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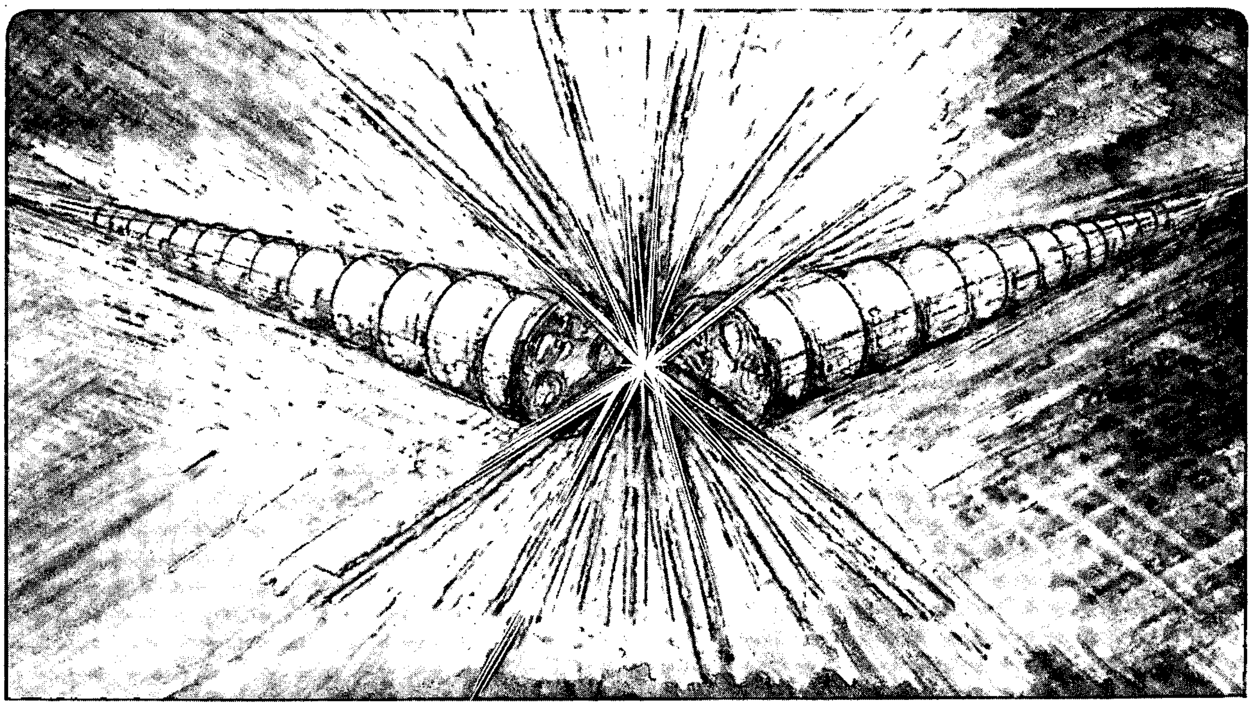

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APPLICATION OF FREQUENCY MAP ANALYSIS TO THE ALS*

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APPLICATION OF FREQUENCY MAP ANALYSIS TO THE ALS.

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Frequency map analysis is a numerical method based on Fourier techniques which provides a clear representation of the global dynamics of many multi-dimensional systems, and which is particularly useful for systems of 3 degrees of freedom and more. The frequency dependence with time also allows refined estimates of the diffusion of the orbits. Here are presented the theoretical foundation of the method, and some applications to the Advanced Light Source, demonstrating how frequency map analysis can be used to understand the limits of the dynamic aperture under various lattice conditions and predict more favorable working points.

KEY WORDS: Frequency map analysis, diffusion, chaos, accelerator dynamics

1 FREQUENCY MAPS

According to the KAM theorem [5,1,14], in the phase space of a sufficiently close to integrable conservative system, many invariant tori will persist. Trajectories starting on one of these tori remain on it thereafter, executing quasiperiodic motion with a fixed frequency vector depending only on the torus. The family of tori is parameterized over a Cantor set of frequency vectors, while in the gaps of the Cantor set chaotic behavior can occur. These slightly deformed tori are fixed structures of the system. It is possible numerically to find them, to straighten them out, and to interpolate between them to form an action-angle coordinate system in which regular (quasiperiodic) motion appears uniformly circular, and weakly chaotic motion stands out as a slight departure [19].

