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INTRODUCTION

During 1950, E. Fermi (1) published an "attempt to develop a crude theoretical approach for calculating the outcome of nuclear collisions with very high energy."[1] In it he proposes a model for the multiple production of mesons "which deviates from the unknown truth in the opposite direction from conventional theory" based on weak coupling expansions. His proposal has some resemblance to the points of view adopted by others "who also stress the importance of the strong coupling for production processes of high multiplicity" and "consists in pushing this point of view to its extreme consequences." In doing so he is motivated by the hope "that it may be possible to bracket the correct state of fact between the two theories" and the belief that it "may perhaps be a fairly good approximation to actual events at very high energy."

Schematically the "crude theoretical approach" may be put in the following form. According to the "Golden Rule", the transition rate $w(f_1, f_2, \ldots \mid i)$ from an initial state (i) to a final state consisting of N mesons $(f_1, f_2, \ldots f_N)$ is given by the expression

$n = N$?

$w(f_1 \; f_2 \ldots f_n \mid i) = (\hbar)^{-1} 2\pi \; \rho_E(f) \mid M_E(f \mid i) \mid^2$, where M_E is the transition matrix with the dimension of energy $[E]$, evaluated on the energy shell E, and ρ_E is the classical extension in phase of this shell, divided by $\hbar^N$, and hence of dimension $[E]^{-1}$. If the usual device of normalizing the wave functions in a finite periodicity volume V is employed, ρ_E may be written as a product $\rho_E = \rho_E^{(p)} V^N$, where $\rho_E^{(p)}$ is the momentum space projection of the phase integral. Since V cannot appear in any physically significant context, this quantity raised to a high power must be cancelled by a similar one coming from the transition matrix. On dimensional grounds we then have

$\left| M_E \right|^2 = V^{-N} v_N(f_1 \; f_2 \ldots f_N \mid i) \left[E \right]^2$ where v_N is a quantity with dimension $\left[V \right]^N$, and is so defined that $\left[E \right]$ is the same for all (f). The expression for the transition rate may then be represented as $w(f \mid i) = \hbar^{-1} 2\pi \; \tilde{\rho}_E(f \mid i)$, where $\tilde{\rho}_E(f \mid i) = \rho_E^{(p)}(f) \; v_N(f_1 \ldots f_N \mid i)$. Let us compare this expression with a distribution function that could be obtained from a solution of the Liouville equation by integrating over the configurational variables and retaining only the variables specifying the momenta of the particles. (A) Unlike V^N appearing in ρ_E, v_N in $\tilde{\rho}$ cannot be represented as a constrained product[2] $\prod_{i=1}^{N *} v_s \left[f_s \right]$ with factors each depending on variables of one particle only. (B) The quantity v_N depends on the variables (i) of the initial state of the system. Property (A) in a solution of a classical statistical problem would indicate that the particles are stochastically dependent in an essential manner; property (B), that the state is not one of equilibrium. The core of Fermi's idea is to argue that at high energy $v_N^{\,o}(f_1 \; f_2 \ldots f_N \mid i) = \prod^{*} v_s(f_s)$ constitutes none the less a legitimate zero point approximation.

Let us note that this involves two completely independent assumptions corresponding to (B) and (A) respectively. It is possible to have $v_N(f_1\ \ f_2 \ldots \mid i) = v'_N(f_1 \ldots f_N)$ implying an equilibrium state in which each particle depends in an essential manner on all the others that emerge during the process; on the other hand one might assume $v_N(f_1 \ldots f_N \mid i) = \prod_s^{*} v_s(f_s \mid i)$ thus expressing a condition of statistical independence without an equilibrium state. Property (B), generally possessed by matrix elements, is argued away as follows. In a high energy nuclear collision, when "the nucleons with their surrounding retinue of pions hit against each other, all the portion of space occupied by the nucleon and their pion fields is suddenly loaded with a very great amount of energy. The interactions of the pion fields being strong and the number of possible states of a given energy, large...this energy will be distributed among the various degrees of freedom according to statistical laws." This is qualified by the remark that "only those states that are easily reachable from the initial state may actually attain statistical equilibrium" and "the only type of transitions that are believed to be fast enough are of the Yukawa type." Property (A) is dropped in a somewhat offhand manner and little is said against it. "Our assumption of statistical equilibrium consists in postulating that the square of the matrix element is merely proportional to the probability that for a state in question all particles" (presumably regarded as independent) "are contained at the same time inside v."

The discussion in this article will be largely limited to this central idea of Fermi. Less extreme proposals aiming at greater realism that have been made in recent years will not be considered. Fermi's own attempt to deduce the form of $v_i(f_i)$ will be briefly presented from a somewhat different point of view in the last section. The principal reason for "bracketing the correct state of fact" between two extremes is the hope of setting up bases in these peripheral territories from which forays into the interior where "the unknown truth" is intrenched might be conducted. The very remoteness of the region charted by Fermi would seem to afford some safety for such an enterprise.

With this in mind we start with a Lorentz covariant formulation of the classical theory of the microcanonical ensemble. The configuration space projections of the phase space expression have to be treated with some care and the requirement of Lorentz invariance seems to lead naturally to a more specific form of the Fermi proposal, a static, spherically symmetric model. Within it we can easily take into account the conservation of the six vector of angular momentum. The classical phase space integral having thus been given a very specific form we proceed to examine the quantum mechanical S matrix expression with a view of finding a corresponding structure. This leads us to what appears to be covariant version of the Wigner coordinate-momentum distribution function [2] used by this author in connection with quantum corrections to classical statistics. This identification permits us also to make a plausible guess on how to include spin effects, statistical correlations due to indistinguishability and interaction in the final state into the Fermi model. A good deal of space is then devoted to the discussion of thermodynamic approximations at very high energies. Not much has been done along these lines and some of the problems encountered are unlike those of the more conventional situations in statistical mechanics. We have tried to follow a classical procedure which would avoid some of the difficulties connected with indistinguishability, and permit us to take into account quantum correlations. Finally against this background we present a very brief discussion of the more interesting and hopeful of the recent work. For less recent work and for most details the reader is being referred to a review article by Milburn [3] which could be read with advantage in conjunction with this paper. Much had to be left out or barely touched upon. It is felt, none the less, that a reasonably unified presentation might be more useful than a comprehensive survey.

RELATIVISTIC DENSITY FUNCTIONS

Although the expression $w(f \mid i) \cong \hbar^{-1} 2\pi [E]^2 \rho^{(p)} v_N$ calculated from a relativistic field theory is an obvious covariant, Fermi's zeroth order approximation to it,

$w_0(f \mid i) = \hbar^{-1} 2\pi [E]^2 \rho^{(p)} \prod_s v_s [f_s]$ is not. This derives from the fact that the transformation properties of $\tilde{\rho}_0 = \rho^{(p)} \prod_s v_s$ are essentially those of $\rho = \rho^{(p)} v^N$, the classical density, which becomes a covariant only when multiplied by $|M|^2$. We accordingly proceed to find a relativistic version of $\rho = \rho^{(p)} v^N$. This will restrict the form of the zeroth order approximation to v_N.

In view of later applications we discuss ρ in a context in which this quantity is meaningful classically, that is, in connection with the microcanonical distribution. The probability of finding a closed system of N distinguishable particles with coordinates $x_1 = q_1$, $x_2 = p_1$, $x_3 = q_2$, $x_4 = p_2$, ... $x_{6N} = p_{3N}$ and a single energy integral H at the point x of phase space after it has reached equilibrium is given by the expression

$$W_E(x) = \frac{\delta(E - H(x))}{\int (dx')^{6N} \, \delta(E - H(x'))} \qquad 1.$$

where E is the value of the energy integral. We now introduce a sequence of density functions

$$\rho_E(x_1 \ldots x_{6N}) = \delta(E - H(x_1 \ldots x_{6N}))$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\rho_E(x_1 \ldots x_{i-1}\, x_{i+1} \ldots x_{6N}) = \int dx_i \;\; \delta(E - H(x_1 \ldots x_{6N}))$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\rho_E(x_1 \ldots x_{i-1}\, x_{i+1} \ldots x_{j-1}\, x_{j+1} \ldots x_{6N}) = \int dx_i\, dx_j \;\; \delta(E - H(x_1 \ldots x_{6N}))$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\rho_E = \int dx_1 \ldots dx_{6N} \;\; \delta(E - H(x_1, \ldots x_{6N})) \quad .$$

The right member of 1 is the ratio of the first to the last member of this sequence. With the aid of the notation just introduced we may write not only for the total but also for the marginal distributions

$$W_E(\underset{\sim}{x}) = \rho_E^{-1} \;\; \rho_E(\underset{\sim}{x}) \qquad 2.$$

where x may now be any subset of the set of the $6N$ variables. Changes in random variables can now also be affected with ease. If instead of x_s we are interested in the probability distribution of $\xi_s = \xi_s(x)$, we obtain it with the aid of the expression

$$W[\xi_s] = \rho_E^{-1} \int \delta(\xi_s - \xi_s(x))(dx)^{6N} \rho(x_1 \; x_2 \dots x_{6N})$$

$$= \rho_E^{-1} \rho(\xi_s) \;,$$

thus defining the density function for the ξ's variables. Of particular interest in thermodynamic applications is the case of an H consisting of several independent parts $H = H^1(x_1) + H^2(x_2) + \dots + H^s(x_s)$. Choosing the random energies of the several parts as new variables, we obtain

$$W_E[\epsilon_1 \; \epsilon_2 \dots \epsilon_s] = \delta(E - \epsilon_1 - \epsilon_2 \dots - \epsilon_s) \rho_E^{-1} \rho_{\epsilon_1} \rho_{\epsilon_2} \cdots \rho_{\epsilon_s}$$

an expression for the distribution of the energy among the several parts of the system.

Since the concept of equilibrium occupies a central position in Fermi's thought, it might be helpful to have a working idea on how the microcanonical distribution 1 associated with it arises from exact mechanics. Being of certain knowledge that at t_0, $x(t_0) = x_0$, this discipline tells us how to express $x(t) = x(t; x_0, t_0)$,—a task which may be broken up into two steps. We express $x(t) = x(t; x(t_0), t_0)$

without regard to our state of knowledge about $x(t_0)$; we average the expression so obtained over a distribution $W[x(t_0)]$ which, in the case of exact mechanics, is an improper function

$W[x(t_0)] = \prod_i \delta(x^i(t_0) - x_0^i)$ reflecting our certainly that at t_0, $x(t_0) = x_0$. The transition from exact to statistical mechanics consists in modifying the second step. Our doubts about the initial values of all the variables might prompt us to replace the infinitely peaked product of delta function by a completely regular distribution. This however would be very bad methodology. Although, in practice, there may be sound grounds for assuming such broad ignorance, the underlying reasons are basically different for different variables. If the total energy and momentum of the system are not known exactly, it is because of practical limitations on a macroscopic laboratory measurement; the vast majority, however, of other variables are uncertain because of the complexity and inaccessibility of the microscopic world. We accordingly separate the specialty of analysis of laboratory data from that of thermodynamics. For the latter we reserve the program of investigating the implications of the replacement

$$w\left[x(t)\right] = \prod \delta\left(x(t_0) - x_0\right) \rightarrow \frac{\delta\left(E - H(x(t_0))\right) w\left[x(t_0)\right]}{\int dx(t_0)\, \delta\left(E - H\left[x(t_0)\right]\right) w\left[x(t_0)\right]} \qquad 4.$$

alone, in which the energy remains certain and the other variables are smeared out. Putting w = const. amounts to the assumption of equal a priori initial weights and makes 4 identical with the microcanonical distribution 1. It is a fundamental (unproven) statement of statistical mechanics that the actual value of $x(t;\, t_0,\, x(t_0))$ averaged over 4 does not differ much from what would be obtained by averaging over 1, if t is appreciably different from t_0. The time interval $t - t_0$ may be regarded as the relaxation time, and the ensuing state, that of equilibrium for the system. Under its regime, macroscopic measurements on systems which started out from vastly different initial conditions yield substantially identical results. In formulas

$$\overline{x(t;\, t_0,\, x(t_0))} = \frac{\int dx(t_0)\, x\left[t;\, t_0, x(t_0)\right]\, \delta\left(E - H\left[x(t_0)\right]\right) w\left[x(t_0)\right]}{\int dx(t_0)\, \delta\left(E - H\left[x(t_0)\right]\right) w\left[x(t_0)\right]}$$

Eq. 5 continued.

$$= \frac{\int dx(t)\, J\begin{bmatrix} x(t_0) \\ x(t) \end{bmatrix} x(t)\ \delta\left[E - H\left[x(t_0;\ t,\ x(t))\right]\right] w\left\{x(t_0;\ t,\ x(t))\right\}}{\int dx(t)\, J\begin{bmatrix} x(t_0) \\ x(t) \end{bmatrix} \delta\left[E - H\left[x(t_0;\ t,\ x(t))\right]\right]\ w\left\{x(t_0;\ t\ \ x(t))\right\}} \tag{5.}$$

$$= \frac{\int d\xi\ \ \xi\ \delta\left[E - H(\xi)\right]\ \ w_t\left[\xi\right]}{\int d\xi\ \ \delta(E - H(\xi))\ w_t(\xi)}\ .$$

The second member is a representation of the first in terms of integrals over $x(t_0)$; the third, over $x(t)$. To reach the fourth we make use of Liouville's theorem which asserts that the Jacobian determinant $J\begin{bmatrix} x(t_0) \\ x(t) \end{bmatrix} = 1$, and define a new probability distribution W_t by means of the relation $w\left[x(t_0)\right] = w\left\{x(t_0;\ t,\ x(t)\right\} = w_t\left[x(t)\right]$. The new function satisfies the Liouville equation and, inserted in 4 instead of $w\left[x(t_0)\right]$, represents its time dependent normalized solution on the energy shell. It will in general depend on the initial conditions of the system, as will the sequence of densities ρ_E, $\rho_E(x_1) \ldots \rho_E(x_1 \ldots x_{6N}) = \delta(E - H(x))\ w_t(x)$, derived from it. It was a similarity to such a ρ_E that was suggested in the introduction in connection with the quantum mechanical transition probability. In formula 5, to reach the last member we also had to limit ourselves to conservative dynamical systems for which $H_t(x) = H(x)$. Observe that if $w[x(t_0)] = \text{const}$, then also $w_t[x] = \text{const}$. A microcanonical distribution is thus seen to be stable with respect to temporal change. A broad class of other distributions is believed to tend to this stable one when permitted to evolve freely in time.

