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Impulse Control and Optimal Stopping

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Impulse Control and Optimal Stopping

by

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Abstract

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This thesis analyzes a class of impulse control problems for multi-dimensional jump diffusions in a finite time horizon. Following the basic mathematical setup from Stroock and Varadhan [44], this paper first establishes rigorously an appropriate form of the Dynamic Programming Principle (DPP). It then shows that the value function is a viscosity solution for the associated Hamilton-Jacobi-Belleman (HJB) equation involving integro-differential operators. Finally, it proves the $W^{(2,1),p}_{loc}$ regularity for $2 \leq p < \infty$ of the viscosity solution for HJB with first-order jump diffusions. Furthermore, it proposes a new regularity framework for second-order jump diffusions in the optimal stopping problem.

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I want to thank my advisors for advising me.

Chapter 1

Introduction

Stochastic control is the subject that involves studying the following problem; given a stochastic process with a control input that can influence the trajectory of the process, we would like to find the best control policy, with respect to certain objective. For this thesis, we will focus on the case of the underling process being a solution to the stochastic differential equation, with various modification, such as Poisson jumps. The two problems we will investigate are the impulse control problem and optimal stopping.

The impulse control problem considers the solution of a stochastic differential equation, in which the only control at a given time is an impulse. An impulse simply push the process to a different point at a chosen stopping time. The process therefore would have jump type of discontinuity whenever such an impulse is applied. The optimal stopping problem considers again the solution of a stochastic differential equation. The only control that we can apply is a stopping time.

The typical method of solving the stochastic control involves looking at the optimal payoff, as a function of the starting time and location. Although we do not a priori know the optimal control policy, the theoretical optimal payoff often times satisfies certain non-linear partial differential equation, usually known as the Hamilton-Jacobi-Bellman's equation (HJB). The study of the existence, uniqueness, and regularity of that HJB would give us clues about the optimal control, and in some cases the precise formula of that optimal control.

We have chosen to study both the impulse control problem and optimal stopping problem because they are closely related, in the sense that the impulse control problem can be thought of as an iterated stopping problem, in which every single impulse to the impulse control problem is a stopping for a stopping problem, along with a new stopping problem starting at the location which the impulse pushes the process to. In this thesis, we first tackle the impulse control problem without Poisson jumps. Then, as we add the Poisson jumps into the mix, the HJB becomes much more complicated, and we go back to the corresponding optimal stopping problem to build a new regularity theory for integro-differential equations, which will be suitable for the impulse control problem setting.

Precise Setup This thesis considers the following class of impulse control problem for an n-dimensional diffusion process X_t. In the absence of control, X_t is governed by an Itô's stochastic differential equation

$$\begin{aligned} X_t =& x_0 + \int_{t_0}^t b(X_{s-},s)ds + \int_{t_0}^t \sigma(X_{s-},s)dW_s \\ &+ \int_{t_0}^t \int j_1(X_{s-},s,z)N(dz,dt) + \int_{t_0}^t \int j_2(X_{s-},s,z)\widetilde{N}(dz,dt). \end{aligned}$$

here W is a standard Brownian motion, $\widetilde{N} = N(dt,dz) - \rho(dz)dt$ with N a Poisson point process on $[0,T] \times \mathbb{R}^k$ with density $\rho(dz)dt$, W and N are independent in an appropriate filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and b, σ, j_1, j_2 satisfy suitable regularity conditions to be specified later. If a control policy $V = (\tau_1, \xi_1; \tau_2, \xi_2; \ldots)$ is adopted, then X_t evolves as

$$\begin{aligned} X_t =& x_0 + \int_{t_0}^t b(X_{s-},s)ds + \int_{t_0}^t \sigma(X_{s-},s)dW_s + \sum_i \xi_i 1_{(\tau_i \le t)} \\ &+ \int_{t_0}^t \int j_1(X_{s-},s,z)N(dz,dt) + \int_{t_0}^t \int j_2(X_{s-},s,z)\widetilde{N}(dz,dt). \end{aligned}$$

Here the control $(\tau_i, \xi_i)_i$ is of an impulse type such that $\tau_1, \tau_2, \ldots$ is an increasing sequence of stopping times with respect to $\mathcal{F}_t^{W,N}$, the natural filtration generated by W and N, and ξ_i is an $\mathbb{R}^n$-valued, $\mathcal{F}_{\tau_i}^{W,N}$-measurable random variable.

The objective is to choose an appropriate impulse control $(\tau_i, \xi_i)_i$ so that the following cost function is minimized:

$$J[x_0, t_0, \tau_i, \xi_i] = \mathbb{E}\left[\int_{t_0}^T f(X_t^{x_0,t_0,\tau_i,\xi_i}, t)dt + \sum_i B(\xi_i, \tau_i)1_{\{t_0 \le \tau_i \le T\}} + g(X_T^{x_0,t_0,\tau_i,\xi_i})\right].$$

Here $f : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ is the running cost function, $B : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ is the transaction cost function, and $g : \mathbb{R}^n \times \mathbb{R}$ is the terminal cost function.

Literature Review. Impulse control, in contrast to regular and singular controls, allows the state space to be discontinuous and is a more natural mathematical framework for many applied problems in engineering and economics. Examples in financial mathematics include portfolio management with transaction costs [5, 29, 30, 15, 35, 40], insurance models [26, 9], liquidity risk [33], optimal control of exchange rates [27, 36, 8], and real options [46, 34].

There is a rich literature on stochastic control problems. [19] is one of the earliest references. For impulse control problem, the earliest work is the well-known book by Bensoussan and Lions [4], where value functions of the control problems for diffusions without jumps were shown to be the solutions of quasi-variational-inequalities and where their regularity properties were established when the control is strictly positive and the state space is in a bounded region. It points out a clear connection between the impulse control problem and the optimal stopping problem, and its corresponding variational inequality, which is studied in [3] by the same authors. These books were written before the notion of viscosity solution was invented.

After the development of viscosity solutions, there are renewed interest in stochastic control. The viscosity solution framework fits perfectly into the control problem, both deterministic and stochastic types. For reference on viscosity solution, see [12], [13], and [11]. The book [17] has a proof on the value function of the deterministic control problem being the viscosity solution of the Hamilton-Jacobi PDE. A standard reference on control problems in general and its relation to viscosity solution is [20].

Although the connection between deterministic control and Hamilton-Jacobi equation can be proved without much of difficulty, the stochastic control counterpart, including the optimal stopping, regular/singular/impulse control problems, is highly non-trivial. There are mainly two approaches to this. One approach is to focus on solving for the value function the associated (quasi)-variational inequalities or Hamilton-Jacobian-Bellman (HJB) integro-differential equations, and then establishing the optimality of the solution by a verification type theorem. (See Øksendal and Sulem [39].) Another approach is to characterize the value function of the control problem as a (unique) viscosity solution to the associated PDEs, and/or to study their regularities. The former approach assumes sufficient regularity of the value function (usually C^2), which in some cases are not fully established. The latter approach requires at least certain version of the Dynamic Programming Principle (DPP), which is again often times assumed without explicit proof, or only proved for the class of feedback controls. [20] has a proof of DPP for the regular control problem restricted to the feedback controls. Yong and Zhou [47] contains a more general proof of DPP, although without all the delicate details. Tang and Yong [45] stated the DPP and refers the proof to [21], which contains some but not all the details. On the other hand, Ishikawa [25] established some version of the DPP and the uniqueness of the viscosity solution for diffusions without jumps. More recently, Seydel [43] used a version of the DPP for feedback controls to study the viscosity solution of control problems on jump diffusions.

The regularity of the impulse control problem is built on top of the regularity for the corresponding optimal stopping problem, as the connection is already exploited in [4]. More recent literature tend to adopt the notion of the viscosity solution. Guo and Wu [24] proves the regularity for the elliptic case without jumps. Based on the restrictive setup of [43], Davis, Guo and Wu [14] proves the case with first-order jump diffusions in an infinite time horizon. However, the regularity for the jump case with non-local operator above order one is significantly more difficult. Bayraktar and Xing [2] proved it for the parabolic case, for operator strictly less than 2, under the assumption that the diffusion coefficients are constant in time and space.

The difficulty of the higher order operator lies in the fundamental theory of linear integro-differential equation on bounded domain. The book by Garroni and Menaldi [23] gives a rather comprehensive account for the integro-differential operator, but we find many of the regularity results unsuitable for our purpose, because it imposes conditions on the integral operator near the boundary of a bounded domain, and these assumptions do not make any sense for the general unbounded domain problem. In this thesis, we attempt to develop new regularity theory for the linear integro-differential equation. Then we apply the theory to the optimal stopping problem, which is a the building block for regularity for the impulse control problem. Our approach completely solves the regularity for optimal stopping and impulse control problem in unbounded domain in one dimension, and the regularity for the optimal stopping on bounded domain.

There is also a rich literature on the optimal stopping problem. [3] showed that the value function of the optimal stopping problem satisfies the variational inequality, before the notion of viscosity solution was developed. Another valuable source is [22]. [38] proves that the value function is the viscosity solution to the variational inequality for the elliptic case without jumps on unbounded domain. There are a few that discusses optimal stopping with jump diffusion in particular. Also see [37] and Pham [41]. Pham [42] proved that the value function for the parabolic case on unbounded domain is Holder continuous.

Our Results. This thesis is organized as the following.

- In the second chapter, we follow the classical setup of Stroock and Varadhan [44] and work on the natural filtration of the underlying Brownian motion and the Poisson process, instead of the "usual hypothesis", i.e., the completed right continuous filtration adopted in previous work. Within this framework and based on the estimation techniques developed in Tang and Yong [45] for diffusion processes without jumps, we prove a general form of the DPP.

  We remark that various forms of the DPP for impulse controls of jump diffusions have been exploited quite literally in the stochastic control literature, and their proofs can be found for several cases, yet not with the full generality needed in this thesis. For instance, our result includes those in [45] and [43] as special cases and includes non-Markov controls. Because of the inclusion of the jumps in the diffusion processes and the possibility of non-Markov controls, there are essential mathematical subtlety and difficulties, hence the necessity to adopt the classical and framework of [44]. This framework ensures certain properties of the regular conditional probability, and ensures that the controlled jump diffusions are well defined. These properties are crucial for rigorously establishing the DPP. In a way, our approach to the DPP is in the similar spirit of Yong and Zhou [47] for one-dimensional regular controls. For more generality, we have added regular control into our problem, although it is not needed in other chapters. We also want to point out that the proof can easily be modified for the optimal stopping problem.

  Note that there are separate lines of research on the DPP, including the weak DPP formulation by Bouchard and Touzi [7] and Bouchard and Nutz [6], as well as the classical work by El Kaouri [16]. However, it does not seem easy for us to fit their results to our problem and setup.

- In the third chapter, we show that the value function is a viscosity solution in the sense of [1]. This form of viscosity solution is convenient for the HJB equations involving integro-differential operators, which is the key for analyzing control problems on jump diffusions. Note that [1] contains a very general uniqueness proof of the viscosity solution.

  Closely related to our work in this aspect are the works of [43] and [45]. The former allowed only Markov controls and the latter did not deal with jump diffusions.

- In the fourth chapter, we prove the $W_{loc}^{(2,1),p}$ regularity for the value function with first-order jumps. We will also provide a proof of local uniqueness of the viscosity solution, which is appropriate to study the regularity property. We also include a further extension, $W_{loc}^{(2,1),\infty}$, with some rather mild additional assumptions. Some of the methods here are proceeded by [22].

  Compared to [14] for an infinite horizon problem, this thesis is on a finite time horizon which requires different PDEs techniques. Moreover, [14] did not study the DPP, nor the uniqueness of the viscosity solution, and was restricted to Markov controls. Thus it built partial results in a restrictive setting.

- In the last chapter, we attempt to develop a new regularity theory for linear elliptic integro-differential equations. Along with Lenhart [31] thesis providing us the L^p-estimates for the Dirichlet problem, we prove the regularity for the elliptic optimal stopping problem on bounded domain for second-order non-local operators. Our result can be extended to the parabolic equation, and ultimately the impulse control problem on unbounded domain in one space dimension.

Chapter 2

Dynamic programming principle and viscosity solutions

2.1 Problem formulation and main results

Filtration Fix a time $T > 0$. For each $t_0 \in [0, T]$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space that supports a Brownian motion $\{W_s\}_{t_0 \leq s \leq T}$ starting at t_0, and an independent Poisson point process $N(dt, dz)$ on $([t_0, T], \mathbb{R}^k \setminus \{0\})$ with intensity $L \otimes \rho$. Here L is the Lebesgue measure on $[t_0, T]$ and ρ is a measure defined on $\mathbb{R}^k \setminus \{0\}$. For each $t \in [t_0, T]$, define $\{\mathcal{F}_t^{W,N}\}_{t_0 \leq t \leq T}$ to be the natural filtration of the Brownian motion W and the Poisson process N, define $\{\mathcal{F}_{t_0,t}[t_0, T]\}$ to be $\{\mathcal{F}_t^{W,N}\}_{t_0 \leq t \leq T}$ restricted to the interval $[t_0, t]$.

Throughout the paper, we will use this uncompleted natural filtration $\{\mathcal{F}_{t_0,t}[t_0, T]\}$. This specification ensures that the stochastic integration and therefore the controlled jump diffusion to be well defined. (See Lemma 4.3.3 from Stroock & Varadhan [44]).

Now, we can define mathematically the mixed control problem, starting with the set of admissible controls.

Definition 1. *The set of admissible regular control* $\mathcal{U}[t_0, T]$ *consists of all previsible process* $u : \Omega \times [t_0, T] \to U$ *with respect to the filtration* $\{\mathcal{F}_{t_0,s}^{W,N}\}_{t_0 \leq s \leq T}$ *for some separable metric space* U*.*

Definition 2. *The set of admissible impulse control* $\mathcal{V}[t_0, T]$ *consists of pairs of sequences* $\{\tau_i, \xi_i\}_{1 \leq i < \infty}$ *such that*

1. $\tau_i : \Omega \to [t_0, T] \cup \{\infty\}$ *such that* τ_i *are stopping times with respect to the filtration* $\{\mathcal{F}_{t_0,s}^{W,N}\}_{t_0 \leq s \leq T}$*,*

2. $\tau_i \leq \tau_{i+1}$ *for all* i*,*

3. $\xi_i : \Omega \to \mathbb{R}^n \setminus \{0\}$ *is a random variable such that* $\xi_i \in \mathcal{F}_{t_0,\tau_i}^{W,N}$*.*

Now, given an admissible impulse control $\{\tau_i, \xi_i\}_{1\le i<\infty}$, a stochastic process $(X_t)_{t\ge 0}$ follows a stochastic differential equation with jumps,

$$\begin{aligned} X_t =& x_0 + \int_{t_0}^t b(X_{s-}, s, u_s)ds + \int_{t_0}^t \sigma(X_{s-}, s, u_s)dW_s + \sum_i \xi_i 1_{(\tau_i \le t)} \\ &+ \int_{t_0}^t \int j_1(X_{s-}, s, u_s, z)N(dz, dt) + \int_{t_0}^t \int j_2(X_{s-}, s, u_s, z)\widetilde{N}(dz, dt). \end{aligned} \tag{2.1}$$

Here $\widetilde{N} = N(dt, dz) - \rho(dz)dt$, $b : \mathbb{R}^n \times [0, T] \times \mathbb{R}^n$, $\sigma : \mathbb{R}^n \times [0, T] \to \mathbb{R}^{n\times m}$, and $j_1, j_2 : \mathbb{R}^n \times [0, T] \times \mathbb{R}^k \to \mathbb{R}^n$. For each $(\tau_i, \xi_i)_i \in \mathcal{V}[t_0, T]$, $u. \in \mathcal{U}[t_0, T]$ and $(x_0, t_0) \in \mathbb{R}^n \times [t_0, T]$, denote $X = X^{t_0,x_0,u.,\tau_i,\xi_i}$.

The stochastic control problem is to

$$\text{(Problem)} \quad \text{Minimize} \quad J[x_0, t_0, u., \tau_i, \xi_i] \quad \text{over all } (\tau_i, \xi_i) \in \mathcal{V}[t, T],\ u. \in \mathcal{U}[t_0, T], \tag{2.2}$$

subject to Eqn. (2.1) with

$$J[x_0, t_0, u., \tau_i, \xi_i] = \mathbb{E}\left[\int_{t_0}^T f(s, X_s^{t_0,x_0,u.,\tau_i,\xi_i})ds + g(X_T^{t_0,x_0,u.,\tau_i,\xi_i})\right] + \mathbb{E}\left[\sum_i B(\xi_i, \tau_i)1_{\{t_0\le\tau_i\le T\}}\right]. \tag{2.3}$$

Here we denote V for the associated value function

$$\text{(Value Function)} \quad V(x, t) = \inf_{(\tau_i,\xi_i)\in\mathcal{V}[t,T]} J[x, t, u., \tau_i, \xi_i]. \tag{2.4}$$

In order for J and V to be well defined, and for the Brownian motion W and the Poisson process N as well as the controlled jump process $X^{x_0,t_0,u.\tau_i,\xi_i}$ to be unique at least in a distribution sense, we shall specify some assumptions in Section 2.2.

The focus of the paper is to analyze the following HJB equation associated with the value function

$$\text{(HJB)} \quad \begin{cases} \max\{-u_t + Lu - f - Iu, u - Mu\} = 0 & \text{in } \mathbb{R}^n \times (0, T), \\ u = g & \text{on } \mathbb{R}^n \times \{t = T\}. \end{cases}$$

Here

$$\begin{aligned} I\phi(x, t) &= \int \phi(x + j_1(x, t, z), t) - \phi(x, t)\rho(dz) \\ &+ \int \phi(x + j_2(x, t, z), t) - \phi(x, t) - D\phi(x, t) \cdot j_2(x, t, z)\rho(dz), \end{aligned} \tag{2.5}$$

$$Mu(x, t) = \inf_{\xi\in\mathbb{R}^n}(u(x + \xi, t) + B(\xi, t)), \tag{2.6}$$

$$Lu(x, t) = -tr\left[A(x, t) \cdot D^2u(x, t)\right] - b(x, t) \cdot Du(x, t) + ru(x, t). \tag{2.7}$$

Main result. Our main result states that the value function $V(x,t)$ is a unique $W^{(2,1),p}_{loc}(\mathbb{R}^n \times (0,T))$ viscosity solution to the (HJB) equation with $2 \le p < \infty$. In particular, for each $t \in [0,T)$, $V(\cdot,t) \in C^{1,\gamma}_{loc}(\mathbb{R}^n)$ for any $0 < \gamma < 1$.

The main result is established in three steps.

- First, in order to connect the (HJB) equation with the value function, we prove an appropriate form of the DPP. (Theorem 1).
- Then, we show that the value function is a continuous viscosity solution to the (HJB) equation in the sense of [1]. (Theorem 2).
- Finally, we show that the value function is $W^{(2,1),p}_{loc}$ for $2 \le p < \infty$, and in fact a unique viscosity solution to the (HJB) equation. (Theorem 8).

  All results, unless otherwise specified, are built under the assumptions specified in Section 2.2.

2.2 Standing assumptions

Assumption 1. *Given $t_0 \le T$, assume that*

$$\begin{aligned}(\Omega_{t_0,T}, \mathcal{F}, \{\mathcal{F}_{t_0,t}[t_0,T]\}_{t_0\le t\le T}) =&(C[t_0,T]\times M[t_0,T],\\ &\mathcal{B}_{t_0,T}[t_0,T]\otimes \mathcal{M}_{t_0,T}[t_0,T],\\ &\{\mathcal{B}_{t_0,t}[t_0,T]\otimes \mathcal{M}_{t_0,t}[t_0,T]\}_{t_0\le t\le T})\end{aligned}$$

such that the projection map $(W,N)(x.,n) = (x.,n)$ is the Brownian motion and the Poisson point process with density $\rho(dz)\times dt$ under $\mathbb{P}$, and for $t_0 \le s \le t \le T$,

$$\begin{aligned}C[t_0,T] =&\{x. : [t_0,T]\to\mathbb{R}^n, x_{t_0} = 0\},\\ M[t_0,T] =&\textit{the class of locally finite measures on } [t_0,T]\times\mathbb{R}^k\setminus\{0\},\\ \mathcal{B}_{s,t}[t_0,T] =&\sigma(\{x_r : x. \in C[t_0,T], s\le r\le t\}),\\ \mathcal{M}_{s,t}[t_0,T] =&\sigma(\{n(B) : B\in\mathcal{B}([s,t]\times\mathbb{R}^k\setminus\{0\}), n\in M[t_0,T]\}).\end{aligned}$$

Assumption 2. *(Lipschitz Continuity.) The functions b, σ, and j are deterministic measurable functions such that there exists constant C independent of $t\in[t_0,T]$, $z\in\mathbb{R}^k\setminus\{0\}$, $u\in U$, such that*

$$\begin{aligned}|b(x,t,u)-b(y,t,u)| &\le C|x-y|,\\ |\sigma(x,t,u)-\sigma(y,t,u)| &\le C|x-y|,\\ \int_{|z|\ge 1}|j_1(x,t,u,z)-j_1(x,t,u,z)|\rho(dz) &\le C|x-y|,\\ \int_{|z|<1}|j_2(x,t,u,z)-j_2(x,t,u,z)|^2\rho(dz) &\le C|x-y|^2.\end{aligned}$$

Assumption 3. *(Growth Condition.) There exists constant $C > 0$, $\nu \in [0,1)$, such that for any $x, y \in \mathbb{R}^n$,*

$$\begin{aligned} |b(t,x,u)| \leq & L(1+|x|^{\nu}), \\ |\sigma(t,x,u)| \leq & L(1+|x|^{\nu/2}), \\ \int_{|z|\geq 1} |j_1(x,t,u,z)|\rho(dz) \leq & C(1+|x|^{\nu}), \\ \int_{|z|<1} |j_2(x,t,u,z)|^2\rho(dz) \leq & C(1+|x|^{\nu}). \end{aligned}$$

Assumption 4. *(Hölder Continuity.) $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are measurable functions such that there exists $C > 0$, $\delta \in (0,1]$, $\gamma \in (0,\infty)$ such that*

$$\begin{aligned} |f(t,x,u) - f(t,\hat{x},u)| &\leq C(1+|x|^{\gamma}+|\hat{x}|^{\gamma})|x-\hat{x}|^{\delta}, \\ |g(x) - g(\hat{x})| &\leq C(1+|x|^{\gamma}+|\hat{x}|^{\gamma})|x-\hat{x}|^{\delta}, \end{aligned}$$

for all $t \in [0,T]$, $x, \hat{x} \in \mathbb{R}^n$, $uinU$.

Assumption 5. *(Lower Boundedness) There exists an $L > 0$ and $\mu \in (0,1]$ such that*

$$\begin{aligned} f(t,x,u) \geq & -L, \\ h(x) \geq & -L, \\ B(\xi,t) \geq & L + C|\xi|^{\mu}, \end{aligned}$$

for all $t \in [0,T]$, $x \in \mathbb{R}^n$, $u \in U$, $\xi \in \mathbb{R}^n$.