The frequency analysis method [3,6,7,10,11,12] also relies on a fixed feature of the model system, but one which is simpler to compute; namely, the frequency vectors associated with each of the invariant tori. Although the frequencies are strictly speaking only defined and fixed on these tori, the frequency analysis algorithm will numerically compute over a finite time span a frequency vector for any initial condition. On the KAM tori, this frequency vector will be a very accurate approximation of the actual frequencies, while in the chaotic regions, it will provide a natural interpolation between these fixed frequencies.

Let us consider a n–DOF Hamiltonian system close to integrable in the form $H(I,\theta) = H_0(I) + \varepsilon H_1(I,\theta)$ where H is real analytic for $(I,\theta) \in B^n \times \mathbb{T}^n$, where B^n is a domain of $\mathbb{R}^n$ and $\mathbb{T}^n$ is the n-dimensional torus. For $\varepsilon = 0$, the Hamiltonian reduces to $H_0(J)$ and is integrable. The equations of motion are then

$$\dot{I}_j = 0, \qquad \dot{\theta}_j = \frac{\partial H_0(I)}{\partial I_j} = \nu_j(I) \ ; \quad j = 1,\ldots,n \ , \tag{1}$$

which gives in the complex variables $z_j = I_j \exp i\theta_j$; $z_j(t) = z_{j0}e^{i\nu_j t}$, where $z_{j0} = z_j(0)$. The motion in phase space takes place on tori, products of true circles with constant radii $I_j = |z_j(0)|$, which are described at constant velocity $\nu_j(I)$. If the system is nondegenerate, that is if

$$det\left(\frac{\partial \nu(I)}{\partial I}\right) = det\left(\frac{\partial^2 H_0(I)}{\partial I^2}\right) \neq 0 \tag{2}$$

the frequency map $F : B^n \longrightarrow \mathbb{R}^n; \quad (I) \longrightarrow (\nu)$ is a diffeomorphism on its image Ω, and the tori are as well described by the action variables $(I) \in B^n$ or in an equivalent manner by the frequency vector $(\nu) \in \Omega$ (in the case of an isoenergetically nondegenerate system[2], one will consider the frequency map $\mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$, $(I_i)_{i=1,n-1} \rightarrow (\nu_i/\nu_n)_{i=1,n-1}$). For a nondegenerate system (or for an isoenergetically nondegenerate system), when ε is nonzero, the KAM theorem[2] still asserts that for sufficiently small values of ε , there exists a Cantor set Ω_ε of values of (ν), satisfying a Diophantine condition of the form

$$|(k,\nu)| > \frac{\kappa_\varepsilon}{|k|^m} \tag{3}$$

for which the perturbed system still possess smooth invariant tori with linear flow (the KAM tori). Moreover, according to Pöschel[18], there exists a diffeomorphism

$$\Psi : \mathbb{T}^n \times \Omega \longrightarrow \mathbb{T}^n \times B^n; \qquad (\varphi,\nu) \longrightarrow (\theta, I) \tag{4}$$

which is analytical with respect to φ and C^∞ in ν and on $\mathbb{T}^n \times \Omega_\varepsilon$ transforms the Hamiltonian equations into : $\dot{\nu}_j = 0; \quad \dot{\varphi}_j = \nu_j$. For frequency vectors (ν) in Ω_ε, the solution lies on a torus and is given in complex form by its Fourier series

$$z_j(t) = z_{j0}e^{i\nu_j t} + \sum_m a_m(\nu)e^{i<m,\nu>t} \tag{5}$$

where the coefficients $a_m(\nu)$ depend smoothly on the frequencies (ν). If we fix $\theta \in T^n$, to some value $\theta = \theta_0$, we obtain a frequency map on B^n defined as