We shall now examine the simplest of the ρ's, ρ_E, with a view of finding its relativistic generalization. The Hamiltonian of the complete system is assumed to be the sum of the Hamiltonian of free particles composing it and ρ_E is written as

$$\rho_E = \int \prod_{(m)}^{*} \frac{d^3p^{(m)}}{(2\pi\hbar)^3} \int \prod_{(n)}^{*} d^3x^{(n)} \chi_n(x^n)$$

$$= \int \prod_{(m)}^{*} \frac{d^3p^{(m)}}{(2\pi\hbar)^3} \, \delta\left[E - c \sum_s \sqrt{p_s^2 + m_s^2 c^2}\right] \int \prod_n^{*} d^3x^{(n)} \chi_n(x^n). \tag{6}$$

The symbol $\chi_n(x)$ denotes the characteristic function of the finite domain of the configuration space which is available to the particle n. Its value is one when x is inside the domain, zero when outside. It will be convenient to extend this definition and include other functions which vanish sufficiently rapidly at infinity. The symbol χ_n will from now on denote members of this broader class. Introducing new variables $p = \hbar k$, $mc = \hbar \kappa$, $E = \hbar c K_0$, $\rho_E = (\hbar c)^{-1} \mathring{\rho}_K$ we rewrite 6 as

$$\mathring{\rho}_K = \int \prod_m \frac{d^3\vec{k}^{(m)}}{(2\pi)^3} \delta\left[K_0 - \sum_s \sqrt{k_s^2 + \kappa_s^2}\right] \int \prod_n^* d^3x^{(n)} \chi_n(x^n) , \tag{7.}$$

which is readily seen to be equivalent to

$$\mathring{\rho}_K = \int \prod_m \left[d^4k^{(m)} \Delta\left[k^{(m)}, \kappa^{(m)}\right]\right] \delta\left[K_0 - \sum_s k_0^{(s)}\right] \cdot$$

$$\cdot \int \prod_s^* \frac{k_0^s}{\kappa_s} d^3\vec{x}^{(s)} \chi_s(x^s) , \tag{8.}$$

if[3]

$$\Delta(k, \kappa) = \frac{\kappa}{(2\pi)^3} \theta(k_0) \frac{\delta\left(k_0 - \sqrt{k^2 + \kappa^2}\right)}{\sqrt{k^2 + \kappa^2}} ;$$

$$\begin{aligned} \theta(k_0) &= 1, \quad k_0 > 0 \\ &= 0, \quad k_0 < 0 . \end{aligned} \tag{9.}$$

The momentum integration in 8 will be invariant if not only the energy but also momentum is conserved. We accordingly replace $\overset{\circ}{\rho}_K$ by ρ_K in which

$$\delta(K_0 - \sum_s k_0^{(s)}) \rightarrow \delta(K_0 - \sum_s k_0^{(s)})\, \delta\left[\vec{K} - \sum_s \vec{k}^{(s)}\right] = \delta(K - \sum k^{(s)}) .$$

The integral then becomes

$$\rho_K = \int \prod_m d^4k^{(m)} \Delta^+ \; (k^m,\ \chi^{(m)})\ \delta(K - \sum_s k^{(s)})$$

$$\int \prod^{*} \frac{k_0^{(s)}}{\chi^{(s)}} \quad d^3\vec{x}^{(s)}\, \mathcal{X}_{(s)}(x^s) \quad . \tag{10}$$

It is now a simple matter to make 10 form invariant and even manifestly so. We declare $\mathcal{X}_s$ to be a scalar, that is to have the same numerical value at the same geometrical point x without regard to the coordinatization $(x_1,\ x_2,\ x_3,\ x_4)$ adapted for the point x. This does not involve any assumption about the functional dependence of $\mathcal{X}$ on x. We further assume $\chi^{-1}\, k_0\ d^3\vec{x}$ to be a scalar product of two four vectors $\chi^{-1}\, k_\mu$ and $d\sigma^\mu$, the latter of which happens to reduce in our coordinate system to $\left[0,\ 0,\ 0,\ -\ i d^3\vec{x}\,\right]$. Equation 10 may then be written

$$\rho_K = \int \prod_{(m)}{}^{*} d^4k^{(m)}\Delta^{+}\ (k^{(m)}\ \chi^{(m)}) \int_{\{\sigma\}} \prod{}^{*} d\sigma_{(s)}{}^{\mu}\ (x)\ \chi_{\mu}^{(s)}(x)$$

11.

where $\underset{(s)}{\chi}{}_{\mu} = \underset{(s)}{k}{}_{\mu}\ \underset{(s)}{\chi}$. The restriction to flat surfaces (σ) in carrying out the integration in 11 can also be removed. Nor need the confinguration space of every particle be on the same space-like surface. It is, however, in line with the idea of equilibrium to require that ρ_K does not depend on the system of surfaces adopted. This requirement is satisfied if

$$\frac{\partial \underset{(s)}{\chi}{}_{\mu}\,(x)}{\partial\ x^{\mu}} = 0\ . \qquad 12.$$

We shall designate with the superscript zero quantities evaluated in a frame, $k_{\mu}^{0} = (0,\ 0,\ 0,\ k_0^{0})$, in which the particle is at rest. Because of the special form $\underset{(s)}{\chi}{}_{\mu} = \underset{(s)}{k}{}_{\mu}\ \underset{(s)}{\chi}$, condition 12 reduces in the rest frame of the s'th particle to

$$\frac{\partial}{\partial\ \overset{\circ}{t}}\ \overset{\circ}{\underset{(s)}{\chi}}\ (\overset{\circ}{\vec{x}}\ ,\ \overset{\circ}{t}\ )\ . \qquad 13.$$

A model satisfying this condition will be called static.

In order to narrow down the range of possible choices of χ_μ, we impose a more stringent requirement of covariance on ρ_K. This quantity shall not only transform as a scalar under the Lorentz group but it shall also admit the group. This requirement reduces to a condition on χ_s which we previously defined to be a scalar. χ shall be the same function of the coordinate variables (x_1, x_2, x_3, x_4) of the point x in whatever Lorentz frame these are evaluated. Thus if in one frame we have $\{x_\mu\}$ and in another $\{x'_\mu\} = \{L_{\mu\nu} x^\nu\}$, we must still have $\chi(x'_\mu) = \chi(L_{\mu\nu} x^\nu) = \chi'(x_\mu) = \chi(x_\mu)$. In going from the first to the second member we express the new coordinate variables x'_μ in terms of the old ones x by means of transformation coefficients L; the second equality sign is a definition of χ'; the the third expresses our demand. It will be satisfied if $\chi(x)$ commutes with the generators of the Lorentz group. Observe that the characteristic function appropriate to a large finite box which is usually employed does not satisfy this requirement. It is in the sense of group invariance that the integration over a finite volume is a noncovariant operation.

To proceed further we need some additional assumption. Keeping in mind that ρ_K will eventually be accepted as a zeroth order approximation to a quantum mechanical problem, it seems reasonable to require that χ_s be spherically symmetric in the frame in which the s'th particle is at rest. In symbols, $\underset{(s)}{\chi}(x_\mu) = \underset{(s)}{\overset{\circ}{\chi}}(\overset{\circ}{x}_\mu) = \underset{(s)}{\chi}(\overset{\circ}{x}) = \underset{(s)}{\chi}\left(\chi_s\sqrt{\overset{\circ}{x}_1{}^2+\overset{\circ}{x}_2{}^2+\overset{\circ}{x}_3{}^2}\right)$.

The first equaltity sign is a definition of $\overset{\circ}{\chi}$; the second makes use of the imposed group invariance; the third expresses our demand. We regard χ_s to be a function of the dimensionless variable

$$\chi_s\,\overset{\circ}{r} = \chi_s\sqrt{\overset{\circ}{x}_1{}^2+\overset{\circ}{x}_2{}^2+\overset{\circ}{x}_3{}^2}\,.$$

This imposes no additional restriction, since we qualify it with the subscript s on χ_s. It is now readily seen that

$$\mathcal{L}^{(s)} = \sqrt{-\tfrac{1}{2}\,L_{\mu\nu}^{(s)}\,L_{\mu\nu}^{(s)}} = \sqrt{-\tfrac{1}{2}\,\overset{\circ}{L}_{\mu\nu}^{(s)}\,\overset{\circ}{L}_{\mu\nu}^{(s)}} = \chi_s\,\overset{\circ}{r}_{(s)}\,. \qquad 14.$$

The symbol $L_{\mu\nu}^{(s)}$ in 14 is an abbreviation for the component $x_\mu^{(s)}k_\nu^{(s)} - x_\nu^{(s)}k_\mu^{(s)}$ of the angular momentum six vector of the s'th particle. We may accordingly write

$$\chi_{(s)}(x) = \chi_{(s)}\left[\mathcal{L}^{(s)}\right] \qquad 15.$$

and expression 11 becomes

$$\rho_K = \int \prod_m^* d^4k_{(m)}\, \Delta\,(k^{(m)},\ \chi^{(m)}) \int_\sigma \prod_{(s)}^* d\sigma_{(s)}^{\ \mu}(x_s)\, \chi_\mu^{(s)}\left[\mathcal{L}^{(s)}\right] . \qquad 16$$

The characteristic function χ limiting the configuration space avilable to a particle thus depends on its energy momentum four vector through $\mathcal{L}_{(s)}$. A model of this form may be called static, spherically symmetric. Observe that there need not exist a single frame in which all the $\chi_{(s)}$ have this symmetry.

An obvious constraint restricting the range of integration in configuration space arises from the conservation of the six vector of angular momentum. In the absence of other constraints we have

$$\Sigma(L) = \int \prod_s^* d\sigma_{(s)}^{\ \mu}\, \chi_{(s)}^{\ \mu} = \int \prod_s d\sigma_{(s)}^{\ \mu}\, \chi_{(s)}^{\ \mu}\, \delta(L - \sum_n L^{(n)}) \qquad 17.$$

where L denotes the values of components of the six vector, and the delta symbol stands for a product of six delta functions. For a static spherically symmetric model, $\left[\chi_s = \chi_s\left(-\sqrt{-\tfrac{1}{2}L_{(s)}^{2}}\right)\right]$, the multiple integrations may be carried out and the result represented in simple parametric form. To illustrate, we do it for a gaussian

$$\chi_\mu = v\,\frac{k_\mu}{\kappa}\left(\frac{\kappa}{2\sqrt{\pi}}\right)^3 e^{\frac{1}{8}L_{\mu\nu}L^{\mu\nu}} \qquad 18.$$

normalized in the rest frame to volume v. Fourier analyzing the delta functions in 17 we write

$$\Sigma(L) = \frac{1}{(2\pi)^6}\int d^6\Lambda\; e^{\frac{1}{2}i\Lambda_{\mu\nu}L^{\mu\nu}}\;\prod_s{}' I_s \qquad 19.$$

where

$$I_s = \int d\sigma^\lambda\;\underset{(s)}{\chi}{}_\lambda\left[\mathcal{L}_s\right] e^{-\frac{1}{2}\Lambda_{\mu\nu}L_{\mu\nu}^{(s)}} \quad . \qquad 20.$$

Since every I_s is an invariant we may also say that

$$\begin{aligned} I_s &= \int d\underset{(0)}{\sigma}{}^\mu\;\underset{(0)}{\chi}{}_\lambda\left[\overset{(o)}{\mathcal{L}}_s\right] e^{-\frac{1}{2}\overset{(o)}{\Lambda}_{\mu\nu}\overset{(o)}{L}_{(s)}^{\mu\nu}} \\ &= \int \frac{d^3r}{i}\;\underset{0}{\chi}{}_4\left[\underset{(s)}{\kappa}\;\underset{(s)}{\overset{o}{r}}\right] e^{+i\kappa x_i\overset{o}{\Lambda}_{io}} \quad . \end{aligned} \qquad 21.$$

Exploiting the fact that χ is group invariant $\underset{0}{\chi} = \chi$, we can express I_s in terms of the Fourier image of χ and do so in a particularly simple manner for 18, obtaining $I_s = v\, e^{+\overset{\circ}{\Lambda}_{14}\overset{\circ}{\Lambda}_{14}}$.

A little reflection will show that in an arbitrary frame $\overset{\circ}{\Lambda}_{14}\overset{\circ}{\Lambda}_{14} =$ $-\chi^{-2}(\Lambda_{\mu\nu}k_\nu)^2 \doteq +\chi^{-2}(k, \Lambda^2 k)$, the square length of a space-like vector $\chi^{-1}\Lambda_{\mu\nu}k_\nu$ which we indicated in bracket notation. Collecting the terms of the product we have

$$\Sigma(\mathcal{L}) = \frac{1}{(2\pi)^6}\int d^6\Lambda\, e^{-i(\Lambda,\, L) + \sum_s \chi_s^{-2}(k^{(s)},\, \Lambda^2 k^{(s)})} ,$$

22.

where we have also written $(\Lambda, \mathcal{L})$ for $(-\tfrac{1}{2})\Lambda_{\mu\nu}\mathcal{L}^{\mu\nu}$, and finally

$$\rho_{\chi,\mathcal{L}} = \frac{v^N}{(2\pi)^6}\int d^6\Lambda\, e^{-i(\Lambda,\, L)}\int \prod{}^{*} d^4k_{(s)}\, e^{\chi_s^{-2}(k^{(s)},\, \Lambda^2 k^{(s)})}\,\Delta_s(k_s, \chi_s).$$

23.

Making use of specific features of the spherically symmetric model, we are thus able to represent the 3N fold configurational integral by means of a six fold integral with the angular momentum conservation law explicitly taken into account. It is hoped to investigate the restrictions imposed by this integral of motion in a future publication. For the remainder of this article we shall neglect its effect.

Equation 11 is a relativistic version of the last member of the sequence of density functions following 1. The form of other members of this sequence can be inferred from it. Thus

$$\rho^{\mu_1 \cdots \mu_N}(k_1, \ldots k_N; x_1 \ldots x_N) = \prod_{s=1}^{N}{}^{*} \Delta_s(k_s) \, \chi^{\mu_s}(k_s, x_s) \quad , \tag{24}$$

a tensor of rank N divergenceless in every index is the analogue of the first member. The symbols k_s and x_s in 24 stand for the momentum and position four vectors of the s'th particle. The total number of particles present is taken to be N. A statistical mechanics based on the microcanonical distribution of Gibbs is, however, too restrictive to be appropriate to a relativistic high energy situation. It is predicated on the idea that the number of particles is fixed and its distribution function envisages only possibilities that fall within this narrow range. Not only the four vector K but also the number of particles, N, should appear as a subscript of the density function we have written before. The distribution function that is needed to take into account the great variation in this number at high energy might be called the grand microcanonical distribution. The letter n is no longer fixed, but should appear as a variable in the argument of the density function: $\rho_K(k_1, \ldots k_n; n)$. This distribution should be distinguished from the grand canonical distribution where the number of particles in a closed system is fixed and only for subsystems is it a random variable. The new ρ_K to be used in normalizing the densities in order to convert them to probabilities is now

$$\rho_K = \sum_n \int \prod_s dk_s \; \rho(k_1, \ldots k_n; n) \quad . \tag{25.}$$

A greater variety of marginal distribution and marginal densities is also possible. Important will be $\rho_K(n) = \int \prod_s (d^4k^{(s)}) \; \rho(k_1, \ldots k_n)n)$, the density for the appearance of n particle. In terms of it, $\rho_K = \sum_n \rho_K(n)$. The density for the first particle having momentum k when n others are also present is given by

$$\rho_K(k_1; n) = \int \prod_{s=2}^{n} d^4k^{(s)} \; \rho(k_1 \, k_2 \ldots k_n; n) \,,$$

the momentum density of the first particle regardless of the number of others is $\rho_K(k) = \sum_n \rho_K(k; n)$.

