}_{\mathcal{L}^{\kappa}_{\mathsf{PAL}}}(\mathrm{Mod}(\mathbf{K}_\kappa)).$$

The same result holds for $\mathtt{T}_\kappa/\mathbf{T}_\kappa$, $\mathtt{S4}_\kappa/\mathbf{S4}_\kappa$, and $\mathtt{S5}_\kappa/\mathbf{S5}_\kappa$ in place of $\mathtt{K}_\kappa/\mathbf{K}_\kappa$.

[2] If $\mathtt{L}_\kappa$ contains the rule of uniform substitution, then we must either restrict this rule so that in $\mathtt{PAL}\text{-}\mathtt{L}_\kappa$ we can only substitute into formulas φ with $\mathsf{An}(\varphi) = \emptyset$, or remove the rule and add for each axiom of $\mathtt{L}_\kappa$ all substitution instances of that axiom with formulas in $\mathcal{L}^{\kappa}_{\mathsf{PAL}}$. Either way, we take the rules of modus ponens and $\Box_a$-necessitation from $\mathtt{L}_\kappa$ to apply in $\mathtt{PAL}\text{-}\mathtt{L}_\kappa$ to all formulas. Finally, for $\varphi, \psi \in \mathcal{L}^{\kappa}_{\mathsf{PAL}}$ and $p \in \mathsf{At}(\varphi)$, $\varphi(\psi/p)$ is the formula obtained by replacing all occurrences of p in φ by ψ. For alternative axiomatizations of PAL, see [32,33].

Although we have taken diamond operators as primitive for convenience in later sections, typically the PAL axiomatization is stated in terms of box operators by replacing axiom schemas ii - v by the following: $[\varphi]p \leftrightarrow (\varphi \to p)$; $[\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi)$; $[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$; $[\varphi]\Box_a\psi \leftrightarrow (\varphi \to \Box_a[\varphi]\psi)$.

1.2 Introduction to UPAL

As noted above, one of the striking features of PAL is that it is not closed under uniform substitution. In the terminology of Goldblatt [20], PAL is not a *uniform* modal logic. For example, the valid atomic reduction axiom has invalid substitution instances, e.g., $\langle p\rangle\Box_a p \leftrightarrow (p\wedge\Box_a p)$. Given this observation, a distinction arises between PAL and its *substitution core*, defined as follows.

Definition 1.5 A substitution is any $\sigma\colon \mathsf{At} \to \mathcal{L}^{\kappa}_{\mathsf{PAL}}$; and $(\cdot)^{\sigma}\colon \mathcal{L}^{\kappa}_{\mathsf{PAL}} \to \mathcal{L}^{\kappa}_{\mathsf{PAL}}$ is the extension such that $(\varphi)^{\sigma}$ is obtained from φ by replacing each $p \in \mathsf{At}(\varphi)$ by $\sigma(p)$ [14, Def. 1.18]. The *substitution core* of $\mathrm{Th}_{\mathcal{L}^{\kappa}_{\mathsf{PAL}}}(\mathsf{C})$ is the set

$$\{\varphi \in \mathcal{L}^{\kappa}_{\mathsf{PAL}}\colon (\varphi)^{\sigma} \in \mathrm{Th}_{\mathcal{L}^{\kappa}_{\mathsf{PAL}}}(\mathsf{C}) \text{ for all substitutions } \sigma\}.$$

Formulas in the substitution core of $\mathrm{Th}_{\mathcal{L}^{\kappa}_{\mathsf{PAL}}}(\mathsf{C})$ are *schematically valid* over C.

Examples of formulas that are in $\mathrm{Th}_{\mathcal{L}^{\kappa}_{\mathsf{PAL}}}(\mathrm{Mod}(\mathbf{K}_{\kappa}))$ but are not in the substitution core of $\mathrm{Th}_{\mathcal{L}^{\kappa}_{\mathsf{PAL}}}(\mathrm{Mod}(\mathbf{K}_{\kappa}))$ include the following (for $\kappa \geq 1$): [3]

$$\begin{array}{ll}
[p]p & \Box_a p \to [p]\Box_a p \\
[p]\Box_a p & \Box_a p \to [p](p \to \Box_a p) \\
[p](p \to \Box_a p) & \Box_a(p \to q) \to (\langle q\rangle\Box_a r \to \langle p\rangle\Box_a r) \\
[p \wedge \neg\Box_a p]\neg(p \wedge \neg\Box_a p) & (\langle p\rangle\Box_a r \wedge \langle q\rangle\Box_a r) \to \langle p \vee q\rangle\Box_a r.
\end{array}$$

We discuss the epistemic significance of such failures of uniformity in [23]. Burgess [15] explains the logical significance of uniformity as follows:

> The standard aim of logicians at least from Russell onward has been to characterize the class [of] all formulas *all of whose instantiations are true*. Thus, though Russell was a logical atomist, when he endorsed $p \vee \sim p$ as [a] law of logic, he did not mean to be committing himself only to the view that the disjunction of any *logically atomic* statement with its negation is true, but rather to be committing himself to the view that the disjunction of *any statement whatsoever* with its negation is true .... This has remained the standard employment of statement letters ever since, not only among Russell's successors in the classical tradition, but also among the great majority of formal logicians who have thought classical logic to be in need of additions and/or amendments, including C. I. Lewis, the founder of modern modal logic. With such an understanding of the role of statement letters, it is clear that if A is a law of logic, and B is any substitution in A, then B also is a law of logic .... Thus it is that the rule of substitution applies not

[3] The first two principles in the second column are schematically valid over transitive single-agent models, but not over all single-agent models or over transitive multi-agent models.

> only in classical logic, but in standard, Lewis-style modal logics (as well as in intuitionistic, temporal, relevance, quantum, and other logics). None of this is meant to deny that there may be circumstances where it is legitimate to adopt some other understanding of the role of statement letters. If one does so, however, it is indispensable to note the conceptual distinction, and highly advisable to make a notational and terminological distinction. (147-148)

In PAL, an atomic sentence p has the same truth value at any pointed models $\mathcal{M}, w$ and $\mathcal{M}_{|\varphi}, w$, whereas a formula containing a modal operator may have different truth values at $\mathcal{M}, w$ and $\mathcal{M}_{|\varphi}, w$, which is why uniform substitution does not preserve PAL-validity. Hence in PAL an atomic sentence cannot be thought of as a *propositional variable* in the ordinary sense of something that stands in for any proposition. By contrast, if we consider the substitution core of PAL as a logic in its own right, for which semantics will be given in §2, then we can think of the atomic sentences as genuine propositional variables.

The distinction between PAL and its substitution core leads to Question 1 in van Benthem's list of "Open Problems in Logical Dynamics" [3]:

Question 1 ([2,3,4]) Is the substitution core of PAL axiomatizable?

To answer this question, we will introduce a new framework of Uniform Public Announcement Logic (UPAL), which we use to prove the following result.

Theorem 1.6 (Axiomatization of the PAL Substitution Core) Let $\mathtt{UPAL}\text{-}\mathtt{L}_\kappa$ be the system extending $\mathtt{L}_\kappa$ with the following rules and axioms: [4]

1. (uniformity) $\dfrac{\varphi}{(\varphi)^\sigma}$ for any substitution σ
2. (necessitation) $\dfrac{\varphi}{[p]\varphi}$
3. (extensionality) $\dfrac{\varphi \leftrightarrow \psi}{\langle\varphi\rangle p \leftrightarrow \langle\psi\rangle p}$
4. (distribution) $[p](q \to r) \to ([p]q \to [p]r)$
5. (p-seriality) $p \to \langle p\rangle\top$
6. (truthfulness) $\langle p\rangle\top \to p$
7. ($\top$-reflexivity) $p \to \langle\top\rangle p$
8. (functionality) $\langle p\rangle q \to [p]q$
9. (pa-commutativity) $\langle p\rangle\Diamond_a q \to \Diamond_a\langle p\rangle q$
10. (ap-commutativity) $\Diamond_a\langle p\rangle q \to [p]\Diamond_a q$
11. (composition) $\langle p\rangle\langle q\rangle r \leftrightarrow \langle\langle p\rangle q\rangle r$.