$$F_{\theta_0} : B^n \longrightarrow \Omega; \qquad I \longrightarrow p_2(\Psi^{-1}(\theta_0, I)) \tag{6}$$

where p_2 is the projection on Ω ($p_2(\phi, \nu) = \nu$). It should be noted that for sufficiently small ε, the torsion condition (2) ensures that the frequency map F_{θ_0} is a diffeomorphism. The frequency map analysis consists to obtain directly, in a numerical manner, a natural frequency map F, defined on a whole domain B^n, which coincide, up to numerical accuracy, with F_{θ_0} (Eq. 6) on the set of the KAM tori. The frequency map F is obtained by searching for quasiperiodic approximations of the solutions, over a finite time span, in the form of a finite number of terms

$$z_j(t) = z_{j0} e^{i\nu_j t} + \sum_{k=1}^{N} a_{m_k} e^{i<m_k,\nu>t} \ . \tag{7}$$

Once the quasiperiodic approximation (7) is obtained, the construction of the frequency map can be made and the study of the global dynamics of the system (1) will then be possible in a very effective manner by the analysis of the regularity of this frequency map [3,6,7,10,11,12].

Let $\mathcal{A}$ be the subset of B^n of the values of (I) such that (I, θ_0) belongs to a KAM torus of dimension n. In this case, we can assume that, up to the numerical accuracy of our numerical procedure, the rotation vector (ν) is the true rotation vector of the considered torus. We thus assume that on $\mathcal{A}$, F_T is a very good approximation of the frequency map F_{θ_0} defined in (6) and the restriction of the frequency map F_T to $\mathcal{A}$ will have thus the following properties : a) If $(I) \in \mathcal{A}$, then $F_T((I), \cdot)$ is constant on $\mathbb{R}$; b) For any given τ, the map $F_T^\tau : \mathcal{A} \longrightarrow \mathbb{R}^n$ $(I) \longrightarrow F_T((I), \tau)$ is regular in some sense, as it coincides on $\mathcal{A}$ with the restriction to $\mathcal{A}$ of a smooth diffeomorphism.

The criterion (b) ensures that when the frequency map is not regular, the corresponding KAM tori are destroyed. In case of a two degrees of freedom Hamiltonian, we can even obtain a more precise criterion. Indeed, in this case, and under some condition of non-degeneracy, the frequency map $F_T : \mathbb{R} \longrightarrow \mathbb{R}$ should be monotonic. As soon as this is not verified for two values of the action like variables I_{10} and I'_{10}, we can conclude to the destruction of invariant KAM tori in all the corresponding interval of frequencies $[F_T(I_{10}), F_T(I'_{10})]$ [10,7].

2 NUMERICAL ANALYSIS OF THE FREQUENCIES (NAFF)

The frequency map analysis relies heavily on the observation that when a quasiperiodic function $f(t)$ in the complex domain $\mathbb{C}$ is given numerically, it is possible to recover a quasiperiodic approximation of $f(t)$ in a very precise way over a finite time span $[-T, T]$, several orders of magnitude more precisely than what is given by simple Fourier series. Indeed, when one computes the Fourier series of $f(t)$ over the finite interval $[-T, T]$, one assumes that $f(t)$ is periodic of period $2T$, which is obviously not true. We make here a different hypothesis, which is dictated by the knowledge of the regular dynamics of our system, and we search for quasiperiodic approximations. We briefly describe here the numerical algorithm (NAFF),

