Title
On Distributed Observers for Linear Time-invariant Systems Under Intermittent Information Constraints**This research has been partially supported by the National Science Foundation under CAREER Grant no. ECS-1450484 and Grant no. CNS-1544396, and by...

Journal
IFAC-PapersOnLine, 49(18)

ISSN
2405-8963

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Publication Date
2016

DOI
10.1016/j.ifacol.2016.10.240

Peer reviewed

On Distributed Observers for Linear Time-invariant Systems Under Intermittent Information Constraints ⋆

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Abstract: We study the problem of estimating the state of a linear time-invariant plant in a distributed fashion over networks allowing only intermittent transmission of information. By attaching to each node an observer that employs information received from its neighbors in an intermittent fashion, we propose a distributed state observer that guarantees global asymptotic stability of the zero estimation error set. We also characterize the proposed observer's robustness to measurement and communication noise in terms of ISS. The design of parameters is formulated as matrix inequalities. The properties of the proposed observer are shown analytically and validated numerically.

Keywords: Distributed estimation, Intermittent communication, Hybrid systems

1. INTRODUCTION

State estimation in multi-agent systems over generic connected networks has seen increased attention recently. A typical challenge in this setting comes in the form of each agent being unable to generate an estimate alone, therefore, communication to other systems is needed. One methodology to solve this problem is to design an appropriate *information fusion protocol* which uses the local information available to an agent so as to guarantee convergence of the local state estimates to the state of the plant of interest. For example, the estimation of the trajectories of a moving target can be solved by using distributed sensor networks, see, e.g., Wang and Ren (2015) and Kamal et al. (2013). In Li and Sanfelice (2016), robust decentralized estimation with performance guarantees is explored under the assumption that information is available continuously. The distributed Kalman filtering is employed for achieving spatially-distributed estimation tasks in Cortes (2009). In Hong et al. (2008), switching inter-agent topologies are used to design distributed observers for a leader-follower problem in multi-agent systems.

In this paper, we consider the problem of designing a distributed estimation algorithm to estimate the state of the linear time-invariant plant

$$\dot{x} = Ax \qquad (1)$$

when measurement of its output is only available intermittently, i.e., at isolated (potentially nonperiodic) time instances. For the case of a single estimator, Ferrante et al. (2013) proposes a hybrid observer guaranteeing global exponential stability of the zero estimation error set. In the context of multi-agent systems, Cheng and Xie (2014) proposes a distributed observer with undirected fixed communication topology and switching communication topology for periodic sampling time/communication events. In contrast to the latter work, in this paper, we consider the case where the agents may only receive information from their neighbors intermittently, say, to reduce the cost of communication and computation. In our setting, agents are only allowed to take measurements and communicate at isolated (potentially nonperiodic) events, where the time elapsed between two consecutive events are governed by a random variable. Our contributions include the following:

1. We establish a hybrid model for a distributed state observer under intermittent information transmission that is well-posed;
2. We propose sufficient conditions to guarantee uniform global asymptotic stability of the zero estimation error set. Moreover, robustness of the stability property of the proposed observer with respect to communication and measurement noise is characterized in terms of input-to-state stability (ISS).

The remainder of this paper is organized as follows. In Section 2, some preliminaries on the hybrid systems framework used and graph theory are briefly discussed. The main results are presented in Section 3. In Section 4, the scenario of asynchronous event times is studied. Numerical and analytical examples illustrating the results are discussed throughout the paper.

2. PRELIMINARIES

2.1 Notation

Given a matrix A, the set $\mathrm{eig}(A)$ contains all eigenvalues of A and $|A| := \max\{|\lambda|^{\frac{1}{2}} : \lambda \in \mathrm{eig}(A^\top A)\}$. Given two vectors $u, v \in \mathbb{R}^n$, $|u| := \sqrt{u^\top u}$ and notation $[u^\top \; v^\top]^\top$ is equivalent to (u, v). Given a function $m : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $|m|_\infty := \sup_{t\geq 0} |m(t)|$. $\mathbb{Z}_{\geq 1}$ denotes the set of positive integers, i.e., $\mathbb{Z}_{\geq 1} := \{1, 2, 3, \dots\}$. $\mathbb{N}$ denotes the set of natural numbers including zero, i.e., $\mathbb{N} := \{0, 1, 2, 3, \dots\}$. Given a symmetric matrix P, $\overline{\lambda}(P) := \max\{\lambda : \lambda \in \mathrm{eig}(P)\}$ and $\underline{\lambda}(P) := \min\{\lambda : \lambda \in \mathrm{eig}(P)\}$. Given matrices A, B with proper dimensions, we define the operator