[4] As in $\mathtt{PAL}\text{-}\mathtt{L}_\kappa$, in $\mathtt{UPAL}\text{-}\mathtt{L}_\kappa$ we take the rules of modus ponens and $\Box_a$-necessitation from $\mathtt{L}_\kappa$ to apply to all formulas in $\mathcal{L}^\kappa_{\mathsf{PAL}}$.

For all $\varphi \in \mathcal{L}^{\omega}_{\mathsf{PAL}}$,

$$\vdash_{\mathtt{UPAL\text{-}K}_\omega} \varphi \text{ iff } \varphi \text{ is in the substitution core of } \mathrm{Th}_{\mathcal{L}^{\omega}_{\mathsf{PAL}}}(\mathrm{Mod}(\mathbf{K}_\omega)).$$

The same result holds for $\mathtt{T}_\omega/\mathbf{T}_\omega$, $\mathtt{S4}_\omega/\mathbf{S4}_\omega$, and $\mathtt{S5}_\omega/\mathbf{S5}_\omega$ in place of $\mathtt{K}_\omega/\mathbf{K}_\omega$, with only minor adjustments to the proof (see note 5).

Theorem 1.7 (Axiomatization of the PAL Substitution Core cont.)

1. $\vdash_{\mathtt{UPAL\text{-}T}_\omega} \varphi$ iff φ is in the substitution core of $\mathrm{Th}_{\mathcal{L}^{\omega}_{\mathsf{PAL}}}(\mathrm{Mod}(\mathbf{T}_\omega))$;
2. $\vdash_{\mathtt{UPAL\text{-}S4}_\omega} \varphi$ iff φ is in the substitution core of $\mathrm{Th}_{\mathcal{L}^{\omega}_{\mathsf{PAL}}}(\mathrm{Mod}(\mathbf{S4}_\omega))$;
3. $\vdash_{\mathtt{UPAL\text{-}S5}_\omega} \varphi$ iff φ is in the substitution core of $\mathrm{Th}_{\mathcal{L}^{\omega}_{\mathsf{PAL}}}(\mathrm{Mod}(\mathbf{S5}_\omega))$.

Unless the specific base system $\mathtt{L}_\kappa$ matters, we simply write '`UPAL`' and '`PAL`'. It is easy to check that all the axioms of `PAL` except atomic reduction are derivable in `UPAL`, and the rule of replacement is an admissible rule in `UPAL`. Another system with the same theorems as `UPAL`, but presented in a format closer to that of the typical box version of `PAL`, is the following (with $\bot := \neg\top$):

I.	(uniformity)	$\dfrac{\varphi}{(\varphi)^\sigma}$ for any substitution σ
II.	(RE)	$\dfrac{\varphi \leftrightarrow \psi}{[p]\varphi \leftrightarrow [p]\psi}$
III.	([ ]-extensionality)	$\dfrac{\varphi \leftrightarrow \psi}{[\varphi]p \leftrightarrow [\psi]p}$
IV.	(N)	$[p]\top$
V.	($\top$-reflexivity)	$[\top]p \to p$
VI.	($\bot$-reduction)	$[p]\bot \leftrightarrow \neg p$
VII.	($\neg$-reduction)	$[p]\neg q \leftrightarrow (p \to \neg[p]q)$
VIII.	($\wedge$-reduction)	$[p](q \wedge r) \leftrightarrow ([p]q \wedge [p]r)$
IX.	($\Box_a$-reduction)	$[p]\Box_a q \leftrightarrow (p \to \Box_a[p]q)$
X.	([ ]-composition)	$[p][q]r \leftrightarrow [p \wedge [p]q]r.$

We have formulated `UPAL` as in Theorem 1.6 to make clear the correspondence between axioms and the semantic conditions in Definition 2.3 below, as well as to make clear the specific properties used in the steps of our main proof.

2 Semantics for UPAL

In this section we introduce semantics for Uniform Public Announcement Logic, for which the system of `UPAL` is shown to be sound and complete in §3.

Definition 2.1 Models for UPAL are tuples $\mathfrak{M}$ of the form $\langle M, \{\mathcal{R}_a\}_{a\in\mathsf{Agt}}, \{\mathcal{R}_\varphi\}_{\varphi\in\mathcal{L}^{\kappa}_{\mathsf{PAL}}}, \mathcal{V}\rangle$, where M is a non-empty set, $\mathcal{R}_a$ and $\mathcal{R}_\varphi$ are binary relations on M, and $\mathcal{V}\colon \mathsf{At} \to \mathcal{P}(M)$.

Unlike in the PAL truth definition, in the UPAL truth definition we treat $\langle\varphi\rangle$ like any other modal operator.

Definition 2.2 Given a UPAL model $\mathfrak{M} = \langle M, \{\mathcal{R}_a\}_{a\in\mathsf{Agt}}, \{\mathcal{R}_\varphi\}_{\varphi\in\mathcal{L}^\kappa_{\mathsf{PAL}}}, \mathcal{V}\rangle$ with $w \in M$, $\varphi, \psi \in \mathcal{L}^\kappa_{\mathsf{PAL}}$, and $p \in \mathsf{At}$, we define $\mathfrak{M}, w \Vdash \varphi$ as follows:

$$
\begin{array}{lll}
\mathfrak{M}, w \Vdash \top; & & \\
\mathfrak{M}, w \Vdash p & \text{iff} & w \in \mathcal{V}(p); \\
\mathfrak{M}, w \Vdash \neg\varphi & \text{iff} & \mathfrak{M}, w \nVdash \varphi; \\
\mathfrak{M}, w \Vdash \varphi \wedge \psi & \text{iff} & \mathfrak{M}, w \Vdash \varphi \text{ and } \mathfrak{M}, w \Vdash \psi; \\
\mathfrak{M}, w \Vdash \Diamond_a\varphi & \text{iff} & \exists v \in M\colon w\mathcal{R}_a v \text{ and } \mathfrak{M}, v \Vdash \varphi; \\
\mathfrak{M}, w \Vdash \langle\varphi\rangle\psi & \text{iff} & \exists v \in M\colon w\mathcal{R}_\varphi v \text{ and } \mathfrak{M}, v \Vdash \psi.
\end{array}
$$

We use the notation $\|\varphi\|^{\mathfrak{M}} = \{v \in M \mid \mathfrak{M}, v \Vdash \varphi\}$.

Instead of giving the $\langle\varphi\rangle$ operators a special truth clause, we ensure that they behave in a PAL-like way by imposing constraints on the $\mathcal{R}_\varphi$ relations in Definition 2.3 below. Wang and Cao [33] have independently proposed a semantics for PAL in this style, with respect to which they prove that `PAL` is complete. The difference comes in the specific constraints for UPAL vs. PAL.

Definition 2.3 A UPAL model $\mathfrak{M} = \langle M, \{\mathcal{R}_a\}_{a\in\mathsf{Agt}}, \{\mathcal{R}_\varphi\}_{\varphi\in\mathcal{L}^\kappa_{\mathsf{PAL}}}, \mathcal{V}\rangle$ is *legal* iff the following conditions hold for all $\psi, \chi \in \mathcal{L}^\kappa_{\mathsf{PAL}}$, $w, v \in M$, and $a \in \mathsf{Agt}$:

(**extensionality**) if $\|\psi\|^{\mathfrak{M}} = \|\chi\|^{\mathfrak{M}}$, then $\mathcal{R}_\psi = \mathcal{R}_\chi$;

(ψ-**seriality**) if $w \in \|\psi\|^{\mathfrak{M}}$, then $\exists v$: $w\mathcal{R}_\psi v$;

(**truthfulness**) if $w\mathcal{R}_\psi v$, then $w \in \|\psi\|^{\mathfrak{M}}$;

($\top$-**reflexivity**) $w\mathcal{R}_\top w$;

(**functionality**) if $w\mathcal{R}_\psi v$, then for all $u \in M$, $w\mathcal{R}_\psi u$ implies $u = v$;

(ψa-**commutativity**) if $w\mathcal{R}_\psi v$ and $v\mathcal{R}_a u$, then $\exists z$: $w\mathcal{R}_a z$ and $z\mathcal{R}_\psi u$;

($a\psi$-**commutativity**) if $w\mathcal{R}_a v$, $v\mathcal{R}_\psi u$ and $w \in \|\psi\|^{\mathfrak{M}}$, then $\exists z$: $w\mathcal{R}_\psi z$ and $z\mathcal{R}_a u$;

(**composition**) $\mathcal{R}_{\langle\psi\rangle\chi} = \mathcal{R}_\psi \circ \mathcal{R}_\chi$.