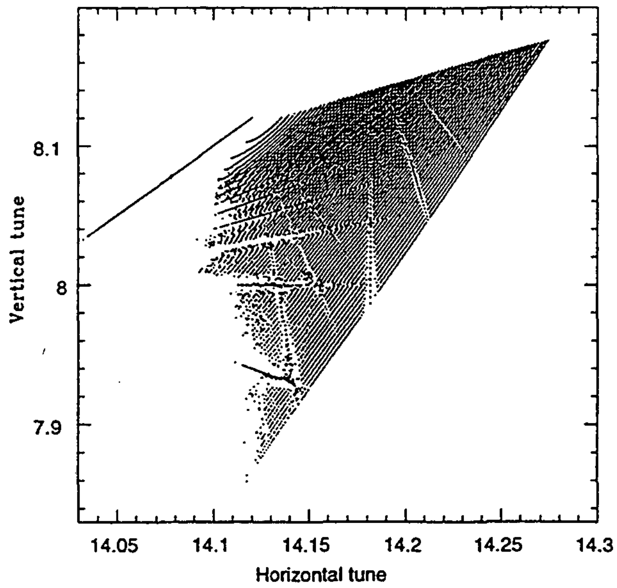

FIGURE 1: Ideal ALS lattice

which is effectively used for the determination of these quasiperiodic approximations required by the frequency map analysis. More details can be found in [6,10]. Let

$$f(t) = e^{i\nu_1 t} + \sum_{k \in \mathbb{Z}^n - (1,0,\dots,0)} a_k e^{i\langle k,\nu \rangle t}; \quad a_k \in \mathbb{C} \tag{8}$$

be a KAM quasiperiodic solution of an Hamiltonian system in $B^n \times \mathbb{T}^n$, where the frequency vector (ν) satisfies a Diophantine condition (3). The frequency analysis algorithm NAFF will provide an approximation $f'(t) = \sum_{k=1}^{N} a'_k e^{i\omega'_k t}$ of $f(t)$ from its numerical knowledge over a finite time span $[-T, T]$. The frequencies ω'_k and complex amplitudes a'_k are found with an iterative scheme. To determine the first frequency ω'_1, one searches for the maximum amplitude of $\phi(\sigma) = \langle f(t), e^{i\sigma t} \rangle$ where the scalar product $\langle f(t), g(t) \rangle$ is defined by

$$\langle f(t), g(t) \rangle = \frac{1}{2T} \int_{-T}^{T} f(t)\bar{g}(t)\chi(t)dt \ , \tag{9}$$

and where $\chi(t)$ is a weight function, that is, a positive and even function with $1/2T \int_{-T}^{T} \chi(t)dt = 1$. In all our computations, we used the Hanning window filter, that is $\chi_1(t) = 1 + \cos(\pi t/T)$. Once the first periodic term $e^{i\omega'_1 t}$ is found, its complex amplitude a'_1 is obtained by orthogonal projection, and the process is started again on the remaining part of the function $f_1(t) = f(t) - a'_1 e^{i\omega'_1 t}$. It is also necessary to orthogonalize the set of functions $(e^{i\omega'_k t})_k$, when projecting f iteratively on these $e^{i\omega'_k t}$. For a KAM solution, the frequency analysis algorithm allows a very accurate determination of the frequencies over the time span $[-T, T]$, several orders of magnitude better than with simple FFTs. This was rigorously established by a general theorem[7] which can be stated as following for a weight function of the form

$$\chi_p(t) = \frac{2^p (p!)^2}{(2p)!}(1 + \cos \pi t)^p \tag{10}$$

Proposition. 1. *For a KAM solution $f(t)$ of the form (11), and using the weight function $\chi(t) = \chi_p(t)$, the application of the frequency analysis algorithm over the time span $[-T, T]$, as described above, provides a determination ν_1^T of the frequency ν_1 with a precision $\nu_1 - \nu_1^T$ having the asymptotic expression for $T \longrightarrow +\infty$*

$$\nu_1 - \nu_1^T = \frac{(-1)^{p+1}\pi^{2p}(p!)^2}{A_p T^{2p+2}} \sum_k \frac{\Re(a_k)}{\Omega_k^{2p+1}} \cos(\Omega_k T) + o\left(\frac{1}{T^{2p+2}}\right) \tag{11}$$

$$\textit{with} \qquad \Omega_k = \langle k, \nu \rangle - \nu_1 \ ; \qquad A_p = -\frac{2}{\pi^2}\left(\frac{\pi^2}{6} - \sum_{k=1}^{p} \frac{1}{k^2}\right) \tag{12}$$