Assumption 6. *(Monotonicity and Subadditivity) $B : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ is a continuous function such that for any $0 \leq s \leq t \leq T$, $B(t,\xi) \leq B(s,\xi)$, and for $(t,\xi), (t,\hat{\xi})$ being in a fixed compact subset of $\mathbb{R}^n \times [0,T)$, there exists constant $K > 0$ such that*

$$B(t,\xi+\hat{\xi}) + K \leq B(t,\xi) + B(t,\hat{\xi}).$$

Assumption 7. *(Dominance) The growth of B exceeds the growth of the cost functions f and g so that*

$$\begin{aligned} \delta + \gamma < & \mu, \\ \nu \leq & \mu. \end{aligned}$$

Assumption 8. *(No Terminal Impulse) For any $x, \xi \in \mathbb{R}^n$,*

$$g(x) \leq \inf_{\xi} g(x+\xi) + B(\xi,T).$$

Assumption 9. *Suppose that there exists a measurable map $M : \mathbb{R}^n \times [0,T] \to M(\mathbb{R}^n \setminus \{0\})$, in which $M(\mathbb{R}^n \setminus \{0\})$ is the set of locally finite measure on $\mathbb{R}^n \setminus \{0\}$, such that one has the following representation of the integro operator:*

$$I\phi(x,t) = \int [\phi(x+z,t) - \phi(x,t) - D\phi(x,t) \cdot z 1_{|z|\leq 1}] M(x,t,dz).$$

Also, assume that for (x,t) in some compact subset of $\mathbb{R}^n \times [0,T]$, there exists C such that

$$\int_{|z|<1} |z|^2 M(x,t,dz) + \int_{|z|\geq 1} |z|^{\gamma+\delta} M(x,t,dz) \leq C.$$

Notations Throughout the paper, unless otherwise specified, we will use the following notations.

- $0 < \alpha \leq 1$.
- $$A(x,t) = (a_{ij})_{n\times n}(x,t) = \frac{1}{2}\sigma(x,t)\sigma(x,t)^T.$$
- $\Xi(x,t)$ is the set of points ξ for which MV achieves the value, i.e.,
$$\Xi(x,t) = \{\xi \in \mathbb{R}^n : MV(x,t) = V(x+\xi,t) + B(\xi,t)\}.$$
- The continuation region $\mathcal{C}$ and the action region $\mathcal{A}$ are
$$\mathcal{C} := \{(x,t) \in \mathbb{R}^n \times [0,T] : V(x,t) < MV(x,t)\}, \tag{2.8}$$
$$\mathcal{A} := \{(x,t) \in \mathbb{R}^n \times [0,T] : V(x,t) = MV(x,t)\}. \tag{2.9}$$
- Let Ω be a bounded open set in $\mathbb{R}^{n+1}$. Denote $\partial_P\Omega$ to be the parabolic boundary of Ω, which is the set of points $(x_0,t_0) \in \overline{\Omega}$ such that for all $R > 0$, $Q(x_0,t_0;R) \nsubseteq \overline{\Omega}$. Here $Q(x_0,t_0;R) = \{(x,t) \in \mathbb{R}^{n+1} : \max\{|x-x_0|, |t-t_0|^{1/2}\} < R, t < t_0\}$.

  Note that $\overline{\Omega}$ is the closure of the open set Ω in $\mathbb{R}^{n+1}$. In the special case of a cylinder, $\Omega = Q(x_0,t_0;R)$, the parabolic boundary $\partial_P\Omega = (B(0,R) \times \{t=0\}) \cup (\{|x| = R\} \times [0,T])$.

- Function spaces for Ω being a bounded open set,

$$
\begin{aligned}
&W^{(1,0),p}(\Omega) = \{u \in L^p(\Omega) : u_{x_i} \in L^p(\Omega)\},\\
&W^{(2,1),p}(\Omega) = \{u \in W^{(1,0),p}(\Omega) : u_{x_i x_j} \in L^p(\Omega)\},\\
&C^{2,1}(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : u_t, u_{x_i x_j} \in C(\overline{\Omega})\},\\
&C^{0+\alpha,0+\frac{\alpha}{2}}(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : \sup_{(x,t),(y,s)\in\Omega,(x,t)\neq(y,s)} \frac{|u(x,t)-u(y,s)|}{(|x-y|^2+|t-s|)^{\alpha/2}} < \infty\},\\
&C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : u_{x_i x_j}, u_t \in C^{0+\alpha,0+\frac{\alpha}{2}}(\overline{\Omega})\},\\
&L^p_{loc}(\Omega) = \{u|_U \in L^p(U) \ \forall \text{open } U \text{ such that } \overline{U} \subset \overline{\Omega} \setminus \partial_P \Omega\},\\
&W^{(1,0),p}_{loc}(\Omega) = \{u \in L^p_{loc}(\Omega) : u \in W^{(1,0),p}(U) \ \forall \text{open } U \text{ such that } \overline{U} \subset \overline{\Omega} \setminus \partial_P \Omega\},\\
&W^{(2,1),p}_{loc}(\Omega) = \{u \in L^p_{loc}(\Omega) : u \in W^{(2,1),p}(U) \ \forall \text{open } U \text{ such that } \overline{U} \subset \overline{\Omega} \setminus \partial_P \Omega\}.
\end{aligned}
$$

2.3 Dynamic programming principle

Theorem 1. *(Dynamic Programming Principle) Assuming (A1), (A2), (A3), (A4), and (A5). For $t_0 \in [0,T]$, $x_0 \in \mathbb{R}^n$, let τ be a stopping time on $(\Omega_{t_0,T}, \{\mathcal{F}_{t_0,t}\}_{t\leq T})$, we have*

$$
\begin{aligned}
V(t_0,x_0) = &\inf_{\substack{u.\in\mathcal{U}[t_0,T]\\(\tau_i,\xi_i)\in\mathcal{V}[t_0,T]}} \mathbb{E}\left[\int_{t_0}^{\tau\wedge T} f(s, X_s^{t_0,x_0,u.,\tau_i,\xi_i})ds\right]\\
&+\mathbb{E}\left[\sum_i B(\xi_i,\tau_i)1_{\tau_i\leq\tau\wedge T} + V(\tau\wedge T, X_{\tau\wedge T}^{t_0,x_0,u.,\tau_i,\xi_i})\right]. \qquad (2.10)
\end{aligned}
$$

In order to establish the DPP, the first key issue is: given a stopping time τ, understand how the martingale property and the stochastic integral change under the regular conditional probability distribution $(\mathbb{P}|\mathcal{F}_\tau)$. The next key issue is the continuity of the value function, which will ensure that a countable selection is adequate without the abstract measurable selection theorem. (See [18]).

To start, let us first introduce a new function that connects two Brownian paths which start from the origin at different times into a single Brownian path. This function also combines two Poisson measures on different intervals into a single Poisson measure.

Definition 3. *For each $t \in [t_0,T]$, define a map $\Pi^t = (\Pi^t_1, \Pi^t_2) : C[t_0,T] \times M[t_0,T] \to C[t_0,t] \times M[t_0,t] \times C[t,T] \times M[t,T]$ such that*

$$
\begin{aligned}
\Pi^t_1(x.,n) =&(x|_{[t_0,t]}, n|_{[t_0,t]\times\mathbb{R}^k\setminus\{0\}}),\\
\Pi^t_2(x.,n) =&(x|_{[t,T]} - x_t, n|_{(t,T]\times\mathbb{R}^k\setminus\{0\}}).
\end{aligned}
$$

Note that this is an $\mathcal{F}_{t_0,T}[t_0,T]/\mathcal{F}_{t_0,t}[t_0,t] \otimes \mathcal{F}_{t,T}[t,T]$-measurable bijection. Therefore, for fixed $(y.,m) \in C[t_0,T] \times M[t_0,T]$, the map from $C[t,T] \times M[t,T] \to C[t_0,T] \times M[t_0,T]$

defined by

$$\begin{aligned}(x_\cdot, n) \mapsto &(\Pi^t)^{-1}(\Pi_1^t(y_\cdot, m), \Pi_2^t(x_\cdot, n)) \\ &= (x_{\cdot\vee t} - x_t + y_{\cdot\wedge t}, m|_{[t_0,t]\times\mathbb{R}^k\setminus\{0\}} + n|_{(t,T]\times\mathbb{R}^k\setminus\{0\}})\end{aligned}$$

is $\mathcal{F}_{t,s}[t,T]/\mathcal{F}_{t_0,s}[t_0,T]$-measurable for each $s \in [t,T]$.

Next, we need two technical lemmas regarding $(\mathbb{P}|\mathcal{F}_\tau)$. Specifically, the first lemma states that the local martingale property is preserved, and the second one ensures that the stochastic integration is well defined under $(\mathbb{P}|\mathcal{F}_\tau)$.

According to Theorem 1.2.10 of [44],

Lemma 1. *Given a filtered space, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$, and an associated martingale $\{M_t\}_{0\le t\le T}$. Let τ be an $\mathcal{F}$-stopping time. Assume $(\mathbb{P}|\mathcal{F}_\tau)$ exists. Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$, $N_t = M_t - M_{t\wedge\tau}$ is a local martingale under $(\mathbb{P}|\mathcal{F}_\tau)(\omega,\cdot)$.*

Lemma 2. *Given a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$, a stopping time τ, a previsible process $H : (0,T] \times \Omega \to \mathbb{R}^n$, a local martingale $M : [0,T] \times \Omega \to \mathbb{R}^n$ such that*

$$\int_\tau^T |H_s|^2 d[M]_s < \infty$$

$\mathbb{P}$-almost surely, and $N_t = \int_\tau^t H_s dM_s$ (a version of the stochastic integral that is right-continuous on all paths). Assume that $(\mathbb{P}|\mathcal{F}_\tau)$ exists. Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$, N_t is also the stochastic integral $\int_\tau^t H_s dM_s$ under the new probability measure $(\mathbb{P}|\mathcal{G})(\omega,\cdot)$.

Proof. First assume that $\int_\tau^T H^2 d[M]$ is bounded and M is a L^2-martingale. The conclusion is clearly true when H is an elementary previsible process of the form $\sum_i^m Z_i 1_{(S_i,T_i]}$. Now let $H^{(n)}$ be a sequence of such elementary previsible process that converges to H uniformly on $\Omega \times (0,T]$. Let $N_t^{(n)} = \int_\tau^t H_s^{(n)} dM_s$, and $N_t = \int_\tau^t H_s dM_s$.

First we show that quadratic variation is preserved under regular conditional probability distribution. By Lemma 1, $Q_t = M_t - M_{t\wedge\tau}$ is a martingale. Consider the quadratic variation $[Q]$ under $\mathbb{P}$. By definition, $Q_t^2 - [Q]_t$ is a martingale. Thus by Lemma 1, for almost every ω, $Q_t^2 - [Q]_t - (Q_{t\wedge\tau}^2 - [Q]_{t\wedge\tau})$ is martingale under $(\mathbb{P}|\mathcal{F}_\tau)(\omega)$.

$$\begin{aligned}&Q_t^2 - [Q]_t - (Q_{t\wedge\tau}^2 - [Q]_{t\wedge\tau}) \\ =&(M_t - M_{t\wedge\tau})^2 - ([M]_t - [M]_{t\wedge\tau}) \\ &- \big((M_{t\wedge\tau} - M_{t\wedge\tau})^2 - ([M]_{t\wedge\tau} - [M]_{t\wedge\tau})\big) \\ =&Q_t^2 - ([M]_t - [M]_{t\wedge\tau})\end{aligned}$$

This shows that under $(\mathbb{P}|\mathcal{F}_\tau)(\omega)$, $[Q]_t^{(\mathbb{P}|\mathcal{F}_\tau)(\omega)} = [M]_t - [M]_{t\wedge\tau}$. This allows us to simply write $[\cdot]$ instead of $[\cdot]^{(\mathbb{P}|\mathcal{F}_\tau)(\omega)}$. Hence,

$$\int_\tau^T H^2 d[M] = \int_\tau^T H^2 d[N]$$

is bounded under $(\mathbb{P}|\mathcal{F}_\tau)(\omega)$.

By the definition of $H^{(n)}$,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[\liminf_n \mathbb{E}^{(\mathbb{P}|\mathcal{F}_\tau)(\omega)}\left[\int_\tau^T |H_s^{(n)} - H_s|^2 d[Q]_s\right]\right] \\
\leq &\liminf_n \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{(\mathbb{P}|\mathcal{F}_\tau)(\omega)}\left[\int_\tau^T |H_s^{(n)} - H_s|^2 d[Q]_s\right]\right] \\
= &\liminf_n \mathbb{E}^{\mathbb{P}}\left[\int_\tau^T |H_s^{(n)} - H_s|^2 d[M]_s\right]^2 = 0.
\end{aligned}$$

Thus, $\liminf_n \mathbb{E}^{(\mathbb{P}|\mathcal{F}_\tau)(\omega)}\left[\int_\tau^T |H_s^{(n)} - H_s|^2 d[Q]_s\right] = 0$ for almost every ω. On the other hand,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[\liminf_n \mathbb{E}^{(\mathbb{P}|\mathcal{G})(\omega)}\left[N_T^n - N_T\right]^2\right] \\
\leq &\liminf_n \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{(\mathbb{P}|\mathcal{G})(\omega)}\left[N_T^n - N_T\right]^2\right] \\
\leq &\liminf_n \mathbb{E}^{\mathbb{P}}\left[N_T^n - N_T\right]^2 = 0.
\end{aligned}$$

So we have

$$\liminf_n \mathbb{E}^{(\mathbb{P}|\mathcal{G})(\omega)}\left[(N^n - N)_T\right]^2 = 0,$$

for $\mathbb{P}$-a.e. ω. This proves the claim. The general case follows from the localization technique.

□

Now, we establish the first step of the Dynamic Programming Principle.

Proposition 1. *Let τ be a stopping time defined on some setup $(\Omega, \{\mathcal{F}_{t_0,s}\})$. For any impulse control $(\tau_i, \xi_i) \in \mathcal{V}[t_0, T]$,*

$$\begin{aligned}
J[t_0, x_0, u_\cdot, \tau_i, \xi_i] = &\mathbb{E}\left[\int_{t_0}^{\tau\wedge T} f(s, X_s^{t_0,x_0,u_\cdot,\tau_i,\xi_i}, u_s) ds + \sum_i B(\xi_i, \tau_i) 1_{\tau_i < \tau\wedge T}\right] \\
&+ \mathbb{E}\left[J[\tau\wedge T, X_{\tau\wedge T-}^{t_0,x_0,u_\cdot,\tau_i^\omega,\xi_i^\omega}, u_\cdot^\omega, \tau_i^\omega, \xi_i^\omega]\right]. \qquad (2.11)
\end{aligned}$$

Here $\tau_i^\omega, \xi_i^\omega$ are defined as follows. For $t \in [t_0, T]$, for each $(y_\cdot, m) \in C[t_0, T] \times M([t_0, T] \times \mathbb{R}^k \setminus \{0\})$,

$$\begin{aligned}
u_\cdot^{t,y_\cdot,m}(x_\cdot, n) = &u_\cdot((\Pi^t)^{-1}(\Pi_1^t(y_\cdot, m), \Pi_2^t(x_\cdot, n))), \\
\tau_i^{y_\cdot,m}(x_\cdot, n) = &\tau_i((\Pi^t)^{-1}(\Pi_1^t(y_\cdot, m), \Pi_2^t(x_\cdot, n))), \\
\xi_i^{y_\cdot,m}(x_\cdot, n) = &\xi_i((\Pi^t)^{-1}(\Pi_1^t(y_\cdot, m), \Pi_2^t(x_\cdot, n))).
\end{aligned}$$

And for each ω*,*

$$\begin{aligned} u_\cdot^\omega &= u^{\tau(\omega),W_\cdot(\omega),N(\omega)} \\ \tau_i^\omega &= \tau_i^{\tau(\omega),W_\cdot(\omega),N(\omega)}, \\ \xi_i^\omega &= \xi_i^{\tau(\omega),W_\cdot(\omega),N(\omega)}. \end{aligned}$$

Proof. Consider $(\mathbb{P}|\mathcal{F}_{t_0,\tau})$ on $(\Omega_{t_0,t}, \{\mathcal{F}_{t_0,t}\})$. Since we are working with canonical spaces, the sample space is in fact a Polish space (see [28] Theorem A2.1 and A2.3), and the regular conditional probability exists by Theorem 6.3 of [28]. Since Polish spaces are completely separable metric spaces and have countably generated σ-algebra, $\mathcal{F}_{t_0,\tau}$ is countably generated. By Lemma 1.3.4 from Stroock & Varadhan [44], there exists some null set N_0 such that if $(x_\cdot, n) \notin N_0$, then

$$(\mathbb{P}|\mathcal{F}_{t_0,\tau})((x_\cdot, n), \{(y_\cdot, m) : \Pi_1^{\tau(x_\cdot,n)}(y_\cdot, m) = \Pi_1^{\tau(x_\cdot,n)}(x_\cdot, n)\}) = 1.$$

Therefore, for $\omega = (x_\cdot, n) \notin N_0$, $\tau_i = \tau_i^\omega$, and $\xi_i = \xi_i^\omega$ almost surely.

Moreover, by Lemma 2, the stochastic integrals are preserved. Therefore, for $\omega \notin N_0$, the solution to Eq. (2.1) remains a solution to the same equation on the interval $[\tau(\omega), T]$ with $(\tau_i^\omega, \xi_i^\omega) \in \mathcal{V}[\tau(\omega), T]$. So $X^{t_0,x_0,u_\cdot,\tau_i,\xi_i}$ on the interval $[\tau(\omega), T]$ has the same distribution as $X^{\tau(\omega),y,\tau_i^\omega,\xi_i^\omega}$ for $y = X_{\tau(\omega)}^{t_0,x_0,u_\cdot,\tau_i,\xi_i}(\omega)$ under $(\mathbb{P}|\mathcal{F}_{t_0,\tau})(\omega, \cdot)$ for $\omega \notin N_0$. □

Now, to obtain the Dynamic Programming Principle, one needs to take the infimum on both sides of Eq. (2.11). The part of "$\leq$" is immediate, but the opposite direction is more delicate. At the stopping time τ, for each ω, one needs to choose a good control so that the cost J is close to the optimal V. To do this, one needs to show that the functional J is continuous in some sense, and therefore a countable selection is adequate.

The following result, the Hölder continuity of the value function, is essentially Theorem 3.1 of Tang & Yong [45]. The major difference is that their work is for diffusions without jumps, therefore some modification in terms of estimation and adaptedness are needed, as outlined in the proof.

Lemma 3. *There exists constant* $C > 0$ *such that for all* $t, \hat{t} \in [0, T]$*,* $x, \hat{x} \in \mathbb{R}^n$*,*

$$\begin{aligned} -C(T+1) \leq & V(t,x) \leq C(1 + |x|^{\gamma+\delta}), \\ |V(t,x) - V(\hat{t},\hat{x})| \leq & C[(1 + |x|^\mu + |\hat{x}|^\mu)|t - \hat{t}|^{\delta/2} + (1 + |x|^\gamma + |\hat{x}|^\gamma)|x - \hat{x}|^\delta]. \end{aligned}$$

Proof. To include the jump terms, it suffices to note the following inequalities,

$$\begin{aligned} \mathbb{E}\left|\int_{t_0}^t \int j_1(s, X_s, u_s, z) N(dz, ds)\right|^\beta \leq & \mathbb{E}\left(\int_{t_0}^t \int |j_1(s, X_s, u_s, z)| \rho(dz) ds\right)^\beta, \\ \mathbb{E}\left|\int_{t_0}^t \int j_2(s, X_s, u_s, z) \widetilde{N}(dz, ds)\right|^\beta \leq & \mathbb{E}\left(\int_{t_0}^t \int |j_2(s, X_s, u_s, z)|^2 \rho(dz) ds\right)^{\beta/2}. \end{aligned}$$

Moreover, in our framework, $\bar{\xi}(\cdot)$ and $\hat{\xi}(\cdot)$ would not be in $\mathcal{V}[\hat{t},T]$ because it is adapted to the filtration $\{\mathcal{F}^{W,N}_{t,s}\}_{t\leq s\leq T}$ instead of $\{\mathcal{F}^{W,N}_{\hat{t},s}\}_{\hat{t}\leq s\leq T}$. To fix this, consider for each $\omega\in\Omega_{t_0,T}$,

$$\begin{aligned}\bar{\xi}^{\omega}(\cdot) =&\bar{\xi}((\Pi^{\hat{t}})^{-1}(\Pi^{\hat{t}}_1(\omega),\Pi^{\hat{t}}_2(\cdot))),\\ \hat{\xi}^{\omega}(\cdot) =&\hat{\xi}((\Pi^{\hat{t}})^{-1}(\Pi^{\hat{t}}_1(\omega),\Pi^{\hat{t}}_2(\cdot))),\end{aligned}$$

and consequently use $\mathbb{E}\left[J[\hat{t},x,\bar{\xi}^{\omega}]\right]$ instead of $J[\hat{t},x,u.,\bar{\xi}]$. □

Given that the value function V is continuous, we can prove Theorem 1.

Proof. (Dynamic Programming Principle) Without loss of generality, assume that $\tau\leq T$.

$$\begin{aligned}J[t_0,x_0,u.,\tau_i,\xi_i] =&\mathbb{E}\left[\int_{t_0}^{\tau} f(s,X_s^{t_0,x_0,u.,\tau_i,\xi_i})ds+\sum_i B(\tau_i,\xi_i)1_{\tau_i<\tau}\right]\\ &+\mathbb{E}\left[J[\tau,X_{\tau-}^{t_0,x_0,u.,u^{\omega}_{.},\tau_i^{\omega},\xi_i^{\omega}},u^{\omega}_{.},\tau_i^{\omega},\xi_i^{\omega}]\right]\\ \geq&\mathbb{E}\left[\int_{t_0}^{\tau} f(s,X_s^{t_0,x_0,u.,\tau_i,\xi_i},u_s)ds+\sum_i B(\tau_i,\xi_i)1_{\tau_i<\tau}\right]\\ &+\mathbb{E}\left[V(\tau-,X_{\tau-}^{t_0,x_0,u.,\tau_i,\xi_i})\right].\end{aligned}$$

Taking infimum on both sides, we get

$$V(t_0,x_0)\geq \inf_{(\tau_i,\xi_i)\in\mathcal{V}_{t_0}}\mathbb{E}\left[\int_{t_0}^{\tau} f(s,X_s^{t_0,x_0,u.,\tau_i,\xi_i},u_s)ds+\sum_i B(\tau_i,\xi_i)1_{\tau_i\leq\tau}+V(\tau,X_{\tau}^{t_0,x_0,u.,\tau_i,\xi_i})\right].$$

Now we are to prove the reverse direction for the above inequality. Fix $\epsilon>0$. Divide $\mathbb{R}^n\times[t_0,T)$ into rectangles $\{R_j\times[s_j,t_j)\}$ disjoint up to boundaries, such that for any $x,\hat{x}\in R_j$ and $t,\hat{t}\in[s_j,t_j)$,

$$\begin{aligned}|V(x,t)-V(\hat{x},\hat{t})| <&\epsilon,\\ |t_j-s_j| <&\epsilon,\\ diam(R_j) <&\epsilon.\end{aligned}$$

For each R_j, pick $x_j\in R_j$. For each (x_j,t_j), choose $u^j\in\mathcal{U}[t_j,T]$, $(\tau_k^j,\xi_k^j)\in\mathcal{V}[t_j,T]$, such that $V(x_j,t_j)+\epsilon>J[t_j,x_j,\tau_i^j,\xi_i^j]$. Let

$$A_j=\{(X_{\tau}^{t_0,x_0,u.,\tau_i,\xi_i},\tau)\in R_j\times[s_j,t_j)\}.$$

And, $A_0=\{\tau=T\}$. Note that $\{A_j\}_j$ partitions the sample space $C[t_0,T]\times M[t_0,T]$. Define a new stopping time $\hat{\tau}$ by:

$$\hat{\tau}=t_j \quad\text{on } A_j, j>0$$

and $\hat{\tau} = T$ on A_0. Note that $\hat{\tau} \geq \tau$.