In §4, we will also refer to weaker versions of the first and third conditions:

(**extensionality for** φ) if $\psi, \chi \in \mathsf{An}(\varphi) \cup \{\top\}$ and $\|\psi\|^{\mathfrak{M}} = \|\chi\|^{\mathfrak{M}}$, then $\mathcal{R}_\psi = \mathcal{R}_\chi$;

(**truthfulness for** φ) if $\psi \in \mathsf{An}(\varphi) \cup \{\top\}$ and $w\mathcal{R}_\psi v$, then $w \in \|\psi\|^{\mathfrak{M}}$.

It is easy to see that each of the axioms of `UPAL` in Theorem 1.6 corresponds to the condition of the same name written in boldface in Definition 2.3.

3 Completeness of `UPAL`

In this section, we take our first step toward proving Theorem 1.6 by proving:

Theorem 3.1 (Soundness and Completeness) The system of $\texttt{UPAL-K}_\omega$ given in Theorem 1.6 is sound and complete for the class of legal UPAL models.

Soundness is straightforward. To prove completeness, we use the standard canonical model argument.

Definition 3.2 The canonical model $\mathfrak{M}^c = \langle M^c, \{\mathcal{R}^c_a\}_{a \in \mathsf{Agt}}, \{\mathcal{R}^c_\varphi\}_{\varphi \in \mathcal{L}^\kappa_{\mathsf{PAL}}}, \mathcal{V}^c \rangle$ is defined as follows:

1. $M^c = \{\Gamma \mid \Gamma$ is a maximally $\texttt{UPAL-K}_\omega$-consistent set$\}$;
2. $\Gamma \mathcal{R}^c_a \Delta$ iff $\psi \in \Delta$ implies $\Diamond_a \psi \in \Gamma$;
3. $\Gamma \mathcal{R}^c_\varphi \Delta$ iff $\psi \in \Delta$ implies $\langle \varphi \rangle \psi \in \Gamma$;
4. $\mathcal{V}^c(p) = \{\Gamma \in M^c \mid p \in \Gamma\}$.

The following fact, easily shown, will be used in the proof of Lemma 3.5.

Fact 3.3 For all $\Gamma \in M^c$, $\varphi \in \mathcal{L}^\kappa_{\mathsf{PAL}}$, if $\langle \varphi \rangle \top \in \Gamma$, then $\{\psi \mid \langle \varphi \rangle \psi \in \Gamma\} \in M^c$.

The proof of the truth lemma is completely standard [14, §4.2].

Lemma 3.4 (Truth) For all $\Gamma \in M^c$ and $\varphi \in \mathcal{L}^\kappa_{\mathsf{PAL}}$,

$$\mathfrak{M}^c, \Gamma \Vdash \varphi \text{ iff } \varphi \in \Gamma.$$

To complete the proof of Theorem 3.1, we need only check the following.

Lemma 3.5 (Legality) $\mathfrak{M}^c$ is a legal model.

Proof. Suppose $\|\varphi\|^{\mathfrak{M}^c} = \|\psi\|^{\mathfrak{M}^c}$, so by Lemma 3.4 and the properties of maximally consistent sets, $\varphi \leftrightarrow \psi \in \Gamma$ for all $\Gamma \in M^c$. Hence $\vdash_{\texttt{UPAL-K}_\omega} \varphi \leftrightarrow \psi$, for if $\neg(\varphi \leftrightarrow \psi)$ is $\texttt{UPAL-K}_\omega$-consistent, then $\neg(\varphi \leftrightarrow \psi) \in \Delta$ for some $\Delta \in M^c$, contrary to what was just shown. It follows that for any $\alpha \in \mathcal{L}^\kappa_{\mathsf{PAL}}$, $\vdash_{\texttt{UPAL-K}_\omega} \langle \varphi \rangle \alpha \leftrightarrow \langle \psi \rangle \alpha$, given the extensionality and uniformity rules of $\texttt{UPAL-K}_\omega$. Hence if $\Gamma_1 \mathcal{R}^c_\varphi \Gamma_2$, then for all $\alpha \in \Gamma_2$, $\langle \varphi \rangle \alpha \in \Gamma_1$ and $\langle \psi \rangle \alpha \in \Gamma_1$ by the consistency of Γ_1, which means $\Gamma_1 \mathcal{R}^c_\psi \Gamma_2$. The argument in the other direction is the same, whence $\mathcal{R}^c_\varphi = \mathcal{R}^c_\psi$. $\mathfrak{M}^c$ satisfies **extensionality**.

Suppose $\Gamma_1 \mathcal{R}^c_{\langle \varphi \rangle \psi} \Gamma_2$, so for all $\alpha \in \Gamma_2$, $\langle \langle \varphi \rangle \psi \rangle \alpha \in \Gamma_1$. Hence $\langle \varphi \rangle \langle \psi \rangle \alpha \in \Gamma_1$ given the composition axiom and uniformity rule of $\texttt{UPAL-K}_\omega$, so $\langle \varphi \rangle \top \in \Gamma_1$ by normal modal reasoning with the distribution axiom. It follows by Fact 3.3 and Definition 3.2.3 that there is some Σ_1 such that $\Gamma_1 \mathcal{R}_\varphi \Sigma_1$ and $\langle \psi \rangle \alpha \in \Sigma_1$, and by similar reasoning that there is some Σ_2 such that $\Sigma_1 \mathcal{R}_\psi \Sigma_2$ and $\alpha \in \Sigma_2$. Hence $\Gamma_2 \subseteq \Sigma_2$, so $\Gamma_2 = \Sigma_2$ given that Γ_2 is maximal. Therefore, $\mathcal{R}^c_{\langle \varphi \rangle \psi} \subseteq \mathcal{R}^c_\varphi \circ \mathcal{R}^c_\psi$. The argument in the other direction is similar. $\mathfrak{M}^c$ satisfies **composition**.

We leave the other legality conditions to the reader. □

4 Bridging UPAL and PAL

In this section, we show that $\texttt{UPAL}$ axiomatizes the substitution core of PAL. It is easy to check that all of the axioms of $\texttt{UPAL}$ are PAL schematic validities, and all of the rules of $\texttt{UPAL}$ preserve schematic validity, so $\texttt{UPAL}$ derives only PAL schematic validities. To prove that $\texttt{UPAL}$ derives all PAL schematic validities, we show that if φ is not derivable from UPAL, so by Theorem 3.1 there is a legal UPAL model falsifying φ, then there is a substitution τ and a PAL model falsifying $(\varphi)^\tau$, in which case φ is not schematically valid over PAL models.

Proposition 4.1 For any $\varphi \in \mathcal{L}^\omega_{\mathsf{PAL}}$, if there is a legal UPAL model $\mathfrak{M} = \langle M, \{\mathcal{R}_a\}_{a\in\mathsf{Agt}}, \{\mathcal{R}_\psi\}_{\psi\in\mathcal{L}^\omega_{\mathsf{PAL}}}, \mathcal{V}\rangle$ with $w_0 \in M$ such that $\mathfrak{M}, w_0 \nVdash \varphi$, then there is a PAL model $\mathcal{N} = \langle N_0, \{S_a\}_{a\in\mathsf{Agt}}, U\rangle$ with $w_0 \in N_0$ and a substitution τ such that $\mathcal{N}, w_0 \nvDash (\varphi)^\tau$.

Our first step in proving Proposition 4.1 is to show that we can reduce φ to a certain simple form, which will help us in constructing the substitution τ.

Definition 4.2 The set of *simple* formulas is generated by the grammar

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond_a\varphi \mid \langle\varphi\rangle p,$$

where $p \in \mathsf{At}$ and $a \in \mathsf{Agt}$.

Proposition 4.3 For every $\varphi \in \mathcal{L}^\kappa_{\mathsf{PAL}}$, there is a simple formula $\varphi' \in \mathcal{L}^\kappa_{\mathsf{PAL}}$ that is equivalent to φ over legal UPAL models (and all PAL models).