In particular, the use of a Hanning data window ($p = 1$) ensures that for a KAM solution, the accuracy of determining the main frequencies will be proportional to $1/T^4$, instead of $1/T^2$ without the Hanning window ($p = 0$), while for an ordinary FFT method, this accuracy will only be proportional to $1/T$. The frequency analysis will then easily allow recovery of the frequency vector $(\nu_1, \nu_2, \ldots, \nu_n)$. It should also be stressed that a lot of understanding can already be gained from the examination of the quasiperiodic approximation of the solutions, expressed as (8) [7,16]. It is also clear that one can investigate the improvements which could result from using larger values of p or other weight functions.

The frequency map can also be used to analyze in a very precise way the diffusion of the orbits in the frequency space [7,8,4]. In this case, the initial condition in action (I) is also fixed, and the frequency vector (ν) is evaluated over the time interval $[\tau, \tau + T]$ for different values of τ. The time evolution of the numerical frequencies ν is used as a measure of the diffusion of the orbit. Indeed, for a KAM tori, the frequency vector is fixed, up to numerical accuracy, while for a non regular orbit, the frequency vector (ν) will evolve with time, revealing the chaotic diffusion of the orbit. In a 2-DOF Hamiltonian system, for a fixed level of energy, the frequency space will be a line, so the regular KAM solutions are fixed dots which separates the space, and the chaotic zones are confined by the existing KAM tori. On the contrary, in higher dimension, the KAM tori are still represented as dots in the frequency space, but they no longer prevent the chaotic trajectories to wander in the frequency space. Nevertheless, the diffusion is supposed to be extremely small in their vicinity [15,13]. Thus, for practical view, over finite time, they can be thought to have some width, and regions which are densely filled with such tori will act as effective barriers for limiting the diffusion [8].

3 STUDY OF ERRORS IN THE ALS

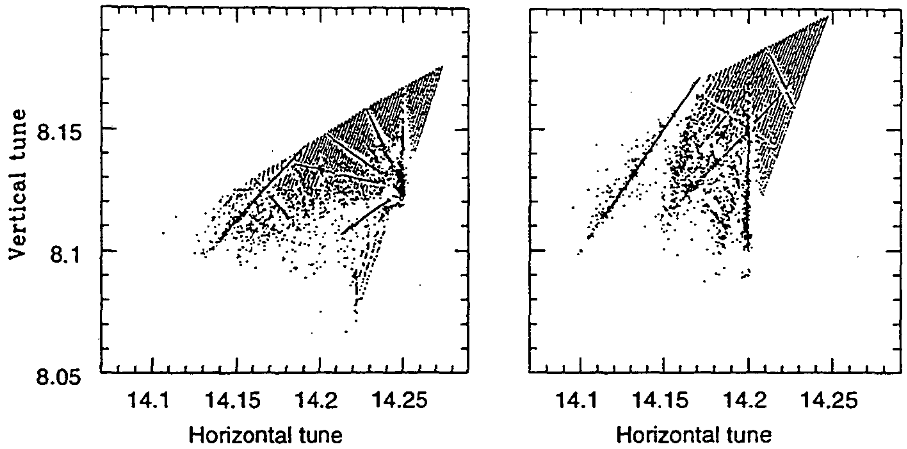

FIGURE 2: One quadrupole detuned by 5%. Original working point (left). New working point (right).

Dumas and Laskar have first applied the frequency map analysis techniques in accelerator dynamics to the study of the global dynamics and diffusion of a modelized cell of the Advanced Light Source (ALS)[3]. In this paper we chose to study the full lattice of the ALS[20] in order to investigate the application of frequency map analysis for finding strategies for the improvement the design of a real machine.