Define a new strategy $(\hat{\tau}_i, \xi_i) \in \mathcal{V}[t_0, T]$ $\hat{u}. \in \mathcal{U}[t_0, T]$ and by the following,

$$\hat{u}(t, W., N) = \begin{cases} u(t, W., N) & \text{if } t_0 \leq t \leq \hat{\tau} \\ u^j(t, \Pi_2^{t_j}(W., N)) & \text{if } \hat{\tau} = t_j < t, X_{\hat{\tau}}^{t_0, x_0, u.} \in R_i \end{cases}$$

$$\sum_i \hat{\xi}_i 1_{\hat{\tau}_i \leq t} = \begin{cases} \sum_i \xi_i 1_{\tau_i \leq t} & \text{if } t \leq \tau, \\ \sum_i \xi_i 1_{\tau_i \leq \tau} + \sum_{i,j} 1_{A_j}(\xi_i^j 1_{\tau_i^j \leq t})(\Pi_2^{t_j}(W, N)) & \text{if } t > \tau. \end{cases}$$

In other word, once τ is reached, the impulse will be modified so that there would be no impulses on $[\tau, \hat{\tau})$, and starting at $\hat{\tau}$, the impulse follows the rule (τ_i^j, ξ_i^j) on the set A_j. Now we have,

$$\begin{aligned}
&V(t_0, x_0) \leq J[t_0, x_0, \hat{u}., \hat{\tau}_i, \hat{\xi}_i] \\
=&\mathbb{E}\left[\int_{t_0}^{\hat{\tau}} f(s, X_s^{t_0, x_0, u., \hat{\tau}_i, \hat{\xi}_i}, \hat{u}_s))ds + \sum_i B(\hat{\tau}_i, \hat{\xi}_i) 1_{\hat{\tau}_i < \hat{\tau}} + J\left[\hat{\tau}, \hat{u}^\omega_\cdot, X_{\hat{\tau}-}^{t_0, x_0, \hat{\tau}_i, \hat{\xi}_i}, \hat{\tau}_i^\omega, \hat{\xi}_i^\omega\right]\right] \\
=&\mathbb{E}\left[\int_{t_0}^{\tau} f(s, X_s^{t_0, x_0, u., \tau_i, \xi_i}, u_s))ds\right] + \mathbb{E}\left[\int_{\tau}^{\hat{\tau}} f(s, X_s^{t_0, x_0, u., \hat{\tau}_i, \hat{\xi}_i}), \hat{u}_s)ds\right] + \mathbb{E}\left[\sum_i B(\tau_i, \xi_i) 1_{\tau_i < \tau}\right] \\
&+ \mathbb{E}\left[\sum_j J\left[t_j, X_{t_j-}^{t_0, x_0, \hat{u}^\omega_\cdot, \hat{\tau}_i, \hat{\xi}_i}, \hat{\tau}_i^\omega, \hat{\xi}_i^\omega\right] 1_{A_j}\right].
\end{aligned}$$

The last equality follows from the fact that, $\hat{\tau}_i$ is either $< \tau$, or $\geq \hat{\tau}$, so $\hat{\tau}_i < \hat{\tau}$ implies that $\hat{\tau}_i = \tau_i < \tau$. Since $\hat{u}^\omega_\cdot = u.(\Pi^{-1}(\Pi_1^{\hat{\tau}}(W(\omega), N(\omega)), \Pi_2(\cdot))) = u^j_\cdot(\cdot)$ on the set A_j for $(\hat{\tau}_j, T]$, and any fixed u_0 on $(\tau, \hat{\tau}]$.

$$\begin{aligned}
V(t_0, x_0) \leq&\mathbb{E}\left[\int_{t_0}^{\tau} f(s, X_s^{t_0, x_0, \hat{u}., \tau_i, \xi_i}, u_s))ds\right] + \mathbb{E}\left[\int_{\tau}^{\hat{\tau}} f(s, X_s^{t_0, x_0, \hat{u}., \hat{\tau}_i, \hat{\xi}_i}), \hat{u}_s)ds\right] \\
&+ \mathbb{E}\left[\sum_i B(\tau_i, \xi_i) 1_{\tau_i \leq \tau}\right] + \mathbb{E}\left[\sum_j J\left[t_j, X_{t_j-}^{t_0, x_0, \hat{\tau}_i, \hat{\xi}_i}, u^j_\cdot, \tau_i^j, \xi_i^j\right] 1_{A_j}\right] \\
\leq&\mathbb{E}\left[\int_{t_0}^{\tau} f(s, X_s^{t_0, x_0, u., \tau_i, \xi_i}, u_s)ds\right] + \mathbb{E}\left[\int_{\tau}^{\hat{\tau}} f(s, X_s^{t_0, x_0, \hat{u}., \hat{\tau}_i, \hat{\xi}_i}, \hat{u}_s))ds\right] \\
&+ \mathbb{E}\left[\sum_i B(\tau_i, \xi_i) 1_{\tau_i \leq \tau}\right] + \mathbb{E}\left[\sum_j V(t_j, X_{t_j-}^{t_0, x_0, \hat{u}., \hat{\tau}_i, \hat{\xi}_i}))1_{A_j}\right] + \epsilon.
\end{aligned}$$

Now, for the second term in the last expression, we see

$$
\begin{aligned}
&\mathbb{E}\left[\int_\tau^{\hat\tau} f(s, X_s^{t_0,x_0,\hat u_\cdot,\hat\tau_i,\hat\xi_i}, \hat u_s)ds\right] \le \mathbb{E}\left[\int_\tau^{\hat\tau} C(1+|X_s|^{\gamma+\delta})ds\right]\\
\le&\mathbb{E}\left[\int_\tau^{\hat\tau} C(1+\mathbb{E}^\omega|X_s-X_\tau|^{\gamma+\delta}+|X_\tau|^{\gamma+\delta})ds\right]\\
\le&\mathbb{E}\left[\int_\tau^{\hat\tau} C(1+|X_\tau|^{\mu})ds\right] \le \mathbb{E}\left[\mathbb{E}^\omega\left[\int_\tau^{\hat\tau} C(1+|X_\tau|^{\mu})ds\right]\right]\\
\le&\mathbb{E}\left[\epsilon C(1+|X_\tau|^{\mu})\right] \le C\epsilon(1+|x_0|^\mu). \qquad (2.12)
\end{aligned}
$$

Therefore, it suffices to bound the following expression,

$$
\mathbb{E}\left[\sum_j [V(t_j, X_{t_j-}^{t_0,x_0,\hat u_\cdot,\hat\tau_i,\hat\xi_i}) - V(\tau, X_{\tau-}^{t_0,x_0,u_\cdot,\tau_i,\xi_i})]1_{A_j}\right].
$$

First, note that on the interval $[\tau,\hat\tau)$, $X = X^{t_0,x_0,\hat u_\cdot,\hat\tau_i,\hat\xi_i}$ solves the jump SDE with no impulse:

$$
\begin{aligned}
X_{t\wedge\hat\tau} - X_\tau =& \int_\tau^{t\wedge\hat\tau} b(s,X_s,\hat u_s)dt + \int_\tau^{t\wedge\hat\tau}\sigma(s,X_s,\hat u_s)dW\\
&+\int_\tau^{t\wedge\hat\tau}\int j_1(s,X_s,\hat u_s,z)N(dz,ds)\\
&+\int_\tau^{t\wedge\hat\tau}\int j_2(s,X_s,\hat u_s,z)\tilde N(dz,ds).
\end{aligned}
$$

In particular, under $(\mathbb{P}|(\mathcal{F}^\circ)^{W,N}_{t_0,\tau})$, τ, $X_\tau^{t_0,x_0,\hat u_\cdot,\hat\tau_i,\hat\xi_i}$ and $\hat\tau$ are all deterministic, hence the following estimates

$$
\begin{aligned}
\mathbb{E}^{(\mathbb{P}|(\mathcal{F}^\circ)^{W,N}_{t_0,\tau})(\omega)}|X_{\hat\tau(\omega)}|^\beta \le& C(1+|X_{\tau(\omega)}|^\beta), && \text{if } \beta>0,\\
\mathbb{E}^{(\mathbb{P}|(\mathcal{F}^\circ)^{W,N}_{t_0,\tau})(\omega)}|X_{\hat\tau(\omega)}-X_{\tau(\omega)}|^\beta \le& C(1+|X_{\tau(\omega)}|^\beta)(\hat\tau(\omega)-\tau(\omega))^{\beta/2\wedge1}\\
\le& C(1+|X_{\tau(\omega)}|^\beta)\epsilon^{\beta/2\wedge1}, && \text{if } \beta\ge\nu.
\end{aligned}
$$

Thus, let $E^\omega = \mathbb{E}^{(\mathbb{P}|(\mathcal{F}^\circ)^{W,N}_{t_0,\tau})(\omega)}$, we see

$$
\begin{aligned}
&\mathbb{E}^\omega[V(t_j, X_{t_j-}^{t_0,x_0,\hat u_\cdot,\hat\tau_i,\hat\xi_i}) - V(\tau, X_{\tau-}^{t_0,x_0,u_\cdot,\tau_i,\xi_i})]\\
\le&\mathbb{E}^\omega\left[C(1+|X_\tau(\omega)|^\mu+|X_{\hat\tau}|^\mu)|\hat\tau-\tau(\omega)|^{\delta/2} + C(1+|X_\tau(\omega)|^\gamma+|X_{\hat\tau}|^\gamma)|X_{\hat\tau}-X_\tau(\omega)|^\delta\right]\\
\le&C(1+|X_\tau(\omega)|^\mu+\mathbb{E}^\omega|X_{\hat\tau}|^\mu)\epsilon^{\delta/2} + C\mathbb{E}^\omega\left[(1+|X_\tau(\omega)|^\gamma+|X_{\hat\tau}|^\gamma)|X_{\hat\tau}-X_\tau(\omega)|^\delta\right]\\
\le&C(1+|X_\tau(\omega)|^\mu)\epsilon^{\delta/2} + C\left(\mathbb{E}^\omega\left[(1+|X_\tau(\omega)|^\gamma+|X_{\hat\tau-}|^\gamma]^{p'}\right)^{1/p'}\left(\mathbb{E}^\omega\left[|X_{\hat\tau}-X_\tau(\omega)|^\delta\right]^p\right)^{1/p}
\end{aligned}
$$

(where $p = \mu/\delta > 0$, and $1/p + 1/p' = 1$)

$$
\begin{aligned}
&\leq C(1+|X_\tau(\omega)|^\mu)\epsilon^{\delta/2} + C\left(1+|X_\tau(\omega)|^\gamma + \left(\mathbb{E}^\omega|X_{\hat{\tau}-}|^{\gamma p'}\right)^{1/p'}\right)\left(\mathbb{E}^\omega|X_{\hat{\tau}} - X_\tau(\omega)|^\mu\right)^{\delta/\mu} \\
&\leq C(1+|X_\tau(\omega)|^\mu)\epsilon^{\delta/2} + C\left(1+|X_\tau(\omega)|^\gamma\right)\left(1+|X_{\tau(\omega)}|^\mu\right)^{\delta/\mu}\epsilon^{(\nu/2\wedge 1)\frac{\delta}{\mu}} \\
&\leq C(1+|X_\tau(\omega)|^\mu)\epsilon^{\delta/2} + C\left(1+|X_\tau(\omega)|^\mu\right)\epsilon^{(\nu/2\wedge 1)\frac{\delta}{\mu}}.
\end{aligned}
$$

Taking expectation, we get

$$
\begin{aligned}
&\mathbb{E}\left[\sum_j [V(t_j, X_{t_j-}^{t_0,x_0,\hat{u}.,\hat{\tau}_i,\hat{\xi}_i}) - V(\tau, X_{\tau-}^{t_0,x_0,u.,\tau_i,\xi_i})]1_{A_j}\right] \\
\leq &C(1+\mathbb{E}|X_\tau|^\mu)\epsilon^{\delta/2} + C\left(1+\mathbb{E}|X_\tau|^\mu\right)\epsilon^{(\nu/2\wedge 1)\frac{\delta}{\mu}} \\
\leq &C(1+|x_0|^\mu)\epsilon^{\delta/2} + C\left(1+|x_0|^\mu\right)\epsilon^{(\nu/2\wedge 1)\frac{\delta}{\mu}}.
\end{aligned}
$$

The last inequality follows from Corollary 3.7 in Tang & Yong [45].

With these two bounds, and taking $\epsilon \to 0$, we get the desired inequality and the DPP.

□

Chapter 3

The value function as a viscosity solution

In this section, we establish the value function $V(x,t)$ as a viscosity solution to the (HJB) equation in the sense of [1].

Notation 1.

$$I_\theta^1[\phi](x,t) = \int_{|z|<\theta} \phi(x+z,t) - \phi(x,t) - D\phi(x,t)\cdot z1_{|z|<1}, \rho(dz),$$
$$I_\theta^2[u](x,t) = \int_{|z|\geq\theta} u(x+z,t) - u(x,t) - D\phi(x,t)\cdot z1_{|z|<1}\rho(dz),$$

with the boundary condition $u = g$ on $\mathbb{R}^n \times \{t = T\}$.

Theorem 2. *(Value Function as Viscosity Solution) The value function $V(x,t)$ is a continuous viscosity solution to the (HJB) equation in the following sense: if for any $\phi \in C^2(\mathbb{R}^n \times [0,T])$,*

1. *$u - \phi$ achieves a local maximum at $(x_0,t_0) \in B(x_0,\theta) \times [t_0, t_0+\theta)$ with $u(x_0,t_0) = \phi(x_0,t_0)$, then V is a subsolution*

$$\max\{-\phi_t + L\phi - f - I_\theta^1[\phi] - I_\theta^2[u], u - Mu\}(x_0,t_0) \leq 0.$$

2. *$u \geq \phi$ and $u - \phi$ achieves a local minimum at $(x_0,t_0) \in B(x_0,\theta) \times [t_0, t_0+\theta)$ with $u(x_0,t_0) = \phi(x_0,t_0)$, then V is a supersolution*

$$\max\{-\phi_t + L\phi - f - I_\theta^1[\phi] - I_\theta^2[u], u - Mu\}(x_0,t_0) \geq 0.$$

Proof. Step 1. Suppose $V - \phi$ achieves a local maximum in $B(x_0,\theta) \times [t_0, t_0+\theta)$ with $V(x_0,t_0) = \phi(x_0,t_0)$, we prove by contradiction that $(-\phi_t + L\phi - I_\theta^1[\phi] - I_\theta^2[V] - f)(x_0,t_0) \leq 0$.

Suppose otherwise, i.e. $(-\phi_t + L\phi - I_\theta^1[\phi] - I_\theta^2[V] - f)(x_0, t_0) > 0$. Then without loss of generality we can assume that $-\phi_t + L\phi - I_\theta^1[\phi] - I_\theta^2[V] - f > 0$ on $B(x_0, \theta) \times [t_0, t_0 + \theta)$. Since the definition of viscosity solution does not concern the value of ϕ outside of $B(x_0, \theta) \times [t_0, t_0 + \theta)$, we can assume that ϕ is bounded by multiples of $|V|$. Let $X^0 = X^{x_0,t_0,\infty,0}$ and

$$\tau = \inf\{t \in [t_0, T] : X_t \notin B(x_0, \theta) \times [t_0, t_0 + \theta)\} \wedge T.$$

By Ito's formula,

$$\mathbb{E}\left[\phi(X^0_\tau, \tau)\right] - \phi(x_0, t_0) = \mathbb{E}\left[\int_{t_0}^{\tau} (\phi_t - L\phi + I_\theta^1[\phi] + I_\theta^2[\phi])(X^0_{s-}, s)ds\right].$$

Meanwhile, by Theorem1,

$$V(x_0, t_0) \leq \mathbb{E}\left[\int_{t_0}^{\tau} f(X^0_s, s)ds + V(X^0_\tau, \tau)\right].$$

Combining these two inequalities, we get

$$\begin{aligned}
&\mathbb{E}\left[V(X^0_\tau, \tau)\right] - \mathbb{E}\left[\int_{t_0}^{\tau} (\phi_t - L\phi + I_\theta^1[\phi] + I_\theta^2[\phi])(X^0_{s-}, s)ds\right] \\
\leq&\mathbb{E}\left[\phi(X^0_\tau, \tau)\right] - \mathbb{E}\left[\int_{t_0}^{\tau} (\phi_t - L\phi + I_\theta^1[\phi] + I_\theta^2[\phi])(X^0_{s-}, s)ds\right] \\
=&\phi(x_0, t_0) = V(x_0, t_0) \leq \mathbb{E}\left[\int_{t_0}^{\tau} f(X^0_s, s)ds + V(X^0_\tau, \tau)\right].
\end{aligned}$$

That is,

$$\mathbb{E}\left[\int_{t_0}^{\tau} (-\phi_t + L\phi - I_\theta^1[\phi] - I_\theta^2[\phi] - f)(X^0_{s-}, s)ds\right] \leq 0.$$

Again by modifying the value of ϕ outside of $B(x_0, \theta) \times [t_0, t_0 + \theta)$, and since $V \leq \phi$ in $B(x_0, \theta) \times [t_0, t_0 + \theta)$, we can take a sequence of $\phi_k \geq V$ dominated by multiples of $|V|$ such that it converges to V outside of $B(x_0, \theta) \times [t_0, t_0 + \theta)$ from above. By the dominated convergence theorem, $I_\theta^2[\phi]$ converges to $I_\theta^2[V]$. Thus,

$$\mathbb{E}\left[\int_{t_0}^{\tau} (-\phi_t + L\phi - I_\theta^1[\phi] - I_\theta^2[V] - f)(X^0_{s-}, s)ds\right] \leq 0,$$

which is a contradiction. Therefore, we must have $(-\phi_t + L\phi - I_\theta^1[\phi] - I_\theta^2[V] - f)(x_0, t_0) \leq 0$, and since $V \leq MV$, we have $\max\{-\phi_t + L\phi - I_\theta^1[\phi] - I_\theta^2[V] - f, V - MV\}(x_0, t_0) \leq 0$.

Step 2. Suppose $V - \phi$ achieves local minimum in $B(x_0, \theta) \times [t_0, t_0 + \theta)$ with $V(x_0, t_0) = \phi(x_0, t_0)$. Then if $(V - MV)(x_0, t_0) = 0$, then we already have the desired inequality. Now suppose $V - MV \leq -\epsilon < 0$ and $-\phi_t + L\phi - I_\theta^1[\phi] - I_\theta^2[V] - f < 0$ on $B(x_0, \theta) \times [t_0, t_0 + \theta)$.

Assuming as before that ϕ is bounded by multiples of $|V|$ outside of $B(x_0, \theta) \times [t_0, t_0 + \theta)$. By Ito's formula

$$\mathbb{E}\left[\phi(X^0_\tau, \tau)\right] - \phi(x_0, t_0) = \mathbb{E}\left[\int_{t_0}^{\tau} (\phi_t - L\phi + I^1_\theta[\phi] + I^2_\theta[V])(X^0_{s-}, s)ds\right].$$

Consider the no impulse strategy $\tau_i^* = \infty$ and let $X^0 = X^{t_0,x_0,u_\cdot,\infty,0}$. Define the stopping time τ as before, i.e,

$$\tau = \inf\{t \in [t_0, T] : X_t \notin B(x_0, \theta) \times [t_0, t_0 + \theta)\} \wedge T.$$

Then for any strategy $(\tau_i, \xi_i) \in \mathcal{V}$,

$$\begin{aligned}
&J[t_0, x_0, \tau_i, \xi_i] \\
=&\mathbb{E}\left[\int_{t_0}^{\tau_1\wedge\tau} f(s, X_s^{t_0,x_0,u_\cdot,\tau_i,\xi_i})ds + B(\tau_1, \xi_1)1_{\{\tau_1 \le \tau\wedge\tau_1\}} + J[\tau_1 \wedge \tau, X_{\tau_1\wedge\tau}^{t_0,x_0,u_\cdot,\tau_i,\xi_i}, \tau_i, \xi_i]\right] \\
\ge&\mathbb{E}\left[\int_{t_0}^{\tau_1\wedge\tau} f(s, X_s^{t_0,x_0,u_\cdot,\tau_i,\xi_i})ds + 1_{\{\tau_1\le\tau\}}(B(\tau_1, \xi_1) + V(X_{\tau_1}^{t_0,x_0,u_\cdot,\tau_i,\xi_i}, \tau_1))\right] \\
&+ \mathbb{E}\left[1_{\{\tau_1>\tau\}}V(X_\tau^{t_0,x_0,u_\cdot,\tau_i,\xi_i}, \tau)\right] \\
\ge&\mathbb{E}\left[\int_{t_0}^{\tau_1\wedge\tau} f(s, X_s^0)ds + 1_{\{\tau_1\le\tau\}}MV(X^0_{\tau_1}, \tau_1)\right] \\
&+ \mathbb{E}\left[1_{\{\tau_1>\tau\}}V(X^0_\tau, \tau)\right] \\
\ge&\mathbb{E}\left[\int_{t_0}^{\tau_1\wedge\tau} f(s, X_s^0)ds + V(X^0_{\tau_1\wedge\tau}, \tau_1 \wedge \tau)\right] + \epsilon \cdot \mathbb{P}(\tau_1 \le \tau) \\
\ge&V(t_0, x_0) + \epsilon \cdot \mathbb{P}(\tau_1 \le \tau).
\end{aligned}$$

Therefore, without loss of generality, we only need to consider $(\tau_i, \xi_i) \in \mathcal{V}$ such that $\tau_1 > \tau$. Now, the Dynamic Programming Principle becomes,

$$u(x_0, t_0) = \mathbb{E}\left[\int_{t_0}^{\tau} f(X^0_s, s)ds + V(X^0_\tau, \tau)\right].$$

Now combining these facts above,

$$\begin{aligned}
&\mathbb{E}\left[V(X^0_\tau, \tau)\right] - \mathbb{E}\left[\int_{t_0}^{\tau} (\phi_t - L\phi + I^1_\theta[\phi] + I^2_\theta[\phi])(X^0_{s-}, s)ds\right] \\
\ge&\mathbb{E}\left[\phi(X^0_\tau, \tau)\right] - \mathbb{E}\left[\int_{t_0}^{\tau} (\phi_t - L\phi + I^1_\theta[\phi] + I^2_\theta[\phi])(X^0_{s-}, s)ds\right] \\
=&\phi(x_0, t_0) = V(x_0, t_0) = \mathbb{E}\left[\int_{t_0}^{\tau} f(X^0_s, s)ds + V(X^0_\tau, \tau)\right] \\
&\mathbb{E}\left[\int_{t_0}^{\tau} (-\phi_t + L\phi - I^1_\theta[\phi] - I^2_\theta[\phi] - f)(X^0_{s-}, s)ds\right] \ge 0.
\end{aligned}$$

Again by modifying the value of ϕ outside of $B(x_0, \theta) \times [t_0, t_0 + \theta)$, and since $V \geq \phi$ in $B(x_0, \theta) \times [t_0, t_0 + \theta)$, we can take a sequence of $\phi_k \leq u$ dominated by multiples of $|V|$ such that it converges to V outside of $B(x_0, \theta) \times [t_0, t_0 + \theta)$ from above. By the dominated convergence theorem, $I^2_\theta[\phi]$ converges to $I^2_\theta[V]$. Hence

$$\mathbb{E}\left[\int_{t_0}^{\tau} (-\phi_t + L\phi - I^1_\theta[\phi] - I^2_\theta[V] - f)(X^0_{s-}, s)ds\right] \geq 0,$$

which contradicts the assumption that $(-\phi_t + L\phi - I^1_\theta[\phi] - I^2_\theta[V] - f)(x_0, t_0) < 0$. Therefore,

$$\max\{-\phi_t + L\phi - I^1_\theta[\phi] - I^2_\theta[V] - f, V - MV\}(x_0, t_0) \geq 0.$$

□

Chapter 4

Regularity of the value function

To study the regularity of the value function, we will consider the time-inverted value function $u(x,t) = V(x, T-t)$. Accordingly, we will assume that a_{ij}, b_i, f, B and j are all time-inverted. This is to be consistent with the standard PDE literature for easy references to some of its classical results, where the value is specified at the initial time instead of the terminal time.