⋆ This research has been partially supported by the National Science Foundation under CAREER Grant no. ECS-1450484 and Grant no. CNS-1544396, and by the Air Force Office of Scientific Research under Grant no. FA9550-16-1-0015.

$\text{He}(A,B) := A^\top B + B^\top A$; $A \otimes B$ defines the Kronecker product; $\text{diag}(A,B)$ denotes a 2×2 block matrix with A and B being the diagonal entries; and $A * B$ defines the Khatri-Rao product between A and B. Given $N \in \mathbb{N}$, $I_N \in \mathbb{R}^{N\times N}$ defines the identity matrix and $\mathbf{1}_N$ is the vector of N ones.

2.2 Preliminaries on Hybrid Systems

In this paper, a hybrid system $\mathcal{H}$ has data (C, f, D, G) and is defined by

$$\begin{aligned} \dot{z} &= f(z) \quad z \in C, \\ z^+ &\in G(z) \quad z \in D, \end{aligned} \tag{2}$$

where $z \in \mathbb{R}^n$ is the state, f defines the flow map capturing the continuous dynamics and C defines the flow set on which f is effective. The map G defines the jump map and models the discrete behavior, while D defines the jump set, which is the set of points from where jumps are allowed. A solution ϕ to $\mathcal{H}$ is parametrized by $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where t denotes ordinary time and j denotes jump time. The domain $\text{dom}\,\phi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if for every $(T, J) \in \text{dom}\,\phi$, the set $\text{dom}\,\phi \cap ([0, T] \times \{0, 1, \ldots, J\})$ can be written as the union of sets $\cup_{j=0}^{J}(I_j \times \{j\})$, where $I_j := [t_j, t_{j+1}]$ for a time sequence $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{J+1}$. The t_j's with $j > 0$ define the time instants when the state of the hybrid system jumps and j counts the number of jumps. A solution to $\mathcal{H}$ is called maximal if it cannot be extended, i.e., it is not a truncated version of another solution. It is called complete if its domain is unbounded. A solution is Zeno if it is complete and its domain is bounded in the t direction. A solution is precompact if it is complete and bounded. The set $\mathcal{S}_\mathcal{H}$ contains all maximal solutions to $\mathcal{H}$, and the set $\mathcal{S}_\mathcal{H}(\xi)$ contains all maximal solutions to $\mathcal{H}$ from ξ.

We consider the definition of uniform global asymptotic stability (UGAS) for a set given in (Goebel et al., 2012, Definition 3.6). A sufficient condition for a set $\mathcal{A}$ to be UGAS for $\mathcal{H}$ is given in (Goebel et al., 2012, Proposition 3.27). A hybrid system is said to satisfy the hybrid basic conditions if (Goebel et al., 2012, Assumption 6.5) holds. We refer the reader to Goebel et al. (2012) for more details on these notions and the hybrid systems framework.

2.3 Preliminaries on Graph Theory

A directed graph (digraph) is defined as $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{G})$. The set of nodes of the digraph are indexed by the elements of $\mathcal{V} = \{1, 2, \ldots, N\}$, and the edges are the pairs in the set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. Each edge directly links two nodes, i.e., an edge from i to k, denoted by (i, k), implies that agent i can receive information from agent k. The adjacency matrix of the digraph Γ is denoted by $\mathcal{G} \in \mathbb{R}^{N\times N}$, where its (i, k)-th entry g_{ik} is equal to one if $(i, k) \in \mathcal{E}$ and zero otherwise. A digraph is undirected if $g_{ik} = g_{ki}$ for all $i, k \in \mathcal{V}$. Without loss of generality, we assume that $g_{ii} = 0$ for all $i \in \mathcal{V}$. The in-degree and out-degree of agent i are defined by $d_i^{in} = \sum_{k=1}^{N} g_{ik}$ and $d_i^{out} = \sum_{k=1}^{N} g_{ki}$. The in-degree matrix $\mathcal{D}$ is the diagonal matrix with entries $D_{ii} = d_i^{in}$ for all $i \in \mathcal{V}$. The Laplacian matrix of the graph Γ, denoted by $\mathcal{L} \in \mathbb{R}^{N\times N}$, is defined as $\mathcal{L} = \mathcal{D} - \mathcal{G}$. The set of indices corresponding to the neighbors that can send information to the i-th agent is denoted by $\mathcal{N}(i) := \{k \in \mathcal{V} : (i, k) \in \mathcal{E}\}$.