S-MATRIX FORMULATION

In the last section we arrived at an expression for the relativistic current function $\rho_K^{\mu_1\cdots\mu_n}(k_1\ldots k_n;\, x_1\ldots x_n)$ which might be regarded as a generalization of the nonrelativistic density $\rho_E(\vec{k}_1,\ldots\vec{k}_n;\, \vec{r}_1\ldots\vec{r}_n)$. With its aid we can now construct a relativistic probability current

$$W_K^{\mu_1\cdots\mu_n}(k_1\ldots k_n,\, x_1\ldots x_n) \;=\; \rho_K^{-1}\;\rho_K^{\mu_1\cdots\mu_n}(k_1\ldots k_n;\, x_1\ldots x_n) \tag{26.}$$

wich is the generalization of the microcanonical distribution of Gibbs. It is only the marginal distribution

$$W_K(k_1\ldots k_n) \;=\; \rho_K^{-1}\;\rho_K(k_1\ldots k_n) \;=\; \rho_K^{-1}\int \prod_{s=1}^{n} d\sigma^{\mu_s}\;\rho_K^{\mu_1\cdots\mu_n}\cdot(k_1\ldots k_n;\, x_1\ldots x_n) \tag{27.}$$

that is relevant to quantum theory.

In this section we undertake a detailed comparison of 27 with the corresponding quantum mechanical expression. This permits us to give a wave mechanical interpretation to the Fermi approximation, enables us to see how to take into account exchange effects associated with the indistinguishability of elementary particles, and also how to construct a zeroth order Fermi approximation for emerging particles which are not scalar. What suggests itself in this connection is that we attempt to construct the quantum analogue not only of 27 but also of 26. If the Fermi model is ever to serve as a zeroth order approximation to a quantum mechanical transition probability we should like to discern the outlines of its rather definite mathematical shape in the quantum expression. As auxilliary functions, simultaneous distributions for coordinates and momenta have been introduced by Wigner (2) and used effectively to calculate quantum deviations from classical averages. These quantities are not amenable to a direct physical interpretations. It is felt none the less that they might be of help in constructing a consistent Fermi approximation to field theory in the high energy limit.

For simplicity we consider a final state consisting of two distinguishable mesons with field operator $A(x)$, $B(x)$. Corresponding to each field (charged or neutral) we construct a set of functions $\{f(x)\}$ satisfying the free Gordon-Klein equation [5] and rendering the expression

$$(f,\ f)_D = \frac{1}{\chi} \int_D d\sigma^{\mu}\ \bar{f}\ \overleftrightarrow{\partial_\mu}\ f = \frac{1}{2i\chi} \int_D d\sigma^{\mu} \left[\bar{f}\ \partial_\mu\ f - \partial_\mu\ \bar{f} \cdot f \right] \qquad 28.$$

positive for every domain D. Some point normalization on the set f might also be imposed to make it more definite. The plane wave e^{ikx} satisfies for example

$$f(0) = 1\ ; \qquad \frac{\partial \bar{f}(0)}{\partial x^{\mu}}\ \frac{\partial f(0)}{\partial x^{\mu}} + \chi^2\ \bar{f}(0)\ f(0) = 0.$$

In these formulas $\bar{f}$ denotes the complex conjugate of f. The usual orthonormal set results if we choose a sub-set $f_n(x)$ which vanishes on the boundary of a domain D and normalize it to

$$f'_{n,D} = \frac{f_n(x)}{\sqrt{(f_n,\ f_n)}} \ . \qquad 29.$$

The completeness relation may now be states as

$$\Delta(x\ \ x';\ D) = \sum_n f'_{n,D}(x)\ \bar{f}'_{n,D}(x') = \sum_n \frac{f_n(x)\ \ \bar{f}_n(x')}{(f_n,\ \ f_n)} \ . \qquad 30.$$

In the limit of $D \rightarrow \infty$ we have the usual representation

$$\Delta(x\ x') = \lim_{D \rightarrow \infty} \Delta(x\ x');\ D) = 2\chi \int \frac{d^4 k}{(2\pi)^3}\ \theta(k_o)\ \delta(k^2 + \chi^2) e^{ik(x - x')} . \qquad 31.$$

Expression 28 can be extended in an obvious manner to define symbols like (f, g). It may also be used in connection with functions F which do not satisfy the G.K. equation or do so only asymptotically $F(f) \rightarrow f$. The scalar product $(F\ F)_{D,\sigma}$ will in general depend on the surface σ. For more general F it may exist even for unbounded D. Besides c-quantities we also consider q-quantities of the form of a scalar product. Thus $A_n = \frac{(f_n, A)}{\sqrt{(f_n, f_n)D}}$ is a destruction operator for the A field. In connection with this generalization the following should be noted. For an ordinary scalar product we have $\overline{(f, g)} = (g, f)$. To get a formula of comparable simplicity with operators we should consider $(f_n, A)^+ = (A, f_n)$ where $+$ denotes Hermitian adjoint. The latter symbol may be used in the case of both c and q quantities.

The probability of finding the system in the final state (f) conditional on the hypothesis that it was originally in the initial state (i) may be written as

$$W(f \mid i) = \psi^+(f)\, S\, \psi(i)\ \psi^+(i)\, S^+\, \psi(f) \qquad 32.$$

where ψ^+ and ψ are state bra and kets respectively and S denotes the Heisenberg S operator. Since according to Fermi's idea the final outcome is only weakly conditioned by the original state it is more natural to replace 32 by the final probability $W(f) = \sum_i W(f \mid i)\, w(i)$ where $w(i)$ is the probability of the initial state. Defining the statistical operator for the initial state by $U(i) = \sum w(i)\, \psi(i)\ \psi^+(i)$ we may write

$$W(f) = \psi^+(f)\, S\, U(i)\, S^+\, \psi(f) \quad . \qquad 33.$$

The distribution

$$w[f] = \frac{\psi^+(f)\ \delta(P - \mathcal{P})\ \psi(f)}{Sp\ \delta(P - \mathcal{P})}$$

may be taken as the quantum version of the microcanonical ensemble of Gibbs. The letter $\mathcal{P}$ denotes here the second quantized expression for the total energy momentum four vector of the system and P its particular set of eigenvalues. The stability of this distribution may be seen from the fact that for an initial statistical operator

$$U(i) = \frac{\delta(P - \mathcal{P})}{Sp\ \delta(P - \mathcal{P})}$$

we have $S\,U(i)S^{+} = U(i)$, since S commutes with $\mathcal{P}$.

Assume a final state consisting of an A meson in state a and a B meson in state b. Thus

$$\psi(f) = \psi(a, b) = \frac{(A, f_a)(B, f_b)}{\sqrt{(f_a, f_a)_D (f_b, f_b)_D}} \psi_0$$

where ψ_0 denotes the vacuum ket. Equation 33 may then be put in the form

$$W(a, b; D) = \int_D d\xi\, d\bar{\xi}\, d\eta\, d\bar{\eta}\, \frac{f_a(\xi)\,\bar{f}_a(\bar{\xi})}{(f_a, f_a)_D}\, \frac{f_b(\eta)\,\bar{f}_b(\bar{\eta})}{(f_b, f_b)_D}$$

$$\psi_0^{+} K(\xi, \eta)\, U(i)\, K^{+}(\bar{\xi}, \bar{\eta})\psi_0 \qquad 34.$$

defining the kernel K. Summing this expression over the labels a and b we obtain for the probability of finding an A and a B meson in the domain D .

$$W(A, B; D) = \int_D d\xi\, d\bar{\xi}\, d\eta\, d\bar{\eta}\, \Delta(\xi\, \bar{\xi}\, ; D)\, \Delta(\eta\, \bar{\eta}; D)$$

$$\psi_0^+ \, K(\xi\, \eta)\, U(1)\, K^+(\bar{\xi}\, \bar{\eta})\, \psi_0 .$$

Letting D cover all space we have

$$W(A\, B) = \int d\xi\, d\bar{\xi}\, d\eta\, d\bar{\eta}\, \Delta(\xi, \bar{\xi})\, \Delta(\eta, \bar{\eta})\, \psi_0^+ \, K(\xi\, \eta) U(1)\, K^+(\bar{\xi}\, \bar{\eta})\, \psi_0 . \tag{36.}$$

The assumed invariance of the theory under four dimensional translations leads to the kind of restrictions we encountered in the classical phase space expressions. The invariance has two aspects to it: kinematic and dynamic. The first may be interpreted by saying that the field operators are effectively constants. If p_μ denotes the displacement operator $(h/i)(\partial / \partial x_\mu)$, then under a displacement $x \longrightarrow x' = x + a$ any c number function will undergo the transformation $f \longrightarrow f' = e^{iap} f e^{-iap}$. For a constant we have $f' = f$. A field operator is a second rank tensor in the space of occupation numbers. A translation of coordinates $x \longrightarrow x' = x + a$ whose infinitesimal generator is p will induce a corresponding representation of this operation in the occupation number space with generator P. The total transformation may then be denoted as $A \rightarrow A' = e^{ia(p+P)} A\, e^{-ia(p+P)}$.

In a kinematically invariant field theory this representation is so defined that $A' = A$. In this extended sense all field operators may be looked upon as "constants" and this fact expressed by the relation

$$e^{iaP} A(x)\, e^{-iaP} = e^{-iap} A(x)\, e^{+iap} = A(x-a)\,. \qquad 37.$$

Dynamical invariance on the other hand implies that K in 35 and 36 depends on ξ and η through the operators A and B only. Adopting the convention that the vacuum is a state of zero energy and momentum we may write

$$\psi_0^+ K(\xi, \eta)\, U(i)\, K^+(\bar{\xi}, \bar{\eta})\psi_0 = \psi_0^+ e^{i\alpha P} K(\xi, \eta) U(i) K^+(\bar{\xi}, \bar{\eta}) e^{-i\beta P}\psi_0$$

$$= \psi_0^+ e^{-i\alpha p} K(\xi\ \eta) e^{i\alpha p}\, e^{i\alpha P}\, U(i)\, e^{-i\beta P}\, e^{-i\beta p} K^+(\bar{\xi}\ \bar{\eta}) e^{+i\beta p}\psi_0$$

$$= \psi_0^+ K(\xi-\alpha, \eta-\alpha)\, U'(i)\, K^+(\bar{\xi}-\beta, \bar{\eta}-\beta)\psi_0\,.$$

Both assumption of invariance were exploited in going from the second to the third member of this chain. We now make the additional assumption that the initial state is one of definite energy and momentum. In terms of the statistical operator this may be expressed as

$$U'(i) = e^{i\alpha P} U(i) e^{-i\beta P} = e^{i(\alpha - \beta)K} U(i)$$

where K is the value of the energy momentum four vector. Inserting this into 36, subtracting the resulting expression from the original form of 36 and changing variables in an obvious manner we infer

$$0 \equiv \int dx\, dy \left\{ \Delta_A\left[x - (\alpha - \beta)\right] \Delta_B\left[y - (\alpha - \beta)\right] e^{i(\alpha - \beta)K} - \Delta_A(x)\, \Delta_B(y) \right\}$$

$$\cdot \int dX\, dY\, \Psi_0^{+}\, K(X + \tfrac{x}{2}, Y + \tfrac{y}{2})\, U(i)\, K(X - \tfrac{x}{2}, Y - \tfrac{y}{2}) \Psi_0 \qquad 38.$$

for all values of the continuous parameters α, β. This restriction on the form of the vacuum expectation value is particularly simple in the Fourier integral representation of Δ. For every Fourier component we have

$$0 = \Delta_A(k_1)\,\Delta_B(k_2)\left[e^{i(\alpha - \beta)(K - k_1 - k_2)} - 1\right]$$

$$\int dx\,dy\,e^{i k_1 x + i k_2 y}\int dX\,dY\;\psi_0^{+}\;K(X + \tfrac{x}{2},\,Y + \tfrac{y}{2})\;U(i)\;K(X - \tfrac{x}{2},\,Y - \tfrac{y}{2})\;\psi_0 .$$

This identity in two sets of continuous parameters could only be satisfied if the Fourier transform of the expectation value has a $\delta(K - k_1 - k_2)$ as a factor. There must however be two of them corresponding to α and β respectively. One of these is usually interpreted in terms of a time integral and the expression written as

$$\int dx\,dy\,e^{ik_1 x + ik_2 y}\int dX\,dY\;\psi_0^{+}\;K(X + \tfrac{x}{2},\,Y + \tfrac{y}{2})\;U(i)\;K(X - \tfrac{x}{2},\,Y - \tfrac{y}{2})\,\psi_0$$

$$= T\;\delta(K - k_1 - k_2)\;\frac{(2\pi)^4}{\hbar^2 c}\;\Big|\;M_K(k_1\;\;k_2)\Big|^2 ,$$

where T is an infinite time factor and $\left|M_K\right|^2$ is the usual matrix element on the energy shell here already averaged over the initial states. Fourier analyzing 36 and expressing it as a transition rate $w(k_1\;\;k_2)$ (to absorb the infinite time factor) we obtain

$$w(k_1\;\;k_2) = \frac{(2\pi)^4}{\hbar^2 c}\;\prod_s{}^{*}\;\Delta_s^{+}(k_s)\;\Big|M_K(k_1\;\;k_2)\Big|^2 . \qquad 39.$$

The reasons for rederiving this much derived formula were several. The usual versions do not contain the factors Δ which appeared in the classical phase space expression, because the final momentum states are described in terms of three--rather than four--vectors. We also wanted to suggest a quantum interpretation for the integrals $\int d\sigma^{\mu}\chi_{\mu}$ of the classical model. These obviously correspond to the invariant scalar products (f_p, f_p'). The usual normalization factor $(V \frac{k_o}{\chi})^{\frac{1}{2}}$ appearing in 34 is seen to be of that nature. Finally a wave mechanical interpretation of the Fermi model is intended to be suggested.

