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A Tensor Product Operation for Higher Representations

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by

Matthew Ian McMillan

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ABSTRACT OF THE DISSERTATION

A Tensor Product Operation for Higher Representations

by

Matthew Ian McMillan

Doctor of Philosophy in Mathematics

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Professor Raphaël Alexis Rouquier, Chair

We construct an explicit abelian model for the tensor 2-product of 2-representations of $\mathfrak{sl}_2$, specifically the product of a simple 2-representation $\mathcal{L}(1)$ with a given abelian 2-representation $\mathcal{V}$. Both are taken from the 2-category of algebras, and $\mathcal{V}$ is assumed to satisfy two further hypotheses.

The existence of an abelian model like this one, or a generalization of it, was conjectured by Rouquier in 2008.

We study the output of our construction in detail in the case $\mathcal{V} = \mathcal{L}(1)$, and we show that the 2-representation it determines recovers the expected structure of a categorification that is already known for that case.

We form the product construction first for 2-representations of the positive half $\mathcal{U}^+$ (a monoidal category) of the 2-category associated to the Lie algebra $\mathfrak{sl}_2$. In a subsequent chapter we show that the same construction gives a 2-representation of the full 2-category $\mathcal{U}$ when the inputs are also 2-representations of the full 2-category $\mathcal{U}$.

The dissertation of Matthew Ian McMillan is approved.

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CHAPTER 1

Introduction

This dissertation consists in the proof of a single theorem. The theorem establishes the existence of an abelian 2-representation inside the derived 2-representation that is naturally associated to a pair of abelian 2-representations of $\mathfrak{sl}_2$. This 2-representation is a kind of tensor product of the pair of 2-representations, so we call it a 'tensor 2-product'.

We only consider the case where one member of the pair is a certain simple 2-representation called $\mathcal{L}(1)$, and the other member $\mathcal{V}$ is an abelian 2-representation satisfying two additional hypotheses. We do not know how to define the general case. Both 2-representations are given as categories of modules over algebras. The theorem is proved by giving concrete formulas for the structure it postulates, and the formulas are of considerable importance for the potential applications we envision.

The theorem is a major first step toward a fully general construction of an abelian tensor 2-product of 2-representations of Kac-Moody algebras. The existence of an abelian construction (in full generality) was conjectured by Rouquier in 2008, shortly after the modern concept of 2-representation was first defined.

1.1 Categorical representations of Lie algebras

1.1.1 Concepts

Representations of the Lie algebra $\mathfrak{sl}_2$

The enveloping algebra U of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is generated by elements e and f, with the notation $h = [e, f]$. A finite dimensional representation of $\mathfrak{sl}_2$, i.e. a U-module V, may be decomposed into eigenspaces V_λ of h (the 'weight spaces') for $\lambda \in \mathbb{Z}$, so $V = \bigoplus_\lambda V_\lambda$, and the relations $[h, e] = 2e$, $[h, f] = -2f$ imply that e and f restrict to linear maps:

$$e : V_\lambda \rightleftarrows V_{\lambda+2} : f,$$

and from $[e, f] = h$ we have a relation between linear maps on V_λ:

$$e \cdot f - f \cdot e \,|_{V_\lambda} = \lambda \cdot \mathrm{Id}_{V_\lambda}. \tag{1.1.1}$$

Promotion to categorical structures

In a categorical representation, the role of the collection of weight spaces V_λ is played by a collection of additive 'weight categories' $\mathcal{V}_\lambda$, and the role of an element $v \in V_\lambda$ is played by an object $M \in \mathcal{V}_\lambda$. The linear maps e, f are replaced by a pair of linear functors given for each λ:

$$E_\lambda : \mathcal{V}_\lambda \rightleftarrows \mathcal{V}_{\lambda+2} : F_{\lambda+2}.$$

It is useful to package these functors as the restrictions of a single pair of endofunctors E and F of $\mathcal{V} = \bigoplus_\lambda \mathcal{V}_\lambda$. Now, we need a categorical version of the formula (1.1.1) that is induced by the equation $[e, f] = h$.

At least when $\lambda \geqslant 0$, it is natural to replace $\lambda \cdot \mathrm{Id}_{V_\lambda}$ by the functor:

$$\mathrm{Id}_{\mathcal{V}_\lambda}^{\oplus\lambda} : \mathcal{V}_\lambda \to \mathcal{V}_\lambda,$$
$$M \mapsto M^{\oplus\lambda}.$$

Then the formula (1.1.1) can be expressed in the categorical theory by an isomorphism of functors $\rho_\lambda : E_{\lambda-2} \circ F_\lambda \xrightarrow{\sim} F_{\lambda+2} \circ E_\lambda \oplus \mathrm{Id}_{\mathcal{V}_\lambda}^{\oplus\lambda}$. When $\lambda < 0$ we can use the same $\mathrm{Id}_{\mathcal{V}_\lambda}^{\oplus\lambda}$ on the other side and require an isomorphism $\rho_\lambda : E_{\lambda-2} \circ F_\lambda \oplus \mathrm{Id}_{\mathcal{V}_\lambda}^{\oplus-\lambda} \xrightarrow{\sim} F_{\lambda+2} \circ E_\lambda$.

In the general philosophy of categorification, elements are promoted to objects, and one supplies to every pair of objects the additional data of a Hom space such that the Hom spaces jointly have the structure implied by compositionality. The notion of equivalence of objects in a category is more complex than the notion of identity of elements in a vector space, involving as it does the structure of these Hom spaces: a single object may have a variety of isomorphisms to itself, which express its symmetries, and therefore also a variety of isomorphisms to another object. A consequence is that when a categorical structure is defined by generators and relations, it is possible to give a 'weak' or 'naive' definition where the relations simply postulate the existence of isomorphisms between the functors, and a 'strong' or 'genuine' version where the relations stipulate that a certain morphism of functors, which they name, is an isomorphism of functors. Usually the strong definition is preferable. So, to complete the idea of categorical representation of $\mathfrak{sl}_2$, we need names and packaging for many morphisms:

$$\begin{cases} \rho_\lambda : E_{\lambda-2} \circ F_\lambda \to F_{\lambda+2} \circ E_\lambda \oplus \mathrm{Id}_{\mathcal{V}_\lambda}^{\oplus\lambda} & \lambda \geqslant 0 \\ \rho_\lambda : E_{\lambda-2} \circ F_\lambda \oplus \mathrm{Id}_{\mathcal{V}_\lambda}^{\oplus-\lambda} \to F_{\lambda+2} \circ E_\lambda & \lambda < 0. \end{cases}$$

The data of a 2-representation will include these morphisms, and the definition of 2-representation will require that they are isomorphisms.

At this juncture the idea of 'categorifying' the U-module V ceases to be procedural.

The theory of higher representations began in earnest with the discovery of a symmetry describable among these morphisms of functors that may be justified from several points of view. The symmetry is that of a certain Hecke-type algebra. We prepare the way for this algebra next.

Adjunction

Observe first that the data of the morphisms of functors consists of various morphisms $EF \to \mathrm{Id}$, $\mathrm{Id} \to FE$, and $EF \to FE$. With this in view, it is very natural to require that (E, F) be an adjoint pair. (And, notice that with respect to the Shapovalov form on V, e and f act by adjoint linear maps.) Adding this hypothesis allows one to package the data of the various morphisms in a simple way. From the adjunction we have isomorphisms:

$$\mathrm{Hom}(\mathrm{Id}, FE) \cong \mathrm{End}(E) \cong \mathrm{Hom}(EF, \mathrm{Id}),$$

$$\mathrm{Hom}(EF, FE) \cong \mathrm{End}(EE).$$

This means that all the morphisms of functors in question can be determined by the data of various elements of $\mathrm{End}(E)$ and $\mathrm{End}(EE)$. The unit $\eta : \mathrm{Id} \to FE$ and counit $\varepsilon : EF \to \mathrm{Id}$ are determined, of course, by the natural element Id_E. But there is a further aspect, since $\mathrm{End}(E)$ and $\mathrm{End}(EE)$ have natural ring structures using composition for multiplication. This feature implies that a single element $x \in \mathrm{End}(E)$ generates a potentially infinite list of morphisms $EF \to \mathrm{Id}$ and $\mathrm{Id} \to FE$, namely by interpreting the powers $x^n \in \mathrm{End}(E)$ through the above isomorphisms of Hom spaces.

Let us say that the modern definition of categorical representation makes two major moves, both of which concern the assembly of functors we are discussing. The first is to insist that the pair (E, F) form an adjunction, and then to define the constituent maps $EF \to \mathrm{Id}$, $\mathrm{Id} \to FE$, and $EF \to FE$ in the above manner in terms of a single generator $x \in \mathrm{End}(E)$ and another generator $\tau \in \mathrm{End}(EE)$. In precise detail, we let $\sigma : EF \to FE$ by

$\sigma = FE\varepsilon \circ F\tau F \circ \eta EF$, and for $\lambda > 0$ we let $\varepsilon \circ x^i F : EF \to \mathrm{Id}$, $i \in \{0, \ldots, \lambda - 1\}$ give the maps $EF \to \mathrm{Id}$, and for $\lambda < 0$ we let $Fx^i \circ \eta : \mathrm{Id} \to FE$, $i \in \{0, \ldots, -\lambda - 1\}$ give the maps $\mathrm{Id} \to FE$.

The second major move is to introduce relations on the generators. These will be relations between Ex and xE and τ inside $\mathrm{End}(EE)$, as well as relations between $E\tau$ and τE inside $\mathrm{End}(EEE)$. It is much harder to motivate the details of these relations purely from considerations of the original Lie algebra structure, even granting a desire to recast it in categorical terms. The move was not made for a decade after the concept of naive categorical representation was introduced, probably for that reason, and there remains some variety when it comes to the details used in the literature for the relations. The relations do have the common feature of a 'Hecke-type', and that may be, as we have mentioned, justified from several points of view, to which we now turn.

Hecke relations

The categorical version of Lie theory did not originate as a sophistication of classical Lie theory. Some of the momentum it has, to be sure, derives from an ambitious vision of Crane-Frenkel to make Lie theory a suitable receptacle of $4d$ topological information. Were it not for these laudable motivations, namely their extrinsic goals, adherents of the Crane-Frenkel TQFT program would lie somewhat exposed to a charge – not infrequently levied by representation theorists – of dressing classical theory in complicated clothing without increasing its real content. But, we hasten to insert, the origin of the categorical theory, to include, if only implicitly, the 'Hecke-type' actions that are now seen to have central importance, can be found in the geometrical manifestation of the 'canonical bases' in quantum groups. These were discovered by Lusztig, and used by him to prove positivity statements by interpreting integer coefficients as dimensions of spaces of perverse sheaves. An analogous strategy also worked for the positivity conjecture of Kazhdan-Lusztig.

The point, the first point of view, is that one must acknowledge a close connection between the very idea of categorical Lie theory and the passage to geometrical settings, and that the modules playing the role formerly occupied by elements have a natural meaning in the geometrical settings. With this admitted, the Hecke-type relations on x and τ may be justified by their occurrence in the geometrical settings. We will say more about those settings below.

A second point of view is that a definition should be judged by the theorems one can prove using it. The Hecke-type relations on the generators x and τ, when added to the definition of 2-representation, enable one to define 'minimal' 2-representations $\mathcal{L}(\lambda)$ that have the irreducible 1-representations $L(\lambda)$ of $\mathfrak{g}$ as their Grothendieck groups, and that also have a universal property analogous to that of Verma modules. Those 2-representations enable the proof of a categorical analogue of the Jordan-Hölder series decomposition, and this in turn allows one to reduce some statements about general 2-representations to statements about the minimal ones. This technique was used by Chuang-Rouquier in the course of their proof of the symmetric group case of Broué's Abelian Defect Conjecture about finite groups.

Let us say a brief word about how the Hecke-type relations lead to structure theorems in the case of $\mathfrak{sl}_2$. Firstly, the (nil affine) Hecke algebras have the structure of $n! \times n!$ matrix algebras, the idempotents of which yield decompositions of the powers of E, namely $E^n \cong E^{(n)} \oplus \cdots \oplus E^{(n)}$, where $E^{(n)}$ is called a 'divided power'. (Similarly $F^n \cong (F^{(n)})^{\oplus n!}$.) The divided powers are shown to be indecomposable modules, in appropriate circumstances, whose classes give Lusztig's canonical basis elements in the Grothendieck group. If the structure of the Grothendieck group is known in advance, which it typically is, then the divided powers and their endomorphism algebras determine the structure of the whole 2-representation.

We may add a third point of view. In representation theory, one looks for structures with interesting sets of representations. In categorical representation theory, it has proven difficult to say as much as in the classical theory about the (internal) structure of the set of categorical representations of a Lie algebra. (This dissertation adds something to remedy the deficit.) It is difficult partly because a categorical representation is a more complicated structure, owing to the Hecke-type symmetry in the 2-morphisms. On the other hand, one can find interest in the diversity of domains in which categorical representations (with the Hecke symmetry) can be found. We alluded in the previous paragraph to an appearance in connection with the Broué conjecture. This has to do more specifically with the modular representation theory of symmetric groups, where the categorical 'E' is given by a restriction functor and the generator x arises from Jucys-Murphy elements. Perhaps unsurprisingly, considering Schur-Weyl duality, one can also provide the structure of categorical representation to several categories of representations of $\mathfrak{gl}_n$, such as parabolic category $\mathcal{O}$. And, of course, there is thirdly the class of geometrical categorical representations that we will address shortly.

Categorical enveloping algebra $\mathcal{U}(\mathfrak{g})$

Categorical representations can be described with a little more abstraction as '2-representations' of certain 2-categories. Let us illustrate the idea first with the categorical version of the positive half $U(\mathfrak{sl}_2)^+$ of the enveloping algebra associated to $\mathfrak{sl}_2$. One forms a monoidal category $\mathcal{U}^+$ generated by a single object E. The morphisms are generated by $x \in \mathrm{End}(E)$ and $\tau \in \mathrm{End}(EE)$ modulo the Hecke-type relations, see (1.3.1). Then a '2-representation' of $\mathcal{U}^+$ on a monoidal category $\mathcal{V}$ is a strict monoidal functor $\mathcal{U}^+ \to \mathcal{V}$. For example, when $\mathcal{V}$ is the category $\mathsf{Bim}_k(A)$ of (A, A)-bimodules for a k-algebra A, then a 2-representation on $\mathcal{V}$ amounts to the data of such a bimodule ${}_AE_A$, together with bimodule maps $x \in \mathrm{End}(E)$ and $\tau \in \mathrm{End}(EE)$ satisfying the Hecke-type relations. This setup can be generalized to other Lie types by including generators E_i for simple roots i of a root system, and augmenting the set of morphisms and relations accordingly. We generalize 'Hecke-type relations' from 'nil affine Hecke relations' to 'quiver Hecke relations'.

When considering the full algebra $U(\mathfrak{sl}_2)$, and the categorical setting where we are interested in representations having weight decompositions, it is sensible to expand $U(\mathfrak{sl}_2)$ somewhat and form a 1-category $\dot{U}(\mathfrak{sl}_2)$ in which the generators of $U(\mathfrak{sl}_2)$ become arrows, and we include an object 1_λ for each weight λ. Now, instead of forming a monoidal category as for $\mathcal{U}^+$, one forms a 2-category $\mathcal{U}$, where the single object is expanded to the set of weight objects $\{1_\lambda\}$, the monoidal structure is promoted to the arrows of a 1-category, and the old morphisms become the new 2-morphisms. A '2-representation' of $\mathcal{U}$ on a 2-category $\mathcal{V}$, such as the category Cat of categories, is then a strict 2-functor $\mathcal{U} \to \mathcal{V}$. The data of such a map $\mathcal{U} \to \mathsf{Cat}$ consists in the choice of 'weight categories' $\mathcal{V}_\lambda$, i.e. the images of $\{1_\lambda\}$ in Cat, together with 1-morphisms $E_\lambda : \mathcal{V}_\lambda \to \mathcal{V}_{\lambda+2}$ and $F_\lambda : \mathcal{V}_\lambda \to \mathcal{V}_{\lambda-2}$, and 2-morphisms $x_\lambda \in \mathrm{End}(E_\lambda)$ and $\tau_\lambda \in \mathrm{End}(E_{\lambda+2}E_\lambda)$, such that the Hecke-type relations are satisfied, and the maps ρ_λ they determine are isomorphisms. The same upgrade procedure can be performed in other Lie types, and the resulting $\mathcal{U}(\mathfrak{g})$ is said to be the 2-category that categorifies $\dot{U}(\mathfrak{g})$. A technical complication comes in defining the general version of Hecke-type relations. Let us also remark that a graded version '$\mathcal{U}_q(\mathfrak{g})$' or '$\mathcal{U}(\mathfrak{g})$-gr' gives the categorical counterpart of the quantum group $U_q(\mathfrak{g})$.

It turns out that the positive half $\mathcal{U}^+$ has an increased significance in the categorical setting, relative to the full 2-category $\mathcal{U}$, when compared to the significance of $U(\mathfrak{g})^+$ relative to $U(\mathfrak{g})$. In the next section, $\mathcal{U}^+$ will be given a geometric meaning that does not generalize to $\mathcal{U}$. This meaning was essential for the discoveries and constructions in the history of $\mathcal{U}$.

Geometry

One can realize $\mathcal{U}^+$ as the monoidal category of perverse sheaves on a moduli stack of representations of a quiver. While there is not a similar realization known for the whole of $\mathcal{U}$, one can provide a geometric realization of the minimal 2-representations of $\mathcal{U}$, those mentioned above in connection with Jordan-Hölder series, by using Nakajima quiver varieties.

(These varieties were designed to provide a geometric realization of the whole of $U(\mathfrak{g})$, as well as its irreducible integrable representations, i.e. at the level of the Grothendieck ring.) This construction generalizes to a geometric realization of the categorical representations having Grothendieck group a *tensor product of irreducibles*. (More on that later.)

Let Q be a quiver with I its set of vertices, where Q is the Dynkin diagram of $\mathfrak{g}$ with added orientation. Let $\mathrm{Rep}(Q)$ be the moduli stack of representations of Q over $\mathbb{C}$. We have $\mathrm{Rep}(Q) = \sqcup_\nu \mathrm{Rep}(Q)_\nu$ where $\mathrm{Rep}(Q)_\nu$ collects the representations with dimension vector $\nu \in \mathbb{N}I$. The derived category $D(\mathrm{Rep}(Q)) = \oplus_\nu D(\mathrm{Rep}(Q)_\nu)$ has a monoidal structure given in the manner of convolution, as follows. There is a stack of exact sequences of Q-representations, $S = \{0 \to V_1 \to V_3 \to V_2 \to 0\}$, with three projections $\pi_i : S \to \mathrm{Rep}(Q)$. Given $A \in D(\mathrm{Rep}(Q)_\nu)$, $B \in D(\mathrm{Rep}(Q)_\mu)$, we form a product:

$$A * B = \pi_{3*}(\pi_1^* A \otimes \pi_2^* B) \in D(\mathrm{Rep}(Q)_{\nu+\mu}).$$

Let $P(\mathrm{Rep}(Q)_\nu)$ collect the direct sums of shifts of simple perverse sheaves in $D(\mathrm{Rep}(Q)_\nu)$. Then $P(\mathrm{Rep}(Q)) = \oplus P(\mathrm{Rep}(Q)_\nu)$ is a monoidal category by the decomposition theorem, and it has a homological grading. Lusztig showed that the Grothendieck ring of $P(\mathrm{Rep}(Q))$ is isomorphic to $U_q(\mathfrak{g})^+$.

There is an isomorphism of monoidal categories $(\mathcal{U}_q^+)^i \xrightarrow{\sim} P(\mathrm{Rep}(Q))$. (Here $(\mathcal{U}_q^+)^i$ refers to an idempotent-completion of $\mathcal{U}_q^+$.) We do not spell out any details, but this isomorphism entails the presence of the Hecke-type symmetry in $P(\mathrm{Rep}(Q))$. This should be viewed as a fundamental source for the Hecke relations appearing in higher representation theory, and the isomorphism in question may be viewed as contributing to the definition of $\mathcal{U}$.

In the case of $\mathfrak{sl}_2$, Q is a single vertex α, and an object $V \in \mathrm{Rep}(Q)$ is a vector space. Then $\mathrm{Rep}(Q)_{n\alpha}$ is the stack pt/GL_n. One can find the (nil affine) Hecke algebra ${}^0H_n$ in $P(\mathrm{Rep}(Q))$ as follows. Let $Z = \widetilde{\mathrm{Rep}(Q)}_{n\alpha} \times_{\mathrm{Rep}(Q)_{n\alpha}} \widetilde{\mathrm{Rep}(Q)}_{n\alpha}$, where $\widetilde{\mathrm{Rep}(Q)}_{n\alpha}$ is the stack of complete flags of representations of Q with top dimension vector $n\alpha$. So $\widetilde{\mathrm{Rep}(Q)}_{n\alpha}$ is $Fl(\mathbb{C}^n)/GL_n$

and $Z = \big(Fl(\mathbb{C}^n) \times Fl(\mathbb{C}^n)\big)/GL_n$. Then the convolution algebra $H_*(Z)$ is an Ext-algebra in $P(\mathrm{Rep}(Q))$ isomorphic to ${}^0H_n$. It can also be computed as $H_*^{GL_n}\big(Fl(\mathbb{C}^n) \times Fl(\mathbb{C}^n)\big)$. The latter may be viewed as the most basic appearance of Hecke-type relations in geometry.

We briefly turn to the Nakajima quiver varieties. Let $Y = \cup_\mu Y(\lambda, \mu)$ be a Nakajima quiver variety. It is an open subset of $T^*M(\lambda, \mu)$, where $M(\lambda, \mu)$ is a stack naturally defined in Nakajima's context. There is a category $D(\lambda, \mu)$ of constructible sheaves on $M(\lambda, \mu)$. In the case of $\mathfrak{sl}_2$, with $\lambda = n$ and $\mu = n - 2k$, this is $D(\lambda, \mu) = D^b_c(Gr(k, \mathbb{C}^n))$. It is a theorem that, for general $\mathfrak{g}$, the categories $D(\lambda, \mu)$ give the μ-weight categories of a 2-representation of $\mathcal{U}(\mathfrak{g})$ with Grothendieck group the irreducible representation $L(\lambda)$ of $\mathfrak{g}$.

1.1.2 History

So far as the author is able to see into the past, the main idea of categorical representation theory originated with C. Ringel in 1990 [Rin90]. Ringel discovered that the isomorphism classes of representations of a Dynkin quiver, over a finite field $\mathbb{F}_p$, has the structure of the positive half of the quantized enveloping algebra $U_q(\mathfrak{g})^+$, where q is a function of the prime p, and $\mathfrak{g}$ is determined by the quiver. The 'structure of isomorphism classes' refers to the ring structure of extensions that was defined much earlier by P. Hall. This discovery of Ringel inspired[1] G. Lusztig's work using perverse sheaves on the space of representations of a quiver [Lus90, Lus91].[2] The description in terms of perverse sheaves is what gives the canonical bases for $U_q(\mathfrak{g})^+$ and the irreducible $U_q(\mathfrak{g})$-modules.

In 1994, L. Crane and I. Frenkel made explicit a proposal to recast many algebraic structures in representation theory into categorical terms [CF94]. The reason for their interest in this project was that the higher dimension of categorical terms could allow some existing ideas through which $3d$ topological information is rendered into representation-theoretic

[1] Cf. [Lus10, note 5, p. 127].

[2] Another part of Lusztig's innovation was inspired by his own earlier work on character sheaves [Lus85].

structures to be 'upgraded' to ideas through which $4d$ topological information is rendered into categorical analogues of the same representation-theoretic structures. These ideas included, especially, the invariant of 3-manifolds discovered by Witten and Reshetikhin-Turaev (WRT) [Wit89, RT91]. The WRT invariant relies on the Hopf structure of a quantum group $U_q(\mathfrak{g})$, which provides tensor product and dualization structures for the category of representations of $U_q(\mathfrak{g})$. Crane and Frenkel recognized in Lusztig's canonical bases the traces of a world of higher algebras, higher representations, and higher Hopf structure.

M. Broué's Abelian Defect Conjecture [Bro86] motivated J. Chuang and R. Rouquier's study [CR08] (released 2004) of the representation category for symmetric groups, where they incorporated the structure of higher representation we have discussed, describing for the first time endomorphisms $x \in \mathrm{End}(E)$ and $\tau \in \mathrm{End}(EE)$ that carry the data of the maps involved in the categorical commutator relations, as well as the Hecke-type relations that x and τ should satisfy. Rouquier extended this work to a systematic theory of categorical Kac-Moody algebras $\mathcal{U}$ in [Rou08a] (see also [Rou12]), introducing the 'quiver Hecke algebras' that generalize the Hecke-type symmetry of x and τ. Around the same time, independently, M. Khovanov and A. Lauda [KL09, Lau10, KL10] found a special case of the Hecke-type relations in the cohomology of partial flag varieties, and used those to produce categorical representations, as well as to inspire their own definition of essentially the same $\mathcal{U}$, given by diagrammatic generators with the Hecke-type relations that they saw in the cohomology. The connection between these cohomology rings and quantum groups was known from Beilinson-Lusztig-MacPherson's geometric model of $U_q(\mathfrak{sl}_n)$ [BLM90], which had in turn been interpreted categorically already by Grojnowski-Lusztig [GL92]. Khovanov and Lauda recognized that the Hecke-type action on 2-morphisms in $\mathcal{U}$ leads to the existence of commutator isomorphisms ρ_λ, although they did not emphasize the way the ρ_λ are determined by x and τ together with the one adjunction (E, F), as Rouquier had done. In Rouquier's approach only one adjunction is postulated, and the morphisms ρ_λ are inverted formally to define $\mathcal{U}$. In Khovanov-Lauda's approach, a bi-adjunction is postulated, and the existence of commutator isomorphisms is established using the bi-adjunction and the Hecke action. Brundan has shown [Bru16] that the two definitions of $\mathcal{U}$ essentially agree.

The geometric description of categorical representations using the quiver varieties of H. Nakajima [Nak94] was given first for tensor products of $U_q(\mathfrak{sl}_2)$-representations in 2007 by H. Zheng [Zhe07], and extended in 2008 to 2-representations with Grothendieck group either irreducible or a tensor product of irreducibles [Zhe14]. Zheng did not show that the ρ_λ as determined by x and τ were isomorphisms, but he showed that some commutator isomorphisms existed, and Rouquier was able to deduce the former from this fact and his own theory [Rou12, Thm. 5.10].

1.2 Tensor product

1.2.1 Concepts

The operation of tensor product is ubiquitous in representation theory and its applications. The tensor product is a primary means of generating new representations from old ones. In this thesis we develop a tensor product for 2-representations. The nature of the problems arising at the categorical level should be set against the structure of the classical theory, so we summarize the classical theory first.

Let us be given two representations of $\mathfrak{sl}_2(\mathbb{C})$, called V_1 and V_2. Form the vector space $V_1 \otimes_{\mathbb{C}} V_2$. This space has two commuting actions of $\mathfrak{sl}_2$: in the first, $e \in \mathfrak{sl}_2$ acts by $e \otimes 1$, in the second it acts by $1 \otimes e$. The tensor product representation consists in a new, 'diagonal', action in which e acts by $e \otimes 1 + 1 \otimes e$. It is not hard to see that this rule gives an action. It's a little harder to see, but this action has the property that the canonical (trivial) isomorphism $(V_1 \otimes_{\mathbb{C}} V_2) \otimes_{\mathbb{C}} V_3 \xrightarrow{\sim} V_1 \otimes_{\mathbb{C}} (V_2 \otimes_{\mathbb{C}} V_3)$ commutes with the actions determined on each side by applying the above procedure twice. Now there is an anti-automorphism S of $U(\mathfrak{sl}_2)$ given by $X \mapsto -X$. Using this, V_1 can be viewed as a $U(\mathfrak{sl}_2)^{\mathsf{op}}$-module, which is to say, as a *right* $U(\mathfrak{sl}_2)$-module, and we can form the tensor product *over the* $\mathfrak{sl}_2$ *action*, written $V_1 \otimes_U V_2$. This product is smaller, lacking the diagonal symmetry. Now, observe a simple relationship between the diagonal action on $V_1 \otimes_{\mathbb{C}} V_2$ and the smaller product $V_1 \otimes_U V_2$: the latter is the largest quotient of the former on which $\mathfrak{sl}_2$ acts diagonally by zero.

In the case of $U(\mathfrak{g})$ or $U_q(\mathfrak{g})$, the discussion above can be expressed in the language of Hopf algebras. Suppose ${}_HM$ and ${}_HN$ are two representations of a Hopf k-algebra H with coproduct $\Delta : H \to H \otimes H$ and antipode $S : H \to H$. There is a large outer product $M \otimes_k N$ with two commuting actions of H on the two factors, and a third, diagonal, action given by first applying Δ. The coassociativity property of Δ implies that the trivial map $(M \otimes N) \otimes L \to M \otimes (N \otimes L)$ is an isomorphism of H-modules. By using S to view M as a right H-module, we can form the smaller product $M \otimes_H N$. The smaller product is related to the larger as follows: $M \otimes_H N$ is the largest quotient of $M \otimes_k N$ on which H acts through Δ by 0. For $U(\mathfrak{g})$ or $U_q(\mathfrak{g})$, the formulas $\Delta(h) = h \otimes 1 + 1 \otimes h$ and $S(h) = -h$ can be used to write the condition $\Delta(h).(m \otimes n) = 0$ as the equality of elements $m.h \otimes n = m \otimes h.n$.

We do not have a Hopf structure on the categorical analogue $\mathcal{U}$. We are interested in building such a structure, or at least the expression of such structure on the collection of 2-representations. Let $\mathcal{V}_i$ be an abelian category of A_i-modules for $i = 1, 2$, where $\mathcal{V}_i$ is a 2-representation of $\mathcal{U}$ given by the data (E_i, x^i, τ^i). We can easily define a large 'outer product' category $\mathcal{V}_1 \otimes_k \mathcal{V}_2$, with objects generated by pairs of modules $M \otimes_k N$, and it has two commuting actions of $\mathcal{U}$. We seek a kind of diagonal action of $\mathcal{U}$ on $\mathcal{V}_1 \otimes_k \mathcal{V}_2$, but we do not at the outset insist on the coassociativity feature or the use of an antipode.

The fundamental conceptual choice of our method, advocated by Rouquier since at least 2008 [Rou08b], is to define the diagonal 2-representation, call it $\mathcal{V}_1 \bigotimes \mathcal{V}_2$, by first imagining the smaller product taken over $\mathcal{U}$, written perhaps $\mathcal{V}_1 \otimes_{\mathcal{U}} \mathcal{V}_2$, and then defining $\mathcal{V}_1 \bigotimes \mathcal{V}_2$ so that it bears an analogous relation to $\mathcal{V}_1 \otimes_{\mathcal{U}} \mathcal{V}_2$ as $V_1 \otimes_{\mathbb{C}} V_2$ does to $V_1 \otimes_U V_2$ or $M \otimes_k N$ does to $M \otimes_H N$. That is to say, one should be able to realize $\mathcal{V}_1 \otimes_{\mathcal{U}} \mathcal{V}_2$ as the largest quotient of $\mathcal{V}_1 \circledcirc \mathcal{V}_2$ on which $\mathcal{U}$ acts diagonally by zero. Then, if we can define first $\mathcal{V}_1 \otimes_{\mathcal{U}} \mathcal{V}_2$, the relationship will point us toward the definition of $\mathcal{V}_1 \circledcirc \mathcal{V}_2$.

Now there is a natural way to define $\mathcal{V}_1 \otimes_{\mathcal{U}} \mathcal{V}_2$, at least in principle. In this category, there should be isomorphisms of modules $\alpha_M^N : E_1(M) \otimes_k N \xrightarrow{\sim} M \otimes_k E_2(N)$. (And they should be equivariant over the actions of x^i on E_i and τ^i on E_i^2.) These isomorphisms would induce the conditions $\Delta(e).(m \otimes n) = 0$ on the Grothendieck group, where $e = [E]$, $m = [M]$, $n = [N]$. As in our first discussion about the commutator isomorphisms, the category $\mathcal{V}_1 \otimes_{\mathcal{U}} \mathcal{V}_2$ should include the data of the morphisms α_M^N. Of course, the collection should be functorial, so really we want a single Hecke-equivariant morphism of functors $\alpha : E_1 \otimes \mathrm{Id} \to \mathrm{Id} \otimes E_2$. This α should be formally inverted in the definition of the product $\mathcal{V}_1 \otimes_{\mathcal{U}} \mathcal{V}_2$.

At this point one can see a way to define the larger category $\mathcal{V}_1 \circledcirc \mathcal{V}_2$, namely as the outer product $\mathcal{V}_1 \otimes_k \mathcal{V}_2$ with the additional data of a Hecke-equivariant morphism α of functors. The smaller category is determined from this by adding the condition that α is an isomorphism. What is the diagonal action of E on $\mathcal{V}_1 \circledcirc \mathcal{V}_2$? This should be a functor that acts by zero on a simple tensor of modules $M \otimes_k N$ for which α_M^N is an isomorphism. Recall that α_M^N is a quasi-isomorphism if and only if its cone is acyclic. The second basic idea of our method, also due to Rouquier, is to use the *cone of* α for this functor, and to move to a homotopy or derived setting. Needless to say, this move opens the door to myriad technical complications, and it is the procedural reason that the definition of $\mathcal{V}_1 \circledcirc \mathcal{V}_2$ we study in this thesis belongs in its 'full' nature to the homotopy or derived setting.

To complete the concept of $\mathcal{V}_1 \circledcirc \mathcal{V}_2$, it is necessary to supply natural x- and τ-equivariant morphisms $\alpha_{(E_1 \otimes \mathrm{Id})C}^{(\mathrm{Id} \otimes E_2)C}$ (where $C = Cone(\alpha_M^N)$) in order to make $Cone(\alpha_M^N)$ an object in $\mathcal{V}_1 \circledcirc \mathcal{V}_2$, and to supply endomorphisms x and τ of $Cone(\alpha_M^N)$ and $Cone(Cone(\alpha_M^N))$ satisfying Hecke-type relations in order to make a 2-representation of $\mathcal{U}$ using $Cone(\alpha_M^N)$ for the image of E. Here one encounters further technical difficulties. We continue in §1.4 a discussion about how the construction in this thesis overcomes those difficulties.

We may include here an important observation about the idea sketched thus far. In the concept of $\mathcal{V}_1 \otimes_{\mathcal{U}} \mathcal{V}_2$, the isomorphisms are, naturally, bidirectional. In the definition of $\mathcal{V}_1 \bigotimes \mathcal{V}_2$ on the other hand, we have broken the symmetry by preferring a choice of domain and codomain for the morphism α. It may seem at first that this asymmetry is unnatural and undesirable, but deeper consideration suggests that thought to be premature. Recall that the tensor product $M_1 \otimes M_2$ of representations of the quantum group $U_q(\mathfrak{g})$ is isomorphic to the tensor product $M_2 \otimes M_1$, but *not by the trivial exchange of factors*. Indeed the usual isomorphism is by the highly nontrivial action of the quantum R-matrix. In this way the quantum group tensor product is not symmetric. Given that a graded version of categorical tensor product should categorify the quantum tensor product, we might in fact expect asymmetry in the categorical setting for the undeformed Lie algebra.

1.2.2 History

The original Crane-Frenkel program included the idea of building a 'Hopf category' upgrading the Hopf structure of quantum groups that was central to the WRT invariant. Early work on 'categorification' that explicitly participated in this program, such as Bernstein-Frenkel-Khovanov's [BFK99], sought and studied categorical representations including categorifications of tensor products of simple representations of $U(\mathfrak{sl}_2)$. In [FKS07] the authors extended this theory to the quantized case $U_q(\mathfrak{sl}_2)$ using graded versions of similar structures. The main structures used were singular blocks of Harish-Chandra bimodules considered in category $\mathcal{O}$ of $\mathfrak{gl}_n$, and parabolic subcategories of the regular block of the same category $\mathcal{O}$. (They are related by a Koszul duality.) These methods were carried further by Sussan [Sus07] and Mazorchuk-Stroppel [MS09], and then by Sartori-Stroppel [SS15] who formulated categorifications of arbitrary tensor products of finite dimensional simples in type A. All these categorifications were eventually understood to possess the Hecke symmetries of their 2-morphisms that Rouquier had incorporated in his 2004 definition.

The theory of tensor product categorifications with broadest coverage was developed by B. Webster [Web17]. He defined an algebra $T^{\underline{\lambda}}$ depending on a list $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$ of dominant weights of any symmetrizable Kac-Moody algebra $\mathfrak{g}$, and showed that the category $\mathcal{R}ep_{\text{f.d.}}(T^{\underline{\lambda}})$ of its finite dimensional representations categorifies the tensor product $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$ of simple representations with highest weights λ_i. The $\mathcal{R}ep_{\text{f.d.}}(T^{\underline{\lambda}})$ categorifications (and their derived categories $D(T^{\underline{\lambda}})$) were sufficient for Webster to define link invariants for all symmetrizable $\mathfrak{g}$, although these invariants have not been amenable to computation. The algebras $T^{\underline{\lambda}}$ are defined using diagrammatic generators and relations, similarly to the algebras defined by Khovanov and Lauda that categorify quantum groups ([KL09], [KL11]). According to a unicity theorem of Losev-Webster [LW14], Webster's $\mathcal{R}ep_{\text{f.d.}}(T^{\underline{\lambda}})$ in type A is equivalent to the categorifications by Sartori-Stroppel and others.

The categorifications of Stroppel et al. and Webster do not give the categorical analogue of the operation of tensor product: such an operation must by definition start with the abstract data of given 2-representations, and determine a third 2-representation from this data. It is a fact that 2-representations are not in general semisimple, and the tensor 2-product of given simple 2-representations having certain simple 1-representations for their Grothendieck groups is *not* expected to agree with a direct sum of simple 2-representations having for *their* Grothendieck groups the direct summands of the tensor 1-product of those certain simple 1-representations. (Symbolically, if $K_0(\mathcal{V}_1) = V_1$ and $K_0(\mathcal{V}_2) = V_2$ and $K_0(\mathcal{W}_i) = W_i$ with $V_1 \otimes V_2 \cong \oplus_i W_i$, then $\mathcal{V}_1 \bigotimes \mathcal{V}_2$ is not going to agree with $\oplus_i \mathcal{W}_i$.) Furthermore, a natural product operation should be functorial in its arguments, and this desideratum cannot even be *stated* for the ad hoc categorifications of the kind constructed before this thesis. We hasten to add, though, that Losev-Webster's axiomatic description of Webster's 2-representations, and the agreement of the latter with Sartori-Stroppel et al.'s constructions in type A, is very strong evidence that these constructions have the right structure for the tensor 2-product of simple 2-representations with irreducible Grothendieck groups. So, we expect a 2-product definition to determine structures equivalent to these when the factors are simple.

We are informed that Rouquier has a broad definition of tensor 2-product given in an $\mathcal{A}_\infty$ setting that encodes the technical complications as higher homotopies. (We anticipate a future publication [Rou].) This setting brings its own technical complications, and the construction does not provide any explicit formulas for the product action even in that setting. Rouquier has conjectured (we believe in [Rou08b]) that there should exist an abelian subcategory of the derived category of the 2-product he defines, a subcategory which affords an abelian 2-representation. Abelian 2-representations are the sort we handle in this thesis and the sort usually intended in the literature.

The main construction of this thesis partially verifies Rouquier's conjecture by defining an abelian 2-product when one factor is $\mathcal{L}(1)$ and the other factor $\mathcal{V}$ is taken from the 2-category of algebras (and satisfies two further hypotheses). In addition, our construction takes a step toward defining a practically useful 2-product by providing explicit formulas for the component structures.

In certain cases the homotopical complications in Rouquier's $\mathcal{A}_\infty$ approach naturally disappear. This happens in the case of the super Lie algebra $\mathfrak{gl}(1|1)^+$, and A. Manion and Rouquier [MR20] have developed the theory in that case. They show that the 2-product can be used to describe the Bordered Heegaard-Floer theory for surfaces [LOT18].

The Manion-Rouquier work is in the direction of perhaps the most compelling motivation to find a 2-product that is properly native to the theory of 2-representations, and that is the ambition of Crane-Frenkel to build a $4d$ TQFT. In particular, in the case of the 2-algebra $\mathcal{U}_q(\mathfrak{sl}_2)$, such a 2-product may be expected to play a central role in a prospective $4d$ TQFT that extends the Jones polynomial. It may be possible to build this TQFT as a

$4d$ layer on the $3d$ TQFT of Witten-Reshetikhin-Turaev. Glimmers of this $4d$ theory have been seen by physicists [GPV17], and some aspects are defined rigorously in some cases [GM21]. We emphasize that the rank one case of $\mathcal{U}(\mathfrak{sl}_2)$ theory, and the tensor product of simple 2-representations such as the minimal one $\mathcal{L}(1)$, are expected to be sufficient for many topological applications, for the same reason that tensor products of the fundamental representation $L(1)$ of $U_q(\mathfrak{sl}_2)$ were enough for the Jones polynomial and WRT invariants.

1.3 The theorem

Let us be given a field k and the data of a triple (A, E, x, τ) as follows. Let A be a k-algebra and E an (A, A)-bimodule, let $x \in \mathrm{End}(E)$ and $\tau \in \mathrm{End}(E^2)$ be bimodule endomorphisms, and suppose that x and τ generate an action of the nil affine Hecke algebra, that is, that they satisfy the following relations:

$$\begin{gathered}\tau^2 = 0,\\ \tau E \circ E\tau \circ \tau E = E\tau \circ \tau E \circ E\tau,\\ \tau \circ Ex = xE \circ \tau + 1,\ Ex \circ \tau = \tau \circ xE + 1.\end{gathered} \tag{1.3.1}$$

(Here we write xE for the endomorphism $x \otimes \mathrm{Id}_E$ in $\mathrm{End}(E^2)$, and similarly for the others.)

Let $\mathcal{U}^+$ denote the monoidal category associated to the positive half of the enveloping algebra of $\mathfrak{sl}_2$. The data above determines a 2-representation $\mathcal{V}$ of $\mathcal{U}^+$.

We can give such data for a simple 2-representation $\mathcal{L}(1)$ of $\mathcal{U}^+$ whose Grothendieck group is the fundamental representation $L(1)$ of $\mathfrak{sl}_2$. The data is $\big(k[y]_{+1} \times k[y]_{-1}, k[y], y, 0\big)$, where the algebra is given in its weight decomposition. Here $y \in k[y]_{-1}$ acts on $k[y]$ on the right by multiplication, and $y \in k[y]_{+1}$ acts by zero. These roles are reversed for the left action. The endomorphism x acts by multiplication by y.

Let $P_n = k[x_1, \ldots, x_n]$ be the polynomial algebra. Then P_n acts on E^n with $x_i \in P_n$

acting by the endomorphism $E^{n-i}xE^{i-1}$.

The second chapter is organized around a proof of Part I of our main theorem.

Theorem (Main Theorem Part I: Positive Half)**.** *Suppose x and τ satisfy the nil affine Hecke relations, so (A, E, x, τ) gives the data of a 2-representation of $\mathcal{U}^+$, denoted $\mathcal{V}$, and suppose the bimodule E has the following additional properties:*

- *${}_AE$ is finitely generated and projective,*
- *E^n is free as a P_n-module.*

Then we define explicitly:

- *a k-algebra C (Def. 2.2.32),*
- *a (C, C)-bimodule $\tilde{E}$ (Def. 2.2.38),*
- *bimodule endomorphisms $\tilde{x}$ and $\tilde{\tau}$ of $\tilde{E}$ (Def. 2.3.4),*

such that $\tilde{x}$ and $\tilde{\tau}$ satisfy the nil affine Hecke relations, so $(C, \tilde{E}, \tilde{x}, \tilde{\tau})$ gives the data of a 2-representation of $\mathcal{U}^+$ that we denote $\mathcal{L}(1) \circledast \mathcal{V}$.

The constructions defined by Stroppel, Webster, Zheng, Lauda and others include 2-representations having Grothendieck group $L(1) \otimes L(n)$, where $L(n)$ is the irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ of dimension $n + 1$. In the last section of Chapter 2, we study the output of our construction for $\mathcal{V} = \mathcal{L}(1)$, and show that it is equivalent to a known 2-representation having Grothendieck group $L(1) \otimes L(1)$. We can describe the algebra for that known 2-representation.

Let $P_2 = k[x_1, x_2]$ and let $P_2^{S_2}$ be the subalgebra of symmetric polynomials. Let $B_{s_1} = P_2 \otimes_{P_2^{S_2}} P_2$ be the (P_2, P_2) bimodule; it is also a P_2-algebra with structure map $P_2 \to B_{s_1}$ given by $f \mapsto 1 \otimes f$. Now let $T = T_{+2} \oplus T_0 \oplus T_{-2}$ be the P_2-algebra:

$$T_{+2} = P_2,\ T_0 = \mathrm{End}_{B_{s_1}}(P_2 \oplus B_{s_1})^{\mathsf{op}},\ T_{-2} = P_2.$$

One can define a (T, T)-bimodule and bimodule endomorphisms that, together with T, give the data of a 2-representation of $\mathcal{U}$, denoted $\mathscr{T}$. This 2-representation is known to have Grothendieck group $L(1) \otimes L(1)$. See §2.4.1 for more details.

Theorem (Comparison Theorem)**.** *There is an equivalence $\mathcal{L}(1) \bigotimes \mathcal{L}(1) \xrightarrow{\sim} \mathscr{T}$ of 2-representations.*

Now let $\mathcal{U}$ be the 2-category associated with the enveloping algebra of $\mathfrak{sl}_2$, as given in Rouquier [Rou08a] or Vera [Ver20, §3.2]. Since we only work with 2-representations of $\mathcal{U}$ and not $\mathcal{U}$ itself, completeness does not demand a definition of $\mathcal{U}$. See [Rou08a, §5.1.1] for the definition of 2-representation of $\mathcal{U}$.

Assume we are given (A, E, x, τ) as above, determining a 2-representation of $\mathcal{U}^+$. Now assume also that (A, E, x, τ) has a weight decomposition $A = \prod_{\lambda \in \mathbb{Z}} A_\lambda$ (Def. 2.3.25 below). The data (A, E, x, τ) extends to determine a 2-representation of the full 2-category $\mathcal{U}$ when the functor $E \otimes_A -$ admits a right adjoint functor F such that certain maps ρ_λ are isomorphisms in each weight $\lambda \in \mathbb{Z}$. The maps ρ_λ are determined by x and τ. See §3.1.2 below for the definition of ρ_λ.

The simple 2-representation $\mathcal{L}(1)$ of $\mathcal{U}^+$ extends to a 2-representation of the full $\mathcal{U}$ in this way, where the right adjoint is given by tensor product with the bimodule $F = k[y] \cong \mathrm{Hom}_{k[y]}(k[y], k[y])$.

The third chapter is a proof of Part II of our main theorem.

Theorem (Main Theorem Part II: And Negative Half)**.** *Suppose (A, E, x, τ) gives the data of a 2-representation $\mathcal{V}$ of $\mathcal{U}^+$ such that $\mathcal{V}$ has a weight decomposition. Define the left-dual (A, A)-bimodule $F = \mathrm{Hom}_A({}_AE, A)$. Suppose E has the following properties:*

- *${}_AE$ is finitely generated and projective (as in Part I), so $(E \otimes_A -, F \otimes_A -)$ is an adjunction,*
- *E^n is free as a P_n-module (as in Part I),*

- *E and F are locally nilpotent (i.e. the 2-representation is integrable),*

- *The maps ρ_λ defined using x and τ are isomorphisms for each $\lambda \in \mathbb{Z}$, so (A, E, x, τ) gives a 2-representation of $\mathcal{U}$ and ${}_AF$ is finitely generated and projective. (See §3.1.2.)*

Let C be the k-algebra, $\tilde{E}$ the (C, C)-bimodule, and $\tilde{x}$ and $\tilde{\tau}$ the endomorphisms from the Main Thorem Part I. Note that $\tilde{E}$ is locally nilpotent. Let $\tilde{F} = \mathrm{Hom}_C({}_C\tilde{E}, C)$. Then:

- *The unit $\tilde{\eta}$ and counit $\tilde{\varepsilon}$ of the duality pairing give an adjunction $(\tilde{E} \otimes_C -, \tilde{F} \otimes_C -)$,*

- *The new maps $\tilde{\rho}_\lambda$ defined as in §3.1.2 using $\tilde{x}, \tilde{\tau}, \tilde{\varepsilon}, \tilde{\eta}$ are isomorphisms, so:*

- *${}_A\tilde{F}$ is finitely generated and projective, and $\tilde{F}$ is locally nilpotent,*

- *$(C, \tilde{E}, \tilde{x}, \tilde{\tau})$ gives the data of an integrable 2-representation of $\mathcal{U}$ for C.*

We emphasize that for a 2-representation of $\mathcal{U}^+$ (with weight decomposition) given by the data (A, E, x, τ), the fact that the data extends to determine a 2-representation of the full 2-category $\mathcal{U}$ is equivalent to a property of that data: namely that the canonical commutator maps ρ_λ determined by x and τ are isomorphisms. When this holds, then (according to the theorem) the maps $\tilde{\rho}_\lambda$ of the product are also isomorphisms. So the new data $(C, \tilde{E}, \tilde{x}, \tilde{\tau})$ inherits the property of extending to an action of the full $\mathcal{U}$.

In this thesis, a symbol $\mathcal{V}$ is used sometimes to denote a 2-representation of $\mathcal{U}^+$, and sometimes to denote the 'extension' to a 2-representation of $\mathcal{U}$. This is an abuse of notation because in the first instance $\mathcal{V}$ is a monoidal category, and in the second instance it is a 2-category. The meaning of our abuse is that the data determining the monoidal category may also determine a (related) 2-category. Thinking in terms of the underlying 4-tuple of data may prevent misunderstandings.

1.4 Remarks on the method

Assume the setting of the previous section, so $\mathcal{L}(1)$ is given by the data $(A_1, E_1, x^1, \tau^1) = (A^\circ, k[y], y, 0)$ with $A^\circ = k[y]_{+1} \times k[y]_{-1}$, and $\mathcal{V}$ is given by the data $(A_2, E_2, x^2, \tau^2) = (A, E, x, \tau)$. One can define a tensor algebra B':

$$B' = T_{A^\circ \otimes_k A}\big({}^\vee k[y] \otimes_k E\big).$$

There is a canonical isomorphism ${}^\vee k[y] \otimes_k E \xrightarrow{\sim} E[y]$, and another $A^\circ \otimes_k A \xrightarrow{\sim} A[y] \times A[y]$. The data of a B'-module is equivalent to the data of a triple (M, N, α_M^N) where $M, N \in A[y]$-mod and $\alpha_M^N : E[y] \otimes_{A[y]} M \to N$. Since $\tau^1 = 0$ in this case, α is automatically τ-equivariant. We can enforce x-equivariance of α by taking a quotient by $I = \mathrm{Im}(x - y)$, where $x - y$ is understood in $\mathrm{End}_{A[y]}(E[y])$. Define $B = B'/I$. Write E_y for the $(A[y], A[y])$-bimodule $E[y]/(x - y)E[y]$. Then we can present B using matrices by:

$$B = \begin{pmatrix} A[y] & E_y \\ 0 & A[y] \end{pmatrix}.$$

The ring structure is given by matrix multiplication, using the $(A[y], A[y])$-bimodule structure of E_y as well as the algebra multiplication in $A[y]$ to define the multiplication of matrix coefficients. The category B-mod is our initial candidate for the underlying category of $\mathcal{L}(1) \circledotimes \mathcal{V}$.