The regularity study is built in two phases.

First in Section 4.2, we focus on the case without jumps . We will construct a unique $W^{(2,1),p}_{loc}$ regular viscosity solution to a corresponding equation without the integro-differential operator on a fixed bounded domain Q_T with $Q_T = B(0,R) \times (\delta, T]$ for $R > 0$ and $\delta > 0$,

$$\begin{cases} \max\{u_t + Lu - f, u - \Psi\} = 0 & \text{in } Q_T, \\ u = \phi & \text{on } \partial_P Q_T, \end{cases} \tag{4.1}$$

in which $\phi(x,t) = V(x, T-t)$ and $\Psi(x,t) = (Mu)(x,t)$. The local uniqueness of the viscosity solution then implies that this solution must be the time-inverted value function, hence the $W^{(2,1),p}_{loc}$ smoothness for the value function.

Then in Section 4.4, we extend the analysis to the case with a first-order jump and establish the regularity property of the value function.

4.1 Preliminary results

To analyze the value function, we also need some preliminary results, in addition to the DPP.

Lemma 4. *The set*

$$\Xi(x,t) := \{\xi \in \mathbb{R}^n : MV(x,t) = V(x+\xi, t) + B(\xi, t)\}$$

is nonempty. Moreover, for (x,t) in bounded $B' \subset \mathbb{R}^n \times [0,T]$, $\{(y,t) : y = x + \xi, (x,t) \in B', \xi \in \Xi(x,t)\}$ is also bounded.

Proof. This is easy by $B(\xi,t) \geq L + C|\xi|^\mu$, $-C \leq V \leq C(1+|x|^{\gamma+\delta})$, and $\mu > \gamma+\delta$. □

Lemma 5. *(Theorem 4.9 in [32]) Assume that $a_{ij}, b_i, f \in C^\alpha_{loc}(\mathbb{R}^n \times (0,T))$. If $-u_t + Lu = f$ in $\mathcal{C}$ in the viscosity sense, then it solves the PDE in the classical sense as well, and $u(x, T-t) \in C^{2+\alpha, 1+\frac{\alpha}{2}}_{loc}(\mathcal{C})$.*

Lemma 6. *The value function V and MV satisfies $V(x,t) \leq MV(x,t)$ pointwise.*

Lemma 7. *MV is continuous, and there exists C such that for any $x, y \in \mathbb{R}^n$, $s < t$,*

$$\begin{aligned}|MV(x,t) - MV(y,t)| \leq &C(1+|x|^\gamma + |y|^\gamma)|x-y|^\delta,\\ MV(x,t) - MV(x,s) \leq &C(1+|x|^\mu)|t-s|^{\delta/2}.\end{aligned}$$

Proof. First we prove continuity. For each ξ, $V(x,t) + B(\xi,t)$ is a uniformly continuous function on compact sets. And since $\Xi(x,t)$ is bounded for (x,t) on compact sets, taking the infimum over ξ on some fixed compact sets implies that MV is continuous.

For the Hölder continuity in t, let $\xi \in \Xi(x,s)$, then

$$\begin{aligned}&MV(x,t) - MV(x,s)\\ \leq &V(x+\xi,t) + B(\xi,t) - V(x+\xi,s) - B(\xi,s)\\ \leq &C(1+|x|^\mu)|t-s|^{\delta/2},\end{aligned}$$

given that $B(\xi,s) \geq B(\xi,t)$ for $s < t$. □

As a consequence, the continuous region $\mathcal{C}$ is open.

Lemma 8. *Fix x in some bounded $B \subset \mathbb{R}^n$. Let $\epsilon > 0$. For any*

$$\xi \in \Xi^\epsilon(x,t) = \{\xi : V(x+\xi,t) + B(\xi,t) < MV(x,t) + \epsilon\},$$

we have

$$V(x+\xi,t) + K - \epsilon < MV(x+\xi,t). \tag{4.2}$$

In particular, Let $\mathcal{C}^{K/2} = \{(x,t) \in \mathbb{R}^n \times [0,T] : V(x,t) < MV(x,t) - K/2\}$. If $\xi \in \Xi^{K/2}(x,t)$, then $(x+\xi,t) \in \mathcal{C}^{K/2}$.

Proof. Suppose $\xi \in \Xi^\epsilon(x,t)$, i.e.

$$V(x+\xi,t) + B(\xi,t) < MV(x,t) + \epsilon.$$

Then,

$$\begin{aligned} MV(x+\xi,t) =& \inf_\eta V(x+\xi+\eta,t)+B(\eta,t) \\ =& \inf_\eta V(x+\xi+\eta,t)+B(\xi+\eta,t)-B(\xi+\eta,t)+B(\eta,t) \\ \geq& \inf_\eta V(x+\xi+\eta,t)+B(\xi+\eta,t)-B(\xi,t)+K \\ =& \inf_{\eta'} V(x+\eta',t)+B(\eta',t)-B(\xi,t)+K \\ =& MV(x,t)-B(\xi,t)+K \\ >& V(x+\xi,t)-\epsilon+K. \end{aligned}$$

Let $\epsilon = K/2$, we get that $\xi \in \Xi^{K/2}(x,t)$ implies $x+\xi \in \mathcal{C}^{K/2}$. □

Lemma 9. *MV is uniformly semi-concave in x, and MV_t is bounded above in the distributional sense on compact sets away from $t = T$.*

Proof. Let A be a compact subset of $\mathbb{R}^n \times [0, T-\delta]$. For any $\xi \in \Xi(x,t)$ for $(x,t) \in A$, $(x+\xi,t)$ lies in a bounded region B independent of (x,t). For any $|y| = 1$ and $\delta > 0$ sufficiently small,

$$\begin{aligned} & \frac{MV(x+\delta y,t)-2MV(x,t)+M(x-\delta y,t)}{2\delta} \\ \leq& \frac{(V(x+\delta y+\xi,t)+B(\xi,t))-2(V(x+\xi,t)+B(\xi,t))+(V(x-\delta y+\xi,t)+B(\xi,t)))}{2\delta} \\ =& \frac{V(x+\delta y+\xi,t)-2V(x+\xi,t)+V(x-\delta y+\xi,t)}{2\delta} \\ \leq& C\|D^2V\|_{B\cap\mathcal{C}^{K/2}}, \end{aligned}$$

which is bounded by Lemma 5. Similarly,

$$\begin{aligned} & \frac{MV(x,t+\delta)-MV(x,t)}{\delta} \\ \leq& \frac{V(x+\xi,t+\delta)+B(\xi,t+\delta)-(V(x+\xi,t)+B(\xi,t))}{\delta} \\ =& \frac{V(x+\xi,t+\delta)-V(x+\xi,t)}{\delta}+\frac{B(\xi,t+\delta)-B(\xi,t)}{\delta} \\ \leq& C\|V_t\|_{B\cap\mathcal{C}^{K/2}}. \end{aligned}$$

□

4.2 $\mathbf{W}^{(2,1),p}_{loc}$ Regularity for cases without jumps

The key idea is to study a corresponding homogenous HJB, based on the following classical result in PDEs.

Lemma 10. *(Theorem 4.9, 5.9, 5.10, and 6.33 of [32]) Let* $\alpha \in (0,1]$. *Assume that* $a_{ij}, b_i, f \in C^{0+\alpha,0+\frac{\alpha}{2}}(\overline{Q_T})$, a_{ij} *is uniformly elliptic,* $\phi \in C^{0+\alpha,0+\frac{\alpha}{2}}(\partial_P Q_T)$. *Then the linear PDE*

$$\begin{cases} u_t + Lu = f & \text{in } Q_T; \\ u = \phi & \text{on } \partial_P Q_T. \end{cases} \tag{4.3}$$

has a unique solution to (4.3) that lies in $C^{0+\alpha,0+\frac{\alpha}{2}}(\overline{Q_T}) \cap C^{2+\alpha,1+\frac{\alpha}{2}}_{loc}(Q_T)$.

Indeed, given Lemma 10, let u_0 be the unique classical solution to (4.3), with the boundary condition $\phi(x,t) = V(T-t,x)$. Then, our earlier analysis (Lemma 3) of Hölder continuity for the value function implies that $V(x,T-t) - u_0(x,t)$ solves the following "homogenous" HJB,

$$\begin{cases} \max\{u_t + Lu, u - \overline{\Psi}\} = 0 & \text{in } Q_T, \\ u = 0 & \text{on } \partial_P Q_T. \end{cases}$$

for $\overline{\Psi} = \Psi - u_0$. Since $V \le MV$, we have $\overline{\Psi}(x,t) = (\Psi - u_0)(x,t) = (MV - V)(x, T-t) \ge 0$ on $\partial_P Q_T$.

Therefore, our first step is to study the above "homogenous" HJB.

Step I: Viscosity solution of the "homogenous" HJB

Theorem 3. *Assume*

1. $a_{ij}, b_i, \overline{\Psi} \in C^{0+\alpha,0+\frac{\alpha}{2}}(\overline{Q_T})$,
2. (a_{ij}) *uniformly elliptic,*
3. $\overline{\Psi}$ *is semiconcave,*
4. $\overline{\Psi}_t$ *is bounded below, in the distributional sense.*

Then there exists a viscosity solution $u \in W^{(2,1),p}(Q_T)$ *to the homogenous HJB*

$$\begin{cases} \max\{u_t + Lu, u - \overline{\Psi}\} = 0 & \text{in } Q_T, \\ u = 0 & \text{on } \partial_P Q_T. \end{cases} \tag{4.4}$$

Furthermore, $u \in W^{(2,1),p}(Q_T)$ *for any* $p > 1$.

To prove this theorem, we first consider a corresponding penalized version. For every $\epsilon > 0$, let $\beta_\epsilon : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\beta_\epsilon(x) \ge -1$, $\beta_\epsilon(0) = 0$, $\beta' > 0$, $\beta'' \ge 0$, $\beta'_\epsilon(x) \le C/\epsilon$ for $x \ge 0$, $\beta'_\epsilon(0) = 1/\epsilon$ and as $\epsilon \to 0$, $\beta_\epsilon(x) \to \infty$ for $x > 0$, $\beta_\epsilon(x) \to 0$ for $x < 0$. One such example is, $\beta(x) = x/\epsilon$ for $x \ge 0$ and it smooth extension to $x < 0$. We see that there is a classical solution u to the penalized problem, assuming some regularity on the coefficients $a^{ij}, b^i, \overline{\Psi}$.

Lemma 11. *Fix $\epsilon > 0$. Suppose that $a^{ij}, b^i, \overline{\Psi} \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$, and (a^{ij}) is uniformly elliptic. Then exists a unique $u \in C^{4+\alpha,2+\alpha/2}(\overline{Q}_T)$ such that*

$$\begin{cases} u_t + Lu + \beta_\epsilon(u - \overline{\Psi}) = 0 & \text{on } Q_T, \\ u = 0 & \text{on } \partial_P Q_T. \end{cases} \tag{4.5}$$

Note that Friedman [22] proved a similar result for a $W^{2,p}$ solution for the elliptic case using the L^p estimates. He then used the Schauder estimates to bootstrap for the C^2 regularity. Our proof is more elementary using only the Schauder estimates.

Proof. Define the operator $A : C^{2+\alpha,1+\alpha/2}(\overline{Q}_T) \to C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$ by the following: $A[v] = u$ is the solution to the PDE

$$\begin{cases} u_t + Lu + \beta_\epsilon(v - \overline{\Psi}) = 0 & \text{on } Q_T, \\ u = 0 & \text{on } \partial_P Q_T. \end{cases}$$

By the Schauder's estimates (Theorem 4.28 in [32]), we have,

$$\begin{aligned} \|u\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)} &\le C\|\beta_\epsilon(v - \overline{\Psi})\|_{C^{\alpha,\alpha/2}(\overline{Q}_T)} \\ &\le C_\epsilon \|v - \overline{\Psi}\|_{C^{\alpha,\alpha/2}(\overline{Q}_T)}. \end{aligned}$$

Thus the map A is clearly continuous and compact.

The next step is to show that the set $\{u : u = \lambda A[u], 0 \le \lambda \le 1\}$ is bounded. Then we can apply Schaefer's Fixed Point Theorem (Theorem 9.4 in ([17]). Suppose $u = \lambda A[u]$ for some $0 \le \lambda \le 1$. Then,

$$\begin{cases} u_t + Lu + \lambda\beta_\epsilon(u - \overline{\Psi}) = 0 & \text{on } Q_T, \\ u = 0 & \text{on } \partial_P Q_T. \end{cases}$$

Since

$$\begin{aligned} \|u\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)} &\le C_\epsilon \|v - \overline{\Psi}\|_{C^{\alpha,\alpha/2}(\overline{Q}_T)}) \\ &\le C_\epsilon (\|u\|_{C^{\alpha,\alpha/2}(\overline{Q}_T)} + \|\overline{\Psi}\|_{C^{\alpha,\alpha/2}(\overline{Q}_T)}) \\ &\le C_\epsilon (\|u\|^{1/2}_{C(\overline{Q}_T)} \|u\|^{1/2}_{C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)} + \|\overline{\Psi}\|_{C^{\alpha,\alpha/2}(\overline{Q}_T)}) \\ &\le \frac{1}{2}\|u\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)} + C_\epsilon(\|u\|_{C(\overline{Q}_T)} + \|\overline{\Psi}\|_{C^{\alpha,\alpha/2}(\overline{Q}_T)}). \end{aligned}$$

Thus

$$\|u\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)} \le C_\epsilon(\|u\|_{C(\overline{Q}_T)} + \|\overline{\Psi}\|_{C^{\alpha,\alpha/2}(\overline{Q}_T)}).$$

So we only need to bound u independent of λ now.

If $\lambda = 0$, then $u = 0$. So we can assume that $\lambda > 0$. Suppose u has a maximum at $(x_0, t_0) \in Q_T$. Then, $-\lambda\beta_\epsilon(u(x_0,t_0) - \overline{\Psi}(x_0,t_0)) = (u_t + Lu)(x_0,t_0) \ge 0$, $\beta_\epsilon(u(x_0,t_0) - \overline{\Psi}(x_0,t_0)) \le 0$, $u(x_0,t_0) \le \overline{\Psi}(x_0,t_0)$, and $u = 0$ on $\partial_P Q_T$. So we get $u \le \|\overline{\Psi}\|_{L^\infty(Q_T)}$.

For a lower bound, consider the open set $\Omega = \{u < \overline{\Psi}\}$ in $\overline{Q}_T$. Since in Ω, $u_t + Lu \geq 0$, $u \geq \inf_{\partial_P \Omega} u$. Yet, $\partial_P \Omega \subset \partial_P Q_T \cup \{u \geq \overline{\Psi}\}$, and in both cases u is bounded below. Thus we conclude that u is bounded independently of λ, and $\|u\|_{L^\infty(Q_T)} \leq \|\overline{\Psi}\|_{L^\infty(Q_T)}$.

Now Schaefer's Fixed Point Theorem (Theorem 9.4 in [17]) gives us the existence of $u \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$ that solves (4.5). Now we have $-\beta_\epsilon(u - \overline{\Psi}) \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$. By the Schauder's estimates again, we have $u \in C^{4+\alpha,2+\alpha/2}(\overline{Q}_T)$. □

Next, consider the case with $C^{\alpha,\alpha/2}(\overline{Q}_T)$ coefficients. We will smooth out the coefficients first to the above result, and then let $\epsilon \to 0$. More precisely, let $(a^\epsilon)^{ij}, (b^\epsilon)^i, \overline{\Psi}^\epsilon \in C^\infty(\overline{Q}_T)$ be such that they converge to the respective function in $C^{\alpha,\alpha/2}(\overline{Q}_T)$ and $\overline{\Psi}^\epsilon \geq 0$ on $\partial_P Q_T$. This is possible because $\overline{\Psi} \geq 0$ on $\partial_P Q_T$. Define L^ϵ to be the corresponding linear operator and u^ϵ to be the unique solution to

$$\begin{cases} u^\epsilon_t + L^\epsilon u^\epsilon + \beta_\epsilon(u^\epsilon - \overline{\Psi}^\epsilon) = 0 & \text{on } Q_T, \\ u^\epsilon = 0 & \text{on } \partial_P Q_T. \end{cases}$$

Now we establish some bound for $\beta_\epsilon(u^\epsilon - \overline{\Psi}^\epsilon)$, in order to apply an L^p estimate.

Lemma 12. *Assuming $\overline{\Psi}$ is semiconcave in x, i.e.*

$$\frac{\partial^2 \overline{\Psi}}{\partial \xi^2} \leq C,$$

for any direction $|\xi| = 1$, and

$$\frac{\partial \overline{\Psi}}{\partial t} \geq -C,$$

where both derivatives are interpreted in the distributional sense. We have

$$|\beta_\epsilon(u^\epsilon - \overline{\Psi}^\epsilon)| \leq C,$$

with C independent of ϵ.

Proof. Clearly $\beta_\epsilon \geq -1$, so we only need to give an upper bound. The assumption above translates to the same derivative condition on mollified $\overline{\Psi}^\epsilon$, which can be interpreted classically now. Thus we have

$$\overline{\Psi}^\epsilon_t + L^\epsilon \overline{\Psi}^\epsilon \geq -C.$$

Suppose $u^\epsilon - \overline{\Psi}^\epsilon$ achieves maximum at $(x_0, t_0) \in Q_T$, then

$$(u^\epsilon - \overline{\Psi}^\epsilon)_t + L^\epsilon(u^\epsilon - \overline{\Psi}^\epsilon)(x_0, t_0) \geq 0.$$

Hence

$$\begin{aligned}-\beta_\epsilon(u^\epsilon - \overline{\Psi}^\epsilon)(x_0, t_0) &= (u^\epsilon_t + L^\epsilon u^\epsilon)(x_0, t_0)\\ &\geq (\overline{\Psi}^\epsilon_t + L^\epsilon \overline{\Psi}^\epsilon)(x_0, t_0) \geq -C,\end{aligned}$$

in which C is an upper bound independent of ϵ. On the other hand, if it achieves maximum on $\partial_P Q_T$, we get $u^\epsilon - \overline{\Psi}^\epsilon \leq 0$ since $\overline{\Psi}^\epsilon \geq 0$ on $\partial_P Q_T$. Either way we have an upper bound independent of ϵ. □

Now with this estimate of the boundedness of $\beta_\epsilon(u^\epsilon - \overline{\Psi}^\epsilon)$, we are ready to prove Theorem 3.

Proof. Lemma 12 allows us to apply L^p estimate:

$$\|u^\epsilon\|_{W^{(2,1),p}(Q_T)} \leq C\|\beta_\epsilon(u^\epsilon - \overline{\Psi}^\epsilon)\|_{L^p(Q_T)} \leq C,$$

for $p > 1$. Thus there exists a sequence $\epsilon_n \to 0$ and $u \in W^{(2,1),p}(Q_T)$ such that

$$u^{\epsilon_n} \rightharpoonup u$$

weakly in $W^{(2,1),p}(Q_T)$. For p large enough, there exists $\alpha' > 0$ such that $u^\epsilon \to u$ in $C^{\alpha',\alpha'/2}(\overline{Q}_T)$, so $u^\epsilon \to u$ uniformly in $\overline{Q}_T$.

On one hand, since $\beta_\epsilon(u^\epsilon - \overline{\Psi}^\epsilon) \leq C$, yet $\beta_\epsilon(x) \to \infty$ as $x > 0$, hence $u \leq \overline{\Psi}$. Suppose $u - \phi$ achieves a strict local maximum at (x_0, t_0), then $u^\epsilon - \phi$ achieves a strict local maximum at $(x^\epsilon_0, t^\epsilon_0)$ and $(x^\epsilon_0, t^\epsilon_0) \to (x_0, t_0)$ as $\epsilon \to 0$, then

$$\lim_{\epsilon\to 0}(\phi_t + L^\epsilon\phi)(x^\epsilon_0, t^\epsilon_0) \leq \liminf_{\epsilon\to 0} \beta_\epsilon(u^\epsilon(x^\epsilon_0, t^\epsilon_0) - \overline{\Psi}(x^\epsilon_0, t^\epsilon_0)) \leq 0.$$

So $(\phi_t + L\phi)(x_0, t_0) \leq 0$.