3. INTERMITTENT STATE OBSERVERS WITH SYNCHRONOUS COMMUNICATION EVENTS

3.1 Configuration and basic properties

Consider N agents that are connected via a directed graph and where each agent runs a local observer of the state x of (1). Each local observer uses its own measurement and information received from its neighbors. In particular, the local observer at the i-th agent has a local estimate $\hat{x}_i$ and it has a measurement of the state x, which is given by

$$y_i = H_i x \in \mathbb{R}^{p_i} \tag{3}$$

as well as information from all its connected neighbors. However, the availability of such measurements is at isolated time instances $\{t_s\}_{s=1}^{\infty}$, where s is the communication event index. More precisely, the i-th agent receives $y_i(t_s) = H_i x(t_s)$, $y_k(t_s) = H_k x(t_s)$, and $\hat{x}_k(t_s)$ at each communication event, where $k \in \mathcal{N}(i)$. Furthermore, it is assumed that a random variable $\Omega_s \in [T_1, T_2]$ determines the time elapsed between such communication events, i.e.,

$$t_{s+1} - t_s = \Omega_s \tag{4}$$

where $s \in \{1, 2, 3, \cdots\}$ and $T_2 \geq T_1 > 0$. The scalar values T_1 and T_2 define the lower and upper bounds, respectively, of the time allowed to elapse between consecutive communication events. Due to the impulsive nature of such communication mechanism, the communication events are triggered when the timer τ reaches zero, which is then reset to a point in $[T_1, T_2]$, from where it decreases with ordinary time. Namely, the timer τ evolves according to

$$\dot{\tau} = -1 \qquad \tau \in [0, T_2], \tag{5a}$$

$$\tau^+ \in [T_1, T_2] \qquad \tau = 0. \tag{5b}$$

This hybrid system generates any possible sequence of time instances $\{t_s\}_{s=1}^{\infty}$ at which events occur and satisfy (4) with Ω_s given by any probability distribution function that assigns $\Omega_s \in [T_1, T_2]$.

The proposed distributed observer is as follows. The i-th agent runs the following local observer:

$$\dot{\hat{x}}_i = A\hat{x}_i + \eta_i \tag{6}$$

where η_i, referred as the information fusion state, is a variable that stores the information obtained at each communication event. Due to the intermittent nature of availability of information, η_i is updated impulsively when new information arrives. The generic dynamics of η_i can be captured by

$$\dot{\eta}_i = f_{oi}(\hat{x}_i, \eta_i) \qquad \tau \in [0, T_2] \tag{7}$$

$$\eta_i^+ = \sum_{k\in\mathcal{V}} g_{ik} G_{oi}^k(\hat{x}_i, \hat{x}_k, y_i, y_k) \qquad \tau = 0 \tag{8}$$

for each $i \in \mathcal{V}$, where the map $f_{oi} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defines the continuous evolution of the information fusion state and the map $G_{oi}^k : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{p_i} \times \mathbb{R}^{p_k} \to \mathbb{R}^n$ defines the impulsive update law when new information is collected from the k-th neighbor.