To develop a 2-representation, we seek a (B, B)-bimodule for the image of E from $\mathcal{U}$, and bimodule endomorphisms x and τ. There is a natural candidate for the image of E, we will call it E^Δ, but it is a complex of (B, B)-bimodules, not a bimodule. It is given as a complex (in degrees 0 and 1) by:

$$E^\Delta = \begin{pmatrix} E[y] & E[y]E_y \\ 0 & E[y] \end{pmatrix} \xrightarrow{d} \begin{pmatrix} E_y & E_yE_y \\ A[y] & E_y \end{pmatrix}.$$

(The differential and action data are described in Definition 2.2.2. There E^Δ is written E' for convenience, as elsewhere in Chapters 2 and 3 below.) There is also a natural candidate for $x \in \mathrm{End}_{B\text{-cplx}}(E^\Delta)$ arising from the data of $\mathcal{L}(1)$ and $\mathcal{V}$, but that x is *not equivariant* over the action of generators in E_y in B. (It is equivariant in a derived category.) There is no natural candidate for $\tau \in \mathrm{End}_{B\text{-cplx}}(E^\Delta E^\Delta)$, though that appears to be for technical reasons.

Let $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in B$. Our technique in this thesis is to define a new algebra C, derived-equivalent to B, that is the End-algebra of a complex X:

$$X = Be_1 \oplus E^\Delta e_1,$$

$$C = \mathrm{End}_{K^b(B)}(X).$$

In particular, we show that $\mathscr{H}om_B(X, -)$ defines an equivalence of triangulated categories:

$$\mathrm{per}\ B \xrightarrow[\mathscr{H}om_B(X,-)]{\sim} \mathrm{per}\ C,$$

where per B is the full subcategory of the derived category of complexes quasi-isomorphic to strictly perfect complexes of B-modules, and similarly for per C. The bimodule complex E^Δ may be transported through the equivalence, and the result is quasi-isomorphic to a complex $\tilde{E}$ of (C, C)-bimodules that is concentrated in degree 0 and such that ${}_C\tilde{E}$ is projective as a module. We construct explicit bimodule endomorphisms $\tilde{x} \in \mathrm{End}(\tilde{E})$ (compatible with x^1 and x^2) and $\tilde{\tau} \in \mathrm{End}(\tilde{E}^2)$ that satisfy the nil affine Hecke relations. The data $(C, \tilde{E}, \tilde{x}, \tilde{\tau})$ gives a 2-representation.

In order to define $\tilde{x}$ and $\tilde{\tau}$ and verify the relations, we study the tensor powers $\tilde{E}^n$. These powers can be parametrized by explicit models containing $\mathrm{Hom}_{K^b(B)}\big(E^\Delta e_1, (E^\Delta)^n e_1\big)$. We give presentations of these modules by generators and relations for $n = 1, 2, 3, 4$ in the second chapter. To add the structure of the lower half and obtain a 2-representation of the full $\mathcal{U}$, we also need models for $\mathrm{Hom}_{K^b(B)}\big((E^\Delta)^2 e_1, (E^\Delta)^n e_1\big)$ for $n = 1, 2$, and those are developed in the third chapter.

1.5 Directions for future work

Considering our 2-product as part of a 2-representation theory of the 2-category $\mathcal{U}$, several questions are very natural.

- Given the asymmetry entailed by the choice of direction for α, can we define a product $\mathcal{L}(1)\overline{\bigotimes}\mathcal{V}$ by making the opposite choice?
- A small modification of our method should also lead to the reverse product $\mathcal{V} \bigotimes \mathcal{L}(1)$. Is there a natural equivalence between $\mathcal{V} \bigotimes \mathcal{L}(1)$ and $\mathcal{L}(1)\overline{\bigotimes}\mathcal{V}$?
- Is our product functorial in the second argument $\mathcal{V}$?
- Our methods appear amenable to generalization to a product $\mathcal{L}(n) \bigotimes \mathcal{V}$, albeit with a complexity of description that seems unmanageable. Is there a way to simplify or package the technical aspects in order to give explicit formulas for $\mathcal{L}(n) \bigotimes \mathcal{V}$?

The following deeper questions would contribute to a general theory of the 2-product:

- Rouquier's construction gives a derived category $D^b(\mathcal{L}(1) \bigotimes \mathcal{V})$. Can we explain our 2-product as the abelian core of $D^b(\mathcal{L}(1) \bigotimes \mathcal{V})$ determined by a t-structure or something similar? 'Perverse tilts' might give the higher analogues of the crystal bases of $U_q(\mathfrak{g})$-modules. (The crystal basis is compatible with the operation of tensor product, while the canonical basis is not.)
- Could a t-structure enable us to define a 2-product for higher rank Kac-Moody algebras?
- Could a t-structure determine a new class 2-$\mathscr{R}ep(\mathcal{U})$ of 2-representations, for any two members $\mathcal{V}$, $\mathcal{W}$ of which, the 2-product $\mathcal{V} \bigotimes \mathcal{W}$ is defined?

The search for a $4d$ TQFT motivates additional questions that should be reasonably straightforward to answer:

- Does the 2-product output $(C, \tilde{E}, \tilde{x}, \tilde{\tau})$ have the property that $\tilde{E}^n$ is free as a $k[\tilde{x}_1, \ldots, \tilde{x}_n]$-module? (Here $\tilde{x}_i$ acts by $\tilde{E}^{n-i}\tilde{x}\tilde{E}^{i-1}$.) The input is assumed to have this property. The other assumption of left-projectivity is already known to be satisfied by the output.

- Answering the above question affirmatively will enable a definition of the n-fold iterate:

$$\mathcal{T}_n = \mathcal{L}(1) \bigotimes \Big(\mathcal{L}(1) \bigotimes \big(\mathcal{L}(1) \bigotimes \ldots\big)\Big).$$

  This product is of great importance for topological applications. It will be valuable to compare $\mathcal{T}_n$ with the (equivalent) 2-representations already defined by Lauda, Stroppel, Webster, Zheng and others; we expect to find that $\mathcal{T}_n$ is equivalent to theirs.

- When the reverse product $\mathcal{V} \bigotimes \mathcal{L}(1)$ is also defined, we will have in hand the definition of the iterate $\mathcal{L}(1)^{\bigotimes n}$ with any given choice of parenthesization. Obvious questions about associativity of the 2-product will make sense at that point.

- We anticipate a braid group action on $\mathcal{T}_n$, or on a dg or derived precursor. It will be determined by an auto-equivalence $\mathcal{R}_1$ of $\big(\mathcal{L}(1) \bigotimes \mathcal{L}(1)\big) \bigotimes \mathcal{V}$ that is functorial in $\mathcal{V}$.

Some significant questions in low-dimensional topology would be very interesting to address with the 2-product construction:

- By defining 'cup' and 'cap' morphisms and using a braid action on $\mathcal{T}_n$, can we give a new definition of Khovanov homology?

- How fast is the new definition to compute? (Is it comparable to the fast procedure of Bar-Natan?) Fast procedures to compute link homologies in higher rank are not yet known, and this approach using a 2-product might provide them, giving access to a large new volume of computable information about links.

- There is a spectral sequence from Khovanov homology to knot Floer homology. Can it be explained using the 2-product we have developed, in conjunction with the 2-product applied by Manion-Rouquier [MR20] to cast Heegaard-Floer theory in terms of 2-representations of $\mathfrak{gl}(1|1)^+$?

- With a new definition given for Khovanov homology, can we extend it from links to 3-manifolds?

1.6 Outline summary

- In §2.1 we describe some conventions and background theory. The data of a 2-representation of $\mathcal{U}^+$ consists of an algebra A, a bimodule ${}_AE_A$, and endomorphisms $x \in \mathrm{End}(E)$ and $\tau \in \mathrm{End}(E^2)$ satisfying nil affine Hecke relations. This data determines a monoidal category: the object is the bimodule E, tensor product over A gives the monoidal structure, and morphisms are bimodule maps.

- In §2.2 we begin with a naive product algebra B and complex of bimodules ${}_BE'_B$. We construct a derived-equivalent algebra C. We define a (C,C)-bimodule $\tilde{E}$ and study a new class of bimodules we call G_n that arise inside the tensor powers of $\tilde{E}$. This study has a technical and computational flavor.

- In §2.3 we construct the new nil affine Hecke action, with generators $\tilde{x}$ and $\tilde{\tau}$, on powers of the new bimodule $\tilde{E}$. More computations are required to establish the properties we need. They rely on results about G_n proved in §2.2.

- In §2.4 we write out explicit details for the most basic example of our construction: $\mathcal{L}(1) \bigotimes \mathcal{L}(1)$. This product agrees with a well-known categorification of $L(1) \otimes L(1)$, where $L(1)$ is the fundamental representation of $\mathfrak{sl}_2$.

- In §3.1 we discuss the adjunction and the maps ρ_λ that are needed to define the extension of a 2-representation of $\mathcal{U}^+$ to a 2-representation of the full 2-category $\mathcal{U}$.

- In §3.2 we define and study more bimodules, giving concrete algebraic models for them in the manner of §2.2.
- In §3.3 we consider the right adjoint to $\tilde{E} \otimes_C -$, namely $\tilde{F} \otimes_C -$ where $\tilde{F} = \mathrm{Hom}_C({}_C\tilde{E}, C)$, and we show how to describe it concretely by making use of the B side of the equivalence constructed in §2.2.
- In §3.4.1 we compute explicitly the tensor products needed to write explicit formulas for $\tilde{\rho}_\lambda$, namely the products $\tilde{E} \otimes_C \tilde{E}$ and $\tilde{E} \otimes_C \tilde{F}$ and $\tilde{F} \otimes_C \tilde{E}$. In §3.4.2 we compute explicit formulas for $\tilde{\rho}_\lambda$. In §3.4.3 we show directly that $\tilde{\rho}_\lambda$ is an isomorphism for each $\lambda \in \mathbb{Z}$.

CHAPTER 2

Construction of the product: the positive half

2.1 Background structures

Let k be a field.

2.1.1 Nil affine Hecke algebras

The nil affine Hecke algebra ${}^0H_n$ is the k-algebra with generators $x_1, \ldots, x_n, \tau_1, \ldots, \tau_{n-1}$ and relations:

$$x_i x_j = x_j x_i, \tau_i^2 = 0,$$

$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1},$$

$$\tau_i \tau_j = \tau_j \tau_i \text{ if } |i - j| > 1,$$

$$\tau_i x_j = x_j \tau_i \text{ if } j - i \notin \{0, 1\},$$

$$\tau_i x_i = x_{i+1} \tau_i + 1, x_i \tau_i = \tau_i x_{i+1} + 1.$$

Define $s_i = \tau_i (x_i - x_{i+1}) - 1$. Observe that $s_i^2 = 1$ and $s_i \circ \tau_i = \tau_i$.

2.1.2 $\mathcal{U}^+(\mathfrak{sl}_2)$ and its 2-representations

2.1.2.1 Monoidal category $\mathcal{U}^+$

Definition 2.1.1. Let $\mathcal{U}^+(\mathfrak{sl}_2)$ (hereafter '$\mathcal{U}^+$') be the strict monoidal k-linear category generated by an object E and maps $x : E \to E$ and $\tau : E^2 \to E^2$ subject to the relations:

$$\tau^2 = 0, \tag{2.1.1}$$

$$\tau E \circ E\tau \circ \tau E = E\tau \circ \tau E \circ E\tau, \tag{2.1.2}$$

$$\tau \circ Ex = xE \circ \tau + 1,\ Ex \circ \tau = \tau \circ xE + 1. \tag{2.1.3}$$

We write $s = \tau \circ (Ex - xE) - 1$. Observe that $s^2 = 1$ and $s \circ \tau = \tau$.

One easily checks that non-trivial Hom spaces of $\mathcal{U}^+$ are Hecke algebras:

Proposition 2.1.2. *The objects of $\mathcal{U}^+$ are the E^n for $n \in \mathbb{Z}^{\geqslant 0}$, and*

$$\mathrm{Hom}(E^n, E^m) \cong \begin{cases} {}^0H_n & n = m \\ 0 & n \neq m \end{cases}$$

with the isomorphism from ${}^0H_n$ given by $x_i \mapsto E^{n-i}xE^{i-1}$, $\tau_i \mapsto E^{n-i-1}\tau E^{i-1}$. Using the obvious morphism ${}^0H_n \otimes {}^0H_m \to {}^0H_{n+m}$, the diagram commutes:

$$\begin{array}{ccc} {}^0H_n \otimes {}^0H_m & \longrightarrow & {}^0H_{n+m} \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{End}(E^n) \otimes \mathrm{End}(E^m) & \xrightarrow{\ \otimes\ } & \mathrm{End}(E^{n+m}). \end{array}$$

2.1.2.2 2-representations of $\mathcal{U}^+$

Definition 2.1.3. A 2-representation of $\mathcal{U}^+$ on a category $\mathcal{V}$ is a strict monoidal functor $\mathcal{U}^+ \to \mathrm{End}(\mathcal{V})$. The data of such a functor consists of an endofunctor E of $\mathcal{V}$ and natural transformations $x \in \mathrm{End}(E)$, $\tau \in \mathrm{End}(E^2)$ satisfying (2.1.1)–(2.1.3). A morphism of 2-representations $(\mathcal{V}, E, x, \tau) \to (\mathcal{V}', E', x', \tau')$ consists of a functor $\Phi : \mathcal{V} \to \mathcal{V}'$ and an isomorphism of functors $\varphi : \Phi E \xrightarrow{\sim} E'\Phi$ such that:

$$\varphi \circ \Phi x = x'\Phi \circ \varphi : \Phi E \to E'\Phi,$$

$$E'\varphi \circ \varphi E \circ \Phi\tau = \tau'\Phi \circ E'\varphi \circ \varphi E : \Phi E^2 \to E'^2\Phi.$$

Note that $\mathrm{End}(\mathcal{V})$ is the full sub-2-category of the 2-category of categories Cat generated by the object $\mathcal{V}$. One can define $\mathcal{U}^+$ as a 2-category with a single object, so that the data of 2-representation is the data of 2-functor from $\mathcal{U}^+$ to Cat. This justifies our '2' prefixes.

In this chapter we study monoidal functors from $\mathcal{U}^+$ to monoidal categories of the form $\mathsf{Bim}_k(A)$ which are defined for k-algebras A as follows: the objects of $\mathsf{Bim}_k(A)$ are (A, A)-bimodules, and the morphisms of $\mathsf{Bim}_k(A)$ are bimodule maps. The monoidal structure on $\mathsf{Bim}_k(A)$ is given by tensor product of bimodules over A.

Note that there is a 2-category Alg_k with k-algebras, bimodules, and bimodule maps as the objects, 1-morphisms, and 2-morphisms. Then $\mathsf{Bim}_k(A)$ is the full sub-2-category of Alg_k generated by the object A.

Proposition 2.1.4. *The data of a 2-representation $\mathcal{U}^+ \to \mathsf{Bim}_k(A)$ for a k-algebra A consists of a bimodule ${}_AE_A$ and bimodule maps $x \in \mathrm{End}(E)$, $\tau \in \mathrm{End}(E^2)$ that satisfy (strictly) the relations of $\mathcal{U}^+$.*

We will use 'x_i' and 'τ_i' to denote the generators in any ${}^0H_n$ (where $i \leqslant n$ for x_i and $i < n$ for τ_i are assumed). Given a 2-representation for a k-algebra A with bimodule E, these symbols are also used to denote the corresponding elements in each $\mathrm{End}(E^n)$.

2.1.2.3 The 2-representation $\mathcal{L}(1)$

A simple 2-representation of $\mathcal{U}^+$ is given for the algebra $A = A_{+1} \times A_{-1}$, $A_i = k[y]$, by the bimodule $E = k[y]$, where $y \in A_{-1}$ acts on the left by 0 and on the right by multiplication by y, and $y \in A_{+1}$ acts on the right by 0 and the left by y. The Hecke actions are generated by $x \in \mathrm{End}(E)$ acting by multiplication by y, and $\tau \in \mathrm{End}(E^2)$ satisfies $\tau = 0$ because $E^2 = 0$.

2.1.3 Further conventions

Assume we are given a k-algebra A and a 2-representation for A with data $({}_AE_A, x, \tau)$, and fix these through §4. Assume that ${}_AE$ is finitely generated projective and that E^n is free as a P_n-module.

Consider the endomorphism $x-y$ of the $(A[y], A[y])$-bimodule $E[y]$. Its image $(x-y)E[y]$ is a sub-bimodule of $E[y]$. Write E_y for the quotient $E[y]/(x-y)E[y]$. (Alternatively: E_y is E extended to an $(A[y], A[y])$-bimodule by specifying that y acts on both sides by x.) The projection

$$\begin{aligned} \pi : E[y] &\to E_y \\ ey^n &\mapsto x^n(e) \end{aligned}$$

is a surjection of bimodules.

We simplify notation for tensor products by adopting a convention that concatenation indicates the tensor product over an algebra that is clear from the context. Sometimes it will be unclear whether a tensor product is meant over A or over $A[y]$, so we further stipulate that if the expression for a module contains 'y', it will be understood as an $A[y]$-module, and if the expression lacks 'y', it will be understood as an A-module. Concatenation will indicate tensor product over $A[y]$ if both are $A[y]$-modules, otherwise it will indicate tensor product over A.

We will tacitly use canonical isomorphisms such as

$$M[y] \otimes_{A[y]} N[y] \xrightarrow{\sim} M[y] \otimes_A N \xrightarrow{\sim} (MN)[y]$$

for M a right A-module and N a left A-module. For example, EE_y denotes $E \otimes_A E_y$ according to our convention, but this is canonically isomorphic to $E[y] \otimes_{A[y]} E_y$, and the latter may be written $E[y]E_y$. So we may write either EE_y or $E[y]E_y$ with equivalent meanings.

Extend x to $\mathrm{End}(E[y])$ by $x : ey^n \mapsto x(e)y^n$ and τ to $\mathrm{End}(E^2[y])$ by $\tau : eey^n \mapsto \tau(ee)y^n$. The map s defined above in terms of x and τ extends in the same manner to a map in $\mathrm{End}(E^2[y])$. Note that we denote an arbitrary element of $E[y]$ by the single letter 'e'. Similarly an arbitrary element of $E^2[y]$ is denoted by the doubled symbol 'ee', which may well not be a simple tensor of the form $e \otimes e$. Later we will use 'eee' or 'eee_i' as suggestive notation for elements of $E^3[y]$, and so on.

Define $\delta = \tau \circ (Ex - y) \in \mathrm{End}(E^2[y])$. We also consider the extensions of x_i and τ_i to $E^n[y]$, and then s_i and δ_i defined by their same formulas but replacing x with x_i and τ with τ_i. Some important identities are quickly verified:

Lemma 2.1.5. *We have*

- $s^2 = 1$, *so s is an isomorphism*
- $\delta^2 = \delta$, *so δ is an idempotent,*

and we also have $s_i^2 = 1$ and $\delta_i^2 = \delta_i$.

We adopt a flexible notation $y_i = x_i - y$ until §5. Here y_i indicates $\left(E^j x E^{i-1} - y\right)$ for some j, and context will determine the value of j. Note that $\delta_i = \tau_i y_i$.

One may check that $s \circ x_2 = x_1 \circ s$ and $s \circ x_1 = x_2 \circ s$. It follows that s exchanges y_2 and y_1 and descends to a map:

$$s : E_y \otimes_{A[y]} E[y] \to E[y] \otimes_{A[y]} E_y.$$

So we have $s : E^2 \to E^2$ a map of (A, A)-bimodules, and this induces $s : E^2[y] \to E^2[y]$ as well as $s : E_yE \to EE_y$, maps of $(A[y], A[y])$-bimodules. Context will determine the domain and codomain for the symbol s.

Lemma 2.1.6. *We also have:*

- $\pi_1 \circ \delta = s \circ \pi_2 : E^2[y] \to EE_y$.

We define projections $\pi_i : E^n[y] \to E^{n-i}E_yE^{i-1} = E^n[y]/(y_i)$ by $\pi_i = E^{n-i}\pi E^{i-1}$. The same names may be used for maps between products with E_y factors, for example $\pi_2 : EE_y \to E_yE_y$.

Given a module ${}_AM$, its algebra of endomorphisms $\mathrm{End}_A({}_AM)$ will use the traditional order of composition for multiplication: $(f \circ g)(m) = f(g(m))$. Typically, but not always, '$\circ$' is written to emphasize this convention. A consequence is that for a ring A, the algebra $\mathrm{End}_A({}_AA)$ is identified with A^{op}.

Given two complexes M, N of A-modules, we will write $\mathscr{H}om_A(M, N)$ for the complex generated by homogeneous A-module homomorphisms from M to N. In degree n it is given by homogeneous maps of degree n, and the differential is $d(f) = d \circ f - (-1)^{|f|} f \circ d$ for f a homogeneous map of degree $|f|$. The notation Z^iM refers to the degree i part of the kernel of d.

Given an algebra R, we write $D^b(R)$ for the derived category of bounded complexes of left R-modules. A strictly perfect complex of left R-modules is a bounded complex of finitely generated projective R-modules. The category per $R \subset D^b(R)$ is the full subcategory of complexes quasi-isomorphic to strictly perfect complexes. Given $M \in D^b(R)$, we write $\langle M \rangle_\Delta$ for the smallest triangulated strictly full subcategory of $D^b(R)$ closed under direct summands and containing M.

Lemma 2.1.7. *We have* $\langle R \rangle_\Delta = \mathit{per}\ R$.

2.1.4 Generalized matrix algebras and tensor product

Suppose we are given k-algebras A and D, bimodules ${}_AB_D$ and ${}_DC_A$, and bimodule maps

$$ {}_AB \otimes_D C_A \xrightarrow{\gamma_1} A $$
$$ {}_DC \otimes_A B_D \xrightarrow{\gamma_2} D. $$

With this data we can define a new k-algebra R:

$$ R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, $$

where multiplication of matrices is defined with the customary formulas using the above bimodule structures and maps.

A right R-module consists of the data of M_1 a right A-module, M_2 a right D-module, a map $M_1 \otimes_A B \xrightarrow{\alpha} M_2$ of right D-modules, and a map $M_2 \otimes_D C \xrightarrow{\beta} M_1$ of right A-modules, such that the latter two maps are compatible with γ_1 and γ_2. Here compatibility with γ_1, for example, means that the following compositions agree:

$$ M_1 \otimes_A (B \otimes_D C) \xrightarrow{\mathrm{Id}_{M_1} \otimes \gamma_1} M_1 \otimes_A A \xrightarrow{\sim} M_1 $$
$$ (M_1 \otimes_A B) \otimes_D C \xrightarrow{\alpha \otimes \mathrm{Id}_C} M_2 \otimes_D C \xrightarrow{\beta} M_1. $$

The data of a left R-module may be given in a similar form.

Let

$$ M = \begin{pmatrix} M_1 & M_2 \end{pmatrix} $$

be a right R-module, and

$$ N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} $$

a left R-module. Their tensor product $M \otimes_R N$ may be formed as follows. Consider the pair of maps given by the R action data:

$$M_1 \otimes_A B \otimes_D N_2 \xrightarrow{I_B} M_1 \otimes_A N_1 \oplus M_2 \otimes_D N_2$$

$$M_2 \otimes_D C \otimes_A N_1 \xrightarrow{I_C} M_1 \otimes_A N_1 \oplus M_2 \otimes_D N_2$$

by $I_B(m \otimes b \otimes n) = m \otimes b.n - m.b \otimes n$ and likewise for I_C. Then we have an isomorphism:

$$(M_1 \otimes_A N_1 \oplus M_2 \otimes_D N_2) \big/ (I_B + I_C) \xrightarrow{\sim} M \otimes_R N.$$

Now let $F \in \mathrm{End}_R(N)$ be an endomorphism of left R-modules. It determines an endomorphism $\mathrm{Id}_M \otimes_R F \in \mathrm{End}_k(M \otimes_R N)$ which will be denoted MF. We can study this on components as follows. There are induced endomorphisms $F_1 \in \mathrm{End}_A(N_1)$ and $F_2 \in \mathrm{End}_D(N_2)$ given by restriction of F. These determine endomorphisms $M_1F_1 \in \mathrm{End}_k(M_1 \otimes_A N_1)$ and $M_2F_2 \in \mathrm{End}_k(M_2 \otimes_D N_2)$, and these in turn provide together an endomorphism $\begin{pmatrix} M_1F_1 & 0 \\ 0 & M_2F_2 \end{pmatrix}$ of $M_1 \otimes_A N_1 \oplus M_2 \otimes_D N_2$. The property of full R-linearity of F implies that this morphism preserves the submodules I_B and I_C, and descends to the quotient $M \otimes_R N$ where it agrees with MF.

Lemma 2.1.8. *In the notations used above, an element of* $\mathrm{End}_k(M \otimes_R N)$ *of the form* MF *for* $F \in \mathrm{End}_R(N)$ *is uniquely determined by the induced maps* M_1F_1 *and* M_2F_2*.*

2.2 Product category

Given a 2-representation $\mathcal{V}$ for A with $\mathcal{U}^+$-action data (E, x, τ), we seek a 2-representation for C with data $(\tilde{E}, \tilde{x}, \tilde{\tau})$ to serve as the tensor 2-product $\mathcal{L}(1) \circledast \mathcal{V}$. In this section we describe our proposal for the algebra C and data $(\tilde{E}, \tilde{x}, \tilde{\tau})$, and in the next section we study this data and verify that the nil affine Hecke relations hold for $\tilde{x}$ and $\tilde{\tau}$.

2.2.1 Naive product category

2.2.1.1 Naive product algebra B

Definition 2.2.1. Let B be the k-algebra:

$$B = \begin{pmatrix} A[y] & E_y \\ 0 & A[y] \end{pmatrix}.$$

Here the algebra structure of B is given by matrix multiplication, with the $(A[y], A[y])$-bimodule structure of E_y contributing for products with generators in B_{12}.

A left B-module consists of a pair $\binom{M_1}{M_2}$ of left $A[y]$-modules, together with a morphism $\alpha : E_y \otimes_{A[y]} M_2 \to M_1$ of left $A[y]$-modules. A right B-module is the data of a pair $\begin{pmatrix} N_1 & N_2 \end{pmatrix}$ of right $A[y]$-modules, together with a morphism $\beta : N_1 \otimes_{A[y]} E_y \to N_2$ of right $A[y]$-modules. It follows that a (B, B)-bimodule can be written as a matrix of $(A[y], A[y])$-bimodules with accompanying maps α and β giving left and right actions of E_y. Such a matrix with α, β determines a (B, B)-bimodule only if the actions commute. Usually this commutativity is obvious and we do not bother to check it.

A complex of left B-modules is the same data as a pair of complexes of $A[y]$-modules together with a morphism α of complexes; note that the differential of $E_y \otimes M_2$ for a complex (M_2, d) is just $E_y \otimes d$. Similarly for right B-module complexes.

2.2.1.2 Endofunctor E' of B-cplx

Definition 2.2.2. Let E' be the following bounded complex of (B, B)-bimodules concentrated in degrees 0 and 1:

$$E' = \begin{pmatrix} E[y] & E[y]E_y \\ 0 & E[y] \end{pmatrix} \xrightarrow{d} \begin{pmatrix} E_y & E_y E_y \\ A[y] & E_y \end{pmatrix}.$$

Here the left action data 'α' for B generators in E_y is given on the degree 0 part as a matrix using the decompositions $0 \oplus E_y E[y]$ and $E[y] \oplus E[y]E_y$ by $\left(\begin{smallmatrix} 0 & 0 \\ 0 & s \end{smallmatrix}\right)$, and on the degree 1 part by $\begin{pmatrix} \mathrm{Id}_{E_y} & 0 \\ 0 & \mathrm{Id}_{E_y E_y} \end{pmatrix}$. The right action on the degree 0 part is by $\begin{pmatrix} \mathrm{Id}_{E[y]E_y} & 0 \\ 0 & 0 \end{pmatrix}$ and on degree 1 it is by $\begin{pmatrix} \mathrm{Id}_{E_y E_y} & 0 \\ 0 & \mathrm{Id}_{E_y} \end{pmatrix}$. The differential d is given componentwise by $\left(\begin{smallmatrix} \pi & \pi \otimes \mathrm{Id}_{E_y} \\ 0 & \pi \end{smallmatrix}\right)$.

Tensoring by E' on the left gives an endofunctor ${}_B E' \otimes_B -$ of the category of complexes of B-modules. It is convenient to have a formula for the action of this endofunctor on an arbitrary complex of modules:

Lemma 2.2.3. *Let $M = \left(\left(\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\right), \alpha\right)$ be a complex of B-modules. The functor $E' \otimes_B -$ acts on M by:*

$$\left(\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}, \alpha\right) \xmapsto{E'} \left(\begin{pmatrix} E[y]M_1 \overset{\pi M_1}{\overset{\frown}{\oplus}} E_y M_1[-1] \\ E[y]M_2 \overset{\alpha \circ \pi M_2}{\overset{\frown}{\oplus}} M_1[-1] \end{pmatrix}, \begin{pmatrix} E[y]\alpha \circ sM_2 & 0 \\ 0 & Id_{E_y} M_1 \end{pmatrix}\right).$$

Here the top and bottom rows express cocones of the maps πM_1 and $\alpha \circ \pi M_2$.

Remark 2.2.4. It may help motivation to consider the effect of E' at the level of the Grothendieck group when M_1 and M_2 are just modules, not complexes. The following discussion is not intended to be precise or complete.

Suppose M_1' and M_2' are projective left A-modules, and R_1 and R_2 are projective left $k[y]$-modules. Consider the projective left $A[y]$-modules $M_1 = R_1 \otimes_k M_1'$ and $M_2 = R_2 \otimes_k M_2'$. These are elements of the outer product of categories ($k[y]$-proj) $\boxtimes_k$ (A-proj). Suppose $\alpha : E_y M_2 \to M_1$ is given. Apply E' to $\left(\left(\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\right), \alpha\right)$. The upper row is quasi-isomorphic to:

$$\ker\left(E[y]M_1 \xrightarrow{\pi M_1} E_y M_1\right) \xrightarrow{\sim} (y_1 E[y])M_1 \xrightarrow{\sim} E[y]M_1 \xrightarrow{\sim} R_1 \otimes_k (E \otimes_A M_1'),$$

where the first isomorphism follows by flatness of M_1. Letting e denote the action of E on the Grothendieck group, we have $(1 \otimes e)\left([R_1] \otimes_k [M_1']\right)$ for the upper row in the Grothendieck group. The lower row is the cocone of α, which contributes $[E[y]M_2] + [M_1]$

in the Grothendieck group. Now recall that the raising functor for $\mathcal{L}(1)$ is just $k[y]$. So:

$$M_1 \xrightarrow{\sim} (k[y] \otimes 1)\big(R_1 \otimes_k M_1'\big), \quad [M_1] = (e \otimes 1)\big([R_1] \otimes_k [M_1']\big),$$

and we should interpret the copy of M_1 coming from the lower row in this way, since the factor of $k[y]$ in the $A[y] \cong k[y] \otimes_k A$ of the lower left corner of B is the higher weight copy. We also have $[E[y]M_2] = (e \otimes 1)\big([R_2] \otimes_k [M_2']\big)$. Finally, it is a fact that $(e \otimes 1)\big([R_2] \otimes_k [M_2']\big) = 0$ because $\mathcal{L}(1)$ has only two weight categories. It follows from these calculations that the action of $e' = [E']$ on the Grothendieck group of the derived category has the form:

$$\begin{aligned} e'[(\left(\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\right), \alpha)] &:= [E' \left(\left(\begin{smallmatrix} M_1 \\ M_2 \end{smallmatrix}\right), \alpha\right)] \\ &= (e \otimes 1 + 1 \otimes e)\big([M_1'] \otimes_k [R_1] + [M_2'] \otimes_k [R_2]\big). \end{aligned}$$

This agrees with the Hopf coproduct formula $\Delta(e) = e \otimes 1 + 1 \otimes e$.

Proof of the lemma. We first check that the matrix specifying the new E_y action gives a morphism of complexes. The diagonal coefficients of the matrix give morphisms of the separate summands, and these commute with the differentials on the separate summands. It remains to see that $\pi M_1 \circ E[y]\alpha \circ sM_2 = \mathrm{Id}_{E_y} M_1 \circ E_y(\alpha \circ \pi M_2)$, and these agree because $\pi E_y \circ s = E_y \pi$.

Now we compute the tensor product following the recipe of §2.1.4. We have:

$$E' \otimes_B M = \begin{pmatrix} \Big(E[y]M_1 \oplus E[y]E_y M_2\Big)/I_1 \overset{\pi M_1}{\overset{\curvearrowright}{\oplus}} \Big(\Big(E_y M_1 \oplus E_y E_y M_2\Big)/I_1'\Big)[-1] \\ \Big(0 \oplus E[y]M_2\Big)/I_2 \overset{\alpha \circ \pi M_2}{\overset{\curvearrowright}{\oplus}} \Big(\Big(A[y]M_1 \oplus E_y M_2\Big)/I_2'\Big)[-1] \end{pmatrix}.$$

Here the submodule I_1 is generated by all terms of the form $e \otimes \alpha(e', m_2) - e \otimes e' \otimes m_2$ for $e \in E[y], e' \in E_y, m_2 \in M_2$. So every element of the quotient has a canonical representative in $E[y]M_1$, and the quotient is isomorphic to $E[y]M_1$. With analogous reasoning we see that the quotient by I_1' is isomorphic to $E_y M_1$, that by I_2 is isomorphic to $E[y]M_2$, and that by

I_2' is isomorphic to M_1. The differential may be written before taking quotients as dM_1 on the top and dM_2 on the bottom. The images of dM_2 in E_yM_2 represent elements in M_1 by way of α, and this determines the differential component $\alpha \circ \pi M_2$ between summands of the bottom row.

Now we calculate the new E_y action in order to view this as a complex of B-modules. Using the description of the left B-action on E', one sees that the action on the left summand is by sM_2, which is represented in $E[y]M_1$ through α, so the action written on the quotients as described above is given by $E[y]\alpha \circ sM_2$. The action is obvious on the right summand. $\square$

2.2.1.3 Category per B and generator X

Definition 2.2.5. Let X be the following complex of B-modules:

$$X = X_1 \oplus X_2$$

$$X_1 = \begin{pmatrix} A[y] \\ 0 \end{pmatrix}$$

$$X_2 = E'(X_1) = \begin{pmatrix} E[y] & \xrightarrow{\pi} & E_y \\ 0 & \longrightarrow & A[y] \end{pmatrix}$$

where X_1 lies in degree 0 and X_2 in degrees 0 and 1. The E_y action on X_2 is by $E_y \otimes_{A[y]} A[y] \xrightarrow{\sim} E_y$, $e \otimes 1 \mapsto e$.

One can see that $X_1 = Be_1$ and $X_2 = E'e_1$, with $e_i \in B$ the standard matrix idempotent. Observe that there is a canonical right $A[y]$ action on Be_i and on X_i given componentwise.

Proposition 2.2.6. *The complex X is strictly perfect and generates per B.*

Proof. We can write X in terms of B:

$$X_1 = Be_1$$

$$X_2 = Be_1 \otimes_A E \to Be_2,$$

where the differential is by π on the upper row. This is a complex of finitely generated projective B-modules because ${}_AE$ is finitely generated and projective. So X is strictly perfect. To see that X generates per B, first note that $Be_1 = X_1 \in \langle X \rangle_\Delta$. Now consider $Be_1 \otimes_A E$ as a complex in degree 0. There is a map of complexes $X_2 \to Be_1 \otimes_A E$ given by the identity in degree 0 and by 0 in degree 1. Then $Be_2[-1]$ (a complex in degree 1) is quasi-isomorphic to the cocone of this map. So $Be_2 \in \langle X \rangle_\Delta$. □

Recall our notation $\pi_i = E^{n-i}\pi E^{i-1} : E^n[y] \to E^{n-i}E_yE^{i-1}$.

Lemma 2.2.7. *The kernel of* $\varphi : E^n[y] \xrightarrow{(\pi_i)_i} \bigoplus_{i=1}^n E^{n-i}E_yE^{i-1}$ *is* $(y_1 \dots y_n)E^n[y]$.

Proof. We have assumed that E^n is free as a P_n-module. It follows that $E^n[y]$ is free as a $P_n[y]$-module. Let $e \in \ker\varphi$. So $\pi_i(e) = 0$ and therefore $e \in y_iE^n[y]$ for each $i \in \{1, \dots, n\}$. Let B be a basis of $E^n[y]$ over $P_n[y]$. Write

$$e = y_i \sum_{j=1}^{\ell} f_j^i(x_1, \dots, x_n, y) \cdot b_j$$

for $b_j \in B$ distinct and $f_j^i \in P_n[y]$. It follows that $y_if_j^i = y_kf_j^k$ in $P_n[y]$ for each $(i,k) \in \{1, \dots, n\}^{\times 2}$ and $j \in \{1, \dots, \ell\}$. Then $e = y_1 \dots y_n e^\circ$ for some $e^\circ \in E^n[y]$ because $P_n[y]$ is a unique factorization domain and each y_i is irreducible. □

Lemma 2.2.8. *The complex* $E'X_2$ *is concentrated in degrees* 0, 1, *and* 2*:*

$$E'X_2 = \left(\begin{pmatrix} E^2[y] \xrightarrow{(\pi_2,\pi_1)} E_yE \oplus EE_y \xrightarrow{(-\pi_1,\pi_2)} E_yE_y \\ 0 \longrightarrow E[y] \oplus E[y] \xrightarrow{(-\pi,\pi)} E_y \end{pmatrix}, \alpha \right),$$

where

$$\alpha_0 = 0$$

$$\alpha_1 = \left(\begin{smallmatrix} Id_{E_yE} & 0 \\ 0 & s \end{smallmatrix}\right)$$

$$\alpha_2 = Id_{E_yE_y}.$$

Proof. Computation. The minus signs arise from shifting differentials. □

Proposition 2.2.9. *The complex $E'X$ is quasi-isomorphic to a finite direct sum of summands of X.*

We define two complexes of B-modules before proving the proposition.

Definition 2.2.10. Let $R, X_2' \in B$-cplx be given by

$$R = \begin{pmatrix} E^2[y] \xrightarrow{\left(\begin{smallmatrix} \pi_2 \\ \pi_2 \circ \tau \end{smallmatrix}\right)} E_yE \oplus E_yE \\ 0 \to E[y] \oplus E[y] \end{pmatrix},$$

$$X_2' = \begin{pmatrix} \tau y_1 E^2[y] & \xrightarrow{\pi_2} & E_yE \\ 0 & \longrightarrow & E[y] \end{pmatrix},$$

both lying in degrees 0 and 1, and the E_y action on R is by the canonical map

$$E_y \otimes (E[y] \oplus E[y]) \to E_yE \oplus E_yE,$$

and on X_2' by the canonical map $E_y \otimes E[y] \to E_yE$.

Lemma 2.2.11. *We have that X_2' is a finite direct sum of summands of X.*

Proof. Observe first that $X_2 \otimes_A E$ is a finite direct sum of summands of X because ${}_AE$ is finitely generated projective. (Here we use the componentwise right A-action on X_2.) Using the formulas

$$\pi_2 \circ \delta = \pi_2,$$
$$\pi_2 \circ (1-\delta) = 0,$$

and $\delta \cdot (1-\delta) = 0$, one has the decomposition of $X_2 \otimes_A E$:

$$X_2 \otimes_A E = \begin{pmatrix} E^2[y] & \xrightarrow{\pi_2} & E_y E \\ 0 & \longrightarrow & E[y] \end{pmatrix}$$
$$= \begin{pmatrix} \delta \cdot E^2[y] & \xrightarrow{\pi_2} & E_y E \\ 0 & \longrightarrow & E[y] \end{pmatrix} \oplus \begin{pmatrix} (1-\delta) \cdot E^2[y] \\ 0 \end{pmatrix}.$$

□

The matrix algebra structure of the nil-affine Hecke algebra gives the following isomorphism of left $A[y]$-modules:

$$E^2[y] \xrightarrow[\left(\begin{smallmatrix} \tau y_1 \\ \tau \end{smallmatrix}\right)]{\sim} \tau y_1 E^2[y] \oplus \tau y_1 E^2[y].$$

Lemma 2.2.12. *There is an isomorphism $R \xrightarrow{\sim} X_2' \oplus X_2'$ in B-cplx given by the above isomorphism on the degree 0 term of the upper row, and the identity on all other terms. So R is a finite direct sum of summands of X_2, and hence of X. In particular, R is strictly perfect.*

Lemma 2.2.13. *There is a quasi-isomorphism $R \xrightarrow{q.i.} E'X_2$ determined by $Id_{E^2[y]}$ on the degree 0 term of the upper row and $\left(\begin{smallmatrix} 1 & 0 \\ 1 & -y_1 \end{smallmatrix}\right)$ on the degree 1 term of the lower row.*

Proof. We first check that the map is a morphism in B-cplx. The matrix of the morphism on the degree 1 part of the upper row, as determined by equivariance over generators of B

in E_y, is given by $\left(\begin{smallmatrix} \mathrm{Id} & 0 \\ s & s\circ(x_2-x_1)\end{smallmatrix}\right)$. Observe that:

$$\begin{aligned}
&\mathrm{Id}\circ\pi_2 + 0\circ\pi_2\circ\tau = \pi_2;\\
&s\circ\pi_2 + s\circ(x_2-x_1)\circ\pi_2\circ\tau\\
&\ = \pi_1\circ s + (x_1-x_2)\circ s\circ\pi_2\circ\tau\\
&\ = \pi_1\circ s + \pi_1\circ(x_1-x_2)\circ s\circ\tau\\
&\ = \pi_1\circ\Big((x_2-x_1)\circ\tau + \mathrm{Id}\\
&\qquad + (x_1-x_2)\circ\big((x_2-x_1)\circ\tau+\mathrm{Id}\big)\circ\tau\Big) = \pi_1.
\end{aligned}$$

This shows compatibility with the differential from degree 0 in the upper row. The other compatibility checks are easier.

Now we show that the map is a quasi-isomorphism. The lower row of $E'X_2$ has H^1 given by:

$$\{(e_1,e_2)\in E[y]^{\oplus 2} \mid e_1-e_2 = y_1e \text{ for some } e\in E[y]\}.$$

This is also the image of the (injective) map from R in degree 1 of the lower row. The upper row of $E'X_2$ has $H^0 = \ker(d^0) = y_1y_2E^2[y]$ by Lemma 2.2.7. The cohomology of the upper row of R is computed as follows. We have an isomorphism:

$$E^2[y] \xrightarrow{\sim} \tau y_1E^2[y] \oplus -y_2\tau E^2[y].$$

Notice that $\pi_2\circ\tau$ vanishes on the first summand, and π_2 vanishes on the second. Then one may compute:

$$\ker\big(\tau y_1E^2[y] \xrightarrow{\pi_2} E_yE\big) = \tau y_1y_2E^2[y] \subset y_1y_2E^2[y]$$

and

$$\ker\big(-y_2\tau E^2[y] \underset{\sim}{\xrightarrow{\tau}} \tau y_1E^2[y] \xrightarrow{\pi_2} E_yE\big) = -y_2\tau y_1y_2E^2[y] \subset y_1y_2E^2[y].$$

So

$$\ker\left(\left(\begin{smallmatrix}\pi_2\\ \pi_2\circ\tau\end{smallmatrix}\right)\right) \subset y_1y_2E^2[y].$$

The reverse inclusion is obvious, so H^0 of the upper row is $y_1y_2E^2[y]$. This shows that $\mathrm{Id}_{E^2[y]}$ induces an isomorphism on homology in degree 0 of the upper row. Using the decomposition and inspecting the maps above, we also see that d^0 on the upper row of R is surjective. Finally we consider H^1 of the upper row of $E'X_2$ and show it is zero. (Clearly the H^2 is zero.) Let $(ee_1, ee_2) \in E_yE \oplus EE_y$ be in $\ker(d^1)$, i.e. such that $\pi_1(ee_1) = \pi_2(ee_2)$. Then $ee_1 = ee_2 + (Ex - xE)ee^\circ$ for some $ee^\circ \in E^2$. (Note that $E_yE_y \cong E^2/(Ex - xE)$ where y acts by Ex or xE.) Then consider $ee_2 + (Ex - y)ee^\circ \in E^2[y]$. The differential d^0 sends this to ee_1 in E_yE and to ee_2 in EE_y. □

Proof of Proposition 2.2.9. The proposition follows from the preceding three lemmas. □

Corollary 2.2.14. *Tensoring with ${}_BE'_B$ gives an endofunctor $E' \otimes_B -$ of per B.*

Proof. We know that $X \in \mathrm{per}\, B$, and it follows from Prop. 2.2.9 that $E' \otimes_B X \in \mathrm{per}\, B$. The corollary follows because X generates per B. □

Remark 2.2.15. We do not know that $E' \otimes_B -$ on $K^b(B)$ is exact, so we do not know that it descends to an endofunctor defined on all of $D^b(B)$.

2.2.2 Bimodules G_n

The constructions of this chapter make use of certain bimodules that we describe next.

Definition 2.2.16. Let G_n denote $\mathrm{Hom}_{K^b(B)}(X_2, E'^n X_1)$.

Every G_n has the structure of $(G_1^{\mathsf{op}}, A[y])$-bimodule by pre- and post-composition. Here we understand $A[y] \cong \mathrm{End}_{K^b(B)}(X_1)^{\mathsf{op}}$ and use functoriality of E' for the action. Note that

$G_1 = \mathrm{Hom}_{K^b(B)}(X_2, X_2)$ has an algebra structure, and the right regular action of G_1^{op} on G_1 extends the right $A[y]$ action.

In this section we gather some facts regarding these bimodules and give concrete presentations in small cases that are easier to handle. Given $n \in \{1, 2, 3, 4\}$, we define $\bar{G}_n$ as an $(A[y], A[y])$-sub-bimodule of $E^{n-1}[y]^{\oplus n} \oplus \mathrm{Hom}_A({}_AE, E^n)[y]$. (By $E^0[y]$ we mean $A[y]$.) We give isomorphisms $\bar{G}_n \xrightarrow{\sim} G_n$ for such n. These isomorphisms induce left G_1^{op}-actions on $\bar{G}_n$ that extend the left $A[y]$-actions. In future sections we do not distinguish G_n from $\bar{G}_n$ and write only the former.

Definition 2.2.17. Define the following $(A[y], A[y])$-sub-bimodule of $A^{\mathsf{op}}[y] \oplus \mathrm{End}_A({}_AE)[y]$:

$$\bar{G}_1 = \Big\langle (\theta, \varphi) \in A^{\mathsf{op}}[y] \oplus \mathrm{End}_A({}_AE)[y] \Big| \; \varphi = \_.\theta + y_1\varphi_1 \text{ for some } \varphi_1 \in \mathrm{End}_A({}_AE)[y] \Big\rangle.$$

This bimodule also has a k-algebra structure with componentwise multiplication (using the opposite multiplication on generators in $A[y]$).

Note that $\bar{G}_1$ contains a copy of $A^{\mathsf{op}}[y]$, namely the subspace with $\varphi = \_.\theta$.

Proposition 2.2.18. *There is an isomorphism of $(A[y], A[y])$-bimodules $\bar{G}_1 \xrightarrow{\sim} G_1$ determined by:*

$$(\theta, \varphi) \mapsto \left(\begin{pmatrix} (e, 0) \\ (0, 1) \end{pmatrix} \mapsto \begin{pmatrix} (\varphi(e), 0) \\ (0, \theta) \end{pmatrix} \right).$$

Here $(e, 0) \in E[y] \oplus E_y$ is an element of the upper row of X_2, with e in degree 0 and 0 in degree 1. Analogously with the lower row. This isomorphism respects the k-algebra structure.

Proof. The condition $\varphi = \_.\theta + y_1\varphi_1$ in the definition of $\bar{G}_1$ is equivalent to the statement that the morphism given as the image of (θ, φ) defined in the proposition has zero differential. □

Definition 2.2.19. Define the following $(A[y], A[y])$-sub-bimodule of $E[y]^{\oplus 2} \oplus \mathrm{Hom}_A({}_AE, E^2)[y]$:

$$\begin{aligned}\bar{G}_2 = \Big\langle (e_1, e_2, \xi) \in E[y]^{\oplus 2} \oplus \mathrm{Hom}_A({}_AE, E^2)[y] \Big| \\ e_1 - e_2 = y_1 e' \\ \xi = {}_{-} \otimes e_1 + y_2 \xi_1 \\ = \delta({}_{-} \otimes e_2) + y_1 \xi_2 \\ \text{for some } e' \in E[y] \text{ and } \xi_\ell \in \mathrm{Hom}_A({}_AE, E^2)[y] \Big\rangle.\end{aligned}$$

Proposition 2.2.20. *There is an isomorphism of* $(A[y], A[y])$*-bimodules* $\bar{G}_2 \xrightarrow{\sim} G_2$ *determined by:*

$$(e_1, e_2, \xi) \mapsto \left(\begin{pmatrix} (e, 0) \\ (0, 1) \end{pmatrix} \mapsto \begin{pmatrix} (\xi(e), 0, 0) \\ (0, \left(\begin{smallmatrix} e_1 \\ e_2 \end{smallmatrix}\right), 0) \end{pmatrix} \right).$$

Proof. Use the description of $E'X_2$ in Lemma 2.2.8. As in Prop. 2.2.18, the condition of the definition of $\bar{G}_2$ is equivalent to the statement that the image of (e_1, e_2, ξ) has zero differential. □

In order to parametrize G_3, we compute the components of $E'^2X_2 = E'^3X_1$ in degrees 0, 1, and 2:

$$\begin{pmatrix} E^3[y] & \to & E_yEE \oplus EE_yE \oplus EEE_y & \to & E_yE_yE \oplus E_yEE_y \oplus EE_yE_y \to \dots \\ 0 & \to & E^2[y] \oplus E^2[y] \oplus E^2[y] & \to & E_yE \oplus EE_y \oplus EE_y \to \dots \end{pmatrix}.$$

The upper left differential map is (π_3, π_2, π_1). We don't make use of the upper right. The bottom right differential map is given by the matrix:

$$\begin{pmatrix} -\pi_2 & \pi_2 & 0 \\ -\pi_1 & 0 & \pi_1 \circ \delta \\ 0 & -\pi_1 & \pi_1 \end{pmatrix}.$$

Definition 2.2.21. Define the following $(A[y], A[y])$-sub-bimodule of $E^2[y]^{\oplus 3} \oplus \mathrm{Hom}_A({}_AE, E^3)[y]$:

$$\bar{G}_3 = \Big\langle (ee_1, ee_2, ee_3, \chi) \in E^2[y]^{\oplus 3} \oplus \mathrm{Hom}_A({}_AE, E^3)[y] \Big|$$

$$\begin{aligned} ee_1 - ee_2 &= y_2 ee' \\ ee_3 - ee_2 &= y_1 ee'' \\ \delta(ee_3) - ee_1 &= y_1 ee''', \end{aligned}$$

$$\begin{aligned} \chi &= {}_{-} \otimes ee_1 + y_3 \chi_1 \\ &= \delta E({}_{-} \otimes ee_2) + y_2 \chi_2 \\ &= E\delta \circ \delta E({}_{-} \otimes ee_3) + y_1 \chi_3 \\ &\text{for some } ee^k \in E^2[y] \text{ and } \chi_\ell \in \mathrm{Hom}_A({}_AE, E^3)[y] \Big\rangle. \end{aligned}$$

Proposition 2.2.22. *There is an isomorphism of $(A[y], A[y])$-bimodules $\bar{G}_3 \xrightarrow{\sim} G_3$ determined by:*

$$(ee_1, ee_2, ee_3, \chi) \mapsto \left(\begin{pmatrix} (e, 0) \\ (0, 1) \end{pmatrix} \mapsto \begin{pmatrix} (\chi(e), 0, \dots) \\ (0, \left(\begin{smallmatrix} ee_1 \\ ee_2 \\ ee_3 \end{smallmatrix}\right), \dots) \end{pmatrix} \right).$$

Proof. The condition of the definition of $\bar{G}_3$ is equivalent to the statement that the image of (ee_1, ee_2, ee_3, χ) has zero differential. □

Definition 2.2.23. Define the following $(A[y], A[y])$-sub-bimodule of $E^3[y]^{\oplus 4} \oplus \mathrm{Hom}_A({}_AE, E^4)[y]$:

$$\begin{aligned}
\bar{G}_4 = \Big\langle (eee_1, eee_2, eee_3, eee_4, \psi) \in E^3[y]^{\oplus 4} \oplus \mathrm{Hom}_A({}_AE, E^4)[y] \Big| \\
eee_3 - eee_4 &= y_1 eee^{(1)} \\
eee_2 - eee_3 &= y_2 eee^{(2)} \\
E\delta(eee_4) - eee_2 &= y_1 eee^{(3)} \\
eee_1 - eee_2 &= y_3 eee^{(4)} \\
eee_1 - \delta E(eee_3) &= y_2 eee^{(5)} \\
eee_1 - \delta E \circ E\delta(eee_1) &= y_1 eee^{(6)} \\
\psi &= {}_{-} \otimes eee_1 + y_4 \psi_1 \\
&= \delta E^2({}_{-} \otimes eee_2) + y_3 \psi_2 \\
&= E\delta E \circ \delta E^2({}_{-} \otimes eee_3) + y_2 \chi_3 \\
&= E^2\delta \circ E\delta E \circ \delta E^2({}_{-} \otimes eee_4) + y_1 \chi_4 \\
&\text{for some } eee^k \in E^3[y] \text{ and } \psi_\ell \in \mathrm{Hom}_A({}_AE, E^4)[y] \Big\rangle.
\end{aligned}$$

Lemma 2.2.24. *Under the conditions on eee_i in the definition, there is a unique $\overline{eee} \in E^3[y]$ such that:*

$$eee^{(5)} - eee^{(2)} = y_3 \overline{eee},$$
$$eee^{(4)} - \tau E(eee_3) = y_2 \overline{eee}.$$

Proof. Subtracting two equations from those conditions:

$$\begin{aligned}
y_2 \left(eee^{(5)} - eee^{(2)} \right) &= eee_1 - eee_2 - y_3 \tau E(eee_3) \\
&= y_3 \left(eee^{(4)} - \tau E(eee_3) \right)
\end{aligned}$$

By Lemma 2.2.7 we know there is some $\overline{eee}$ satisfying the claim. It is unique because the y_i

are injective. ☐

Proposition 2.2.25. *There is an isomorphism of* $(A[y],A[y])$*-bimodules* $\bar{G}_4 \xrightarrow{\sim} G_4$ *determined by:*

$$(eee_1, eee_2, eee_3, eee_4, \psi) \mapsto \left(\begin{pmatrix} (e,0) \\ \\ (0,1) \end{pmatrix} \mapsto \begin{pmatrix} (\psi(e), 0, \dots) \\ (0, \begin{pmatrix} eee_1 \\ eee_2 \\ eee_3 \\ eee_4 \end{pmatrix}, \dots) \end{pmatrix} \right).$$

Proof. The reader may compute the first terms of $E'^4 X_1$ and show that the condition of the definition of $\bar{G}_4$ is equivalent to the statement that the image of $(ee_1, ee_2, ee_3, ee_4, \psi)$ defined in the proposition has zero differential. There is some ambiguity in the order of summands in degree 1 of the lower row. The convention we have used is that the first summand arises from the latest application of E' which moves a term from degree 0 of the upper row to degree 1 of the lower (and increments the exponents on existing terms in the lower row). ☐

It will be useful to describe alternative, equivalent, conditions defining $\bar{G}_2$ and $\bar{G}_3$. It is sometimes easier to work with them.