On the other hand, if $u - \phi$ achieves a strict local minimum at (x_0, t_0), then $u^\epsilon - \phi$ achieves a strict local maximum at $(x^\epsilon_0, t^\epsilon_0)$ and $(x^\epsilon_0, t^\epsilon_0) \to (x_0, t_0)$ as $\epsilon \to 0$. If $u(x_0, t_0) < \overline{\Psi}(x_0, t_0)$, then for small ϵ, $u(x^\epsilon_0, t^\epsilon_0) < \overline{\Psi}(x^\epsilon_0, t^\epsilon_0)$,

$$\lim_{\epsilon\to 0}(\phi_t + L^\epsilon\phi)(x^\epsilon_0, t^\epsilon_0) \geq \limsup_{\epsilon\to 0} \beta_\epsilon(u^\epsilon(x^\epsilon_0, t^\epsilon_0) - \overline{\Psi}(x^\epsilon_0, t^\epsilon_0)) \geq 0.$$

□

Step II: Uniqueness of the HJB equation without jump terms

Proposition 2. *Assuming that a_{ij}, b_i, f, Ψ, f are continuous in $\overline{Q}_T$, and ϕ continuous on $\partial_P Q_T$, the viscosity solution to the following HJB equation is unique.*

$$\begin{cases} \max\{u_t + Lu - f, u - \Psi\} = 0 & \text{in } Q_T,\\ u(t,x) = \phi & \text{on } \partial_P Q_T.\end{cases} \tag{4.6}$$

Remark. Note that this is a local uniqueness of the viscosity solution. We later apply $\phi(x,t) = V(x, T-t)$ and $\Psi(x,t) = (Mu)(x,t)$ to our original control problem.

Proof. Let W, U be a viscosity subsolution and supersolution to (4.6) respectively. Then W is clearly a viscosity subsolution to $v_t + Lv - f = 0$, with $W \leq \overline{\Psi}$. On the other hand, at any fixed point (x_0, t_0), either $U(x_0, t_0) = \overline{\Psi}(x_0, t_0)$ or U satisfies the viscosity supersolution property at (x_0, t_0).

Define

$$W^\epsilon(x,t) = W(x,t) + \frac{\epsilon}{t-\delta}$$

for $\epsilon > 0$. Note that W^ϵ is still a viscosity subsolution of $v_t + Lv - f = 0$. For fixed ϵ, α, β, define

$$\Phi(t,x,y) = W^\epsilon(x,t) - U(x,t) - \alpha|x-y|^2 - \beta(t-\delta).$$

Denote $B = B(0,R)$. Suppose $\max_{(x,t)\in\overline{Q}_T} W^\epsilon(x,t) - U(x,t) \geq c > 0$. There exist $\alpha_0, \beta_0, \epsilon_0$, such that for $\alpha \geq \alpha_0$, $\beta \leq \beta_0$, and $\epsilon \leq \epsilon_0$, we have

$$\max_{(t,x,y)\in[\delta,T)\times\overline{B}\times\overline{B}} \Phi(t,x,y) \geq c/2 > 0.$$

Let $(\bar{t}, \bar{x}, \bar{y}) \in (\delta, T) \times B \times B$ be the point where Φ achieves the maximum. Since $\Phi(\delta, 0, 0) \leq \Phi(\bar{t}, \bar{x}, \bar{y})$, we get

$$\alpha|\bar{x}-\bar{y}|^2 \leq h(|\bar{x}-\bar{y}|),$$

in which h is the modulus of continuity of U. Since the domain is bounded, $\alpha|\bar{x}-\bar{y}|^2 \leq K$ for some fixed constant K independent of α, ϵ, β. We have $|\bar{x}-\bar{y}| \leq \sqrt{K/\alpha}$, which implies

$$\alpha|\bar{x}-\bar{y}|^2 \leq \omega(\sqrt{\frac{K}{\alpha}}).$$

Denote ω as the modulus of continuity of $\overline{\Psi}$. We have two cases:

1. $U(\bar{y}, \bar{t}) = \overline{\Psi}(\bar{y}, \bar{t})$. We have

$$\begin{aligned} W^\epsilon(\bar{x}, \bar{t}) \leq & \overline{\Psi}(\bar{x}, \bar{t}) + \frac{\epsilon}{\bar{t}-\delta} \\ \leq & \omega(|\bar{x}-\bar{y}|) + \overline{\Psi}(\bar{y}, \bar{t}) + \frac{\epsilon}{\bar{t}-\delta} \\ = & \omega(|\bar{x}-\bar{y}|) + U(\bar{y}, \bar{t}) + \frac{\epsilon}{\bar{t}-\delta}. \end{aligned}$$

   Thus

$$W^\epsilon(\bar{x}, \bar{t}) - U(\bar{y}, \bar{t}) \leq \omega(|\bar{x}-\bar{y}|) + \frac{\epsilon}{\bar{t}-\delta}.$$

2. $U(\bar{y},\bar{t}) < \overline{\Psi}(\bar{y},\bar{t})$. By the same analysis as Theorem V.8.1 in [20],

$$\beta \leq \omega(\alpha|\bar{x}-\bar{y}|^2 + |\bar{x}-\bar{y}|).$$

Fix $\epsilon \leq \epsilon_0$, $\beta \geq \beta_0$. For each $\alpha \leq \alpha_0$, one of the two cases is true. If case 2 occurs infinitely many times as $\alpha \to \infty$, we have a contradiction, thus case 1 must occur infinitely many times as $\alpha \to \infty$. We have the inequality

$$\begin{aligned} & W^\epsilon(x,t) - U(x,t) - \beta(t-\delta) \\ =&\Phi(x,x,t) \leq \Phi(\bar{x},\bar{y},\bar{t}) \\ \leq& W^\epsilon(\bar{x},\bar{t}) - U(\bar{y},\bar{t}) \\ \leq& \omega(\sqrt{\frac{1}{\alpha}h(\sqrt{\frac{K}{\alpha}})}) + \frac{\epsilon}{\bar{t}-\delta}. \end{aligned}$$

Let $\alpha \to \infty$, then $W^\epsilon(x,t) - U(x,t) - \beta(t-\delta) \leq \frac{\epsilon}{\bar{t}-\delta}$. Let $\beta \to 0$, then $\epsilon \to 0$, we get $W(x,t) \leq U(x,t)$. □

Now, combining Theorem 3 and Proposition 2, together with Lemma 10 for the $C(\overline{Q_T}) \cap C_{loc}^{2+\alpha,1+\frac{\alpha}{2}}(Q_T)$ solution to (4.3), we have

Step III: Regularity of the (HJB) equation without jump terms

Proposition 3. *Assume*

1. $a_{ij}, b_i, \Psi \in C^{0+\alpha,0+\frac{\alpha}{2}}(\overline{Q_T})$,
2. (a_{ij}) *uniformly elliptic,*
3. Ψ *is semiconcave,*
4. Ψ_t *is bounded below, in the distributional sense,*
5. $\phi \in C^{0+\alpha,0+\alpha/2}(\overline{Q}_T)$.

Then there exists a unique viscosity solution $u \in W_{loc}^{(2,1),p}(Q_T) \cap C(\overline{Q}_T)$ *to the PDE*

$$\begin{cases} \max\{u_t + Lu - f, u - \Psi\} = 0 & \text{in } Q_T, \\ u = \phi & \text{on } \partial_P Q_T, \end{cases}$$

for any $p > 1$.

Finally, since $Mu(x,t)$ is semi-concave from Lemma 9, replacing $\Psi(x,t)$ by $Mu(x,t)$ gives us the regularity property of the value function.

Theorem 4. *Assuming $\rho = 0$, the value function $V(x,t)$ is a $W^{2,p}_{loc}(\mathbb{R}^n \times (0,T))$ viscosity solution to the (HJB) equation with $2 \leq p < \infty$. In particular, for each $t \in [0,T)$, $V(\cdot,t) \in C^{1,\gamma}_{loc}(\mathbb{R}^n)$ for any $0 < \gamma < 1$.*

In fact, if one adds additional assumption of a_{ij} and b_i in $W^{(2,1),\infty}_{loc}(\mathbb{R}^n \times [0,T])$, then with more detailed and somewhat tedious analysis, one can establish $W^{(2,1),\infty}_{loc}$ regularity for the value function. For more details, see Chen [10].

4.3 $\mathbf{W}^{(2,1),\infty}_{loc}$ Regularity for cases without jumps

In fact, with an extra assumption, one can establish $W^{(2,1),\infty}_{loc}$ regularity.

Assumption 10. *($W^{(2,1),\infty}_{loc}$ Regularity) The functions a_{ij} and b_i are in $W^{(2,1),\infty}_{loc}(\mathbb{R}^n \times [0,T])$.*

First we start with the solution to (4.5). Differentiate it against t, we get

$$u^\epsilon_{tt} + L^\epsilon u^\epsilon_t + L^\epsilon_t u^\epsilon + \beta'_\epsilon(u^\epsilon - \Psi^\epsilon)(u^\epsilon_t - \Psi^\epsilon_t) = 0$$

in which $L^\epsilon_t u = -(a^\epsilon_{ij})_t u^\epsilon_{x_i x_j} + (b^\epsilon_i)_t u^\epsilon_{x_i}$.

Fix $r > 0$, and let ϕ be a cutoff function that is $= 1$ on Q'_T, defined as:

$$Q'_T = \{(x,t) \in Q_T : (|x-y|^2 + |t-s|) > r, \text{ for all } (y,s) \in \partial_P Q_T\}$$

and 0 within $r/2$ distance from the boundary $\partial_P Q_T$. Compute $(\phi u^\epsilon_t)_t - L^\epsilon(\phi u^\epsilon_t)$:

$$\begin{aligned}
&(\phi u^\epsilon_t)_t - L^\epsilon(\phi u^\epsilon_t)\\
=&(\phi_t + L^\epsilon \phi)u^\epsilon + \phi(u^\epsilon_{tt} + L^\epsilon u^\epsilon_t) - 2a^\epsilon_{ij}\phi_{x_i}u_{x_j t}\\
=&(\phi_t + L^\epsilon \phi)u^\epsilon + \phi(-L^\epsilon_t u^\epsilon - \beta'_\epsilon(u^\epsilon - \Psi^\epsilon)(u^\epsilon_t - \Psi^\epsilon_t)) - 2a^\epsilon_{ij}\phi_{x_i}u_{x_j t}\\
=&(\phi_t + L^\epsilon \phi)u^\epsilon + \phi(-L^\epsilon_t u^\epsilon - \beta'_\epsilon(u^\epsilon - \Psi^\epsilon)(u^\epsilon_t - \Psi^\epsilon_t)) - 2(a^\epsilon_{ij}\phi_{x_i}u_t)_{x_j} + 2(a^\epsilon_{ij}\phi)_{x_j}u_t
\end{aligned}$$

The following classical result is useful here. (See [32] Theorem 6.15 and Corollary 6.16.)

Theorem 5. *Suppose $a_{ij}, b_i \in L^\infty(Q_T)$, (a_{ij}) uniformly elliptic, $h_1, h^i_2 \in L^p(Q_T)$ for $p > n+2$, there exists a solution u to the following PDE:*

$$\begin{cases} u_t - a_{ij}u_{x_i x_j} + b_i u_{x_i} = h_1 + (h^i_2)_{x_i} & \text{in } Q_T \\ u = 0 & \text{on } \partial_P Q_T \end{cases}$$

with the following bound:

$$\|u\|_{L^\infty(Q_T)} \leq C(\|h_1\|_{L^{p/2}} + \|h_2\|_{L^p})$$

In order to apply the above Lemma to ϕu_t^ϵ, we need to show that $\beta'(u^\epsilon - \Psi^\epsilon)$ is bounded, independently of ϵ. Recall that $|\beta_\epsilon(u^\epsilon - \Psi^\epsilon)| \leq C$, and since for $x \geq 0$, $\beta'(x) \geq \beta'(0) = 1/\epsilon$, so when $u^\epsilon > \Psi^\epsilon$,

$$\frac{1}{\epsilon}(u^\epsilon - \Psi^\epsilon) \leq \beta_\epsilon(u^\epsilon - \Psi^\epsilon) \leq C$$

And we know that $\beta'_\epsilon(x) \leq x/\epsilon$ for $x \geq 0$,

$$\beta'_\epsilon(u^\epsilon - \Psi^\epsilon) \leq \frac{(u^\epsilon - \Psi^\epsilon)^+}{\epsilon} \leq C$$

The lower bound on β' is clear, so $\beta'(u^\epsilon - \Psi^\epsilon)$ is bounded independently of ϵ.

We arrive the following conclusion:

Lemma 13. *u_t is bounded in Q'_T independently of ϵ*

Back to the solution to (4.5). Differentiate both sides against direction $\eta \in \mathbb{R}^n$ with $|\eta| = 1$, we get

$$\begin{aligned} &u^\epsilon_{t\eta\eta} + L^\epsilon u^\epsilon_{\eta\eta} + \beta''_\epsilon(u^\epsilon - \Psi^\epsilon)(u^\epsilon_\eta - \Psi^\epsilon_\eta)^2 + \beta'_\epsilon(u^\epsilon - \Psi^\epsilon)(u^\epsilon_{\eta\eta} - \Psi^\epsilon_{\eta\eta}) \\ =&(a^\epsilon_{ij})_{\eta\eta} u^\epsilon_{x_i x_j} + 2(a^\epsilon_{ij})_\eta u^\epsilon_{x_i x_j \eta} - (b^\epsilon_i)_{\eta\eta} u^\epsilon_{x_i} - 2(b^\epsilon_i)_\eta u^\epsilon_{x_i \eta} \\ =&(a^\epsilon_{ij})_{\eta\eta} u^\epsilon_{x_i x_j} + 2\left((a^\epsilon_{ij})_\eta u^\epsilon_{x_j \eta}\right)_{x_i} - 2(a^\epsilon_{ij})_{\eta x_i} u^\epsilon_{x_j \eta} - (b^\epsilon_i)_{\eta\eta} u^\epsilon_{x_i} - 2(b^\epsilon_i)_\eta u^\epsilon_{x_i \eta} \end{aligned}$$

Note that the right hand side can be written as $h^\epsilon_1 + (h^{\epsilon,i}_2)_{x_i}$, in which for every $Q'_T \subset Q_T$, with a positive distance away from the parabolic boundary, $\|h^\epsilon_1\|_{L^p(Q'_T)} \leq C$, $\|h^{\epsilon,i}_2\|_{L^p(Q'_T)} \leq C$ for some C independent of ϵ.

Fix $r > 0$, and let ϕ be a cutoff function that is $= 1$ on Q'_T, which is defined as:

$$Q'_T = \{(x,t) \in Q_T : (|x-y|^2 + |t-s|) > r, \text{ for all } (y,s) \in \partial_P Q_T\}$$

and 0 within $r/2$ distance from the boundary $\partial_P Q_T$. Compute $(\phi u^\epsilon_{\eta\eta})_t - L^\epsilon(\phi u^\epsilon_{\eta\eta})$:

$$\begin{aligned} &(\phi u^\epsilon_{\eta\eta})_t + L^\epsilon(\phi u^\epsilon_{\eta\eta}) \\ =&(\phi_t + L^\epsilon \phi) u^\epsilon_{\eta\eta} + \phi(u^\epsilon_{\eta\eta t} + L^\epsilon u^\epsilon_{\eta\eta}) - 2(a^\epsilon_{ij})_{x_i} u^\epsilon_{x_j \eta} \\ =&(\phi_t + L^\epsilon \phi) u^\epsilon_{\eta\eta} - 2(a^\epsilon_{ij})_{x_i} u^\epsilon_{x_j \eta} \\ &+ \phi\left(h^\epsilon_1 + (h^{\epsilon,i}_2)_{x_i} - \beta''_\epsilon(u^\epsilon - \Psi^\epsilon)(u^\epsilon_\eta - \Psi^\epsilon_\eta)^2 - \beta'_\epsilon(u^\epsilon - \Psi^\epsilon)(u^\epsilon_{\eta\eta} - \Psi^\epsilon_{\eta\eta})\right) \\ =&(\phi_t + L^\epsilon \phi) u^\epsilon_{\eta\eta} - 2(a^\epsilon_{ij})_{x_i} u^\epsilon_{x_j \eta} + \phi h^\epsilon_1 + (\phi h^{\epsilon,i}_2)_{x_i} - \phi_{x_i} h^{\epsilon,i}_2 \\ &- \phi \beta''_\epsilon(u^\epsilon - \Psi^\epsilon)(u^\epsilon_\eta - \Psi^\epsilon_\eta)^2 - \phi \beta'_\epsilon(u^\epsilon - \Psi^\epsilon)(u^\epsilon_{\eta\eta} - \Psi^\epsilon_{\eta\eta}) \end{aligned}$$

The right hand side, excluding β_ϵ terms, can now as before, written as $\tilde{h}^\epsilon_1 + (\tilde{h}^{\epsilon,i}_2)_{x_i}$, in which $\|\tilde{h}^\epsilon_1\|_{L^p(Q_T)} \leq C$, $\|\tilde{h}^{\epsilon,i}_2\|_{L^p(Q_T)} \leq C$ for some C independent of ϵ. Note that $\phi u^\epsilon_{\eta\eta} = 0$

on $\partial_P Q_T$. We can apply Theorem 5 again and obtain bounded solution v^ϵ for the respective ϵ. We find that $w^\epsilon = \phi u^\epsilon_{\eta\eta} - v^\epsilon$ solves the following equation:

$$\begin{cases} w^\epsilon_t + L^\epsilon w^\epsilon + \phi\beta''_\epsilon(u^\epsilon - \Psi^\epsilon)(u^\epsilon_\eta - \Psi^\epsilon_\eta)^2 + \phi\beta'_\epsilon(u^\epsilon - \Psi^\epsilon)(u^\epsilon_{\eta\eta} - \Psi^\epsilon_{\eta\eta}) = 0 & \text{in } Q_T \\ w^\epsilon = 0 & \text{on } \partial_P Q_T \end{cases}$$

We know that $|v^\epsilon| \le C$ independent of ϵ. Now we will try to bound w^ϵ.

Lemma 14. *w^ϵ, defined as above, is bounded above, independently of ϵ. And thus, $u_{\eta\eta}$ is bounded above on Q'_T.*

Proof. Suppose w^ϵ has a maximum at $(x_0, t_0) \in Q_T$, then $(w^\epsilon_t + L^\epsilon w^\epsilon)(x_0, t_0) \ge 0$, thus

$$\phi\beta'_\epsilon(u^\epsilon - \Psi^\epsilon)(u^\epsilon_{\eta\eta} - \Psi^\epsilon_{\eta\eta}) \le 0$$

If $\phi(x_0, t_0) = 0$, then $w^\epsilon(x_0, t_0) = -v^\epsilon(x_0, t_0)$, which is bounded. On the other hand if $\phi(x_0, t_0) > 0$, then since $\beta'(x) > 0$, we get $u^\epsilon_{\eta\eta}(x_0, t_0) \le \Psi^\epsilon_{\eta\eta}(x_0, t_0)$. Then $w^\epsilon(x_0, t_0) = \phi(x_0, t_0)u^\epsilon_{\eta\eta}(x_0, t_0) - v^\epsilon(x_0, t_0) \le \Psi^\epsilon_{\eta\eta}(x_0, t_0) - v^\epsilon(x_0, t_0)$. The initial assumption that Ψ being semiconcave gives us a bound on $\Psi^\epsilon_{\eta\eta}$ independent of ϵ. Either way w^ϵ is bounded above independently of ϵ. So now we have a uniform upper bound on $u^\epsilon_{\eta\eta} = w^\epsilon + v^\epsilon$ in Q'_T. □

Now, for each $(x_0, t_0) \in Q'_T$, we can find a new coordinate system with orthogonal transofrmation such that $a^\epsilon_{ij}(x_0, t_0)u_{x_i x_j} = \lambda^i u_{y_i y_i}$ with λ^i bounded above and below away from 0, independent of (x_0, t_0) and ϵ. Then, we have

$$|\lambda^i u^\epsilon_{y_i y_i}(x_0, t_0)| \le |u^\epsilon_t(x_0, t_0)| + |\beta_\epsilon(u^\epsilon(x_0, t_0) - \Psi^\epsilon(x_0, t_0))| \le C$$

in which C is independent of ϵ. For $u^\epsilon_{y_1 y_1}$, we have

$$-C \le \lambda^1 u^\epsilon_{y_1 y_1} + \sum_{i \ne 1} \lambda^i u_{y_i y_i} \le C$$

$$\lambda^1 u^\epsilon_{y_1 y_1} \ge -C - \sum_{i \ne 1} \lambda^i u_{y_i y_i} \ge -C'$$

Proceed similarly for other components, we get $u^\epsilon_{y_i y_i}$ being bounded uniformly in ϵ and $(x_0, t_0) \in Q'_T$. Since mixed second derivative can be written as a linear combination of the pure second derivatives against a basis, we get the conclusion:

Lemma 15.

$$\|u^\epsilon_t\| \le C$$
$$\|D^2 u^\epsilon\| \le C$$

in Q'_T, for C not depending on ϵ.

Therefore, we have

Theorem 6. *With additional assumption 10 and assuming there is no jump term $\rho = 0$, the value function $V(x, t)$ is in $W^{2,p}_{loc}(\mathbb{R}^n \times (0, T))$ for any $2 \le p < \infty$. In particular, for each $t \in [0, T)$, $V(\cdot, t) \in C^{1,\gamma}_{loc}(\mathbb{R}^n)$ for any $0 < \gamma < 1$.*

4.4 Regularity for value function of first-order jump diffusion

Careful checking of our analysis for the case without jumps in Section 4.2 reveals that we could relax our assumption on f to be bounded instead of Hölder continuous, and take f^ϵ to be Hölder continuous and converge to f in L^∞. This observation is key for the regularity with jumps.

To proceed, we will add two new assumptions in this subsection.

Assumption 11. *The operator $I[\phi]$ is of order-δ, i.e, for (x,t) in any compact subset of $\mathbb{R}^n \times [0,T]$, there exists C such that*

$$\int_{|z|<1} |z|^\delta M(x,t,dz) < C < \infty.$$

Remark. Since $\delta \in (0,1]$, this implies that $\int_{|z|<1} |z| M(x,t,dz) < C$.

Assumption 12. *The measure $M(x,t,dz)$ is continuous with respect to the weighted total variation, i.e., for $(x_n,t_n) \to (x_0,t_0)$,*

$$\int \left(|z|^\gamma + |z|^\delta\right) |M(x_n,t_n,dz) - M(x_0,t_0,dz)| \to 0.$$

Proposition 4. *With additional assumptions 11 and 12, there exists a unique $u \in W^{(2,1),p}(Q_T)$ viscosity solution of the following equations,*

$$\begin{cases} \max\{-u_t + Lu - f - Iu, u - MV\} = 0 & \text{in } Q_T, \\ u = V(x, T-t) & \text{on } \partial_P Q_T, \end{cases} \tag{4.7}$$

in the following sense. For any $\phi \in C^2(\mathbb{R}^n \times [0,T])$,

1. *If $u - \phi$ achieves a local maximum at $(x_0,t_0) \in Q_T$, then*

$$\max\{-\phi_t + L\phi - f - I^0[V], u - MV\}(x_0,t_0) \le 0;$$

2. *If $u - \phi$ achieves a local minimum at $(x_0,t_0) \in Q_T$, then*

$$\max\{-\phi_t + L\phi - f - I^0[V], u - MV\}(x_0,t_0) \ge 0.$$

Here

$$I^0[V](x,t) = \int V(x+z,t) - V(x,t) - D\phi(x,t) \cdot z 1_{|z|<1} \rho(dz),$$

with the boundary condition $u = g$ on $\mathbb{R}^n \times \{t = T\}$.