The resulting hybrid closed-loop system is given by the interconnection between the plant in (1), all local observers in form (6), and the dynamics of the information fusion state in (7)-(8). We denote the closed-loop hybrid system by $\mathcal{H}_o = (C, f, D, G)$, which has state $z = (x, \xi_1, \ldots, \xi_N, \tau) \in \mathcal{X} := \mathbb{R}^n \times S^N \times [0, T_2]$, where $S := \mathbb{R}^n \times \mathbb{R}^n$ and $\xi_i = (\hat{x}_i, \eta_i)$ for each $i \in \mathcal{V}$. Its continuous dynamics are given by

$$\dot{z} = f(z) := (Ax, f_1(\xi_1), \ldots, f_N(\xi_N), -1) \quad z \in C \tag{9}$$

where $C := \mathcal{X}$ and $f_i(\xi_i) := (A\hat{x}_i + \eta_i, f_{oi}(\hat{x}_i, \eta_i))$ for each $i \in \mathcal{V}$. Since the communication and update events are induced by the timer with dynamics as in (5), the jump set D for $\mathcal{H}_o$ is defined as $D := \{z \in \mathcal{X} : \tau = 0\}$. Then

$$z^+ \in G(z) := (x, G_1(z), \ldots, G_N(z), [T_1, T_2]) \quad z \in D, \tag{10}$$

where $G_i(z) = \left(\hat{x}_i, \sum_{k\in\mathcal{V}} g_{ik} G^k_{oi}(\hat{x}_i, \hat{x}_k, y_i, y_k)\right)$ for all $i \in \mathcal{V}$.

With the construction above, the data of the hybrid closed-loop system $\mathcal{H}_o$ satisfies the following property.

Lemma 3.1. Suppose that, for each $i \in \mathcal{V}$, the flow map $f_{oi} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous; and for each $(k, i) \in \mathcal{E}$, the map $G^k_{oi} : \mathbb{R}^{2n} \times \mathbb{R}^{p_i+p_k} \to \mathbb{R}^n$ is continuous. Then, the hybrid system $\mathcal{H}_o$ satisfies the hybrid basic conditions.

Remark 3.2. Note that satisfying the hybrid basic conditions implies that the hybrid system $\mathcal{H}_o$ is well-posed and, with asymptotic stability of a compact set, leads to robustness to small enough perturbations; see Goebel et al. (2012) for more information. □

3.2 Sample-and-hold estimation protocol

In this section, we consider the following dynamics for η_i defining a specific hybrid information fusion strategy:

$$f_{oi}(\hat{x}_i, \eta_i) = 0 \tag{11}$$

for all $(\hat{x}_i, \eta_i) \in \mathbb{R}^n \times \mathbb{R}^n$, and, for all $(\hat{x}_i, \hat{x}_k, y_i, y_k) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{p_i} \times \mathbb{R}^{p_k}$,

$$G^k_{oi}(\hat{x}_i, \hat{x}_k, y_i, y_k) = \frac{1}{d^{in}_i} K_{ii} y^e_i + K_{ik} y^e_k + \gamma(\hat{x}_i - \hat{x}_k) \tag{12}$$

where, for each $i, k \in \mathcal{V}$, $y^e_i = H_i\hat{x}_i - y_i$, $K_{ik} \in \mathbb{R}^{n\times p_k}$, and $\gamma \in \mathbb{R}$ are the gain matrices for the i-th agent. Note that due to the specific update law in (8) and the definition of g_{ik}, the second term in (12) uses the output error of each k-th agent that is a neighborhood of the i-th agent, and the third term in (12) uses the difference between the estimates $\hat{x}_i$ and $\hat{x}_k$. These are the quantities that are transmitted at communication events.

Remark 3.3. The hybrid information fusion strategy proposed in (11)-(12) falls into the category of zero-order sample-and-hold control; see, e.g., Naghshtabrizi et al. (2008) and Raff et al. (2008). Note that the work in Naghshtabrizi et al. (2008) and Raff et al. (2008) pertain to single-agent systems and that robustness to measurement and communication noise is not studied. □

To analyze the hybrid system $\mathcal{H}_o$, for each $i \in \mathcal{V}$, denote the local estimation error $e_i = \hat{x}_i - x$. Then, it follows that the continuous dynamics of e_i and η_i are given by

$$\dot{e}_i = Ae_i + \eta_i, \tag{13}$$

$$\dot{\eta}_i = 0, \tag{14}$$

for each $\tau \in [0, T_2]$, and, when $\tau = 0$, the discrete dynamics are

$$e^+_i = e_i, \tag{15}$$

$$\eta^+_i = K_{ii}H_ie_i + \sum_{k\in\mathcal{V}} g_{ik}K_{ik}H_ke_k + \gamma \sum_{k\in\mathcal{V}} g_{ik}(e_i - e_k). \tag{16}$$