Proposition 2.2.26. *Given* $(e_1, e_2, \xi) \in E[y]^{\oplus 2} \oplus \mathrm{Hom}_A({}_A E, E^2)[y]$ *with* $e_1 - e_2 = y_1 e'$*, the following conditions are equivalent:*

$$\begin{aligned} \xi &= {}_\otimes e_1 + y_2 \xi_1 \\ &= \delta({}_\otimes e_2) + y_1 \xi_2 \\ &\quad \textit{for some } \xi_\ell \in \mathrm{Hom}_A({}_A E, E^2)[y] \end{aligned}$$

and

$$\begin{aligned} \xi &= {}_\otimes e_1 + y_2 \xi_1 \\ \xi_1 &= \tau({}_\otimes e_2) + y_1 \xi' \\ &\quad \textit{for some } \xi' \in \mathrm{Hom}_A({}_A E, E^2)[y]. \end{aligned}$$

When these conditions hold, the ξ_ℓ *and* ξ' *are uniquely determined by the data* (e_1, e_2, ξ)*, and*

$\xi_2 = {}_{-} \otimes e' + y_2 \xi'$.

Proof. Suppose the first condition holds. Using $\delta = y_2\tau + \mathrm{Id}$ and $e_1 - e_2 = (x-y)e'$, we can rearrange the first equality:

$$ {}_{-} \otimes e_1 + y_2\xi_1 = y_1\xi_2 + y_2\tau({}_{-} \otimes e_2) + {}_{-} \otimes e_2, $$

from which

$$ y_2\Big(\xi_1 - \tau({}_{-} \otimes e_2)\Big) = y_1\Big(\xi_2 - {}_{-} \otimes e'\Big). $$

By Lemma 2.2.7, the image of $\xi_1 - \tau({}_{-} \otimes e_2)$ is in $y_1 y_2 E^2[y]$. We can then make the following definition:

$$ \xi' = y_1^{-1}(\xi_1 - \tau({}_{-} \otimes e_2)). $$

The second condition and the final claim follow from this.

Starting now with the second condition, plugging the second equation into the first, we find:

$$ \begin{aligned} \xi &= {}_{-} \otimes e_1 + y_2\big(\tau({}_{-} \otimes e_2) + y_1\xi'\big) \\ &= \delta({}_{-} \otimes e_2) + {}_{-} \otimes (e_1 - e_2) + y_2 y_1 \xi' \\ &= \delta({}_{-} \otimes e_2) + y_1\big({}_{-} \otimes e' + y_2\xi'\big). \end{aligned} $$

This is the second line of the first condition, and it establishes the final claim.

The uniqueness claims are clear. □

Proposition 2.2.27. *Given* $(ee_1, ee_2, ee_3, \chi) \in E^2[y]^{\oplus 3} \oplus \mathrm{Hom}_A({}_A E, E^3)[y]$ *with*

$$ ee_1 - ee_2 = y_2 ee' \tag{2.2.1} $$

$$ ee_3 - ee_2 = y_1 ee'' \tag{2.2.2} $$

$$ \delta(ee_3) - ee_1 = y_1 ee''', \tag{2.2.3} $$

the following conditions are equivalent:

$$
\begin{aligned}
\chi &= {}_{-}\otimes ee_1 + y_3\chi_1 \\
&= \delta E({}_{-}\otimes ee_2) + y_2\chi_2 \\
&= E\delta \circ \delta E({}_{-}\otimes ee_3) + y_1\chi_3 \\
&\quad \textit{for some } \chi_\ell \in \operatorname{Hom}_A({}_AE, E^3)[y]
\end{aligned}
$$

and

$$
\begin{aligned}
\chi &= {}_{-}\otimes ee_1 + y_3\chi_1 \\
\chi_1 &= \tau E({}_{-}\otimes ee_2) + y_2\chi_1' \\
\chi_1' &= E\tau \circ \tau E({}_{-}\otimes ee_3) + y_1\chi'' \\
&\quad \textit{for some } \chi'' \in \operatorname{Hom}_A({}_AE, E^3)[y].
\end{aligned}
$$

When the conditions hold, the χ_ℓ and χ'' are uniquely determined by the data (ee_1, ee_2, ee_3, χ), and there is a unique $\overline{ee} \in E^2[y]$ such that

$$
\begin{aligned}
\tau(ee_3) - ee' &= y_1\overline{ee} \\
ee''' - ee'' &= y_2\overline{ee}.
\end{aligned}
$$

Define a map $\chi_2' = -{}_{-}\otimes \overline{ee} + y_3\chi''$. Then we also have

$$
\chi_2 = E\tau \circ \delta E({}_{-}\otimes ee_3) + y_1\chi_2'
$$

and

$$
\chi_3 = -\delta E({}_{-}\otimes ee'') + y_2\chi_2'.
$$

Assuming $\chi = {}_{-}\otimes ee_1 + y_3\chi_1$, the other two conditions together are equivalent to a single *condition on χ_1:*

$$\chi_1 = -\tau E y_1(\_ \otimes ee'') + E\delta \circ \tau E(\_ \otimes ee_3) + y_2 y_1 \chi''.$$

Proof. Suppose the first condition holds. Equating the first two formulas for χ in the first condition and using $\delta E = y_3 \tau E + \mathrm{Id}$ gives:

$$\_ \otimes ee_1 + y_3\chi_1 = y_3\tau E(\_ \otimes ee_2) + \_ \otimes ee_2 + y_2\chi_2$$

thus

$$y_3\big(\chi_1 - \tau E(\_ \otimes ee_2)\big) = y_2\big(\chi_2 - \_ \otimes ee'\big).$$

By Lemma 2.2.7 again, the image of this function lies in $y_2 y_3 E^3[y]$, and since each y_i is injective, we can define a new function χ_1' such that:

$$\chi_1 = \tau E(\_ \otimes ee_2) + y_2\chi_1'$$
$$\chi_2 = \_ \otimes ee' + y_3\chi_1'.$$

Equating now the second and third formulas, we have:

$$y_2 E\tau \circ \delta E(\_ \otimes ee_3) + \delta E(\_ \otimes ee_3) + y_1\chi_3 = \delta E(\_ \otimes ee_2) + y_2\chi_2$$

so

$$y_2\big(\chi_2 - E\tau \circ \delta E(\_ \otimes ee_3)\big) = y_1\big(\chi_3 + \delta E(\_ \otimes ee'')\big),$$

so for some χ_2' we can write:

$$\chi_2 = E\tau \circ \delta E(\_ \otimes ee_3) + y_1\chi_2'$$
$$\chi_3 = -\delta E(\_ \otimes ee'') + y_2\chi_2'.$$

We will need a fact derived from the relations (2.2.1)–(2.2.3) of the ee^k. Adding the first and third relations and subtracting the second yields

$$y_1\big(ee''' - ee''\big) = y_2\big(\tau(ee_3) - ee'\big),$$

from which we see there must be a (unique) $\overline{ee}$ with

$$\tau(ee_3) - ee' = y_1\overline{ee}$$
$$ee''' - ee'' = y_2\overline{ee}.$$

This gives the third claim of the proposition.

Equating now the two formulas we derived for χ_2:

$$y_3E\tau \circ \tau E(\_ \otimes ee_3) + E\tau(\_ \otimes ee_3) + y_1\chi_2' = \_ \otimes ee' + y_3\chi_1'$$

so

$$y_3\big(\chi_1' - E\tau \circ \tau E(\_ \otimes ee_3)\big) = y_1\big(\chi_2' + \_ \otimes \overline{ee}\big).$$

Therefore

$$\chi_1' = E\tau \circ \tau E(\_ \otimes ee_3) + y_1\chi''$$
$$\chi_2' = -\_ \otimes \overline{ee} + y_3\chi''$$

for some χ'', as desired.

In the reverse direction, starting with the second condition, plugging the χ_1 and χ_1' formulas into the first χ formula gives:

$$\chi = \_ \otimes ee_1 + y_3\Big(\tau E(\_ \otimes ee_2) + y_2\big(E\tau \circ \tau E(\_ \otimes ee_3) + y_1\chi''\big)\Big),$$

so

$$\begin{aligned}\chi - \delta E(\_\otimes ee_2) &= \_\otimes (ee_1 - ee_2) \\ &\qquad + y_2\big(E\tau \circ \tau E(\_\otimes ee_3) + y_1\chi''\big) \\ &= y_2\Big(\_\otimes ee' + E\tau \circ \tau E(\_\otimes ee_3) + y_1\chi''\Big),\end{aligned}$$

as desired. Similarly:

$$\begin{aligned}\chi - E\delta \circ \delta E(\_\otimes ee_3) &= \chi - y_3 y_2 E\tau \circ \tau E(\_\otimes ee_3) \\ &\qquad - y_3\tau E(\_\otimes ee_3) - E\delta(\_\otimes ee_3) \\ &= \_\otimes ee_1 + y_3\big(\tau E(\_\otimes ee_2) + y_1 y_2\chi''\big) \\ &\qquad - y_3\tau E(\_\otimes ee_3) - E\delta(\_\otimes ee_3) \\ &= \_\otimes \big(ee_1 - \delta(ee_3)\big) + y_1\Big(-y_3\tau E(\_\otimes ee'') + y_2 y_3\chi''\Big) \\ &= y_1\Big(-\_\otimes ee''' - y_3\tau E(\_\otimes ee'') + y_2 y_3\chi''\Big).\end{aligned}$$

The final statement of the proposition is a rearrangement of the second and third equalities of the second condition. □

Remark 2.2.28. We will not need to use alternative conditions for G_n for $n \geqslant 4$.

2.2.3 Product category C-mod

Let $C = \mathrm{End}_{\mathrm{per}\, B}(X)^{\mathsf{op}}$. We 'change basis' from $Be_1 \oplus Be_2$ to $X_1 \oplus X_2$, i.e. from complexes of modules over B to complexes of modules over C. This is performed by $\mathscr{H}om_B(X,-)$:

$$\mathrm{per}\ B \xrightarrow[\mathscr{H}om_B(X,-)]{\sim} \mathrm{per}\ C,$$

which is a restricted Rickard (derived Morita) equivalence. It has an inverse given by $X \otimes_C -$. Under this equivalence, the action of ${}_B E' \otimes_B -$ on per B translates to ${}_C\tilde{E} \otimes_C -$ on per C, where $\tilde{E}$ is a (C, C)-bimodule that is finitely generated and projective on the left. Our main theorem says that $\mathsf{Bim}_k(C)$ has the structure of 2-representation of $\mathcal{U}^+$ using $\tilde{E}$. In this section we describe C and the derived equivalence in more detail.

2.2.3.1 New algebra C

Let $\mathscr{C} = \mathscr{E}nd_B(X_1 \oplus X_2)^{\mathsf{op}}$ be the dg-algebra of endomorphisms of X (with left-to-right composition).

Definition 2.2.29. Define two $(A[y], A[y])$-bimodules:

$$G_1' = A[y] \oplus \mathrm{Hom}_{A[y]}({}_{A[y]}E[y], E[y])$$

and

$$G_1'' = \mathrm{Hom}_{A[y]}({}_{A[y]}E[y], E_y).$$

The complex $\mathscr{E}nd_B(X_2)$ is given in degrees 0 and 1 by

$$G_1' \xrightarrow{d^0} G_1''$$

where

$$d^0\big((\theta(y), \varphi)\big) = \pi \circ \varphi - \pi(-).\theta(x).$$

The direct sum decomposition $X_1 \oplus X_2$ provides a matrix presentation for $\mathscr{C}$ with $\mathscr{C}_{ij} = \mathscr{H}om_B(X_i, X_j)$.

Definition 2.2.30. Let F denote the (A, A)-bimodule

$$F = \mathrm{Hom}_A({}_AE, A).$$

Note the canonical isomorphism

$$\mathrm{Hom}_A({}_AE, A)[y] \xrightarrow{\sim} \mathrm{Hom}_{A[y]}({}_{A[y]}E[y], A[y])$$

that exists because ${}_AE$ is finitely generated. Since ${}_AE$ and ${}_{A[y]}E[y]$ are both finitely generated projective, we also have canonical isomorphisms of functors:

$$\mathrm{Hom}_A({}_AE, -) \xrightarrow{\sim} \mathrm{Hom}_A({}_AE, A) \otimes_A -$$

$$\mathrm{Hom}_{A[y]}({}_{A[y]}E[y], -) \xrightarrow{\sim} \mathrm{Hom}_{A[y]}({}_{A[y]}E[y], A[y]) \otimes_{A[y]} -.$$

Proposition 2.2.31. *The algebra $\mathscr{C}$ is isomorphic to a generalized matrix algebra of complexes concentrated in degrees* 0 *and* 1*:*

$$\begin{pmatrix} A[y] & E[y] \xrightarrow{\pi} E_y \\ F[y] & G_1'^{\mathsf{op}} \xrightarrow{d^0} G_1''^{\mathsf{op}} \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} \mathscr{C}_{11} & \mathscr{C}_{12} \\ \mathscr{C}_{21} & \mathscr{C}_{22} \end{pmatrix}.$$

The map is given on components by:

- *for* $\mathscr{C}_{11}$*:*

$$A[y] \ni a \mapsto \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} a \\ 0 \end{pmatrix} \right)$$

- *for* $\mathscr{C}_{12}$*:*

$$(E[y] \to E_y) \ni (e, e') \mapsto \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} (e, e') \\ 0 \end{pmatrix} \right)$$

- *for* $\mathscr{C}_{21}$*:*

$$F[y] \ni f \mapsto \left(\begin{pmatrix} (e, 0) \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} f(e) \\ 0 \end{pmatrix} \right)$$

- *for* $\mathscr{C}_{22}$*:*

$$\left(G_1'^{\mathsf{op}} \to G_1''^{\mathsf{op}}\right) \ni \left((\theta, \varphi'), \varphi''\right) \mapsto \left(\begin{pmatrix} (e, 0) \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} \left(\varphi'(e), (\pi \circ \varphi'')(e)\right) \\ \theta \end{pmatrix} \right).$$

Proof. Computation. □

Definition 2.2.32. Let C denote the k-algebra $\mathrm{End}_{K^b(B)}(X)^{\mathsf{op}}$.

Sometimes we consider C to be a dg-algebra concentrated in degree 0.

Lemma 2.2.33. *The projection $Z^0(\mathscr{C}) \to H^0(\mathscr{C}) = C$ is an isomorphism. Its inverse gives an injection $C \hookrightarrow \mathscr{C}$ which is a quasi-isomorphism of dg-algebras.*

Proof. The first claim follows because $\mathscr{C}$ lies in degrees 0 and 1. For the second claim we just need that $H^1(\mathscr{C}) = 0$. It is clear that the map $\pi : E[y] \to E_y$ is surjective. We can see that d^0 is surjective as well: since ${}_{A[y]}E[y]$ is projective, $\mathrm{Hom}_{A[y]}({}_{A[y]}E[y], -)$ is exact, so

$$\mathrm{Hom}_{A[y]}({}_{A[y]}E[y], \pi) : \mathrm{Hom}_{A[y]}({}_{A[y]}E[y], E[y]) \to \mathrm{Hom}_{A[y]}({}_{A[y]}E[y], E_y)$$

is surjective. □

The injection of the lemma gives a right action of C on X.

Lemma 2.2.34. *The algebra C is isomorphic to a generalized matrix algebra:*

$$\begin{pmatrix} A[y] & y_1 E[y] \\ F[y] & G_1^{\mathsf{op}} \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$

with component maps given by (restrictions of) those in Proposition 2.2.31.

Proof. We have $d^0\big((\theta, \varphi)\big) = 0$ exactly when $\varphi = \_.\theta + y_1\varphi'$ for some $\varphi' \in \mathrm{Hom}_{A[y]}({}_{A[y]}E[y], E[y])$, and it follows that the map to C_{22} is an isomorphism. □

2.2.3.2 Derived equivalence

Since X is strictly perfect, the triangulated functor

$$\mathscr{H}om_B(X, -) : K^b(B) \to K^b(C)$$

descends to the derived categories and resolutions are not needed:

$$\mathscr{H}om_B(X, -) : D^b(B) \to D^b(C).$$

Since X generates per B, it is perfect as a right $\mathscr{C}$-dg-module, and then also as a complex of C-modules because the inclusion $C \hookrightarrow \mathscr{C}$ is a quasi-isomorphism. It follows that the functor restricts to a functor

$$\mathscr{H}om_B(X, -) : \text{per } B \to \text{per } C,$$

and this is essentially surjective because C is in the essential image. To show that the functor is fully faithful, it is enough to check endomorphisms of X and its translates, since X generates per B. The induced map:

$$\text{Hom}_{D^b(B)}(X, X[i]) \to \text{Hom}_{D^b(C)}(\mathscr{E}nd_B(X), \mathscr{E}nd_B(X)[i])$$

is an isomorphism for all i: with $i = 0$ both sides are canonically isomorphic to C, and the map induces the identity on C; with $i \neq 0$ both sides are 0.

The endofunctor $E' \otimes_B -$ on per B induces an endofunctor on per C using this equivalence: first apply $X \otimes_C -$, then $E' \otimes_B -$, then $\mathscr{H}om_B(X, -)$. Since X is finitely generated and strictly perfect, this induced endofunctor is isomorphic to $\mathscr{H}om_B(X, E'X) \otimes_C -$.

Remark 2.2.35. In the above context a theorem of Rickard shows that $\mathscr{H}om_B(X, -) : D^b(B) \to D^b(C)$ is also an equivalence of categories. We do not know $E' \otimes_B -$ to be exact, however, so we use the restricted equivalence of perfect complexes, and the full version of Rickard's theorem is not needed.

Definition 2.2.36. In §2.2, let $\mathscr{E}$ denote the (C, C)-bimodule complex $\mathscr{H}om_B(X, E'X)$.

Then we have the following:

Proposition 2.2.37. *For each n, the morphism of (C, C) bimodule complexes*

$$\overbrace{\mathscr{E} \otimes_C \cdots \otimes_C \mathscr{E}}^{n\text{-times}} \to \mathscr{H}om_B(X, E'^n X)$$

given by

$$f_1 \otimes \cdots \otimes f_n \mapsto E'^{n-1}(f_n) \circ E'^{n-2}(f_{n-1}) \circ \cdots \circ f_1$$

is a quasi-isomorphism. These maps give the vertical maps in diagrams of the following form, which commute:

$$\begin{array}{ccc} \mathscr{H}om_B(X,E'X)^{\otimes n}\otimes_C \mathscr{H}om_B(X,E'X)^{\otimes m} & \longrightarrow & \mathscr{H}om_B(X,E'X)^{\otimes n+m} \\ \downarrow & & \downarrow \\ \mathscr{H}om_B(X,E'^{n}X)\otimes_C \mathscr{H}om_B(X,E'^{m}X) & \xrightarrow{f\otimes g\mapsto E'^{n}(g)\circ f} & \mathscr{H}om_B(X,E'^{n+m}X). \end{array}$$

Proof. All diagrams contained in the following diagram commute, up to canonical isomorphisms in per B and per C:

$$\begin{array}{ccc} \text{per } B & \underset{X\otimes_C -}{\overset{\mathscr{H}om_B(X,-)}{\rightleftarrows}} & \text{per } C \\ {\scriptstyle E'\otimes_B -}\downarrow & & \downarrow{\scriptstyle \mathscr{E}\otimes_C -} \\ \text{per } B & \underset{X\otimes_C -}{\overset{\mathscr{H}om_B(X,-)}{\rightleftarrows}} & \text{per } C \\ {\scriptstyle E'\otimes_B -}\downarrow & & \downarrow{\scriptstyle \mathscr{E}\otimes_C -} \\ \text{per } B & \underset{X\otimes_C -}{\overset{\mathscr{H}om_B(X,-)}{\rightleftarrows}} & \text{per } C. \end{array}$$

This gives the first statement of the proposition. The diagrams commute by functoriality of E'. □

2.2.4 New bimodule $\tilde{E}$

2.2.4.1 Definition of $\tilde{E}$

Now we define the lead actor of this chapter.

Definition 2.2.38. Define a (C, C)-bimodule:

$$\tilde{E} = \operatorname{Hom}_{K^b(B)}(X, E'X),$$

with left C action given by precomposition with $\varphi \in C$, and right C action given by post-composition with $E'(\varphi)$ for $\varphi \in C$.

Lemma 2.2.39. *For each n, the complex $\mathscr{H}om_B(X, E'^n X)$ of (C, C)-bimodules is concentrated in nonnegative degree.*

Proof. The lower row of $E'^n X$ has components in degrees at least 1, and the upper row has components in degrees at least 0. This is shown by a simple inductive argument using the formulas for X and E' in §2.2.1.2. It follows that there are no nonzero morphisms in $\mathscr{H}om_B(X, E'^n X)$ of negative degree. □

Proposition 2.2.40. *The complex $\mathscr{E} = \mathscr{H}om_B(X, E'X)$ of (C, C)-bimodules has cohomology concentrated in degree 0.*

Proof. We consider separately the matrix components $\mathscr{H}om_B(X_i, E'X_j)$:

- $\mathscr{H}om_B(X_1, E'X_1)$: since $X_1 = Be_1$ this is isomorphic to $e_1 E' X_1$ which is $E[y] \xrightarrow{\pi} E_y$, and π is surjective.
- $\mathscr{H}om_B(X_1, E'X_2)$: this is isomorphic to $e_1 E'^2 X_1$, which is

$$E^2[y] \xrightarrow{\begin{pmatrix} \pi_2 \\ \pi_1 \end{pmatrix}} E_y E \oplus E E_y \xrightarrow{(-\pi_1 \ \pi_2)} E_y E_y.$$

  The second map is clearly surjective. Its kernel consists of pairs $(ee_1, ee_2) \in E^2$ such that $ee_1 - ee_2 = (Ex - xE)ee^\circ$ for some $ee^\circ \in E^2$. Such a pair is the image of $ee_2 + (Ex - y)ee^\circ$ in $E^2[y]$.

- $\mathscr{H}om_B(X_2, E'X_1)$: this is isomorphic to $\mathscr{C}_{22}$, and we saw that d^0 is surjective.
- $\mathscr{H}om_B(X_2, E'X_2)$: this is isomorphic to $G'_2 \xrightarrow{d^0} G''_2 \xrightarrow{d^1} G'''_2$, where

$$G'_2 = E[y]^{\oplus 2} \oplus \mathrm{Hom}_{A[y]}({}_{A[y]}E[y], E^2[y])$$

$$G''_2 = E_y \oplus \mathrm{Hom}_{A[y]}({}_{A[y]}E[y], E_y E \oplus E E_y)$$

$$G'''_2 = \mathrm{Hom}_{A[y]}({}_{A[y]}E[y], E_y E_y),$$

with

$$d^0 : (e_1, e_2, \xi) \mapsto \big(\pi(e_2 - e_1), (\pi_2 \circ \xi; \pi_1 \circ \xi)\big)$$

$$d^1 : \big(e, (\xi'; \xi'')\big) \mapsto -\pi_1 \circ \xi' + \pi_2 \circ \xi''.$$

It is easy to see that $H^1 = 0$ and $H^2 = 0$ by applying the exact functor $\mathrm{Hom}_{A[y]}({}_{A[y]}E[y], -)$ to the sequence considered in the second bullet.

□

Corollary 2.2.41. *The surjection*

$$Z^0 \mathscr{H}om_B(X, E'X) \to H^0 \mathscr{H}om_B(X, E'X) = \tilde{E}$$

is an isomorphism. Its inverse gives an injection

$$\tilde{E} \hookrightarrow \mathscr{E}$$

which is a quasi-isomorphism of complexes of (C, C)-bimodules.

Remark 2.2.42. Whereas E' is a complex of bimodules, $\tilde{E}$ is just a bimodule. This observation is the starting point for our construction. The basis $X_1 \oplus X_2$ is designed to be more compatible with the $\mathcal{U}^+$ action in this sense.

Lemma 2.2.43. *As a left C-module, $\tilde{E}$ is finitely generated and projective.*

Proof. In Prop. 2.2.9 we saw that $E'X$ is quasi-isomorphic to a finite direct sum of summands of X, so ${}_C\tilde{E}$ is a finite direct sum of summands of C. □

Lemma 2.2.44. *The map $\tilde{E}^n \to \mathscr{H}om_B(X, E'^n X)$ of complexes of (C, C)-bimodules given by*

$$f_1 \otimes \cdots \otimes f_n \mapsto E'^{n-1}(f_n) \circ E'^{n-2}(f_{n-1}) \circ \cdots \circ f_1$$

is a quasi-isomorphism.

Proof. Use a copy of the morphism

$$\tilde{E} \overset{q.i.}{\hookrightarrow} \mathscr{E}$$

from Corollary 2.2.41 onto each factor of the product on the left in Proposition 2.2.37, and the fact that $\tilde{E}$ is finitely generated and projective on the left. □

Lemma 2.2.45. *The maps of Lemma 2.2.44 induce isomorphisms of* (C, C)*-bimodules*

$$\tilde{E}^n \xrightarrow{\sim} \mathrm{Hom}_{K^b(B)}(X, E'^n X)$$

making the following diagrams commute:

$$\begin{array}{ccc} \tilde{E}^n \otimes_C \tilde{E}^m & \xrightarrow{\sim} & \tilde{E}^{n+m} \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ \mathrm{Hom}_{K(B)}(X, E'^n(X)) \otimes_C \mathrm{Hom}_{K(B)}(X, E'^m(X)) & \xrightarrow{\sim} & \mathrm{Hom}_{K(B)}(X, E'^{n+m}(X)). \end{array}$$

Proof. By Lemma 2.2.44, the cohomology of $\mathscr{H}om_B(X, E'^n X)$ is concentrated in degree 0. By Lemma 2.2.39,

$$Z^0 \mathscr{H}om_B(X, E'^n X) = H^0 \mathscr{H}om_B(X, E'^n X).$$

So the degree 0 part of the map of Lemma 2.2.44 is an isomorphism from $\tilde{E}^n$ to $Z^0 \mathscr{H}om_B(X, E'^n X)$, which is $\mathrm{Hom}_{K^b(B)}(X, E'^n X)$. The diagrams commute because the morphisms are restrictions of the morphisms of Proposition 2.2.37. □

Definition 2.2.46. We let $\tilde{E}^n_{ij}$ denote $\mathrm{Hom}_{K^b(B)}(X_i, E'^n X_j)$.

Defined in this way, $\tilde{E}^n_{ij}$ lies in $\mathrm{Hom}_{K^b(B)}(X, E'^n X)$, not in $\tilde{E}^n$, but we consider it also in the latter through the isomorphism of Lemma 2.2.45.

2.2.4.2 Some low powers of $\tilde{E}$

The bimodule $\tilde{E}$ can be presented as a matrix with ij-component $\tilde{E}_{ij}$ given by $\mathrm{Hom}_{K^b(B)}(X_i, E'X_j)$. This component is an $(\mathrm{End}(X_i)^{\mathsf{op}}, \mathrm{End}(X_j)^{\mathsf{op}})$-bimodule. Recall that $\mathrm{End}(X_1)^{\mathsf{op}} \cong A[y]$ and $\mathrm{End}(X_2)^{\mathsf{op}} \cong G_1^{\mathsf{op}}$.

Lemma 2.2.47. *We have*

$$(y_1 \dots y_n)E^n[y] \xrightarrow{\sim} \mathrm{Hom}_{K^b(B)}(X_1, E'^n X_1),$$

where $y_1 \dots y_n e$ is sent to the map in $K^b(B)$ determined by:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} (y_1 \dots y_n e, 0, \dots, 0) \\ 0 \end{pmatrix}.$$

Proof. Computation. Note that $E'^n X_1$ has just one term in degree 0, which is $E^n[y]$ in the upper row. The differential of $E'^n X_1$ out of this term is the map whose kernel is computed in Lemma 2.2.7. □

Proposition 2.2.48. *We have:*

$$\begin{pmatrix} y_1 \dots y_n E^n[y] & y_1 \dots y_{n+1} E^{n+1}[y] \\ G_n & G_{n+1} \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} \tilde{E}_{11}^n & \tilde{E}_{12}^n \\ \tilde{E}_{21}^n & \tilde{E}_{22}^n \end{pmatrix},$$

where the maps on the upper row are from Lemma 2.2.47, and on the lower they are from the definition of G_n.

Together with Lemma 2.2.45, this gives a parametrization of $\tilde{E}^n$. We may record the matrix presentations for the first few powers:

$$\begin{pmatrix} y_1 E[y] & y_1 y_2 E^2[y] \\ G_1 & G_2 \end{pmatrix} \xrightarrow{\sim} \tilde{E},$$

$$\begin{pmatrix} y_1y_2E^2[y] & y_1y_2y_3E^3[y] \\ G_2 & G_3 \end{pmatrix} \xrightarrow{\sim} \tilde{E}^2,$$

$$\begin{pmatrix} y_1y_2y_3E^3[y] & y_1y_2y_3y_4E^4[y] \\ G_3 & G_4 \end{pmatrix} \xrightarrow{\sim} \tilde{E}^3.$$

2.3 Hecke action

In this section we introduce (C,C)-bimodule endomorphisms $\tilde{x}$ of $\tilde{E}$ and $\tilde{\tau}$ of $\tilde{E}^2$, and show that they satisfy the relations of $\mathcal{U}^+$.

2.3.1 Definition of the action

In §2.3.1.1 we give formulas for endomorphisms of the separate components of $\tilde{E}$ and $\tilde{E}^2$. A few lemmas are needed first in order to show that the formulas are well-defined on components of the form G_n, $n = 1,2,3$. Then in §2.3.1.2 we argue that these componentwise definitions jointly determine a morphism of (C,C)-bimodules.

2.3.1.1 Formulas for $\tilde{x}$ and $\tilde{\tau}$

Lemma 2.3.1. *Let* $(\theta,\varphi) \in G_1 \subset A^{\mathsf{op}}[y] \oplus \mathrm{Hom}_A({}_AE, E)[y]$. *Then* $(y\theta, x\circ\varphi) \in G_1$.

Proof. Compute:

$$\begin{aligned} x\circ\varphi - y\theta &= x(\_.\theta + y_1\varphi_1) - y\theta \\ &= y_1(\_.\theta + x\varphi_1). \end{aligned}$$

□

Lemma 2.3.2. *Let* $(e_1, e_2, \xi) \in G_2 \subset E[y]^{\oplus 2} \oplus \mathrm{Hom}_A({}_AE, E^2)[y]$. *Then* $(ye_1, xe_2, xE \circ \xi) \in G_2$ *and* $(e', e', \tau \circ \xi) \in G_2$.

Proof. For the first claim, compute:

$$\begin{aligned} xE \circ \xi - {}_{-} \otimes ye_1 &= xE \circ ({}_{-} \otimes e_1 + y_2 \xi_1) - {}_{-} \otimes ye_1 \\ &= y_2({}_{-} \otimes e_1 + xE \circ \xi_1), \end{aligned}$$

and

$$\begin{aligned} xE \circ \xi - \delta({}_{-} \otimes xe_2) &= xE \circ \big(\delta({}_{-} \otimes e_2) + y_1 \xi_2\big) - \delta({}_{-} \otimes xe_2) \\ &= \delta \circ Ex({}_{-} \otimes e_2) - y_1({}_{-} \otimes e_2) \\ &\quad + y_1 xE \circ \xi_2 - \delta({}_{-} \otimes xe_2) \\ &= y_1(-{}_{-} \otimes e_2 + xE \circ \xi_2). \end{aligned}$$

For the second claim, use the alternative characterization of G_2 as given in Prop. 2.2.26, and compute:

$$\begin{aligned} \tau \circ \xi &= \tau({}_{-} \otimes e_1) + \tau y_2 \xi_1 \\ &= \tau({}_{-} \otimes e_1) + y_1 \tau \xi_1 - \xi_1 \\ &= \tau({}_{-} \otimes e_1) + y_1 \tau y_1 \xi' - \xi_1 \\ &= \tau\big({}_{-} \otimes (e_1 - e_2)\big) + y_1 y_2 \tau \xi' \\ &= \tau y_1({}_{-} \otimes e') + y_1 y_2 \tau \xi' \\ &= {}_{-} \otimes e' + y_2\big(\tau({}_{-} \otimes e') + y_1 \tau \xi'\big). \end{aligned}$$

The last line has the form of an element of G_2. □

Lemma 2.3.3. *Let* $(ee_1, ee_2, ee_3, \chi) \in G_3 \subset E^2[y]^{\oplus 3} \oplus \mathrm{Hom}_A({}_AE, E^3)[y]$. *Then* $(ee', ee', \tau(ee_3), \tau E \circ \chi) \in G_3$.

Proof. We use the alternative characterization of G_3 as given in Prop. 2.2.27, and compute:

$$\begin{aligned}
\tau E \circ \chi &= \tau E({}_{-} \otimes ee_1) + \tau E y_3 \chi_1 \\
&= \tau E({}_{-} \otimes ee_1) - \chi_1 + y_2 \tau E \circ \chi_1 \\
&= \tau E({}_{-} \otimes ee_1) - \chi_1 + y_2 \tau E y_2 \big(E\tau \circ \tau E({}_{-} \otimes ee_3) + y_1 \chi''\big) \\
&= \tau E({}_{-} \otimes ee_1) - \chi_1 \\
&\qquad + \big(y_2 y_3 \tau E + y_2\big) \cdot \big(E\tau \circ \tau E({}_{-} \otimes ee_3) + y_1 \chi''\big) \\
&= \tau E({}_{-} \otimes (ee_1 - ee_2)) \\
&\qquad + y_2 y_3 \big(\tau E \circ E\tau \circ \tau E({}_{-} \otimes ee_3) + y_1 \tau E \circ \chi''\big) \\
&= \tau E y_2 ({}_{-} \otimes ee') \\
&\qquad + y_2 y_3 \Big(E\tau \circ \tau E({}_{-} \otimes ee') + E\delta \circ \tau E({}_{-} \otimes \overline{ee}) + y_1 \tau E \circ \chi''\Big) \\
&= {}_{-} \otimes ee' + y_3 \cdot \\
&\qquad \Big(E\delta \circ \tau E({}_{-} \otimes ee') + y_2 \big(E\delta \circ \tau E({}_{-} \otimes \overline{ee}) + y_1 y_2 \tau E \circ \chi''\big)\Big)
\end{aligned}$$

$$\begin{aligned}
&= {}_{-} \otimes ee' + y_3 \cdot \\
&\qquad \Big(-\tau E y_1 ({}_{-} \otimes \overline{ee}) + E\delta \circ \tau E({}_{-} \otimes \tau(ee_3)) + y_1 y_2 \tau E \circ \chi''\Big).
\end{aligned}$$

The last line has the form of an element of G_3, namely $(ee', ee', \tau(ee_3), \tau E \circ \chi)$. □

The element $(ee_1, ee_2, ee_3, \chi) \in G_3$ is associated (by Prop. 2.2.27) with further data that has been notated ee^ℓ, $\overline{ee}$, χ_ℓ, χ_1', and χ''. We record the corresponding data associated with

$(ee', ee', \tau(ee_3), \tau E \circ \chi)$ using the notation $\bar{e}$ and $\bar{\chi}$ for the new versions:

$$\bar{e}\bar{e}' = 0$$

$$\bar{e}\bar{e}'' = \overline{ee}$$

$$\bar{e}\bar{e}''' = \overline{ee}$$

$$\overline{\bar{e}\bar{e}} = 0,$$

and

$$\begin{aligned}
\bar{\chi} &= (ee', ee', \tau(ee_3), \tau E \circ \chi) \\
\bar{\chi}_1 &= -\tau E y_1(_ \otimes \overline{ee}) + E\delta \circ \tau E \circ E\tau(_ \otimes ee_3) + y_1 y_2 \tau E \circ \chi'' \\
\bar{\chi}_2 &= E\tau \circ \delta E \circ E\tau(_ \otimes ee_3) + y_1 y_3 \tau E \circ \chi'' \\
\bar{\chi}_3 &= -\delta E(_ \otimes \overline{ee}) + y_2 y_3 \tau E \circ \chi'' \\
\bar{\chi}_1' &= E\tau \circ \tau E \circ E\tau(_ \otimes ee_3) + y_1 \tau E \circ \chi'' \\
\bar{\chi}'' &= \tau E \circ \chi''.
\end{aligned}$$

Now we give componentwise formulas for $\tilde{x}$ and $\tilde{\tau}$. These formulas are well-defined on $\tilde{E}_{21}$, $\tilde{E}_{22}$, $\tilde{E}^2_{21}$, and $\tilde{E}^2_{22}$ by the lemmas above.

Definition 2.3.4. We define the action of $\tilde{x}$ on $\tilde{E}$ as follows:

- on $\tilde{E}_{11}$: $\tilde{x}$ acts by x
- on $\tilde{E}_{12}$: $\tilde{x}$ acts by xE
- on $\tilde{E}_{21}$: $\tilde{x}$ acts by $(\theta, \varphi) \mapsto (y\theta, x \circ \varphi)$
- on $\tilde{E}_{22}$: $\tilde{x}$ acts by $(e_1, e_2, \xi) \mapsto (ye_1, xe_2, xE \circ \xi)$.

We define the action of $\tilde{\tau}$ on $\tilde{E}^2$ as follows:

- on $\tilde{E}^2_{11}$: $\tilde{\tau}$ acts by τ

- on $\tilde{E}^2_{12}$: $\tilde{\tau}$ acts by τE
- on $\tilde{E}^2_{21}$: $\tilde{\tau}$ acts by $(e_1, e_2, \xi) \mapsto (e', e', \tau \circ \xi)$
- on $\tilde{E}^2_{22}$: $\tilde{\tau}$ acts by $(ee_1, ee_2, ee_3, \chi) \mapsto (ee', ee', \tau(ee_3), \tau E \circ \chi)$.

Lemma 2.3.5. *The formulas for $\tilde{x}$ give a (C, C)-bimodule endomorphism of $\tilde{E}$.*

Proof. Recall the definition of the complex E' of (B, B)-bimodules in §2.2.1.2. There is an $\left(\begin{pmatrix} A[y] & 0 \\ 0 & A[y] \end{pmatrix}, \begin{pmatrix} A[y] & 0 \\ 0 & A[y] \end{pmatrix}\right)$-bimodule endomorphism x' of E' given componentwise in degrees 0 and 1 by $(A[y], A[y])$-bimodule endomorphisms:

$$x'_0 = \begin{pmatrix} x & xE_y \\ 0 & x \end{pmatrix}, \quad x'_1 = \begin{pmatrix} x & xE_y \\ y & x \end{pmatrix}.$$

The relation $s \circ E_y x = xE_y \circ s$ may be used to check that x'_0 and x'_1 together give a morphism of complexes of (B, B)-bimodules. This map induces a (C, C)-bimodule endomorphism of $\mathrm{Hom}_{K^b(B)}(X, E'X)$ that agrees with the definition of $\tilde{x}$. □

It follows that $\tilde{x}$ induces endomorphisms $\tilde{x}\tilde{E}$ and $\tilde{E}\tilde{x}$. For future reference we write the formulas for those:

Proposition 2.3.6. *The formulas for $\tilde{x}$ determine the following formulas for $\tilde{x}\tilde{E}$ and $\tilde{E}\tilde{x}$ on $\tilde{E}^2$:*

- on $\tilde{E}^2_{11}$: $\tilde{x}\tilde{E}$ acts by xE and $\tilde{E}\tilde{x}$ acts by Ex
- on $\tilde{E}^2_{12}$: $\tilde{x}\tilde{E}$ acts by xE^2 and $\tilde{E}\tilde{x}$ by ExE
- on $\tilde{E}^2_{21}$: $\tilde{x}\tilde{E}$ acts by

  $$(e_1, e_2, \xi) \mapsto (ye_1, xe_2, xE \circ \xi)$$

  and $\tilde{E}\tilde{x}$ by

  $$(e_1, e_2, \xi) \mapsto (xe_1, ye_2, Ex \circ \xi)$$

- on $\tilde{E}^2_{22}$: $\tilde{x}\tilde{E}$ acts by

$$(ee_1, ee_2, ee_3, \chi) \mapsto (yee_1, xE(ee_2), xE(ee_3), xE^2 \circ \chi)$$

and $\tilde{E}\tilde{x}$ by

$$(ee_1, ee_2, ee_3, \chi) \mapsto (xE(ee_1), yee_2, Ex(ee_3), ExE \circ \chi).$$

Proof. Use Lemma 2.2.45, in particular the diagram in the case $n = m = 1$. □

2.3.1.2 Bimodule structure of $\tilde{E}^2$ and equivariance of $\tilde{\tau}$

Lemma 2.3.7. *The formulas for $\tilde{\tau}$ give a (C, C)-bimodule endomorphism of $\tilde{E}^2$.*

For the maps we defined on components of $\tilde{E}^2$ to determine jointly a (C, C)-bimodule endomorphism $\tilde{\tau}$, they must be equivariant with respect to the left and right C-actions. In order to check equivariance, we write formulas for the actions of the generators in C in the following four lemmas. The reader may verify these formulas from the various definitions.