Proof. The proof is similar to the standard PAL reduction argument [9, §7.4], only we do not perform atomic reduction steps, and we use the composition axiom of UPAL to eliminate consecutive occurrences of dynamic operators. □

By Proposition 4.3, given that $\mathfrak{M}$ is legal, we can assume that φ is simple. Before constructing $\mathcal{N}$ and τ, we show that our initial model $\mathfrak{M}$ can be transformed into an intermediate model $\mathfrak{N}$ that satisfies a property (part 2 of Lemma 4.4) that we will take advantage of in our proofs below. We will return to the role of this property in relating UPAL to PAL in Example 5.2 and §6.

For what follows, we need some new notation. First, let

$$\mathcal{R}_{\mathsf{Agt}} = \bigcup_{a\in\mathsf{Agt}} \mathcal{R}_a;$$

$\mathcal{R}^*$ is the reflexive transitive closure of $\mathcal{R}$; and $\mathcal{R}(w) = \{v \in M \mid w\mathcal{R}v\}$.

Lemma 4.4 For any legal model $\mathfrak{M} = \langle M, \{\mathcal{R}_a\}_{a\in\mathsf{Agt}}, \{\mathcal{R}_\varphi\}_{\varphi\in\mathcal{L}^\omega_{\mathsf{PAL}}}, \mathcal{V}\rangle$ with $w_0 \in M$ such that $\mathfrak{M}, w_0 \Vdash \varphi$, there is a model $\mathfrak{N} = \langle N, \{\mathcal{S}_a\}_{a\in\mathsf{Agt}}, \{\mathcal{S}_\varphi\}_{\varphi\in\mathcal{L}^\omega_{\mathsf{PAL}}}, \mathcal{U}\rangle$ with $w_0 \in N$ such that

1. $\mathfrak{N}, w_0 \Vdash \varphi$;
2. if $\alpha, \beta \in \mathsf{An}(\varphi) \cup \{\top\}$ and $\|\alpha\|^{\mathfrak{N}} \neq \|\beta\|^{\mathfrak{N}}$, then

$$\|\alpha\|^{\mathfrak{N}} \cap \mathcal{S}^*_{\mathsf{Agt}}(w_0) \neq \|\beta\|^{\mathfrak{N}} \cap \mathcal{S}^*_{\mathsf{Agt}}(w_0).$$

3. $\mathfrak{N}$ satisfies $\top$**-reflexivity**, **functionality**, **extensionality for** φ and **truthfulness for** φ.

Proof. Consider some $\alpha, \beta \in \mathsf{An}(\varphi) \cup \{\top\}$ such that $\|\alpha\|^{\mathfrak{M}} \neq \|\beta\|^{\mathfrak{M}}$. Hence there is some $v \in M$ such that $\mathfrak{M}, v \nVdash \alpha \leftrightarrow \beta$. Let $\mathfrak{M}'$ be exactly like $\mathfrak{M}$

except that for some $x \notin \mathsf{Agt}(\varphi)$, $w_0 \mathcal{R}'_x v$.[5] Then it is easy to show that for all $\psi \in \mathsf{Sub}(\varphi)$ and $u \in M$,

$$\mathfrak{M}', u \Vdash \psi \text{ iff } \mathfrak{M}, u \Vdash \psi.$$

Hence $\mathfrak{M}', w_0 \Vdash \varphi$ and $\mathfrak{M}', v \nVdash \alpha \leftrightarrow \beta$. Then given $w_0 \mathcal{R}'_x v$, we have

$$\|\alpha\|^{\mathfrak{M}'} \cap \mathcal{R}'^{*}_{\mathsf{Agt}}(w_0) \neq \|\beta\|^{\mathfrak{M}'} \cap \mathcal{R}'^{*}_{\mathsf{Agt}}(w_0).$$

Finally, one can check that $\mathfrak{M}'$ satisfies $\top$-**reflexivity**, **functionality**, **extensionality for** φ and **truthfulness for** φ by the construction. By repeating this procedure, starting now with $\mathfrak{M}'$, for each of the finitely many α and β as described above, one obtains a model $\mathfrak{N}$ as described in Lemma 4.4 □

Obtaining $\mathfrak{N}$ from $\mathfrak{M}$ as in Lemma 4.4, we now define our PAL model $\mathcal{N} = \langle N_0, \{S_a\}_{a \in \mathsf{Agt}}, U\rangle$. Let $N_0 = \mathcal{S}^*_{\mathsf{Agt}}(w_0)$; for some $z \notin \mathsf{Agt}(\varphi)$, let S_z be the universal relation on N_0; and for each $a \in \mathsf{Agt}$ with $a \neq z$, let S_a be the restriction of $\mathcal{S}_a$ to N_0. We will define the valuation U after constructing the substitution τ. The following facts will be used in the proof of Lemma 4.8.

Fact 4.5

1. For all $a \in \mathsf{Agt}$ and $w \in N_0$, $S_a(w) = \mathcal{S}_a(w)$.
2. if $\|\alpha\|^{\mathfrak{N}} \cap N_0 = \|\beta\|^{\mathfrak{N}} \cap N_0$, then for all $u \in N_0$,

$$\mathfrak{N}, u \Vdash \langle\alpha\rangle\chi \text{ iff } \mathfrak{N}, u \Vdash \langle\beta\rangle\chi.$$

Proof. Part 1 is obvious. For part 2, if $\|\alpha\|^{\mathfrak{N}} \cap N_0 = \|\beta\|^{\mathfrak{N}} \cap N_0$, then $\|\alpha\|^{\mathfrak{N}} = \|\beta\|^{\mathfrak{N}}$ by Lemma 4.4.2, so $\mathcal{S}_\alpha = \mathcal{S}_\beta$ by Lemma 4.4.3 (**extensionality for** φ).□

Remark 4.6 There is another way of transforming the UPAL model $\mathfrak{M} = \langle M, \{\mathcal{R}_a\}_{a \in \mathsf{Agt}}, \{\mathcal{R}_\varphi\}_{\varphi \in \mathcal{L}^\omega_{\mathsf{PAL}}}, \mathcal{V}\rangle$ into a PAL model $\mathcal{N}$ sufficient for our purposes. First, let $\mathfrak{N} = \langle N, \{\mathcal{S}_a\}_{a \in \mathsf{Agt}}, \{\mathcal{S}_\varphi\}_{\varphi \in \mathcal{L}^\omega_{\mathsf{PAL}}}, \mathcal{U}\rangle$ be exactly like $\mathfrak{M}$ except that for some $z \notin \mathsf{Agt}(\varphi)$, $\mathcal{S}_z$ is the universal relation on N, and observe that $\mathfrak{N}$ satisfies the conditions of Lemma 4.4. Second, take $\mathcal{N} = \langle N_0, \{S_a\}_{a \in \mathsf{Agt}}, U\rangle$ such that $N_0 = N$, $S_a = \mathcal{S}_a$, and U is defined as below, and observe that Fact 4.5 holds. Then the proof can proceed as below. The difference is that this approach takes the domain of the PAL model to be the entire domain of the UPAL model $\mathfrak{N}$, with S_z as the universal relation on this entire domain, whereas our approach takes the domain of the PAL model to be just that of the "epistemic submodel" *generated by* w_0 *in* $\mathfrak{N}$, $\mathcal{S}^*_{\mathsf{Agt}}(w_0)$, with S_z as the universal relation on this set. We prefer the latter approach because it allows us to work with smaller PAL models when we carry out the construction with concrete examples as in §5.

[5] As noted after Theorem 1.6, we can modify our proof for other models classes. For example, for the class of models with equivalence relations, in this step we can define $\mathcal{R}'_x$ to be the smallest equivalence relation extending $\mathcal{R}_x$ such that $w_0 \mathcal{R}'_x v$. Note that since $\alpha, \beta \in \mathsf{An}(\varphi) \cup \{\top\}$ and $x \notin \mathsf{Agt}(\varphi)$, no matter how we define $\mathcal{R}'_x$, the following claim in the text still holds.