By denoting $e = (e_1, e_2, \cdots, e_N)$ and $\eta = (\eta_1, \eta_2, \cdots, \eta_N)$, it follows that

$$\dot{e} = (I_N \otimes A)e + \eta,$$
$$\dot{\eta} = 0,$$

for each $\tau \in [0, T_2]$, while when $\tau = 0$,

$$e^+ = e, \tag{17}$$

$$\eta^+ = (K_gH_g * (I_N + \mathcal{G}) + \gamma\mathcal{L} \otimes I_n)e, \tag{18}$$

where the matrix $H_g = \mathrm{diag}(H_1, H_2, \ldots, H_N)$ is block diagonal and $K_g \in \mathbb{R}^{nN\times p}$ is a $N \times N$ block matrix with the (i, k)-th entry given by $K_{ik} \in \mathbb{R}^{n\times p_k}$ for all $i, k \in \mathcal{V}$ with $p = \sum_{i\in\mathcal{V}} p_i$. Note that the operation "$*$" denotes the Khatri-Rao product, where the matrix K_gH_g is treated as a $N \times N$ block matrix.

Then, in the (e, η) coordinates, the system $\mathcal{H}_o$ can be written as a hybrid system $\mathcal{H}_s = (C_s, f_s, D_s, G_s)$ with state $\chi = (\sigma, \tau) \in \mathcal{X}_s := \mathbb{R}^{nN} \times \mathbb{R}^{nN} \times [0, T_2]$ with $\sigma = (e, \eta)$, and data given by

$$f_s(\chi) := (A_f\sigma, -1) \qquad \forall\chi \in C_s \tag{19}$$

$$G_s(\chi) := (A_g\sigma, [T_1, T_2]) \qquad \forall\chi \in D_s, \tag{20}$$

where $C_s := \mathcal{X}_s$, $D_s := \mathbb{R}^{nN} \times \mathbb{R}^{nN} \times \{0\}$,

$$A_f := \begin{bmatrix} I_N \otimes A & I_{nN} \\ 0 & 0 \end{bmatrix}, \tag{21}$$

$$A_g := \begin{bmatrix} I_{nN} & 0 \\ K_gH_g * (I_N + \mathcal{G}) + \gamma\mathcal{L} \otimes I_n & 0 \end{bmatrix}, \tag{22}$$

Note that $\mathcal{H}_s$ is autonomous; in particular, its dynamics are independent of the dynamics of $\mathcal{H}_o$.

Remark 3.4. Due to the jumps being triggered by the timer τ with dynamics (5) reaching zero, the hybrid time domain of any maximal solution to $\mathcal{H}_o$ (or to $\mathcal{H}_s$) satisfies the following property. For any given $\phi \in \mathcal{S}_{\mathcal{H}_o}$ (or $\phi \in \mathcal{S}_{\mathcal{H}_s}$), $T_1 \le t_{j+1} - t_j \le T_2$ for all $j \ge 1$ and $0 \le t_1 \le T_2$, and

$$(j - 1)T_1 \le t \le (j + 1)T_2 \qquad \forall\, j \ge 1 \tag{23}$$

for all $(t, j) \in \mathrm{dom}\,\phi$. □

As the objective of each agent is to estimate the state x of the plant, i.e., for each $i \in \mathcal{V}$, make $\hat{x}_i$ converge to x asymptotically, and since η_i approaches zero as e_i approaches zero, the set of interest is defined as

$$\mathcal{A} = \{0_{nN}\} \times \{0_{nN}\} \times [0, T_2]. \tag{24}$$

The following result provides a sufficient condition for uniform global asymptotic stability of $\mathcal{A}$.