Lemma 2.3.8. *Generators in $A[y] \subset C$ act on the right on $\tilde{E}^2$, in terms of the separate bimodule structures of $\tilde{E}^2_{ij}$, as follows:*

- $\tilde{E}^2_{11} \otimes A[y] \to \tilde{E}^2_{11}$ *by*

$$y_1y_2E^2[y] \otimes_{A[y]} A[y] \longrightarrow y_1y_2E^2[y]$$
$$y_1y_2ee \otimes \theta \mapsto y_1y_2ee.\theta.$$

- $\tilde{E}^2_{21} \otimes A[y] \to \tilde{E}^2_{21}$ *by*

$$G_2 \otimes_{A[y]} A[y] \longrightarrow G_2$$
$$(e_1, e_2, \xi) \otimes \theta \mapsto (e_1.\theta, e_2.\theta, \xi(-).\theta)\,.$$

They act on the left as follows:

- $A[y] \otimes \tilde{E}^2_{11} \to \tilde{E}^2_{11}$ *by*

$$A[y] \otimes_{A[y]} y_1y_2E^2[y] \longrightarrow y_1y_2E^2[y]$$

$$\theta \otimes y_1y_2ee \mapsto y_1y_2\theta.ee.$$

- $A[y] \otimes \tilde{E}^2_{12} \to \tilde{E}^2_{12}$ *by*

$$A[y] \otimes_{A[y]} y_1y_2y_3E^3[y] \longrightarrow y_1y_2y_3E^3[y]$$

$$\theta \otimes y_1y_2y_3eee \mapsto y_1y_2y_3\theta.eee.$$

Remark. We may confirm that the image of the action map $\tilde{E}^2_{21} \to \tilde{E}^2_{21}$ preserves the conditions for G_2:

$$\xi.\theta - \_ \otimes e_1.\theta = y_2\xi_1.\theta,$$

$$\xi_1.\theta = \delta(\_ \otimes e_2).\theta + (y_1\xi_2).\theta$$

$$= \delta(\_ \otimes e_2.\theta) + y_1(\xi_2.\theta),$$

and the e_ℓ relation:

$$e_1.\theta - e_2.\theta = y_1e'.\theta.$$

Lemma 2.3.9. *Generators in $G_1^{\mathsf{op}} \subset C$ act on the right on $\tilde{E}^2$ as follows:*

- $\tilde{E}^2_{12} \otimes G_1^{\mathsf{op}} \to \tilde{E}^2_{12}$ *by*

$$y_1y_2y_3E^3[y] \otimes_{G_1^{\mathsf{op}}} G_1^{\mathsf{op}} \longrightarrow y_1y_2y_3E^3[y]$$

$$y_1y_2y_3eee \otimes (\theta, \varphi) \mapsto E^2\varphi(y_1y_2y_3eee)$$

- $\tilde{E}^2_{22} \otimes G_1^{\mathsf{op}} \to \tilde{E}^2_{22}$ *by*

$$G_3 \otimes_{G_1^{\mathsf{op}}} G_1^{\mathsf{op}} \longrightarrow G_3$$
$$(ee_1, ee_2, ee_3, \chi) \otimes (\theta, \varphi) \mapsto \big(E\varphi(ee_1), E\varphi(ee_2), ee_3.\theta, E^2\varphi \circ \chi\big)\,.$$

They act on the left as follows:

- $G_1^{\mathsf{op}} \otimes \tilde{E}^2_{21} \to \tilde{E}^2_{21}$ *by*

$$G_1^{\mathsf{op}} \otimes_{G_1^{\mathsf{op}}} G_2 \longrightarrow G_2$$
$$(\theta, \varphi) \otimes (e_1, e_2, \xi) \mapsto (\theta.e_1, \theta.e_2, \xi \circ \varphi)$$

- $G_1^{\mathsf{op}} \otimes \tilde{E}^2_{22} \to \tilde{E}^2_{22}$ *by*

$$G_1^{\mathsf{op}} \otimes_{G_1^{\mathsf{op}}} G_3 \longrightarrow G_3$$
$$(\theta, \varphi) \otimes (ee_1, ee_2, ee_3, \chi) \mapsto (\theta.ee_1, \theta.ee_2, \theta.ee_3, \chi \circ \varphi)\,.$$

Remark 2.3.10. We may confirm that the image of the right action map $\tilde{E}^2_{22} \otimes G_1^{\mathsf{op}} \to \tilde{E}^2_{22}$ preserves the conditions for G_3:

$$\begin{aligned}
E^2\varphi \circ \chi &= {_} \otimes E\varphi(ee_1) + E^2\varphi(\chi - {_} \otimes ee_1) \\
&= {_} \otimes E\varphi(ee_1) + y_3\big(E^2\varphi \circ \chi_1\big), \\
E^2\varphi \circ \chi_1 &= \tau E({_} \otimes E\varphi(ee_2)) + y_2 E^2\varphi \circ \chi_1' \\
&= \tau E \circ E^2({_}.\theta + y_1\varphi_1) \circ ({_} \otimes ee_2) + y_2 E^2\varphi \circ \chi_1', \\
E^2\varphi \circ \chi_1' &= E^2({_}.\theta) \circ E\tau \circ \tau E({_} \otimes ee_3) + y_1 E^2\varphi_1 \circ \chi_1' + y_1\chi''.\theta \\
&= E\tau \circ \tau E({_} \otimes ee_3.\theta) + y_1\big(\chi''.\theta + E^2\varphi_1 \circ \chi_1'\big).
\end{aligned}$$

And the ee_ℓ relations:

$$\begin{aligned}
E\varphi(ee_1) - E\varphi(ee_2) &= y_2 E\varphi(ee'),\\
ee_3.\theta - E\varphi(ee_2) &= (ee_3 - ee_2).\theta - y_1 E\varphi_1(ee_2)\\
&= y_1\left(ee''.\theta - E\varphi_1(ee_2)\right),\\
\delta(ee_3.\theta) - E\varphi(ee_1) &= y_2\tau(ee_3).\theta + (ee_3 - ee_1).\theta - y_1 E\varphi_1(ee_1)\\
&= y_2\tau(ee_3).\theta + y_1 ee''.\theta - y_2 ee'.\theta - y_1 E\varphi_1(ee_1)\\
&= y_1\left(y_2\overline{ee}.\theta + ee''.\theta - E\varphi_1(ee_1)\right).
\end{aligned}$$

Similarly we may confirm that the image of the left action map $G_1^{\mathsf{op}} \otimes \tilde{E}^2_{21} \to \tilde{E}^2_{21}$ lies in G_2:

$$\begin{aligned}
\xi\circ\varphi &= \varphi(-)\otimes e_1 + y_2\xi_1\circ\varphi\\
&= {}_{-}\otimes\theta.e_1 + y_2\big(\varphi_1(-)\otimes e_1 + \xi_1\circ\varphi\big),\\
\xi_1\circ\varphi + \varphi_1(-)\otimes e_1 &= \tau({}_{-}\otimes e_2)\circ\varphi + y_1\xi'\circ\varphi + \varphi_1(-)\otimes e_1\\
&= \tau({}_{-}\otimes\theta.e_2) + \tau y_2(\varphi_1(-)\otimes e_2) + y_1\xi'\circ\varphi + \varphi_1(-)\otimes e_1\\
&= \tau({}_{-}\otimes\theta.e_2) + y_1\left(\tau(\varphi_1(-)\otimes e_2) + \varphi_1(-)\otimes e' + \xi'\circ\varphi\right).
\end{aligned}$$

And the e_ℓ relation:

$$\theta.e_1 - \theta.e_2 = y_1\theta.e'.$$

And the image of the left action map $G_1^{\mathsf{op}} \otimes \tilde{E}^2_{22} \to \tilde{E}^2_{22}$ lies in G_3:

$$\begin{aligned}
\chi\circ\varphi &= \varphi(-)\otimes ee_1 + y_3\chi_1\circ\varphi\\
&= {}_{-}\otimes\theta.ee_1 + y_3\big(\varphi_1\otimes ee_1 + \chi_1\circ\varphi\big),\\
\chi_1\circ\varphi &= \tau E({}_{-}\otimes\theta.ee_2) + \tau E y_3(\varphi_1\otimes ee_2) + y_2\chi_1'\circ\varphi,\\
\varphi_1\otimes ee_1 + \chi_1\circ\varphi &= \tau E({}_{-}\otimes\theta.ee_2) + y_2\Big(\tau E(\varphi_1\otimes ee_2) + \varphi_1\otimes ee' + \chi_1'\circ\varphi\Big),
\end{aligned}$$

$$
\begin{aligned}
\chi_1' \circ \varphi &= E\tau \circ \tau E(\_ \otimes \theta.ee_3) + E\tau \circ \tau E \circ y_3(\varphi_1 \otimes ee_3) + y_1\chi'' \circ \varphi \\
&= E\tau \circ \tau E(\_ \otimes \theta.ee_3) + y_1(E\tau \circ \tau E)(\varphi_1 \otimes ee_3) \\
&\quad - \tau E(\varphi_1 \otimes ee_3) - E\tau(\varphi_1 \otimes ee_3) + y_1\chi'' \circ \varphi,
\end{aligned}
$$

and

$$
\begin{aligned}
&\tau E(\varphi_1 \otimes ee_2) + \varphi_1 \otimes ee' + \chi_1' \circ \varphi = \\
&\qquad E\tau \circ \tau E(\_ \otimes \theta.ee_3) + y_1\Big((E\tau \circ \tau E)(\varphi_1 \otimes ee_3) - \tau E(\varphi_1 \otimes ee'') - \varphi_1 \otimes \overline{ee} + \chi'' \circ \varphi\Big).
\end{aligned}
$$

And the ee_ℓ relations:

$$
\begin{aligned}
\theta.ee_1 - \theta.ee_2 &= y_2\theta.ee' \\
\theta.ee_3 - \theta.ee_2 &= y_1\theta.ee'' \\
\delta(\theta.ee_3) - \theta.ee_1 &= y_1\theta.ee'''.
\end{aligned}
$$

Lemma 2.3.11. *Generators in $y_1E[y] \subset C$ act on the right on $\tilde{E}^2$ as follows:*

- $\tilde{E}^2_{11} \otimes y_1E[y] \to \tilde{E}^2_{12}$ *by*

$$
\begin{aligned}
y_1y_2E^2[y] \otimes_{A[y]} y_1E[y] &\longrightarrow y_1y_2y_3E^3[y] \\
y_1y_2ee \otimes y_1e &\mapsto y_1y_2y_3(ee \otimes e)
\end{aligned}
$$

- $\tilde{E}^2_{21} \otimes y_1E[y] \to \tilde{E}^2_{22}$ *by*

$$
\begin{aligned}
&G_2 \otimes_{A[y]} y_1E[y] \longrightarrow G_3 \\
&(e_1, e_2, \xi) \otimes y_1e \mapsto (e_1 \otimes y_1e, e_2 \otimes y_1e, 0, \xi(-) \otimes y_1e)\,.
\end{aligned}
$$

They act on the left as follows:

- $y_1E[y] \otimes \tilde{E}^2_{21} \to \tilde{E}^2_{11}$ *by*

$$y_1E[y] \otimes_{G_1^{\mathsf{op}}} G_2 \longrightarrow y_1y_2E^2[y]$$
$$y_1e \otimes (e_1, e_2, \xi) \mapsto \xi(y_1e)$$

- $y_1E[y] \otimes \tilde{E}^2_{22} \to \tilde{E}^2_{12}$ *by*

$$y_1E[y] \otimes_{G_1^{\mathsf{op}}} G_3 \longrightarrow y_1y_2y_3E^3[y]$$
$$y_1e \otimes (ee_1, ee_2, ee_3, \chi) \mapsto \chi(y_1e).$$

Remark. We may confirm that the image of the right action map $\tilde{E}^2_{21} \otimes y_1E[y] \to \tilde{E}^2_{22}$ preserves the conditions for G_3:

$$\begin{aligned} \chi &= \xi \otimes y_1e, \\ \chi - {}_{-}\otimes e_1 \otimes y_1e &= y_1y_3(\xi_1 \otimes e), \\ \chi - \delta E({}_{-}\otimes e_2 \otimes y_1e) &= \big(\xi - \delta({}_{-}\otimes e_2)\big) \otimes y_1e \\ &= y_1y_2(\xi_2 \otimes e). \end{aligned}$$

Similarly we may confirm that the image of the left action map $y_1E[y] \otimes \tilde{E}^2_{21} \to \tilde{E}^2_{11}$ lies in $y_1y_2E^2[y]$:

$$\begin{aligned} \xi \circ y_1 &= y_2\big({}_{-}\otimes e_1 + \xi_1 \circ y_1\big), \\ \xi_1 \circ y_1 &= \tau y_2({}_{-}\otimes e_2) + y_1\xi' \circ y_1 \\ &= y_1\big(\tau({}_{-}\otimes e_2) + \xi' \circ y_1\big) - {}_{-}\otimes e_2, \\ \xi \circ y_1 &= y_2\Big(y_1\big(\tau({}_{-}\otimes e_2) + \xi' \circ y_1\big) + {}_{-}\otimes (e_1 - e_2)\Big) \\ &= y_1y_2\Big(\tau({}_{-}\otimes e_2) + {}_{-}\otimes e' + \xi' \circ y_1\Big). \end{aligned}$$

And the image of the left action map $y_1 E[y] \otimes \tilde{E}^2_{22} \to \tilde{E}^2_{12}$ lies in $y_1 y_2 y_3 E^3[y]$:

$$
\begin{aligned}
\chi \circ y_1 &= y_3\big(\_ \otimes ee_1 + \chi_1 \circ y_1\big) \\
\chi_1 \circ y_1 &= -\tau E y_3 y_1 (\_ \otimes ee'') \\
&\quad + E\delta \circ \tau E y_3 (\_ \otimes ee_3) + y_1 y_2 \chi'' \circ y_1 \\
&= -\tau E y_3 y_1 (\_ \otimes ee'') + E\delta \circ y_2 \tau E (\_ \otimes ee_3) \\
&\quad - E\delta(\_ \otimes ee_3) + y_1 y_2 \chi'' \circ y_1 \\
&= -y_2 \tau E y_1 (\_ \otimes ee'') + y_1(\_ \otimes ee'') + y_1 y_2 E\tau \circ \tau E(\_ \otimes ee_3) \\
&\quad - y_1(\_ \otimes ee''') - \_ \otimes ee_1 + y_1 y_2 \chi'' \circ y_1 \\
\chi \circ y_1 &= y_3 y_2 y_1 \Big(-\tau E(\_ \otimes ee'') + E\tau \circ \tau E(\_ \otimes ee_3) + \chi'' \circ y_1\Big) \\
&\quad + y_3 y_1 \big(\_ \otimes ee'' - \_ \otimes ee'''\big) \\
&= y_3 y_2 y_1 \Big(-\tau E(\_ \otimes ee'') + E\tau \circ \tau E(\_ \otimes ee_3) - \_ \otimes \overline{ee} + \chi'' \circ y_1\Big).
\end{aligned}
$$

Lemma 2.3.12. *Generators in $F[y] \subset C$ act on the right on $\tilde{E}^2$ as follows:*

- $\tilde{E}^2_{12} \otimes F[y] \to \tilde{E}^2_{11}$ *by*

$$
\begin{aligned}
y_1 y_2 y_3 E^3[y] \otimes_{G_1^{\mathsf{op}}} F[y] &\longrightarrow y_1 y_2 E^2[y] \\
y_1 y_2 y_3 eee \otimes f &\mapsto y_1 y_2 E^2 f(y_1 eee)
\end{aligned}
$$

- $\tilde{E}^2_{22} \otimes F[y] \to \tilde{E}^2_{21}$ *by*

$$
\begin{aligned}
G_3 \otimes_{G_1^{\mathsf{op}}} F[y] &\longrightarrow G_2 \\
(ee_1, ee_2, ee_3, \chi) \otimes f &\mapsto \big(Ef(ee_1), Ef(ee_2), E^2 f \circ \chi\big)\,.
\end{aligned}
$$

They act on the left as follows:

- $F[y] \otimes \tilde{E}^2_{11} \to \tilde{E}^2_{21}$ *by*

$$\begin{aligned} F[y] \otimes_{A[y]} y_1 y_2 E^2[y] &\longrightarrow G_2 \\ f \otimes y_1 y_2 ee &\mapsto (0, 0, f(-).y_1 y_2 ee) \end{aligned}$$

- $F[y] \otimes \tilde{E}^2_{12} \to \tilde{E}^2_{22}$ *by*

$$\begin{aligned} F[y] \otimes_{A[y]} y_1 y_2 y_3 E^3[y] &\longrightarrow G_3 \\ f \otimes y_1 y_2 y_3 eee &\mapsto (0, 0, 0, f(-).y_1 y_2 y_3 eee)\,. \end{aligned}$$

Remark. We may observe that the image of the right action map $\tilde{E}^2_{22} \otimes F[y] \to \tilde{E}^2_{21}$ preserves the conditions for G_3:

$$\begin{aligned} E^2 f \circ \chi - {}_{-} \otimes Ef(ee_1) &= E^2 f \circ (\chi - {}_{-} \otimes ee_1) \\ &= y_2 E^2 f \circ \chi_1, \\ E^2 f \circ \chi - \delta({}_{-} \otimes Ef(ee_2)) &= E^2 f \circ (\chi - \delta E({}_{-} \otimes ee_2)) \\ &= E^2 f \circ y_2 \chi_2 \\ &= y_1 E^2 f \circ \chi_2, \end{aligned}$$

and the ee_ℓ relation:

$$\begin{aligned} Ef(ee_1 - ee_2) &= Ef(y_2 ee'') \\ &= y_1 Ef(ee''). \end{aligned}$$

It is trivial to check the conditions for the images of the left action maps $F[y] \otimes \tilde{E}^2_{11} \to \tilde{E}^2_{21}$ and $F[y] \otimes \tilde{E}^2_{12} \to \tilde{E}^2_{22}$.

Proof of Lemma 2.3.7. The reader may now check that $\tilde{\tau}$ defined in §2.3.1.1 is equivariant over the left and right C actions. These checks are completely mechanical using the formulas just given. □

2.3.2 Hecke relations

2.3.2.1 $\tilde{x}$ and $\tilde{\tau}$ satisfy Hecke relations

These checks are also mechanical, but we write them out because they are important.

Proposition 2.3.13. *On each component $\tilde{E}^2_{ij}$, the maps $\tilde{x}$ and $\tilde{\tau}$ defined in §2.3.1.1 satisfy*

$$\tilde{E}\tilde{x} \circ \tilde{\tau} - \tilde{\tau} \circ \tilde{x}\tilde{E} = Id$$

$$\tilde{\tau} \circ \tilde{E}\tilde{x} - \tilde{x}\tilde{E} \circ \tilde{\tau} = Id.$$

Proof. On the first row, $\tilde{E}^2_{11}$ and $\tilde{E}^2_{12}$, the relations follow from the corresponding relations between x and τ.

On $\tilde{E}^2_{21}$ presented as G_2, we have:

$$\tilde{E}\tilde{x} \circ \tilde{\tau} : (e_1, e_2, \xi) \mapsto (xe', ye', Ex \circ \tau \circ \xi)$$

$$\tilde{\tau} \circ \tilde{x}\tilde{E} : (e_1, e_2, \xi) \mapsto (ye' - e_2, ye' - e_2, \tau \circ xE \circ \xi)$$

$$\tilde{\tau} \circ \tilde{E}\tilde{x} : (e_1, e_2, \xi) \mapsto (e_2 + xe', e_2 + xe', \tau \circ Ex \circ \xi)$$

$$\tilde{x}\tilde{E} \circ \tilde{\tau} : (e_1, e_2, \xi) \mapsto (ye', xe', xE \circ \tau \circ \xi),$$

from which

$$\tilde{E}\tilde{x} \circ \tilde{\tau} - \tilde{\tau} \circ \tilde{x}\tilde{E} : (e_1, e_2, \xi) \mapsto \big(y_1 e' + e_2, e_2, (Ex \circ \tau - \tau \circ xE) \circ \xi\big)$$

$$= (e_1, e_2, \xi),$$

and similarly for the other relation.

On $\tilde{E}^2_{22}$ presented as G_3, we have:

$$\tilde{E}\tilde{x} \circ \tilde{\tau} : (ee_1, ee_2, ee_3, \chi) \mapsto (xE(ee'), yee', Ex \circ \tau(ee_3), ExE \circ \tau E \circ \chi)$$

$$\tilde{\tau} \circ \tilde{x}\tilde{E} : (ee_1, ee_2, ee_3, \chi) \mapsto (yee' - ee_2, yee' - ee_2, \tau \circ xE(ee_3), \tau E \circ xE^2 \circ \chi)$$

$$\tilde{\tau} \circ \tilde{E}\tilde{x} : (ee_1, ee_2, ee_3, \chi) \mapsto (ee_1 + yee', ee_1 + yee', \tau \circ Ex(ee_3), \tau E \circ ExE \circ \chi)$$

$$\tilde{x}\tilde{E} \circ \tilde{\tau} : (ee_1, ee_2, ee_3, \chi) \mapsto (yee', xE(ee'), xE \circ \tau(ee_3), xE^2 \circ \tau E \circ \chi),$$

and so

$$\begin{aligned} \tilde{E}\tilde{x} \circ \tilde{\tau} - \tilde{\tau} \circ \tilde{x}\tilde{E} : (ee_1, ee_2, ee_3, \chi) \mapsto & \\ \big(y_2 ee' + ee_2, ee_2, (Ex \circ \tau - \tau \circ xE)(ee_3), (ExE \circ \tau E - \tau E \circ xE^2) \circ \chi\big) & \\ = (ee_1, ee_2, ee_3, \chi), & \end{aligned}$$

and similarly for the other relation. □

2.3.2.2 $\tilde{\tau}^2 = 0$

This is clear.

2.3.2.3 $\tilde{\tau}$ satisfies the braid relation

In this section we give formulas defining k-module endomorphisms $\tilde{\tau}_1$ and $\tilde{\tau}_2$ of the components of the matrix parametrization of $\tilde{E}^3$. We show that these endomorphisms satisfy the braid relations. Then we argue that they correspond to the maps $\tilde{E}\tilde{\tau}$ and $\tilde{\tau}\tilde{E}$ induced on the same bimodule components. This will complete our proof that $\tilde{x}$ and $\tilde{\tau}$ satisfy the nil affine Hecke relations in $\mathcal{U}^+$.

Lemma 2.3.14. *Let us be given $(ee_1, ee_2, ee_3, \chi) \in G_3$ with ee'' defined as in §2.2.21. Then*

$$(\tau(ee_1), -ee'', -ee'', E\tau \circ \chi) \in E^2[y]^{\oplus 3} \oplus \mathrm{Hom}_A({}_AE, E^3)[y]$$

also lies in G_3.

Proof. The reader may check this directly. In Prop. 2.3.18 we will interpret this element as the image of (ee_1, ee_2, ee_3, χ) under $\tilde{E}\tilde{\tau}$, and it must therefore lie in G_3. $\square$

Lemma 2.3.15. *Let us be given $(eee_1, eee_2, eee_3, eee_4, \psi) \in G_4$ with $eee^{(\ell)}$ defined as in §2.2.23. Then the following elements of $E^3[y]^{\oplus 4} \oplus \mathrm{Hom}_A({}_AE, E^4)[y]$ also lie in G_4:*

$$(\tau E(eee_1), eee^{(2)}, eee^{(2)}, E\tau(eee_4), E\tau E \circ \psi),$$

$$(eee^{(4)}, eee^{(4)}, \tau E(eee_3), \tau E(eee_4), \tau E^2 \circ \psi).$$

Proof. The reader may check this directly. In Prop. 2.3.18 we will interpret these elements as the images of $(eee_1, eee_2, eee_3, eee_4, \psi)$ under $\tilde{E}\tilde{\tau}$ and $\tilde{\tau}\tilde{E}$ respectively, and they must therefore lie in G_4. $\square$

Definition 2.3.16. Let $\tilde{\tau}_1, \tilde{\tau}_2$ be k-module maps defined on $\tilde{E}^3_{ij}$, presented as in §2.2.4.2, as follows:

- on $\tilde{E}^3_{11}$:
  - $\tilde{\tau}_1$ acts by $E\tau$
  - $\tilde{\tau}_2$ by τE
- on $\tilde{E}^3_{12}$:
  - $\tilde{\tau}_1$ by $E\tau E$
  - $\tilde{\tau}_2$ by τE^2
- on $\tilde{E}^3_{21}$:
  - $\tilde{\tau}_1$ by $(ee_1, ee_2, ee_3, \chi) \mapsto (\tau(ee_1), -ee'', -ee'', E\tau \circ \chi)$

– $\tilde{\tau}_2$ by $(ee_1, ee_2, ee_3, \chi) \mapsto (ee', ee', \tau(ee_3), \tau E \circ \chi)$,

i.e. $\tilde{\tau}$ as defined above on G_3 considered as $\tilde{E}^2_{22}$

- on $\tilde{E}^3_{22}$:

– $\tilde{\tau}_1$ by $(eee_1, eee_2, eee_3, eee_4, \psi) \mapsto$

$$(\tau E(eee_1), eee^{(2)}, eee^{(2)}, E\tau(eee_4), E\tau E \circ \psi)$$

– $\tilde{\tau}_2$ by $(eee_1, eee_2, eee_3, eee_4, \psi) \mapsto$

$$(eee^{(4)}, eee^{(4)}, \tau E(eee_3), \tau E(eee_4), \tau E^2 \circ \psi).$$

Proposition 2.3.17. *The $\tilde{\tau}_i$ satisfy $\tilde{\tau}_1 \circ \tilde{\tau}_2 \circ \tilde{\tau}_1 = \tilde{\tau}_2 \circ \tilde{\tau}_1 \circ \tilde{\tau}_2$.*

Proof. On $\tilde{E}^2_{1j}$ the claim follows from the τ_i braid relation. On $\tilde{E}^2_{21} = G_3$ we have:

$$(ee_1, ee_2, ee_3, \chi) \xmapsto{\tilde{\tau}_1}$$
$$(\tau(ee_1), -ee'', -ee'', E\tau \circ \chi) \xmapsto{\tilde{\tau}_2}$$
$$(-\overline{ee} - \tau(ee'''), -\overline{ee} - \tau(ee'''), -\tau(ee''), \tau E \circ E\tau \circ \chi) \xmapsto{\tilde{\tau}_1}$$
$$(-\tau(\overline{ee}), -\tau(\overline{ee}), -\tau(\overline{ee}), E\tau \circ \tau E \circ E\tau \circ \chi)$$

and

$$(ee_1, ee_2, ee_3, \chi) \xmapsto{\tilde{\tau}_2}$$
$$(ee', ee', \tau(ee_3), \tau E \circ \chi) \xmapsto{\tilde{\tau}_1}$$
$$(\tau(ee'), -\overline{ee}, -\overline{ee}, E\tau \circ \tau E \circ \chi) \xmapsto{\tilde{\tau}_2}$$
$$(-\tau(\overline{ee}), -\tau(\overline{ee}), -\tau(\overline{ee}), \tau E \circ E\tau \circ \tau E \circ \chi).$$

On $\tilde{E}^3_{22} = G_4$ we have:

$$(eee_1, eee_2, eee_3, eee_4, \psi) \xmapsto{\tilde{\tau}_1}$$
$$(\tau E(eee_1), eee^{(2)}, eee^{(2)}, E\tau(eee_4), E\tau E \circ \psi) \xmapsto{\tilde{\tau}_2}$$
$$(\tau E(eee^{(5)}) + \overline{eee}, \tau E(eee^{(5)}) + \overline{eee}, \tau E(eee^{(2)}), \tau E \circ E\tau(eee_4), \tau E^2 \circ E\tau E \circ \psi) \xmapsto{\tilde{\tau}_1}$$
$$(\tau E(\overline{eee}), \tau E(\overline{eee}), \tau E(\overline{eee}), E\tau \circ \tau E \circ E\tau(eee_4), E\tau E \circ \tau E^2 \circ E\tau E \circ \psi)$$

and

$$(eee_1, eee_2, eee_3, eee_4, \psi) \xmapsto{\tilde{\tau}_2}$$
$$(eee^{(4)}, eee^{(4)}, \tau E(eee_3), \tau E(eee_4), \tau E^2 \circ \psi) \xmapsto{\tilde{\tau}_1}$$
$$(\tau E(eee^{(4)}), \overline{eee}, \overline{eee}, E\tau \circ \tau E(eee_4), E\tau E \circ \tau E^2 \circ \psi) \xmapsto{\tilde{\tau}_2}$$
$$(\tau E(\overline{eee}), \tau E(\overline{eee}), \tau E(\overline{eee}), \tau E \circ E\tau \circ \tau E(eee_4), \tau E^2 \circ E\tau E \circ \tau E^2 \circ \psi).$$

□

The remaining goal of this section is to relate the $\tilde{\tau}_i$ just defined to the $\tilde{\tau}$ acting on $\tilde{E}$ as described in §2.3.1.1. The latter is known to be a (C, C)-bimodule morphism.

Proposition 2.3.18. *Under the isomorphism of Lemma 2.2.45, namely*

$$\tilde{E}^3 \xrightarrow{\sim} \mathrm{Hom}_{K^b(B)}(X, E'^3 X),$$

the maps $\tilde{E}\tilde{\tau}$ and $\tilde{\tau}\tilde{E}$ on $\tilde{E}^3$ correspond to $\tilde{\tau}_1$ and $\tilde{\tau}_2$ of Definition 2.3.16.

Corollary 2.3.19. *Lemmas 2.3.14 and 2.3.15 follow.*

Corollary 2.3.20. *Proposition 2.3.17 implies $\tilde{E}\tilde{\tau} \circ \tilde{\tau}\tilde{E} \circ \tilde{E}\tilde{\tau} = \tilde{\tau}\tilde{E} \circ \tilde{E}\tilde{\tau} \circ \tilde{\tau}\tilde{E}$.*

Proof of the proposition. We consider the tensor product $\tilde{E} \otimes_C \tilde{E}^2$ formed according to the procedure of §2.1.4, and study the endofunctor $\tilde{E}\tilde{\tau}$ as in Lemma 2.1.8, and similarly for

$\tilde{E}^2 \otimes_C \tilde{E}$ and $\tilde{\tau}\tilde{E}$. From Lemma 2.2.45, we have isomorphisms

$$\mathrm{Hom}_{K^b(B)}(X, E'X) \otimes_C \mathrm{Hom}_{K^b(B)}(X, E'^2X) \xrightarrow{\sim} \mathrm{Hom}_{K^b(B)}(X, E'^3X)$$
$$\mathrm{Hom}_{K^b(B)}(X, E'^2X) \otimes_C \mathrm{Hom}_{K^b(B)}(X, E'X) \xrightarrow{\sim} \mathrm{Hom}_{K^b(B)}(X, E'^3X)$$

associated with the products

$$\tilde{E} \otimes_C \tilde{E}^2 = \tilde{E}^3$$
$$\tilde{E}^2 \otimes_C \tilde{E} = \tilde{E}^3.$$

The maps are given by

$$f \otimes g \mapsto E'g \circ f$$
$$f \otimes g \mapsto E'^2 g \circ f.$$

These isomorphisms determine actions of $\tilde{E}\tilde{\tau}$ and $\tilde{\tau}\tilde{E}$ on $\mathrm{Hom}_{K^b(B)}(X, E'^3X)$ that we may compare to the $\tilde{\tau}_1$ and $\tilde{\tau}_2$ defined there by components.

The components $\tilde{E}_{ij}$ and $\tilde{E}^2_{ij}$ are $(\mathrm{End}(X_i)^{\mathsf{op}}, \mathrm{End}(X_j)^{\mathsf{op}})$-bimodules, and $\tilde{\tau}$ gives bimodule endomorphisms $\tilde{\tau}_{|ij}$ of the latter. These induce endomorphisms $(\tilde{E}\tilde{\tau})^{1|2}_{|ijk}$ of

$$\tilde{E}^{1|2}_{ijk} = \tilde{E}_{ij} \otimes_{\mathrm{End}(X_j)^{\mathsf{op}}} \tilde{E}^2_{jk},$$

as in §2.1.4. We know that $\tilde{E}^3_{ik}$ is canonically isomorphic to a quotient of $\tilde{E}^{1|2}_{i1k} \oplus \tilde{E}^{1|2}_{i2k}$, and that $\begin{pmatrix} (\tilde{E}\tilde{\tau})^{1|2}_{|i1k} & 0 \\ 0 & (\tilde{E}\tilde{\tau})^{1|2}_{|i2k} \end{pmatrix}$ acting on $\tilde{E}^{1|2}_{i1k} \oplus \tilde{E}^{1|2}_{i2k}$ descends to $\tilde{E}^3_{ik}$, where it gives the components of $\tilde{E}\tilde{\tau}$. Here it may be compared directly with $\tilde{\tau}_1$ that we defined on $\tilde{E}^3_{ik}$. It therefore suffices for our objective to check commutativity of the following diagrams labeled $D_{1|2}(i,j,k)$, indexed by triples $(i,j,k) \in \{1,2\}^3$:

$$D_{1|2}(i,j,k): \quad \begin{array}{ccc} \tilde{E}_{ij} \otimes_{\mathrm{End}(X_j)^{\mathsf{op}}} \tilde{E}^2_{jk} & \xrightarrow[f\otimes g \mapsto E'g\circ f]{} & \tilde{E}^3_{ik} \\ \Big\downarrow {\scriptstyle (\tilde{E}\tilde{\tau})^{1|2}_{|ijk}} & & \Big\downarrow {\scriptstyle \tilde{\tau}_{1|ik}} \\ \tilde{E}_{ij} \otimes_{\mathrm{End}(X_j)^{\mathsf{op}}} \tilde{E}^2_{jk} & \xrightarrow{f\otimes g \mapsto E'g\circ f} & \tilde{E}^3_{ik}. \end{array}$$

Exactly parallel considerations apply to the study of $\tilde{\tau}\tilde{E}$, where the diagrams for (i,j,k), now labeled $D_{2|1}(i,j,k)$, instead involve maps $(\tilde{E}\tilde{\tau})^{2|1}_{|ijk}$ and $\tilde{\tau}_{2|ik}$.

Checking the diagrams will occupy the next three pages.

Lemma 2.3.21. *The diagrams $D_{1|2}(i,j,k)$ commute.*

Proof. We consider the diagrams in turn:

- Diagram $D_{1|2}(1,1,1)$:

Consider $(\tilde{E}\tilde{\tau})^{1|2}_{|111} \in \mathrm{End}(\tilde{E}_{11}\otimes\tilde{E}^2_{11})$. Let $y_1e \in \tilde{E}_{11}$ and $y_1y_2ee \in \tilde{E}^2_{11}$. The image of $y_1e\otimes y_1y_2ee$ in the top right corner of the diagram is

$$E'(y_1y_2ee) \circ y_1e = y_1y_2y_3(e \otimes ee) \in \tilde{E}^3_{11}.$$

Here we can write out $E'(y_1y_2ee) = (y_1y_2ee, 0, 0, \_ \otimes y_1y_2ee) \in G_3$. On the other hand, $\tilde{\tau}(y_1y_2ee) = y_1y_2\tau(ee)$, so the image of $(\tilde{E}\tilde{\tau})^{1|2}_{|111}\big(y_1e \otimes y_1y_2ee\big)$ is $y_1y_2y_3(e\otimes\tau(ee)) \in \tilde{E}^3_{11}$, which agrees with $\tilde{\tau}_1(y_1y_2y_3(e\otimes ee))$.

- Diagram $D_{1|2}(1,2,1)$:

Consider $(\tilde{E}\tilde{\tau})^{1|2}_{|121} \in \mathrm{End}(\tilde{E}_{12}\otimes\tilde{E}^2_{21})$. Let $y_1y_2ee \in \tilde{E}_{12}$ and $(e_1,e_2,\xi) \in \tilde{E}^2_{21}$. We have no established notation for $E'\big((e_1,e_2,\xi)\big) \in \mathrm{Hom}_{K^b(B)}(E'X_2, E'^2X_1)$. It is nevertheless easy to check, by tracking 'leading terms' of the upper rows, that

$$E'\big((e_1,e_2,\xi)\big) \circ y_1y_2ee = E\xi(y_1y_2ee) \in \tilde{E}^3_{11}.$$

This lies in $y_1y_2y_3E^3[y]$. Then $\tilde{\tau}\big((e_1,e_2,\xi)\big) = (e', e', \tau\circ\xi)$, so $(\tilde{E}\tilde{\tau})^{1|2}_{|121}$ applied to $y_1y_2ee\otimes(e_1,e_2,\xi)$ and viewed in $\tilde{E}^3_{11}$ is $E\tau\circ E\xi(y_1y_2ee)$.

- Diagram $D_{1|2}(2,1,1)$:

Consider $(\tilde{E}\tilde{\tau})^{1|2}_{|211} \in \mathrm{End}(\tilde{E}_{21}\otimes\tilde{E}^2_{11})$. Let $(\theta,\varphi)\in\tilde{E}_{21}$ and $y_1y_2ee\in\tilde{E}^2_{11}$. This time we can write $E'(y_1y_2ee) = (y_1y_2ee, 0, 0, \_\otimes y_1y_2ee)$. Then

$$E'(y_1y_2ee)\circ(\theta,\varphi) = (\theta y_1y_2ee, 0, 0, \varphi\otimes y_1y_2ee)\in\tilde{E}^3_{21}.$$

Going around the diagram in either direction yields $(\theta y_1y_2\tau(ee), 0, 0, \varphi\otimes y_1y_2\tau(ee))$.

- Diagram $D_{1|2}(2,2,1)$:

Consider $(\tilde{E}\tilde{\tau})^{1|2}_{|221} \in \mathrm{End}(\tilde{E}_{22}\otimes\tilde{E}^2_{21})$. Let $(e_1,e_2,\xi)\in\tilde{E}_{22}$ and $(\bar{e}_1,\bar{e}_2,\bar{\xi})\in\tilde{E}^2_{21}$. We have no notation for $E'\left((\bar{e}_1,\bar{e}_2,\bar{\xi})\right)$. One computes that

$$E'\left((\bar{e}_1,\bar{e}_2,\bar{\xi})\right)\circ(e_1,e_2,\xi) = (\bar{\xi}(e_1), e_2\otimes\bar{e}_1, e_2\otimes\bar{e}_2, E\bar{\xi}\circ\xi)\in\tilde{E}^3_{21}.$$

Traversing the diagram in either direction gives $(\tau\circ\bar{\xi}(e_1), e_2\otimes\bar{e}', e_2\otimes\bar{e}', E\tau\circ E\bar{\xi}\circ\xi)$.

- Diagram $D_{1|2}(1,1,2)$:

Consider $(\tilde{E}\tilde{\tau})^{1|2}_{|112} \in \mathrm{End}(\tilde{E}_{11}\otimes\tilde{E}^2_{12})$. Let $y_1e\in\tilde{E}_{11}$ and $y_1y_2y_3eee\in\tilde{E}^2_{12}$. Again by tracking 'leading terms', one checks that

$$E'(y_1y_2y_3eee)\circ y_1e = y_1\ldots y_4(e\otimes eee)\in\tilde{E}^3_{12}.$$

Traversing the diagram in either direction gives $E\tau E(y_1\ldots y_4e\otimes eee)$ which is $y_1\ldots y_4(e\otimes\tau E(eee))$.

- Diagram $D_{1|2}(1,2,2)$:

Consider $(\tilde{E}\tilde{\tau})^{1|2}_{|122} \in \mathrm{End}(\tilde{E}_{12}\otimes\tilde{E}^2_{22})$. Let $y_1y_2ee\in\tilde{E}_{12}$ and $(ee_1, ee_2, ee_3,\chi)\in\tilde{E}^2_{22}$. Then check that

$$E'\left((ee_1,ee_2,ee_3,\chi)\right)\circ y_1y_2ee = E\chi(y_1y_2ee)\in\tilde{E}^3_{12}.$$

Traversing the diagram in either direction gives $(E\tau E \circ E\chi)(y_1 y_2 ee)$.

- Diagram $D_{1|2}(2, 1, 2)$:

Consider $(\tilde{E}\tilde{\tau})^{1|2}_{|212} \in \mathrm{End}(\tilde{E}_{21} \otimes \tilde{E}^2_{12})$. Let $(\theta, \varphi) \in \tilde{E}_{21}$ and $y_1 y_2 y_3 eee \in \tilde{E}^2_{12}$. Then check that

$$E'(y_1 y_2 y_3 eee) \circ (\theta, \varphi) = (\theta y_1 y_2 y_3 eee, 0, 0, 0, \varphi \otimes y_1 y_2 y_3 eee) \in \tilde{E}^3_{22}.$$

Traversing the diagram in either direction gives

$$(\tau E(\theta y_1 y_2 y_3 eee), 0, 0, 0, E\tau E \circ (\varphi \otimes y_1 y_2 y_3 eee))\,.$$

- Diagram $D_{1|2}(2, 2, 2)$:

Consider $(\tilde{E}\tilde{\tau})^{1|2}_{|222} \in \mathrm{End}(\tilde{E}_{22} \otimes \tilde{E}^2_{22})$. Let $(e_1, e_2, \xi) \in \tilde{E}_{22}$ and $(ee_1, ee_2, ee_3, \chi) \in \tilde{E}^2_{22}$. Then check that

$$E'\big((ee_1, ee_2, ee_3, \chi)\big) \circ (e_1, e_2, \xi) = (\chi(e_1), e_2 \otimes ee_1, e_2 \otimes ee_2, e_2 \otimes ee_3, E\chi \circ \xi) \in \tilde{E}^3_{22}.$$

Traversing the diagram in either direction gives

$$\big(\tau E(\chi(e_1)), e_2 \otimes ee', e_2 \otimes ee', E\tau(e_2 \otimes ee_3), E\tau E \circ E\chi \circ \xi\big).$$

□

Lemma 2.3.22. *The diagrams $D_{2|1}(i, j, k)$ commute.*

Proof. We consider the diagrams in turn:

- Diagram $D_{2|1}(1, 1, 1)$:

Consider $(\tilde{\tau}\tilde{E})^{2|1}_{|111} \in \mathrm{End}(\tilde{E}^2_{11} \otimes \tilde{E}_{11})$. Let $y_1 y_2 ee \in \tilde{E}^2_{11}$ and $y_1 e \in \tilde{E}_{11}$. Then check that

$$E'^2(y_1 e) \circ y_1 y_2 ee = y_1 y_2 y_3 ee \otimes e \in \tilde{E}^3_{11}.$$

Traversing the diagram in either direction gives

$$y_1 y_2 y_3(\tau(ee) \otimes e).$$

- Diagram $D_{2|1}(1,2,1)$:

Consider $(\tilde{\tau}\tilde{E})^{2|1}_{|121} \in \mathrm{End}(\tilde{E}^2_{12} \otimes \tilde{E}_{21})$. Let $y_1y_2y_3eee \in \tilde{E}^2_{12}$ and $(\theta,\varphi) \in \tilde{E}_{21}$. Then check that

$$E'^2\big((\theta,\varphi)\big) \circ y_1y_2y_3eee = E^2\varphi(y_1y_2y_3eee) \in \tilde{E}^3_{11}.$$

Traversing the diagram in either direction gives

$$(\tau E \circ E^2\varphi)(y_1y_2y_3eee).$$

- Diagram $D_{2|1}(2,1,1)$:

Consider $(\tilde{\tau}\tilde{E})^{2|1}_{|211} \in \mathrm{End}(\tilde{E}^2_{21} \otimes \tilde{E}_{11})$. Let $(e_1,e_2,\xi) \in \tilde{E}^2_{21}$ and $y_1e \in \tilde{E}_{11}$. Then check that

$$E'^2(y_1e) \circ (e_1,e_2,\xi) = (e_1 \otimes y_1e, e_2 \otimes y_1e, 0, \xi \otimes y_1e) \in \tilde{E}^3_{21}.$$

Traversing the diagram in either direction gives

$$(e' \otimes y_1e, e' \otimes y_1e, 0, (\tau \circ \xi) \otimes y_1e).$$

- Diagram $D_{2|1}(2,2,1)$:

Consider $(\tilde{\tau}\tilde{E})^{2|1}_{|221} \in \mathrm{End}(\tilde{E}^2_{22} \otimes \tilde{E}_{21})$. Let $(ee_1,ee_2,ee_3,\chi) \in \tilde{E}^2_{22}$ and $(\theta,\varphi) \in \tilde{E}_{21}$. Then check that

$$E'^2\big((\theta,\varphi)\big) \circ (ee_1,ee_2,ee_3,\chi) = (E\varphi(ee_1), E\varphi(ee_2), \theta ee_3, E^2\varphi \circ \chi) \in \tilde{E}^3_{21}.$$

Traversing the diagram in either direction gives

$$\big(E\varphi(ee'), E\varphi(ee'), \theta\tau(ee_3), E^2\varphi \circ \tau E \circ \chi\big).$$

- Diagram $D_{2|1}(1,1,2)$:

Consider $(\tilde{\tau}\tilde{E})^{2|1}_{|112} \in \mathrm{End}(\tilde{E}^2_{11} \otimes \tilde{E}_{12})$. Let $y_1y_2ee \in \tilde{E}^2_{11}$ and $y_1y_2\bar{e}\bar{e} \in \tilde{E}_{12}$. Then check that

$$E'^2(y_1y_2\bar{e}\bar{e}) \circ y_1y_2ee = (y_1y_2ee) \otimes (y_1y_2\bar{e}\bar{e}) = y_1 \dots y_4(ee \otimes \bar{e}\bar{e}) \in \tilde{E}^3_{12}.$$

Traversing the diagram in either direction gives

$$y_1 \dots y_4(\tau(ee) \otimes \bar{e}\bar{e}).$$

- Diagram $D_{2|1}(1,2,2)$:

Consider $(\tilde{\tau}\tilde{E})^{2|1}_{|122} \in \mathrm{End}(\tilde{E}^2_{12} \otimes \tilde{E}_{22})$. Let $y_1y_2y_3eee \in \tilde{E}^2_{12}$ and $(e_1, e_2, \xi) \in \tilde{E}_{22}$. Then check that

$$E'^2\big((e_1, e_2, \xi)\big) \circ y_1y_2y_3eee = E^2\xi(y_1y_2y_3eee) \in \tilde{E}^3_{12}.$$

Traversing the diagram in either direction gives

$$(\tau E^2 \circ E^2\xi)(y_1y_2y_3eee).$$

- Diagram $D_{2|1}(2,1,2)$:

Consider $(\tilde{\tau}\tilde{E})^{2|1}_{|212} \in \mathrm{End}(\tilde{E}^2_{21} \otimes \tilde{E}_{12})$. Let $(e_1, e_2, \xi) \in \tilde{E}^2_{21}$ and $y_1y_2ee \in \tilde{E}_{12}$. Then check that

$$E'^2(y_1y_2ee) \circ (e_1, e_2, \xi) = (e_1 \otimes y_1y_2ee, e_2 \otimes y_1y_2ee, 0, 0, \xi \otimes y_1y_2ee) \in \tilde{E}^3_{22}.$$

Traversing the diagram in either direction gives

$$(e' \otimes y_1y_2ee, e' \otimes y_1y_2ee, 0, 0, (\tau \circ \xi) \otimes y_1y_2ee)\,.$$

- Diagram $D_{2|1}(2,2,2)$:

Consider $(\tilde{\tau}\tilde{E})^{2|1}_{|222} \in \mathrm{End}(\tilde{E}^2_{22} \otimes \tilde{E}_{22})$. Let $(ee_1, ee_2, ee_3, \chi) \in \tilde{E}^2_{22}$ and $(e_1, e_2, \xi) \in \tilde{E}_{22}$. Then check that

$$E'^2\big((e_1, e_2, \xi)\big) \circ (ee_1, ee_2, ee_3, \chi) =$$
$$\big(E\xi(ee_1), E\xi(ee_2), ee_3 \otimes e_1, ee_3 \otimes e_2, E^2\xi \circ \chi\big) \in \tilde{E}^3_{22}.$$

Traversing the diagram in either direction gives

$$\big(E\xi(ee'), E\xi(ee'), \tau(ee_3) \otimes e_1, \tau(ee_3) \otimes e_2, \tau E^2 \circ E^2\xi \circ \chi\big)\,.$$

□

The proposition that $\tilde{E}\tilde{\tau}$ and $\tilde{\tau}\tilde{E}$ correspond to $\tilde{\tau}_1$ and $\tilde{\tau}_2$ is now proved. □

2.3.3 Definition of the 2-product

Definition 2.3.23. Let $\mathcal{V}$ be a 2-representation of $\mathcal{U}^+$ given by the data (E, x, τ) for a k-algebra A such that ${}_AE$ is finitely generated and projective and E^n is free as a P_n-module. We define $\mathcal{L}(1) \bigotimes \mathcal{V}$ to be the 2-representation of $\mathcal{U}^+$ given for the k-algebra C by the data $(\tilde{E}, \tilde{x}, \tilde{\tau})$.

Proposition 2.3.24. *If E is locally nilpotent, then $\tilde{E}$ is locally nilpotent.*

Proof. Note that in our setting of bimodules, local nilpotence of $E \otimes_A -$ is equivalent to nilpotence of E, meaning that $E^n \cong 0$ for some n. This is because local nilpotence implies $E^n \otimes_A A \cong 0$ for some n, but that is just E^n as a bimodule.

Recall the expression for $\tilde{E}^n$ as a matrix of $(A[y], A[y])$-bimodules:

$$\begin{pmatrix} y_1 \dots y_n E^n[y] & y_1 \dots y_{n+1} E^{n+1}[y] \\ G_n & G_{n+1} \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} \tilde{E}^n_{11} & \tilde{E}^n_{12} \\ \tilde{E}^n_{21} & \tilde{E}^n_{22} \end{pmatrix}.$$

The method we used to compute a model for G_n for $n = 1, 2, 3$ also shows that G_n for any n can be described as a sub-bimodule of $E^{n-1}[y]^{\oplus n} \oplus \mathrm{Hom}_A({}_AE, E^n)[y]$, given by the elements satisfying a certain set of conditions. It follows that G_n vanishes for large n if E^n does. Also $y_1 \dots y_n E^n[y]$ vanishes for large n because E^n does. It follows that $\tilde{E}$ is nilpotent. □

2.3.3.1 Weights and gradings for the 2-product

It frequently happens that a 2-representation has additional structure, and we may ask whether our 2-product inherits that structure. A 2-representation of $\mathcal{U}^+$ may have a weight decomposition, or its algebra may have a grading.

Definition 2.3.25. A 2-representation $\mathcal{V}$ of $\mathcal{U}^+$ given for k-algebra A by the data (E, x, τ) is said to have a weight decomposition when A has the form $A = \prod_{i \in \mathbb{Z}} A_i$ with units $e_i \in A_i$, and $e_j E e_i = \delta_{i+2,j} \cdot e_{i+2} E e_i$.

Proposition 2.3.26 (weight decomposition)**.** *Let A and (E, x, τ) satisfy the conditions of Def. 2.3.23, and let $\mathcal{V}$ be the 2-representation they determine. Suppose that $\mathcal{V}$ has a weight decomposition with units $e_i \in A_i$. Let C and $(\tilde{E}, \tilde{x}, \tilde{\tau})$ give the data of $\mathcal{L}(1) \bigotimes \mathcal{V}$. Then C has a weight decomposition $C = \prod_{i \in \mathbb{Z}} C_i$ with $C_i = f_i C f_i$ where the units $f_i \in C_i \subset C$ are given in matrix form as follows:*

$$f_i = \begin{pmatrix} e_{i+1} & 0 \\ 0 & (e_{i-1}, \_.e_{i-1}) \end{pmatrix}.$$

Proof. The elements f_i are clearly idempotent and orthogonal, and they sum to the identity. We have for the matrix components of $f_j \tilde{E} f_i$:

$$[f_j \tilde{E} f_i]_{11} = e_{j+1}.y_1 E[y].e_{i+1}$$

$$[f_j \tilde{E} f_i]_{12} = e_{j+1}.y_1 y_2 E^2[y].e_{i-1}$$

$$[f_j \tilde{E} f_i]_{21} = G_1 \bigcap \Big(e_{j-1} A[y] e_{i+1} \oplus e_{j-1}.\mathrm{Hom}_A({}_A E, E).e_{i+1}[y] \Big)$$

$$[f_j \tilde{E} f_i]_{22} = G_2 \bigcap \Big(e_{j-1}.E[y].e_{i-1} \oplus e_{j-1}.E[y].e_{i-1} \oplus e_{j-1}.\mathrm{Hom}_A({}_A E, E^2).e_{i-1}[y] \Big).$$

These are all zero unless $j = i + 2$. ∎

Definition 2.3.27 (graded case)**.** A 2-representation $\mathcal{V}$ of $\mathcal{U}^+$ given for k-algebra A by the data (E, x, τ) is said to be a $\mathbb{Z}$-graded 2-representation when A is a $\mathbb{Z}$-graded k-algebra, E is a graded bimodule, and x and τ are graded endomorphisms with $\deg x = +2$ and $\deg \tau = -2$.

Proposition 2.3.28. *Let A and (E, x, τ) satisfy the conditions of Def. 2.3.23, and let $\mathcal{V}$ be the 2-representation they determine. Suppose that $\mathcal{V}$ is a $\mathbb{Z}$-graded 2-representation. Let C and $(\tilde{E}, \tilde{x}, \tilde{\tau})$ give the data of $\mathcal{L}(1) \bigotimes \mathcal{V}$. Then $\mathcal{L}(1) \bigotimes \mathcal{V}$ is a $\mathbb{Z}$-graded 2-representation. The* *gradings on generators in C and $\tilde{E}$ are inherited from the gradings on A and E with the assumption that* $\deg y = +2$.

Proof. It is trivial to check that C is graded and $\tilde{E}$ is a graded bimodule. The formulas for $\tilde{x}$ and $\tilde{\tau}$ in Def. 2.3.4 show that they have the right degrees. □

2.4 Comparison: $\mathcal{V} = \mathcal{L}(1)$

In §2.4.1 we describe a well-known 2-representation of $\mathcal{U}^+$ categorifying $L(1) \otimes L(1)$ using Soergel bimodules. In §2.4.2 we describe our product explicitly for $\mathcal{V} = \mathcal{L}(1)$, and in §2.4.3 we show that the result is equivalent to the known one. The reader is warned that notations in this section will diverge from the previous sections.

Let $P_2 = k[y_1, y_2]$. Let S_2 denote the symmetric group on 2 letters, generated by t_1, and acting on P_2 by permutation of the y_i. Let $P_2^{S_2}$ be the subalgebra generated by invariant homogeneous polynomials.

2.4.1 A categorification of $L(1) \otimes L(1)$

Definition 2.4.1. We define:

- a (P_2, P_2)-bimodule $B_{s_1} = P_2 \otimes_{P_2^{S_2}} P_2$
  - and observe that B_{s_1} is also a P_2-algebra with structure map $P_2 \to B_{s_1}$ given by $f \mapsto 1 \otimes f$
  - and that P_2 is a left B_{s_1}-module by $(f \otimes g).\theta = fg\theta$
- a P_2-algebra $T = T_{+2} \oplus T_0 \oplus T_{-2}$ by

$$T_{+2} = P_2,\ T_0 = \mathrm{End}_{B_{s_1}}(P_2 \oplus B_{s_1})^{\mathsf{op}},\ T_{-2} = P_2$$

- a (T,T)-bimodule $\mathscr{E} = {}_{+2}\mathscr{E}_0 \oplus {}_0\mathscr{E}_{-2}$ by

$$
{}_0\mathscr{E}_{-2} = \begin{pmatrix} P_2 \\ B_{s_1} \end{pmatrix} \cong T_0 e_2
$$
$$
{}_{+2}\mathscr{E}_0 = \begin{pmatrix} P_2 & B_{s_1} \end{pmatrix} \cong e_2 T_0
$$

  for e_2 the projection onto B_{s_1}

  – and observe the canonical isomorphism

$$
{}_{+2}\mathscr{E}^2_{-2} = e_2 T_0 \otimes_{T_0} T_0 e_2 \xrightarrow{\sim} B_{s_1}
$$

- a bimodule endomorphism $x \in \operatorname{End}(\mathscr{E})$ by

$$
{}_{+2}x_0 = \begin{pmatrix} y_2 & y_2 \otimes 1 \end{pmatrix}, \quad {}_0x_{-2} = \begin{pmatrix} y_1 \\ y_1 \otimes 1 \end{pmatrix}
$$

  (acting by multiplication)

- a bimodule endomorphism $\tau \in \operatorname{End}(\mathscr{E}^2)$ by

$$
{}_{+2}\tau_{-2} : f \otimes g \mapsto \partial_{t_1}(f) \otimes g
$$

  where $\partial_{t_1} \in \operatorname{End}_k(P_2)$ is a Demazure operator:

$$
\partial_{s_1} : f \mapsto \frac{f - f^{t_1}}{y_1 - y_2}.
$$

The next theorem is well-known. Cf., for example, Lauda [Lau09], Webster [Web16, §2.3], Stroppel [Str03, §5.1.1], Sartori-Stroppel [SS15]:

Theorem 2.4.2. *The k-algebra T and triple $(\mathscr{E}, x, \tau)$ defined above gives a 2-representation of $\mathcal{U}^+$, called $\mathscr{T}$ below, that categorifies the tensor product $L(1) \otimes L(1)$ of fundamental representations of $\mathfrak{sl}_2$.*

2.4.2 $\mathcal{L}(1)\bigotimes\mathcal{L}(1)$

We notate both factors as in §2.1.2.3 except that on the right factor we use y_1 in place of y, and on the left factor we use y_2 in place of y. We write E_i, x_i, τ_i, $i = 1, 2$ for the 2-representation data on the right ($i = 1$) and on the left ($i = 2$).

In the formulas we have given for the product, the algebra A, now A_1, becomes $k[y_1]_{+1} \times k[y_1]_{-1}$ (in its weight decomposition), E becomes $k[y_1]$, x becomes y_1, and y becomes y_2. Let $\omega = y_1 - y_2 \in P_2$. So ω will take over the role of '$y_1 = x - y$' that was written in previous sections. Write $\pi : P_2 \to P_2/(\omega)$ for the projection.

We let B, X, E', C, $\tilde{E}$, $\tilde{x}$, and $\tilde{\tau}$ be defined as above. The algebra B and complex X have nonzero elements only in weights -2, 0, $+2$. These are given as follows:

$$B_{-2} = \begin{pmatrix} P_2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad X_{1_{-2}} = \begin{pmatrix} P_2 \\ 0 \end{pmatrix}, \qquad X_{2_{-2}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} P_2 & k[y] \\ 0 & P_2 \end{pmatrix}, \qquad X_{1_0} = \begin{pmatrix} P_2 \\ 0 \end{pmatrix}, \qquad X_{2_0} = \begin{pmatrix} P_2 \xrightarrow{\pi} P_2/(\omega) \\ 0 \to P_2 \end{pmatrix},$$

$$B_{+2} = \begin{pmatrix} 0 & 0 \\ 0 & P_2 \end{pmatrix}, \qquad X_{1_{+2}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad X_{2_{+2}} = \begin{pmatrix} 0 \\ 0 \to P_2 \end{pmatrix}.$$

Here the action of $P_2/(\omega)$ from the upper right of B_0 on X_{2_0} is $P_2/(\omega) \otimes_{P_2} P_2 \to P_2/(\omega)$ given by $f \otimes 1 \mapsto f$. The complexes for X start in degree 0 on the left. The matrix coefficients are in each case from the -1 weight space of A_2 in the upper left corner.