The ALS is a third generation synchrotron light source, and like other third generation light sources, the ALS storage ring is built up of strongly focussing quadrupoles that are necessary to reduce the beam's emittance. These quadrupoles generate large chromaticity that needs to be corrected with sextupole magnets. The sextupoles excite many nonlinear resonances and have a major impact on the single particle dynamics

The storage ring lattice is constructed of twelve identical sectors. This 12-fold symmetry helps suppress many nonlinear resonances. The reason being that if a storage ring has no symmetry, resonances can be excited when the condition $N_x\nu_x + N_y\nu_y + N_s\nu_s = R$ is satisfied, where ν_x, ν_y, and ν_s are the horizontal, vertical, and synchrotron tunes respectively and N_x, N_y, N_s, and R are all integers. However if the lattice is M-fold symmetric, there is an additional constraint for resonance excitation — R must be evenly divisible by M (12 in the case of the ALS). So the ALS's 12-fold symmetry is beneficial in suppressing many resonances. If the symmetry of the lattice is broken by magnetic field errors, resonances that are "not allowed" by symmetry can become excited, influencing the stability of the particle motion. This has been observed experimentally[21].

It is desirable to understand the affect of resonances on the ALS particle dynamics. From short term particle tracking alone it is very difficult to gain much insight into the dynamics. Frequency map analysis enhances our understanding by providing us with a simple global picture of the dynamics. We use frequency map analysis to illustrate the effect of symmetry breaking on the beam dynamics.

We combined a particle tracking code[22] with the NAFF frequency package. Particles are launched with different initial horizontal and vertical transverse momentums but with zero transverse offsets. The particles are then tracked on-energy and without synchrotron oscillations for 2048 turns or until lost. If the particle survives all 2048 turns, then the tunes and diffusion rate are calculated.

The first example that is tracked is the ideal lattice with perfect symmetry. In Figure 1 the frequency map in tunespace is presented. The horizontal and vertical tune of each surviving particle is plotted as a single dot. The working point of machine is $\nu_x = 14.28$ and $\nu_y = 8.18$ (upper right corner of Figure 1). Small amplitude particles tend to oscillate with tunes close to the working point. Particles with large amplitudes oscillate with tunes that are shifted left and downward from the working point. This is due to the negative tune shift with amplitude that is generated by the sextupoles.

We also see the influence of different resonances in Figure 1. In regions of tunespace where there are no strongly excited resonances, the tunes are evenly spaced. In regions of the map where resonances are strongly excited, the tunes become unevenly spaced. In these regions, particles have larger calculated diffusion rates. We have tracked particles in these regions for a greater number of turns and found that many of the particles eventually become unstable. It may seem strange that there are stable particles with tunes below the integer resonance ($\nu_y = 8.0$). This is because in a 12-fold machine, this resonance is not excited.

To illustrate the effect of symmetry breaking we created a lattice in which one quadrupole is detuned by 5%. The remaining quadrupoles have been adjusted to restore the global tunes. All this had the effect of introducing a beating of the betatron function of about 20% which is similar in magnitude to the beta-beating that has been measured in the actual machine. The results of the frequency map are seen in Figure 2 (left side). Comparing Figure 2 with Figure 1 we see that the stable area or dynamic aperture has shrunk drastically. In particular it is clear that the fourth order resonance $\nu_x = 14.25$ is excited and limits the dynamic aperture. This resonance was previously unallowed by the ideal ring's 12-fold symmetry.

We can increase the dynamic aperture by retuning the machine to a different working point. It is remarkable that a simple examination of the frequency map plot (Fig. 2) provided a very good guess for a better working point. We moved the working point to $\nu_x = 14.25$ and $\nu_y = 8.20$ and tracked the lattice again. The results can be seen in Figure 2 (right side). Now the limit to the dynamic aperture is the fifth order resonance $\nu_x = 14.20$. Comparing right side and the left side of Figure 2 we see that stable region of the map has indeed increased. In this case of large symmetry breaking the new working point is a more desirable operating point.

This example illustrates that the frequency map provides much more insight into the dynamics than just particle tracking alone. It gives a global picture of the dynamics that helped us locate a good working point. It contains information about the long term stability of the particles. Since more that 99% of the time is spent in the tracking portion of the code, the frequency map information is obtained at little cost in CPU time.

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