Theorem 3.5. Let $0 < T_1 \le T_2$ be given. Suppose there exist matrices $K_g \in \mathbb{R}^{nN\times p}$ and $P \in \mathbb{R}^{2nN\times 2nN}$ such that $P = P^\top > 0$ and

$$A_g^\top \exp(A_f^\top\nu)P\exp(A_f\nu)A_g - P < 0 \tag{25}$$

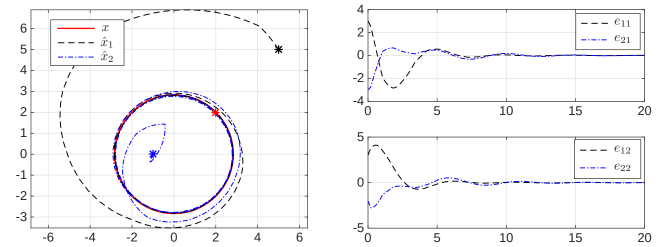
for all $\nu \in [T_1, T_2]$. Then, the set $\mathcal{A}$ in (24) is UGAS for the hybrid system $\mathcal{H}_s$.

Remark 3.6. Note that when condition (25) holds, Theorem 3.5 guarantees that e and η converges to zero asymptotically. Recalling that $e_i = \hat{x}_i - x$, this implies that $\hat{x}_i$ converges to x asymptotically for all $i \in \mathcal{V}$. Furthermore, due to the multiplication of the parameters ν, P, and K_g in the first term of (25), it might be difficult to efficiently check (25) numerically. It is worth pointing out that condition (25) is satisfied if, for all $\nu \in [T_1, T_2]$, the eigenvalues of $\exp(A_f\nu)A_g$ are contained in the unit circle. □

Note that the pair (H_i, A) is not explicitly assumed to be detectable in this work. In fact, as we will show in the following example, even when (H_i, A) are not detectable for each $i \in \mathcal{V}$, we can still guarantee UGAS of $\mathcal{A}$ by satisfying (25).

Example 3.7. Consider an oscillatory plant as in (1) with

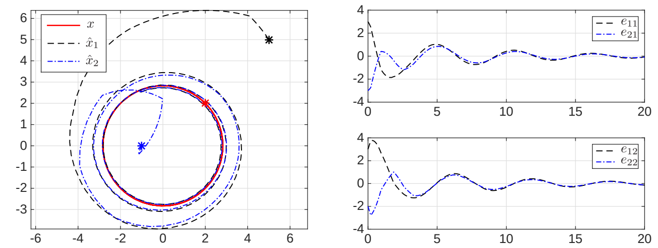
$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{26}$$



(a) Phase plots of $x, \hat{x}_1, \hat{x}_2$ from initial conditions that are denoted by $*$.

(b) Estimation error, where $x = (x_1, x_2)$, $\hat{x}_i = (\hat{x}_{i1}, \hat{x}_{i2})$, $e_{i1} = \hat{x}_{i1} - x_1$, $e_{i2} = \hat{x}_{i2} - x_2$, for $i \in \{1, 2\}$.

Fig. 1. Phase portraits and estimation errors of two agents that are all-to-all connected for the observer in Example 3.7. Initial conditions are $x(0,0) = (2,2)$, $\hat{x}_1(0,0) = (5,5)$, $\hat{x}_2(0,0) = (-1,0)$, $\eta_1(0,0) = (1,1)$, $\eta_2(0,0) = (-1,-1)$, $\tau(0,0) = 0.2$.



(a) Phase plots of $x, \hat{x}_1, \hat{x}_2$ from initial conditions that are denoted by $*$.

(b) Estimation errors for both components.

Fig. 2. Phase portraits and estimation errors for the observer in Example 3.7. The gain used is $\gamma = -0.4$ while the rest of the parameters are the same as those used for the simulations in Figure 1.

Consider the case of two agents that are all-to-all connected, namely,

$$\mathcal{G} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Let the measurements of x at agent 1 be

$$y_1 = H_1 x, \quad H_1 = [1 \;\; 0]$$

and the measurements at agent 2 be

$$y_2 = H_2 x, \quad H_2 = [0 \;\; 1].$$

Let $T_1 = 0.5$ and $T_2 = 1$. By solving inequality[1] (25) in Theorem 3.5, we obtain the following parameters:

$$K_{11} = [\,-0.5 \;\; -0.2\,]^\top, \quad K_{12} = [\,-0.2 \;\; -0.2\,]^\top,$$
$$K_{21} = [\,0.2 \;\; 0.3\,]^\top, \quad K_{22} = [\,-0.1 \;\; -0.5\,]^\top,$$

with $\gamma = -0.1$. A simulation with these parameters is shown in Figure 1. The estimates $\hat{x}_1$ and $\hat{x}_2$ converge to x asymptotically as guaranteed by Theorem 3.5. [2]