To compute $\tilde{E}$ we will also need $E'X_2$, which is:

$$
{}_0E'_{-2}(X_{2_{-2}}) = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix},
$$

$$
{}_{+2}E'_0(X_{2_0}) = \begin{pmatrix} 0 \\ \\ 0 \to P_2 \oplus P_2 \xrightarrow{(-\pi,\pi)} P_2/(\omega) \end{pmatrix}.
$$

Next we compute C:

$$
[C_{+2}] = \begin{pmatrix} 0 & 0 \\ \\ 0 & P_2 \end{pmatrix},\ [C_0] = \begin{pmatrix} P_2 & \omega P_2 \\ \\ P_2 & Q_1^{\mathsf{op}} \end{pmatrix},\ [C_{-2}] = \begin{pmatrix} P_2 & 0 \\ \\ 0 & 0 \end{pmatrix}.
$$

Here $Q_1^{\mathsf{op}} \subset P_2 \oplus P_2$ is the (commutative) algebra of all (θ, φ) such that $\varphi - \theta \in \omega P_2$, with componentwise multiplication. It is a P_2-algebra by $P_2 \ni f \mapsto (f, f) \in Q_1$. The algebra structure of C_0 (cf. §2.1.4) may be described as follows. The upper right term, ωP_2, is a left P_2-module by multiplication. It is a right Q_1^{op}-module with (θ, φ) acting by multiplication by φ. The lower left P_2 is a left Q_1^{op}-module with the same action. It has a right P_2 action by multiplication. The remaining structure maps are:

$$
\omega P_2 \otimes_{P_2} P_2 \to P_2 \tag{2.4.1}
$$

$$
\text{by } \omega\theta' \otimes \theta \mapsto \omega\theta\theta'
$$

and

$$
P_2 \otimes_{P_2} \omega P_2 \to Q_1^{\mathsf{op}} \tag{2.4.2}
$$

$$
\text{by } \theta \otimes \omega\theta' \mapsto (0, \omega\theta\theta').
$$

Now compute $\tilde{E}$ and the endomorphisms $\tilde{x}$ by components:

$$
{}_0[\tilde{E}]_{-2} = \begin{pmatrix} \omega P_2 & 0 \\ Q_1 & 0 \end{pmatrix}, \qquad {}_0[\tilde{x}]_{-2} = \begin{pmatrix} y_1 & 0 \\ (y_2, y_1) & 0 \end{pmatrix},
$$

$$
{}_{+2}[\tilde{E}]_0 = \begin{pmatrix} 0 & 0 \\ P_2 & Q_2 \end{pmatrix}, \qquad {}_{+2}[\tilde{x}]_0 = \begin{pmatrix} 0 & 0 \\ y_2 & (y_2, y_1) \end{pmatrix},
$$

where $Q_2 \subset P_2 \oplus P_2$ is the (P_2, Q_1^{op})-bimodule containing all (e_1, e_2) such that $e_1 - e_2 \in \omega P_2$; Q_1^{op} acts on Q_2 on the right by $(e_1, e_2).(\theta, \varphi) = (e_1\varphi, e_2\theta)$ (note the swap), and P_2 on the left by diagonal multiplication.

In the next section it will be useful to view ${}_0\tilde{E}_{-2}$ as $C_0 q_2$ using the idempotent $q_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in [C_0]$, and to view ${}_{+2}\tilde{E}_0$ as $q_2 C_0$ using the isomorphism of (P_2, Q_1^{op})-bimodules $\sigma : Q_1 \xrightarrow{\sim} Q_2$ by $(\theta, \varphi) \mapsto (\varphi, \theta)$. Viewing them in this way, we may write ${}_0\tilde{x}_{-2}$ as multiplication on $C_0 q_2$ on the left by $\left(\begin{smallmatrix} y_1 & 0 \\ 0 & (y_2, y_1) \end{smallmatrix}\right) \in C_0$, and ${}_{+2}\tilde{x}_0$ as multiplication on $q_2 C_0$ on the right by $\left(\begin{smallmatrix} y_2 & 0 \\ 0 & (y_1, y_2) \end{smallmatrix}\right) \in C_0$ (note the swap).

Finally, compute $\tilde{E}^2$ and $\tilde{\tau}$ by components:

$$
{}_{+2}[\tilde{E}^2]_{-2} = \begin{pmatrix} 0 & 0 \\ Q_2 & 0 \end{pmatrix}, \qquad {}_{+2}[\tilde{\tau}]_{-2} = \begin{pmatrix} 0 & 0 \\ t_{21} & 0 \end{pmatrix},
$$

where

$$
t_{21} : (e_1, e_2) \mapsto (\omega^{-1}(e_1 - e_2), \omega^{-1}(e_1 - e_2)).
$$

2.4.3 Comparison

Theorem 2.4.3. *There is an equivalence $\mathcal{L}(1) \circledast \mathcal{L}(1) \xrightarrow{\sim} \mathscr{T}$ of 2-representations.*

We will use a few intermediate steps.

Define a new algebra R:

$$
R = P_2[e]/(e^2 - \omega e).
$$

There is a map of P_2-algebras $R \xrightarrow{\gamma} B_{s_1}$ given by $e \mapsto 1 \otimes y_1 - y_1 \otimes 1$. There is another map of P_2-algebras $R \xrightarrow{\gamma'} Q_1^{\mathsf{op}}$ given by $P_2 \ni f \mapsto (f, f) \in Q_1^{\mathsf{op}}$ and $e \mapsto (\omega, 0)$.

Lemma 2.4.4. *The maps γ and γ' are isomorphisms of P_2-algebras.*

Proof. Straightforward. □

We will also use the composition $\sigma \circ \gamma'$ to obtain an isomorphism of (P_2, P_2)-bimodules $R \xrightarrow{\sim} Q_2$ given by $f \mapsto (f, f)$ and $e \mapsto (0, \omega)$.

Now we translate $\mathscr{T}$ using γ. The action of B_{s_1} on P_2 induces an action of R on P_2 through γ, according to which $P_2 \hookrightarrow R$ acts on P_2 by multiplication, and e acts by zero. We have an isomorphism of R-modules $P_2 \xrightarrow{\sim} R/(e)$ using this action on P_2. In the remainder of this section we assume this isomorphism and write R in place of B_{s_1} everywhere in the 2-representation $\mathscr{T}$. Under this translation, and using the decomposition $R \xrightarrow{\sim} P_2 \oplus P_2 e$ as P_2-modules, we have:

$$ {}_{+2}x_0 = \begin{pmatrix} y_2 & y_2 + e \end{pmatrix}, \quad {}_0x_{-2} = \begin{pmatrix} y_1 \\ y_1 - e \end{pmatrix}, $$

and

$$ {}_{+2}\tau_{-2} = \big(p_1 + p_2 e \mapsto -p_2\big). $$

Lemma 2.4.5. *The matrix presentation of T_0 is given by:*

$$ \begin{pmatrix} P_2 & P_2 \\ P_2 & R \end{pmatrix} \xrightarrow{\sim} T_0, $$

where:

- *for $[T_0]_{11}$ the map sends $\theta \in P_2$ to $(1 \mapsto \theta) \in \mathrm{End}_R(P_2)^{\mathsf{op}}$*
- *for $[T_0]_{21}$ the map sends $\theta \in P_2$ to $(1 \mapsto \theta) \in \mathrm{Hom}_R(R, P_2)$*
- *for $[T_0]_{12}$ the map sends $\theta \in P_2$ to $(1 \mapsto \theta\omega - \theta e) \in \mathrm{Hom}_R(P_2, R)$*

- *for* $[T_0]_{22}$ *the map sends* $r \in R$ *to* $(1 \mapsto r) \in \mathrm{End}_R(R,R)^{\mathsf{op}}$.

The algebra structure maps (cf. §2.1.4) are given as follows:

- $[T_0]_{11} \circlearrowright [T_0]_{12}$ *by* $\theta.\theta' = \theta\theta'$
- $[T_0]_{21} \circlearrowleft [T_0]_{11}$ *by* $\theta'.\theta = \theta'\theta$
- $[T_0]_{12} \circlearrowleft [T_0]_{22}$ *by* $\theta.(p_1 + p_2 e) = \theta p_1$
- $[T_0]_{22} \circlearrowright [T_0]_{21}$ *by* $(p_1 + p_2 e).\theta = p_1\theta$
- $[T_0]_{12} \otimes [T_0]_{21} \to [T_0]_{11}$ *by* $\theta \otimes \theta' \mapsto \omega\theta\theta'$
- $[T_0]_{21} \otimes [T_0]_{12} \to [T_0]_{22}$ *by* $\theta' \otimes \theta \mapsto \omega\theta'\theta - \theta'\theta e$.

Proof. Let us explain the map to $[T_0]_{12}$. Recall that $P_2 \cong R/(e)$. An element of $\mathrm{Hom}_R(R/(e), R)$ is given by the image $r = p_1 + p_2 e$ of 1, which may be anything satisfying $e.r = 0$, and that condition is equivalent to $p_1 = -p_2\omega$. The other morphisms and the structure maps are easily computed. □

Lemma 2.4.6. *Let* $\Phi_0 : T_0 \to C_0$ *be given on components by:*

$$\begin{pmatrix} Id_{P_2} & \omega \\ Id_{P_2} & \gamma' \end{pmatrix}.$$

Then Φ_0 *is an isomorphism of* P_2*-algebras.*

Proof. The specified maps give algebra isomorphisms on the diagonal components, and k-module isomorphisms on the off-diagonal components. Now we check equivariance under the bimodule structure maps. The only nonobvious cases concern maps involving the lower right component.

An element of Q_1^{op} may be written uniquely as a sum $(\omega\theta, 0) + (\varphi, \varphi)$. This is sent by γ'^{-1} to $\varphi + \theta e \in R$. So the action of (θ, φ) by multiplication by φ agrees with the action of

$p_1 + p_2 e$ by multiplication by p_1. The structure map $[T_0]_{12} \otimes [T_0]_{21} \to [T_0]_{11}$ clearly agrees with Eq. 2.4.1 through Φ_0. The map $[T_0]_{21} \otimes [T_0]_{12} \to [T_0]_{22}$ agrees with Eq. 2.4.2 through Φ_0 because $\gamma' : \omega\theta'\theta - \theta'\theta e \mapsto (0, \omega\theta\theta')$. $\square$

Proof of Theorem 2.4.3. Extend Φ_0 to an algebra isomorphism $\Phi : T \xrightarrow{\sim} C$ by $\Phi_{+2} = \mathrm{Id}_{P_2}$ and $\Phi_{-2} = \mathrm{Id}_{P_2}$. It remains to check compatibility with the actions of E, x, and τ in $\mathcal{U}^+$, and this poses no difficulty. We summarize that now.

We have in $\mathscr{T}$ that ${}_0\mathscr{E}_{-2} \xrightarrow{\sim} T_0 r_2$ for $r_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in [T_0]$, and similarly ${}_0\tilde{E}_{-2} = C_0 q_2$ in $\mathcal{L}(1) \circledast \mathcal{L}(1)$; and we have $q_2 = \Phi_0(r_2)$. The action of ${}_0x_{-2}$ on ${}_0\mathscr{E}_{-2}$ in $\mathscr{T}$ can be written in $T_0 r_2$ as multiplication on the left by $\left(\begin{smallmatrix} y_1 & 0 \\ 0 & y_1 - e \end{smallmatrix}\right) \in [T_0]$. In $\mathcal{L}(1) \circledast \mathcal{L}(1)$ it is written as multiplication on the left by $\left(\begin{smallmatrix} y_1 & 0 \\ 0 & (y_2, y_1) \end{smallmatrix}\right)$. These correspond using $\gamma' : R \xrightarrow{\sim} Q_1^{\mathsf{op}}$. Similarly for ${}_{+2}x_0$ since $\gamma' : R \ni y_2 + e \mapsto (y_1, y_2) \in Q_1^{\mathsf{op}}$. Finally, the action of τ in R by ${}_{+2}\tau_{-2} = \big(p_1 + p_2 e \mapsto -p_2\big)$ corresponds to ${}_{+2}\tilde{\tau}_{-2}$, now using $\sigma \circ \gamma' : R \xrightarrow{\sim} Q_2$. $\square$

CHAPTER 3

Construction of the product: adding the negative half

This chapter provides a proof of the Main Theorem, Part II.

3.1 Additional background

3.1.1 Adding a dual

We begin with a description of 2-representations of the full 2-category $\mathcal{U}$ associated to the Lie algebra $\mathfrak{sl}_2$. The 2-category $\mathcal{U}$ that we mean is defined in [Rou08a, §4.1.3], but with τ replaced by $-\tau$ in the Hecke relations. We do not repeat that definition here since we work with the concrete data of 2-representations and not with the 2-category $\mathcal{U}$ itself.

A 2-representation of the full $\mathcal{U}$ is defined in terms of weights (see Def. 2.3.25). The monoidal category $\mathsf{Bim}_k(A)$ may be interpreted as a 2-category with a single object A. When A is provided with a weight decomposition $A = \prod_{\lambda\in\mathbb{Z}} A_\lambda$, then $\mathsf{Bim}_k(A)$ may be interpreted as a 2-category with objects given by the weight algebras A_λ, morphisms given by (A_μ, A_λ)-bimodules, and 2-morphisms given by bimodule maps. With this interpretation, we may describe a 2-representation of $\mathcal{U}$ as a strict 2-functor $\mathcal{U} \to \mathsf{Bim}_k(A)$ given on objects by $\mathbf{1}_\lambda \mapsto A_\lambda$.

In Chapter 2, we considered 2-representations of $\mathcal{U}^+$ in $\mathsf{Bim}_k(A)$. These were determined by a choice of bimodule ${}_AE_A \in \mathsf{Bim}_k(A)$ and bimodule endomorphisms x and τ. Every bimodule ${}_AE_A$ has left- and right-dual bimodules, respectively:

$$\begin{aligned} {}^\vee E &= \mathrm{Hom}_A({}_AE, A), \\ E^\vee &= \mathrm{Hom}_A(E_A, A). \end{aligned}$$

Now, when ${}_AE$ is f.g. projective, the canonical morphism ${}^\vee E \otimes_A E \to \mathrm{Hom}_A({}_AE, E)$ is an isomorphism of bimodules. More generally, the canonical morphism of functors ${}^\vee E \otimes_A - \to \mathrm{Hom}_A({}_AE, -)$ is an isomorphism. In this situation, the endofunctor ${}^\vee E \otimes_A -$ of the category mod-A is right adjoint to the endofunctor $E \otimes_A -$ of the same category. The triple $({}^\vee E, \varepsilon, \eta)$ gives the right-dual object for E in the monoidal category $\mathsf{Bim}_k(A)$, where $\varepsilon : E \otimes_A {}^\vee E \to A$ and $\eta : A \to {}^\vee E \otimes_A E$ are given by evaluation and right A-action ($a \mapsto \_.a$), respectively. (Note that ${}^\vee E$ is the *left*-dual bimodule but it gives the *right*-dual object.)

Now assume only that $(E \otimes_A -, {}^\vee E \otimes_A -)$ is an adjoint pair for some bimodule ${}_AE_A$. The adjunction gives equivalences of functors:

$$\mathrm{Hom}_A({}_AE, -) \cong \mathrm{Hom}_A({}_AA, {}^\vee E \otimes_A -) \cong {}^\vee E \otimes_A -,$$

so all three are both right- and left-exact functors. So ${}_AE$ is projective. Furthermore, these functors commute with infinite direct sums, so ${}_AE$ is finitely generated as well.

In this chapter we consider 2-representations for which the image of F in $\mathsf{Bim}_k(A)$, i.e. the bimodule ${}_AF_A$, is identically the left-dual bimodule ${}^\vee E$. There is no loss of generality because any 2-representation of $\mathcal{U}$ in $\mathsf{Bim}_k(A)$ is equivalent to one of these. (For any 2-representation in $\mathsf{Bim}_k(A)$, the endofunctor ${}_AF \otimes_A -$ of mod-A is right adjoint to ${}_AE \otimes_A -$, and is therefore unique up to unique isomorphism.) A 2-representation of $\mathcal{U}$ given by the data $(A, E, F, x, \tau, \varepsilon, \eta)$ in $\mathsf{Bim}_k(A)$ is said to *extend* a 2-representation (A, E, x, τ) of $\mathcal{U}^+$ when $F = {}^\vee E$ and ε, η arise from the duality.

It was a hypothesis of the Main Theorem, Part I that ${}_AE$ is f.g. projective. This assumption was needed in order to show that $E'X$ is a perfect complex (for example). In light of the above, we see that this condition is also necessitated by the existence of an extension of the 2-representation of $\mathcal{U}^+$ to a 2-representation of $\mathcal{U}$ in $\mathsf{Bim}_k(A)$.

The following lemma is a consequence of the foregoing discussion.

Lemma 3.1.1. *Suppose the data (A, E, x, τ) determines a 2-representation of $\mathcal{U}^+$ in $\mathsf{Bim}_k(A)$ having a weight decomposition. This data extends to determine a 2-representation of $\mathcal{U}$ if and only if ${}_AE$ is f.g. projective and the commutator morphisms ρ_λ determined by x and τ are isomorphisms.*

In Chapter 2 we already established that ${}_C\tilde{E}$ is f.g. projective. Here in Chapter 3 we show that $(C, \tilde{E}, \tilde{x}, \tilde{\tau})$ extends to determine a 2-representation of $\mathcal{U}$ (assuming (A, E, x, τ) does and that E^n is free over P_n) by showing that $\tilde{\rho}_\lambda$ are isomorphisms and then applying this lemma.

3.1.2 Commutator morphisms

We define the commutator morphisms more precisely.

Assume we are given the data (A, E, x, τ) of a 2-representation of $\mathcal{U}^+$ in $\mathsf{Bim}_k(A)$ such that ${}_AE$ is f.g. projective so $(E, {}^\vee E)$ is an adjunction. Write $F = {}^\vee E$ and write $\eta : A \to FE$ and $\varepsilon : EF \to A$ for the unit and counit of the adjunction given by the duality. Assume further that this 2-representation has a weight decomposition $A = \prod_{\lambda \in \mathbb{Z}} A_\lambda$. We use the notation $E_\lambda = E \cdot A_\lambda$ and ${}_\mu E_\lambda = A_\mu \cdot E \cdot A_\lambda$. So $E = \bigoplus_{\lambda,\mu} {}_\mu E_\lambda$.

In this chapter we use a convention that '$\oplus$' and '$\sum$' denote the components of a map to and from a direct sum, respectively.

We define $\sigma : EF \to FE$ by:

$$\sigma = (FE\varepsilon) \circ (F\tau F) \circ (\eta EF) : EF \to FE.$$

For $\lambda \in \mathbb{Z}^{\geqslant 0}$ we define:

$$\rho_\lambda = \sigma \oplus \bigoplus_{i=0}^{\lambda-1} \varepsilon \circ x^i F : EF_\lambda \to FE_\lambda \oplus A_\lambda^{\oplus \lambda}, \tag{3.1.1}$$

and for $\lambda \in \mathbb{Z}^{\leqslant 0}$:

$$\rho_\lambda = \left(\sigma, \sum_{i=0}^{-\lambda-1} Fx^i \circ \eta \right) : EF_\lambda \oplus A_\lambda^{\oplus -\lambda} \to FE_\lambda. \tag{3.1.2}$$

The summation terms are neglected when $\lambda = 0$.

The data determines a 2-representation of the full $\mathcal{U}$ using $F = {}^\vee E$ when ρ_λ is an isomorphism of (A, A)-bimodules for each λ.

3.1.3 Integrability

In the literature, a 2-representation is typically defined in terms of weight categories $\mathcal{C}_\lambda$ and functors E and F between them, whereas we have framed our results entirely in terms of bimodules E and F. One reason for this is that the operation of tensoring with chosen bimodules may determine various functors that act on various reasonable categories of modules. The most important ones are mod-A and proj-A.

The distinction between mod-A and proj-A interacts with our results and the hypothesis of integrability in an interesting way. This interaction is mediated by the property of 'second adjunction' that a 2-representation of $\mathcal{U}$ may possess. We explain this next. Note that some authors include the second adjunction in their definition of a 2-representation, and for them, this discussion will be of minor significance. It may be interesting for them to observe, though, that in our construction of tensor product, the hypothesis of integrability passes from the factors to the product quite easily, while it is not clear that the second adjunction on its own passes from the factors to the product at all.

Every 2-representation of $\mathcal{U}$ given with functors E and F comes with one adjunction

(E, F), and with the data of a 'candidate' unit and counit pair for a second adjunction (F, E). When the 2-representation acts on a category mod-A and E and F are given by tensoring with bimodules, the first adjunction implies that ${}_A E$ is f.g. projective. In this case, the upper half $\mathcal{U}^+$ also acts on the smaller category proj-A. If the 2-representation is assumed to be integrable, and the full $\mathcal{U}$ acts, i.e. the ρ_λ are isomorphisms, then by [Rou08a, Thm. 5.16], the given candidates do provide a second adjunction (F, E). This adjunction implies that ${}_A F$ is also f.g. projective, and now the full $\mathcal{U}$ action may be restricted to proj-A.

Given only the first adjunction with an action of $\mathcal{U}^+$, so ${}_A E$ is f.g. projective, together with the hypothesis that E^n is free over P_n, we can form the 2-representation of $\mathcal{U}^+$ called $\mathcal{L}(1) \circledast \mathcal{V}$ in Chapter 2. In the course of forming $\mathcal{L}(1) \circledast \mathcal{V}$ we found that ${}_C \tilde{E}$ is f.g. projective, so it may be interpreted either in an action on mod-C or else in an action restricted to proj-C. Given also a second adjunction $({}^\vee E, E)$ with an action of the full $\mathcal{U}$, we know that $\mathcal{U}$ acts on proj-A through E and ${}^\vee E$ in the 2-representation $\mathcal{V}$, but we are not (currently) able to show from this alone that $\mathcal{U}$ acts on proj-C through $\tilde{E}$ and ${}^\vee \tilde{E}$ since we do not know that ${}^\vee \tilde{E}$ is f.g. projective.

Given the first adjunction $(E, {}^\vee E)$ and also the hypothesis of integrability of an action of the full $\mathcal{U}$, we know that there is a second adjunction $({}^\vee E, E)$. Now the hypothesis of integrability itself passes to the product bimodule $\tilde{E}$. (See Prop. 2.3.24.) Given that we can also show (below in this chapter) that the product maps $\tilde{\rho}_\lambda$ are isomorphisms, so we have an action of the full $\mathcal{U}$ on mod-C, it follows from integrability that there is a second adjunction $({}^\vee \tilde{E}, \tilde{E})$ for the product. This implies, in turn, that ${}_C {}^\vee \tilde{E}$ is f.g. projective and that the $\mathcal{U}$ action may be restricted to the category proj-C.

To summarize, second adjunctions enable restriction of the full $\mathcal{U}$ action to the subcategories proj-A and proj-C. The existence of a second adjunction $({}^\vee E, E)$ in $\mathcal{V}$ is *not enough* (to our knowledge, according to our argument) to guarantee a second adjunction $({}^\vee \tilde{E}, \tilde{E})$ in $\mathcal{L}(1) \circledast \mathcal{V}$. But *integrability* of $\mathcal{V}$ is enough to guarantee *integrability* of $\mathcal{L}(1) \circledast \mathcal{V}$, as well as to give *both* second adjunctions $({}^\vee E, E)$ and $({}^\vee\tilde{E}, \tilde{E})$.

3.2 More bimodules

Definition 3.2.1. Let L_n denote $\mathrm{Hom}_{D^b(B)}(E'^n X_1, X_2)$.

Note that $L_1 = G_1$. We will only need L_1 and L_2 in what follows. Observe that L_n has a right G_1^{op}-module structure given by post-composition. We now study L_2 and provide it with the structure of $(G_1^{\mathsf{op}}, G_1^{\mathsf{op}})$-bimodule.

Recall from Def. 2.2.10 the two complexes of B-modules:

$$R = \begin{pmatrix} E^2[y] \xrightarrow{(\pi_2, \pi_2\circ\tau)} E_yE \oplus E_yE \\ 0 \to E[y] \oplus E[y] \end{pmatrix},$$

$$X_2' = \begin{pmatrix} \tau y_1 E^2[y] & \xrightarrow{\pi_2} & E_yE \\ 0 & \longrightarrow & E[y] \end{pmatrix},$$

where E_y acts by the obvious canonical maps. Recall that X_2' is a finite direct sum of summands of X, and hence strictly perfect.

The matrix algebra structure of the nil-affine Hecke algebra implies that the map $\left(\begin{smallmatrix}\tau y_1\\ \tau\end{smallmatrix}\right)$ gives a decomposition of left $A[y]$-modules:

$$E^2[y] \xrightarrow[\left(\begin{smallmatrix}\tau y_1\\ \tau\end{smallmatrix}\right)]{\sim} \tau y_1 E^2[y] \oplus \tau y_1 E^2[y].$$

Recall that we have an isomorphism $R \xrightarrow{\sim} X_2' \oplus X_2'$ in B-cplx given by the above isomorphism on the degree 0 term of the upper row, and the identity on all other terms. (Lemma 2.2.12.) So $R \in \mathrm{per}\, B$ and is strictly perfect. Recall that there is a quasi-isomorphism $R \xrightarrow{q.i.} E'X_2$ determined by $\mathrm{Id}_{E^2[y]}$ on the degree 0 term of the upper row and $\left(\begin{smallmatrix}1 & 0\\ 1 & -y_1\end{smallmatrix}\right)$ on the degree 1 term of the lower row. (Lemma 2.2.13.) In this chapter we need an additional feature of R.

Lemma 3.2.2. *The complex R carries a right action of the algebra G_1^{op}, where $(\theta, \varphi) \in G_1^{\mathsf{op}}$ acts by post-composition with $E\varphi \in \mathrm{End}(E^2[y])$ on the degree 0 term $E^2[y]$ in the top row, and by the matrix*

$$\Phi = \begin{pmatrix} \varphi & 0 \\ \varphi_1 & \theta \end{pmatrix}$$

on the degree 1 term $E[y]^{\oplus 2}$ in the bottom row, and by $E_y\Phi$ on the degree 1 term $E_yE^{\oplus 2}$ in the top row. Through the quasi-isomorphism of the previous lemma, this action induces the canonical action of $G_1^{\mathsf{op}} = \mathrm{End}_{K^b(B)}(X_2)^{\mathsf{op}}$ on $E'X_2$ given by functoriality of E'.

Proof. First we check that the right action of (θ, φ) described in the lemma gives a morphism of complexes of left B-modules. The action is clearly $A[y]$-linear in the top and bottom rows, and it is clearly linear over the off-diagonal generators in $E_y \subset B$. The action commutes with the differential on the bottom row. We check the top row:

$$\begin{aligned}\begin{pmatrix} E_y\varphi & 0 \\ E_y\varphi_1 & E_y\theta \end{pmatrix} \cdot \begin{pmatrix} \pi E \\ \pi E \circ \tau \end{pmatrix} &= \begin{pmatrix} E_y\varphi \circ \pi E \\ E_y\varphi_1 \circ \pi E + E_y\theta \circ \pi E \circ \tau \end{pmatrix} \\ &= \begin{pmatrix} \pi E \circ E\varphi \\ \pi E \circ E\varphi_1 + \pi E \circ \tau \circ E\theta \end{pmatrix} \\ &= \begin{pmatrix} \pi E \\ \pi E \circ \tau \end{pmatrix} \circ E\varphi.\end{aligned}$$

Next we check that the action commutes with multiplication in the algebra. In the algebra we have $(\theta, \varphi) \cdot (\theta', \varphi') = (\theta\theta', \varphi' \circ \varphi)$, while the action of the product in the degree 1 term of the bottom row is given by:

$$\begin{pmatrix} \varphi' & 0 \\ \varphi_1' & \theta' \end{pmatrix} \cdot \begin{pmatrix} \varphi & 0 \\ \varphi_1 & \theta \end{pmatrix} = \begin{pmatrix} \varphi' \circ \varphi & 0 \\ \varphi_1' \circ \varphi + (\_\theta') \circ \varphi_1 & \theta\theta' \end{pmatrix}.$$

Note that

$$\varphi_1' \circ \varphi + (\_\theta') \circ \varphi_1 = \theta.\varphi_1' + \varphi_1' \circ y_1\varphi_1 + \varphi_1.\theta',$$

$$\varphi' \circ \varphi - \_.\theta\theta' = y_1\big((\_\theta') \circ \varphi_1 + \varphi_1' \circ \varphi\big),$$

so the composition of the actions agrees with the action of the product on that term. The other terms are trivial to check.

Lastly we check that through the quasi-isomorphism of Lemma 2.2.13, this action is compatible with the canonical action on $E'X_2$. Start with the degree 1 term in the bottom row:

$$\begin{pmatrix} 1 & 0 \\ 1 & -y_1 \end{pmatrix} \cdot \begin{pmatrix} \varphi & 0 \\ \varphi_1 & \theta \end{pmatrix} = \begin{pmatrix} \varphi & 0 \\ \varphi - y_1\varphi_1 & -y_1\theta \end{pmatrix},$$

$$\begin{pmatrix} \varphi & 0 \\ 0 & \theta \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & -y_1 \end{pmatrix} = \begin{pmatrix} \varphi & 0 \\ \theta & -\theta y_1 \end{pmatrix}.$$

These agree because $\varphi - y_1\varphi_1 = \theta$. The other terms are trivial to check. □

Now we compute a model for L_2 using the strictly perfect R as a replacement for $E'X_2$.

Definition 3.2.3. Define the following $(A[y], A[y])$-sub-bimodule of $F[y]^{\oplus 2} \oplus \mathrm{Hom}_A({}_AE^2, E)[y]$:

$$\bar{L}_2 = \Big\langle (f', f, \rho) \in F[y]^{\oplus 2} \oplus \mathrm{Hom}_A({}_AE^2, E)[y] \,\Big|\, \rho = Ef + Ef' \circ \tau + y_1 \circ \rho' \text{ for some } \rho' \in \mathrm{Hom}_A({}_AE^2, E)[y] \Big\rangle.$$

Proposition 3.2.4. *There is an isomorphism of $(A[y], A[y])$-bimodules $\bar{L}_2 \xrightarrow{\sim} \mathrm{Hom}_{K^b(B)}(R, X_2)$ determined by:*

$$(f', f, \rho) \mapsto \left(\begin{pmatrix} (ee, (\begin{smallmatrix} 0 \\ 0 \end{smallmatrix})) \\ (0, (\begin{smallmatrix} e \\ e' \end{smallmatrix})) \end{pmatrix} \mapsto \begin{pmatrix} (\rho(ee), 0) \\ (0, f(e) + f'(e')) \end{pmatrix} \right).$$

Proof. The proof is seen by directly computing $Z^0\mathscr{H}om_B(R, X_2)$. It is easy to check that the morphism given as the image of (f', f, ρ) is a morphism of complexes of left B-modules. The condition $\rho = Ef + Ef' \circ \tau + y_1 \circ \rho'$ is equivalent to the statement that this morphism has zero differential. □

Corollary 3.2.5. *The isomorphism above, followed by the canonical isomorphism of functors* $\mathrm{Hom}_{K^b(B)}(R, -) \xrightarrow{\sim} \mathrm{Hom}_{D^b(B)}(R, -)$ *applied to* X_2*, gives an isomorphism* $\bar{L}_2 \xrightarrow{\sim} L_2$ *of* $(A[y], A[y])$*-bimodules.*

Proposition 3.2.6. *There is an isomorphism of* $(A[y], A[y])$*-bimodules*
$F^2[y] \xrightarrow{\sim} \mathrm{Hom}_{K^b(B)}(R, X_1)$ *given by*

$$F^2[y] \ni ff \mapsto \left(\begin{pmatrix} (ee, (\begin{smallmatrix} ee \\ ee' \end{smallmatrix})) \\ \\ (0, (\begin{smallmatrix} e \\ e' \end{smallmatrix})) \end{pmatrix} \mapsto \begin{pmatrix} ff(ee) \\ \\ 0 \end{pmatrix} \right).$$

Proof. The proof is seen by directly computing $Z^0\mathscr{H}om_B(R, X_1)$. □

(Recall the meaning of the notation: ee is an arbitrary element of $E^2[y]$ (and ff of $F^2[y]$), not a simple tensor. It is unrelated to e, which is an arbitrary element of $E[y]$.)

It is useful to give a model of G_2 that is compatible with this model of L_2 by using the replacement R for $E'X_2$.

Definition 3.2.7. Define the following $(A[y], A[y])$-sub-bimodule of $E[y]^{\oplus 2} \oplus \mathrm{Hom}_A({}_AE, E^2)[y]$:

$$\bar{G}'_2 = \Big\langle (e', e, \xi) \in E[y]^{\oplus 2} \oplus \mathrm{Hom}_A({}_AE, E^2)[y] \Big| \\ \xi = {}_{-} \otimes e + y_2\tau\left({}_{-} \otimes (e - y_1 e')\right) + y_1 y_2 \xi' \\ \text{for some } \xi' \in \mathrm{Hom}_A({}_AE, E^2)[y] \Big\rangle.$$

Proposition 3.2.8. *There is an isomorphism of* $(A[y], A[y])$*-bimodules* $\bar{G}'_2 \xrightarrow{\sim} \mathrm{Hom}_{K^b(B)}(X_2, R)$

determined by:

$$(e', e, \xi) \mapsto \left(\begin{pmatrix} (e,0) \\ (0,1) \end{pmatrix} \mapsto \begin{pmatrix} (\xi(e),0) \\ (0, (\begin{smallmatrix} e \\ e-y_1e' \end{smallmatrix})) \end{pmatrix} \right).$$

Proof. The proof is seen by directly computing $Z^0\mathscr{H}om_B(X_2, R)$. □

The quasi-isomorphism $R \xrightarrow{q.i.} E'X_2$ determines an isomorphism $\bar{G}_2' \xrightarrow{\sim} \bar{G}_2$, since X_2 is strictly perfect, given by $(e', e, \xi) \mapsto (e, e - y_1e', \xi)$, with inverse given by $(e_1, e_2, \xi) \mapsto (y_1^{-1}(e_1 - e_2), e_1, \xi)$. In the remainder of this chapter we will use $\bar{G}_2'$ instead of $\bar{G}_2$ as a model for G_2.

Definition 3.2.9. Let U denote $\mathrm{Hom}_{K^b(B)}(R, R)$. It is canonically isomorphic to $\mathrm{Hom}_{D^b(B)}(E'X_2, E'X_2)$.

As in other cases, we describe a model for U and work with the model in what follows.

Definition 3.2.10. Define the following $(A[y], A[y])$-sub-bimodule of $FE[y]^{\oplus 4} \oplus \mathrm{Hom}_A({}_AE^2, E^2)[y]$:

$$\bar{U} = \Big\langle (\Phi_{11}, \Phi_{21}, \Phi_{12}, \Phi_{22}, \Lambda) \in FE[y]^{\oplus 4} \oplus \mathrm{Hom}_A({}_AE^2, E^2)[y] \Big| \\ \Lambda = \tau y_1(E\Phi_{11} + E\Phi_{12} \circ \tau) - y_2\tau y_1(E\Phi_{21} + E\Phi_{22} \circ \tau) + y_1y_2\Lambda^\circ \\ \text{for some } \Lambda^\circ \in \mathrm{Hom}_A({}_AE^2, E^2)[y] \Big\rangle.$$

Here Φ_{ij} give the components of the matrix $[\Phi]$ of a map $\Phi \in \mathrm{End}_A({}_AE[y] \oplus E[y])$. Note that because y_1y_2 is injective, Λ° is uniquely determined by (Φ, Λ).

Proposition 3.2.11. *There is an isomorphism of $(A[y], A[y])$-bimodules $\bar{U} \xrightarrow{\sim} U$ determined by*

$$(\Phi, \Lambda) \mapsto \left(\begin{pmatrix} (ee, (\begin{smallmatrix} 0 \\ 0 \end{smallmatrix})) \\ (0, (\begin{smallmatrix} e \\ e' \end{smallmatrix})) \end{pmatrix} \mapsto \begin{pmatrix} (\Lambda(ee), (\begin{smallmatrix} 0 \\ 0 \end{smallmatrix})) \\ (0, [\Phi] \cdot (\begin{smallmatrix} e \\ e' \end{smallmatrix})) \end{pmatrix} \right).$$

Proof. The proof is seen by directly computing $U = Z^0 \mathscr{H}om_B(R, R)$. We must show that the condition on Λ is equivalent to the statement that the image of (Φ, Λ) has zero differential. One computes directly that the morphism given as this image has zero differential if and only if the following pair of equations holds:

$$\begin{cases} \pi E \circ \Lambda = E_y \Phi_{11} \circ \pi E + E_y \Phi_{12} \circ \pi E \circ \tau \\ \pi E \circ \tau \Lambda = E_y \Phi_{21} \circ \pi E + E_y \Phi_{22} \circ \pi E \circ \tau. \end{cases}$$

That pair is manifestly equivalent to the condition:

$$\begin{cases} \Lambda = E\Phi_{11} + E\Phi_{12} \circ \tau + y_2 \Lambda' \\ \tau\Lambda = E\Phi_{21} + E\Phi_{22} \circ \tau + y_2 \Lambda'' \end{cases} \tag{3.2.1}$$

for some $\Lambda', \Lambda'' \in \mathrm{Hom}_A({}_A E^2, E^2)[y]$.

Claim 3.2.12. Suppose (Φ, Λ) is given such that (3.2.1) holds for some Λ', Λ''. Then there is $\Lambda^\circ \in \mathrm{Hom}_A({}_A E^2, E^2)[y]$ such that

$$\Lambda = \tau y_1 (E\Phi_{11} + E\Phi_{12} \circ \tau) - y_2 \tau y_1 (E\Phi_{21} + E\Phi_{22} \circ \tau) + y_1 y_2 \Lambda^\circ. \tag{3.2.2}$$

Proof. Multiply the second equation of the pair by τ and obtain:

$$-\tau y_2 \Lambda'' = \tau \circ E\Phi_{21} + \tau \circ E\Phi_{22} \circ \tau.$$

Multiply the first by τ and the second by τy_1 and identify the results to obtain:

$$\tau y_2 \Lambda' = y_1 y_2 \tau \Lambda'' + \tau y_1 \circ \big(E\Phi_{21} + E\Phi_{22} \circ \tau\big) - \tau \circ \big(E\Phi_{11} + E\Phi_{12} \circ \tau\big).$$

Then:

$$\begin{aligned} \Lambda' &= (y_1 \tau - \tau y_2) \circ \Lambda' \\ &= y_1 \tau \Lambda' - y_1 y_2 \tau \Lambda'' - \tau y_1 \circ \big(E\Phi_{21} + E\Phi_{22} \circ \tau\big) + \tau \circ \big(E\Phi_{11} + E\Phi_{12} \circ \tau\big) \\ &= y_1 \big(\tau \Lambda' - y_2 \tau \Lambda''\big) - \tau y_1 \circ \big(E\Phi_{21} + E\Phi_{22} \circ \tau\big) + \tau \circ \big(E\Phi_{11} + E\Phi_{12} \circ \tau\big). \end{aligned}$$

Let $\Lambda^\circ = \tau\Lambda' - y_2\tau\Lambda''$. Then:

$$\begin{aligned}\Lambda &= E\Phi_{11} + E\Phi_{12}\circ\tau + y_1y_2\Lambda^\circ \\ &\qquad - y_2\tau y_1\circ\big(E\Phi_{21} + E\Phi_{22}\circ\tau\big) + y_2\tau\circ\big(E\Phi_{11} + E\Phi_{12}\circ\tau\big) \\ &= \tau y_1\circ(E\Phi_{11} + E\Phi_{12}\circ\tau) - y_2\tau y_1\circ(E\Phi_{21} + E\Phi_{22}\circ\tau) + y_1y_2\Lambda^\circ,\end{aligned}$$

as desired. □

Claim 3.2.13. Now suppose (Φ, Λ) and Λ° are given such that (3.2.2) holds. Then there are Λ', Λ'' such that (3.2.1) holds.

Proof. Let

$$\begin{aligned}\Lambda' &= \tau\circ\big(E\Phi_{11} + E\Phi_{12}\circ\tau\big) - \tau y_1\circ\big(E\Phi_{21} + E\Phi_{22}\circ\tau\big) + y_1\Lambda^\circ, \\ \Lambda'' &= \tau\circ\big(E\Phi_{21} + E\Phi_{22}\circ\tau\big) + y_1\tau\Lambda^\circ.\end{aligned}$$

Multiplying the first by y_2, adding $E\Phi_{11} + E\Phi_{12}\circ\tau$, and simplifying with (3.2.2), we find:

$$y_2\Lambda' + E\Phi_{11} + E\Phi_{12}\circ\tau = \Lambda.$$

Multiplying the second by y_2 and adding $E\Phi_{21} + E\Phi_{22}\circ\tau$, we find:

$$y_2\Lambda'' + E\Phi_{21} + E\Phi_{22}\circ\tau = \tau y_1\circ\big(E\Phi_{21} + E\Phi_{22}\circ\tau\big) + \tau y_1y_2\Lambda^\circ,$$

while

$$\begin{aligned}\tau\Lambda &= -\tau y_2\tau y_1\circ\big(E\Phi_{21} + E\Phi_{22}\circ\tau\big) + \tau y_1y_2\Lambda^\circ \\ &= \tau y_1\circ\big(E\Phi_{21} + E\Phi_{22}\circ\tau\big) + y_1y_2\tau\Lambda^\circ\end{aligned}$$

using (3.2.2). So the pair of equations (3.2.1) is satisfied. □

The proposition follows. □

We will need one more description of U:

Lemma 3.2.14. *The composition map $L_2 \otimes_{G_1^{\mathsf{op}}} G_2 \to U$ is an isomorphism.*

Proof. Consider the triangulated functor:

$$\mathscr{H}om_B(X_2, -) : K^b(B) \to K^b(G_1^{\mathsf{op}}).$$

By the same reasoning as in §2.2.3.2, this functor descends to the derived categories

$$\mathscr{H}om_B(X_2, -) : D^b(B) \to D^b(G_1^{\mathsf{op}}),$$

it is fully faithful when restricted to $\langle X_2 \rangle_\Delta$, and it is essentially surjective from $\langle X_2 \rangle_\Delta$ (because the image of X_2 is quasi-isomorphic to G_1^{op}). The inverse is given by $X_2 \otimes_{G_1^{\mathsf{op}}} -$. It follows from $R \in \langle X_2 \rangle_\Delta$ (Lemma 2.2.12) and

$$\begin{aligned} \operatorname{Hom}_{K^b(B)}(X_2, R) &\xrightarrow{\sim} \operatorname{Hom}_{K^b(B)}(X_2, E'X_2) \\ &\xrightarrow{q.i.} \mathscr{H}om_B(X_2, E'X_2) \\ &\xrightarrow{q.i.} \mathscr{H}om_B(X_2, R) \end{aligned}$$

that the evaluation map is an isomorphism:

$$X_2 \otimes_{G_1^{\mathsf{op}}} \operatorname{Hom}_{K^b(B)}(X_2, R) \xrightarrow{\sim} R.$$

This shows that the map in the lemma statement is an isomorphism:

$$\begin{aligned} \operatorname{Hom}_{K^b(B)}(R, X_2) \otimes_{G_1^{\mathsf{op}}} \operatorname{Hom}_{K^b(B)}(X_2, R) &\xrightarrow{\sim} \operatorname{Hom}_{K^b(B)}\big(R, X_2 \otimes_{G_1^{\mathsf{op}}} \operatorname{Hom}_{K^b(B)}(X_2, R)\big) \\ &\xrightarrow{\sim} \operatorname{Hom}_{K^b(B)}(R, R). \end{aligned}$$

□

We will need to know the $(A[y], A[y])$-bimodule structure of the components of $\tilde{E}$ and $\tilde{E}^2$ and $\tilde{F}$. This information is implicit in the calculations in Chapter 2 for the first three bimodules of the next proposition. The structure of the fourth bimodule is easy to compute using the map provided.

Proposition 3.2.15. *We have isomorphisms of $(A[y], A[y])$-bimodules:*

- $y_1 \dots y_n E^n[y] \xrightarrow{\sim} E^n[y]$ *given by composing with* $(y_1 \dots y_n)^{-1}$.

- $L_1 = G_1 \xrightarrow{\sim} A[y] \oplus FE[y]$ *given by* $(\theta, \varphi) \mapsto (\theta, \varphi_1)$, *where*

$$\varphi_1 = y_1^{-1}(\varphi - \theta)$$

  is interpreted in $FE[y]$. *Note that the summand* $FE[y]$ *is a left* G_1^{op}*-submodule of* G_1.

- $G_2 \xrightarrow{\sim} E[y] \oplus E[y] \oplus FE^2[y]$ *given by* $(e', e, \xi) \mapsto (e', e, \xi')$, *where*

$$\xi' = (y_1 y_2)^{-1}\big(\xi - {}_{-} \otimes e - y_2 \tau({}_{-} \otimes (e - y_1 e'))\big)$$

  is interpreted in $FE^2[y]$. *Note that the summand* $FE^2[y]$ *is not only a left* $A[y]$*-submodule of* G_2, *but moreover a left* G_1^{op}*-submodule of* G_2.

- $L_2 \xrightarrow{\sim} F[y] \oplus F[y] \oplus F^2E[y]$ *given by* $(f', f, \rho) \mapsto (f', f, \rho_1)$, *where*

$$\rho_1 = y_1^{-1}\big(\rho - Ef - Ef' \circ \tau\big)$$

  is interpreted in $F^2E[y]$. *Note that the summand* $F^2E[y]$ *is a left* G_1^{op}*-submodule of* L_2.

- $U \xrightarrow{\sim} FE[y]^{\oplus 4} \oplus F^2E^2[y]$ *given by*

$$(\Phi_{11}, \Phi_{21}, \Phi_{12}, \Phi_{22}, \Lambda) \mapsto (\Phi_{11}, \Phi_{21}, \Phi_{12}, \Phi_{22}, \Lambda^\circ),$$

  where

$$\Lambda = \tau y_1(E\Phi_{11} + E\Phi_{12} \circ \tau) - y_2 \tau y_1 (E\Phi_{21} + E\Phi_{22} \circ \tau) + y_1 y_2 \Lambda^\circ \tag{3.2.3}$$

  determines Λ°, *which is interpreted in* $F^2E^2[y]$. *Note that the summand* $F^2E^2[y]$ *is a left* G_1^{op}*-submodule of* U.

In what follows we will frequently use the bimodule descriptions on the right side of these isomorphisms for the components of $\tilde{E}$ and $\tilde{F}$. Sometimes, to avoid confusion, we will use the shorthand expressions 'submodule form' and 'bimodule form' to distinguish the two sides. The word 'submodule' suggests '$(A[y], A[y])$-sub-bimodule', and the word 'bimodule' suggests 'structure as $(A[y], A[y])$-bimodule'. The data of an element given in the two forms will differ only in the last component: in the 'submodule form' the last component gives the morphism restricted to the degree 0 part of the top row of the B-module complexes, and in the 'bimodule form' the last component gives just the remainder term 'φ_1', 'ξ'', 'ρ_1', 'χ''', or 'Λ°'.

3.3 Adjunction

Definition 3.3.1. Let $\tilde{F}$ denote the (C, C)-bimodule ${}^\vee\tilde{E}$, that is, $\mathrm{Hom}_C({}_C\tilde{E}, C)$.

We have seen that, under the hypotheses of the Main Theorem, ${}_C\tilde{E}$ is f.g. projective. It follows that the right adjoint functor $\mathrm{Hom}_C({}_C\tilde{E}, -)$ of $\tilde{E} \otimes_C -$ is canonically isomorphic to $\tilde{F} \otimes_C -$. We have already defined $\tilde{x}$ and $\tilde{\tau}$. We define $\tilde{\varepsilon} : \tilde{E}\tilde{F} \to C$ and $\tilde{\eta} : C \to \tilde{F}\tilde{E}$ using the duality, and then $\tilde{\sigma}$ and $\tilde{\rho}_\lambda$ using the formulas in §3.1.2 with $(A, E, F, x, \tau, \varepsilon, \eta)$ replaced by $(C, \tilde{E}, \tilde{F}, \tilde{x}, \tilde{\tau}, \tilde{\varepsilon}, \tilde{\eta})$. Sometimes we view $\tilde{F}\tilde{E}$ through the canonical isomorphism $\mathrm{Hom}(\tilde{E}, C) \otimes_C \tilde{E} \xrightarrow{\sim} \mathrm{Hom}(\tilde{E}, \tilde{E})$.

Now we construct an isomorphism of (C, C)-bimodules:

$$\tilde{F} \xrightarrow{\sim} \mathrm{Hom}_{K^b(B)}(X_2 \oplus R, X_1 \oplus X_2),$$

as follows:

$$\begin{aligned}
\tilde{F} = {} & \operatorname{Hom}_C({}_C\tilde{E}, C) \\
\xrightarrow{\sim} {} & \operatorname{Hom}_{D^b(C)}(\tilde{E}, C) \\
\xrightarrow{\sim} {} & \operatorname{Hom}_{D^b(C)}(\mathscr{E}, \mathscr{C}) \\
= {} & \operatorname{Hom}_{D^b(C)}(\mathscr{H}om_B(X, E'X), \mathscr{H}om_B(X, X)) \\
\xrightarrow{\sim} {} & \operatorname{Hom}_{D^b(B)}(E'X, X) \\
\xrightarrow{\sim} {} & \operatorname{Hom}_{D^b(B)}(X_2 \oplus R, X_1 \oplus X_2) \\
\xrightarrow{\sim} {} & \operatorname{Hom}_{K^b(B)}(X_2 \oplus R, X_1 \oplus X_2).
\end{aligned}$$

(The third arrow comes from the Rickard equivalence.)

With this description of $\tilde{F}$, and using the direct sum decompositions, we can express $\tilde{F}$ as a 2×2 matrix of $(A[y], A[y])$-bimodules:

$$\tilde{F} \xrightarrow{\sim} \begin{pmatrix} F[y] & L_1 \\ F^2[y] & L_2 \end{pmatrix}.$$

We have $C = \operatorname{End}_{K^b(B)}(X_1 \oplus X_2)$, and the right action of C on $\tilde{F}$ is given by post-composition. The left action of C is by pre-composition, but one must first apply functoriality of E' and use the quasi-isomorphism from Lemma 2.2.13, which we write $\gamma : R \xrightarrow{q.i.} E'X_2$.

- A generator $\phi \in Z^0\mathscr{H}om_B(X_1, X_1)^{\mathsf{op}} \cong A[y] \subset C$ determines $E'\phi \in \operatorname{Hom}_{K^b(B)}(X_2, X_2)$ that acts on $\tilde{F}$ (on the top row) by pre-composition. An element $\phi = \theta \in A[y]$ acts in the obvious way on the left on $F[y]$ and L_1.

- A generator $\phi \in Z^0\mathscr{H}om_B(X_2, X_1) \cong F[y] \subset C$ determines

$$E'\phi \in \operatorname{Hom}_{D^b(B)}(E'X_2, E'X_1) \xrightarrow[\sim]{-\circ\gamma} \operatorname{Hom}_{K^b(B)}(R, X_2).$$

  So ϕ acts on $\tilde{F}$ (on the top row) by pre-composition with $E'\phi \circ \gamma : R \to X_2$. Recall that we have the model $\bar{L}_2$ for $\mathrm{Hom}_{K^b(B)}(R, X_2)$. An element $\phi = f \in F[y]$ acts by pre-composition with the morphism determined by $E'\phi \circ \gamma = (0, f, 0) \in \bar{L}_2$.