Proof. Let $\bar{b}_i = b_i - \int z_i M(x,t,dz)$ and $\overline{f} = \int u(x+z,t) - u(x,t)M(x,t,dz)$. It suffices to show that $\bar{b}_i$ and $\overline{f}$ are bounded. In fact we will show that they are continuous.

Step 1, $\overline{f}$ is continuous:
Let $x_n \to x_0$, then

$$\begin{aligned}
&\left|\int V(x_n+z,t) - V(x_n,t)M(x_n,t,dz) - \int V(x_0+z,t) - V(x_0,t)M(x_0,t,dz)\right| \\
\leq &\left|\int \left(V(x_n+z,t) - V(x_n,t)\right)\left(M(x_n,t,dz) - M(x_0,t,dz)\right)\right| \\
&+ \left|\int \left(V(x_n+z,t) - V(x_n,t)\right) - \left(V(x_0+z,t) - V(x_0,t)\right) M(x_0,t,dz)\right|.
\end{aligned}$$

For the first term,

$$\begin{aligned}
&\left|\int \left(V(x_n+z,t) - V(x_n,t)\right)\left(M(x_n,t,dz) - M(x_0,t,dz)\right)\right| \\
\leq &C\int \left(1+|x_n|^\gamma + |z|^\gamma\right)|z|^\delta |M(x_n,t,dz) - M(x_0,t,dz)| \\
\leq &C\int |z|^\gamma + |z|^\delta |M(x_n,t,dz) - M(x_0,t,dz)| \to 0.
\end{aligned}$$

For the second term, the integrand $\to 0$ as $n \to \infty$. So by the dominated convergence theorem,

$$\begin{aligned}
&\left|\int \left(V(x_n+z,t) - V(x_n,t)\right) - \left(V(x_0+z,t) - V(x_0,t)\right) M(x_0,t,dz)\right| \\
\leq &\int C(1+|x_n|^\gamma + |z|^\gamma)|z|^\delta + C(1+|x_0|^\gamma + |z|^\gamma)|z|^\delta M(x_0,t,dz) \\
\leq &C\int |z|^\gamma + |z|^\delta M(x_0,t,dz) < \infty.
\end{aligned}$$

Therefore $\overline{f}$ is continuous in x. Now let $t_n \to t_0$,

$$\begin{aligned}
&\left|\int V(x+z,t_n) - V(x,t_n)M(x,t_n,dz) - \int V(x+z,t_0) - V(x,t_0)M(x,t_0,dz)\right| \\
\leq &\left|\int \left(V(x+z,t_n) - V(x,t_n)\right)\left(M(x,t_n,dz) - M(x,t_0,dz)\right)\right| \\
&+ \left|\int \left(V(x+z,t_n) - V(x,t_n)\right) - \left(V(x+z,t_0) - V(x,t_0)\right) M(x_0,t,dz)\right|.
\end{aligned}$$

For the first term,

$$
\begin{aligned}
&\left|\int (V(x+z,t_n) - V(x,t_n))\,(M(x,t_n,dz) - M(x,t_0,dz))\right| \\
\leq &\int C(1+|x|^\gamma + |z|^\gamma)|z|^\delta\,|M(x,t_n,dz) - M(x,t_0,dz)| \\
\leq &C\int |z|^\gamma + |z|^\delta\,|M(x,t_n,dz) - M(x,t_0,dz)| \to 0,
\end{aligned}
$$

as $t_n \to t_0$. For the second term, the dominated convergence theorem implies

$$
\begin{aligned}
&\left|\int (V(x+z,t_n) - V(x,t_n)) - (V(x+z,t_0) - V(x,t_0))\,M(x_0,t,dz)\right| \\
\leq &\int C(1+|x|^\gamma + |z|^\gamma)|z|^\delta M(x,t_0,dz) < \infty.
\end{aligned}
$$

Therefore $\overline{f}$ is continuous in t.

Step 2, $\bar{b}_i$ is continuous:
This follows easily from Assumption 12. Let $(x_n,t_n) \to (x_0,t_0)$,

$$
\begin{aligned}
\left|\int_{|z|<1} z[M(x_n,t_n,dz) - M(x_0,t_0,dz)]\right| &\leq \int_{|z|<1} |z||M(x_n,t_n,dz) - M(x_0,t_0,dz)| \\
&\leq \int |z|^\delta |M(x_n,t_n,dz) - M(x_0,t_0,dz)|,
\end{aligned}
$$

which goes to 0 as $n \to \infty$.

Step 3, replace b_i by $\bar{b}_i = b_i - \int z_i M(x,t,dz)$ and f by $\overline{f} = f + \int u(x+z,t) - u(x,t)M(x,t,dz)$, and follow the same line of reasoning in the proof for Proposition 3.

□

Notice, however, the "apparent" difference between the two types of viscosity solutions: the one in the above proposition, and the one in Theorem 2. Therefore, we need to show that the viscosity solution in Theorem 2 is also a viscosity solution of Eqn. (4.7). Then, with the standard local uniqueness of HJB of equation (4.7), the regularity of value function is obtained.

Theorem 7. *V is also a solution of equation (4.7), with additional assumptions 11 and 12.*

Proof. Suppose $V - \phi$ has a local minimum in $B(x_0,\theta_0) \times [t_0, t+\theta_0)$. Then we know that

$$\max\{-\phi_t + L\phi - f - I^1_\theta[\phi] - I^2_\theta[V], V - MV\} \leq 0$$

for any $0 < \theta < \theta_0$. And with the additional assumptions, $I^1_\theta[\phi] + I^2_\theta[V] \to I^0[V]$ as $\theta \to 0$. Therefore we have

$$\max\{-\phi_t + L\phi - f - I^0[V], V - MV\} \leq 0$$

The other inequalities can be derived similarly. □

In summary,

Theorem 8. *(Regularity of the Value Function and Uniqueness) With additional assumptions 11 and 12, the value function* $V(x,t)$ *is a unique* $W^{(2,1),p}_{loc}(\mathbb{R}^n \times (0,T))$ *viscosity solution to the (HJB) equation with* $2 \le p < \infty$. *In particular, for each* $t \in [0,T)$, $V(\cdot,t) \in C^{1,\gamma}_{loc}(\mathbb{R}^n)$ *for any* $0 < \gamma < 1$.

Chapter 5

Optimal stopping for second order jump diffusions

5.1 Integro-differential equations

Let $M(x, dz)$ be the Levy measure on $\mathbb{R}^n \setminus \{0\}$, such that

$$\int_{|z|>0} (|z|^2 \wedge 1) M(x; dz)$$

is bounded in $x \in \mathbb{R}^n$, and for each $\Gamma \in \mathcal{B}(\mathbb{R}^n \setminus \{0\})$,

$$\int_{\Gamma} (|z|^2 \wedge 1) M(x; dz)$$

is uniformly continuous for $x \in \Omega$. (These are the set of conditions imposed to ensure the existence of weak solution to the jump diffusion).

We would like to solve the equation

$$\begin{cases} Lu - Iu = f & \text{in } \Omega \\ u = g & \text{on } \Omega^c \end{cases}$$

in which $Lu = -a_{ij}u_{x_i x_j} + b_i u_{x_i} + cu$, and

$$Iu(x) = \int_{0<|z|} u(x+z) - u(x) - Du(x) \cdot z 1_{|z|<1} \; M(x, dz)$$

First, we will break up Iu into three pieces: Let $r = \text{diam}(\Omega)$.

$$\begin{aligned} Iu(x) = &\int_{|z|\geq r} u(x+z) - u(x) M(x; dz) - Du(x) \cdot \int_{r \leq |z| < 1} z \; M(x; dz) \\ &+ \int_{0<|z|<1} u(x+z) - u(x) - Du(x) \cdot z 1_{|z|<1} \; M(x, dz) \end{aligned}$$

If $r < 1$, then we simply don't have the second term. For the first term, $x + z$ lies outside of Ω, thus we have

$$\int_{|z|\geq r} \varphi(x+z)M(x;dz) - u(x)\left(\int_{|z|\geq r} M(x;dz)\right) - Du(x)\cdot\int_{r\leq|z|<1} z\ M(x;dz)$$

$\int_{|z|\geq r} M(x;dz)$ and $\int_{r\leq|z|<1} zM(x;dz)$ are uniformly continuous in x, these two terms can be absorbed into c and (b_i) in the elliptic operator. For the problem to be well-defined, we need $\int_{|z|\geq r} g(x+z)M(x;dz)$ to be at least bounded. Thus we will need to impose extra growth assumption on g and M to fix this. For the rest of the section, we will assume the following,

Assumption 13. *$M(x,dz)$ is supported on some ball $B(0,r)\setminus\{0\}$, and thus*

$$\begin{aligned}Iu(x) =& \int_{0<|z|<1} u(x+z) - u(x) - Du(x)\cdot z1_{|z|<1}\ M(x,dz)\\ &+ \int_{1\leq|z|<r} u(x+z) - u(x)\ M(x,dz)\\ =& I^1u(x) + I^2u(x)\end{aligned}$$

Notation 2.

$$\begin{aligned}d_x =& d(x,\partial\Omega) = \inf\{|x-y| : y\in\partial\Omega\}\\ d_{x,y} =& \min\{d(x,\partial\Omega), d(y,\partial\Omega)\}\\ \Omega_\delta =& \{y\in\Omega : d(y,\partial\Omega)\geq\delta\}\\ \Omega_I =& \{x + supp\, M(x,dz) : x\in\Omega\} = \{x\in\mathbb{R}^n : d(x,\overline{\Omega})\leq r\}\\ [u]_{\alpha;\Omega} =& \sup_{x,y\in\Omega}\frac{|u(x)-u(y)|}{|x-y|^\alpha}\\ |u|_{0;\Omega} =& \sup_{x\in\Omega}|u(x)|\\ |u|_{\alpha;\Omega} =& |u|_{0;\Omega} + [u]_{\alpha;\Omega}\\ |u|^{(\beta)}_{0;\Omega} =& \sup_{x\in\Omega} d_x^\beta|u(x)|\\ [u]^{(\beta)}_{0;\Omega} =& \sup_{x\in\Omega} d_{x,y}^{\beta+\alpha}\frac{|u(x)-u(y)|}{|x-y|^\alpha}\\ |u|^{(\beta)}_{\alpha;\Omega} =& |u|^{(\beta)}_{0;\Omega} + [u]^{(\beta)}_{0;\Omega}\\ |u|^{(\beta)}_{k+\alpha;\Omega} =& \sum_{j=0}^{k}|D^k u|^{(\beta+k)}_{\alpha;\Omega}\end{aligned}$$

5.2 $C^{2+\alpha}$ Theory

We need to make the following assumption:

Assumption 14. *1. For $\delta > 0$, let*

$$K_1(\delta) = \sup_{x\in\Omega} \int_{0<|z|<\delta} |z|^2 \wedge 1 M(x;dz)$$

Then $K_1(\delta)$ is finite for each $\delta > 0$, and $K_1(\delta) \to 0$ as $\delta \to 0$. Let K_1 denote $K_1(r)$.

2. For $\delta > 0$,

$$K_2(\delta) = \sup_{x,y\in\Omega} \frac{\int_{0<|z|<\delta} |z|^2 \wedge 1 |M(x;dz) - M(y;dz)|}{|x-y|^\alpha}$$

Then $K_2(\delta)$ is finite for each $\delta > 0$, and $K_2(\delta) \to 0$ as $\delta \to 0$. Let K_2 denote $K_2(r)$.

In other words, M is Holder continuous uniformly in Ω, and as we reduce the integration radius δ, we can make this Holder constant as small as we want.

Lemma 16. *Under the Assumption 13 and 14, let $\varphi \in C^{2+\alpha}(\overline{\Omega}_I)$, then $I\varphi \in C^\alpha(\overline{\Omega})$, and for $\epsilon > 0$, we have the following estimates:*

$$|I\varphi|_{0;\Omega} \le \epsilon |D^2\varphi|_{0;\Omega_I} + C(|\varphi|_{0;\Omega_I} + |D\varphi|_{0;\Omega_I})$$
$$[I\varphi]_{\alpha;\Omega} \le \epsilon [D^2\varphi]_{\alpha;\Omega_I} + C([\varphi]_{\alpha;\Omega_I} + [D\varphi]_{\alpha;\Omega_I})$$

and combined together, with interpolation inequality,

$$|I\varphi|_{\alpha;\Omega} \le \epsilon |D^2\varphi|_{\alpha;\Omega_I} + C(|\varphi|_{\alpha;\Omega_I})$$

for $C = C(\epsilon, K_1, K_2)$.

Proof. Let $0 < \delta < 1$.

$$\begin{aligned} I\varphi(x) = &\int_0^1 (1-s) \int_{0<|z|<\delta} z^T \cdot D^2\varphi(x+sz) \cdot z\, M(x;dz) ds \\ &+ \int_{|z|\ge\delta} \varphi(x+z) - \varphi(x) - z \cdot D\varphi(x) 1_{|z|<1}\, M(x;dz) ds \end{aligned}$$

Thus for δ small enough,

$$|I\varphi|_{0;\Omega} \le \epsilon |D^2\varphi|_{0;\Omega_I} + C(|D\varphi|_{0;\Omega_I} + |\varphi|_{0;\Omega_I})$$

On the other hand,

$$
\begin{aligned}
&|I\varphi(x) - I\varphi(y)| \\
\leq &\left| \int_0^1 (1-s) \int_{0<|z|<\delta} z^T \cdot (D^2\varphi(x+sz) - D^2\varphi(y+sz)) \cdot z \, M(x;dz) ds \right| \\
&+ \left| \int_0^1 (1-s) \int_{0<|z|<\delta} z^T \cdot D^2\varphi(y+sz) \cdot z \, (M(x;dz) - M(y;dz) ds \right| \\
&+ \left| \int_{|z|\geq\delta} (\varphi(x+z) - \varphi(y+z)) - (\varphi(x) - \varphi(y)) - z \cdot (D\varphi(x) - D\varphi(y)) 1_{|z|<1} \, M(x;dz) ds \right| \\
&+ \left| \int_{|z|\geq\delta} (\varphi(y+z) - \varphi(y) - z \cdot D\varphi(y) 1_{|z|<1}) \, (M(x;dz) - M(y;dz)) ds \right| \\
\leq &K_1(\delta)[D^2\varphi]_{\alpha;\Omega_I} + K_2(\delta)|D^2\varphi|_{0;\Omega_I} \\
&+ 2[\varphi]_{\alpha;\Omega_I} \int_{|z|\geq\delta} \frac{|z|^2}{\delta^2} M(x;dz) + [D\varphi]_{\alpha;\Omega_I} \int_{|z|\geq\delta} \frac{|z|^2}{\delta} M(x;dz) \\
&+ \int_{|z|\geq\delta} \left(2|\varphi|_{0;\Omega_I} \frac{|z|^2}{\delta^2} + |z||D\varphi|_{0;\Omega_I} \frac{|z|}{\delta} \right) |M(x;dz) - M(y;dz)| \\
\leq &\left[(K_1(\delta) + K_2(\delta))[D^2\varphi]_{\alpha;\Omega_I} + C([\varphi]_{\alpha;\Omega_I} + [D\varphi]_{\alpha;\Omega_I}) \right] |x-y|^\alpha
\end{aligned}
$$

By choosing δ sufficiently small, we get the desired result. □

Lemma 17. *Under the Assumption 13 and 14, and assume $\partial\Omega \in C^{2+\alpha}$, if $\varphi \in C^{2+\alpha}(\overline{\Omega}) \cap C^\alpha(\overline{\Omega}_I)$, then for $\epsilon > 0$, there exists $C = C(\epsilon, K_1, K_2)$ such that*

$$
|I\varphi|^{(2-\alpha)}_{\alpha;\Omega} \leq \epsilon |D^2\varphi|_{\alpha;\Omega} + C(|\varphi|_{\alpha;\Omega_I \setminus \overline{\Omega}} + |D\varphi|_{\alpha;\Omega})
$$

Proof. Given $\varphi \in C^{2+\alpha}(\overline{\Omega}) \cap C^\alpha(\overline{\Omega}_I)$, we first extends φ to Ω_I such that the extension $\tilde{\varphi}$ is in $C^{2+\alpha}(\overline{\Omega}_I)$ with $\|\tilde{\varphi}\|_{C^{2+\alpha}(\overline{\Omega}_I)} \leq C\|\varphi\|_{C^{2+\alpha}(\overline{\Omega})}$. So we need to estimate $I(\varphi - \tilde{\varphi})$ to get the conclusion. Thus, let $\varphi' = \varphi - \tilde{\varphi}$, we know that $\varphi' = 0$ in Ω, and $\varphi' \in C^\alpha(\overline{\Omega}_I)$, with $|\varphi'|_{\alpha;\Omega_I} \leq C|\varphi|_{\alpha;\Omega}$.

Step 1. For any $x \in \Omega$ such that $d(x, \partial\Omega) < 1$,

$$
\begin{aligned}
I\varphi'(x) &= \int_{x+z\notin\Omega} \varphi'(x+z) M(x,dz) \\
&= \int_{x+z\notin\Omega} \varphi'(x+z) - \varphi'(x + t(x,z)z) M(x,dz)
\end{aligned}
$$

in which $t(x,z)=\inf\{t\geq 0: x+tz\in\overline{\Omega}\}$.

$$\begin{aligned}|I\varphi'(x)| \leq&[\varphi']_{\alpha;\Omega_I}\int_{x+z\notin\Omega}|z-t(x,z)z|^\alpha M(x,dz)\\ \leq&[\varphi']_{\alpha;\Omega_I}\int_{|z|>d(x,\partial\Omega)}|z|^\alpha M(x,dz)\\ \leq&[\varphi']_{\alpha;\Omega_I}\frac{1}{d(x,\partial\Omega)^{2-\alpha}}\int|z|^2M(x,dz)\\ \leq&C[\varphi']_{\alpha;\Omega_I}\frac{1}{d(x,\partial\Omega)^{2-\alpha}}\end{aligned}$$

Step 2. Let $x,y\in\Omega$. Denote $d_{x,y}=\min\{d(x,\partial\Omega),d(y,\partial\Omega)\}$.

$$\begin{aligned}|I\varphi'(x)-I\varphi'(y)|\leq&\left|\int_{|z|>d_{x,y}}\varphi'(x+z)(M(x,dz)-M(y,dz))\right|\\ &+\int_{|z|>d_{x,y}}|\varphi'(x+z)-\varphi'(y+z)|M(x,dz)\\ \leq&|\varphi'|_{0;\Omega_I}\int_{d_{x,y}<|z|}\frac{|z|^2}{d_{x,y}^2}|M(x,dz)-M(y,dz)|\\ &+\int_{|z|>d_{x,y}}|x-y|^\alpha[\varphi']_{\alpha;\Omega_I}\frac{|z|^2}{d_{x,y}^2}M(x,dz)\\ \leq&\frac{|\varphi'|_{0;\Omega_I}+[\varphi']_{\alpha;\Omega_I}}{d_{x,y}^2}|x-y|^\alpha\end{aligned}$$

Combine these pieces, we have

$$\begin{aligned}|I\varphi|^{(2-\alpha)}_{\alpha;\Omega}\leq&|I\tilde{\varphi}|^{(2-\alpha)}_{\alpha;\Omega}+|I\varphi'|^{(2-\alpha)}_{\alpha;\Omega}\\ \leq&C|I\tilde{\varphi}|_{\alpha;\Omega}+C|\varphi'|_{\alpha;\Omega_I}\\ \leq&\epsilon|D^2\tilde{\varphi}|_{\alpha;\Omega_I}+C|\tilde{\varphi}|_{\alpha;\Omega_I}+C|\varphi'|_{\alpha;\Omega_I}\\ \leq&\epsilon|D^2\varphi|_{\alpha;\Omega_I}+C|\varphi|_{\alpha;\Omega_I}\end{aligned}$$

□

Notation 3. *Let $\varphi\in C^0(\overline{\Omega}_I\setminus\Omega)$, define the operator $I_\varphi:\{u\in C^2(\Omega): u=\varphi$ on $\partial\Omega\}\to C(\Omega)$ by defining the extension*

$$\bar{u}=\begin{cases}u & x\in\Omega\\ \varphi & x\in\Omega_I\setminus\Omega\end{cases}$$

and define $I_\varphi u=I\bar{u}$. If $\varphi=0$, we simply write $I_0u=I_\varphi u$.