To construct $\tau(p)$ for $p \in \mathsf{At}(\varphi)$, let $B_1, \dots, B_m$ be the sequence of all B_i such that $\langle B_i \rangle p \in \mathsf{Sub}(\varphi)$, and let $B_0 := \top$. For $0 \leq i, j \leq m$, if $\|B_i\|^{\mathfrak{N}} \cap N_0 = \|B_j\|^{\mathfrak{N}} \cap N_0$, delete one of B_i or B_j from the list (but never B_0), until there is no such pair. Call the resulting sequence $A_0, \dots, A_n$, and define

$$s(i) = \{j \mid 0 \leq j \leq n \text{ and } \|A_j\|^{\mathfrak{N}} \cap N_0 \subsetneq \|A_i\|^{\mathfrak{N}} \cap N_0\}.$$

Extend the language with new variables $p_0, \dots, p_n$ and $a_0, \dots, a_n$, and define $\tau(p) = \gamma_0 \wedge \dots \wedge \gamma_n$ such that

$$\gamma_i := (\Box_z a_i \wedge \bigwedge_{j \in s(i)} \neg \Box_z a_j) \to p_i.$$

Having extended the language for each $p \in \mathsf{At}(\varphi)$, define the valuation U for N_0 such that for each $p \in \mathsf{At}(\varphi)$, $U(p) = \mathcal{U}(p) \cap N_0$, and for the new variables:

(a) $U(p_i) = \{w \in N_0 \mid \exists u \colon w \mathcal{S}_{A_i} u \text{ and } u \in \mathcal{U}(p)\}$;

(b) $U(a_i) = \|A_i\|^{\mathfrak{N}} \cap N_0$.

Hence:

(**a**) $[\![p_i]\!]^{\mathcal{N}} = \{w \in N_0 \mid \exists u \colon w \mathcal{S}_{A_i} u \text{ and } u \in \mathcal{U}(p)\}$;

(**b**) $[\![a_i]\!]^{\mathcal{N}} = \|A_i\|^{\mathfrak{N}} \cap N_0$.

Note that it follows from (**a**) and the UPAL truth definition that

(**c**) $[\![p_i]\!]^{\mathcal{N}} = \|\langle A_i \rangle p\|^{\mathfrak{N}} \cap N_0$.

Using these facts, we will show that $\mathfrak{N}, w_0 \nVdash \varphi$ implies $\mathcal{N}, w_0 \nvDash \tau(\varphi)$.

Lemma 4.7 For all $0 \leq i \leq n$,

$$[\![\tau(p)]\!]^{\mathcal{N}_{|a_i}} = [\![p_i]\!]^{\mathcal{N}}.$$

Proof. We first show that for $0 \leq i, j \leq n$, $i \neq j$:

(i) $[\![\gamma_i]\!]^{\mathcal{N}_{|a_i}} = [\![p_i]\!]^{\mathcal{N}_{|a_i}}$;

(ii) $[\![\gamma_j]\!]^{\mathcal{N}_{|a_i}} = [\![a_i]\!]^{\mathcal{N}_{|a_i}}$ $(= N_{0|a_i})$.

For (i), we claim that

$$[\![\Box_z a_i \wedge \bigwedge_{k \in s(i)} \neg \Box_z a_k]\!]^{\mathcal{N}_{|a_i}} = N_{0|a_i}.$$

Since a_i is atomic, $[\![\Box_z a_i]\!]^{\mathcal{N}_{|a_i}} = N_{0|a_i}$. By definition of the s function and (**b**), for all $k \in s(i)$, $[\![a_k]\!]^{\mathcal{N}} \subsetneq [\![a_i]\!]^{\mathcal{N}}$, so $[\![\neg \Box_z a_k]\!]^{\mathcal{N}_{|a_i}} = N_{0|a_i}$. Hence the claimed equation holds, so $[\![\gamma_i]\!]^{\mathcal{N}_{|a_i}} = [\![p_i]\!]^{\mathcal{N}_{|a_i}}$ given the structure of γ_i.

For (ii), we claim that for $j \neq i$,

$$[\![\Box_z a_j \wedge \bigwedge_{k \in s(j)} \neg \Box_z a_k]\!]^{\mathcal{N}_{|a_i}} = \emptyset.$$

By construction of the sequence $A_0, \ldots, A_n$ for p and (**b**), $[\![a_j]\!]^{\mathcal{N}} \neq [\![a_i]\!]^{\mathcal{N}}$. Hence if not $[\![a_i]\!]^{\mathcal{N}} \subsetneq [\![a_j]\!]^{\mathcal{N}}$, then $[\![a_i]\!]^{\mathcal{N}} \not\subseteq [\![a_j]\!]^{\mathcal{N}}$, so $[\![\Box_z a_j]\!]^{\mathcal{N}_{|a_i}} = \emptyset$ because S_z is the universal relation on N_0. If $[\![a_i]\!]^{\mathcal{N}} \subsetneq [\![a_j]\!]^{\mathcal{N}}$, then by (**b**) and the definition of s, $i \in s(j)$; since a_i is atomic, $[\![\neg\Box_z a_i]\!]^{\mathcal{N}_{|a_i}} = \emptyset$. In either case the claimed equation holds, so $[\![\gamma_j]\!]^{\mathcal{N}_{|a_i}} = N_{0|a_i}$ given the structure of γ_j.

Given the construction of τ, (i) and (ii) imply:

$$[\![\tau(p)]\!]^{\mathcal{N}_{|a_i}} = [\![\gamma_i]\!]^{\mathcal{N}_{|a_i}} \cap \bigcap_{j\neq i} [\![\gamma_j]\!]^{\mathcal{N}_{|a_i}} = [\![p_i]\!]^{\mathcal{N}_{|a_i}} \cap [\![a_i]\!]^{\mathcal{N}_{|a_i}} = [\![p_i]\!]^{\mathcal{N}},$$

where the last equality holds because $[\![p_i]\!]^{\mathcal{N}} \subseteq [\![a_i]\!]^{\mathcal{N}}$, which follows from (**a**), (**b**), and the fact that $\mathfrak{N}$ satisfies **truthfulness for** φ. $\Box$

We now establish the connection between the UPAL model $\mathfrak{N}$ on the one hand and the PAL model $\mathcal{N}$ and substitution τ on the other.

Lemma 4.8 For all simple subformulas χ of φ,

$$[\![(\chi)^\tau]\!]^{\mathcal{N}} = \|\chi\|^{\mathfrak{N}} \cap N_0.$$

Proof. By induction on χ. For the base case, we must show $[\![(p)^\tau]\!]^{\mathcal{N}} = \|p\|^{\mathfrak{N}} \cap N_0$ for $p \in \mathsf{At}(\varphi)$. By construction of the sequence $A_0, \ldots, A_n$ for p, $A_0 = \top$, so $\|A_0\|^{\mathfrak{N}} \cap N_0 = N_0$. Then by (**b**), $[\![a_0]\!]^{\mathcal{N}} = N_0$, and hence

$$\begin{aligned}
[\![(p)^\tau]\!]^{\mathcal{N}} &= [\![(p)^\tau]\!]^{\mathcal{N}_{|a_0}} \\
&= [\![p_0]\!]^{\mathcal{N}} && \text{by Lemma 4.7} \\
&= \{w \in N_0 \mid \exists u : w\mathcal{S}_{A_0}u \text{ and } u \in \mathcal{U}(p)\} && \text{by } (\mathbf{a}) \\
&= \{w \in N_0 \mid w \in \mathcal{U}(p)\} && \text{by } \top\textbf{-reflexivity} \\
& && \text{and } \textbf{functionality} \\
&= \|p\|^{\mathfrak{N}} \cap N_0.
\end{aligned}$$

The boolean cases are straightforward. Next, we must show $[\![(\Box_a\varphi)^\tau]\!]^{\mathcal{N}} = \|\Box_a\varphi\|^{\mathfrak{N}} \cap N_0$. For the inductive hypothesis, $[\![(\varphi)^\tau]\!]^{\mathcal{N}} = \|\varphi\|^{\mathfrak{N}} \cap N_0$, so

$$\begin{aligned}
[\![(\Box_a\varphi)^\tau]\!]^{\mathcal{N}} &= [\![\Box_a(\varphi)^\tau]\!]^{\mathcal{N}} \\
&= \{w \in N_0 \mid\ S_a(w) \subseteq [\![(\varphi)^\tau]\!]^{\mathcal{N}}\} \\
&= \{w \in N_0 \mid\ S_a(w) \subseteq \|\varphi\|^{\mathfrak{N}} \cap N_0\} \\
&= \{w \in N_0 \mid\ S_a(w) \subseteq \|\varphi\|^{\mathfrak{N}}\} && \text{given } S_a \subseteq N_0 \times N_0 \\
&= \{w \in N_0 \mid\ \mathcal{S}_a(w) \subseteq \|\varphi\|^{\mathfrak{N}}\} && \text{by Fact 4.5.1} \\
&= \|\Box_a\varphi\|^{\mathfrak{N}} \cap N_0.
\end{aligned}$$