- A generator $\phi \in Z^0\mathscr{H}om_B(X_1, X_2) \cong y_1E[y] \subset C$ determines

$$E'\phi \in \mathrm{Hom}_{K^b(B)}(E'X_1, E'X_2) \xleftarrow[\sim]{\gamma\circ -} \mathrm{Hom}_{K^b(B)}(X_2, R).$$

  Recall that we have the models $\bar{G}_2$ for $\mathrm{Hom}_{K^b(B)}(X_2, E'X_2)$ and $\bar{G}_2'$ for $\mathrm{Hom}_{K^b(B)}(X_2, R)$, and the isomorphism $\bar{G}_2 \xrightarrow{\sim} \bar{G}_2'$ given by $(e_1, e_2, \xi') \mapsto (y_1^{-1}(e_1 - e_2), e_1, \xi')$ (in bimodule forms). An element $\phi = y_1 e \in y_1E[y]$ determines $E'\phi = (y_1e, 0, 0) \in \bar{G}_2$, so this acts on $\tilde{F}$ by pre-composition with the morphism determined by $(e, y_1e, 0) \in \bar{G}_2'$.

- A generator $\phi \in Z^0\mathscr{H}om_B(X_2, X_2)^{\mathsf{op}} \cong G_1^{\mathsf{op}} \subset C$ determines $\phi_R \in \mathrm{Hom}_{K^b(B)}(R, R)$ from the right action of G_1^{op} on R. In terms of the model $\bar{U}$, we have $\phi_R = (\varphi, \varphi_1, 0, \theta, E\varphi)$ (in submodule form), determined by $\phi = (\theta, \varphi) \in G_1^{\mathsf{op}}$. This acts on $\tilde{F}$ (on the bottom row) by pre-composition.

3.4 Isomorphisms $\tilde{\rho}_\lambda$

3.4.1 Some tensor products of (C, C)-bimodules

In this section we compute three tensor products of bimodules over C, namely $\tilde{E}\tilde{E}$, $\tilde{F}\tilde{E}$, and $\tilde{E}\tilde{F}$, and describe the products in each case as matrices of $(A[y], A[y])$-bimodules. These calculations are used in the remaining sections to verify that $\tilde{\rho}_\lambda$ are isomorphisms. Note that the product $\tilde{E}\tilde{E} = \tilde{E}^2$ is already given (Prop. 2.2.48) a description as a matrix of $(A[y], A[y])$-bimodules using the identification with $\mathrm{Hom}_{K^b(B)}(X, E'^2X)$, but for a computation of $\tilde{\sigma}$ it is also necessary to realize the matrix description of $\tilde{E}^2$ as it arises from the matrix description of $\tilde{E}$ as a (C, C)-bimodule.

These tensor products are computed according to the general formulation described in

§2.1.4. First we take the tensor product over the subalgebra $\Delta := \left(\begin{smallmatrix} A[y] & 0 \\ 0 & G_1^{\mathsf{op}} \end{smallmatrix}\right) \subset C$. This product is given on components by matrix multiplication and tensor product over $A[y]$ or G_1^{op}. After this we must take the quotient by the image of the map "$I_B + I_C$" (cf. §2.1.4) that is produced using the action of the off-diagonal generators in C. This quotient may be taken separately on each coefficient of the product over Δ.

The simplest technique for computing a quotient by the image of (say) I_B is to identify one of its projections as an isomorphism. (In §2.1.4, there is a projection of I_B to $M_1 \otimes_A N_1$ and another projection to $M_2 \otimes_D N_2$.) In this situation the quotient by $\mathrm{Im}(I_B)$ reduces to the summand of the other projection, because every element of the first summand (in the quotient) has a unique representative in the second. The basic technique for computing the quotient by $\mathrm{Im}(I_B) + \mathrm{Im}(I_C)$ is to show that the projections of I_C have a compatibility with those of I_B. Many of the components computed below are found in this way, but a few of them require more complicated reasoning.

Let us write, in general, I'_β for the projection of I_B to the first summand, and $-I''_\beta$ for the projection to the second. Similarly write I'_δ and $-I''_\delta$ for the projections of I_C. Here 'first' and 'second' summand and 'I_B' and 'I_C' are understood as in §2.1.4. In a tensor product of (C, C)-bimodules, each of the four coefficients will have its own set of maps I'_β, I''_β, I'_δ, I''_δ.

3.4.1.1 $\tilde{E}\tilde{E}$

For the product $\tilde{E}\tilde{E}$, we already know the structure of the coefficients of the matrix presentation. We will need to compute the action of $\tilde{\tau}$ on elements of $\tilde{E}\tilde{E}$ in order to compute $\tilde{\sigma}$, and for this it will be enough to compute the map from the tensor product over Δ to the product over C, i.e. to the quotient by $\mathrm{Im}(I'_\beta - I''_\beta) + \mathrm{Im}(I'_\delta - I''_\delta)$. Write Γ for this map. Let the subscript 'G' between concatenated modules indicate the tensor product over G_1^{op}. (An empty subscript indicates the product over $A[y]$.) So we have:

$$\begin{aligned}\tilde{E}\otimes_\Delta \tilde{E} &\cong \begin{pmatrix} E[y] & E^2[y] \\ G_1 & G_2 \end{pmatrix} \otimes_\Delta \begin{pmatrix} E[y] & E^2[y] \\ G_1 & G_2 \end{pmatrix} \\ &\cong \begin{pmatrix} EE[y]\oplus E^2[y]_G G_1 & EE^2[y]\oplus E^2[y]_G G_2 \\ G_1E[y]\oplus (G_2)_G G_1 & G_1E^2[y]\oplus (G_2)_G G_2 \end{pmatrix} \\ &\xrightarrow{\Gamma} \begin{pmatrix} E^2[y] & E^3[y] \\ G_2 & G_3 \end{pmatrix} \cong [\tilde{E}^2],\end{aligned} \tag{3.4.1}$$

and we wish to understand the map Γ on each component. In subsequent sections we must determine the structure of the quotient and then write Γ, but here, since we know the image to be $\mathrm{Hom}_{K^b(B)}\big(X, E'^2(X)\big)$, we simply compute Γ by composing elements of $\mathrm{Hom}_{K^b(B)}(X, E'X)$, using functoriality of E' and applying Lemmas 2.2.44 and 2.2.45.

- For Γ_{11}, we have:
  - $\Gamma_{11}\mid_{EE[y]}$ is given by $\mathrm{Id}_{EE[y]}$,
  - $\Gamma_{11}\mid_{E^2[y]_G G_1}$ is given as the inverse of $E^2[y] \xrightarrow{\sim} E^2[y]_G G_1$, $ee \mapsto ee\otimes 1_{G_1}$.
- For Γ_{21}, we have:
  - $\Gamma_{21}\mid_{G_1E[y]}$ is given (in bimodule forms) by

$$(\theta, \varphi_1)\otimes e \mapsto (\theta e, \theta y_1 e, \varphi_1(-)\otimes e) \in G_2,$$

  - $\Gamma_{21}\mid_{(G_2)_G G_1}$ is given as the inverse of $G_2 \xrightarrow{\sim} (G_2)_G G_1$, $g_2 \mapsto g_2\otimes 1_{G_1}$.
- For Γ_{12}, we have:
  - $\Gamma_{12}\mid_{EE^2[y]}$ is given by $\mathrm{Id}_{E^3[y]}$,

– $\Gamma_{12}\mid_{E^2[y]_G G_2}$ is given (in bimodule forms) by

$$ee \otimes (e', e, \xi') \mapsto (y_1y_2y_3)^{-1}(E\xi)(y_1y_2ee) \in E^3[y].$$

- For Γ_{22}, we have:

– $\Gamma_{22}\mid_{G_1E^2[y]}$ is given by

$$(\theta, \varphi_1) \otimes ee \mapsto (\theta y_1y_2ee, 0, 0, \varphi_1(-) \otimes ee)$$

(c.f. Diagram $D_{1|2}(2,1,1)$ in §2.3.2.3),

– $\Gamma_{22}\mid_{(G_2)_G G_2}$ is given (in submodule forms) by

$$(e_1, e_2, \xi) \otimes (\bar{e}_1, \bar{e}_2, \bar{\xi}) \mapsto (\bar{\xi}(e_1), e_2 \otimes \bar{e}_1, e_2 \otimes \bar{e}_2, E\bar{\xi} \circ \xi)$$

(c.f. Diagram $D_{1|2}(2,2,1)$). We need to compute this map on the bimodule forms. First compute $E\bar{\xi} \circ \xi$:

$$\begin{aligned}
E\bar{\xi} \circ \xi &= \big(_\,_ \otimes \bar{e} + y_2\tau(_\,_ \otimes (\bar{e} - y_1\bar{e}')) + y_1y_2E\bar{\xi}'\big) \\
&\quad \circ \big(_ \otimes e + y_2\tau(_ \otimes (e - y_1e')) + y_1y_2\xi'\big) \\
&= _ \otimes \big(e\bar{e} + y_2\tau(e\bar{e} - y_1e\bar{e}') + y_1y_2\bar{\xi}'(e)\big) \\
&\quad + y_3 \circ \tau E\big(_ \otimes (e\bar{e} - y_2e'\bar{e})\big) \\
&\quad + y_2y_3 \circ E\tau \circ \tau E\big(_ \otimes (e - y_1e')(\bar{e} - y_1\bar{e}')\big) \\
&\quad + y_1y_2y_3\big(E\bar{\xi}' \circ \tau(_ \otimes (e - y_1e')) + E\tau(\xi' \otimes (\bar{e} - y_1\bar{e}')) + \xi' \otimes \bar{e}'\big).
\end{aligned}$$

Using Prop. 2.2.26, we can read off the data of the bimodules formulation we want:

$$\begin{aligned}
&(e', e, \xi') \otimes (\bar{e}', \bar{e}, \bar{\xi}') \mapsto \\
&\big(e\bar{e} + y_2\tau(e\bar{e} - y_1e\bar{e}') + y_1y_2\bar{\xi}'(e), e\bar{e} - y_2e'\bar{e}, (e - y_1e')(\bar{e} - y_1\bar{e}'), \\
&\qquad E\bar{\xi}' \circ \tau(_ \otimes (e - y_1e')) + E\tau(\xi' \otimes (\bar{e} - y_1\bar{e}')) + \xi' \otimes \bar{e}'\big).
\end{aligned}$$

3.4.1.2 $\tilde{F}\tilde{E}$

For the product $\tilde{F}\tilde{E}$, we can find the $(A[y], A[y])$-bimodule structure of the components of its matrix presentation using the same technique as for $\tilde{F}$ and $\tilde{E}^2$. We have:

$$\begin{aligned}
\tilde{F}\tilde{E} = \;& \mathrm{Hom}_C({}_C\tilde{E}, C) \otimes_C \tilde{E} \\
& \xrightarrow{\sim} \mathrm{Hom}_C({}_C\tilde{E}, \tilde{E}) \xrightarrow{\sim} \mathrm{Hom}_{D^b(C)}(\tilde{E}, \tilde{E}) \xrightarrow{\sim} \mathrm{Hom}_{D^b(C)}(\mathscr{E}, \mathscr{E}) \\
& \xrightarrow{\sim} \mathrm{Hom}_{D^b(C)} \left(\mathscr{H}om_B(X, E'X), \mathscr{H}om_B(X, E'X)\right) \\
& = \mathrm{Hom}_{D^b(B)}(E'X, E'X) \\
& \xrightarrow{\sim} \mathrm{Hom}_{K^b(B)}(E'X_1 \oplus R, E'X_1 \oplus R).
\end{aligned}$$

So the matrix presentation is:

$$[\tilde{F}\tilde{E}] = \begin{pmatrix} G_1 & G_2 \\ L_2 & U \end{pmatrix}.$$

As we did for $\tilde{E}^2$, we study the map Γ from the components of the product over Δ to those of the product over C:

$$\begin{aligned}
\tilde{F} \otimes_\Delta \tilde{E} &\cong \begin{pmatrix} F[y] & L_1 \\ F^2[y] & L_2 \end{pmatrix} \otimes_\Delta \begin{pmatrix} E[y] & E^2[y] \\ G_1 & G_2 \end{pmatrix} \\
&\cong \begin{pmatrix} FE[y] \oplus (L_1)_G G_1 & FE^2[y] \oplus (L_1)_G G_2 \\ F^2E[y] \oplus (L_2)_G G_1 & F^2E^2[y] \oplus (L_2)_G G_2 \end{pmatrix} \\
&\cong \begin{pmatrix} FE[y] \oplus G_1 & FE^2[y] \oplus G_2 \\ F^2E[y] \oplus L_2 & F^2E^2[y] \oplus (L_2)_G G_2 \end{pmatrix} \xrightarrow{\Gamma} \begin{pmatrix} G_1 & G_2 \\ L_2 & U \end{pmatrix}.
\end{aligned} \tag{3.4.2}$$

The bulleted claims below are justified in the paragraphs following them.

- We have $\Gamma_{11} : FE[y] \oplus G_1 \to G_1$ given by $(\iota, \mathrm{Id}_{G_1})$.

  Here the map $\iota : FE[y] \hookrightarrow L_1 = G_1$ is the inclusion of the second summand as written in

Prop. 3.2.15.

- $I'_\beta : FE[y]_G G_1 \xrightarrow{\sim} FE[y]$ given as the inverse of the isomorphism $\big(fe \mapsto fe \otimes 1_{G_1}\big)$,
- $I''_\beta : FE[y]_G G_1 \xrightarrow{\iota \otimes G_1} (L_1)_G G_1 \cong G_1$,
- $I'_\delta : (G_1)_G FE[y] \xrightarrow{\sim} FE[y]$ given as the inverse of the isomorphism $\big(fe \mapsto 1_{G_1} \otimes fe\big)$,
- $I''_\delta : (G_1)_G FE[y] \xrightarrow{G_1 \otimes \iota} (G_1)_G L_1 \cong G_1$.

Using either $I''_\beta \circ I'^{-1}_\beta$, or $I''_\delta \circ I'^{-1}_\delta$, one associates a unique representative in $(L_1)_G G_1 \cong G_1$ to each element of $FE[y]$. We see that $I''_\beta \circ I'^{-1}_\beta = I''_\delta \circ I'^{-1}_\delta$, so the two associate the same representatives. It follows that the quotient projection Γ_{11} is given by the proposed formula.

- We have $\Gamma_{21} : F^2E[y] \oplus L_2 \to L_2$ given by $(\iota', \mathrm{Id}_{L_2})$.

  Here the map $\iota' : F^2E[y] \hookrightarrow L_2$ is the inclusion of the third summand as written in Prop. 3.2.15.

  - $I'_\beta : F^2E[y]_G G_1 \xrightarrow{\sim} F^2E[y]$ given as the inverse of $\big(ffe \mapsto ffe \otimes 1_{G_1}\big)$,
  - $I''_\beta : F^2E[y]_G G_1 \xrightarrow{\iota' \otimes G_1} (L_2)_G G_1 \cong L_2$,
  - $I'_\delta : (L_2)_G FE[y] \to F^2E[y]$ given by

$$(f', f, \rho') \otimes \bar{f}\bar{e} \mapsto (\bar{f} \circ \rho) \otimes \bar{e} = \big(\bar{f} \circ (Ef + Ef' \circ \tau + y_1\rho')\big) \otimes \bar{e},$$

  - $I''_\delta : (L_2)_G FE[y] \xrightarrow{L_2 \otimes \iota} (L_2)_G G_1 \cong L_2$.

  Consider the first two maps. We have that $I''_\beta \circ I'^{-1}_\beta = \iota'$ as maps $F^2E[y] \to L_2$. Consider the last two maps. One may check that $\iota' \circ I'_\delta = I''_\delta$. It follows that $\mathrm{Im}(I'_\delta - I''_\delta) \subset \mathrm{Im}(I'_\beta - I''_\beta)$, so in the quotient every element of $F^2E[y]$ is associated to a unique element of L_2, given by applying the map ι'.

- We have $\Gamma_{12} : FE^2[y] \oplus G_2 \to G_2$ given by $(\iota'', \mathrm{Id}_{G_2})$.

  Here the map $\iota'' : FE^2[y] \hookrightarrow G_2$ is the inclusion of the third summand as written in Prop. 3.2.15.

  - $I'_\beta : FE[y]_G G_2 \to FE^2[y]$ given by

  $$\bar{f}\bar{e} \otimes (e', e, \xi') \mapsto \bar{f} \otimes (y_1 y_2)^{-1} \xi(y_1 \bar{e}) = \bar{f} \otimes \big(\tau(\bar{e} \otimes e) - y_2 \tau(\bar{e} \otimes e') + \xi'(y_1 \bar{e})\big),$$

  - $I''_\beta : FE[y]_G G_2 \xrightarrow{\iota \otimes G_2} (L_1)_G G_2 \cong G_2$,
  - $I'_\delta : (G_1)_G FE^2[y] \xrightarrow{\sim} FE^2[y]$ given as the inverse of $\big(fee \mapsto 1_{G_1} \otimes fee\big)$,
  - $I''_\delta : (G_1)_G FE^2[y] \xrightarrow{G_1 \otimes \iota''} (L_1)_G G_2 \cong G_2$.

  Consider the last two maps. We have that $I''_\delta \circ I'^{-1}_\delta = \iota''$ as maps $FE^2[y] \to G_2$. Now consider the first two maps. Observe that $I''_\beta = \iota'' \circ I'_\beta$. It follows that $\mathrm{Im}(I'_\beta - I''_\beta) \subset \mathrm{Im}(I'_\delta - I''_\delta)$, so every element of $FE^2[y]$ is associated in the quotient to a unique element of G_2 by applying the map ι''.

- We have $\Gamma_{22} : F^2E^2[y] \oplus (L_2)_G G_2 \to U$ given by $(\iota''', \mathrm{Id}_U)$.

  Here the map $\iota''' : F^2E^2[y] \to U$ is the inclusion of the fifth summand as written in Prop. 3.2.15.

  - $I'_\beta : F^2E[y]_G G_2 \to F^2E^2[y]$ given by $\overline{ff}\bar{e} \otimes (e', e, \xi') \mapsto \overline{ff} \otimes (y_1 y_2)^{-1} \xi(y_1 \bar{e})$,
  - $I''_\beta : F^2E[y]_G G_2 \xrightarrow{\iota' \otimes G_2} (L_2)_G G_2 \cong U$,
  - $I'_\delta : (L_2)_G FE^2[y] \to F^2E^2[y]$ given by $(f', f, \rho') \otimes \bar{f}\overline{ee} \mapsto (\bar{f} \circ \rho) \otimes \overline{ee}$,
  - $I''_\delta : (L_2)_G FE^2[y] \xrightarrow{L_2 \otimes \iota''} (L_2)_G G_2 \cong U$.

  Consider the first two maps. Observe that

  $$I'_\beta\big(\overline{ff}\bar{e} \otimes (e, y_1 e, \xi' = 0)\big) = \overline{ff} \otimes (\bar{e} \otimes e) \in F^2E^2[y].$$

It follows that I'_β is surjective. Now we show that $\iota''' \circ I'_\beta = I''_\beta$ and that $\iota''' \circ I'_\delta = I''_\delta$ using the bimodule forms:

$$\begin{aligned}
\iota'''\big(\overline{ff} \otimes (y_1y_2)^{-1}\xi(y_1\bar{e})\big) &= \big(0,0,0,0,\Lambda^\circ = \overline{ff} \otimes (y_1y_2)^{-1}\xi(y_1\bar{e})\big), \\
I''_\beta\big(\overline{ff}\bar{e} \otimes (e', e, \xi')\big) &= (0,0,\overline{ff}\bar{e}) \otimes_{G_1^{\mathsf{op}}} (e', e, \xi') \\
&\mapsto \big(0,0,0,0,\overline{ff} \otimes (y_1y_2)^{-1}\xi(y_1\bar{e})\big) \in U,
\end{aligned}$$

and

$$\begin{aligned}
\iota'''\big((\bar{f} \circ \rho) \otimes \overline{ee}\big) &= \big(0,0,0,0,(\bar{f} \circ \rho) \otimes \overline{ee}\big), \\
I''_\delta\big((f', f, \rho') \otimes \bar{f}\overline{ee}\big) &= (f', f, \rho') \otimes_{G_1^{\mathsf{op}}} (0,0,\bar{f}\overline{ee}) \\
&\mapsto \big(0,0,0,0,(\bar{f} \circ \rho) \otimes \overline{ee}\big).
\end{aligned}$$

It follows that every element of $F^2E^2[y]$ is associated in the quotient to a unique representative in U by applying ι'''.

Remark 3.4.1. The map ι''' describes the inclusion of the morphisms of $\mathrm{Hom}_{K^b(B)}(R,R)$ that factor through X_1. The maps I'_β and I'_δ are in fact isomorphisms, as can be seen using isomorphisms:

$$\begin{aligned}
&\mathrm{Hom}_{K^b(B)}(X_1,X_2) \otimes_{G_1^{\mathsf{op}}} \mathrm{Hom}_{K^b(B)}(X_2,R) \xrightarrow{\sim} \mathrm{Hom}_{K^b(B)}(X_1,R), \\
&\mathrm{Hom}_{K^b(B)}(R,X_2) \otimes_{G_1^{\mathsf{op}}} \mathrm{Hom}_{K^b(B)}(X_2,X_1) \xrightarrow{\sim} \mathrm{Hom}_{K^b(B)}(R,X_1),
\end{aligned}$$

which are produced by reasoning as in Lemma 3.2.14 using that R is a finite direct sum of summands of X_2.

3.4.1.3 $\tilde{E}\tilde{F}$

We do not have a matrix presentation of the components of the product $\tilde{E}\tilde{F}$ from the Rickard equivalence. Instead, in this section, we proceed by studying the quotient directly, by components, determining the quotient projection Γ from the tensor product over Δ to the tensor product over C, as well as the structure of the quotient itself.

As before, in each bulleted section we propose a component of Γ. Here the arguments following a bulleted line also must justify the structure of the codomain of the Γ component written in that bulleted line. The domains are known, and in each case the annihilated submodule $\mathrm{Im}(I'_\beta - I''_\beta) + \mathrm{Im}(I'_\delta - I''_\delta)$ is defined already. Our method is to write down a map called Γ_{ij} from the appropriate domain, show that it is surjective, and show that its kernel is $\mathrm{Im}(I'_\beta - I''_\beta) + \mathrm{Im}(I'_\delta - I''_\delta)$. The codomain of Γ can be summarized in a matrix:

$$\begin{aligned}\tilde{E} \otimes_\Delta \tilde{F} &\cong \begin{pmatrix} E[y] & E^2[y] \\ G_1 & G_2 \end{pmatrix} \otimes_\Delta \begin{pmatrix} F[y] & L_1 \\ F^2[y] & L_2 \end{pmatrix} \\ &\cong \begin{pmatrix} EF[y] \oplus E^2[y]_G F^2[y] & E[y]G_1 \oplus E^2[y]_G L_2 \\ G_1 F[y] \oplus (G_2)_G F^2[y] & G_1 G_1 \oplus (G_2)_G L_2 \end{pmatrix} \\ &\xrightarrow{\Gamma} \begin{pmatrix} EF[y] & E[y]G_1 \\ G_1 F[y] & G_1 G_1 \oplus EF[y] \end{pmatrix}.\end{aligned} \tag{3.4.3}$$

- We have $\Gamma_{11} : EF[y] \oplus E^2[y]_G F^2[y] \to EF[y]$ given by $(\mathrm{Id}_{EF[y]}, \omega)$.

  Define a map $\omega : E^2[y] \otimes_{A[y]} F^2[y] \to EF[y]$ by:

  $$e_1 e_2 \otimes f_2 f_1 \mapsto e_1.f_2(y_1 e_2) \otimes f_1 = e_1 \otimes f_2(y_1 e_2).f_1.$$

  Let $\varphi_1 \in FE[y]$ be given in the second summand of (the bimodule form) $G_1^{\mathsf{op}} \cong A[y] \oplus FE[y]$. Observe that $\big(e_1 \otimes \varphi_1(y_1 e_2)\big) \otimes f_2 f_1$ and $e_1 e_2 \otimes \big((f_2 \circ y_1 \varphi_1) \otimes f_1\big)$ are both sent by ω to $e_1.(f_2 \circ y_1 \varphi_1)(y_1 e_2) \otimes f_1$. This means ω is middle-linear over generators in both summands of G_1^{op}, so it descends to a map, also called ω, from the tensor product $E^2[y]_G F^2[y]$ taken over G_1^{op}.

– $I'_\beta : EE[y]_G F^2[y] \to EF[y]$ given (using bimodule forms) by

$$\begin{aligned} e_1 \otimes e_2 \otimes f_2 f_1 \mapsto\ & e_1 \otimes \big(f_1 \circ Ef_2 \circ (e_2, y_1 e_2, 0)\big) \\ &= e_1 \otimes \big(f_1 \circ Ef_2 \circ ({}_- \otimes y_1 e_2)\big) \\ &= e_1 \otimes f_1({}_-.f_2(y_1 e_2)) \\ &= e_1 \otimes f_2(y_1 e_2).f_1 \end{aligned}$$

(observe the notation $f_1 \circ Ef_2 : E'X_2 \to X_1$ in the first line),

– $I''_\beta : EE[y]_G F^2[y] \xrightarrow{\mathrm{Id}} E^2[y]_G F^2[y]$,

– $I'_\delta : E^2[y]_G FF[y] \to EF[y]$ given (using bimodule forms) by

$$\begin{aligned} e_1 e_2 \otimes f_2 \otimes f_1 \mapsto\ & \big((0, f_2, 0) \circ e_1 e_2\big) \otimes f_1 \\ &= y_1^{-1}(Ef_2)(y_1 y_2(e_1 e_2)) \otimes f_1 \\ &= e_1.f_2(y_1 e_2) \otimes f_1 \end{aligned}$$

(observe the notation $e_1 e_2 : X_1 \to E'X_2$ in the first line),

– $I''_\delta : E^2[y]_G FF[y] \xrightarrow{\mathrm{Id}} E^2[y]_G F^2[y]$.

We see that $I'_\beta = \omega$ and $I'_\delta = \omega$ after identifying $EE[y] \cong E^2[y]$ and $FF[y] \cong F^2[y]$. It follows that the kernel of Γ_{11} is the image of $I'_\beta - I''_\beta$, which is also the image of $I'_\delta - I''_\delta$, and thus $\ker(\Gamma_{11}) = \mathrm{Im}(I'_\beta - I''_\beta) + \mathrm{Im}(I'_\delta - I''_\delta)$ as desired.

Remark 3.4.2. The map ω corresponds on the models to the map given by composition:

$$\mathrm{Hom}_{K^b(B)}(X_2, R) \otimes_{G_1^{\mathrm{op}}} \mathrm{Hom}_{K^b(B)}(R, X_2) \to \mathrm{Hom}_{K^b(B)}(X_2, X_2).$$

- We have $\Gamma_{21} : G_1 F[y] \oplus (G_2)_G F^2[y] \to G_1 F[y]$ given by $(\mathrm{Id}_{G_1 F[y]}, \omega')$.

Let $\omega' : (G_2)_G F^2[y] \to G_1 F[y]$ be defined (using bimodule forms) by

$$\begin{aligned}(e', e, \xi') \otimes f_2 f_1 &\mapsto \big((0, f_2, 0) \circ (e', e, \xi')\big) \otimes f_1 \\ &= \Big(f_2(e), y_1^{-1} E f_2 \circ \big(y_2 \tau({}_{-} \otimes (e - y_1 e')) + y_1 y_2 \xi'\big)\Big) \otimes f_1 \\ &= \big(f_2(e), E f_2 \circ \tau({}_{-} \otimes (e - y_1 e')) + E(f_2 \circ y_1) \circ \xi'\big) \otimes f_1.\end{aligned}$$

– $I'_\beta : G_1 E[y]_G F^2[y] \to G_1 F[y]$ given (using bimodule form) by

$$\begin{aligned} ge \otimes f_2 f_1 &\mapsto g \otimes \big((f_1 \circ E f_2) \circ (e, y_1 e, 0)\big) \\ &= g \otimes f_2(y_1 e).f_1, \end{aligned}$$

– $I''_\beta : G_1 E[y]_G F^2[y] \to (G_2)_G F^2[y]$ given (using bimodule forms) by

$$\begin{aligned} (\theta, \varphi_1) \otimes e \otimes f_2 f_1 &\mapsto \big((e, y_1 e, 0) \circ (\theta, \varphi_1)\big) \otimes f_2 f_1 \\ &= (\theta e, \theta y_1 e, \varphi_1({}_{-}) \otimes e) \otimes f_2 f_1, \end{aligned}$$

– $I'_\delta : (G_2)_G F F[y] \to G_1 F[y]$ given by the map ω' (after identifying $FF[y]$ with $F^2[y]$),

– $I''_\delta : (G_2)_G F F[y] \xrightarrow{\mathrm{Id}} (G_2)_G F^2[y]$.

We show that $\omega' \circ I''_\beta = I'_\beta$:

$$\begin{aligned} \omega'\big((\theta e, \theta y_1 e, \varphi_1 \otimes e) \otimes f_2 f_1\big) &= \big(f_2(\theta y_1 e), E(f_2 \circ y_1) \circ (\varphi_1 \otimes e)\big) \otimes f_1 \\ &= \big(\theta f_2(y_1 e), \varphi_1.f_2(y_1 e)\big) \otimes f_1 \\ &= (\theta, \varphi_1).f_2(y_1 e) \otimes f_1 \\ &= I'_\beta\big((\theta, \varphi_1) \otimes e \otimes f_2 f_1\big). \end{aligned}$$

Thus $I'_\beta - I''_\beta = (\omega' - \mathrm{Id}) I''_\beta$, and therefore $\mathrm{Im}(I'_\beta - I''_\beta) \subset \mathrm{Im}(I'_\delta - I''_\delta)$. It follows that $\ker(\Gamma_{21}) = \mathrm{Im}(I'_\beta - I''_\beta) + \mathrm{Im}(I'_\delta - I''_\delta)$, as desired.

- We have $\Gamma_{12} : E[y] G_1 \oplus E^2[y]_G L_2 \to E[y] G_1$ given by $(\mathrm{Id}_{E[y]G_1}, \omega'')$.

Let $\omega'' : E^2[y]_G L_2 \to E[y]G_1$ be defined (using bimodule forms) by

$$\begin{aligned} e_1e_2 \otimes (f', f, \rho') &\mapsto e_1 \otimes \big((f', f, \rho') \circ (e_2, y_1e_2, 0)\big) \\ &= e_1 \otimes \big(f(y_1e_2) + f'(e_2), Ef' \circ \tau(\_ \otimes e_2) + \rho'(\_ \otimes y_1e_2)\big). \end{aligned}$$

– $I'_\beta : EE[y]_G L_2 \to E[y]G_1$ given by the map ω'' (after identifying $EE[y]$ with $E^2[y]$),

– $I''_\beta : EE[y]_G L_2 \xrightarrow{\text{Id}} E^2[y]_G L_2$,

– $I'_\delta : E^2[y]_G F[y]G_1 \to E[y]G_1$ given (borrowing from I'_δ of Γ_{11}) by

$$e_1e_2 \otimes f_2 \otimes g \mapsto e_1 \otimes f_2(y_1e_2).g,$$

– $I''_\delta : E^2[y]_G F[y]G_1 \to E^2[y]_G L_2$ given (using bimodule forms) by

$$\begin{aligned} e_1e_2 \otimes f \otimes (\theta, \varphi_1) &\mapsto e_1e_2 \otimes \big((\theta, \varphi_1) \circ (0, f, 0)\big) \\ &= e_1e_2 \otimes (0, f.\theta, f \otimes \varphi_1). \end{aligned}$$

We show that $\omega'' \circ I''_\delta = I'_\delta$:

$$\begin{aligned} \omega''\big(e_1e_2 \otimes (0, f.\theta, \varphi_1 \circ Ef)\big) &= e_1 \otimes \big(f(y_1e_2).\theta, (\varphi_1 \circ Ef)(\_ \otimes y_1e_2)\big) \\ &= e_1 \otimes \big(f(y_1e_2).\theta, \varphi_1(\_.f(y_1e_2))\big) \\ &= e_1 \otimes \big(f(y_1e_2).\theta, f(y_1e_2).\varphi_1\big) \\ &= e_1 \otimes f(y_1e_2).(\theta, \varphi_1) \\ &= I'_\delta\big(e_1e_2 \otimes f \otimes (\theta, \varphi_1)\big). \end{aligned}$$

Thus $I'_\delta - I''_\delta = (\omega'' - \text{Id})I''_\delta$, and therefore $\text{Im}(I'_\delta - I''_\delta) \subset \text{Im}(I'_\beta - I''_\beta)$. It follows that $\ker(\Gamma_{12}) = \text{Im}(I'_\beta - I''_\beta) + \text{Im}(I'_\delta - I''_\delta)$, as desired.

- We have $\Gamma_{22} : G_1G_1 \oplus (G_2)_G L_2 \to G_1G_1 \oplus EF[y]$ given by $\begin{pmatrix} \text{Id}_{G_1G_1} & \omega''' \\ 0 & \kappa \end{pmatrix}$.

Below we describe the maps I'_β, I''_β, I'_δ, I''_δ, and define a map $\omega''' : (G_2)_G L_2 \to G_1 G_1$, and we show that $\omega''' \circ I''_\beta = I'_\beta$ and $\omega''' \circ I''_\delta = I'_\delta$. Then we describe a decomposition of $(G_2)_G L_2$ into $(A[y], A[y])$-sub-bimodules $(G_2)_G L_2 \cong H \oplus EF[y]$ where $H = \mathrm{Im}(I''_\beta) + \mathrm{Im}(I''_\delta)$. The projection onto $EF[y]$ is called κ. (This copy of $EF[y]$ lies in the kernel of ω'''.) From all this it follows that $\ker(\Gamma_{22}) = \mathrm{Im}(I'_\beta - I''_\beta) + \mathrm{Im}(I'_\delta - I''_\delta)$ and Γ_{22} describes the projection to the quotient.

– $I'_\beta : G_1 E[y]_G L_2 \to G_1 G_1$ given (borrowing from I'_β of Γ_{12}) by

$$\begin{aligned} &g \otimes e \otimes (f', f, \rho') \mapsto \\ &\quad g \otimes \big(f'(e) + f(y_1 e), Ef' \circ \tau(\_ \otimes e) + \rho'(\_ \otimes y_1 e)\big), \end{aligned}$$

– $I''_\beta : G_1 E[y]_G L_2 \to (G_2)_G L_2$ given (borrowing from I''_β of Γ_{21}) by

$$(\theta, \varphi_1) \otimes e \otimes \ell \mapsto (\theta e, \theta y_1 e, \varphi_1(\_) \otimes e) \otimes \ell,$$

– $I'_\delta : (G_2)_G F[y] G_1 \to G_1 G_1$ given (borrowing from I'_δ of Γ_{21}) by

$$\begin{aligned} &(e', e, \xi') \otimes f \otimes g \mapsto \\ &\quad \big(f(e), Ef \circ \tau(\_ \otimes (e - y_1 e')) + E(f \circ y_1) \circ \xi'\big) \otimes g, \end{aligned}$$

– $I''_\delta : (G_2)_G F[y] G_1 \to (G_2)_G L_2$ given (borrowing from I''_δ of Γ_{12}) by

$$g \otimes f \otimes (\theta, \varphi_1) \mapsto g \otimes (0, f.\theta, f \otimes \varphi_1).$$

Now we define a morphism of $(A[y], A[y])$-bimodules $\omega''' : G_2 \otimes_{A[y]} L_2 \to G_1 G_1$, and then we show that ω''' descends to a morphism $\omega''' : G_2 \otimes_{G_1^{\mathsf{op}}} L_2 \to G_1 G_1$ by showing that it is middle-linear over generators of G_1^{op} in $FE[y]$. Let $(e', e, \xi') \otimes (f', f, \rho') \in G_2 \otimes_{A[y]} L_2$ be an arbitrary simple tensor. We define:

$$\begin{aligned}\omega''' : (e', e, \xi') \otimes (f', f, \rho') \mapsto \Big(&\varepsilon(e' \otimes f') + \varepsilon(e \otimes f), FE(\varepsilon \circ y_1 F)(\xi' \otimes f) \\ &+ FE\varepsilon(\xi' \otimes f') + \sigma(e \otimes f) - \sigma(y_1 e' \otimes f)\Big) \otimes (1, 0) \\ &+ (1, 0) \otimes \Big(0, \varepsilon FE(e \otimes \rho') + \sigma(e' \otimes f')\Big) \\ &+ \sigma FE(e \otimes \rho') - \sigma FE(y_1 e' \otimes \rho') + FE\sigma(\xi' \otimes f') + FE(\varepsilon \circ y_1 F) FE(\xi' \otimes \rho').\end{aligned}$$

The last four terms, beginning with $\sigma FE(e \otimes \rho')$, are elements of $FEFE[y]$. They are interpreted in the last summand of $G_1 G_1$ using the decomposition of bimodules:

$$\begin{aligned} G_1 \otimes_{A[y]} G_1 &\xrightarrow{\sim} A[y] \oplus FE[y] \oplus FE[y] \oplus FEFE[y], \\ (\theta, \varphi_1) \otimes (\theta', \varphi_1') &\mapsto \big(\theta\theta', \theta.\varphi_1', \varphi_1.\theta', \varphi_1 \otimes \varphi_1'\big). \end{aligned} \tag{3.4.4}$$

We can also give a decomposition of $G_2 \otimes_{A[y]} L_2$ into $(A[y], A[y])$-bimodules:

$$\begin{aligned} G_2 \otimes_{A[y]} L_2 &\xrightarrow{\sim} EF[y]^{\oplus 4} \oplus FE^2F[y]^{\oplus 2} \oplus EF^2E[y]^{\oplus 2} \oplus FE^2F^2E[y], \\ (e', e, \xi') \otimes (f', f, \rho') &\mapsto (e' \otimes f', e' \otimes f, e \otimes f', e \otimes f) \\ &\qquad \oplus (e' \otimes \rho', e \otimes \rho') \oplus (\xi' \otimes f', \xi' \otimes f) \oplus (\xi' \otimes \rho'). \end{aligned}$$

Each of the terms in the formula for ω''' is a morphism of $(A[y], A[y])$-bimodules.

Definition 3.4.3. Using the two ordered decompositions above, the map $\omega''' : G_2 \otimes_{A[y]} L_2 \to G_1 G_1$ is given by the following matrix:

$$\begin{pmatrix} \varepsilon & 0 & 0 & \varepsilon & 0 & 0 & 0 & 0 & 0 \\ \sigma & 0 & 0 & 0 & 0 & \varepsilon FE & 0 & 0 & 0 \\ 0 & -\sigma \circ y_1 F & 0 & \sigma & 0 & 0 & FE\varepsilon & FE(\varepsilon \circ y_1 F) & 0 \\ 0 & 0 & 0 & 0 & -(\sigma \circ y_1 F)FE & \sigma FE & FE\sigma & 0 & FE(\varepsilon \circ y_1 F)FE \end{pmatrix}.$$

Lemma 3.4.4. *The map ω''' is middle-linear over the action of generators of the summand $FE[y] \subset G_1^{\mathsf{op}}$.*

Proof. We first compute the middle actions $(e', e, \xi').\varphi_1$ and $\varphi_1.(f', f, \rho')$ for $\varphi_1 \in FE[y] \subset G_1^{\mathsf{op}}$, $(e', e, \xi') \in G_2$, and $(f', f, \rho') \in L_2$, both in bimodule form. These are:

$$(e', e, \xi').\varphi_1 = \big(\varphi_1(e), y_1\varphi_1(e), E\varphi_1 \circ \tau({}_{-} \otimes (e - y_1 e')) + E(\varphi_1 y_1) \circ \xi'\big)$$

$$\varphi_1.(f', f, \rho') = \big(0, f \circ y_1\varphi_1 + f' \circ \varphi_1, Ef' \circ \tau \circ E\varphi_1 + \rho' \circ E(y_1\varphi_1)\big).$$

Using the formulas above, one easily computes the images under ω''' of $(e', e, \xi').\varphi_1 \otimes (f', f, \rho')$ and $(e', e, \xi') \otimes \varphi_1.(f', f, \rho')$ and checks that they agree. □

Corollary 3.4.5. *It follows from Lemma 3.4.4 that ω''' determines a morphism of $(A[y], A[y])$-bimodules $\omega''' : (G_2)_G L_2 \to G_1 G_1$.*

We show next that $\omega''' \circ I''_\beta = I'_\beta$ and $\omega''' \circ I''_\delta = I'_\delta$. The formula for ω''' is determined by these conditions and may be derived from them. Evaluating the right side of the first equation:

$$\begin{aligned}
&\omega''' \circ I''_\beta\big((\theta, \varphi_1) \otimes e \otimes (f', f, \rho')\big) \\
&= \omega'''\big((\theta e, \theta y_1 e, \varphi_1 \otimes e) \otimes (f', f, \rho')\big) \\
&= \big(f'(\theta e) + f(\theta y_1 e), \varphi_1.f(y_1 e) + \varphi_1.f'(e)\big) \otimes (1, 0) \\
&\quad + (1, 0) \otimes \big(0, \rho'({}_{-} \otimes \theta y_1 e) + Ef' \circ \tau({}_{-} \otimes \theta e)\big) \\
&\quad + (0, \varphi_1) \otimes \big(0, Ef' \circ \tau({}_{-} \otimes e) + \rho'({}_{-} \otimes y_1 e)\big) \\
&= \big(\theta.(f'(e) + f(y_1 e)), \varphi_1.(f'(e) + f(y_1 e))\big) \otimes (1, 0) \\
&\quad + (\theta, \varphi_1) \otimes \big(0, Ef' \circ \tau({}_{-} \otimes e) + \rho'({}_{-} \otimes y_1 e)\big) \\
&= (\theta, \varphi_1) \otimes \big(f'(e) + f(y_1 e), Ef' \circ \tau({}_{-} \otimes e) + \rho'({}_{-} \otimes y_1 e)\big) \\
&= I'_\beta\big((\theta, \varphi_1) \otimes e \otimes (f', f, \rho')\big).
\end{aligned}$$

Now evaluating the right side of the second equation:

$$\omega''' \circ I''_\delta\big((e', e, \xi') \otimes f \otimes (\theta, \varphi_1)\big)$$

$$= \omega'''\big((e', e, \xi') \otimes (0, f.\theta, f \otimes \varphi_1)\big)$$

$$= \big(f(e).\theta, E(f.\theta \circ y_1) \circ \xi' + E(f.\theta) \circ \tau(\_ \otimes (e - y_1 e'))\big) \otimes (1, 0)$$

$$+ (1, 0) \otimes \big(0, f(e).\varphi_1\big) + \big(0, Ef \circ \tau(\_ \otimes (e - y_1 e')\big) \otimes (0, \varphi_1)$$

$$+ \big(0, E(f \circ y_1) \circ \xi'\big) \otimes (0, \varphi_1)$$

$$= \big(f(e), E(f \circ y_1) \circ \xi' + Ef \circ \tau(\_ \otimes (e - y_1 e'))\big) \otimes (\theta, 0)$$

$$+ \big(f(e), Ef \circ \tau(\_ \otimes (e - y_1 e')) + E(f \circ y_1) \circ \xi'\big) \otimes (0, \varphi_1)$$

$$= \big(f(e), Ef \circ \tau(\_ \otimes (e - y_1 e')) + E(f \circ y_1) \circ \xi'\big) \otimes (\theta, \varphi_1)$$

$$= I'_\delta\big((e', e, \xi') \otimes f \otimes (\theta, \varphi_1)\big).$$

Now the product $(G_2)_G L_2$ is the quotient of the product $(G_2)_{A[y]} L_2$ by the image of $\gamma' - \gamma''$, where:

– $\gamma' : \big(G_2 \otimes_{A[y]} FE[y]\big) \otimes_{A[y]} L_2 \to (G_2)_{A[y]} L_2$ given by

$$(e', e, \xi') \otimes \varphi_1 \otimes \ell \mapsto$$

$$\big(\varphi_1(e), y_1\varphi_1(e), E\varphi_1 \circ \tau(\_ \otimes (e - y_1 e')) + E(\varphi_1 \circ y_1) \circ \xi'\big) \otimes \ell,$$

– $\gamma'' : G_2 \otimes_{A[y]} \big(FE[y] \otimes_{A[y]} L_2\big) \to (G_2)_{A[y]} L_2$ given by

$$g \otimes \varphi_1 \otimes (f', f, \rho') \mapsto$$

$$g \otimes \big(0, f' \circ \varphi_1 + f \circ y_1\varphi_1, Ef' \circ \tau \circ E\varphi_1 + \rho' \circ E(y_1\varphi_1)\big).$$

There is a copy of $EF[y]$ in $(G_2)_{A[y]} L_2$ generated by terms of the form $(0, e, 0) \otimes (f', 0, 0)$.

Let $\bar{H}$ be its direct complement. The images of γ' and γ'' lie in $\bar{H}$, so $(G_2)_G L_2 \cong H \oplus EF[y]$, where H is the quotient of $\bar{H}$ by the image of $\gamma' - \gamma''$.

The image of I''_β includes every term of the form $(e, y_1 e, \varphi_1 \otimes e) \otimes \ell$, and the image of I''_δ includes every term of the form $g \otimes (0, f, f \otimes \varphi_1)$. By adding appropriate linear combinations of terms of the first form, one obtains any element $(e, y_1 e, \xi') \otimes \ell$, and similarly from terms of the second form one obtains any $g \otimes (0, f, \rho')$. It follows that $\mathrm{Im}(I''_\beta + I''_\delta) = H$.

3.4.2 Maps $\tilde{\rho}_\lambda$: computation

In this section we derive formulas for the maps $\tilde{\rho}_\lambda$ in terms of the $(A[y], A[y])$-bimodule decompositions of the four components of the matrix expressions of $\tilde{E}\tilde{F}$ and $\tilde{F}\tilde{E}$ and C.

3.4.2.1 Map $\tilde{\sigma}$

We begin by computing the map $\tilde{\sigma} : \tilde{E}\tilde{F} \to \tilde{F}\tilde{E}$. Recall that $\tilde{\sigma}$ is defined by $\tilde{\sigma} = \tilde{F}\tilde{E}\tilde{\varepsilon} \circ \tilde{F}\tilde{\tau}\tilde{F} \circ \tilde{\eta}\tilde{E}\tilde{F}$, and $\tilde{\eta}$, $\tilde{\varepsilon}$, and $\tilde{\tau}$ are determined already. We will need formulas for each component of $\tilde{\sigma}$ in its matrix presentation.

We use the following technique to derive the formulas. We start with an appropriate matrix coefficient of the element $[\tilde{\eta}(1)] \in [\tilde{F}\tilde{E}]$, together with an arbitrary generator of a component of the matrix $[\tilde{E}\tilde{F}]$. Then we write the latter as a sum of simple tensor products of elements of $[\tilde{E}]$ with elements of $[\tilde{F}]$. As a point of notation, this will be said to lie in $[\tilde{E}] \cdot [\tilde{F}]$ (and similarly for other matrix products). Then we write $[\tilde{\eta}(1)]$ in $[\tilde{F}] \cdot [\tilde{E}]$, and taking another tensor product we have an element we can write in $[\tilde{F}] \cdot [\tilde{E}] \cdot [\tilde{E}] \cdot [\tilde{F}]$. Upon this we apply $[\tilde{F}] \cdot [\tilde{\tau}] \cdot [\tilde{F}]$ using the formulas from Def. 2.3.4. We view the result in $[\tilde{F}] \cdot [\tilde{E}] \cdot [\tilde{E}\tilde{F}]$, apply $[\tilde{F}] \cdot [\tilde{E}] \cdot [\tilde{\varepsilon}]$ to obtain an element of $[\tilde{F}] \cdot [\tilde{E}] \cdot [C]$, view this in $[\tilde{F}\tilde{E}] \cdot [C]$, and allow the coefficient in $[C]$ to act on the right on the coefficient in $[\tilde{F}\tilde{E}]$.

The result is the image under $[\tilde{\sigma}]$ of the arbitrary generator in $[\tilde{E}\tilde{F}]$ with which we began.

The following bulleted lines state the results of this procedure, and the procedure itself is carried out in detail in the paragraphs below those lines.

- We have $[\tilde{\sigma}]_{11} : [\tilde{E}\tilde{F}]_{11} \to [\tilde{F}\tilde{E}]_{11}$ given by $\left(\begin{smallmatrix} \varepsilon \\ \sigma \end{smallmatrix}\right)$ using the decompositions:

  - $[\tilde{E}\tilde{F}]_{11} \cong EF[y]$,
  - $[\tilde{F}\tilde{E}]_{11} \cong (G_1)_G G_1 \cong G_1 \cong A[y] \oplus FE[y]$.

  We take $[\tilde{\eta}(1)]_{11} = (1,0) \otimes (1,0) \in (G_1)_G G_1 \cong [\tilde{F}\tilde{E}]_{11}$ (using bimodule form), and an arbitrary generator $e \otimes f \in EF[y] \cong [\tilde{E}\tilde{F}]_{11}$. The product of these in $[\tilde{F}\tilde{E}] \cdot [\tilde{E}\tilde{F}]$ can be represented in $[\tilde{F}] \cdot [\tilde{E}] \cdot [\tilde{E}] \cdot [\tilde{F}]$ by:

$$\left(\begin{smallmatrix} 0 & (1,0) \\ 0 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ (1,0) & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} f & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \left(\begin{smallmatrix} F[y] & L_1 \\ F^2[y] & L_2 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} E[y] & E^2[y] \\ G_1 & G_2 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} E[y] & E^2[y] \\ G_1 & G_2 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} F[y] & L_1 \\ F^2[y] & L_2 \end{smallmatrix}\right).$$

  The middle factors give $(1,0) \otimes e \in G_1 \otimes_{A[y]} E[y]$. Passing through Γ_{21} of Eq. 3.4.1, this represents $(e, y_1 e, 0) \in G_2 \cong [\tilde{E}^2]_{21}$. Applying $[\tilde{\tau}]_{21}$ yields $(0, e, 0) \in G_2$, which may be represented by:

$$\left(\begin{smallmatrix} 0 & 0 \\ 0 & (0,e,0) \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ (1,0) & 0 \end{smallmatrix}\right) \in [\tilde{E}] \cdot [\tilde{E}].$$

  Then:

$$\left(\begin{smallmatrix} 0 & 0 \\ (1,0) & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} f & 0 \\ 0 & 0 \end{smallmatrix}\right) \xmapsto{\tilde{\varepsilon}} \left(\begin{smallmatrix} 0 & 0 \\ f & 0 \end{smallmatrix}\right) \in [C]$$

  and

$$\left(\begin{smallmatrix} 0 & (1,0) \\ 0 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ 0 & (0,e,0) \end{smallmatrix}\right) \xmapsto{\Gamma_{12}} \left(\begin{smallmatrix} 0 & (0,e,0) \\ 0 & 0 \end{smallmatrix}\right) \in \left(\begin{smallmatrix} G_1 & G_2 \\ L_2 & U \end{smallmatrix}\right) = [\tilde{F}\tilde{E}].$$

  Finally letting $f \in C$ act on the right, we have:

$$\left(\begin{smallmatrix} 0 & (0,e,0) \\ 0 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ f & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \left(f(e), Ef \circ \tau(\_ \otimes e)\right) & 0 \\ 0 & 0 \end{smallmatrix}\right) \in [\tilde{F}\tilde{E}].$$

  The nonzero coefficient may be interpreted as $\big(\varepsilon(e \otimes f), \sigma(e \otimes f)\big)$.

- We have $[\tilde{\sigma}]_{21} : [\tilde{E}\tilde{F}]_{21} \to [\tilde{F}\tilde{E}]_{21}$ given by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & F\varepsilon \\ 0 & F\sigma \end{smallmatrix}\right)$ using the decompositions:

  - $[\tilde{E}\tilde{F}]_{21} \cong G_1F[y] \cong F[y] \oplus FEF[y]$,
  - $[\tilde{F}\tilde{E}]_{21} \cong L_2 \cong F[y] \oplus F[y] \oplus F^2E[y]$.