We can derive more delicate estimates in the case when $u = 0$ on $\partial\Omega$:
First we need an interpolation inequality:

Lemma 18. *Let $\varphi \in C^2(\Omega)$, let $\alpha \in (0,1]$,*

$$|D\varphi|^{(1-\alpha)}_{0;\Omega} \leq C[\varphi]_{\alpha;\Omega} + \epsilon |D^2\varphi|^{(2-\alpha)}_{0;\Omega}$$

Proof. Fix $x \in \Omega$, and let $d = \mu d_x = d(x, \partial\Omega)$, for $\mu \in (0,1)$. Let x' and x'' be the two endpoints of the segment of length $2d$ parallel to the x_i-axis, with x being the center of the segment. By the mean value theorem, there exists $\bar{x}$ on the line segment such that

$$|\varphi_{x_i}(\bar{x})| = \frac{|\varphi(x') - \varphi(x'')|}{2d} \leq \frac{[\varphi]_{\alpha;\Omega}(2d)^\alpha}{2d} = \frac{[\varphi]_{\alpha;\Omega}}{2^{1-\alpha}d^{1-\alpha}}$$

Thus

$$\begin{aligned} |u_{x_i}(x)| =& \left| u_{x_i}(\bar{x}) + \int_0^1 Du_{x_i}(\bar{x} + s(x - \bar{x})) \cdot (x - \bar{x}) ds \right| \\ \leq& |u_{x_i}(\bar{x})| + d|D^2u|_{0;\Omega_{(1-\mu)d_x}} \\ \leq& \frac{[\varphi]_{\alpha;\Omega}}{2^{1-\alpha}d^{1-\alpha}} + d\frac{|D^2u|^{(2-\alpha)}_{0;\Omega}}{(1-\mu)^{2-\alpha}d_x^{2-\alpha}} \\ \leq& \frac{[\varphi]_{\alpha;\Omega}}{2^{1-\alpha}\mu^{1-\alpha}d_x^{1-\alpha}} + \frac{\mu}{(1-\mu)^{2-\alpha}} \frac{|D^2u|^{(2-\alpha)}_{0;\Omega}}{d_x^{1-\alpha}} \end{aligned}$$

Thus, for μ sufficiently small,

$$d_x^{1-\alpha}|\varphi_{x_i}(x)| \leq \epsilon |D^2\varphi|^{(2-\alpha)}_{0;\Omega} + C[\varphi]_{\alpha;\Omega}$$

Take supremum over x and sum over i, we get the desired inequality. □

Lemma 19. *If $\varphi \in C^{2+\alpha,(-\alpha)}(\Omega)$, then $I_0\varphi \in C^{\alpha,(2-\alpha)}(\Omega)$ and for each $\epsilon > 0$, there exists $C = C(\epsilon, K_1, K_2)$ such that*

$$|I\varphi|^{(2-\alpha)}_{\alpha;\Omega} \leq \epsilon |D^2\varphi|^{(2-\alpha)}_{\alpha;\Omega} + C[\varphi]_{\alpha;\Omega}$$

and since $[\varphi]_{\alpha;\Omega} = [\varphi]^{(-\alpha)}_{\alpha;\Omega}$, we can use interpolation to get

$$|I\varphi|^{(2-\alpha)}_{\alpha;\Omega} \leq \epsilon |\varphi|^{(-\alpha)}_{2+\alpha;\Omega} + C|\varphi|^{(-\alpha)}_{0;\Omega}$$

Proof. Step 1. Given $x \in \Omega$, fix $\mu \in (0,1)$, and let $d = \mu d_x = d(x, \partial\Omega)$, for $\mu \in (0,1)$. Let $d' = d_x - \mu d_x = (1-\mu)d_x$.

$$
\begin{aligned}
|I\varphi(x)| \leq & \left| \int_{|z|<d} \int_0^1 (1-s) z^T D^2 \varphi(x+sz) z \, ds \, M(x;dz) \right| \\
& + \left| \int_{|z|\geq d} \varphi(x+z) - \varphi(x) M(x;dz) \right| + \left| \int_{|z|\geq d} z \cdot D\varphi(x) M(x;dz) \right| \\
\leq & K_1(d) |D^2\varphi|_{0;\Omega_{d'}} + [\varphi]_{\alpha;\Omega} \int_{|z|\geq d} |z|^\alpha M(x;dz) + |D\varphi|_{0;\Omega_{d_x}} \int_{|z|\geq d} |z| M(x;dz) \\
\leq & K_1(d) \frac{|D^2\varphi|^{(2-\alpha)}_{0;\Omega}}{d'^{2-\alpha}} + [\varphi]_{\alpha;\Omega} \int_{|z|\geq d} \frac{|z|^2}{d^{2-\alpha}} M(x;dz) + \frac{|D\varphi|^{(1-\alpha)}_{0;\Omega}}{d_x^{1-\alpha}} \int_{|z|\geq d} \frac{|z|^2}{d} M(x;dz) \\
\leq & K_1(\mu \cdot \operatorname{diam}(\Omega)) \frac{|D^2\varphi|^{(2-\alpha)}_{0;\Omega}}{(1-\mu)^{2-\alpha} d_x^{2-\alpha}} + K_1 \frac{[\varphi]_{\alpha;\Omega}}{\mu^{2-\alpha} d_x^{2-\alpha}} + K_1 \frac{|D\varphi|^{(1-\alpha)}_{0;\Omega}}{\mu d_x^{2-\alpha}}
\end{aligned}
$$

Pick μ small enough such that $K_1(\mu \cdot \operatorname{diam}(\Omega)) \leq \epsilon$, and by the interpolation inequality Lemma 18, we get

$$
|I\varphi|^{(2-\alpha)}_{0;\Omega} \leq \epsilon |D^2\varphi|^{(2-\alpha)}_{0;\Omega} + C[\varphi]_{\alpha;\Omega}
$$

in which $C = C(\epsilon, K_1, \Omega)$.

Step 2. Fix $x, y \in \Omega$. Let $d_{x,y} = \min\{d(x,\partial\Omega), d(y,\partial\Omega)\}$, and $d = \mu d_{x,y}$ for $\mu \in (0,1)$, $d' = (1-\mu)d_{x,y}$.

$$
\begin{aligned}
|I\varphi(x) - I\varphi(y)| \leq & \left| \int_{|z|<d} \int_0^1 (1-s) z^T (D^2\varphi(x+sz) - D^2\varphi(y+sz)) z \, ds \, M(x;dz) \right| \\
& + \left| \int_{|z|<d} \int_0^1 (1-s) z^T D^2\varphi(y+sz) z \, ds \, (M(x;dz) - M(y;dz)) \right| \qquad (5.1) \\
& + \left| \int_{|z|\geq d} \varphi(x+z) - \varphi(y+z) - \varphi(x) + \varphi(y) M(x;dz) \right| \\
& + \left| \int_{|z|\geq d} (\varphi(y+z) - \varphi(y))(M(x;dz) - M(y;dz)) \right| \qquad (5.2) \\
& + \left| \int_{|z|\geq d} z \cdot (D\varphi(x) - D\varphi(y)) M(x;dz) \right| \\
& + \left| \int_{|z|\geq d} z \cdot D\varphi(y)(M(x;dz) - M(y;dz)) \right| \qquad (5.3)
\end{aligned}
$$

We will break up these six terms into three groups.

Part a). Bound the first two terms (5.1).

$$
\begin{aligned}
&\left|\int_{|z|<d}\int_0^1 (1-s)z^T(D^2\varphi(x+sz)-D^2\varphi(y+sz))z\,ds\,M(x;dz)\right| \\
&\quad+\left|\int_{|z|<d}\int_0^1 (1-s)z^T D^2\varphi(y+sz)z\,ds\,(M(x;dz)-M(y;dz))\right| \\
\leq& K_1(d)[D^2\varphi]_{\alpha;\Omega_{d'}}|x-y|^\alpha \\
&\quad+|D^2\varphi|_{0;\Omega_{d'}}\int_{|z|<d}|z|^2|M(x;dz)-M(y;dz)| \\
\leq& K_1(d)\frac{[D^2\varphi]^{(2-\alpha)}_{\alpha;\Omega}}{d'^2}|x-y|^\alpha \\
&\quad+K_2(d)\frac{|D^2\varphi|^{(2-\alpha)}_{0;\Omega}}{d'^{2-\alpha}}|x-y|^\alpha\cdot\frac{\mathrm{diam}(\Omega)^\alpha}{d^\alpha_{x,y}}
\end{aligned}
$$

Part b). Bound the second two terms (5.2).

$$
\begin{aligned}
&\left|\int_{|z|\geq d}\varphi(x+z)-\varphi(y+z)-\varphi(x)+\varphi(y)M(x;dz)\right| \\
&\quad+\left|\int_{|z|\geq d}(\varphi(y+z)-\varphi(y))(M(x;dz)-M(y;dz))\right| \\
\leq& 2\frac{[\varphi]_{\alpha;\Omega}|x-y|^\alpha}{d^2}\int_{|z|\geq d}|z|^2M(x;dz) \\
&\quad+[\varphi]_{\alpha;\Omega}\int_{|z|\geq d}|z|^\alpha\frac{|z|^{2-\alpha}}{d^{2-\alpha}}|M(x;dz)-M(y;dz)| \\
\leq& \frac{2K_1}{d^2}[\varphi]_{\alpha;\Omega}|x-y|^\alpha \\
&\quad+\frac{K_2}{d^{2-\alpha}}[\varphi]_{\alpha;\Omega}|x-y|^\alpha\cdot\frac{\mathrm{diam}(\Omega)^\alpha}{d^\alpha_{x,y}}
\end{aligned}
$$

Part c). Bound the last two terms (5.3).

$$
\begin{aligned}
&\left|\int_{|z|\geq d} z\cdot(D\varphi(x)-D\varphi(y))M(x;dz)\right| \\
&\quad+\left|\int_{|z|\geq d} z\cdot D\varphi(y)(M(x;dz)-M(y;dz))\right| \\
\leq&[D\varphi]_{\alpha;\Omega_{d_{x,y}}}\int_{|z|\geq d}\frac{|z|^2}{d}M(x;dz)|x-y|^\alpha \\
&\quad+|D\varphi|_{0;\Omega_{d_{x,y}}}\int_{|z|\geq d}\frac{|z|^2}{d}|M(x;dz)-M(y;dz)| \\
\leq&\frac{K_1}{\mu d_{x,y}^2}[D\varphi]^{(1-\alpha)}_{\alpha;\Omega}|x-y|^\alpha \\
&\quad+\frac{K_2}{\mu d_{x,y}^2}|D\varphi|^{(1-\alpha)}_{0;\Omega}|x-y|^\alpha\cdot\operatorname{diam}(\Omega)^\alpha
\end{aligned}
$$

By interpolation, we know that $[D\varphi]^{(1-\alpha)}_{\alpha;\Omega}\leq\epsilon|D^2\varphi|^{(2-\alpha)}_{0;\Omega}+C|D\varphi|^{(1-\alpha)}_{0;\Omega}$, and by Lemma 18, $|D\varphi|^{(1-\alpha)}_{0;\Omega}\leq C[\varphi]_{\alpha;\Omega}+\epsilon|D^2\varphi|^{(2-\alpha)}_{0;\Omega}$, thus, by picking μ small enough, we can get

$$[I\varphi]^{(2-\alpha)}_{\alpha;\Omega}\leq\epsilon|D^2\varphi|^{(2-\alpha)}_{\alpha;\Omega}+C[\varphi]_{\alpha;\Omega}$$

□

Lemma 20. *If $\varphi\in C^{2+\alpha,(0)}(\Omega)\cap C^\alpha(\overline{\Omega}_I)$, then $I\varphi\in C^{\alpha,(2)}(\Omega)$, and for each $\epsilon>0$, there exists $C=C(\epsilon,K_1,K_2)$ such that*

$$|I\varphi|^{(2)}_{\alpha;\Omega}\leq\epsilon|D^2\varphi|^{(2)}_{\alpha;\Omega}+C|\varphi|_{0;\Omega}+C[\varphi]_{\alpha;\Omega_I}$$

and thus,

$$|I\varphi|^{(2)}_{\alpha;\Omega}\leq\epsilon|\varphi|^{(0)}_{2+\alpha;\Omega}+C|\varphi|_{0;\Omega}+C[\varphi]_{\alpha;\Omega_I}$$

Proof. Step 1. Given $x\in\Omega$, fix $\mu\in(0,1)$, and let $d=\mu d_x=d(x,\partial\Omega)$, for $\mu\in(0,1)$. Let $d'=d_x-\mu d_x=(1-\mu)d_x$.

$$
\begin{aligned}
|I\varphi(x)| \leq & \left| \int_{|z|<d} \int_0^1 (1-s) z^T D^2\varphi(x+sz) z \, ds \, M(x;dz) \right| \\
& + \left| \int_{|z|\geq d} \varphi(x+z) - \varphi(x) M(x;dz) \right| + \left| \int_{|z|\geq d} z \cdot D\varphi(x) M(x;dz) \right| \\
\leq & K_1(d) |D^2\varphi|_{0;\Omega_{d'}} + 2|\varphi|_{0;\Omega_I} \int_{|z|\geq d} M(x;dz) + |D\varphi|_{0;\Omega_{d_x}} \int_{|z|\geq d} |z| M(x;dz) \\
\leq & K_1(d) \frac{|D^2\varphi|^{(2)}_{0;\Omega}}{d'^2} + 2|\varphi|_{0;\Omega_I} \int_{|z|\geq d} \frac{|z|^2}{d^2} M(x;dz) + \frac{|D\varphi|^{(1)}_{0;\Omega}}{d_x} \int_{|z|\geq d} \frac{|z|^2}{d} M(x;dz) \\
\leq & K_1(\mu d_x) \frac{|D^2\varphi|^{(2)}_{0;\Omega}}{(1-\mu)^2 d_x^2} + 2K_1 \frac{|\varphi|_{0;\Omega_I}}{\mu^2 d_x^2} + K_1 \frac{|D\varphi|^{(1)}_{0;\Omega}}{\mu d_x^2}
\end{aligned}
$$

And by the interpolation inequality $|D\varphi|^{(1)}_{0;\Omega} \leq \epsilon |D^2\varphi|^{(2)}_{0;\Omega} + C|\varphi|_{0;\Omega}$ for C depending on ϵ, we get

$$
\leq 2K_1(\mu d_x) \frac{|D^2\varphi|^{(2)}_{0;\Omega}}{(1-\mu)^2 d_x^2} + (2K_1 + C) \frac{|\varphi|_{0;\Omega_I}}{\mu^2 d_x^2}
$$

Take supremum over x, we get

$$
|\varphi|^{(2)}_{0;\Omega} \leq \frac{K_1(\mu \cdot \mathrm{diam}(\Omega))}{(1-\mu)^2} |D^2\varphi|^{(2)}_{0;\Omega} + \frac{2K_1 + C}{\mu^2} |\varphi|_{0;\Omega_I}
$$

By picking μ small enough so that $K_1(\mu \cdot \mathrm{diam}(\Omega)) \leq \epsilon$, we get

$$
|I\varphi|^{(2)}_{0;\Omega} \leq \epsilon |D^2\varphi|^{(2)}_{0;\Omega} + C|\varphi|_{0;\Omega_I}
$$

in which $C = C(\epsilon, K_1, \Omega)$.

Step 2. Fix $x, y \in \Omega$. Let $d_{x,y} = \min\{d(x,\partial\Omega), d(y,\partial\Omega)\}$, and $d = \mu d_{x,y}$ for $\mu \in (0,1)$,

$d' = (1-\mu)d_{x,y}$.

$$
\begin{aligned}
|I\varphi(x) - I\varphi(y)| \leq & \left|\int_{|z|<d}\int_0^1 (1-s)z^T(D^2\varphi(x+sz) - D^2\varphi(y+sz))z\,ds\,M(x;dz)\right| \\
& + \left|\int_{|z|<d}\int_0^1 (1-s)z^T D^2\varphi(y+sz)z\,ds\,(M(x;dz) - M(y;dz))\right| \qquad (5.4)\\
& + \left|\int_{|z|\geq d} \varphi(x+z) - \varphi(y+z) - \varphi(x) + \varphi(y) M(x;dz)\right| \\
& + \left|\int_{|z|\geq d} (\varphi(y+z) - \varphi(y))(M(x;dz) - M(y;dz))\right| \qquad (5.5)\\
& + \left|\int_{|z|\geq d} z\cdot(D\varphi(x) - D\varphi(y))M(x;dz)\right| \\
& + \left|\int_{|z|\geq d} z\cdot D\varphi(y)(M(x;dz) - M(y;dz))\right| \qquad (5.6)
\end{aligned}
$$

Part a) Bound (5.4).

$$
\begin{aligned}
& \left|\int_{|z|<d}\int_0^1 (1-s)z^T(D^2\varphi(x+sz) - D^2\varphi(y+sz))z\,ds\,M(x;dz)\right| \\
& \quad + \left|\int_{|z|<d}\int_0^1 (1-s)z^T D^2\varphi(y+sz)z\,ds\,(M(x;dz) - M(y;dz))\right| \\
\leq & K_1(d)[D^2\varphi]_{\alpha;\Omega_{d'}}|x-y|^\alpha \\
& \quad + |D^2\varphi|_{0;\Omega_{d'}}\int_{|z|<d}|z|^2|M(x;dz) - M(y;dz)| \\
\leq & K_1(d)\frac{[D^2\varphi]^{(2)}_{\alpha;\Omega}}{d'^{2+\alpha}}|x-y|^\alpha \\
& \quad + K_2(d)\frac{|D^2\varphi|^{(2)}_{0;\Omega}}{d'^2}|x-y|^\alpha\cdot\frac{\mathrm{diam}(\Omega)^\alpha}{d^\alpha_{x,y}} \\
\leq & K_1(\mathrm{diam}(\Omega)d_{x,y})\frac{[D^2\varphi]^{(2)}_{\alpha;\Omega}}{(1-\mu)^{2+\alpha}d^{2+\alpha}_{x,y}}|x-y|^\alpha \\
& \quad + K_2(\mathrm{diam}(\Omega)d_{x,y})\cdot\mathrm{diam}(\Omega)^\alpha\frac{|D^2\varphi|^{(2)}_{0;\Omega}}{(1-\mu)^2 d^{2+\alpha}_{x,y}}|x-y|^\alpha
\end{aligned}
$$

Part b) Bound (5.5).

$$
\begin{aligned}
&\left|\int_{|z|\geq d} \varphi(x+z)-\varphi(y+z)-\varphi(x)+\varphi(y) M(x;dz)\right| \\
&\quad + \left|\int_{|z|\geq d} (\varphi(y+z)-\varphi(y))(M(x;dz)-M(y;dz))\right| \\
\leq & 2[\varphi]_{\alpha;\Omega_I}|x-y|^\alpha \int_{|z|\geq d} \frac{|z|^2}{d^2} M(x;dz) \\
&\quad + 2|\varphi|_{0;\Omega_I} \int_{|z|\geq d} \frac{|z|^2}{d^2} |M(x;dz)-M(y;dz)| \\
\leq & \frac{2K_1}{d^2}[\varphi]_{\alpha;\Omega_I}|x-y|^\alpha \cdot \frac{\operatorname{diam}(\Omega)^\alpha}{d_{x,y}^\alpha} \\
&\quad + \frac{2K_2}{d^2}|\varphi|_{0;\Omega_I}|x-y|^\alpha \cdot \frac{\operatorname{diam}(\Omega)^\alpha}{d_{x,y}^\alpha} \\
\leq & \frac{2K_1 \cdot \operatorname{diam}(\Omega)^\alpha}{\mu^2 d_{x,y}^{2+\alpha}}[\varphi]_{\alpha;\Omega_I}|x-y|^\alpha \\
&\quad + \frac{2K_2 \cdot \operatorname{diam}(\Omega)^\alpha}{\mu^2 d_{x,y}^{2+\alpha}}|\varphi|_{0;\Omega_I}|x-y|^\alpha
\end{aligned}
$$

Part c) Bound (5.6).

$$
\begin{aligned}
&\left|\int_{|z|\geq d} z\cdot (D\varphi(x)-D\varphi(y)) M(x;dz)\right| \\
&\quad + \left|\int_{|z|\geq d} z\cdot D\varphi(y)(M(x;dz)-M(y;dz))\right| \\
\leq & [D\varphi]_{\alpha;\Omega_d}|x-y|^\alpha \int_{|z|\geq d} \frac{|z|^2}{d} M(x;dz) \\
&\quad + |D\varphi|_{0;\Omega_d} \int_{|z|\geq d} \frac{|z|^2}{d} |M(x;dz)-M(y;dz)| \\
\leq & \frac{K_1}{d}\frac{[D\varphi]^{(1)}_{\alpha;\Omega}}{d^{1+\alpha}}|x-y|^\alpha \\
&\quad + \frac{|D\varphi|^{(1)}_{0;\Omega}}{d}\frac{K_2}{d}|x-y|^\alpha \cdot \frac{\operatorname{diam}(\Omega)^\alpha}{d_{x,y}^\alpha} \\
\leq & K_1 \frac{[D\varphi]^{(1)}_{\alpha;\Omega}}{\mu^{2+\alpha} d_{x,y}^{2+\alpha}}|x-y|^\alpha \\
&\quad + K_2\cdot \operatorname{diam}(\Omega)^\alpha \frac{|D\varphi|^{(1)}_{0;\Omega}}{\mu^2 d_{x,y}^{2+\alpha}}|x-y|^\alpha
\end{aligned}
$$

By picking a sufficiently small μ, and by the interpolation inequality $|D\varphi|^{(1)}_{\alpha;\Omega} \leq \epsilon|D^2\varphi|^{(2)}_{\alpha;\Omega} + C|\varphi|_{0;\Omega_I}$, we get

$$[I\varphi]^{(2)}_{\alpha;\Omega} \leq \epsilon|D^2\varphi|^{(2)}_{\alpha;\Omega} + C|\varphi|_{0;\Omega_I} + C[\varphi]_{\alpha;\Omega_I}$$

□

Now we proceeds to the existence of the solution to the integro-differential equation:

Assumption 15. *Let a^{ij}, b^i, c be functions defined on Ω such that*

1. $$|a^{ij}|_{\alpha;\Omega}, |b^i|_{\alpha;\Omega}, |c|_{\alpha;\Omega} \leq \Lambda$$

2. $$a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2$$

3. $$c \geq 0$$

Lemma 21. *Under the Assumption 13, 14, and 15, suppose $\Omega = B = B(x_0, R)$, $f \in C^{\alpha,(2-\beta)}(\overline{B})$, and $u \in C^0(\overline{B}) \cap C^2(B)$ solves the following equation*

$$\begin{cases} Lu - I_0 u = f & \text{in } B \\ u = 0 & \text{on } \overline{B}_I \setminus B \end{cases}$$

Then for any $\beta \in (0,1)$,

$$|u|^{(-\beta)}_{0;B} \leq C|f|^{(2-\beta)}_{0;B}$$

for some $C = C(\beta, n, R, \lambda, \Lambda)$.

Proof. The proof is a direct extension of Lemma 6.21 in Gilbarg & Trudinger's. Without loss of generality, assume that $x_0 = 0$. Let $w_1 = (R^2 - |x|^2)^\beta$. We know that

$$\begin{aligned} Lw_1 \geq& (R^2 - |x|^2)^{\beta-2}\left[4\beta(1-\beta)\lambda|x|^2 + 2\beta(n\lambda - b(x)\cdot x)(R^2 - |x|^2)\right] \\ \geq& \beta(R^2 - |x|^2)^{\beta-2}\left[4(1-\beta)\lambda|x|^2 + 2(n\lambda - \sqrt{n}\Lambda|x|)(R^2 - |x|^2)\right] \end{aligned}$$

For some R_0, $0 \leq R_0 < R$, the expression in the brackets is positive if $R_0 \leq |x| \leq R$. Thus, there exists c_1 and c_2, depending on $\beta, n, R, \lambda, \Lambda$ such that

$$Lw_1(x) \geq \begin{cases} c_1(R - |x|)^{\beta-2} & \text{if } R_0 \leq |x| < R \\ -c_2(R - |x|)^{\beta-2} & \text{if } 0 \leq |x| < R_0 \end{cases}$$

On the other hand,

$$\begin{aligned}I_0 w_1 =& \int_0^1 \int_{x+sz\in B} (1-s) z^T \cdot D^2 w_1(x+sz)\cdot z \ M(x;dz)\, ds \\ =& \int_0^1 (1-s) \int_{x+sz\in B} \beta (R^2-|x|^2)^{\beta-2} \left[-4(1-\beta) z^T (x+sz)(x+sz)^T z \right. \\ & \left. +(R^2-|x|^2)(-2)|z|^2\right] M(x;dz) ds \\ \leq& 0\end{aligned}$$

Now let $w_2 = e^{\gamma R} - e^{\alpha x_1}$, for $\gamma \geq 1 + \Lambda/\lambda$.

$$\begin{aligned}Lw_2 =& a_1 1 \alpha^2 e^{\alpha x_1} - \alpha e^{\alpha x_1} b_1 + c(e^{\alpha R} - e^{\alpha x_1}) \\ \geq& \lambda e^{-\alpha R}\end{aligned}$$

Then

$$Lw_2(x) \geq \begin{cases} 0 & \text{if } R_0 \leq |x| < R \\ c_3 (R-|x|)^{\beta-2} & \text{if } 0 \leq |x| < R_0 \end{cases}$$

where $c_3 = \lambda e^{-\alpha R}(R-R_0)^{2-\beta}$. For the integral part,

$$I_0 w_2 = -\int_0^1 \int_{x+sz\in B} z_1^2 \alpha^2 e^{\alpha x_1 + s z_1} M(x;dz) ds \leq 0$$

Let $w = \gamma_1 w_1 + \gamma_2 w_2$, for $\gamma_1 = 1/c_1$, and $\gamma_2 = (1+c_2/c_1)/c_3$. We get

$$Lw - I_0 w \geq (R-|x|)^{\beta-2}$$

Thus applying the maximum principle on $\pm u - |f|_{0;\Omega}^{(2-\beta)} w(x)$, we get $|u(x)| \leq |f|_{0;\Omega}^{(2-\beta)} w(x)$.