Let $f_1 \ldots f_n$ be the wave function for the particles in the final state. With every f associate an $F(f)$, a wave packet built about f and "diffusing" into it. Fermi's approximation is then schematically

$$(f_1, f_2, \ldots f_n \mid M \mid i)(i \mid M \mid f_1 \ldots f_n) \rightarrow (F_1, F_1)(F_2, F_2) \ldots (F_n, F_n).$$

What would seem to be involved here is an attempt to analyze the effect of the interaction in terms of diffusion characteristics of individual wave packet, one for each emerging particle. To the lowest approximation only their length (F, F) seems to be of moment. The high energy collisions are then interpreted on the basis of such an individual packet model.

We shall now try to arrive at the quantum analogue of $W^{\mu_1 \ldots \mu_n}(k_1 \ldots k_n;\ x_1 \ldots x_n)$. In deducing the nonrelativistic expression $W(\vec{k}, \vec{r})$, Wigner demanded that $W(\vec{k})$ and $W(\vec{r})$ obtained from it be the probability densities in momentum and configuration space respectively. These two requirements turn out to be incompatible with the positive definiteness of $W(\vec{k}, \vec{r})$ and even overlooking this unpleasant fact do not determine $W(\vec{k}, \vec{r})$ uniquely. In our case the requirement that $W(k_1 \ldots k_n)$ be the transition probability into state (k) will be satisfied trivially. The quantity $W^{\mu_1 \mu_2 \cdots \mu_n}(x_1\, x_2 \ldots x_n)$ does not seem to have any clear-cut meaning in a relativistic field theory. We shall therefore be missing the condition that would correspond to the one of Wigner's in configuration space. We may, however, demand that $\dfrac{\partial}{\partial x^{\mu_s}}\, W^{\mu_1 \ldots \mu_s \ldots \mu_n} = 0$.

It will now be convenient to change to the mixed OUT-IN representation of the S matrix. In it the S operator is the identity; $S(f \mid i) = \psi_{OUT}^{+}(f)\psi_{IN}(i)$. We have labelled the outgoing bra and the incoming ket with appropriate subscripts. In our previous work we employed the pure IN-IN representation, since the S operator may be defined by the relation $\psi_{OUT} = S^{-1}\, \psi_{IN}$. The expression for the transition probability now has the form

$$W(f) = \psi_{OUT}^{+}(f)\ U_{IN}(i)\ \psi_{OUT}(f) = Sp\ U_{IN}(i)\ U_{OUT}(f) \qquad 40.$$

with U_{IN} and U_{OUT} related by $S^{-1}\ U_{IN}\ S = U_{OUT}$ reflecting the well-known reversal of equivalence characteristic of U. The final state may be constructed from the vacuum state: $\psi_{OUT}(a, b) = A_{OUT}^{+}(a)\ B_{OUT}^{+}(b)\psi_0$ where $A_{OUT}(a) = (f_a, A_{OUT})$, $B_{OUT}(b) = (f_b, B_{OUT})$. The functions f_a and f_b form an orthonormal set of free solutions of the G.K. equation. Equation 40 now takes the form

$$W(a, b) = \psi_0^{+}\ A_{OUT}(a)\ B_{OUT}(b)\ U_{IN}(i)\ A_{OUT}^{+}(a)\ B_{OUT}^{+}(b)\ \psi_0 \ . \qquad 41.$$

The four scalar products between wave functions and operators implicit in 41 do not depend on the space-like surface on which they are defined. This is a consequence of the fact that the outgoing fields and the orthonormal sets satisfy the free particle G.K. equations. A simple (although not unique) definition of a divergenceless probability tensor is then

$$W^{\mu\nu}(a\ \ b;\ x\ \ y) = \psi_0^{+}\ A_{OUT}(a)\ B_{OUT}(b)\ \ U_{IN}(i)\ \chi^{\mu\nu}(a, b;\ x\ y)\ \psi_0 \qquad 42.$$

where

$$\chi^{\mu\nu}(a,\ b;\ x\ y) = -\frac{1}{\chi_a \chi_b}\ f_a(x)\ f_b(y)\ \frac{\overleftrightarrow{\partial}}{\partial x_\mu}\ \frac{\overleftrightarrow{\partial}}{\partial y_\nu}\ A_{OUT}(x)\ B_{OUT}(y) \quad . \tag{42'}$$

The configurational transition current is most easily constructed if the remaining two surface integrals implicit in 42 are converted to volume integrations. Assuming that the particles present in the final state are absent in the same state initially we have

$$\begin{aligned} A_{OUT}(a)\ U_{IN}(i) &= \left[A_{OUT}(a) - A_{IN}(a)\right]\ U_{IN}(i) \\ &= \frac{1}{2i\chi_a}\left[\lim_{\sigma\to+\infty} - \lim_{\sigma\to-\infty}\right]\int_\sigma d\sigma^\mu\ \bar{f}_a\ \overleftrightarrow{\partial_\mu}\ A\ U_{IN}(i) \\ &= \frac{1}{2i\ \chi_a}\int d^4\xi\ \ \bar{f}_a(\Box^2 - \chi_a^{\ 2})\ A\ U_{IN}(i) \quad . \end{aligned}$$

In going from the first to the second member we made use of the fact that $A_{IN}(a)$ is a destruction operator for the initial state; from the second to the third, the quantum analogue of the classical Sommerfeld radiation condition, $\lim_{x_0 \to \pm\infty} A(x) = A_{\substack{OUT \\ IN}}(x)$, was taken for granted. The essential point in the next step is to write

$$B_{OUT}(b)\ A_{OUT}(a)\ U_{IN}(i) = \frac{1}{2i\ \chi_a}\int d^4\xi\ f_a(\xi)(\Box_\xi^{\ 2} - \chi_a^{\ 2})$$

$$\left[B_{OUT}(b)\ A(\xi) - A(\xi)\ B_{IN}(b)\right]\ U_{IN}(i)$$

$$= \frac{\lim\limits_{\sigma\to+\infty} - \lim\limits_{\sigma\to-\infty}}{1/\left[(2i\ \chi_a)(2i\ \chi_b)\right]}\int d^4\xi \int_\sigma d\sigma^\mu(\eta) f_a(\xi) f_b(\eta)(\Box_\xi^{\ 2} - \chi_a^{\ 2})\frac{\overleftrightarrow{\partial}}{\partial\eta_\mu}$$

$$T(A(\xi)\ B(\eta))\ U_{IN}\quad ,$$

where T is the symbol of chronological ordering. The last expression is readily converted into a volume integral and with the aid of the suggestive abbreviation

$$J(\xi\ \eta) = \frac{1}{(2i\ \chi_a)}\ \frac{1}{(2i\ \chi_b)}\ (\Box_\xi^{\ 2} - \chi_a^{\ 2})(\Box_\xi^{\ 2} - \chi_b^{\ 2})\ T(A(\xi)\ B(\eta)) \qquad 43.$$

expression 42 becomes

$$W^{\mu\nu}(a, b; x, y) = \int d^4\xi\ d^4\eta\ \psi_0^{\ +}\ J(\xi\ \eta) U_{IN}(i) \chi^{\mu\nu}(a, b; x\ y)\ \bar{f}_a(\xi)\ \bar{f}_b(\eta).$$

Summing over the labels a and b we finally obtain for the configurational transition current

$$W^{\mu\nu}(x\ y) = \int d^4\xi\ d^4\eta\ \psi_0^{\ +}\ J(\xi\ \eta)\ U_{IN}(i) \chi^{\mu\nu}(\xi, \eta; x\ y)\ \psi_0 \qquad 44a.$$

where

$$\chi^{\mu\nu}(\xi\ \eta\ ;\ x\ y) = \frac{1}{(2i\ \chi_a)(2i\ \chi_b)}\ A_{OUT}(x)\ B_{OUT}(y)\ \frac{\overleftrightarrow{\partial}}{\partial x_\mu}\ \frac{\overleftrightarrow{\partial}}{\partial y_\nu}\ \Delta_A(x,\xi)\,\Delta_B(y,\eta). \qquad \text{44b.}$$

This tensor is seen to be divergenceless because of the free particle character of the outgoing operators and the Δ functions. Fourier analyzing the Δ functions in 44 we have in an obvious notation

$$W^{\mu\nu}(\vec{k},\vec{\ell}\ ;\ x\ y) = \psi_0^+\ J(k\ \ell)\ U_{IN}\ \chi^{\mu\nu}(k\ \ell\ ;\ x\ y)\ \psi_0\ \Delta(k)\ \Delta(\ell)\ . \qquad \text{45.}$$

The constraints inherent in the vacuum expectation value as a consequence of invariance under displacements are readily deduced by the method outlined previously and the Fermi approximation seen to amount to the replacement

$$\psi_0^+\ J(k\ \ell)\ U_{IN}\ \chi^{\mu\nu}(k\ \ell\ x\ y)\ \psi_0 \longrightarrow \delta(K - k - \ell)\ \chi_\mu(x\ \ k)\ \chi_\nu(y\ \ \ell).$$

A much more detailed investigation is clearly needed before one could decide whether a consistent approximation could be based on this procedure.

Having seen how to obtain the Fermi approximation from a field theoretical expression, we can now exploit the apparatus of field theory in order to include into the Fermi model typical quantum effects. The expression for W (x y), as given in 44, appears to be suitable for this purpose. The modifications that are called for in the case when two mesons are identical are best exhibited in terms of the "vacuum representations" of the Δ functions [4]. Thus

$$\left(\frac{1}{2\chi_a}\right)\Delta(x\ \xi)\left(\frac{1}{2\chi_a}\right)\Delta(y\ \eta) =$$

$$= \frac{1}{\hbar^2\ c^2}\ \psi_0^+\ A_+(x)\ A_+^+(\xi)\psi_0\ \psi_0^+\ B_+(y)\ B_+^+(\eta)\psi_0$$

$$= \frac{1}{\hbar^2\ c^2}\ \psi_0^+\ A_+(x)\ \ B_+(y)\ A_+^+(\xi)\ B_+^+(\eta)\ \psi_0 \quad ,$$

where $A_+(x)$ is the "destruction" part of field A and $A_+^+(x)$ its Hermitian adjoint and similarly for the B field. It is not too hard to see that for identical fields this expression should be replaced by

$$\psi_0^+\ A_+(x)\ B_+(y)\ A_+^+(\xi)\ B_+^+(\eta)\psi_0 \longrightarrow \tfrac{1}{2}\psi_0^+\ A_+(x)\ A_+(y)\ A_+^+(\xi)\ A_+^+(\eta)\psi_0 \ .$$

The Wick contraction rules 4 may now be used to decompose this into a sum of products of Δ functions and the Fermi approximation carried out. It is also easy to see how spin effects may be included. The scalar A and B fields should be replaced in this representation with appropriate spinor field operators and the definition of the scalar product (f, f) modified in an obvious manner. To include final state interaction we could replace the free particle f_m's in the bilinear representation 30 by wave functions depending on a few parameters fixed by experiment.

III. THERMODYNAMIC LIMIT

A. Introduction

Statistical models are based on the close structural resemblence between the marginal classical distribution over momentum variables and the expression for the quantum mechanical transition probability. Both are of the form $\prod_s' \Delta_s(k_s)\,\delta(K - \sum_s k^{(s)})\,\mathcal{M}(k_1 \ldots k_s)$, where $\mathcal{M}$ is the square of the matrix element in the quantum case. Classically, $\mathcal{M}$ results from integration over configurational variables and will not in general be factorable. However for essentially uncorrelated systems we may write $\mathcal{M} = \prod^{*} \Omega_s(k_s)$ where the Ω_s have the character of volumes. The classical distribution then becomes $\prod_s'^{*} \Delta_s(k_s)\Omega_s(k_s)$. If the number of factors in the product is large, it may be treated by the methods of statistics of independent random variables or "statistics" for short. In this asymptotic limit there is a marked lack of sensitivity in the detailed nature of the Ω_s.

What one usually understands by a model is an attempt to infer the form of $\mathcal{M} = |M|^2$ without resort to detailed dynamical theory. An equilibrium model would argue that $|M|^2$ is independent of initial conditions beyond restrictions imposed by conservation laws, a statistical one would favor the view that $|M|^2 = \prod_s^* \Omega_s$. The two viewpoints are frequently combined. The crudest version of an equilibrium statistical model is obtained by setting the matrix element equal to one.

In this chapter, one shall be concerned with the techniques employed in handling integrals over products of the type $\prod_s^* \Delta_s(k_s)\Omega_s$, that is, with the calculational machinery of statistical mechanics. Special emphasis will be placed on the limit in which the number N of factors becomes large—the thermodynamic limit. Because of the observed copious production of particles at high energies, it is likely to be relevant. In this limit new qualitative features involving concepts of temperature and entropy come into play. It can be specified somewhat more closely. Besides $W\left[k_1 \ldots k_n\right]$ we need marginal distributions $W\left[k_1 \ldots k_s\right]$ derived from it. The set of variables $\left[k_1 \ldots k_s\right]$ may be thought of as referring to the system of interest, the other, integrated out set $\left[k_{s+1} \ldots k_n\right]$, to the "bath." For systems of size comparable to their baths, one is lead to Gaussian distribution if both system and both are large; for small system in large baths, to Gibbsian distributions. It is the latter limit which will interest us here and the density ρ_K, all bath--no system, is the one on which we shall expand most of our computational efforts. In particular we discuss those features which are important in the high energy limit and are not treated adequately in standard textbooks. The cases dealt with are intended primarily to illustrate procedures and have been chosen from the point of view of simplicity. It is hoped that the reader will see how to apply these techniques singly or in necessary combinations to the more complex situations encountered in practice.