  Let us choose an expression for $\eta(1) = \mathrm{Id}_E \in \mathrm{Hom}_A({}_AE, E)$ as a sum of simple tensors in FE:

$$\eta(1) = \sum_{a \in Q} f_a \otimes e_a \in FE \subset FE[y],$$

where Q is some finite index set. Using f_a, e_a for $a \in Q$, we find an expression for $[\tilde{\eta}(1)]_{22}$ in $(L_2)_G G_2$:

Lemma 3.4.6. *The element*

$$\sum_{a \in Q} (f_a, 0, 0) \otimes (e_a, 0, 0) + \sum_{b \in Q} (0, f_b, 0) \otimes (0, e_b, 0) \in (L_2)_G G_2$$

(written using bimodule forms) is sent to $Id_R \in U$ under the composition morphism $(L_2)_G G_2 \xrightarrow{\sim} U$ of Lemma 3.2.14. We write $[\tilde{\eta}(1)]_{22}$ for this element.

Proof. We first take composition of the first sum, and then of the second.

Claim 3.4.7. Under the map $(L_2)_G G_2 \xrightarrow{\sim} U$, we have:

$$\sum_{a \in Q} (f_a, 0, 0) \otimes (e_a, 0, 0) \mapsto (0, 0, 0, \mathrm{Id}_{E[y]}, 0).$$

Proof of Claim. The matrix $[\Phi]$ giving the degree 1 lower row part of the image, which is a morphism in $\mathrm{Hom}_{K^b(B)}(R, R)$ written in U, is $\sum_{a \in Q} \left(\begin{smallmatrix} 0 & 0 \\ 0 & f_a(\text{-}) \otimes e_a \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. To compute the fifth coefficient Λ° of the image, we find the degree 0 part Λ of the map on the upper row, given by taking the composition $E^2[y] \to A[y] \to E^2[y]$:

$$\begin{aligned}&\sum_{a\in Q} -y_2\tau(\_\otimes y_1 e_a)\circ(Ef_a\circ\tau)\\&=\sum_{a\in Q, d\in P} -y_2\tau y_1\big(\tau(\_\,\_)_{(1d)}\otimes f_a(\tau(\_\,\_)_{(2d)}).e_a\big)\\&= -y_2\tau y_1\tau = -y_2\tau\end{aligned}$$

(in the second line we introduce notation for a decomposition $\tau(ee) = \sum_{d\in P}\tau(ee)_{(1d)}\otimes\tau(ee)_{(2d)}$ for some choices of $\tau(ee)_{(id)}$, $i = 1, 2$ and finite index set P, and in the third line we use that $\sum_{a\in Q} f_a(e^*).e_a = e^*$ for any $e^* \in E[y]$). Then $\Lambda^\circ = 0$ is determined by Eq. (3.2.3) with this Λ and Φ. □

Claim 3.4.8. Under the map $(L_2)_G G_2 \xrightarrow{\sim} U$, we have:

$$\sum_{b\in Q}(0, f_b, 0)\otimes(0, e_b, 0)\mapsto(\mathrm{Id}_{E[y]}, 0, 0, 0, 0).$$

Proof of Claim. Computing as above, the matrix $[\Phi]$ is given by $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$, and we have:

$$\begin{aligned}&\sum_{b\in Q}\big(\_\otimes e_b + y_2\tau(\_\otimes e_b)\big)\circ Ef_b\\&=\sum_{b\in Q}\tau y_1(\_\otimes e_b)\circ Ef_b\\&=\tau y_1(\sum_{b\in Q} f_b(\_).e_b)\\&=\tau y_1.\end{aligned}$$

Again, $\Lambda^\circ = 0$ is determined by Eq. (3.2.3) with this Λ and Φ. □

So $[\tilde\eta(1)]_{22}$ is sent to $(1, 0, 0, 1, 0) \in U$, which indeed corresponds to Id_R. □

Then we take an arbitrary generator $(\theta, \varphi_1)\otimes f \in G_1F[y] \cong [\tilde E\tilde F]_{21}$. Expressing the product $(\theta,\varphi_1)\otimes f\otimes\tilde\eta(1)$ in $[\tilde F]\cdot[\tilde E]\cdot[\tilde E]\cdot[\tilde F]$, we have:

$$\begin{aligned}
&\sum_{a\in Q}\left(\begin{smallmatrix}0&0\\0&(f_a,0,0)\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}0&0\\0&(e_a,0,0)\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}0&0\\(\theta,\varphi_1)&0\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}f&0\\0&0\end{smallmatrix}\right)\\
&+\sum_{b\in Q}\left(\begin{smallmatrix}0&0\\0&(0,f_b,0)\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}0&0\\0&(0,e_b,0)\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}0&0\\(\theta,\varphi_1)&0\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}f&0\\0&0\end{smallmatrix}\right)\\
&\in\begin{pmatrix}F[y]&L_1\\F^2[y]&L_2\end{pmatrix}\cdot\begin{pmatrix}E[y]&E^2[y]\\G_1&G_2\end{pmatrix}\cdot\begin{pmatrix}E[y]&E^2[y]\\G_1&G_2\end{pmatrix}\cdot\begin{pmatrix}F[y]&L_1\\F^2[y]&L_2\end{pmatrix}.
\end{aligned}$$

Now we interpret $\left(\begin{smallmatrix}0&0\\0&(e_a,0,0)\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}0&0\\(\theta,\varphi_1)&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&0\\0&(0,e_b,0)\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}0&0\\(\theta,\varphi_1)&0\end{smallmatrix}\right)$ in $[\tilde E^2]$ and apply $[\tilde\tau]$:

$$\begin{aligned}
\Gamma_{21}:(e_a,0,0)\otimes(\theta,\varphi_1)&\mapsto(e_a,0,0).(\theta,\varphi_1)\\
&=\big(e_a.\theta,0,-E\varphi_1\circ\tau(_\otimes y_1e_a)\big)\in G_2=[\tilde E^2]_{21}\\
&\overset{\tilde\tau}{\mapsto}\big(0,e_a.\theta,-\tau\circ E\varphi_1\circ\tau(_\otimes y_1e_a)\big)\in[\tilde E^2]_{21},\\
\Gamma_{21}:(0,e_b,0)\otimes(\theta,\varphi_1)&\mapsto(0,e_b,0).(\theta,\varphi_1)\\
&=\big(\varphi_1(e_b),\varphi(e_b),E\varphi_1\circ\tau(_\otimes e_b)\big)\in[\tilde E^2]_{21}\\
&\overset{\tilde\tau}{\mapsto}\big(0,\varphi_1(e_b),\tau\circ E\varphi_1\circ\tau(_\otimes e_b)\big)\in[\tilde E^2]_{21}.
\end{aligned}$$

We can represent these in $[\tilde E]\cdot[\tilde E]$ using the isomorphism $G_2\xrightarrow{\sim}(G_2)_GG_1$, $g\mapsto g\otimes(1,0)$. So, after applying $[\tilde F]\cdot[\tilde\tau]\cdot[\tilde F]$ to the middle terms, we have:

$$\begin{aligned}
&\sum_{a\in Q}\left(\begin{smallmatrix}0&0\\0&(f_a,0,0)\end{smallmatrix}\right)\cdot\begin{pmatrix}0&0\\0&\big(0,e_a.\theta,-\tau\circ E\varphi_1\circ\tau(_\otimes y_1e_a)\big)\end{pmatrix}\cdot\left(\begin{smallmatrix}0&0\\(1,0)&0\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}f&0\\0&0\end{smallmatrix}\right)\\
&+\sum_{b\in Q}\left(\begin{smallmatrix}0&0\\0&(0,f_b,0)\end{smallmatrix}\right)\cdot\begin{pmatrix}0&0\\0&\big(0,\varphi_1(e_b),\tau\circ E\varphi_1\circ\tau(_\otimes e_b)\big)\end{pmatrix}\cdot\left(\begin{smallmatrix}0&0\\(1,0)&0\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}f&0\\0&0\end{smallmatrix}\right).
\end{aligned}$$

Then $\tilde\varepsilon:\left(\begin{smallmatrix}0&0\\(1,0)&0\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}f&0\\0&0\end{smallmatrix}\right)\mapsto f\in F[y]\cong[C]_{21}$, so by applying $[\tilde F]\cdot[\tilde E]\cdot[\tilde\varepsilon]$ and viewing the first two factors in $(L_2)_GG_2\subset[\tilde F\tilde E]_{22}$ we obtain:

$$\begin{pmatrix}0&0\\0&\begin{smallmatrix}\sum_{a\in Q}(f_a,0,0)\otimes\big(0,e_a.\theta,-\tau\circ E\varphi_1\circ\tau(\_\otimes y_1e_a)\big)\\+\sum_{b\in Q}(0,f_b,0)\otimes\big(0,\varphi_1(e_b),\tau\circ E\varphi_1\circ\tau(\_\otimes e_b)\big)\end{smallmatrix}\end{pmatrix}\cdot\begin{pmatrix}0&0\\f&0\end{pmatrix}\in[\tilde F\tilde E]\cdot[C].$$

Now we express this element in $L_2=[\tilde F\tilde E]_{21}$ by applying the composition map $(L_2)_GG_2\xrightarrow{\sim}$

U and then evaluating the action of $f \in [C]_{21}$ on the right. The latter may be computed by embedding f in L_2 as $(0, f, 0)$ and post-composing with this element.

Passing first through the composition map $(L_2)_G G_2 \xrightarrow{\sim} U$, we have:

$$\sum_{a \in Q} \big(0, e_a.\theta, -\tau \circ E\varphi_1 \circ \tau(\_ \otimes y_1 e_a)\big) \circ (f_a, 0, 0) \mapsto (0, 0, \theta, 0, -\tau \circ E\varphi_1 \circ \tau),$$

where for the last component we have used:

$$\begin{aligned}
&\sum_{a \in Q} \big(_ \otimes e_a.\theta + y_2 \tau(_ \otimes e_a.\theta) - y_1 y_2 \tau \circ E\varphi_1 \circ \tau(_ \otimes y_1 e_a)\big) \circ (Ef_a \circ \tau) \\
&= \tau.\theta - y_1 y_2 \tau \circ E\varphi_1 \circ \tau \\
&= \tau y_1 (E\theta \circ \tau) + y_1 y_2 (-\tau \circ E\varphi_1 \circ \tau),
\end{aligned}$$

and the fact that $\Lambda^\circ = -\tau \circ E\varphi_1 \circ \tau$ can be deduced by comparing with Eq. (3.2.3) where $[\Phi] = \left(\begin{smallmatrix} 0 & E\theta \\ 0 & 0 \end{smallmatrix}\right)$. Similarly, we have:

$$\sum_{b \in Q} \big(0, \varphi_1(e_b), \tau \circ E\varphi_1 \circ \tau(\_ \otimes e_b)\big) \circ (0, f_b, 0) \mapsto (\varphi_1, 0, 0, 0, \tau \circ E\varphi_1 \circ \tau),$$

where again we have used:

$$\begin{aligned}
&\sum_{b \in Q} \big(_ \otimes \varphi_1(e_b) + y_2 \tau(_ \otimes \varphi_1(e_b)) + y_1 y_2 \tau \circ E\varphi_1 \circ \tau(_ \otimes e_b)\big) \circ Ef_b \\
&= \tau y_1 \circ E\varphi_1 + y_1 y_2 \tau \circ E\varphi_1 \circ \tau \\
&= \tau y_1 (E\varphi_1) + y_1 y_2 (\tau \circ E\varphi_1 \circ \tau),
\end{aligned}$$

so $\Lambda^\circ = \tau \circ E\varphi_1 \circ \tau$. For the sum of the images, we have $(\varphi_1, 0, \theta, 0, 0) \in U$. Next we compute the right action of $f \in [C]_{21}$ on this element:

$$(0, f, 0) \circ (\varphi_1, 0, \theta, 0, 0) = (\theta.f, f \circ \varphi_1, Ef \circ \tau \circ E\varphi_1),$$

where we have used:

$$\begin{aligned}&Ef \circ \big(\tau y_1(E\varphi_1 + E\theta \circ \tau)\big)\\&= Ef \circ \big(\tau y_1 \circ E\varphi_1 + E\theta \circ \tau\big)\\&= E(\theta.f \circ \tau) + E(f \circ \varphi_1) + y_1 \circ (Ef \circ \tau \circ E\varphi_1).\end{aligned}$$

Our final expression for the image of $\left(\begin{smallmatrix} 0 & 0 \\ 0 & (\theta,\varphi_1)\otimes f \end{smallmatrix}\right)$ under $[\tilde{\sigma}]_{21}$ is therefore:

$$\begin{pmatrix} 0 & 0 \\ \big(\theta.f, f \circ \varphi_1, Ef \circ \tau \circ E\varphi_1\big) & 0 \end{pmatrix} \in \begin{pmatrix} G_1 & G_2 \\ L_2 & U \end{pmatrix} = [\tilde{F}\tilde{E}].$$

The bulleted statement follows from the fact that $f \circ \varphi_1 = F\varepsilon(\varphi_1 \otimes f)$ and $Ef \circ \tau \circ E\varphi_1 = F\sigma(\varphi_1 \otimes f)$.

- We have $[\tilde{\sigma}]_{12} : [\tilde{E}\tilde{F}]_{12} \to [\tilde{F}\tilde{E}]_{12}$ given by $\left(\begin{smallmatrix} 0 & \varepsilon E \\ 1 & y_1 \circ \varepsilon E \\ 0 & \sigma E \end{smallmatrix}\right)$ using the decompositions:

– $[\tilde{E}\tilde{F}]_{12} \cong E[y]G_1 \cong E[y] \oplus EFE[y]$,

– $[\tilde{F}\tilde{E}]_{12} \cong G_2 \cong E[y] \oplus E[y] \oplus FE^2[y]$.

We take $[\tilde{\eta}(1)]_{11} = (1,0)\otimes(1,0) \in G_1G_1 \cong [\tilde{F}\tilde{E}]_{11}$, and an arbitrary generator $e\otimes(\theta,\varphi_1) \in E[y]G_1 \cong [\tilde{E}\tilde{F}]_{12}$. The product of these in $[\tilde{F}\tilde{E}]\cdot[\tilde{E}\tilde{F}]$ can be expressed in $[\tilde{F}]\cdot[\tilde{E}]\cdot[\tilde{E}]\cdot[\tilde{F}]$ by:

$$\left(\begin{smallmatrix} 0 & (1,0) \\ 0 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ (1,0) & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & (\theta,\varphi_1) \\ 0 & 0 \end{smallmatrix}\right),$$

and application of $[\tilde{F}] \cdot [\tilde{\tau}] \cdot [\tilde{F}]$ gives:

$$\left(\begin{smallmatrix} 0 & (1,0) \\ 0 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ 0 & (0,e,0) \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ (1,0) & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & (\theta,\varphi_1) \\ 0 & 0 \end{smallmatrix}\right).$$

This is sent by $[\tilde{F}\tilde{E}] \cdot [\tilde{\varepsilon}]$ to

$$\left(\begin{smallmatrix} 0 & (0,e,0) \\ 0 & 0 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ 0 & (\theta,\varphi_1) \end{smallmatrix}\right) \in [\tilde{F}\tilde{E}] \cdot [C]$$

which, after computing the action, gives

$$\big(\varphi_1(e), \varphi(e), E\varphi_1 \circ \tau(\_ \otimes e)\big) \in G_2 \cong [\tilde{F}\tilde{E}]_{12}.$$

The result follows from the observation that $E\varphi_1 \circ \tau(\_ \otimes e) = \sigma E(e \otimes \varphi_1)$.

- We have $[\tilde{\sigma}]_{22} : [\tilde{E}\tilde{F}]_{22} \to [\tilde{F}\tilde{E}]_{22}$ given by:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & F\varepsilon E & 0 \\ \eta & y_1 & 0 & 0 & \sigma \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F\sigma E & 0 \end{pmatrix}$$

using the ordered decompositions from Eq. 3.4.4 and Prop. 3.2.15:

- $[\tilde{E}\tilde{F}]_{22} \cong G_1G_1 \oplus EF[y] \cong A[y] \oplus FE[y] \oplus FE[y] \oplus FEFE[y] \oplus EF[y]$,
- $[\tilde{F}\tilde{E}]_{22} \cong U \cong FE[y]^{\oplus 4} \oplus F^2E^2[y]$.

We compute $[\tilde{\sigma}]_{22}$ first on G_1G_1, and afterwards on $EF[y]$. We can use the same presentation for $[\tilde{\eta}(1)]_{22}$ as in the calculations for $[\tilde{\sigma}]_{21}$. Let $(\theta, \varphi_1) \otimes (\theta', \varphi_1') \in G_1G_1$ be an arbitrary generator. Then the presentation for the product in $[\tilde{F}] \cdot [\tilde{E}] \cdot [\tilde{E}] \cdot [\tilde{F}]$ is:

$$\begin{aligned} &\sum_{a\in Q} \begin{pmatrix} 0 & 0 \\ 0 & (f_a,0,0) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & (e_a,0,0) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ (\theta,\varphi_1) & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & (\theta',\varphi_1') \\ 0 & 0 \end{pmatrix} \\ &+ \sum_{b\in Q} \begin{pmatrix} 0 & 0 \\ 0 & (0,f_b,0) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & (0,e_b,0) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ (\theta,\varphi_1) & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & (\theta',\varphi_1') \\ 0 & 0 \end{pmatrix} \\ &\in \begin{pmatrix} F[y] & L_1 \\ F^2[y] & L_2 \end{pmatrix} \cdot \begin{pmatrix} E[y] & E^2[y] \\ G_1 & G_2 \end{pmatrix} \cdot \begin{pmatrix} E[y] & E^2[y] \\ G_1 & G_2 \end{pmatrix} \cdot \begin{pmatrix} F[y] & L_1 \\ F^2[y] & L_2 \end{pmatrix}. \end{aligned}$$

Using again the calculations for $[\tilde{\sigma}]_{21}$, we see that application of $[\tilde{F}\tilde{E}\tilde{\varepsilon}] \circ [\tilde{F}\tilde{\tau}\tilde{F}]$ yields:

$$\begin{pmatrix} 0 & 0 \\ 0 & (\varphi_1, 0, \theta, 0, 0) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & (\theta', \varphi_1') \end{pmatrix} \in [\tilde{F}\tilde{E}] \cdot [C].$$

This time we compute the action of (θ', φ_1') on the right on U using the action of G_1^{op} on the right on R (Lemma 3.2.2). For the matrix part $[\Phi]$, we have:

$$\begin{pmatrix} \varphi' & 0 \\ \varphi_1' & \theta' \end{pmatrix} \cdot \begin{pmatrix} \varphi_1 & \theta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \varphi' \circ \varphi_1 & \theta.\varphi' \\ \varphi_1' \circ \varphi_1 & \theta.\varphi_1' \end{pmatrix}.$$

The submodule form of $(\varphi_1, 0, \theta, 0, 0)$ is $(\varphi_1, 0, \theta, 0, \tau \circ E\varphi)$ using:

$$\begin{aligned} \Lambda &= \tau y_1(E\varphi_1 + E\theta \circ \tau) \\ &= \tau \circ E(y_1\varphi_1) + \tau \circ E\theta \\ &= \tau \circ E\varphi. \end{aligned}$$

Then after taking the action, the last coefficient of the submodule form is given by post-composing with $E\varphi'$:

$$\begin{aligned} \Lambda &= E\varphi' \circ \tau \circ E\varphi \\ &= E\theta' \circ \tau \circ E\theta + E\theta' \circ \tau \circ E(y_1\varphi_1) + E(y_1\varphi_1') \circ \tau \circ E\theta + E(y_1\varphi_1') \circ \tau \circ E(y_1\varphi_1). \end{aligned}$$

To compute the bimodule form, we evaluate Eq. (3.2.3) using the values of $[\Phi]$:

$$\begin{aligned} \Lambda &= \tau y_1\big(E(\varphi' \circ \varphi_1) + E(\theta.\varphi') \circ \tau\big) - y_2\tau y_1\big(E(\varphi_1' \circ \varphi_1) + E(\theta.\varphi_1') \circ \tau\big) + y_1y_2\Lambda^\circ \\ &= \tau y_1 \circ E\varphi' \circ \big(E\varphi_1 + E\theta \circ \tau\big) - y_2\tau \circ E(y_1\varphi_1') \circ \big(E\varphi_1 + E\theta \circ \tau\big) + y_1y_2\Lambda^\circ \\ &= \tau y_1 \circ E\theta' \circ \big(E\varphi_1 + E\theta \circ \tau\big) + E(y_1\varphi_1') \circ \big(E\varphi_1 + E\theta \circ \tau\big) + y_1y_2\Lambda^\circ. \end{aligned}$$

By identifying the two expressions we can solve to find $\Lambda^\circ = E\varphi_1' \circ \tau \circ E\varphi_1$. So the image is given using the bimodule form of U by:

$$\begin{pmatrix} 0 & 0 \\ 0 & \big(\varphi' \circ \varphi_1, \varphi_1' \circ \varphi_1, \theta.\varphi', \theta.\varphi_1', E\varphi_1' \circ \tau \circ E\varphi_1\big) \end{pmatrix} \in \begin{pmatrix} G_1 & G_2 \\ L_2 & U \end{pmatrix} = [\tilde{F}\tilde{E}].$$

Using the fact that $E\varphi_1' \circ \tau \circ E\varphi_1 = F\sigma E(\varphi_1 \otimes \varphi_1')$ and $\varphi_1' \circ \varphi_1 = F\varepsilon E(\varphi_1 \otimes \varphi_1')$, one recovers the first four columns of the matrix of $[\tilde{\sigma}]_{22}$.

For the fifth column of $[\tilde{\sigma}]_{22}$, we start with an arbitrary generator $e \otimes f' \in EF[y] \subset [\tilde{E}\tilde{F}]_{22}$. The element $(0,e,0) \otimes (f',0,0) \in (G_2)_G L_2$ is sent by Γ_{22} of $[\tilde{E}\tilde{F}]$ to $e \otimes f'$. So we consider the element:

$$\sum_{a \in Q} \left(\begin{smallmatrix} 0 & 0 \\ 0 & (f_a,0,0) \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ 0 & (e_a,0,0) \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ 0 & (0,e,0) \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ 0 & (f',0,0) \end{smallmatrix}\right)$$
$$+ \sum_{b \in Q} \left(\begin{smallmatrix} 0 & 0 \\ 0 & (0,f_b,0) \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ 0 & (0,e_b,0) \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ 0 & (0,e,0) \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ 0 & (f',0,0) \end{smallmatrix}\right)$$
$$\in \begin{pmatrix} F[y] & L_1 \\ F^2[y] & L_2 \end{pmatrix} \cdot \begin{pmatrix} E[y] & E^2[y] \\ G_1 & G_2 \end{pmatrix} \cdot \begin{pmatrix} E[y] & E^2[y] \\ G_1 & G_2 \end{pmatrix} \cdot \begin{pmatrix} F[y] & L_1 \\ F^2[y] & L_2 \end{pmatrix},$$

and we compute its image under $\tilde{F}\tilde{E}\tilde{\varepsilon} \circ \tilde{F}\tilde{\tau}\tilde{F}$. First apply Γ_{22} of $[\tilde{E}\tilde{E}]$ to $(e_a,0,0) \otimes (0,e,0)$ and $(0,e_b,0) \otimes (0,e,0)$, using the rule for bimodule forms on p. 117:

$$(e_a,0,0) \otimes (0,e,0) \xmapsto{\Gamma_{22}} (0, -y_2(e_a \otimes e), -y_2(e_a \otimes e), 0) \in G_3,$$
$$(0,e_b,0) \otimes (0,e,0) \xmapsto{\Gamma_{22}} (\tau y_1(e_b \otimes e), e_b \otimes e, e_b \otimes e, 0) \in G_3.$$

Next we apply $[\tilde{\tau}]_{22}$ to these elements:

$$(0, -y_2(e_a \otimes e), -y_2(e_a \otimes e), 0) \xmapsto{[\tilde{\tau}]_{22}} \big(e_a \otimes e, e_a \otimes e, -\tau y_2(e_a \otimes e), 0\big),$$
$$(\tau y_1(e_b \otimes e), e_b \otimes e, e_b \otimes e, 0) \xmapsto{[\tilde{\tau}]_{22}} \big(\tau(e_b \otimes e), \tau(e_b \otimes e), \tau(e_b \otimes e), 0\big).$$

Note that the formula in Def. 2.3.4 is given for the submodule form of G_3. Using Prop. 2.2.27, one defines a bimodule form in the usual way, where the last coefficient is χ'' instead of χ. By studying the proof of Lemma 2.3.3, one observes that the action of $\tilde{\tau}$ on the last coefficient in this bimodule form is (also) given by post-composition with τE, whence the final zeros above.

The next step is to express $\big(e_a e, e_a e, -\tau y_2(e_a e), 0\big)$ and $\big(\tau(e_b e), \tau(e_b e), \tau(e_b e), 0\big)$ back in $(G_2)_G G_2$ (i.e. find a preimage under $\Gamma_{22}|_{(G_2)_G G_2}$) in order to view them in $[\tilde{E}] \cdot [\tilde{E}]$. We will need the notation $\tau(ee) = \sum_{d \in P} \tau(ee)_{(1d)} \otimes \tau(ee)_{(2d)}$ introduced to compute $[\tilde{\sigma}_{21}]$ above.

Claim. We have:

$$\sum_{d\in P} \begin{smallmatrix} \big(0,\tau(e_ae)_{(1d)},0\big)\otimes\big(\tau(e_ae)_{(2d)},y_1\tau(e_ae)_{(2d)},0\big) \\ -\big(0,\tau y_1(e_ae)_{(1d)},0\big)\otimes\big(0,\tau y_1(e_ae)_{(2d)},0\big) \end{smallmatrix} \xmapsto{\Gamma_{22}} \big(e_ae, e_ae, -\tau y_2(e_ae), 0\big),$$

$$\sum_{d\in P}\big(0,\tau(e_be)_{(1d)},0\big)\otimes\big(0,\tau(e_be)_{(2d)},0\big) \xmapsto{\Gamma_{22}} \big(\tau(e_be),\tau(e_be),\tau(e_be),0\big).$$

Proof. The proof is a direct calculation using the bimodules formulation of $\Gamma_{22}|_{(G_2)_G G_2}$ on p. 117. □

Thus, after applying $\tilde{F}\tilde{\tau}\tilde{F}$, we have the element:

$$\sum_{a\in Q,d\in P} \left(\begin{smallmatrix} 0 & 0 \\ 0 & (f_a,0,0)\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix} 0 & 0 \\ 0 & \big(0,\tau(e_ae)_{(1d)},0\big)\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix} 0 & 0 \\ 0 & \big(\tau(e_ae)_{(2d)},y_1\tau(e_ae)_{(2d)},0\big)\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix} 0 & 0 \\ 0 & (f',0,0)\end{smallmatrix}\right)$$
$$+\sum_{a\in Q,d\in P} \left(\begin{smallmatrix} 0 & 0 \\ 0 & (f_a,0,0)\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix} 0 & 0 \\ 0 & -\big(0,\tau y_1(e_ae)_{(1d)},0\big)\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix} 0 & 0 \\ 0 & \big(0,\tau y_1(e_ae)_{(2d)},0\big)\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix} 0 & 0 \\ 0 & (f',0,0)\end{smallmatrix}\right)$$
$$+\sum_{b\in Q,d\in P} \left(\begin{smallmatrix} 0 & 0 \\ 0 & (0,f_b,0)\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix} 0 & 0 \\ 0 & \big(0,\tau(e_be)_{(1d)},0\big)\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix} 0 & 0 \\ 0 & \big(0,\tau(e_be)_{(2d)},0\big)\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix} 0 & 0 \\ 0 & (f',0,0)\end{smallmatrix}\right),$$

and we need to apply $[\tilde{F}\tilde{E}]\cdot\tilde{\varepsilon}$ and then realize the result in $[\tilde{F}\tilde{E}]$. Observe that:

$$\big(0,\tau y_1(e_ae)_{(2d)},0\big)\otimes(f',0,0)\overset{\tilde{\varepsilon}}{\mapsto} 0,$$
$$\big(0,\tau(e_be)_{(2d)},0\big)\otimes(f',0,0)\overset{\tilde{\varepsilon}}{\mapsto} 0.$$

Therefore only the top row will remain. We have in submodule form:

$$\big(\tau(e_ae)_{(2d)},y_1\tau(e_ae)_{(2d)},0\big)\otimes(f',0,0)\xmapsto{[\tilde{\varepsilon}]_{22}}\big(f'(\tau(e_ae)_{(2d)}),Ef'\circ\tau\circ\big(\_\otimes y_1\tau(e_ae)_{(2d)}\big)\big)\in G_1.$$

We convert to bimodule form and give this a name:

$$(\theta,\varphi_1)_{a,d} := \big(f'(\tau(e_ae)_{(2d)}),Ef'\circ\tau(\_\otimes\tau(e_ae)_{(2d)})\big)\in G_1.$$

Observe that under the composition isomorphism $(L_2)_G G_2 \xrightarrow{\sim} U$ we have:

$$(f_a,0,0)\otimes\big(0,\tau(e_ae)_{(1d)},0\big)\mapsto\big(0,0,f_a(\_).\tau(e_ae)_{(1d)},0,0\big)\in U.$$

We are therefore left with:

$$\sum_{a\in Q, d\in P}\begin{pmatrix}0 & 0\\ 0 & \big(0,0,f_a(\_).\tau(e_a e)_{(1d)},0,0\big)\end{pmatrix}\cdot\begin{pmatrix}0 & 0\\ 0 & (\theta,\varphi_1)_{a,d}\end{pmatrix}\in[\tilde{F}\tilde{E}]\cdot[C].$$

It remains to use the right action of G_1^{op} on R (Lemma 3.2.2) to compute the action of $(\theta,\varphi_1)_{a,d}$. The new matrix is given for each term of the sum by:

$$\begin{pmatrix}Ef'\circ\tau y_1\big(\_\otimes\tau(e_a e)_{(2d)}\big) & 0\\ Ef'\circ\tau(\_\otimes\tau(e_a e)_{(2d)}\big) & f'(\tau(e_a e)_{(2d)})\end{pmatrix}\cdot\begin{pmatrix}0 & f_a(\_).\tau(e_a e)_{(1d)}\\ 0 & 0\end{pmatrix}$$
$$=\begin{pmatrix}0 & Ef'\circ\tau y_1\Big(f_a(\_).\tau(e_a e)_{(1d)}\otimes\tau(e_a e)_{(2d)}\Big)\\ 0 & Ef'\circ\tau\Big(f_a(\_).\tau(e_a e)_{(1d)}\otimes\tau(e_a e)_{(2d)}\Big)\end{pmatrix}.$$

After summing over a and d this becomes:

$$\overset{\sum_{a,d}}{\rightsquigarrow}\begin{pmatrix}0 & Ef'\circ\tau\circ y_1\tau(\_\otimes e)\\ 0 & Ef'\circ\tau\big(\tau(\_\otimes e)\big)\end{pmatrix}=\begin{pmatrix}0 & Ef'\circ\tau(\_\otimes e)\\ 0 & 0\end{pmatrix}.$$

This matrix gives the first four components of the final element of U. To find the fifth, first in the submodule form, we compute the submodule form of $\big(0,0,f_a(\_).\tau(e_a e)_{(1d)},0,0\big)$ and post-compose with $E\varphi$:

$$\begin{aligned}
&E\varphi\circ\Big(\tau y_1\circ E\big(f_a(_).\tau(e_a e)_{(1d)}\big)\circ\tau\Big)\\
&=E\varphi\circ\Big(\tau y_1\big(Ef_a\circ\tau(_\,_)\otimes\tau(e_a e)_{(1d)}\big)\Big)\\
&=\Big(E^2f'\circ E\tau\circ y_1\big(_\,_\otimes\tau(e_a e)_{(2d)}\big)\Big)\\
&\quad\circ\Big(\tau y_1\big(Ef_a\circ\tau(_\,_)\otimes\tau(e_a e)_{(1d)}\big)\Big)\\
&=E^2f'\circ E\tau\circ y_1\big(\tau y_1\big(Ef_a\circ\tau(_\,_)\otimes\tau(e_a e)_{(1d)}\big)\otimes\tau(e_a e)_{(2d)}\big)\\
&=E^2f'\circ E\tau\circ\tau E\circ y_2y_1\big(Ef_a\circ\tau(_\,_)\otimes\tau(e_a e)_{(1d)}\otimes\tau(e_a e)_{(2d)}\big).
\end{aligned}$$

Summing over d and a we obtain:

$$\overset{\sum_{a,d}}{\rightsquigarrow} E^2 f' \circ E\tau \circ \tau E \circ y_2 y_1 \big(E\tau \circ \tau E(\_\_ \otimes e)\big)$$

$$= E^2 f' \circ E\tau \circ \tau E \circ E\tau(\_\_ \otimes y_1 e)$$

$$= E^2 f' \circ E\tau \circ \tau E \circ y_2 E\tau(\_\_ \otimes e)$$

$$+ E^2 f' \circ E\tau \circ \tau E(\_\_ \otimes e)$$

$$= E^2 f' \circ E\tau \circ y_3 \tau E \circ E\tau(\_\_ \otimes e)$$

$$+ E^2 f' \circ E\tau \circ E\tau(\_\_ \otimes e)$$

$$+ E^2 f' \circ E\tau \circ \tau E(\_\_ \otimes e)$$

$$= y_2 E^2 f' \circ E\tau \circ \tau E \circ E\tau(\_\_ \otimes e)$$

$$+ E^2 f' \circ E\tau \circ \tau E(\_\_ \otimes e).$$

Now to find the bimodule form of the fifth component we consider:

$$\tau y_1 \circ \big(E^2 f' \circ E\tau(\_\_ \otimes e) \circ \tau\big)$$

$$= \tau y_1 \circ E^2 f' \circ E\tau \circ \tau E(\_\_ \otimes e)$$

$$= y_2 E^2 f' \circ \tau E \circ E\tau \circ \tau E(\_\_ \otimes e)$$

$$+ E^2 f' \circ E\tau \circ \tau E(\_\_ \otimes e),$$

and since this agrees with the expression before it, Eq. (3.2.3) implies that the fifth component in bimodule form is zero. Note that we have used the fact that $\tau E \circ E\tau \circ \tau E = E\tau \circ \tau E \circ E\tau$. The final expression is $\big(0, 0, Ef' \circ \tau(\_ \otimes e), 0, 0\big) \in U \cong [\tilde{F}\tilde{E}]_{22}$. Observe that $Ef' \circ \tau(\_ \otimes e) = \sigma(e \otimes f')$. This gives the fifth column of the matrix of $[\tilde{\sigma}]_{22}$, and we have now justified all components of that matrix.

3.4.2.2 Maps $\tilde{\varepsilon} \circ \tilde{x}^i \tilde{F}$ and $\tilde{F}\tilde{x}^i \circ \tilde{\eta}$

We continue by computing the maps $\tilde{\varepsilon} \circ \tilde{x}^i \tilde{F}$ and $\tilde{F}\tilde{x}^i \circ \tilde{\eta}$ on the various components of the matrices $[\tilde{E}\tilde{F}]$, $[\tilde{F}\tilde{E}]$, and $[C]$. As before, we propose these maps in the bulleted lines and justify them in the paragraphs following.

- We have $[\tilde{\varepsilon} \circ \tilde{x}^i \tilde{F}]_{11} : [\tilde{E}\tilde{F}]_{11} \to [C]_{11}$ given by $\varepsilon \circ x^i y_1 F$ using the decompositions:

  - $[\tilde{E}\tilde{F}]_{11} \cong EF[y]$,
  - $[C]_{11} \cong A[y]$.

  The endomorphism $\tilde{x} \in \mathrm{End}(\tilde{E})$ given in Def. 2.3.4 determines an endomorphism of $[\tilde{E}\tilde{F}]_{11}$ given by xF on $EF[y]$. The morphism $\tilde{\varepsilon}$ composes elements of $\tilde{E}$ with those of $\tilde{F}$ when they are interpreted in $\mathrm{Hom}_{D^b(B)}(X, E'X)$ and $\mathrm{Hom}_{D^b(B)}(E'X, X)$. In particular, $e \in E[y] \cong [\tilde{E}]_{11}$ represents the morphism $X_1 \to E'X_1$ given by $1 \mapsto y_1 e$ in degree 0 of the top row, and $f \in F[y] \cong [\tilde{F}]_{11}$ represents the morphism given by $e \mapsto f(e)$ in degree 0 of the top row.

- We have $[\tilde{F}\tilde{x}^i \circ \tilde{\eta}]_{11} : [C]_{11} \to [\tilde{F}\tilde{E}]_{11}$ given by $\begin{pmatrix} y^i \\ Fh_{i-1}(x,y)\circ\eta \end{pmatrix}$ using the decompositions:

  - $[C]_{11} \cong A[y]$,
  - $[\tilde{F}\tilde{E}]_{11} \cong G_1 \cong A[y] \oplus FE[y]$.

  Here $h_i(z_1, \ldots, z_n)$ is the complete homogeneous symmetric polynomial of degree i in the variables $z_1, \ldots, z_n$. Note the small case interpretations:

$$\begin{cases} h_{i-1}(x,y) = 0 & i = 0 \\ h_{i-1}(x,y) = 1 & i = 1 \\ h_{i-1}(x,y) = x + y & i = 2 \\ \cdots & \cdots \end{cases}$$

Observe that $[\tilde{\eta}]_{11}$ is given by $1 \mapsto \mathrm{Id}_{X_2} \in G_1^{\mathsf{op}} \cong \mathrm{End}_{K^b(B)}(X_2)$, and $\mathrm{Id}_{X_2} = (1,0)$ (in bimodule form). More generally $\theta \mapsto \_.\theta \in \mathrm{Hom}_A({}_AE, E)[y] \cong FE[y] \subset G_1^{\mathsf{op}}$. From Def. 2.3.4 we have the action of $[\tilde{x}]_{11}$ on G_1^{op} in submodule form: $\tilde{x}^i.(\theta,\varphi) = (y^i\theta, x^i \circ \varphi)$. We convert this expression to bimodule form:

$$\begin{aligned} x^i \circ \varphi &= x^i \circ \_.\theta + x^i y_1 \varphi_1 \\ &= y^i\theta + (x^i - y^i) \circ \_.\theta + y_1 x^i \varphi_1 \\ &= y^i\theta + y_1\big(h_{i-1}(x,y) \circ \_.\theta + x^i \circ \varphi_1\big), \end{aligned}$$

so $\tilde{x}^i.(\theta,\varphi_1) = (y^i\theta, h_{i-1}(x,y)\circ\_.\theta + x^i\circ\varphi_1)$. In particular, $\tilde{x}^i.(1,0) = (y^i, h_{i-1}(x,y))$, which gives the proposed formula by viewing x, y as endofunctors of E instead of as elements of $FE[y]$.

- We have $[\tilde{\varepsilon}\circ\tilde{x}^i\tilde{F}]_{21} : [\tilde{E}\tilde{F}]_{21} \to [C]_{21}$ given by $\big(x^i, F(\varepsilon\circ x^i y_1 F)\big)$ using the decompositions:

  - $[\tilde{E}\tilde{F}]_{21} \cong G_1F[y] \cong F[y] \oplus FEF[y]$,
  - $[C]_{21} \cong F[y]$.

  (Here $x \in \mathrm{End}(F)[y]$ is given by $x(f) = f \circ x$.) The map $[\tilde{\varepsilon}]_{21} : G_1F[y] \to F[y]$ is given (using submodule form) by $(\theta,\varphi)\otimes f \mapsto f \circ \varphi$. The endomorphism $[\tilde{x}]_{21}$ acts on G_1 as described under the previous bullet: $\tilde{x}^i.(\theta,\varphi_1) = (y^i\theta, h_{i-1}(x,y) \circ \_.\theta + x^i \circ \varphi_1)$. Then $[\tilde{\varepsilon}]_{21} : G_1F[y] \to F[y]$ is given using bimodule form by:

  $$\tilde{x}^i.(\theta,\varphi_1)\otimes f \mapsto f \circ x^i \circ \_.\theta + f \circ x^i y_1 \varphi_1,$$

  and the component data follows from this formula.

- We have $[\tilde{F}\tilde{x}^i\circ\tilde{\eta}]_{21} : [C]_{21} \to [\tilde{F}\tilde{E}]_{21}$ given by $\begin{pmatrix} 0 \\ y^i \\ F(Fh_{i-1}(x,y)\circ\eta) \end{pmatrix}$ using the decompositions:

  - $[C]_{21} \cong F[y]$,
  - $[\tilde{F}\tilde{E}]_{21} \cong L_2 \cong F[y] \oplus F[y] \oplus F^2E[y]$.

Let $\left(\begin{smallmatrix} 0 & 0 \\ f & 0 \end{smallmatrix}\right) \in \left(\begin{smallmatrix} A[y] & E[y] \\ F[y] & G_1^{\mathsf{op}} \end{smallmatrix}\right) = [C]$, and observe that:

$$\begin{aligned} \tilde{\eta}\left(\left(\begin{smallmatrix} 0 & 0 \\ f & 0 \end{smallmatrix}\right)\right) &= \tilde{\eta}\left(\left(\begin{smallmatrix} 0 & 0 \\ f & 0 \end{smallmatrix}\right) . \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 0 & 0 \\ f & 0 \end{smallmatrix}\right) .\tilde{\eta}\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right) \\ &= \left(\begin{smallmatrix} 0 & 0 \\ f & 0 \end{smallmatrix}\right) . \left(\begin{smallmatrix} (1,0) & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ (0,f,0) & 0 \end{smallmatrix}\right) \in \left(\begin{smallmatrix} G_1 & G_2 \\ L_2 & U \end{smallmatrix}\right) = [\tilde{F}\tilde{E}]. \end{aligned}$$

Here $(0, f, 0)$ is written in the bimodule form of L_2. (The action of $f \in F[y] \subset [C]_{21}$ on generators in $G_1 \subset [\tilde{F}\tilde{E}]$ is given by $F[y]G_1 \to L_2$, $f \otimes (\theta, \varphi) \mapsto (0, f \circ \_.\theta, \varphi \circ Ef)$ written in submodule form, and this image is $(0, f \circ \_.\theta, \varphi_1 \circ Ef)$ in bimodule form.)

Now we apply $[\tilde{F}\tilde{x}^i]_{21}$. Consider that:

$$[\tilde{F}] \cdot [\tilde{E}] \supset (L_2)_G G_1 \ni (0, f, 0) \otimes (1, 0) \xmapsto{\Gamma_{21}} (0, f, 0) \in L_2 \subset [\tilde{F}\tilde{E}].$$

We have already seen that $\tilde{x}^i.(1, 0) = (y^i, h_{i-1}(x, y)) \in G_1$, so we have:

$$(0, f, 0) \otimes (1, 0) \xmapsto{[\tilde{F}\tilde{x}^i]_{21}} (0, f, 0) \otimes (y^i, h_{i-1}(x, y)).$$

Then $\Gamma_{21} : (0, f, 0) \otimes (y^i, h_{i-1}(x, y)) \mapsto \big(0, y^i f, x^i \circ Ef\big)$ written in submodule form. In bimodule form the image is:

$$\big(0, y^i f, h_{i-1}(x, y) \circ Ef\big),$$

which we compute using:

$$\begin{aligned} x^i \circ Ef &= \big(y^i + y_1 h_{i-1}(x, y)\big) \circ Ef \\ &= E(y^i f) + y_1 \big(h_{i-1}(x, y) \circ Ef\big). \end{aligned}$$

Note that $F^2E[y] \ni h_{i-1}(x, y) \circ Ef = F\big(h_{i-1}(x, y) \circ \eta\big)(f)$.

- We have $[\tilde{\varepsilon} \circ \tilde{x}^i \tilde{F}]_{12} : [\tilde{E}\tilde{F}]_{12} \to [C]_{12}$ given by $\big(x^i, (\varepsilon \circ x^i y_1 F)E\big)$ using the decompositions:

– $[\tilde{E}\tilde{F}]_{12} \cong E[y]G_1 \cong E[y] \oplus EFE[y]$,

– $[C]_{12} \cong E[y]$.

The endomorphism $[\tilde{x}]_{12}$ acts as x on $E[y] = [\tilde{E}]_{11}$, and thus as xG_1 on $E[y]G_1 = [\tilde{E}\tilde{F}]_{12}$. The map $[\tilde{\varepsilon}]_{12} : E[y]G_1 \to E[y]$ is given (using submodule form) by $e \otimes (\theta, \varphi) \mapsto y_1^{-1}\varphi(y_1 e)$. (Recall that $e \in E[y]$ indicates the map $X_1 \to X_2$ given on the top row by $A[y] \to E[y]$, $1 \mapsto y_1 e$.) So we have:

$$x^i(e) \otimes (\theta, \varphi_1) \xrightarrow{[\tilde{\varepsilon}]_{12}} y_1^{-1}\varphi(x^i y_1 e) = x^i(e).\theta + \varphi_1(x^i y_1 e),$$

and the component data follows from this formula.

- We have $[\tilde{F}\tilde{x}^i \circ \tilde{\eta}]_{12} : [C]_{12} \to [\tilde{F}\tilde{E}]_{12}$ given by $\begin{pmatrix} y^i \\ y^i y_1 \\ (Fh_{i-1}(x,y) \circ \eta)E \end{pmatrix}$ using the decompositions:

– $[C]_{12} \cong E[y]$,

– $[\tilde{F}\tilde{E}]_{12} \cong L_2 \cong E[y] \oplus E[y] \oplus FE^2[y]$.

By reasoning as in the $[\tilde{F}\tilde{x}^i \circ \tilde{\eta}]_{21}$ case, we find:

$$[C] \ni \begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix} \xrightarrow{[\tilde{\eta}]} \begin{pmatrix} 0 & (e, y_1 e, 0) \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} G_1 & G_2 \\ L_2 & U \end{pmatrix} = [\tilde{F}\tilde{E}],$$

using the bimodule form of G_2. Now we apply $[\tilde{F}\tilde{x}^i]_{12}$. Consider that:

$$[\tilde{F}] \cdot [\tilde{E}] \supset (L_1)_G G_2 \ni (1, 0) \otimes (e, y_1 e, 0) \xrightarrow{\Gamma_{12}} (e, y_1 e, 0) \in G_2 \subset [\tilde{F}\tilde{E}].$$

In Def. 2.3.4 we have a formula for the action of $[\tilde{x}^i]_{22}$ on $G_2 \subset [\tilde{E}]$ written in terms of the data e_1, e_2, ξ. The data $(e, y_1 e, 0)$ corresponds to $e_1 = y_1 e$, $e_2 = 0$, $\xi = {}_- \otimes y_1 e$. Applying $[\tilde{x}^i]_{22}$ gives $e_1 = y^i y_1 e$, $e_2 = 0$, $\xi = {}_- \otimes y^i y_1 e + y_1 y_2 h_{i-1}(x_2, y)({}_- \otimes e)$, where to compute ξ we have used:

$$\begin{aligned} x_2^i \circ ({}_- \otimes y_1 e) &= \big(y^i + y_2 h_{i-1}(x_2, y)\big) \circ ({}_- \otimes y_1 e) \\ &= {}_- \otimes y^i y_1 e + y_1 y_2 h_{i-1}(x_2, y)({}_- \otimes e). \end{aligned}$$

This corresponds to the data $\big(y^i e, y^i y_1 e, h_{i-1}(x_2, y)({}_- \otimes e)\big) \in G_2$ in the bimodule form. So we have:

$$(1,0) \otimes (e, y_1 e, 0) \overset{[\tilde{F}\tilde{x}^i]_{12}}{\longmapsto} (1,0) \otimes \big(y^i e, y^i y_1 e, h_{i-1}(x_2, y)(\_ \otimes e)\big)$$
$$\overset{\Gamma_{12}}{\longmapsto} \big(y^i e, y^i y_1 e, h_{i-1}(x_2, y)(\_ \otimes e)\big) \in G_2 \subset [\tilde{F}\tilde{E}].$$

Note that $FE^2[y] \ni h_{i-1}(x_2, y)(\_ \otimes e) = \big((Fh_{i-1}(x,y) \circ \eta)E\big)(e)$.

- We have $[\tilde{\varepsilon} \circ \tilde{x}^i \tilde{F}]_{22} : [\tilde{E}\tilde{F}]_{22} \to [C]_{22}$ given by:

$$\begin{pmatrix} y^i & 0 & 0 & 0 & -\varepsilon \circ h_{i-1}(x,y)F \\ h_{i-1}(x,y) \circ \eta & x^i E & Fx^i & F(\varepsilon \circ x^i y_1 F)E & \substack{-FE\varepsilon \circ F(\tau \circ h_{i-1}(x_1,x_2))F \circ \eta EF \\ -FE\varepsilon \circ F(h_{i-2}(x_1,x_2,y))F \circ \eta EF} \end{pmatrix}$$

using the ordered decompositions (recall Eq. 3.4.4):

– $[\tilde{E}\tilde{F}]_{22} \cong G_1 G_1 \oplus EF[y] \cong A[y] \oplus FE[y] \oplus FE[y] \oplus FEFE[y] \oplus EF[y]$,

– $[C]_{22} \cong G_1 \cong A[y] \oplus FE[y]$.

Consider the first four columns first, i.e. the restriction of the map to $G_1 G_1$. Take an arbitrary generator $(\theta, \varphi_1) \otimes (\theta', \varphi_1')$. Borrowing a calculation from the case $[\tilde{\varepsilon} \circ \tilde{x}^i \tilde{F}]_{21}$ we find:

$$(\theta, \varphi_1) \otimes (\theta', \varphi_1') \overset{[\tilde{x}^i \tilde{F}]_{22}}{\longmapsto} (y^i \theta, h_{i-1}(x,y) \circ \_.\theta + x^i \circ \varphi_1) \otimes (\theta', \varphi_1').$$

Now $[\tilde{\varepsilon}]_{22} : G_1 G_1 \to G_1$ is given by composition, so we have:

$$\begin{aligned} &(y^i \theta, h_{i-1}(x,y) \circ \_.\theta + x^i \circ \varphi_1) \otimes (\theta', \varphi_1') \\ &\overset{[\tilde{\varepsilon}]_{22}}{\longmapsto} \big(y^i \theta\theta', \_.\theta' \circ h_{i-1}(x,y) \circ \_.\theta + (\_.\theta') \circ x^i \circ \varphi_1 \\ &\qquad + \varphi_1' \circ (\_.y^i \theta) + \varphi_1' \circ (x^i - y^i) \circ \_.\theta + \varphi_1' \circ y_1 x^i \circ \varphi_1\big) \\ &\quad = \big(y^i \theta\theta', h_{i-1}(x,y) \circ \_.\theta\theta' + \varphi_1' \circ x^i \circ \_.\theta + x^i \circ \_.\theta' \circ \varphi_1 + \varphi_1' \circ y_1 x^i \circ \varphi_1\big). \end{aligned}$$

The first four columns of the matrix of $[\tilde{\varepsilon} \circ \tilde{x}^i \tilde{F}]_{22}$ can be read off this formula.

The last column gives the restriction of $[\tilde{\varepsilon} \circ \tilde{x}^i \tilde{F}]_{22}$ to a map $EF[y] \to A[y] \oplus FE[y]$. Its computation is more involved. We start with a generator $e \otimes f$, and note that:

$$[\tilde{E}\tilde{F}]_{22} \supset (G_2)_G L_2 \ni (0, e, 0) \otimes (f, 0, 0) \xmapsto{\Gamma_{22}} e \otimes f \in EF[y] \subset [\tilde{F}\tilde{E}]_{22}.$$

Now we must apply $[\tilde{x}]_{22}$ to the first factor, and then compose the factors, thereby applying $[\tilde{\varepsilon}]_{22}$ and giving an element of $G_2 \cong \mathrm{End}_{K^b(B)}(X_2)$.