Finally, we must show $[\![(\langle B_i\rangle p)^\tau]\!]^{\mathcal{N}} = \|\langle B_i\rangle p\|^{\mathfrak{N}} \cap N_0$. For the inductive hypothesis, $[\![(B_i)^\tau]\!]^{\mathcal{N}} = \|B_i\|^{\mathfrak{N}} \cap N_0$. By construction of the sequence $A_0, \ldots, A_n$ for $p \in \mathsf{At}(\varphi)$, there is some A_j such that

$$(\star) \quad \|B_i\|^{\mathfrak{N}} \cap N_0 = \|A_j\|^{\mathfrak{N}} \cap N_0.$$

Therefore,

$$\begin{aligned}
[\![(B_i)^\tau]\!]^{\mathcal{N}} &= \|A_j\|^{\mathfrak{N}} \cap N_0 \\
&= [\![a_j]\!]^{\mathcal{N}} && \text{by } (\mathbf{b}),
\end{aligned}$$

and hence

$$\begin{aligned}
[\![(\langle B_i\rangle p)^\tau]\!]^{\mathcal{N}} &= [\![\langle (B_i)^\tau\rangle (p)^\tau]\!]^{\mathcal{N}} \\
&= [\![\langle a_j\rangle (p)^\tau]\!]^{\mathcal{N}} \\
&= [\![(p)^\tau]\!]^{\mathcal{N}|a_j} \\
&= [\![p_j]\!]^{\mathcal{N}} && \text{by Lemma 4.7} \\
&= \|\langle A_j\rangle p\|^{\mathfrak{N}} \cap N_0 && \text{by } (\mathbf{c}) \\
&= \|\langle B_i\rangle p\|^{\mathfrak{N}} \cap N_0 && \text{given } (\star) \text{ and Fact 4.5.2.}
\end{aligned}$$

The proof by induction is complete. □

With the following fact, we complete the proof of Proposition 4.1.

Fact 4.9 $\mathcal{N}, w_0 \nvDash (\varphi)^\tau$.

Proof. Immediate from Lemma 4.8 given $\mathfrak{N}, w_0 \nVdash \varphi$. □

5 Examples

In this section, we work out two examples illustrating how the techniques of §4 allow us to find, for any formula φ that is valid but not schematically valid in PAL, a PAL model that falsifies a substitution instance of φ. The proof in §4 shows that all we need to do is find a legal UPAL model falsifying φ. However, since legal UPAL models are generally large, we would like to instead find a small UPAL model falsifying φ, from which we can read off a PAL model that falsifies a substitution instance of φ. In fact, we can always do so provided that the model satisfies a weaker condition than legality. For a given $\varphi \in \mathcal{L}^{\kappa}_{\mathsf{PAL}}$, we say that a UPAL model $\mathfrak{M}$ is *φ-legal* iff it satisfies all of the legality conditions of Definition 2.3 when we replace **ψ-seriality** with:

(**ψ-seriality for** φ) if $\psi \in \mathsf{An}(\varphi) \cup \{\top\}$ and $w \in \|\psi\|^{\mathfrak{M}}$, then $\exists v$: $w\mathcal{R}_\psi v$.

Hence in a φ-legal model, we can let all of the infinitely many $\mathcal{R}_\psi$ relations irrelevant to φ be empty, which makes constructing φ-legal models easier. With this new notion, we can state a simple method for finding a PAL model that falsifies a substitution instance of the non-schematically valid φ:

Step 1. Transform φ into an equivalent simple formula φ'.

Step 2. Find a φ'-legal pointed UPAL model $\mathfrak{M}, w_0$ such that $\mathfrak{M}, w_0 \nVdash \varphi'$.

Step 3. Obtain $\mathcal{N}$ and τ from $\mathfrak{M}, w_0$ as in §4 so that $\mathcal{N}, w_0 \nvDash (\varphi')^\tau$.

Since $\varphi \leftrightarrow \varphi'$ is schematically valid in PAL, we have $\mathcal{N}, w \nvDash (\varphi)^\tau$, as desired. The key to this method is that the construction in §4 also establishes the following variant of Proposition 4.1:

Proposition 5.1 For any *simple* $\varphi \in \mathcal{L}^{\omega}_{\mathsf{PAL}}$, if there is a *$\varphi$-legal* UPAL model $\mathfrak{M} = \langle M, \{\mathcal{R}_a\}_{a\in\mathsf{Agt}}, \{\mathcal{R}_\psi\}_{\psi\in\mathcal{L}^{\omega}_{\mathsf{PAL}}}, \mathcal{V}\rangle$ with $w_0 \in M$ such that $\mathfrak{M}, w_0 \nVdash \varphi$, then there is a PAL model $\mathcal{N} = \langle N_0, \{S_a\}_{a\in\mathsf{Agt}}, U\rangle$ with $w_0 \in N_0$ and a substitution τ such that $\mathcal{N}, w_0 \nvDash (\varphi)^\tau$.

This proposition holds because if φ is already simple, then the only properties of $\mathfrak{M}$ used in the proof of Fact 4.9 are $\top$-**reflexivity**, **functionality**, **extensionality for** φ and **truthfulness for** φ, which are part of φ-legality.

Finally, if φ does not contain any occurrence of a dynamic operator in the scope of any other, then we can simply skip Step 1 and do Steps 2 and 3 for φ itself. One can check that the construction in §4 works not only with a simple formula, but more generally with any formula with the scope restriction.

Example 5.2 Consider the PAL-valid formula $\varphi := [p]p$, which is already simple. Let us try to falsify φ in a φ-legal UPAL model. The obvious first try is $\mathfrak{M}$ in Fig. 1, which is indeed a φ-legal UPAL model, in which all $\mathcal{R}_a$ relations are empty. (We simplify the diagrams by omitting all reflexive $\mathcal{R}_\top$ loops.) However, $\mathfrak{M}$ has an un-PAL-like property: although $\|\top\|^{\mathfrak{M}} \cap \mathcal{R}^*_{\mathsf{Agt}}(w_0) = \|p\|^{\mathfrak{M}} \cap \mathcal{R}^*_{\mathsf{Agt}}(w_0)$, we have $w_0 \mathcal{R}_\top w_0$ but not $w_0 \mathcal{R}_p w_0$. (See §6 for why this is un-PAL-like.) To eliminate this property, we modify $\mathfrak{M}$ to $\mathfrak{N} = \langle N, \{\mathcal{S}_a\}_{a\in\mathsf{Agt}}, \{\mathcal{S}_\psi\}_{\psi\in\mathcal{L}^\omega_{\mathsf{PAL}}}, \mathcal{U}\rangle$ in Fig. 1 as in Lemma 4.4.[6] Next, following the procedure in §4, we obtain the PAL model $\mathcal{N} = \langle N_0, \{S_a\}_{a\in\mathsf{Agt}}, U\rangle$ in Fig. 1 and the substitution τ given below.

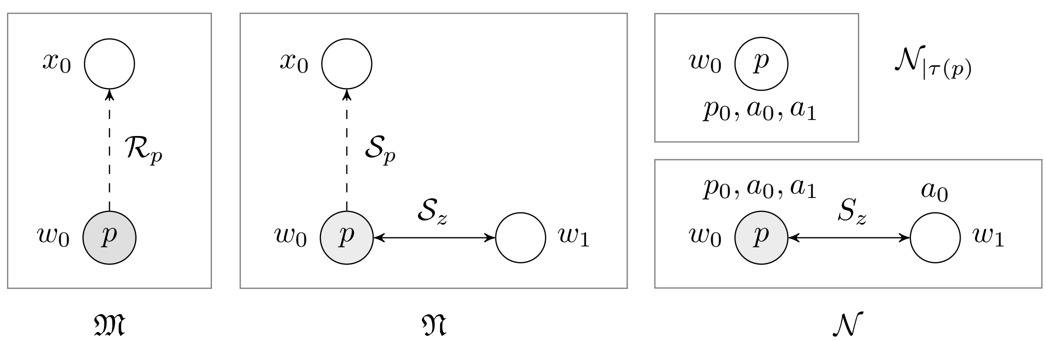

Fig. 1. UPAL and PAL Models for Example 5.2

Where $A_0 := \top$, $A_1 := p$, and a_0, a_1, p_0, and p_1 are the new atoms, we define the valuation U in $\mathcal{N}$ such that:

$U(a_0) = \|A_0\|^{\mathfrak{N}} \cap N_0 = \{w_0, w_1\}$;

$U(a_1) = \|A_1\|^{\mathfrak{N}} \cap N_0 = \{w_0\}$;

$U(p_0) = \{w \in N_0 \mid \exists u\colon w\mathcal{S}_{A_0}u \text{ and } u \in \mathcal{U}(p)\} = \{w_0\}$;

$U(p_1) = \{w \in N_0 \mid \exists u\colon w\mathcal{S}_{A_1}u \text{ and } u \in \mathcal{U}(p)\} = \emptyset$.