B. Approximation to Densities[4]

In this section we should like to outline the techniques that may be employed to obtain approximate expressions for

$$\Omega_s = \int \frac{K_0^{(s)}}{K_\pi^{(s)}} \chi_s(x^{(s)})\, d^3x^{(s)}$$

$$\rho_K = \int (d^4k)^n\, \delta(K - \sum_{s=1}^{n} k^{(s)}) \prod_{s=1}^{n}{}^{*} \Delta[k_s, \chi_s]\, \Omega_s[k_s]$$

$$\Omega_s = \frac{K_0^{(s)}}{K_\pi^{(s)}} \Omega_{usual} = \frac{1}{K_\pi^3} \quad 46.$$

in the limit of large n and to record some of the results so obtained. The simplest reasonable assumption that one can make about Ω_s is $\Omega_s = \chi_s^{-3}$, or that the configurational volume available to each

$\frac{4\pi}{3} K_s^{-3}$?

particle in its own rest frame is of the order of the cube of its Compton wave length. Introducing new dimensionless variables $k_s = \chi_s k'_s$ (and dropping primes), we have

$$\rho_K = \int (d^4k)^n \, \delta(K - \sum \chi^{(s)} k_s) \, \prod^{*} \Delta(k_s, 1)$$

$$= \int \prod_{s=1}^{n} \left[\frac{d^3k^{(s)}}{(2\pi)^3 \sqrt{k_s^2 + 1}} \right] \delta(K_o - \sum_s \chi_s \sqrt{k_s^2 + 1}) \, \delta(\vec{K} - \sum \chi_s \vec{k}_s) \; . \qquad 47.$$

An alternative to 47 has been proposed by Fermi. Its form-invariant version is

$$\Omega_s = \frac{2\chi_N}{\sqrt{-K_\mu K_\mu}} \quad \frac{k_\mu^{(s)} K_\mu}{\chi^{(s)} \sqrt{-K_\mu K_\mu}} \quad \frac{1}{\chi_N^3} \qquad 48.$$

where χ_N is proportional to the mass of one of the initially colliding nucleons and K_μ is the total momentum four vector of the system. In a frame in which $K_\mu = (0, 0, 0, K_o)$, expression 48 reduces to

$$\Omega'_s = \frac{2\chi_N}{K_o} \quad \frac{k_o^{(s)}}{\chi_{(s)}} \quad \frac{1}{\chi_N^3} \; . \qquad 48'.$$

The first factor in 48' expresses the "volume contraction" idea of Fermi. Because of the second factor, the integration will assume an especially simple form

$$ {}^{F}\rho_K = \left(\frac{2\chi_N}{K_o}\right)^{n} {}_{F}\rho_K $$

47'.

$$ {}_{F}\rho_K = \int \prod_{s=1}^{n} \frac{d^3k^{(s)}}{(2\pi)^3} \, \delta\left(K_o - \sum \chi^{(s)} \sqrt{k^2+1}\right) \delta\left(- \sum \chi^{(s)} \vec{k}^{(s)}\right) $$

Frame

in a frame in which the spatial components of the total momentum vanish.

Unlike 47, expression 47' will not retain its form when transferred to a different frame.

In the nonrelativistic limit, 47 and 47' become identical and equal to

$$ {}^{NR}\rho_{EP} = \frac{h_o^4}{(mc)^{3N}} \; {}_{NR}\rho_{E,\vec{P}} $$

49.

$$ {}_{NR}\rho_{E\vec{P}} = \int \prod_s \frac{d^3p^{(s)}}{(2\pi)^3} \, \delta\left(E - \sum \frac{p_s^2}{2M_s}\right) \delta\left(\vec{P} - \sum \vec{P}_{(s)}\right) . $$

They differ radically in the ultra relativistic limit:

$$ {}^{UR}\rho_K = \int \prod_s \frac{d^3k^{(s)}}{(2\pi)^3} \, \frac{1}{|k^{(s)}|} \, \delta\left(K_o - \sum \chi^{(s)} |k^{(s)}|\right) \delta\left(\vec{K} - \sum \chi_s \vec{k}_s\right) $$

50.

$$ {}^{UR}_{F}\rho_K = \int \prod_s \frac{d^3k^{(s)}}{(2\pi)^3} \, \delta\left(K_o - \sum \chi^{(s)} |k^{(s)}|\right) \delta\left(\vec{K} - \sum \chi_s \vec{k}_s\right) . $$

50'.

In the N.R. limit, it is also of some interest to investigate the density function where energy alone is a good constant of motion

$$ {}_{NR}\rho_E = \int \prod \frac{d^3\vec{p}^{(s)}}{(2\pi)^3} \; \delta\left(E - \sum_s \frac{p_s^2}{2m_s}\right) \qquad 49a. $$

Fermi's original calculations were done in the ultra-relativistic domain but with energy alone conserved

$$ {}^{UR}_{F}\rho_{K_o} = \int \prod_s \frac{d^3k^{(s)}}{(2\pi)^3} \; \delta\left(K_o - \sum \chi^{(s)} \left|k^{(s)}\right|\right) \qquad 50'a. $$

We shall first treat the densities involving a single constant of motion ${}^{F}_{NR}\rho_E$ and ${}^{UR}_{F}\rho_{K_o}$. The discussion of ${}_{NR}\rho_E$ is detailed and the simple calculations are used as an opportunity to introduce thermodynamic concepts. The second density ${}^{UR}_{F}\rho_{K_o}$ is dealt with in bare outline. A set of densities involving four delta functions is considered next. The new problem encountered is that of temperatures conjugate to momenta. These are derived in detail for ${}_{NR}\rho_{E,\vec{P}}$ and ${}^{UR}\rho_K$; the result for ${}^{UR}_{F}\rho_K$, only, recorded. The densities mentioned so far are susceptible to both exact and thermodynamic treatment in the energy limits considered. It is hoped that the example presented will illustrate the principles involved in dealing with the more complicated cases which cannot be carried out exactly.

(a) Densities with energy conservation only.

We shall now give an outline of the thermodynamic approximation scheme, uncluttered by details of mathematical rigor. As typical of the two density functions to be considered, we take the somewhat more general expression

$$\rho_E = \int \frac{(d\xi)^N}{(2\pi)^N} \, \delta\left(E - H(\xi_1 \cdots \xi_n)\right) , \qquad 51.$$

where H is regarded to be positive but it need not be a sum of noninteracting Hamiltonians. With a Fourier representation of the delta function we have

$$\rho_E = \frac{1}{2\pi} \int dT \, e^{-iTE} \int \left(\frac{d\xi}{2\pi}\right)^N e^{iTH(\xi_1 \cdots \xi_N)} . \qquad 52.$$

Observe that a new function $\psi[T]$ (the Planck free energy) defined by

$$e^{\psi[-iT]} = \int \left(\frac{d\xi}{2\pi}\right)^N e^{iTH(\xi_1 \cdots \xi_N)} \qquad 53.$$

exists for $\mathrm{Re}\, T > 0$ in the complex T plain. With the aid of another function, the entropy S defined in the same domain

$$S_E[-iT] = -iTE + \psi[-iT] , \qquad 54.$$

we rewrite 52 in the form

$$\rho_E = \frac{1}{2\pi}\int dT\, e^{S_E[-iT]} . \qquad 55.$$

Introducing a Cartezian coordinate system in the complex T plane: $T = t + i\tau$,

we have

$$e^{\psi(\tau)} = \int \left(\frac{d\xi}{2\pi}\right)^N e^{-\tau H(\xi)} \qquad 53'.$$

$$S_E(\tau) = E\tau + \psi(\tau) \qquad 54'.$$

$$\rho_E = \frac{1}{2\pi}\int dt\, e^{S_E(\tau \mp it)} = \frac{1}{2\pi i}\int_C dT\, e^{S_E[T]} , \qquad 55'.$$

with the integration in 55' along a line parallel to the imaginary axis.

The quantities ψ and S are seen to be the real axis.

The basic observation to be made is that on the real axis the real function $S(\tau)$ has a unique minimum. The modulus of S along C perpendicular to it will then have a maximum. We shall therefore be able to collect most of the integrand along a small segment of the line about the real axis. The point on the real axis $\tilde{\tau}$ at which $S(\tau)$ has its minimum depends on the energy of the system and may be called its intrinsic temperature [it turns out to be $(1/kT)$]; the value of S at that point--the proper entropy of the system. Because of this minimal property, the sum of two entropies, one proper to system A, the other, to system B will be less than the entropy proper to the connected system AB. If one is convinced of the existence of a universal tendency towards mergers of small into large systems, one may assert that this quantity tends to increase.

The minimal property of $S(\gamma)$ is readily exhibited in terms of an auxilliary (at this stage) distribution, the canonical. In order to see this, let us differentiate 54' using the definition 53'. Thus

$$S'_E(\gamma) = E - \frac{\int (d\xi)^n H(\xi) e^{-\gamma H(\xi)}}{\int (d\xi)^n e^{-\gamma H(\xi)}} .$$

With the aid of the canonical distribution $f(\xi;\gamma) = e^{-\gamma H(\xi) - \psi(\gamma)}$, depending on an arbitrary parameter γ, this may be written as

$$S'_E(\gamma) = E - \bar{H}(\gamma) , \qquad 56.$$

where the bar denotes an average. Differentiating again we have

$$S''_E(\tau) = \overline{\left[H - \bar{H}^{\tau}\right]^2}^{\,\tau} > 0 \quad . \tag{57.}$$

It is then the canonical distribution labelled with the particular value of the parameter corresponding to the temperature out of a whole one parameter family of distributions which renders the expectation value of the total Hamiltonian equal to the energy of the system: $E - \bar{H}^{\bar{\tau}} = 0$.

For the purpose of expanding close to the real axis, we introduce a new variable $t = \frac{t'}{\sqrt{n}}$, where n is large, and write

$$\rho_E = \frac{1}{2\pi\sqrt{n}} \int_{-\infty}^{\infty} dt \; e^{S_E(\tau - it\, n^{-\frac{1}{2}})} \quad .$$

Expanding S, we have

$$S = S(\tau) - i\frac{t}{\sqrt{n}}\, S'(\tau) - \frac{t^2}{2n}\, S''(\tau) + \frac{it^3}{6n^{3/2}}\, R(\tau).$$

If the remainder $R(\tau)$ happens to be of order not higher than n, we may neglect the last term and carry out the integration. This yields

$$\rho_E(n) \cong \frac{e^{S(\tau)}}{\sqrt{2\pi\, S''(\tau)}}\; e^{-\frac{[S'(\tau)]^2}{2S''(\tau)}} \cong \frac{e^{S(\bar{\tau})}}{\sqrt{2\pi\, S''(\bar{\tau})}} \quad . \tag{58.}$$

The statistical interpretation of $S(\bar{\tau})$ follows from equation 3. Combining it with 58, we have the statement

$$\log W\left[E,\ \epsilon_1,\ \epsilon_2 \ldots \epsilon_n\right] = \sum_s \log \rho_{\epsilon_s} - \log \rho_E$$

$$= \sum S_i(\bar{\tau}_i,\ \epsilon_i) - S(\bar{\tau},\ E) + C \ .$$

The logarithm of the probability for the energy E to be partitioned into amounts $(\epsilon_1,\ \epsilon_2, \ldots \epsilon_n)$ among parts of the system is equal to the difference between the sum of the proper entropies of these parts and the proper entropy of the whole system. In making the statement, we have overlooked an inconsequential small additive term.

(γ) The ${}_{NR}\rho_E$ density.

Using the special form of the Hamiltonian of 49a in 53, we obtain $S(\tau) = E\tau + \frac{3}{2} \sum_s \log \frac{m_s}{2\pi\tau}$. The point $\bar{\tau}$ on the real axis is given by $\bar{\tau}\,(E/\frac{3}{2}\,n) = 1$. We should like to rescale the energy and write this as $\bar{\tau}\epsilon = 1$. The quantity ϵ is the conventional temperature of the system in energy units. Its relation to the total energy will depend on the model used and the energy range in which the approximation is made. In the present case we have the well-known relation $\epsilon = E/\frac{3}{2}n$. The relevant expressions now become

$$S(\bar{\tau}) = \frac{3}{2}n + \frac{3}{2}\sum_s \log\frac{m_s\epsilon}{2\pi} \; ; \qquad S''(\tau) = \frac{3}{2}n\,\epsilon^2$$

and we finally obtain

$$_{NR}\rho_E \cong \frac{1}{\epsilon}\prod_s\left(\frac{m_s\epsilon}{2\pi}\right)^{3/2}\frac{e^{\frac{3}{2}n}}{\sqrt{2\pi\cdot\frac{3}{2}n}} \; . \qquad 58.$$

The integration leading to the asymptotic expression 58 may also be carried out exactly. Substituting the expression for the entropy into 55', we have

$$_{NR}\rho_E = \frac{1}{2\pi}\int dt\; e^{\frac{3}{2}n\epsilon(\bar{\tau}+it)}\prod_s\left[\frac{m_s}{2\pi(\tau\pm it)}\right]^{3/2}$$

$$= \frac{1}{2\pi i}\int_C dz\,\frac{e^{\frac{3}{2}n\epsilon z}}{z^{\frac{3}{2}n}}\prod_s\left(\frac{m_s}{2\pi}\right)^{3/2}$$

$$= \frac{1}{\epsilon}\prod_s\left(\frac{m_s\epsilon}{2\pi}\right)^{3/2}\frac{\left(\frac{3}{2}n\right)^{\frac{3}{2}n-1}}{\left(\frac{3}{2}n-1\right)!} \qquad 58'.$$

In the last step we made use of the fact that $\frac{1}{2\pi i}\int_C d\xi\, \frac{e^{\xi}}{\xi^N} = \frac{1}{(N-1)!}$.

Formula 58' is obviously identical with 58 whenever the factorial may be replaced by its Stirling asymptotic form.

(β) The ${}^{UR}_{P}\rho_{K_o}$ density.

The notation $T = t + i\tau$ is now conveniently replaced by $K_o = \chi_o + i\xi_o$, and the $\bar{\tau}$ by $\bar{\xi}_o$. Expressed in these variables, $S[\xi_o] = K_o\,\xi_o - \sum_s \log\left[\pi^2\ \chi_s^3\ \xi_o^3\right]$. The relation between energy and temperature now how the form $(K_o/3N)\,\bar{\xi}_o = 1$. We denote $K_o/3N$ by m_o and write this as $m_o\,\bar{\xi}_o = 1$. The asymptotic approximation for the density is then

$$ {}^{UR}_{P}\rho_{K_o} \cong \frac{1}{m_o}\ \prod_s\left[\frac{m_o^3}{\pi^2\ \chi_s^3}\right]\ \frac{e^{3N}}{\sqrt{2\pi\cdot 3N}} , \qquad 59. $$

whereas the rigorous expression is given by

$$ {}^{UR}_{P}\rho_{K_o} \cong \frac{1}{m_o}\ \prod_s\left[\frac{m_o^3}{\pi^2\ \chi_s^3}\right]\frac{(3N)^{3N-1}}{(3N-1)!} . \qquad 59'. $$

(b) Densities with energy and momentum conserved.

Corresponding to $\bar{\tau}$ in (a), we shall now have a "vector temperature" with four components, one for each constant of motion.

(α) The density ${}_{NR}\rho_{E\,\vec{P}}$.