The data $(0, e, 0)$ corresponds to $e_1 = e_2 = e$, $\xi = \tau y_1(\_ \otimes e)$. The action of $[\tilde{x}^i]_{22}$ on $G_2 \subset [\tilde{E}]$ then gives $e_1 = y^i e$, $e_2 = x^i e$, $\xi = x_2^i \circ \tau y_1(\_ \otimes e)$. We can compute the composite with $(f, 0, 0)$ directly using this information. It is given in submodule form by:

$$\begin{aligned}
&\big(f \circ y_1^{-1}(y^i e - x^i e), Ef \circ \tau \circ x_2^i \circ \tau y_1(_ \otimes e)\big) \\
&= \big(f(-h_{i-1}(x, y)e), Ef \circ \tau \circ x_2^i \circ \tau y_1(_ \otimes e)\big) \in G_1.
\end{aligned}$$

It remains to convert this to bimodule form. In the calculation we will use three facts, easily checked by the reader:

1. $x_2^i \circ \tau = \tau \circ x_1^i - h_{i-1}(x_1, x_2)$,
2. $x_2^j = y^j + y_2 h_{i-1}(x_2, y)$,
3. $\sum_{j+k=i-1} x_1^j h_{k-1}(x_2, y) = h_{i-2}(x_1, x_2, y)$.

Then we have for the main calculation:

$$\begin{aligned}
&Ef \circ \tau \circ x_2^i \circ \tau y_1(_ \otimes e) \\
&= -Ef \circ \tau y_1 \circ h_{i-1}(x_1, x_2)(_ \otimes e) \\
&= -Ef \circ h_{i-1}(x_1, x_2)(_ \otimes e) - y_1 Ef \circ \tau \circ h_{i-1}(x_1, x_2)(_ \otimes e) \\
&= -Ef \circ \sum_{j+k=i-1} x_1^j \big(y^k + y_2 h_{k-1}(x_2, y)\big)(_ \otimes e) - y_1 \circ Ef \circ \tau \circ h_{i-1}(x_1, x_2)(_ \otimes e) \\
&= -Ef \circ h_{i-1}(x_1, y)(_ \otimes e) - y_1 Ef \circ \big(h_{i-2}(x_1, x_2, y)(_ \otimes e) + \tau \circ h_{i-1}(x_1, x_2)(_ \otimes e)\big).
\end{aligned}$$

Then observe that:

$$\begin{aligned}-Ef\circ h_{i-1}(x_1,y)(\_\otimes e)&=\_\otimes f(-h_{i-1}(x,y)e)\\&=(-\varepsilon\circ h_{i-1}(x,y)F)(e\otimes f),\end{aligned}$$

and that:

$$\begin{aligned}&-Ef\circ\big(h_{i-2}(x_1,x_2,y)(\_\otimes e)+\tau\circ h_{i-1}(x_1,x_2)(\_\otimes e)\big)\\&=\big(-FE\varepsilon\circ F\big(\tau\circ h_{i-1}(x_1,x_2)+h_{i-2}(x_1,x_2,y)\big)F\circ\eta EF\big)(e\otimes f).\end{aligned}$$

The formulas in the last column of $[\tilde{\varepsilon}\circ\tilde{x}^i\tilde{F}]_{22}$ follow.

- We have $[\tilde{F}\tilde{x}^i\circ\tilde{\eta}]_{22}:[C]_{22}\to[\tilde{F}\tilde{E}]_{22}$ given by:

$$\begin{pmatrix}Fy^i\circ\eta & y^iy_1\\ -Fh_{i-1}(x,y)\circ\eta & y^i\\ 0 & 0\\ Fx^i\circ\eta & 0\\ F^2\big(h_{i-1}(x_1,x_2)\circ\tau-h_{i-2}(x_1,x_2,y)\big)\circ\eta^2 & F^2h_{i-1}(x_2,y)\circ F\eta E\end{pmatrix}$$

using the ordered decompositions:

- $[C]_{22}\cong G_1\cong A[y]\oplus FE[y]$,
- $[\tilde{F}\tilde{E}]_{22}\cong U\cong FE[y]^{\oplus 4}\oplus F^2E^2[y]$.

Observe first that $[\tilde{\eta}]_{22}:G_1\to U$ is determined by $(1,0)\mapsto \mathrm{Id}_R=(1,0,0,1,0)\in U$ (using bimodule forms). Recall (Lemma 3.4.6 used for $[\tilde{\sigma}]_{21}$) that:

$$(L_2)_GG_2\ni[\tilde{\eta}(1)]=\sum_{a\in Q}(f_a,0,0)\otimes(e_a,0,0)+\sum_{b\in Q}(0,f_b,0)\otimes(0,e_b,0)\xmapsto{\Gamma_{21}}(1,0,0,1,0)\in U.$$

The map $\Gamma_{21}|_{(L_2)_GG_2}$ is given by composition and hence right G_1^{op}-equivariant, so we can compute any $[\tilde{\eta}]_{22}\big((\theta,\varphi_1)\big)$ as $[\tilde{\eta}(1)].(\theta,\varphi_1)\in(L_2)_GG_2$. The action of $[\tilde{F}\tilde{x}^i]$ is applied to elements of $(L_2)_G G_2$, and after that we pass through Γ_{21} again to obtain the final image in U.

We treat the first column of $[\tilde{F}\tilde{x}^i \circ \tilde{\eta}]_{22}$ first, and consider the second column afterwards. For the first column it is enough to consider the case $(\theta, \varphi_1) = (1, 0)$. Starting with the first term, the data $(e_a, 0, 0)$ corresponds to $e_1 = 0$, $e_2 = -y_1 e_a$, and $\xi = y_2\tau(\_ \otimes (-y_1 e_a))$. Application of the formula for $[\tilde{x}^i]_{22}$ gives $e_1 = 0$, $e_2 = -x^i y_1 e_a$, and $\xi = x_2^i \circ y_2\tau(\_ \otimes (-y_1 e_a))$. Then we convert this to bimodule form, using:

$$\begin{aligned}
&x_2^i \circ y_2\tau(_ \otimes (-y_1 e_a)) \\
&= y_2 \circ x_2^i \tau(_ \otimes (-y_1 e_a)) \\
&= y_2 \circ \tau x_1^i(_ \otimes (-y_1 e_a)) + y_1 y_2 h_{i-1}(x_1, x_2)(_ \otimes e_a) \\
&= y_2\tau(_ \otimes (-y_1 x^i e_a)) + y_1 y_2 h_{i-1}(x_1, x_2)(_ \otimes e_a),
\end{aligned}$$

where in the third line we have used Fact 1 given under the previous bullet. So in bimodule form we have:

$$[\tilde{x}^i]_{22} : (e_a, 0, 0) \mapsto \big(x^i e_a, 0, h_{i-1}(x_1, x_2)(\_ \otimes e_a)\big).$$

Now applying Γ_{21} we obtain:

$$\sum_{a \in Q} \big(x^i e_a, 0, h_{i-1}(x_1, x_2)(\_ \otimes e_a)\big) \circ (f_a, 0, 0) = \big(0, 0, 0, x^i, h_{i-1}(x_1, x_2) \circ \tau\big) \in U,$$

where the last component is computed using:

$$\begin{aligned}
&\Big(y_2\tau(_ \otimes (-y_1 x^i e_a)) + y_1 y_2 h_{i-1}(x_1, x_2)(_ \otimes e_a)\Big) \circ E f_a \circ \tau \\
&= -y_2\tau y_1 x_1^i \tau + y_1 y_2 h_{i-1}(x_1, x_2)\tau,
\end{aligned}$$

together with the facts that $\Phi_{11} = \Phi_{12} = \Phi_{21} = 0$ and $\Phi_{22} = x^i$ so:

$$\Lambda = \tau y_1(0 + 0 \circ \tau) - y_2\tau y_1 \circ (0 + E\Phi_{22} \circ \tau) + y_1 y_2 \Lambda^\circ = -y_2\tau y_1 \circ x_1^i \circ \tau + y_1 y_2 \Lambda^\circ.$$

Continuing with the second term, the data $(0, e_b, 0)$ corresponds to $e_1 = e_b$, $e_2 = e_b$, and $\xi = \tau y_1(\_ \otimes e_b)$. Application of the formula for $[\tilde{x}^i]_{22}$ gives $e_1 = y^i e_b$, $e_2 = x^i e_b$, and $\xi = x_2^i \circ \tau y_1(\_ \otimes e_b)$. Then we convert this to bimodule form, using:

$$\begin{aligned}
&x_2^i \circ \tau y_1(_ \otimes e_b) \\
&= \tau y_1(_ \otimes x^i e_b) - y_1 h_{i-1}(x_1, x_2)(_ \otimes e_b) \\
&= \tau y_1(_ \otimes x^i e_b) - y_1 h_{i-1}(x_1, y)(_ \otimes e_b) \\
&\qquad - y_1 y_2 h_{i-2}(x_1, x_2, y)(_ \otimes e_b) \\
&= _ \otimes x^i e_b + y_2 \tau(_ \otimes x^i e_b) - _ \otimes (x^i - y^i) e_b \\
&\qquad - y_1 y_2 h_{i-2}(x_1, x_2, y)(_ \otimes e_b) \\
&= _ \otimes y^i e_b + y_2 \tau(_ \otimes x^i e_b) - y_1 y_2 h_{i-2}(x_1, x_2, y)(_ \otimes e_b),
\end{aligned}$$

where we have made use of the fact, easily checked by the reader, that:

4. $y_2 h_{i-2}(x_1, x_2, y) = h_{i-1}(x_1, x_2) - h_{i-1}(x_1, y)$.

So in bimodule form we have:

$$[\tilde{x}^i]_{22} : (0, e_b, 0) \mapsto \Big(-h_{i-1}(x_1, y) e_b, y^i e_b, -h_{i-2}(x_1, x_2, y)(\_ \otimes e_b)\Big).$$

Now applying Γ_{21} we obtain:

$$\begin{aligned}
&\sum_{b \in Q} \Big(-h_{i-1}(x_1, y) e_b, y^i e_b, -h_{i-2}(x_1, x_2, y)(_ \otimes e_b)\Big) \circ (0, f_b, 0) \\
&= \big(y^i, -h_{i-1}(x_1, y), 0, 0, -h_{i-2}(x_1, x_2, y)\big) \in U,
\end{aligned}$$

where the last component is computed using:

$$x_2^i \circ \tau y_1(\_ \otimes e_b) \circ E f_b = x_2^i \tau y_1 = \tau x_1^i y_1 - y_1 h_{i-1}(x_1, x_2)$$

together with the facts that $\Phi_{11} = y^i$, $\Phi_{21} = -h_{i-1}(x_1, y)$, $\Phi_{12} = \Phi_{22} = 0$, so:

$$\begin{aligned}
\Lambda &= \tau y_1 \circ \big(y^i + 0 \circ \tau\big) - y_2 \tau y_1 \circ \big(-h_{i-1}(x_1, y) + 0 \circ \tau\big) + y_1 y_2 \Lambda^\circ \\
&= \tau y_1 y^i + y_2 \tau y_1 h_{i-1}(x_1, y) + y_1 y_2 \Lambda^\circ \\
&= \tau y_1 y^i + y_2 \tau (x_1^i - y^i) + y_1 y_2 \Lambda^\circ \\
&= y^i + y_2 \tau x_1^i + y_1 y_2 \Lambda^\circ \\
&= \tau x_1^i y_1 - y_1 h_{i-1}(x_1, y) + y_1 y_2 \Lambda^\circ,
\end{aligned}$$

so using Fact 4 again:

$$\begin{aligned}
y_2 \Lambda^\circ &= -h_{i-1}(x_1, x_2) + h_{i-1}(x_1, y), \\
\Lambda^\circ &= -h_{i-2}(x_1, x_2, y).
\end{aligned}$$

Finally taking the sum of the two terms, we conclude that $[\tilde{F}\tilde{x}^i \circ \tilde{\eta}]_{22} : A[y] \to U$ is determined by:

$$1 \mapsto \big(y^i, -h_{i-1}(x_1, y), 0, x^i, h_{i-1}(x_1, x_2) \circ \tau - h_{i-2}(x_1, x_2, y)\big).$$

By describing these coefficients in $FE[y]$ and $F^2E^2[y]$ instead of in $\mathrm{End}(E[y])$ and $\mathrm{End}(E^2[y])$, we obtain the formulas in the first column of the matrix of $[\tilde{F}\tilde{x}^i \circ \tilde{\eta}]_{22}$.

Now we consider the second column of $[\tilde{F}\tilde{x}^i \circ \tilde{\eta}]_{22}$, a map $FE[y] \to U$. It is found using the same method but with $(\theta, \varphi_1) = (0, \varphi_1)$ for a generator $\varphi_1 \in FE[y]$. We have in bimodule form:

$$\begin{aligned}
(e_a, 0, 0).(0, \varphi_1) &= \big(0, 0, E\varphi_1 \circ \tau(_ \otimes -y_1 e_a)\big) \\
(0, e_b, 0).(0, \varphi_1) &= \big(\varphi_1(e_b), y_1 \varphi_1(e_b), E\varphi_1 \circ \tau(_ \otimes e_b)\big),
\end{aligned}$$

where we have used the calculations:

$$E(y_1 \varphi_1) \circ y_2 \tau(\_ \otimes -y_1 e_a) = y_1 y_2 E\varphi_1 \circ \tau(\_ \otimes -y_1 e_a)$$

and

$$E(y_1\varphi_1)\circ({\_}\otimes e_b + y_2\tau({\_}\otimes e_b)) = {\_}\otimes y_1\varphi_1(e_b) + y_1y_2E\varphi_1\circ\tau({\_}\otimes e_b).$$

Starting with the first term, the data $\big(0,0,E\varphi_1\circ\tau({\_}\otimes -y_1e_a)\big)$ corresponds to $e_1 = e_2 = 0$ and $\xi = y_1y_2E\varphi_1\circ\tau({\_}\otimes -y_1e_a)$. Application of the formula for $[\tilde{x}^i]_{22}$ gives $e_1 = e_2 = 0$ and $\xi = x_2^i\circ y_1y_2E\varphi_1\circ\tau({\_}\otimes -y_1e_a)$. Converting this data to bimodule form is trivial. So we have:

$$[\tilde{x}^i]_{22} : \big(0,0,E\varphi_1\circ\tau({\_}\otimes -y_1e_a)\big)\mapsto\big(0,0,x_2^i\circ E\varphi_1\circ\tau({\_}\otimes -y_1e_a)\big).$$

Now applying Γ_{21} we obtain:

$$\sum_{a\in Q}\big(0,0,x_2^i\circ E\varphi_1\circ\tau({\_}\otimes -y_1e_a)\big)\circ(f_a,0,0) = \big(0,0,0,0,-E\varphi_1\circ x_2^i\tau\big)\in U,$$

where the last component is computed using:

$$\begin{aligned}&y_1y_2x_2^i\circ E\varphi_1\circ\tau({\_}\otimes -y_1e_a)\circ Ef_a\circ\tau\\&= -x_2^i\circ y_1y_2E\varphi_1\circ\tau y_1\circ\tau\\&= -y_1y_2E\varphi_1\circ x_2^i\tau.\end{aligned}$$

Continuing with the second term, the data $\big(\varphi_1(e_b), y_1\varphi_1(e_b), E\varphi_1\circ\tau({\_}\otimes e_b)\big)$ corresponds to $e_1 = y_1\varphi_1(e_b)$, $e_2 = 0$, and $\xi = {\_}\otimes y_1\varphi_1(e_b) + y_1y_2E\varphi_1\circ\tau({\_}\otimes e_b)$. Application of the formula for $[\tilde{x}^i]_{22}$ gives $e_1 = y_1y^i\varphi_1(e_b)$, $e_2 = 0$, and $\xi = x_2^i\circ\big({\_}\otimes y_1\varphi_1(e_b) + y_1y_2E\varphi_1\circ\tau({\_}\otimes e_b)\big)$.

Then we convert this to bimodule form, using:

$$
\begin{aligned}
&x_2^i \circ \big({}_{-} \otimes y_1\varphi_1(e_b) + y_1y_2E\varphi_1 \circ \tau({}_{-} \otimes e_b)\big) \\
&= {}_{-} \otimes y^i y_1\varphi_1(e_b) + y_2h_{i-1}(x_2, y)({}_{-} \otimes y_1\varphi_1(e_b)) + y_1y_2E\varphi_1 \circ x_2^i\tau({}_{-} \otimes e_b) \\
&= {}_{-} \otimes y^i y_1\varphi_1(e_b) + y_1y_2\Big(h_{i-1}(x_2, y) \circ E\varphi_1({}_{-} \otimes e_b) + E\varphi_1 \circ x_2^i\tau({}_{-} \otimes e_b)\Big) \\
&= {}_{-} \otimes y^i y_1\varphi_1(e_b) + y_1y_2E\varphi_1 \circ \big(x_2^i\tau + h_{i-1}(x_2, y)\big)({}_{-} \otimes e_b) \\
&= {}_{-} \otimes y_1y^i\varphi_1(e_b) + y_1y_2\Big(-E\varphi_1 \circ y_1h_{i-2}(x_1, x_2, y)({}_{-} \otimes e_b) + E\varphi_1 \circ \tau \circ x_1^i({}_{-} \otimes e_b)\Big).
\end{aligned}
$$

So in bimodule form we have:

$$
\begin{aligned}
&[\tilde{x}^i]_{22} : \big(\varphi_1(e_b), y_1\varphi_1(e_b), E\varphi_1 \circ \tau({}_{-} \otimes e_b)\big) \mapsto \\
&\Big(y^i\varphi_1(e_b), y_1y^i\varphi_1(e_b), E\varphi_1 \circ \big(x_2^i\tau + h_{i-1}(x_2, y)\big)({}_{-} \otimes e_b)\Big).
\end{aligned}
$$

Now applying Γ_{21} we obtain:

$$
\begin{aligned}
&\sum_{b \in Q}\Big(y^i\varphi_1(e_b), y_1y^i\varphi_1(e_b), E\varphi_1 \circ \big(x_2^i\tau + h_{i-1}(x_2, y)\big)({}_{-} \otimes e_b)\Big) \circ (0, f_b, 0) \\
&= \Big(y^i y_1\varphi_1, y^i\varphi_1, 0, 0, E\varphi_1 \circ \big(x_2^i\tau + h_{i-1}(x_2, y)\big)\Big) \in U,
\end{aligned}
$$

where the last component is computed using:

$$
\begin{aligned}
&\Big({}_{-} \otimes y^i y_1\varphi_1(e_b) + y_1y_2E\varphi_1 \circ \big(x_2^i\tau + h_{i-1}(x_2, y)\big)({}_{-} \otimes e_b)\Big) \circ Ef_b \\
&= y^i y_1E\varphi_1 + y_1y_2E\varphi_1 \circ \big(x_2^i\tau + h_{i-1}(x_2, y)\big)({}_{-} \otimes e_b),
\end{aligned}
$$

together with the facts that $\Phi_{11} = y^i y_1\varphi_1$, $\Phi_{21} = y^i\varphi_1$, $\Phi_{12} = \Phi_{22} = 0$, so:

$$
\Lambda = \tau y_1(y^i y_1E\varphi_1 + 0 \circ \tau) - y_2\tau y_1(y^iE\varphi_1 + 0 \circ \tau) + y_1y_2\Lambda^\circ = y^i y_1E\varphi_1 + y_1y_2\Lambda^\circ.
$$

Taking the sum of the two terms, we conclude that $[\tilde{F}\tilde{x}^i \circ \tilde{\eta}]_{22} : FE[y] \to U$ is given by:

$$
\varphi_1 \mapsto \big(y^i y_1\varphi_1, y^i\varphi_1, 0, 0, E\varphi_1 \circ h_{i-1}(x_2, y)\big).
$$

The last component, an element of $\mathrm{End}_A({}_AE^2)[y]$, is the same as $\big(F^2h_{i-1}(x_2,y)\circ F\eta E\big)(\varphi_1)$. This gives the formulas in the second column of the matrix of $[\tilde{F}\tilde{x}^i\circ\tilde{\eta}]_{22}$.

3.4.3 Maps $\tilde{\rho}_\lambda$: isomorphisms

Now we have formulas by components for the maps $\tilde{\sigma}$, $\tilde{\varepsilon}\circ\tilde{x}^i\tilde{F}$, and $\tilde{F}\tilde{x}^i\circ\tilde{\eta}$ that are used to define the maps $\tilde{\rho}_\lambda$. It remains to make use of the isomorphisms ρ_λ determined by σ, $\varepsilon\circ x^iF$, and $Fx^i\circ\eta$, together with these formulas, to show that $\tilde{\rho}_\lambda$ are isomorphisms. Note that $\tilde{\rho}_\lambda$ are already known to give morphisms of (C,C)-bimodules, so it suffices to show that $\tilde{\rho}_\lambda$ are isomorphisms of sets. We will work again by components and show that $[\tilde{\rho}_\lambda]_{ij}$ is an isomorphism of $(A[y],A[y])$-bimodules for $i,j\in\{1,2\}$.

We remind the reader of our notational convention that $E_\lambda = Ee_\lambda$ for the idempotents $e_\lambda\in A_\lambda$ of a weight decomposition. Recall that the bimodule E satisfies $e_jEe_i=\delta_{i+2,j}\cdot e_{i+2}Ee_i$, and similarly for F but with $i-2$ instead of $i+2$. Finally, recall Prop. 2.3.26 that gives the weight idempotents for the algebra C.

- We have for $[\tilde{\rho}_\lambda]_{11}$, $\lambda\geqslant 0$:

$$[\tilde{\rho}_\lambda]_{11} : EF_{\lambda+1}[y]\to A_{\lambda+1}[y]\oplus FE_{\lambda+1}[y]\oplus A_{\lambda+1}[y]^{\oplus\lambda}$$

  given by:

$$[\tilde{\rho}_\lambda]_{11} = \varepsilon\oplus\sigma\oplus\bigoplus_{i=0}^{\lambda-1}\varepsilon\circ x^iy_1F.$$

- We have for $[\tilde{\rho}_\lambda]_{11}$, $\lambda\leqslant 0$:

$$[\tilde{\rho}_\lambda]_{11} : EF_{\lambda+1}[y]\oplus A_{\lambda+1}[y]^{\oplus-\lambda}\to A_{\lambda+1}[y]\oplus FE_{\lambda+1}[y]$$

  given by:

$$[\tilde{\rho}_\lambda]_{11} = \left(\begin{pmatrix}\varepsilon\\ \sigma\end{pmatrix}, \sum_{i=0}^{-\lambda-1}\begin{pmatrix}y^i\\ Fh_{i-1}(x,y)\circ\eta\end{pmatrix}\right).$$

Proposition 3.4.9. *The morphism of $(A[y], A[y])$-bimodules $[\tilde{\rho}_\lambda]_{11}$ is an isomorphism for all λ.*

Proof. When $\lambda \geqslant 0$ and therefore $\lambda + 1 \geqslant 0$, the map:

$$\sigma \oplus \bigoplus_{i=0}^{\lambda} \varepsilon \circ x^i F : EF_{\lambda+1}[y] \xrightarrow{\sim} FE_{\lambda+1}[y] \oplus A_{\lambda+1}[y]^{\oplus \lambda+1}$$

is just $\rho_{\lambda+1} \otimes_k k[y]$. It is an isomorphism because $\rho_{\lambda+1}$ is an isomorphism.

Claim 3.4.10. When $\lambda \geqslant 0$, the map

$$\sigma \oplus \varepsilon \oplus \bigoplus_{i=0}^{\lambda-1} \varepsilon \circ x^i y_1 F : EF_{\lambda+1}[y] \to FE_{\lambda+1}[y] \oplus A_{\lambda+1}[y]^{\oplus \lambda+1}$$

is also an isomorphism.

Proof. Let $M_{-y} \in \mathrm{End}_{A_{\lambda+1}[y]}\left(A_{\lambda+1}[y]^{\oplus \lambda+1}\right)$ be the endomorphism with matrix coefficients $[M_{-y}] \in \mathrm{Mat}_{(\lambda+1)\times(\lambda+1)}\left(A_{\lambda+1}[y]^{\mathsf{op}}\right)$ given by 1 on the diagonal and $-y$ on the subdiagonal, and 0 elsewhere. This matrix is invertible, and M_{-y} is an isomorphism. Observe that:

$$\varepsilon \circ (-x^{i-1} y F) = -y \cdot \varepsilon \circ x^{i-1} F.$$

Using this we write the map in question as a composition of isomorphisms:

$$\sigma \oplus \varepsilon \oplus \bigoplus_{i=0}^{\lambda-1} \varepsilon \circ x^i y_1 F = \begin{pmatrix} 1 & 0 \\ 0 & M_{-y} \end{pmatrix} \circ \left(\sigma \oplus \bigoplus_{i=0}^{\lambda} \varepsilon \circ x^i F \right).$$

By reordering the first two summands in the codomain, we obtain the map $[\tilde{\rho}_\lambda]_{11}$. □

When $\lambda = 0$, the two formulas for $[\tilde{\rho}_\lambda]_{11}$ agree. Now assume $\lambda < 0$, so $\lambda + 1 \leqslant 0$ and the map:

$$\left(\sigma, \sum_{i=0}^{-(\lambda+1)-1} F x^i \circ \eta \right) : EF_{\lambda+1} \oplus A_{\lambda+1}[y]^{\oplus -(\lambda+1)} \xrightarrow{\sim} FE_{\lambda+1}[y] \tag{3.4.5}$$

is $\rho_{\lambda+1} \otimes_k k[y]$, an isomorphism.

Claim 3.4.11. When $\lambda < 0$, the map:

$$\left(\sigma, \sum_{i=1}^{-\lambda-1} Fh_{i-1}(x,y) \circ \eta\right) : EF_{\lambda+1}[y] \oplus A_{\lambda+1}[y]^{\oplus -(\lambda+1)} \to FE_{\lambda+1}[y]$$

is also an isomorphism.

Proof. This time we define an isomorphism $M_h \in \mathrm{End}_{A_{\lambda+1}[y]}\left(A_{\lambda+1}[y]^{\oplus -(\lambda+1)}\right)$ with components $[M_h]_{ii} = 1$, $[M_h]_{ij} = y^{j-i}$ for $j > i$, and $[M_h]_{ij} = 0$ for $j < i$. This is an upper-triangular invertible matrix:

$$[M_h] = \begin{pmatrix} 1 & y & y^2 & \dots & y^{-(\lambda+1)-1} \\ 0 & 1 & y & \dots & y^{-(\lambda+1)-2} \\ 0 & 0 & 1 & \dots & y^{-(\lambda+1)-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Now observe that $Fx^iy^j \circ \eta = (Fx^i \circ \eta) \cdot y^j$. We use this and write:

$$\begin{aligned} \sum_{i=1}^{-\lambda-1} Fh_{i-1}(x,y) \circ \eta &= \sum_{i=0}^{-\lambda-2} \sum_{j+k=i} Fx^j \circ \eta \cdot y^k \\ &= \left(\sum_{i=0}^{-(\lambda+1)-1} Fx^i \circ \eta \right) \circ M_h, \end{aligned}$$

and it follows from this and the isomorphism above the claim that the map of the claim is an isomorphism. □

By writing out terms, we have:

$$\left(\left(\begin{smallmatrix} \varepsilon \\ \sigma \end{smallmatrix}\right), \sum_{i=0}^{-\lambda-1} \left(\begin{smallmatrix} y^i \\ Fh_{i-1}(x,y)\circ\eta \end{smallmatrix}\right) \right) = \begin{pmatrix} \varepsilon & 1 & y & \dots & y^{-\lambda-1} \\ \sigma & 0 & \eta & \dots & Fh_{-\lambda-2}(x,y) \circ \eta \end{pmatrix}.$$

Interchanging the first two summands of the domain, we obtain the form:

$$\begin{pmatrix} 1 & \left(\varepsilon, y, y^2, \dots, y^{-\lambda-1}\right) \\ 0 & \left(\sigma, \sum_{i=1}^{-\lambda-1} Fh_{i-1}(x,y) \circ \eta\right) \end{pmatrix},$$

which (by the claim) is manifestly an isomorphism. □

- We have for $[\tilde{\rho}_\lambda]_{21}$, $\lambda \geqslant 0$:

$$[\tilde{\rho}_\lambda]_{21} : F_{\lambda+1}[y] \oplus FEF_{\lambda+1}[y] \to F_{\lambda+1}[y] \oplus F_{\lambda+1}[y] \oplus F^2E_{\lambda+1}[y] \oplus F_{\lambda+1}[y]^{\oplus\lambda}$$

given by:

$$[\tilde{\rho}_\lambda]_{21} = \begin{pmatrix} 1 & 0 \\ 0 & F\varepsilon \\ 0 & F\sigma \\ \bigoplus_{i=0}^{\lambda-1} x^i & \bigoplus_{i=0}^{\lambda-1} F(\varepsilon \circ x^i y_1 F) \end{pmatrix}.$$

- We have for $[\tilde{\rho}_\lambda]_{21}$, $\lambda \leqslant 0$:

$$[\tilde{\rho}_\lambda]_{21} : F_{\lambda+1}[y] \oplus FEF_{\lambda+1}[y] \oplus F_{\lambda+1}[y]^{\oplus-\lambda} \to F_{\lambda+1}[y] \oplus F_{\lambda+1}[y] \oplus F^2E_{\lambda+1}[y]$$

given by:

$$[\tilde{\rho}_\lambda]_{21} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & F\varepsilon & \sum_{i=0}^{-\lambda-1} y^i \\ 0 & F\sigma & \sum_{i=0}^{-\lambda-1} F(Fh_{i-1}(x,y) \circ \eta) \end{pmatrix}.$$

Proposition 3.4.12. *The morphism of $(A[y], A[y])$-bimodules $[\tilde{\rho}_\lambda]_{21}$ is an isomorphism for all λ.*

Proof. When $\lambda \geqslant 0$, we have that

$$F\varepsilon \oplus F\sigma \oplus \bigoplus_{i=0}^{\lambda-1} F(\varepsilon \circ x^i y_1 F) : FEF_{\lambda+1}[y] \to F_{\lambda+1}[y] \oplus F^2E_{\lambda+1}[y] \oplus F_{\lambda+1}[y]^{\oplus\lambda}$$

is an isomorphism, using Claim 3.4.10 and the fact that (horizontal) composition of the identity functor on F with an isomorphism gives an isomorphism. Then $[\tilde{\rho}_\lambda]_{21}$ may be compressed to a lower-triangular 2×2 matrix with an isomorphism in position $(2, 2)$, so it is an isomorphism.

When $\lambda = 0$, the two formulas for $[\tilde{\rho}_\lambda]_{21}$ agree. Assume now that $\lambda < 0$, so the map

$$\left(F\sigma, \sum_{i=1}^{-\lambda-1} F\big(Fh_{i-1}(x,y)\circ\eta\big)\right) : FEF_{\lambda+1}[y] \oplus F_{\lambda+1}[y]^{\oplus -(\lambda+1)} \to F^2E_{\lambda+1}[y]$$

is an isomorphism using Claim 3.4.11. Now expand the notation of the map $[\tilde{\rho}_\lambda]_{21}$ in the third row:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & F\varepsilon & 1 & \sum\limits_{i=1}^{-\lambda-1} y^i \\ 0 & F\sigma & 0 & \sum\limits_{i=1}^{-\lambda-1} F\big(Fh_{i-1}(x,y)\circ\eta\big) \end{pmatrix}.$$

After switching the second and third summands of the domain, we obtain an upper-triangular matrix with isomorphisms on the diagonal, so $[\tilde{\rho}_\lambda]_{21}$ is an isomorphism. □

- We have for $[\tilde{\rho}_\lambda]_{12}$, $\lambda \geqslant 0$:

$$[\tilde{\rho}_\lambda]_{12} : E_{\lambda-1}[y] \oplus EFE_{\lambda-1}[y] \to E_{\lambda-1}[y] \oplus E_{\lambda-1}[y] \oplus FE^2_{\lambda-1}[y] \oplus E_{\lambda-1}[y]^{\oplus\lambda}$$

given by:

$$[\tilde{\rho}_\lambda]_{12} = \begin{pmatrix} 0 & \varepsilon E \\ 1 & y_1\circ\varepsilon E \\ 0 & \sigma E \\ \bigoplus\limits_{i=0}^{\lambda-1} x^i & \bigoplus\limits_{i=0}^{\lambda-1} (\varepsilon\circ x^i y_1 F)E \end{pmatrix}.$$

- We have for $[\tilde{\rho}_\lambda]_{12}$, $\lambda \leqslant 0$:

$$[\tilde{\rho}_\lambda]_{12} : E_{\lambda-1}[y] \oplus EFE_{\lambda-1}[y] \oplus E_{\lambda-1}[y]^{\oplus-\lambda} \to E_{\lambda-1}[y] \oplus E_{\lambda-1}[y] \oplus FE^2_{\lambda-1}[y]$$

given by:

$$[\tilde{\rho}_\lambda]_{12} = \begin{pmatrix} 0 & \varepsilon E & \sum\limits_{i=0}^{-\lambda-1} y^i \\ 1 & y_1\circ\varepsilon E & \sum\limits_{i=0}^{-\lambda-1} y^i y_1 \\ 0 & \sigma E & \sum\limits_{i=0}^{-\lambda-1} (Fh_{i-1}(x,y)\circ\eta)E \end{pmatrix}.$$

Proposition 3.4.13. *The morphism of $(A[y], A[y])$-bimodules $[\tilde{\rho}_\lambda]_{12}$ is an isomorphism for all λ.*

Proof. When $\lambda \geqslant 0$, we have that

$$\varepsilon E \oplus \sigma E \oplus \bigoplus_{i=0}^{\lambda-1} (\varepsilon \circ x^i y_1 F)E : EFE_{\lambda-1}[y] \to E_{\lambda-1}[y] \oplus FE^2_{\lambda-1}[y] \oplus E_{\lambda-1}[y]^{\oplus\lambda}$$

is an isomorphism, using Claim 3.4.10 with E applied on the right. Note that E applied on the right here is equivalent to ${}_{\lambda+1}E_{\lambda-1}$ applied on the right, and this raises the weight by 2, so we still invoke the isomorphism $\rho_{\lambda+1}$ for weight $\lambda + 1$.

We perform some row operations on the matrix of $[\tilde{\rho}_\lambda]_{12}$. Subtract y_1 times the first row from the second to eliminate the coefficient $y_1 \circ \varepsilon E$. Then exchange the first and second rows, then exchange the second and third rows, then collapse the second and third into the notation of the fourth. Obtain:

$$\begin{pmatrix} 1 & 0 \\ 0 \oplus 0 \oplus \bigoplus_{i=0}^{\lambda-1} x^i & \sigma E \oplus \varepsilon E \oplus \bigoplus_{i=0}^{\lambda-1} (\varepsilon \circ x^i y_1 F)E \end{pmatrix},$$

which is upper-triangular with isomorphisms on the diagonal, so the original matrix for $[\tilde{\rho}_\lambda]_{12}$ is an isomorphism.

When $\lambda = 0$, the two formulas for $[\tilde{\rho}_\lambda]_{12}$ agree. Assume now that $\lambda < 0$, so the map

$$\left(\sigma E, \sum_{i=1}^{-\lambda-1} \big(Fh_{i-1}(x,y) \circ \eta\big)E\right) : EFE_{\lambda-1}[y] \oplus E_{\lambda-1}[y]^{\oplus-(\lambda+1)} \to FE^2_{\lambda-1}[y]$$

is an isomorphism using Claim 3.4.11. Now expand the notation of the map $[\tilde{\rho}_\lambda]_{12}$ in the third row:

$$[\tilde{\rho}_\lambda]_{12} = \begin{pmatrix} 0 & \varepsilon E & 1 & \sum_{i=1}^{-\lambda-1} y^i \\ 1 & y_1 \circ \varepsilon E & y_1 & \sum_{i=1}^{-\lambda-1} y^i y_1 \\ 0 & \sigma E & 0 & \sum_{i=1}^{-\lambda-1} \big(Fh_{i-1}(x,y) \circ \eta\big)E \end{pmatrix}.$$

Exchange the first and second rows, then the first and third columns, then collapse the third and fourth columns into the notation of the third, and obtain:

$$\begin{pmatrix} 1 & y_1 & \left(y_1 \circ \varepsilon E, \sum_{i=1}^{-\lambda-1} y^i y_1\right) \\ 0 & 1 & \left(\varepsilon E, \sum_{i=1}^{-\lambda-1} y^i\right) \\ 0 & 0 & \left(\sigma E, \sum_{i=1}^{-\lambda-1} \big(F h_{i-1}(x,y) \circ \eta\big) E\right) \end{pmatrix}.$$

Since this is upper-triangular with isomorphisms on the diagonal, the original matrix $[\tilde{\rho}_\lambda]_{12}$ is an isomorphism. □

- We have for $[\tilde{\rho}_\lambda]_{22}$, $\lambda \geqslant 0$:

$$\begin{aligned} [\tilde{\rho}_\lambda]_{22} : A_{\lambda-1}[y] \oplus FE_{\lambda-1}[y]^{\oplus 2} \oplus FEFE_{\lambda-1}[y] \oplus EF_{\lambda-1}[y] \\ \to FE_{\lambda-1}[y]^{\oplus 4} \oplus F^2 E^2_{\lambda-1}[y] \oplus A_{\lambda-1}[y]^{\oplus \lambda} \oplus FE_{\lambda-1}[y]^{\oplus \lambda} \end{aligned} \tag{3.4.6}$$

given by: $[\tilde{\rho}_\lambda]_{22} =$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & F\varepsilon E & 0 \\ \eta & y_1 & 0 & 0 & \sigma \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F\sigma E & 0 \\ \bigoplus_{i=0}^{\lambda-1} y^i & 0 & 0 & 0 & \bigoplus_{i=0}^{\lambda-1} -\varepsilon \circ h_{i-1}(x,y) F \\ \bigoplus_{i=0}^{\lambda-1} h_{i-1}(x,y) \circ \eta & \bigoplus_{i=0}^{\lambda-1} x^i E & \bigoplus_{i=0}^{\lambda-1} F x^i & \bigoplus_{i=0}^{\lambda-1} F(\varepsilon \circ x^i y_1 F) E & \Theta \end{pmatrix},$$

where

$$\Theta = \bigoplus_{i=0}^{\lambda-1} -FE\varepsilon \circ F\big(\tau \circ h_{i-1}(x_1,x_2) - h_{i-2}(x_1,x_2,y)\big) F \circ \eta EF.$$

- We have for $[\tilde{\rho}_\lambda]_{22}$, $\lambda \leqslant 0$:

$$[\tilde{\rho}_\lambda]_{22} : A_{\lambda-1}[y] \oplus FE_{\lambda-1}[y]^{\oplus 2} \oplus FEFE_{\lambda-1}[y] \oplus EF_{\lambda-1}[y] \\ \oplus A_{\lambda-1}[y]^{\oplus -\lambda} \oplus FE_{\lambda-1}[y]^{\oplus -\lambda} \to FE_{\lambda-1}[y]^{\oplus 4} \oplus F^2E^2_{\lambda-1}[y] \quad (3.4.7)$$

given by: $[\tilde{\rho}_\lambda]_{22} =$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & \sum_{i=0}^{-\lambda-1} Fy^i \circ \eta & \sum_{i=0}^{-\lambda-1} y^i y_1 \\ 0 & 0 & 0 & F\varepsilon E & 0 & \sum_{i=0}^{-\lambda-1} -Fh_{i-1}(x,y) \circ \eta & \sum_{i=0}^{-\lambda-1} y^i \\ \eta & y_1 & 0 & 0 & \sigma & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \sum_{i=0}^{-\lambda-1} Fx^i \circ \eta & 0 \\ 0 & 0 & 0 & F\sigma E & 0 & \Theta' & \sum_{i=0}^{-\lambda-1} F^2\big(h_{i-1}(x_2,y)\big) \circ F\eta E \end{pmatrix},$$

where

$$\Theta' = \sum_{i=0}^{-\lambda-1} F^2\big(h_{i-1}(x_1,x_2) \circ \tau - h_{i-2}(x_1,x_2,y)\big) \circ \eta^2.$$

Proposition 3.4.14. *The morphism of $(A[y], A[y])$-bimodules $[\tilde{\rho}_\lambda]_{22}$ is an isomorphism for all λ.*

Proof. When $\lambda > 0$ and therefore $\lambda - 1 \geqslant 0$, the map

$$\sigma \oplus \bigoplus_{i=0}^{\lambda-2} -\varepsilon \circ x^i F : EF_{\lambda-1}[y] \to FE_{\lambda-1}[y] \oplus A_{\lambda-1}[y]^{\oplus \lambda-1}$$

is an isomorphism. (The minus sign does not interfere.)

Claim 3.4.15. When $\lambda > 0$, the map

$$\sigma \oplus \bigoplus_{i=1}^{\lambda-1} -\varepsilon \circ h_{i-1}(x,y)F : EF_{\lambda-1}[y] \to FE_{\lambda-1}[y] \oplus A_{\lambda-1}[y]^{\oplus \lambda-1}$$

is an isomorphism.

Proof. Define an isomorphism $M'_h \in \mathrm{End}_{A_{\lambda-1}[y]}\left(A_{\lambda-1}[y]^{\oplus \lambda-1}\right)$ with components $[M'_h]_{ii} =$

1, $[M'_h]_{ij} = y^{i-j}$ for $i > j$, and $[M'_h]_{ij} = 0$ for $i < j$. This is a lower-triangular invertible matrix:

$$[M'_h] = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ y & 1 & 0 & \dots & 0 \\ y^2 & y & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ y^{\lambda-2} & y^{-\lambda-3} & y^{\lambda-4} & \dots & 1 \end{pmatrix}.$$

Now observe that $\varepsilon \circ x^i y^j F = y^j \cdot \varepsilon \circ x^i F$. Using this, we can write:

$$\begin{aligned} \bigoplus_{i=1}^{\lambda-1} -\varepsilon \circ h_{i-1}(x,y)F &= \bigoplus_{i=0}^{\lambda-2} \sum_{j+k=i} y^k \cdot (-\varepsilon \circ x^j F) \\ &= M'_h \circ \left(\bigoplus_{i=0}^{\lambda-2} -\varepsilon \circ x^j F \right), \end{aligned}$$

and it follows from this and the isomorphism above the claim that the map of the claim is an isomorphism. $\square$

Now assume $\lambda > 0$ and reorder the summands of the domain and codomain to permute the rows and columns of the matrix of $[\tilde{\rho}_\lambda]_{22}$. Let the domain be given in the order:

$$FE_{\lambda-1}[y]^{\oplus 2} \oplus A_{\lambda-1}[y] \oplus EF_{\lambda-1}[y] \oplus FEFE_{\lambda-1}[y],$$

where the first two identical summands appear in the same order as before. Let the codomain be given in the order:

$$\begin{aligned} FE_{\lambda-1}[y]^{\oplus 2} \oplus A_{\lambda-1}[y] \oplus FE_{\lambda-1}[y] \oplus A_{\lambda-1}[y]^{\oplus \lambda-1} \\ \oplus F^2E^2_{\lambda-1}[y] \oplus FE_{\lambda-1}[y] \oplus FE_{\lambda-1}[y]^{\oplus \lambda-1}, \end{aligned}$$

where the new summand number (numbered left to right) and corresponding old summand number are given precisely in the following chart:

new:	1	2	3	4	5	6	7	...	$\lambda+3$	$\lambda+4$	$\lambda+5$	$\lambda+6$	...	$2\lambda+5$
old:	4	1	6	3	7	8	9	...	$\lambda+5$	2	5	$\lambda+6$	...	$2\lambda+5$.

Writing the matrix of $[\tilde{\rho}_\lambda]_{22}$ for $\lambda > 0$, with columns and rows changed by the above permutations, we obtain:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ y_1 & 0 & \eta & \sigma & 0 \\ 0 & 0 & \bigoplus_{i=1}^{\lambda-1} y^i & \bigoplus_{i=1}^{\lambda-1} -\varepsilon \circ h_{i-1}(x,y)F & 0 \\ 0 & 0 & 0 & 0 & F\varepsilon E \\ 0 & 0 & 0 & 0 & F\sigma E \\ \bigoplus_{i=0}^{\lambda-1} x^i E & \bigoplus_{i=0}^{\lambda-1} Fx^i & \bigoplus_{i=0}^{\lambda-1} h_{i-1}(x,y)\circ\eta & \Theta & \bigoplus_{i=0}^{\lambda-1} F(\varepsilon \circ x^i y_1 F)E \end{pmatrix}.$$

After compressing the notation of rows 3 and 4 of this matrix, and also of rows 6-8, we obtain a lower-triangular matrix. The last two diagonal entries are:

$$\begin{pmatrix} \sigma \\ \bigoplus_{i=1}^{\lambda-1} -\varepsilon \circ h_{i-1}(x,y)F \end{pmatrix},$$

which is an isomorphism by the claim, and:

$$\begin{pmatrix} F\varepsilon E \\ F\sigma E \\ \bigoplus_{i=0}^{\lambda-1} F(\varepsilon \circ x^i y_1 F)E \end{pmatrix} : FEFE_{\lambda-1}[y] \to FE_{\lambda-1}[y] \oplus F^2E^2_{\lambda-1}[y] \oplus FE_{\lambda-1}[y]^{\oplus\lambda},$$

which is an isomorphism for $\lambda > 0$, and therefore for $\lambda + 1 \geqslant 0$, using Claim 3.4.10 with F applied on the left and E on the right.

When $\lambda = 0$ the matrix of $[\tilde{\rho}_\lambda]_{22}$ is given by removing rows 3, 5–$(\lambda+3)$, and $(\lambda+6)$–

$(2\lambda + 5)$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ y_1 & 0 & \eta & \sigma & 0 \\ 0 & 0 & 0 & 0 & F\varepsilon E \\ 0 & 0 & 0 & 0 & F\sigma E \end{pmatrix}.$$

When $\lambda = 0$ we also have isomorphisms:

$$(\eta, \sigma) : EF_{\lambda-1}[y] \oplus A[y]_{\lambda-1} \xrightarrow{\sim} FE_{\lambda-1}[y]$$

and

$$\left(\begin{smallmatrix} F\varepsilon E \\ F\sigma E \end{smallmatrix} \right) : F\big(FE_{\lambda+1}\big)E[y] \xrightarrow{\sim} F_{\lambda+1}E[y] \oplus F\big(EF_{\lambda+1}\big)E[y],$$

so we see that again the matrix can be written as a lower-triangular matrix with invertible diagonal entries.

Finally, assume $\lambda < 0$. We have an isomorphism:

$$\left(\sigma, \sum_{i=0}^{-\lambda} Fx^i \circ \eta\right) : EF_{\lambda-1}[y] \oplus A_{\lambda-1}[y]^{\oplus -(\lambda-1)} \xrightarrow{\sim} FE_{\lambda-1}[y],$$

which is the isomorphism $\rho_{\lambda-1} \otimes_k k[y]$. There is a final claim to check:

Claim 3.4.16. When $\lambda < 0$, the map

$$\left(\sigma, \eta, \sum_{i=0}^{-\lambda-1} -Fx^i y_1 \circ \eta\right) : EF_{\lambda-1}[y] \oplus A_{\lambda-1}[y]^{\oplus -(\lambda-1)} \to FE_{\lambda-1}[y]$$

is an isomorphism.

Proof. Define an isomorphism $M'_{-y} \in \mathrm{End}_{A_{\lambda-1}[y]}\left(A_{\lambda-1}[y]^{\oplus \lambda-1}\right)$ with components $[M_h]_{ij}$ given by 1 along the diagonal and $-y$ along the subdiagonal. This is a lower-triangular invertible matrix. We write the map in question as a composition of isomorphisms:

$$\left(\sigma, \eta, \sum_{i=0}^{-\lambda-1} -Fx^i y_1 \circ \eta\right) = \left(\sigma, \eta, \sum_{i=1}^{-\lambda} Fx^i \circ \eta\right)$$

$$\circ \begin{pmatrix} \mathrm{Id}_{EF_{\lambda-1}[y]} & 0 & 0 \\ 0 & \mathrm{Id}_{A_{\lambda-1}[y]} & 0 \\ 0 & 0 & -\mathrm{Id}_{A_{\lambda-1}[y]^{\oplus -\lambda}} \end{pmatrix} \circ \begin{pmatrix} \mathrm{Id}_{EF_{\lambda-1}[y]} & 0 \\ 0 & M'_{-y} \end{pmatrix}.$$

□

Now let W be the endomorphism of the codomain of $[\tilde{\rho}_\lambda]_{22}$ given by the invertible matrix:

$$[W] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -y_1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We show that $[W] \cdot [\tilde{\rho}_\lambda]_{22}$ is equivalent to a lower-triangular matrix after giving a suitable permutation of the domain and codomain summands. Let the domain be given in the order:

$$EF_{\lambda-1}[y] \oplus A_{\lambda-1}[y]^{-(\lambda-1)} \oplus FE_{\lambda-1}[y] \oplus FEFE_{\lambda-1}[y] \oplus FE_{\lambda-1}[y]^{-(\lambda+1)} \oplus FE_{\lambda-1}[y]^{\oplus 2},$$

where the change of summand numbers is given by the following chart:

new:	1	2	3	4	...	$-\lambda+2$	$-\lambda+3$	$-\lambda+4$
old:	5	1	6	7	...	$-\lambda+5$	2	4

new:	$-\lambda+5$	$-\lambda+6$	...	$-2\lambda+4$	$-2\lambda+5$	$-2\lambda+6$
old:	$-\lambda+7$	$-\lambda+8$	...	$-2\lambda+5$	$-\lambda+6$	3.

Let the codomain be given in the order:

$$FE_{\lambda-1}[y]^{\oplus 4} \oplus F^2 E^2_{\lambda-1}[y],$$

where the change of summand numbers is given by the following chart:

$$\begin{array}{cccccc} \text{new:} & 1 & 2 & 3 & 4 & 5 \\ \text{old:} & 3 & 4 & 5 & 2 & 1. \end{array}$$

The matrix of $[W] \cdot [\tilde{\rho}_\lambda]_{22}$ for $\lambda < 0$ agrees with that for $[\tilde{\rho}_\lambda]_{22}$ except in the third row, where it is:

$$\begin{pmatrix} \eta & 0 & 0 & 0 & \sigma & \sum_{i=0}^{-\lambda-1} -Fx^i y_1 \circ \eta & 0 & 0 \end{pmatrix}.$$

Writing now the matrix of $[W] \cdot [\tilde{\rho}_\lambda]_{22}$ with columns and rows changed by the above permutations, and compressing the notation for some columns, we obtain:

$$\begin{pmatrix} \left(\sigma, \eta, \sum_{i=0}^{-\lambda-1} -Fx^i y_1 \circ \eta\right) & 0 & (0,0) & 0 & 0 \\ \left(0, 0, \sum_{i=0}^{-\lambda-1} Fx^i \circ \eta\right) & 1 & (0,0) & 0 & 0 \\ (0,0,\Theta') & 0 & \left(F\sigma E, \sum_{i=1}^{-\lambda-1} F^2 h_{i-1}(x_2, y) \circ F\eta E\right) & 0 & 0 \\ \left(0, 0, \sum_{i=0}^{-\lambda-1} -Fh_{i-1}(x,y) \circ \eta\right) & 0 & \left(F\varepsilon E, \sum_{i=1}^{-\lambda-1} y^i\right) & 1 & 0 \\ \left(0, 0, \sum_{i=0}^{-\lambda-1} Fy^i \circ \eta\right) & 0 & \left(0, \sum_{i=1}^{-\lambda-1} y^i y_1\right) & y_1 & 1 \end{pmatrix}.$$

The upper left map is an isomorphism by the Claim proved above. The middle diagonal map is an isomorphism because it is the isomorphism of Claim 3.4.11 with F applied on the left and E on the right. So the matrix is lower-triangular with isomorphisms along the diagonal. □

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