Defining the function s such that

$$s(i) = \{j \mid 0 \leq j \leq n \text{ and } \|A_j\|^{\mathfrak{N}} \cap N_0 \subsetneq \|A_i\|^{\mathfrak{N}} \cap N_0\},$$

[6] In fact, the construction of Lemma 4.4 would connect w_0 to x_0 by $\mathcal{R}_z$, but note that we can always connect w_0 to a new point falsifying $\alpha \leftrightarrow \beta$ (in this case, $\top \leftrightarrow p$) instead.

we have $s(0) = \{1\}$ and $s(1) = \emptyset$. Defining $\tau(p) = \gamma_0 \wedge \cdots \wedge \gamma_n$ such that

$$\gamma_i := (\Box_z a_i \wedge \bigwedge_{j \in s(i)} \neg \Box_z a_j) \to p_i,$$

we have

$$\tau(p) = ((\Box_z a_0 \wedge \neg \Box_z a_1) \to p_0) \wedge (\Box_z a_1 \to p_1).$$

Observe:

$[\![(\Box_z a_0 \wedge \neg \Box_z a_1) \to p_0]\!]^{\mathcal{N}} = \{w_0\}$;
$[\![\Box_z a_1 \to p_1]\!]^{\mathcal{N}} = \{w_0, w_1\}$;
$[\![\tau(p)]\!]^{\mathcal{N}} = \{w_0\}$.

Hence $\mathcal{N}_{|\tau(p)}$ is the model displayed in the upper-right in Fig. 1. Observe:

$[\![(\Box_z a_0 \wedge \neg \Box_z a_1) \to p_0]\!]^{\mathcal{N}_{|\tau(p)}} = \{w_0\}$;
$[\![\Box_z a_1 \to p_1]\!]^{\mathcal{N}_{|\tau(p)}} = \emptyset$;
$[\![\tau(p)]\!]^{\mathcal{N}_{|\tau(p)}} = \emptyset$.

Hence $\mathcal{N}, w_0 \nvDash ([p]p)^\tau$, so our starting formula φ is not schematically valid over PAL models.

Example 5.3 Consider the PAL-valid formula $\varphi := [p \wedge \neg \Box_b p]\neg(p \wedge \neg \Box_b p)$.[7] Let us try to falsify φ in a φ-legal UPAL model. The obvious first try is the model $\mathfrak{A}$ in Fig. 2. However, $\mathfrak{A}$ is not φ-legal, since it violates ψb-**commutativity** for $\psi := p \wedge \neg \Box_b p$. By modifying $\mathfrak{A}$ to $\mathfrak{N} = \langle N, \{\mathcal{S}_a\}_{a \in \mathsf{Agt}}, \{\mathcal{S}_\psi\}_{\psi \in \mathcal{L}^{\omega}_{\mathsf{PAL}}}, \mathcal{U}\rangle$ in Fig. 2, we obtain a φ-legal UPAL model with $\mathfrak{N}, w_0 \nVdash \varphi$. (In this case, the transformation of Lemma 4.4 is uncecessary, since the condition of Lemma 4.4.2 is already satisfied by $\mathfrak{N}$.) Following the procedure of §4, we obtain the PAL model $\mathcal{N} = \langle N_0, \{S_a\}_{a \in \mathsf{Agt}}, U\rangle$ in Fig. 3 and the substitution τ given below.

Where $A_0 := \top$, $A_1 := p \wedge \neg \Box_b p$, and a_0, a_1, p_0, and p_1 are the new atoms, we define the valuation U in $\mathcal{N}$ such that:

$U(a_0) = \|A_0\|^{\mathfrak{N}} \cap N_0 = \{w_0, w_1, w_2\}$;
$U(a_1) = \|A_1\|^{\mathfrak{N}} \cap N_0 = \{w_0, w_1\}$;
$U(p_0) = \{w \in N_0 \mid \exists u\colon w\mathcal{S}_{A_0}u \text{ and } u \in \mathcal{U}(p)\} = \{w_0, w_1\}$;
$U(p_1) = \{w \in N_0 \mid \exists u\colon w\mathcal{S}_{A_1}u \text{ and } u \in \mathcal{U}(p)\} = \{w_0\}$.

Defining the function s as before, we have $s(0) = \{1\}$ and $s(1) = \emptyset$. Since this is the same s as in Example 5.2, the substitution is also the same:

$$\tau(p) = ((\Box_z a_0 \wedge \neg \Box_z a_1) \to p_0) \wedge (\Box_z a_1 \to p_1).$$

[7] Here we could transform $\varphi := [p \wedge \neg \Box_b p]\neg(p \wedge \neg \Box_b p)$ into the simple

$$\varphi' := (p \wedge \neg \Box_b p) \to \neg([p \wedge \neg \Box_b p]p \wedge ((p \wedge \neg \Box_b p) \to \neg((p \wedge \neg \Box_b p) \to \Box_b [p \wedge \neg \Box_b p]p))),$$

but as noted before Example 5.2, if φ does not contain any occurrence of a dynamic operator in the scope of any other, then we can skip Step 1 and do Steps 2 and 3 for φ itself.

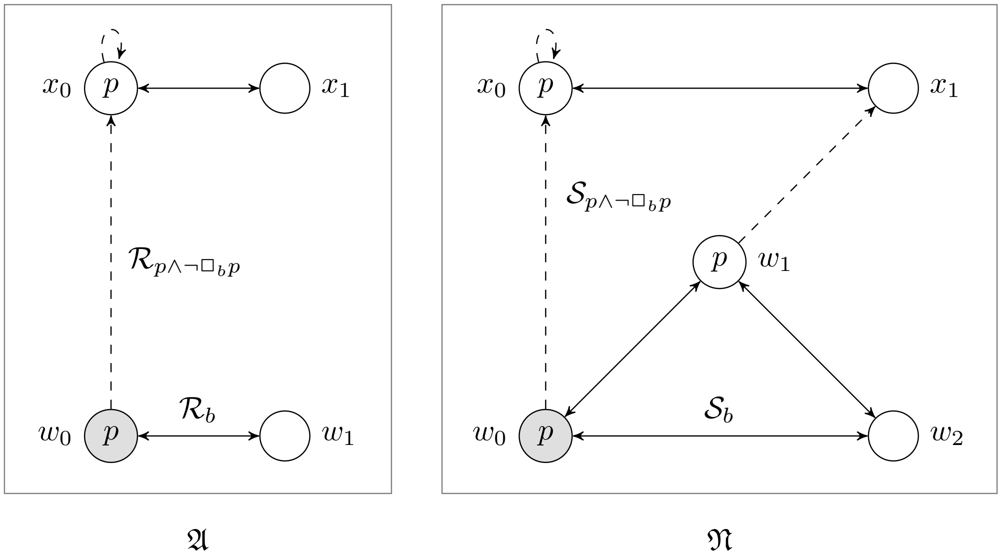

Fig. 2. UPAL Models for Example 5.3

Note that since the construction of $\mathcal{N}$ from $\mathfrak{N}$ is such that $S_z = S_b$, we can simply take $\Box_z$ to be $\Box_b$ in $\tau(p)$, so that $\mathsf{Agt}((\varphi)^\tau) = \mathsf{Agt}(\varphi) = \{b\}$.