Now for each $x \in B$, assume without loss of generality that it lie on the x_1 axis. Then we get the inequality

$$|u(x)| \leq C|f|_{0;\Omega}^{(2-\beta)} (R-|x|)^{\beta} = C|f|_{0;\Omega}^{(2-\beta)} d(x,\partial B)^{\beta}$$

for $C = C(\beta, n, R, \lambda, \Lambda)$. □

Theorem 9. *Under the Assumption 13, 14, 15, suppose Ω is an open ball given $f \in C^{\alpha,(2-\alpha)}(\Omega)$, then there exists a unique $u \in C^{2+\alpha,(-\alpha)}(\overline{\Omega})$ such that*

$$\begin{cases} Lu - I_0 u = f & \text{in } \Omega \\ u = 0 & \text{on } \Omega_I \setminus \Omega \end{cases}$$

and we have the following estimates

$$|u|_{2+\alpha;\Omega}^{(-\alpha)} \leq C|f|_{\alpha;\Omega}^{(2-\alpha)}$$

Proof. Define the operator $A : C^{2+\alpha,(-\alpha)}(\Omega) \to C^{2+\alpha,(-\alpha)}(\Omega)$ such that, $A[v] = u$ if and only if

$$\begin{cases} Lu - I_0 v = f & \text{in } \Omega \\ u = 0 & \text{on } \Omega_I \setminus \Omega \end{cases}$$

The existence and uniqueness of u follows from Lemma 6.21 and Theorem 6.22 of Gilbarg and Tridinger's. And we have the estimates:

$$\begin{aligned} |u|^{(-\alpha)}_{2+\alpha;\Omega} \leq & C(|f|^{(2-\alpha)}_{\alpha;\Omega} + |I_0 v|^{(2-\alpha)}_{\alpha;\Omega}) \\ \leq & C|f|^{(2-\alpha)}_{\alpha;\Omega} + C\epsilon|v|^{(-\alpha)}_{2+\alpha;\Omega} + C|v|^{(-\alpha)}_{0;\Omega} \end{aligned}$$

Now we can apply the Schaefer's Fixed Point Theorem to get the existence. Let $A[v_1] = u_1$ and $A[v_2] = u_2$. Then,

$$|u_1 - u_2|^{(-\alpha)}_{2+\alpha;\Omega} \leq C|I_0(v_1 - v_2)|^{(2-\alpha)}_{\alpha;\Omega} \leq \epsilon|v_1 - v_2|^{(-\alpha)}_{2+\alpha;\Omega} + C|v_1 - v_2|^{(-\alpha)}_{\alpha;\Omega}$$

So A is continuous. Let K be a bounded set in $C^{2+\alpha,(-\alpha)}(\Omega)$. Let $v \in K$, $u = A[v]$,

$$|u|^{(-\alpha)}_{2+\alpha;\Omega} \leq C|f|^{(2-\alpha)}_{\alpha;\Omega} + C\epsilon|v|^{(-\alpha)}_{2+\alpha;\Omega} + C|v|^{(-\alpha)}_{0;\Omega}$$

with the previous lemma, $|v|^{(-\alpha)}_{0;\Omega} \leq C|f|^{(2-\alpha)}_{0;\Omega}$, so the right hand side is bounded on the set K. By Arzela-Ascoli Theorem, the operator A is compact. Now suppose $Lu = \lambda I_0 u + f$ for $\lambda \in [0, 1]$. Then, as before,

$$\begin{aligned} |u|^{(-\alpha)}_{2+\alpha;\Omega} \leq & C|f|^{(2-\alpha)}_{\alpha;\Omega} + C\epsilon|u|^{(-\alpha)}_{2+\alpha;\Omega} + C|u|^{(-\alpha)}_{0;\Omega} \\ \leq & C|f|^{(2-\alpha)}_{\alpha;\Omega} + C\epsilon|u|^{(-\alpha)}_{2+\alpha;\Omega} + C|f|^{(2-\alpha)}_{0;\Omega} \end{aligned}$$

Pick ϵ to be sufficiently small, we get

$$|u|^{(-\alpha)}_{2+\alpha;\Omega} \leq C|f|^{(2-\alpha)}_{\alpha;\Omega}$$

This proves the existence, as well as uniqueness because of linearity and the estimate. □

Theorem 10. *Under the Assumption 13, 14, 15, suppose Ω is an open ball given $f \in C^\alpha(\Omega)$. For each $f \in C^\alpha(\overline{\Omega})$ and $\varphi \in C^{2+\alpha}(\partial\Omega) \cap C^\alpha(\overline{\Omega}_I \setminus \Omega)$, There exists a unique $u \in C^{2+\alpha}_{loc}(\Omega) \cap C^0(\overline{\Omega})$ such that*

$$\begin{cases} Lu - I_\varphi u = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

Moreover, we have the following bound:

$$|u|_{\alpha;\Omega} + [D^2 u]^{(2-\alpha)}_{\alpha;\Omega} \leq C|f|_{\alpha;\Omega} + C|\varphi|_{2+\alpha;\partial\Omega} + C|\varphi|_{\alpha;\overline{\Omega}\setminus\Omega}$$

Proof. First, let u_1 be the solution to

$$\begin{cases} Lu = f & \text{in}\Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

We know that $u_1 \in C^{2+\alpha}(\overline{\Omega})$ with the estimates

$$|u_1|_{2+\alpha;\Omega} \leq C(|f|_{\alpha;\Omega} + |\varphi|_{2+\alpha;\partial\Omega})$$

Then we look for the solution to

$$\begin{cases} Lu - I_0 u = I_\varphi u_1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

in which $I_0 u$ is to extend u to be 0 on $\Omega_I \setminus \Omega$ and apply operator I to it. If we find a solution to this, call it u_2, then $u = u_1 + u_2$ solves the original problem. We know by the previous analysis that,

$$\begin{aligned} |I_\varphi u_1|^{(2-\alpha)}_{\alpha;\Omega} \leq & \epsilon |u_1|_{\alpha;\Omega} + C|\varphi|_{\alpha;\overline{\Omega}_I \setminus \Omega} \\ \leq & C|f|_{\alpha;\Omega} + C|\varphi|_{2+\alpha;\partial\Omega} + C|\varphi|_{\alpha;\overline{\Omega}\setminus\Omega} \end{aligned}$$

Thus, the problem is reduced to, given $\tilde{f} \in C^{\alpha,(2-\alpha)}(\Omega)$, we need to find solution u_2 to the problem

$$\begin{cases} Lu_2 - I_0 u_2 = \tilde{f} & \text{in } \Omega \\ u_2 = 0 & \text{on } \partial\Omega \end{cases}$$

Then the previous theorem gives us the existence, with the inequality:

$$\begin{aligned} |u_2|^{(-\alpha)}_{2+\alpha;\Omega} \leq & C|I_\varphi u_1|^{(2-\alpha)}_{\alpha;\Omega} \\ \leq & C|f|_{\alpha;\Omega} + C|\varphi|_{2+\alpha;\partial\Omega} + C|\varphi|_{\alpha;\overline{\Omega}\setminus\Omega} \end{aligned}$$

Combine u_1 and u_2, and notice that $|u_2|_{\alpha;\Omega} \leq C|u_2|^{(-\alpha)}_{2+\alpha;\Omega}$, and $[D^2 u_1]^{(2-\alpha)}_{\alpha;\Omega} \leq C[D^2 u_1]_{\alpha;\Omega}$, we can obtain the inequality

$$|u|_{\alpha;\Omega} + [D^2 u]^{(2-\alpha)}_{\alpha;\Omega} \leq C|f|_{\alpha;\Omega} + C|\varphi|_{2+\alpha;\partial\Omega} + C|\varphi|_{\alpha;\overline{\Omega}\setminus\Omega}$$

The uniqueness follows from the linearity and maximum principle. □

5.3 Regularity for variational inequality with second-order non-local operator

Suppose Ω is an open ball. Given the obstacle Ψ such that $\varphi \leq \Psi$ on $\partial\Omega$, we will prove a regularity result for the following problem:

$$\begin{cases} \max\{Lu - Iu, u - \Psi\} & \text{in } \Omega \\ u = \varphi & \text{in } \Omega^c \end{cases}$$

Consider the approximation scheme

$$\begin{cases} Lu^\epsilon - Iu^\epsilon + \beta_\epsilon(u^\epsilon - \Psi^\epsilon) = f & \text{in } \Omega \\ u^\epsilon = \varphi & \text{in } \Omega^c \end{cases}$$

in which $\Psi^\epsilon \in C^\infty(\Omega) \cap C^\alpha(\overline{\Omega})$ is a smooth approximation to $\Psi \in C^\alpha(\overline{\Omega})$ with $\Psi^\epsilon \geq \varphi$ on $\partial\Omega$, $\beta_\epsilon : \mathbb{R} \to \mathbb{R}$ is defined such that β_ϵ is smooth, $0 \leq \beta_\epsilon' \leq \frac{1}{\epsilon}$, $\beta_\epsilon \geq -1$, $\beta_\epsilon(0) = 0$, and as $\epsilon \to 0$, $\beta_\epsilon \to 0$ on $(-\infty, 0]$, and $\to \infty$ on $(0, \infty)$.

Instead of solving this directly, we will instead decompose $u^\epsilon = u_1 + u_2^\epsilon$, in which

$$\begin{cases} Lu_1 - Iu_1 = f & \text{in } \Omega \\ u_1 = \varphi & \text{in } \Omega^c \end{cases}$$

$$\begin{cases} Lu_2^\epsilon - Iu_2^\epsilon + \beta_\epsilon(u_2^\epsilon - (\Psi^\epsilon - u_1)) = 0 & \text{in } \Omega \\ u_2^\epsilon = 0 & \text{in } \Omega^c \end{cases}$$

Given the results from the previous section, we know that under the Assumption 13, 14, 15, if $f \in C^\alpha(\overline{\Omega})$, $\varphi \in C^{2+\alpha}(\partial\Omega) \cap C^\alpha(\overline{\Omega}_I \setminus \Omega)$, we can find a unique classical solution $u_1^\epsilon \in C^{2+\alpha}(\Omega) \cap C^\alpha(\overline{\Omega})$. Now we need to show the existence of classical solution u_2^ϵ:

Theorem 11. *Under the Assumption 13, 14, 15, suppose $\widetilde{\Psi}^\epsilon \in C^{2+\alpha}(\overline{\Omega}_I)$ such that $\widetilde{\Psi}^\epsilon \geq 0$ on $\overline{\Omega}_I \setminus \Omega$ and $\widetilde{\Psi}^\epsilon$ is semi-concave, there exists $u_2^\epsilon \in C^{2+\alpha,(-\alpha)}(\overline{\Omega})$ such that*

$$\begin{cases} Lu_2^\epsilon - Iu_2^\epsilon + \beta_\epsilon(u_2^\epsilon - \widetilde{\Psi}^\epsilon) = 0 & \textit{in } \Omega \\ u_2^\epsilon = 0 & \textit{in } \Omega^c \end{cases}$$

Proof. Define the operator $A : C^{2+\alpha,(-\alpha)}(\overline{\Omega}) \to C^{2+\alpha,(-\alpha)}(\overline{\Omega})$, such that $A[v] = u$ if and only if

$$\begin{cases} Lu - Iu + \beta_\epsilon(v - \widetilde{\Psi}^\epsilon) = 0 & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c \end{cases}$$

It is clear that $v - \widetilde{\Psi}^\epsilon \in C^\alpha(\overline{\Omega})$, so $\beta^\epsilon(v - \widetilde{\Psi}^\epsilon) \in C^\alpha(\overline{\Omega})$, so $u \in C^{2+\alpha,(-\alpha)}(\overline{\Omega})$ exists and is unique.

First we show that A is continuous. Let $A[v_1] = u_1$, $A[v_2] = u_2$, then

$$\begin{cases} (L - I)(u_2 - u_1) + \Big(\beta_\epsilon(v_2 - \widetilde{\Psi}^\epsilon) - \beta_\epsilon(v_1 - \widetilde{\Psi}^\epsilon)\Big) = 0 & \text{in } \Omega \\ u_2 - u_1 = 0 & \text{in } \Omega^c \end{cases}$$

By Theorem 9, we know that

$$\begin{aligned}|u_2 - u_1|^{(-\alpha)}_{2+\alpha;\Omega} \leq& C|\beta_\epsilon(v_2 - \widetilde{\Psi}^\epsilon) - \beta_\epsilon(v_1 - \widetilde{\Psi}^\epsilon)|^{(2-\alpha)}_{\alpha;\Omega} \\ \leq& C|\beta'_\epsilon|_0|v_2 - v_1|^{(2-\alpha)}_{\alpha;\Omega} \\ \leq& \frac{C'}{\epsilon}|v_2 - v_1|^{(-\alpha)}_{2+\alpha;\Omega}\end{aligned}$$

in which C' depends on the diameter of Ω. This shows that the map A is continuous.

Next we need to show compactness:

$$\begin{aligned}|u|^{(-\alpha)}_{2+\alpha;\Omega} \leq& C|\beta_\epsilon(v - \widetilde{\Psi}^\epsilon)|^{(2-\alpha)}_{\alpha;\Omega} \\ \leq& C|\beta'_\epsilon|_0|v - \widetilde{\Psi}^\epsilon|^{(2-\alpha)}_{\alpha;\Omega} \\ \leq& \frac{C'}{\epsilon}(|v|^{(-\alpha)}_{2+\alpha;\Omega} + |\widetilde{\Psi}^\epsilon|^{(-\alpha)}_{2+\alpha;\Omega})\end{aligned}$$

again C' depends on the diameter of Ω. Combine this with Arzela-Ascoli's Compactness result, we can easily conclude that A is compact.

We now show that the set $\{u \in C^{2+\alpha,(-\alpha)}(\Omega) : \lambda A[u] = u, 0 \leq \lambda \leq 1\}$ is bounded.

$$\begin{aligned}|u|^{(-\alpha)}_{2+\alpha;\Omega} \leq& C\lambda|\beta_\epsilon(u - \widetilde{\Psi}^\epsilon)|^{(2-\alpha)}_{\alpha;\Omega} \\ \leq& C|\beta'_\epsilon|_0|u - \widetilde{\Psi}^\epsilon|^{(2-\alpha)}_{\alpha;\Omega} \\ \leq& \frac{C'}{\epsilon}(|u|^{(-\alpha)}_{\alpha;\Omega} + |\widetilde{\Psi}^\epsilon|^{(2-\alpha)}_{\alpha;\Omega}) \\ \leq& \frac{C'}{\epsilon}(\delta|u|^{(-\alpha)}_{2+\alpha;\Omega} + C_\delta|u|^{(-\alpha)}_{0;\Omega} + |\widetilde{\Psi}^\epsilon|^{(2-\alpha)}_{\alpha;\Omega})\end{aligned}$$

If δ is small enough, we can subtract it off to the left hand side, and obtain

$$\begin{aligned}|u|^{(-\alpha)}_{2+\alpha;\Omega} \leq& C_\epsilon|u|^{(-\alpha)}_{0;\Omega} + C_\epsilon|\widetilde{\Psi}^\epsilon|^{(2-\alpha)}_{\alpha;\Omega} \\ \leq& C_\epsilon|\beta_\epsilon(u - \widetilde{\Psi}^\epsilon)|^{(2-\alpha)}_{0;\Omega} + C_\epsilon|\widetilde{\Psi}^\epsilon|^{(2-\alpha)}_{\alpha;\Omega}\end{aligned}$$

We will then show that $|\beta_\epsilon(u - \widetilde{\Psi}^\epsilon)|$ is bounded.

First of all, it is bounded below by definition. Suppose $u - \widetilde{\Psi}^\epsilon$ achieves maximum at $x_0 \in \Omega$, then $(L - I_0)(u - \widetilde{\Psi}^\epsilon)(x_0) \geq 0$.

$$\beta_\epsilon(u - \widetilde{\Psi}^\epsilon) \leq \beta_\epsilon(u - \widetilde{\Psi}^\epsilon)(x_0) \leq -(L - I)\widetilde{\Psi}^\epsilon(x_0)$$

And since Ψ^ϵ is semi-concave, we know that $(L-I)\widetilde{\Psi}^\epsilon = (L-I)(\Psi^\epsilon - u_1) = (L-I)\Psi^\epsilon - f \geq -C$ for some constant C. We conclude that $\beta_\epsilon(u - \widetilde{\Psi}^\epsilon)$ is bounded. On the other hand, if $u - \widetilde{\Psi}^\epsilon$ achieves maximum on $\overline{\Omega}_I \backslash \Omega$, since we are given that $u \leq \widetilde{\Psi}^\epsilon$ on $\overline{\Omega}_I \backslash \Omega$, we immediately conclude that $\beta_\epsilon(u - \widetilde{\Psi}^\epsilon) \leq 0$. Either way, we have

$$|u|^{(-\alpha)}_{2+\alpha;\Omega} \leq C_\epsilon + C_\epsilon|\widetilde{\Psi}^\epsilon|^{(2-\alpha)}_{\alpha;\Omega}$$

So the set $\{u \in C^{2+\alpha,(-\alpha)}(\Omega) : \lambda A[u] = u, 0 \leq \lambda \leq 1\}$ is bounded, and we can apply the Schaefer fixed point theorem to obtain the existence of solution. □

Corollary 1. *Under the assumptions of the preceding theorem, $\beta_\epsilon(u_2^\epsilon - \widetilde{\Psi}^\epsilon) \leq C$ for C independent of ϵ.*

Assumption 16. *There exists measure $\nu(dz)$ on $\mathbb{R}^n \setminus \{0\}$ such that $\int |z|^2\nu(dz) < \infty$, and bounded non-negative measurable function $m : \Omega \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ such that $M(x, dz) = m(x, z)\nu(dz)$. Furthermore, $m(x, z)$ is Lipschitz in x, independent of z.*

Note that this assumption implies Assumption 14. The following theorem is proved in [31]:

Theorem 12. *Under the Assumption 13, 14, 15, 16, and in addition, a_{ij}, b_i, c, f are Lipschitz. Then there exists $u \in W_0^{1,p}(\Omega) \cap W^2_{loc}(\Omega)$ such that u solves*

$$\begin{cases} Lu - I_0u = f & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c \end{cases}$$

and, there exists constant C such that

$$|u|_{1,p;\Omega} \leq C|f|_0$$

as well as C' depending on $\Omega' \Subset \Omega$ such that

$$|u|_{2,p;\Omega'} \leq C'$$

Remark. This is not exactly how the theorem is stated in [31], but essentially the same proof would go through for the conditions stated here.

Theorem 13. *Given the Assumption 13, 14, 15, 16, and in addition, a_{ij}, b_i, c are Lipschitz, $\widetilde{\Psi} \in C^\alpha(\overline{\Omega}_I)$ such that $\widetilde{\Psi} \geq 0$ on $\overline{\Omega}_I \setminus \Omega$ and $\widetilde{\Psi}$ is semi-concave, there exists $u_2 \in W_0^{1,p}(\Omega) \cap W^{2,p}_{loc}(\Omega)$ such that*

$$\begin{cases} \max\{Lu_2 - I_0u_2, u_2 - \widetilde{\Psi}\} = 0 & \text{in } \Omega \\ u_2 = 0 & \text{in } \Omega^c \end{cases}$$

Proof. By the preceding corollary, we know that $|\beta_\epsilon(u_2^\epsilon - \widetilde{\Psi}^\epsilon)| \leq C$ independent of ϵ. By the previous theorem, we also know that $|u^\epsilon)2|_{1,p;\Omega} \leq C$, and for compact subsets $\Omega' \Subset \Omega$,

$|u^\epsilon)2|_{2,p;\Omega'} \leq C'$. Therefore, there exists subsequence $\epsilon_k \to 0$ and $u_2 \in W_0^{1,p}(\Omega) \cap W_{loc}^{2,p}(\Omega)$ such that

$$u_2^\epsilon \rightharpoonup u_2 \text{ weakly in } W_0^{1,p}(\Omega)$$
$$u_2^\epsilon \rightharpoonup u_2 \text{ weakly in } W_{loc}^{2,p}(\Omega)$$

Therefore, $u_2^\epsilon \to u_2$ uniformly in certain C^γ Holder space. Therefore, u_2 is a viscosity solution to $\max\{Lu_2 - I_0 u_2, u_2 - \widetilde{\Psi}\} = 0$ with $u_2 = 0$ on $\Omega_I \setminus \Omega$. □

Let $u = u_1 + u_2$, we finally conclude that

Theorem 14. *Given the Assumption 13, 14, 15, 16, and in addition, a_{ij}, b_i, c, f are Lipschitz, $f \in C^\alpha(\overline{\Omega})$, $\Psi \in C^\alpha(\overline{\Omega}_I)$, Ψ is semi-concave, $\varphi \in C^{2+\alpha}(\partial\Omega) \cap C^\alpha(\overline{\Omega}_I \setminus \Omega)$, and $\Psi \geq \varphi$ on $\overline{\Omega}_I \setminus \Omega$, there exists $u \in C(\overline{\Omega}_I) \cap W_{loc}^{2,p}(\Omega)$ such that*

$$\begin{cases} \max\{Lu - Iu - f, u - \Psi\} = 0 & \text{in } \Omega \\ u = \varphi & \text{in } \Omega^c \end{cases}$$

in the viscosity sense.

Remark. For the optimal stopping problem on unbounded domain in one dimension, we can usually prove Holder continuity using moment estimates on the stochastic differential equation. Then we can restrict our equation to any small open interval, and the theorem above along with local uniqueness shows that the function is locally $W_{loc}^{2,p}$, since the boundary of the interval is just two points. In particular, the value function is $C_{loc}^{1,\gamma}$ for some $\gamma > 0$, and the smooth fitting principle holds. We can extend this further to parabolic problems as well.

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