Fourier analyzing the delta functions in 49, we have

$$
\begin{aligned}
{}_{NR}\rho_{E\,\vec{P}} &= \frac{1}{(2\pi)^4} \int dT\, d^3\vec{X}\; e^{-iET + i\vec{P}\cdot\vec{X} + \psi[-iT,\, -i\vec{X}]} \\
&= \frac{1}{(2\pi)^4} \int dT\, d^3X\; e^{S[-iT,\, -i\vec{X}]} ,
\end{aligned}
\qquad 60.
$$

and regarding T and $\vec{X}$ as complex variables $T = t + i\tau$, $\vec{X} = \vec{\chi} + i\vec{\xi}$, we may write

$$
e^{\psi[\tau,\,\vec{\xi}]} = \int \prod_s \left[\frac{d^3p^{(s)}}{(2\pi)^3}\right] e^{-\tau H[\vec{p}_1, \ldots \vec{p}_n] + \vec{\xi}\cdot\sum \vec{p}_s} , \qquad 61a.
$$

$$
S(\tau, \xi) = \tau E - \vec{\xi}\cdot\vec{P} + \psi(\tau, \vec{\xi}) . \qquad 61b.
$$

For Hamiltonians of physical interest, S exists for $-\infty < \xi < +\infty$; $0 < \tau < \infty$. In the case of a system of independent particles, $\psi = \sum_s \psi_s$. The function

$$\phi_s(\tau\,\vec{\xi}\,) = e^{\psi_s(\tau\,\xi)} = \int \frac{d^3p}{(2\pi)^3}\; e^{-\tau H_s(p) + \vec{\xi}\cdot\vec{p}}$$

is readily seen to satisfy a kind of diffusion equation.

$$\left[\frac{\partial}{\partial\tau} + H_s\left(\frac{\partial}{\partial\xi}\right)\right]\;\phi_s(\tau,\vec{\xi}\,) = 0\;.$$

The extremal property of S has now to be proved in a four rather than one dimensional space.

Our previous 56 is replaced by the set

$$S_\tau(\tau,\,\xi_i) = E - \bar{H} \qquad\qquad S_i(\tau,\,\xi_i) = -P_i + \overline{P_i} \tag{62.}$$

and 57, by the even more ample

$$S_{\tau\tau} = \overline{(H - \bar{H})^2}\;,\quad S_{\tau,i} = -\overline{(P_i - \overline{P_i})(H - \bar{H})}\;,\quad S_{ij} = \overline{(P_i - \overline{P_i})(P_j - \overline{P_j})}\;. \tag{63.}$$

The subscripts τ and i denote differentiation with respect to τ and ξ_i, respectively; the bars, averages over the canonical distribution with four free parameters $\exp\left[-\tau H + \vec{\xi}\cdot\vec{p} - \psi(\tau,\vec{\xi}\,)\right]$. To simplify the notation, we have not specified the distribution by indicating the parameters next to the bars. With the aid of 62 and 63, we obtain

$$S\left[\tau+\Delta\tau,\ \xi_i+\Delta\xi_i\right]-S\left[\tau,\xi\right]$$

$$\cong \Delta\tau(E-\bar{H})-\Delta\xi_i(P_i-\bar{P}_i)+\tfrac{1}{2}\overline{\left[\Delta\tau(H-\bar{H})-\Delta\xi_i(P_i-\bar{P}_i)\right]^2} \quad .$$

Thus for the values of the parameters $(\bar{\tau},\ \bar{\xi}_i)$ for which the right members of 62 vanish, we have for a sufficiently small $(\Delta\tau,\ \Delta\xi_i)$ neighborhood $S\left[\tau+\Delta\tau,\ \xi_i+\Delta\xi_i\right]-S\left[\tau,\ \xi_i\right]>0$. In the special cases considered here, the reader will have no difficulty in convincing himself that the $\bar{\xi}_i,\ \bar{\tau}$ point is unique by observing that S in addition to being convex becomes unbounded whenever its argument approaches the boundary of the domain in which the function exists.

In view of the relativistic case to be treated in the next section, it is convenient to introduce a more symmetric four dimensional notation: $x_4=t$, $\xi_4=\tau$, $P_4=-H$. We shall also write for 62 $S_\mu^{(1)}=-(P_\mu-\bar{P}_\mu\xi)$ or $S^{(1)}=-(P-\bar{P}\xi)$. The set 63 then becomes

$$S_{\mu\nu}^{(2)}=\overline{(P_\mu-\bar{P}_\mu\xi)(P_\nu-\bar{P}_\nu\xi)} \quad = \overline{S_\mu^{(1)}\,S_\nu^{(1)}} \quad = (S^{(1)},\ S^{(1)})\ ,$$

and the quadratic form $\chi^\mu S_{\mu\nu}^{(2)}\chi^\nu$ may be written as $(x,\ S^{(2)}x)$.

In this notation the superscripts (1) and (2) indicate that the tensor is of first or second rank whenever this is not obvious. We also introduced a bracket symbol for the scalar product. The positive definite character of $S^{(2)}$ is now expressed by the statement $(x, S^{(2)}x) = (S^{(1)}x)^2$. In this compact notation

$$\rho_{E\vec{P}} = \frac{1}{(2\pi)^4}\int d^4x\; e^{S[\xi + ix]} \cong \frac{1}{(2\pi)^4}\int d^4x\; e^{S(\xi)+i(S^{(1)}x)-\frac{1}{2}(x,S^{(2)}x)}$$

$$= \frac{e^{S[\xi]}}{\sqrt{\det 2\pi\, S^{(2)}}}\; e^{-\frac{1}{2}\left(S^{(1)},\, \left[S^{(2)}\right]^{-1} S^{(1)}\right)} \quad . \qquad 64.$$

At the intrinsic temperature fixed by the requirement $S_\mu^{(1)}(\bar{\xi}) = 0$,

$$\rho_{E,P} = \frac{e^{S[\bar{\xi}]}}{\sqrt{\det 2\pi\, S^{(2)}\,(\bar{\xi})}} \quad . \qquad 64'.$$

For the special case of $H\left[p_1,\ldots p_n\right] = \sum_s \frac{p_s^2}{2m_s}$, the defining equations 61 yield

$$S(\tau, \vec{\xi}) = \tau E - \vec{\xi}\cdot\vec{p} + \frac{M}{2\tau}\,\vec{\xi}^{\,2} + \sum_s \frac{3}{2}\log\frac{m_s}{2\pi\tau} \qquad 65.$$

where $M = \sum_s m_s$. From it, one deduces for the temperatures $(\bar{\tau}, \bar{\xi}_i)$ the relations

$$\bar{\xi}_i = \frac{P_i}{M} \bar{\tau} \qquad \text{66a.}$$

$$\bar{\tau}\left[E - \frac{P^2}{2M}\right]/(\tfrac{3}{2} n) = 1 \ . \qquad \text{66b.}$$

It is thus the internal energy of a system of particles which replaces the energy of the previous section in the definition of temperature. We also see that the temperature conjugate to the momentum is the temperature conjugate to the energy multiplied by the velocity. In terms of a new unit of energy $\epsilon \bar{\tau} = 1$, the asymptotic expression for the ${}_{NR}\rho_{E\,P}$ may be stated as

$$_{NR}\rho_{E\,\vec{P}} \cong \frac{1}{\epsilon} \frac{1}{(M\epsilon)^{3/2}} \prod_s \left(\frac{m_s \epsilon}{2\pi}\right)^{3/2} \frac{e^{\frac{3}{2} n}}{\sqrt{(2\pi)^4 \cdot \frac{3}{2} n}} \ . \qquad \text{67.}$$

The exact expression is 67, modified by the replacement

$$\frac{e^{\frac{3}{2} n}}{\sqrt{2\pi \cdot \frac{3}{2} n}} \longrightarrow \frac{(\frac{3}{2} n)^{\frac{3}{2} n - \frac{5}{2}}}{\left(\frac{3}{2} n - \frac{5}{2}\right)!} \ .$$

Comparing 67 with 58, we conclude that in the nonrelativistic limit the

the conservation of momentum does not produce any drastic changes in the density functions. It is obvious on physical grounds that in 67 ϵ should refer to the internal rather than total energy; on dimensional grounds that an additional energy dependence of the nature $(M\,\epsilon)^{-3/2}$ would have to be introduced because of the three extra delta functions. We shall see later that the modifications are less obvious in the ultrarelativistic limit.

(β) The ${}_{F}^{UR}\rho_{K}$ density.

Subjecting 50' to the treatment of the previous section, we obtain

$$ {}_{e}\Psi[\xi_0, \vec{\xi}] = \frac{1}{\pi^{2n}} \frac{1}{\prod_s \chi_s^3} \frac{\xi_0^{\,n}}{(\xi_0^2 - \xi^2)^{2n}} \quad . \qquad 68. $$

Alternatively, we could start with ψ_s appropriate to ${}_{F}\rho_{K}$:

$$ {}_{e}\psi_s[\xi_0, \vec{\xi}] = \int \frac{d^3k}{(2\pi)^3} \, e^{-\chi_s\left[\xi_0\sqrt{k^2+1} \;-\; \vec{\xi}\cdot\vec{k}\right]} $$

$$ = -\frac{1}{\chi_s} \frac{\partial}{\partial y_0} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{k^2+1}} \, e^{-\chi_s\left[\xi_0\sqrt{k^2+1} \;-\; \vec{\xi}\cdot\vec{k}\right]} $$

Equation 68' continued.

$$= -\frac{1}{\chi_s}\frac{\partial}{\partial y_0}\frac{1}{2\pi^2}\frac{K_1\left[\chi_s\sqrt{\xi_0^2-\vec{\xi}^2}\right]}{\chi_s\sqrt{\xi_0^2-\xi^2}} = \frac{1}{2\pi^2}\chi_s\xi_0\frac{K_2\left[\chi_s\sqrt{\xi_0^2-\vec{\xi}^2}\right]}{\left[\chi_s\sqrt{\xi_0^2-\xi^2}\right]^2}$$

where K_1 denotes a Hankel function of imaginary argument. The function ψ_s is seen to exist for $\xi_0 - |\vec{\xi}| > 0$; $\xi_0 + |\vec{\xi}| > 0$. Expanding this representation about $\chi = 0$, we also obtain 68. The noncovariant nature of these expressions is quite evident. The exact result of the integration is

$$^{UR}_{F}\rho_K = \frac{1}{(4\pi)^{3/2}}\frac{1}{m_0^4}\prod_s^n\left(\frac{m_0^3}{\pi^2\chi_s^3}\right)\frac{(2n-\frac{5}{2})!\;(3n)^{3n-4}}{(2n-1)!\;(3n-4)!} \qquad \text{69a.}$$

where $m_0 = K_0/3n$. For large n this becomes

$$^{UR}_{F}\rho_K \simeq \frac{1}{(8\pi n)^{3/2}}\frac{1}{m_0^4}\prod_s\left(\frac{m_0^3}{\pi^2\chi_s^3}\right)\frac{e^{3n}}{\sqrt{2\pi\cdot 3n}} \qquad \text{69b.}$$

Compared with 59, which was calculated on the basis of energy conservation alone, we notice the factor $(8\pi n)^{-3/2}$ in 69b. The additional three conservation laws thus markedly restrict the phase space available for high multiplicity processes at relativistic energies.

(γ) The density ${}^{UR}\rho_K$.

We shall now evaluate the covariant expression 50. Rigorously

$S[\xi] = -K_\mu \xi^\mu + \sum \psi_s(\xi)$ with

$$\phi_s = e^{\psi_s(\xi)} = \int \frac{d^3k}{(2\pi)^3} \cdot \frac{1}{\sqrt{k^2+1}} \; e^{-\chi_s \xi_0 \sqrt{k^2+1} + \chi_s \vec{\xi} \cdot \vec{k}} \qquad 70.$$

in the domain $\xi_0 - |\xi| > 0$, $\xi_0 + |\xi| > 0$. It is easy to see that ϕ_s obeys the Gordon-Klein equation with "imaginary mass"

$$(\Box^2 + \chi_s^2)\,\phi_s = 0$$

and may be identified with the solution

$$e^{\psi_s[\xi]} = \frac{1}{2\pi^2} \frac{K_1\left[\chi_s\sqrt{\xi_0^2 - \xi_1^2}\right]}{\chi_s \sqrt{\xi_0^2 - \xi_1^2}} \qquad 71.$$

of this equation. In the ultrarelativistic limit (71) simplifies to

$$e^{\psi_s(\xi)} \sim \frac{1}{2\pi^2} \frac{1}{\chi_s^2(\xi_0^2 - \xi_1^2)} = -\frac{1}{2\pi^2} \frac{1}{(\xi \cdot \xi)} \qquad 71'.$$

The relations

$$S = -K_\mu \xi_\mu + \sum_s \log\left[-\frac{1}{2\pi^2} \frac{1}{K_s^2} \frac{1}{(\xi \cdot \xi)}\right]$$

$$S_\mu = -K_\mu - 2n\,\frac{\xi_\mu}{(\xi\cdot\xi)}$$

$$S_{\mu\nu} = \frac{4n}{(\xi\cdot\xi)^2}\left[\xi_\mu\,\xi_\nu - \tfrac{1}{2}\,\delta_{\mu\nu}\,(\xi\cdot\xi)\right]$$

$$\det S = -\left[2N/(\xi\cdot\xi)\right]^4$$

are readily verified. The defining relation for the temperatures may be expressed in terms of $m_\mu = K_\mu/2n$, $m_\mu m^\mu + \mu^2 = 0$ as

$$\bar{\xi}_\mu = \frac{m_\mu}{\mu^2} \quad . \qquad 72.$$

Comparing the present definition of m_o with the one of the last section, we notice that the equipartition law for energy is quite different for the the two types of statistics in the ultrarelativistic domain. The final asymptotic expression for the covariant density turns out to be

$$^{UR}\rho_K \cong \frac{1}{\mu^4}\prod_s\left[\frac{1}{2\pi^2}\;\frac{\mu^2}{\chi_s^{\,2}}\right]\;\frac{e^{2n}}{\sqrt{2\pi\cdot 2n}} \quad . \qquad 73.$$

C. Statistical Correlations

In the previous chapter, we indicated a method based on second quantisation which could be used to take into account statistical correlations having their source in the indistinguishability of elementary particles. In this section, we should like to deal with the problem in a manner closer in spirit to the thinking of Planck in connection with the quantum hypothesis. Because of the somewhat clumsy distinction between generic and specific phase space and the entropy paradox which it entails, we should like to picture the classical situation somewhat differently.

Let $W\left[\vec{x}^{(1)}, \vec{x}^{(2)}, \ldots \vec{x}^{(n)}\right]$ be the probability of finding n distinguishable particles at the following places: the first particle at a particular point of its μ space μ_1 whose coordinate vector is $\vec{\xi}_1 = \vec{x}^{(1)}$, the second in its μ space μ_2 at $\vec{\xi}_2 = \vec{x}^{(2)}$, the n'th at $\vec{\xi}_n = \vec{x}^{(n)}$. Each particle whose coordinate vector appears as an argument of W may be regarded as a single representative of a species of particles indistinguishable among themselves. The conventional Γ space (reduced to n dimensions) is now the space of the distinct species; the μ space of each species is populated by its indistinguishable members.