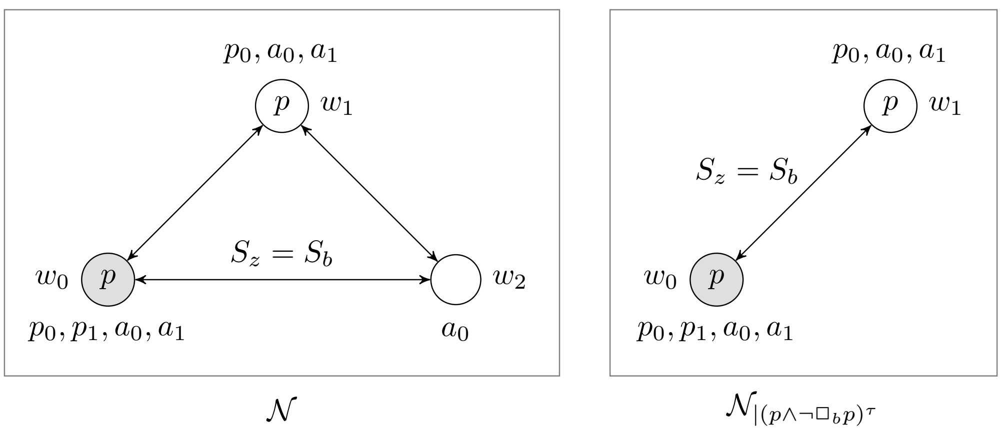

Fig. 3. PAL Models for Example 5.3

Observe:

$[\![(\Box_z a_0 \wedge \neg\Box_z a_1) \to p_0]\!]^{\mathcal{N}} = \{w_0, w_1\}$;
$[\![\Box_z a_1 \to p_1]\!]^{\mathcal{N}} = \{w_0, w_1, w_2\}$;
$[\![\tau(p)]\!]^{\mathcal{N}} = \{w_0, w_1\}$;
$[\![\tau(p) \wedge \neg\Box_b \tau(p)]\!]^{\mathcal{N}} = \{w_0, w_1\}$.

Hence $\mathcal{N}_{|(p\wedge\neg\Box_b p)^\tau}$ is the model displayed on the right in Fig. 3. Observe:

$[\![(\Box_z a_0 \wedge \neg\Box_z a_1) \to p_0]\!]^{\mathcal{N}_{|(p\wedge\neg\Box_b p)^\tau}} = \{w_0, w_1\}$;
$[\![\Box_z a_1 \to p_1]\!]^{\mathcal{N}_{|(p\wedge\neg\Box_b p)^\tau}} = \{w_0\}$;

$$[\![\tau(p)]\!]^{\mathcal{N}|(p\wedge\neg\Box_b p)^\tau} = \{w_0\};$$
$$[\![\tau(p)\wedge\neg\Box_b\tau(p)]\!]^{\mathcal{N}|(p\wedge\neg\Box_b p)^\tau} = \{w_0\}.$$

Hence $\mathcal{N}, w_0 \nvDash ([p\wedge\neg\Box_b p]\neg(p\wedge\neg\Box_b p))^\tau$, so our starting formula φ is not schematically valid over PAL models.

We invite the reader to work out other examples using UPAL, starting from the other valid but not schematically valid PAL principles mentioned in §1.2.

6 Discussion

In this paper, we have shown that `UPAL` axiomatizes the substitution core of PAL with infinitely many agents. In this final section, we briefly discuss the axiomatization question for the single-agent and finite-agent cases. For a given language and class of models, the key question is how close we can come to expressing that two formulas are co-extensional in the epistemic submodel generated by the current point. For example, this condition is expressed by the formula $\Box_a^+(\varphi\leftrightarrow\psi)$ (where $\Box_a^+\alpha := \alpha\wedge\Box_a\alpha$) in *single*-agent PAL over *transitive* models. In this case, we get a new schematic validity in PAL:

(inner extensionality) $\quad \Box_a^+(\varphi\leftrightarrow\psi)\to(\langle\varphi\rangle\alpha\leftrightarrow\langle\psi\rangle\alpha).$

The corresponding legality condition for UPAL models is:

(**inner extensionality**) if $\|\varphi\|^{\mathfrak{M}}\cap\mathcal{R}_a(w) = \|\psi\|^{\mathfrak{M}}\cap\mathcal{R}_a(w)$, then $w\mathcal{R}_\varphi v$ iff $w\mathcal{R}_\psi v$,

which does not follow from any of the other legality conditions.

For multiple agents, we cannot in general express the co-extensionality of two formulas in the epistemic submodel generated by the current point; however, if we allow our models to be *non-serial*, then we do get related schematic validities for the single and finite-agent cases that are not derivable in `UPAL-K`$_n$ (where the antecedent can be written using $\Box_a$ operators and $\bot$):[8]

(FPE) "all $\mathcal{R}_{\mathsf{Agt}}$-paths from the current point are of length $\leq n$" $\to$
$(E^n(\varphi\leftrightarrow\psi)\to(\langle\varphi\rangle\alpha\leftrightarrow\langle\psi\rangle\alpha))$,

where
$$E^0\alpha := \alpha\wedge\bigwedge_{a\in\mathsf{Agt}}\Box_a\alpha \text{ and } E^n\alpha := \alpha\wedge E^0E^{n-1}\alpha.$$

The corresponding legality condition for UPAL is:

(**FPE**) if $\mathcal{R}^*_{\mathsf{Agt}}(w)$ is path-finite and $\|\varphi\|^{\mathfrak{M}}\cap\mathcal{R}^*_{\mathsf{Agt}}(w) = \|\psi\|^{\mathfrak{M}}\cap\mathcal{R}^*_{\mathsf{Agt}}(w)$, then $w\mathcal{R}_\varphi v$ iff $w\mathcal{R}_\psi v$,

[8] The (FPE) axioms are also schematically valid over serial models, because the antecedent is always false, but then they are also derivable using the seriality axiom $\Diamond_a\top$.

where $\mathcal{R}^*_{\mathsf{Agt}}(w)$ is path-finite just in case every $\mathcal{R}_{\mathsf{Agt}}$-path from w ends in a dead-end point in finitely many steps. This shows why the axiomatization of the substitution core of $\mathsf{PAL}\text{-}\mathbf{K}_\omega$ is more elegant than that of $\mathsf{PAL}\text{-}\mathbf{K}_n$: with infinitely many agents we cannot express the "everybody knows" modality E, so we do not need to add to UPAL the infinitely many FPE axioms.

Finally, if we consider PAL with the standard *common knowledge* operator C, then we can express co-extensionality in the generated epistemic submodel using the formula $C(\varphi \leftrightarrow \psi)$, in which case we get the new schematic validity

(common extensionality) $\quad C(\varphi \leftrightarrow \psi) \rightarrow (\langle\varphi\rangle\alpha \leftrightarrow \langle\psi\rangle\alpha).$

The corresponding legality condition in UPAL is:

(**common extensionality**) $\quad$ if $\|\varphi\|^{\mathfrak{M}} \cap \mathcal{R}^*_{\mathsf{Agt}}(w) = \|\psi\|^{\mathfrak{M}} \cap \mathcal{R}^*_{\mathsf{Agt}}(w)$, then $w\mathcal{R}_\varphi v$ iff $w\mathcal{R}_\psi v$.

We leave it to future work to give analyses for the above languages analogous to the analysis we have given here for $\mathcal{L}^{\omega}_{\mathsf{PAL}}$. A natural next step is to axiomatize the substitution core of the system of PAL-RC [6] with relativized common knowledge. Relativized common knowledge $C(\varphi, \psi)$ is interpreted in UPAL models exactly as in PAL models. We conjecture that `UPAL` together with the relativized common knowledge reduction axiom $\langle p\rangle C(q, r) \leftrightarrow C(\langle p\rangle q, \langle p\rangle r)$, the common extensionality axiom above, and the appropriate base logic (see [6]) axiomatizes the substitution core of PAL-RC with finitely or infinitely many agents over any of the model classes we have discussed. Indeed, it can be shown using arguments similar to those of §4 that the set of formulas in the language $\mathcal{L}^{\kappa}_{\mathsf{PAL\text{-}RC}}$ that are valid over legal UPAL models with **common extensionality** is exactly the substitution core of PAL-RC. Hence it only remains to prove that the extended system just described—call it `UPAL-RC`—is sound and complete for this model class. Such a proof requires a finite canonical model construction to deal with common knowledge, and we cannot go into the details here.

Another natural step is to attempt to apply the strategies of this paper to axiomatize the substitution cores of other dynamics epistemic logics, including the full system of DEL [4, Ch. 4]. One may imagine a general program of "uniformizing" dynamic epistemic logics, of which UPAL is only the beginning.

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