We now transcribe $W\left[\vec{x}^{(1)}\right]$, reduced to a single representative of a single species, into this new language. Dividing the μ space of the species into a denumberable set of neighborhoods, each centered about a point $\vec{\xi}_i$, we introduce a neighborhood function $n\left[\xi_i\right]$ with property

$$n\left[\vec{\xi}_i\right] = 0 \qquad \vec{\xi}_i \neq \vec{x}^{(1)}$$

$$n\left[\xi_i\right] = 1 \qquad \xi_i = \vec{x}^{(1)} \tag{76a.}$$

$$\sum_i n\left[\xi_i\right] = 1 \,. \tag{76b.}$$

The probability $W\left[\vec{x}\right]$ may now be replaced by $W\left[n\left[\xi_1\right], n\left[\xi_2\right] \ldots n\left[\xi_s\right] \ldots\right]$, normalized according to

$$\sum_{n[\xi_1]} \quad \sum_{n[\xi_2]} \cdots \sum_{n[\xi_s]} \cdots W\left[n\left[\xi_1\right], n\left[\xi_2\right] \ldots n\left[\xi_s\right] \ldots\right] = 1. \tag{77.}$$

This awkward division into neighborhoods may be dispensed with if we introduce a function $\nu(\xi)$ with the property that its integral over the $\vec{\xi}_i$ neighborhood $c(\xi_i)$ is equal to $n[\xi_i]$. Relations 76 expressed in terms of $\nu(\xi)$ read

$$\lim_{c(\xi_i)\to 0} \int_{c(\xi_i)} d\xi \; \nu(\xi) \;=\; 0 \qquad \vec{\xi}_i \neq \vec{x}^{(1)}$$

$$= \; 1 \qquad \xi_i = \vec{x}^{(1)} \qquad 76a'$$

$$\int d\xi \;\; \nu(\xi) = 1 \; . \qquad 76b'$$

It is clear that 76' will be satisfied if we set

$$\nu(\xi) = \delta(\vec{\xi} - \vec{x}^{(1)}) \; . \qquad 78.$$

The probability distribution function $W[n(\xi_1)\ldots n(\xi_s)\ldots]$ on sets of occupation numbers $\{n(\xi_s)\}$ is now replaced by a functional $W\{\nu(\xi)\}$ on "complexions" $\nu(\xi)$. The sums appearing in the normalization condition 77 will be replaced by a "sum over complexions" (in obvious analogy to Feynman's "integral over paths") and the expression transcribed as

$$\mathcal{E}_{\nu(\xi)} W\{\nu(\xi)\} = 1. \qquad 77'.$$

In actual computations, 77' will be regarded as the limit of 77 when the set $\{\xi_i\}$ becomes very dense.

It is the time dependent quantity, $\nu(\xi, t) = \delta(\xi - x(t))$, which replaces the coordinates and momenta as a new dynamical variable. When dealing with single representatives of each species, its use is optional; otherwise, if not compulsory, at least sometimes desirable. The dynamical equations of $\nu(\xi, t)$ are those of a hydrodynamic density: its time derivative may be expressed as the divergence of a current, the time derivative of the current involves the divergence of a stress tensor, and so forth. Equation 76a', regarded as a restriction on the admissable singularities of a hydrodynamic density, in effect quantizes it. Only a single point contributes to the integral 76b'.

To include more than one member of the species, we retain the functional form of $W\left\{\nu(\xi)\right\}$ but augment its range by modifying 76. Thus for n particles of the species, we should have

$$\int d\xi \quad \nu(\xi) = n \quad . \tag{76b''}$$

To retain the quantization, we permit 76a" to produce only integral contributions from singularities. A crucial point, however, turns out to be whether we retain 76a' in the form

$$\operatorname*{Max\ lim}_{c(\xi_i)\to 0} \int_{c(\xi_i)} d\xi \; \nu(\xi) = 1$$

or replace it with a more general

$$\operatorname*{Max\ lim}_{c(\xi_i)\to 0} \int_{c(\xi_i)} d\xi \; \nu(\xi) = N \;. \tag{76a"}$$

We shall refer to N as the statistical capacity of the species. It appears to assume only two values, $N = 1$ and $N = \infty$. We retain, however, the letter N in order to deal with both cases together. Condition 76b" is not especially pertinent to the high energy domain. A more detailed version of 76a" would be

$$\int_{c(\xi_i)} d\xi \; \nu(\xi) = \nu_i \; ; \qquad \text{Max } \nu_i = N \; ,$$

with the corresponding generalization of the representation 78 to

$$\nu(\xi) = \sum_s \delta(\vec{\xi} - \vec{x}_{(s)})\, \nu_s \; . \tag{78'}$$

For the discussion to follow, it will be convenient not to regard ξ as necessarily the momentum or coordinate variable of a single particles but to admit more general parametrizations of the μ space. We shall thus write $\vec{k}(\xi)$ where, of course, $\vec{k}(k) = \vec{k}$. The parametrizations called for in quantum theory may be so degenerate as to refer to a discrete set of points only. With this added flexibility we can treat the classical and quantum case in a uniform manner.

As an elementary example, let us consider the microcanonical distribution

$$W\{\nu\} = \frac{\delta(E - \int d\xi \; \nu(\xi)\, H(\xi))}{\mathcal{E}_{\nu(\xi)}\; \delta(E - \int d\xi \; \nu(\xi)\, H(\xi))} = \frac{\rho_E\{\nu\}}{\rho_E} \tag{79}$$

for a system of noninteracting identical particles of statistical capacity N having only one constant of motion E. To evaluate ρ_E, we represent it, as before, in the form

$$\rho_E = \frac{1}{2\pi} \int dT\, e^{S[-iT]}$$

where now, however,

$$e^{\psi(\tau)} = \mathcal{E}_{\nu(\xi)}\, e^{-\tau \int H(\xi)\, \nu(\xi) d\xi}$$

$$= \lim \sum_{n(\xi_1)=0}^{N} \; \sum_{n(\xi_2)=0}^{N} \cdots e^{-\tau n(\xi_1) H(\xi_1) - \tau n(\xi_2) H(\xi_2) - \cdots}$$

$$= \prod_s \frac{1 - e^{-(N+1)\tau H[\xi_s]}}{1 - e^{-\tau H[\xi_s]}} = \exp\left[\sum_s \log \frac{1 - e^{-(N+1)\tau H[\xi_s]}}{1 - e^{-\tau H(\xi_s)}}\right.$$

Introducing a new density $\rho(\xi) = \sum_s \delta(\xi - \xi_s)$, we rewrite the last equation as

$$\psi(\tau) = \int d\xi\, \rho(\xi) \log \frac{1 - e^{-(N+1)\tau H(\xi)}}{1 - e^{-\tau(H(\xi))}} \quad . \tag{80.}$$

The form 80 survives the transition from classical to quantum theory. Concepts from the latter are needed only for the parametrization of the μ space and the evaluation of ρ_F and are basically a problem in exact rather than statistical quantum dynamics. Thus if all but a denumerable set $(\epsilon_1, \epsilon_2, \ldots)$ of values of H is excluded by a wave mechanical boundary condition, the μ space becomes discrete. Taking the value of ρ at these points as g_s, the degeneracy of the levels, we have the familiar expression

$$\psi(\tau) = \sum_s g_s \frac{1 - e^{-(N+1)\tau \epsilon_s}}{1 - e^{-\tau \epsilon_s}} \quad . \tag{80'.}$$

For a quasi-continuum of states of free scalar particles, we use

$\xi = k, \quad \rho(\xi) = \dfrac{V}{(2\pi)^3}$ and 80 becomes

$$\psi(\tau) = V \int \frac{d^3k}{(2\pi)^3} \log \frac{1 - e^{-(N+1)\tau H(k)}}{1 - e^{-\tau H(k)}} \quad . \tag{80".}$$

The thermodynamic formalism of the previous section was a simplification resulting from the fact that $\psi(\tau)$ consisted of a sum of a large number n of bounded terms and was in essence an expansion in $(n)^{-\frac{1}{2}}$ as a parameter of smallness. In forms typified by 80" it is the magnitude of $\left(\int d\xi \; \rho(\xi)\right)^{-\frac{1}{2}}$ that is used as a parameter. With this modification, the classical machinery can be taken over and the intrinsic temperature defined as that value of τ for which

$$E = \int d\xi \; \rho(\xi) \, H(\xi) \left[\frac{1}{e^{\tau H(\xi)} - 1} - \frac{N+1}{e^{\tau(N+1)H(\xi)} - 1} \right] . \qquad 81.$$

It is natural to interpret 81 as $E = \int d\xi \; E(\xi)$ where $E(\xi)$ is an energy density, the energy per unit volume of μ space. It then follows that $\rho(\xi)$ is the number of states per unit volume at the point ξ, $H(\xi)$ the value of the energy of a particle when located at that point, and

$$\overline{n(\xi)} = \frac{1}{e^{\tau H(\xi)} - 1} - \frac{N+1}{e^{\tau(N+1)H(\xi)} - 1} \qquad 82.$$

the <u>expected</u> number of particles in a state at the point ξ. Equation 82 contains in it the usual "F.D. and E.B. formulas." What is of greater interest in connection with high energy models is not the expectation value $\overline{n(\xi)}$ but the underlying probability $W\left[n(\xi)\right]$ over which $n(\xi)$ is summed in order to arrive at $\overline{n(\xi)}$

$$\overline{n(\xi)} = \sum n(\xi)\, W\left[n(\xi)\right] \quad .. \qquad 83.$$

An elementary and somewhat loose derivation of 82 may be of help. Let us consider a μ space for a species with statistical capacity N, focussing our attention on a particular point ξ. It is in the spirit of Maxwell-Boltzmann statistics to say that the probability of finding a system at the point ξ, if its energy there is $H(\xi)$, is given by $W_\xi\left[1\right] = A\, e^{-\gamma H(\xi)}$, where A is a normalization constant; if its energy happens to be $2H(\xi)$, it is plausible that $W_\xi\left[2\right] = A\, e^{-\gamma 2H(\xi)}$. Remembering that the capacity of the species is N, we may say $W_\xi\left[s\right] = A\, e^{-\gamma_s H(\xi)}$ if $s \leq N$ and $W_\xi\left[s\right] = 0$ if $s > N$. Thus the correlation introduced by the exclusion principle affects only the normalization cønstant A. We find from $\sum_{s=0}^{N} W_\xi\left[s\right] = 1$ that

$$W_\xi[s] = W_\xi(0)\, e^{-s\tau H(\xi)} \qquad s \leq N$$

$$= 0 \qquad s > 0 \;;$$

$$W_\xi(0) = \frac{1 - e^{-\tau H(\xi)}}{1 - e^{-(N+1) H(\xi)\tau}} \qquad . \qquad 84.$$

This then is the probability distribution which underlies 82 as may be readily verified with the aid of 83.

To make 84 more convincing and to illustrate a point of technique, we shall derive it directly from 79. Let us consider a domain E in μ space with the characteristic function $\chi_D(\xi)$. The probability $W[n;\, D]$ of finding n particles in D is clearly

$$W[n;\, D] = \mathcal{E}_{\nu(\xi)}\, \delta(n - \int d\xi\ \nu(\xi)\, \chi_D(\xi))\, W\{\nu(\xi)\} = \rho_E^{-1}\, \rho_E(n;\, D) \qquad 85a.$$

$$_E\rho(n,\, D) = \mathcal{E}_{\nu(\xi)}\, \delta(n - \int d\xi\ \nu(\xi)\, \chi_D(\xi))\, \delta(E - \int d\xi\, \nu(\xi)\, H(\xi))\ . \qquad 85b.$$

The first δ symbol in the right member of 85b is a Kronecker rather than a Dirac delta. The Fourier parameter associated with its expansion will be denoted by Λ. We thus have

$$\rho_E(n, D) = \int \frac{dT}{2\pi} e^{S[-iT] - G[-iT;\, n,\, D]} . \qquad 86.$$

The entropy function S in 86 is the same as that encountered in connection with the representation of ρ_E. It is thus G which renders the Fourier image of $\rho_E^{(N,D)}$ different from that of ρ_E. We shall shortly relate it to the generating function for the production of various number of mesons. In example 86 $G(\tau;\, n, D)$ is explicitly given by

$$e^{-G[\tau;\, n,\, D]} = \int_0^{2\pi} \frac{d\Lambda}{2\pi} e^{-in\Lambda} e^{-G(\tau;\Lambda,\, D)} , \qquad 87.$$

where $G(\tau;\Lambda, D)$ after the evaluation of the sum over complexions may be written as

$$G(\tau;\Lambda, D) = \int d\xi \; \rho(\xi)\, \chi_D(\xi) \log \frac{1 - e^{-\tau H + i\Lambda}}{1 - e^{-\tau H}} \cdot \frac{1 - e^{-(N+1)\tau H}}{1 - e^{-(N+1)\tau H + i(N+1)\Lambda}} . \qquad 88.$$

In the thermodynamic limit, according to 58,

$$\rho_E[n;\, D] = \frac{e^{S(\bar\tau) - G(\bar\tau)}}{\sqrt{2\pi\left[S''(\bar\tau) - G''(\bar\tau)\right]}} \; e^{\frac{-[G'(\bar\tau)]^2}{2[S''(\bar\tau) - G''(\bar\tau)]}} \qquad 89a.$$

where we have made use of the fact that $S'(\bar{\tau}) = 0$, and

$$\rho_E = \frac{e^{S(\bar{\tau})}}{\sqrt{2\pi S''(\bar{\tau})}} . \tag{89b.}$$

It is at this stage where the assumption, that the system of interest is small compared with the bath, can be used to obtain a simple expression of $W[n; D]$. In this case

$$W[n; D] = \frac{\rho_{E(n, D)}}{\rho_E} = e^{-G(\bar{\tau}; n\ D)} . \tag{90.}$$

In arriving at 90, a certain amount of caution must be exercised in subtracting large quantities not to throw out the system with the bath.

Going back to 87, we transform the right member into a contour integral about the origin

$$e^{-G[\tau; n D]} = \frac{1}{2\pi i} \int_C \frac{dz}{z^{n+1}} e^{-G[\tau ; z, D]} \tag{91.}$$

and separate out from $G[\tau ; z, D]$ a term independent of z:

$$G(\tau; z D) = G(\tau; 0 D) + \mathcal{J}(\tau; z D) \tag{92.}$$

$$G(\tau; 0 D) = \int d\xi\ \rho(\xi)\, \chi_D(\xi)\, \frac{1 - e^{-(N+1)\tau H}}{1 - e^{-\tau H}} \tag{92a.}$$

$$\mathcal{S}(\tau ; z, D) = \int d\xi \; \rho(\xi)\, \chi_D(\xi) \; \frac{1 - z\, e^{-\tau H}}{1 - z^{N+1} \; e^{-\tau(N+1)H}} \; .$$

92b.

The probability $W[n, D]$ may then be expressed as

$$W[n, D] = e^{-G[\bar{\tau};\, 0\, D]} \, \frac{1}{n!} \, \frac{d^N}{dz^N} \, e^{-\mathcal{S}[\tau;\, z\, D]} \Big|_{z=0}$$

or

$$W[n, D] = W[0, D] \, \frac{1}{n!} \, \frac{d^N}{dz^N} \, e^{-\mathcal{S}(\bar{\tau};\, z \;\; D)} \; . \qquad 93.$$

Specializing the domain D to a point one readily recovers 84.

Expressions of the type 93 for the production of mesons at high energies frequently emerge, as a result of certain approximations in field theory. We have given the derivation of 93 in some detail in order to exhibit the very simple statistical assumption that these expressions involve. The relation of 93 to the averaged n of the F.D. and E.D. formula should also be borne in mind as a useful guide to interpretation.

More useful at high energies than the approximation 93 of 85 would be one to a relativistic density with not only energy but also momentum conserved. We shall use Fermi's

$$\rho_K\{\nu\} = \delta\left(K_o - \lambda\int d\xi\ \nu(\xi)(k^2(\xi)+1)^{\frac{1}{2}}\right)\ \delta\left(\vec{K} - \lambda\int d\xi\ \nu(\xi)\,\vec{k}(\xi)\right) \qquad 94.$$

as an example and indicate only the key formulas, since the calculational techniques were illustrated on the previous example. We have in this case

$$\psi(y_o, \vec{y}) = \int d\xi\ \rho(\xi)\ \log \frac{1 - e^{(N+1)\lambda(y\cdot k)}}{1 - e^{\lambda(y\cdot k)}} \ ; \qquad \begin{aligned} y\cdot k &= \vec{y}\cdot\vec{k} - y_o\, k_o \\ k_o &= (k^2+1)^{\frac{1}{2}} \end{aligned} \ . \qquad 95.$$

This expression may be used to define the relativistic temperatures $(\bar{y}_o, \bar{\vec{y}})$. The analogue of 93 emerges without complications and with the altered definitions

$$W[0, D] = e^{\int d\xi\ \chi_D(\xi)\,\rho(\xi)\ \log \frac{1 - e^{(y\cdot k(\xi))\lambda}}{1 - e^{(y\cdot k(\xi))\lambda(N+1)}}} \qquad 96.$$

$$\Delta[y; z, D] = \int d\xi\ \chi_D(\xi)\ \log \frac{1 - z\ e^{\lambda(y\cdot k(\xi))}}{1 - z^{N+1}\ e^{(N+1)\lambda(y\cdot k(\xi))}} \ . \qquad 97.$$

Making the ρ and ξ identification which led from 80 to 80" and denoting by D an angular range, formula 97 could be used to estimate the number of particles scattered into a given solid angle. A great many other examples and applications could be given.

DISCUSSION

It is customary to include in an article of this nature a more or less detailed survey of recent calculations, note their agreement or disagreement with experiment, criticize the logical and physical assumption that went into them. None of this seems to be called for in the present case. The rather detailed assumptions that have to be made in order to obtain definite predictions involve so few intellectual commitments that to note disagreement (or even agreement) with the experiment would hardly be a rewarding experience. The interested reader may be referred to the fine review article by Milburn reflecting the state of affairs until about the middle of 1954 and to a paper by Lindenbaum in this volume for references to more recent work. Neither shall we criticize the various assumptions that were made by various authors. This would hardly be charitable, since most of them are aware of the tentative and exploratory nature of their work and are only too eager to point to its shortcomings. Instead we should like to discuss several themes which have appeared in the literature that seem to be capable of further development.

The Fermi Contraction

Foremost among these is the "Contraction Hypothesis" of Fermi [1]. Its statement in 49 may be written in the form

$$\Omega_s' = \frac{k_o^{(s)}}{K_{(s)}} \Omega_c \quad ; \quad \Omega_c = \frac{2M_N c \cancel{h}}{K_o} \frac{1}{K_\pi^3} \quad . \qquad 49'.$$

The factor (k_o/K) multiplies the covariant element of volume $\frac{K}{k_o}\, d^3\vec{K}$ turning it into the noncovariant $d^3\vec{k}$. It is difficult to see how it could emerge from any of the covariantly-formulated theories; we shall therefore disregard it and focus our attention on Ω_c.

The "Fermi Contraction" derives some support from the idea of the Lorentz contraction. We therefore start by treating the latter in the context of our discussion with some care. We consider $\Omega = k_\mu \Omega_\mu = k_\mu \int_\sigma d\sigma_\mu \chi_\mu$, where σ is a space-like surface. For a flat σ, Ω_o has the character of a volume. It is evident from its construction that Ω is a scalar. Hence for any two frames $\overset{o}{C}$ and C' we must have $\overset{o}{\Omega} = \Omega'$. Explicitly (with some additional equations)

$$\overset{o}{k}_\mu \overset{o}{\Omega}_\mu = k'_\mu \Omega'_\mu \; ; \; \overset{o}{k}_\mu \overset{o}{k}_\mu = k'_\mu k'_\mu \; ; \; \overset{o}{\Omega}_\mu \overset{o}{\Omega}_\mu = \Omega'_\mu \Omega'_\mu \; . \qquad 98.$$

We now choose $\overset{\circ}{C}$ from among the frames in which $\overset{\circ}{k}_\mu = (0, 0, 0, \overset{\circ}{k}_0)$; and C', in which $\Omega'_\mu = (0, 0, 0, \Omega'_0)$. This is possible because of the time-like nature of k and Ω. We can also take $k'_\mu = (k'_1, 0, 0, k'_o)$ and $\overset{\circ}{\Omega}_\mu = (\overset{\circ}{\Omega}_1, 0, 0, \overset{\circ}{\Omega}_o)$ with $k'_1 \neq 0$ and $\overset{\circ}{\Omega}_1 \neq 0$. Transcribing 98 in these specialized frames, we deduce

$$\Omega'_o = \sqrt{\left[1 - (k'_1/k'_o)^2\right]}\ \overset{\circ}{\Omega}_o \ ; \qquad \overset{\circ}{\Omega}_1/\overset{\circ}{\Omega}_o = k'_1/k'_o \ . \qquad 98'$$

The first of these is the usual expression for the Lorentz contractions; the second tells us that the velocity of C' relative to $\overset{\circ}{C}$ must be related to the time-like tilt of the space-like surface in $\overset{\circ}{C}$. We are not free to choose both arbitrarily. In the case of two colliding nucleons a and b, one can show that there exists a single frame which is a $C'(a)$ and $C'(b)$ and at the same time CM frame (that is one in which $k_{a1} + k_{b1} = 0$). Consider a collision along a straight line. For $C'(a)$ we have $\overset{\circ}{\Omega}_{a1}/\overset{\circ}{\Omega}_{ao} = k'_{a1}/k'_{ao}$; for $C'(b)$, $\overset{\circ}{\Omega}_{b}/\overset{\circ}{\Omega}_{bo} = k'_{b1}/k'_{bo}$.

Hence by taking $\overset{\circ}{\Omega}_{a1}/\overset{\circ}{\Omega}_{ao} = -\overset{\circ}{\Omega}_{b1}/\overset{\circ}{\Omega}_{bo}$ we attain the desired frame. In it $k_{a\mu} = (\vec{k}, (k^2 + \chi^2)^{\frac{1}{2}})$, $k_{b\mu} = (-\vec{k}, (k^2 + \chi^2)^{\frac{1}{2}})$, where k is related to the total energy of the system K_o by $k^2 = \frac{1}{4} K_o^2 - \chi^2$.

The contraction factor for each volume is now $\left(1 - \frac{k_1^{\ 2}}{k_o^{\ 2}}\right)^{\frac{1}{2}}$.

In the Fermi scheme, the same Ω^c is taken for every particle emerging from the collision and is also identified with the Ω^c for each colliding nucleon. Thus $\Omega^c = k_\mu^{\ N} \Omega_\mu^{\ N}$. Let us consider the collision in the CM frame which is also a C' frame for each nucleon. In it $\Omega^c = k'^{\,N}_o \Omega'^{\,N}_o$. Fermi pictures the collision as proceeding in three stages. In the first, $k'^{\,N}_o$ is very large because of the "ordered" translational kinetic energy of the nucleon. The two nucleons with very high $k_o^{N\ (ord)}$ collide and are struck. The $k_o^{N\ (ord)}$ is now reduced to $2\ Mc/\hbar$, most of it having turned into disordered energy which is partitioned among various degrees of freedom of the system according to statistical laws. Finally the quasi-equilibrium state breaks up, and the probability of disintegration into various modes is taken to be proportional to their statistical weights. The basic assumption of Fermi is equivalent to postulating an approximate high energy collision invariant which survives the transition between the incident and the "stuck" stage. The invariant in question is expressible as

$$k_o^{ord} \; \Omega_o \Big]_{incident} = k_o^{ord} \Omega_o \Big]_{stuck} \; . \qquad 99.$$

In this expression $k_o^{ord}\Big]_{inc} = k_o^N$; $\Omega_o^{ord}\Big]_{inc}$ is the quantity of interest; $k_o^{ord}\Big]_{stuck} = Mc/\hbar$ because almost all energy is now thermal; finally for $\Omega_o^{ord}\Big]_{stuck}$, we may take a spherical volume of radius of the compton wavelength of the π mesons, "since the pion field surrounding the nucleus extends to this distance." All these quantities are evaluated in the CM system in which the stuck nucleons are at rest. We see no reasons for the volume of the pions surrounding the stuck nucleus to appear contracted for an observer in that frame. The argument sometimes presented, that the contracted kinematic state of a moving nucleon is "frozen in" when the nucleon stops and that its "thawing" time is longer than the life time of the quasi-equilibrium state, is somewhat unconvincing. However with the assumption of a high energy collisional invariant and without this strange kinematics, we deduce from 99

$$\Omega_o\Big]_{inc} \cong \frac{2M_N c/\hbar}{K_o} \quad \frac{1}{\lambda_\pi^3} \; , \qquad 99.$$

where we replaced k_o^N by $\frac{1}{2} K_o$. It is perhaps gratifying to find that the Fermi contraction factor 99' agrees numerically with the Lorentz contraction factor obtained previously. It is well to realize, however, that such agreement is not required by any physical principles and 99' cannot be derived from 98'. Regarded merely as an attempt to invent a new high energy collision invariant, this idea of Fermi merits close study.

<u>Additional constants of motion</u>.

In his original paper Fermi carried out his calculations with 49a for nucleons and 50'a for mesons and tried to allow for the conservation of momentum by an essentially dimensional argument. The explicit calculations of Lepore and Stuart [8] have shown that one must proceed with greater care in the high energy domain. Using 50' rather than 50'a, their results differed considerably from those of Fermi. In addition to pointing out the need for careful treatment of these integrals of motions, these authors also introduced a powerful technique that enables one to do so.

Somewhat later, Lepore and Neuman [9] investigated the effects of including the center of mass of the system among the conserved quantities. For relativistic particles the new conservation law turns out to be quite important, replacing the factor $(k^2+K^2)^{-\frac{1}{2}}$ in 9 by $(k^2+K^2)^{-3/2}$. With this "contraction factor" supplied by relativity these authors felt that they could afford to drop the Fermi contraction hypothesis. This modified model would seem to favor low energy high multiplicity events.

A crude attempt to study the effects of conservation of angular momentum on the angular distribution of particles emerging from a high energy collision was made by Fermi [10]. The subject is of considerable physical interest in connection with high energy stars. It is hoped that the static spherically symmetric model discussed in the first chapter may be of some help in this connection.

<u>Final state interactions.</u>

Retaining the basic statistical outlook and the structure 11 for the density functions, one could modify the propagators Δ^{+} defined by 31 and represented bilinearly in 30. Instead of free particle $f_m(x)$'s, one could insert wave functions depending on a few parameters fixed by experiment. The modified propagators would no longer have to satisfy the Gordon-Klein equations, and through them some of our knowledge about the actual final states of interacting particles could be made to bear on the predictions of the statistical model. Work on this much needed improvement of the statistical model was initiated by Kovac [11] with encouraging results.

<u>The Lindenbaum-Sternheimer isobaric model.</u>

There may be some features of high energy interaction which are too strong not to leave their individual marks on the outcome of the collision process in spite of the randomizing effect of the ample energy and large numbers of degrees of freedom. By singling these out for special attention, one could make the model more suitable for the treatment of the remainder. This is the view represented by Lindenbaum and Sternheimer [12]. In their recent attempt in this direction they single out the isobaric state observed in pion nucleon scattering for special treatment. The reader is referred to a forthcoming publication by these authors for a detailed exposition of their views.

FOOTNOTES

1. The material quoted in this section is from [1].

2. The meaning of the starred product symbol $\prod^{*}$ will best be explained on examples. If an element of volume in phase space is written as a free product

$$\prod_{s=1}^{N} d^3\vec{P}_{(s)}\, d^3\vec{q}_{(s)} = d^3\vec{P}_{(1)}\, d^3\vec{q}_{(1)}\, d^3P_{(2)}\, d^3q_{(2)} \dots d^3P_{(N)}\, d^3q_{(N)}$$

the symbol $\prod_{s=1}^{*} d^3\vec{P}_{(s)}\, d^3\vec{q}_{(s)}$ may denote an expression of the type

$$\delta(E - H(\vec{P}_1\, \vec{q}_1\, \vec{P}_2\, \vec{q}_2 \dots \vec{P}_n\, \vec{q}_n) d^3\vec{P}_1\, d^3q_1\, d^3P_2\, d^3q_2 \dots d^3P_n\, d^3q_n \; .$$

It may also contain more than one delta function. A quantity $v_N\left[f_1 \dots f_N\right]$ will be referred to as essentially factorable if it may be written as

$$\delta\left[F'_1 - F_1(f_1 \dots f_n)\right]\, \delta\left[F'_2 - F_2(f_1 \dots f_n)\right] \dots v_1(f_1)\, v_2(f_2) \dots , .$$

A set of random variables $x_1 \dots x_n$ will be regarded as essentially independent if the joint distribution function $\rho(x_1\;\; x_2 \dots x_n)$ may be put in the form $\delta\left[A_1 - g_1(x_1 \dots x_n)\right] \delta(A_2 - g_2(x_1 \dots\;\;)\, \rho_1(x_1)\, \rho_2(x_2) \dots$.

3. The function Δ^{+} frequently employed in field theory is related to our Δ by $\Delta = 2\kappa\Delta^{+}$. We have chosen the multiplicative constant in this manner in order to have the bilinear representation 30 with f'_n normalized in convection current 28 .

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4. This section leans heavily on 6 and 7.

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