More interestingly, consider the scenario where agent 1 loses the capability of receiving measurements, i.e., $H_1 = 0$. A simulation with same initial conditions and gains as those in Figure 1 is shown in Figure 2. As suggested from the simulation, it can be seen that even though the agent has measurement $y_1 \equiv 0$, through the consensus-like term (the third term in (12)), the components $\hat{x}_1$ and $\hat{x}_2$ reach consensus first and then converge to x asymptotically. This highlights further a benefit of the third term in the dynamics of η in (12). In fact, the third term enforces consensus between the estimates $\hat{x}_1$ and $\hat{x}_2$. Another benefit of using this term will be seen in the next example. △

Next, we establish another sufficient condition to guarantee that the estimation error e converges to zero asymptotically. Following the idea in Ferrante et al. (2015), by defining the variable $\theta_i = \sum_{k \in \mathcal{V}} g_{ik} G^k_{oi}(\hat{x}_i, \hat{x}_k, y_i, y_k) - \eta_i$ and $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$, we have that the continuous evolution of e and θ is given by

$$\dot{e} = A_\theta e - \theta, \tag{27}$$
$$\dot{\theta} = \mathcal{K} A_\theta e - \mathcal{K}\theta, \tag{28}$$

when $\tau \in [0, T_2]$, where $A_\theta = I_N \otimes A + \mathcal{K}$ and $\mathcal{K} = K_g H_g * (I_N + \mathcal{G}) + \gamma \mathcal{L} \otimes I_n$. Moreover, the discrete dynamics of e and θ are given by $e^+ = e$ and $\theta^+ = 0$ when $\tau = 0$. Then, the system in coordinates e and θ can be written as a hybrid system $\mathcal{H}_\theta = (C_\theta, f_\theta, D_\theta, G_\theta)$ with state $\chi_\theta = (\sigma_\theta, \tau) \in \mathcal{X}_s$, $\sigma_\theta = (e, \theta)$, and data given by

$$f_\theta(\chi_\theta) := (A_{f\theta}\sigma_\theta, -1)$$

for each $\chi_\theta \in C_\theta := \mathbb{R}^{nN} \times \mathbb{R}^{nN} \times [0, T_2]$, and

$$G_\theta(\chi_\theta) := (A_{g\theta}\sigma_\theta, [T_1, T_2])$$

when $\chi_\theta \in D_\theta := \mathbb{R}^{nN} \times \mathbb{R}^{nN} \times \{0\}$, where

$$A_{f\theta} = \begin{bmatrix} A_\theta & -I_{nN} \\ \mathcal{K}A_\theta & -\mathcal{K} \end{bmatrix}, \quad A_{g\theta} = \begin{bmatrix} I_{nN} & 0 \\ 0 & 0 \end{bmatrix}. \tag{29}$$

Since the state e denotes the estimation error, the variable θ denotes the difference between the current information fusion state (η) and the new value of the fusion state (η^+), we are also interested in guaranteeing asymptotic stability of the set $\mathcal{A}$ defined in (24). In these new coordinates, we establish the following result.

Theorem 3.8. Let $0 < T_1 \leq T_2$ be given. Suppose there exist $\delta > 0$ and matrices $K_g \in \mathbb{R}^{nN \times p}$, $P, Q \in \mathbb{R}^{nN \times nN}$ satisfying $P = P^\top > 0$, $Q = Q^\top > 0$, and

$$\begin{bmatrix} \mathrm{He}(A_\theta, P) & -P + \exp(\delta\nu) A_\theta^\top \mathcal{K}^\top Q \\ \star & -\exp(\delta\nu)(\delta Q + \mathrm{He}(\mathcal{K}, Q)) \end{bmatrix} < 0 \tag{30}$$

for all $\nu \in [0, T_2]$. Then, the set $\mathcal{A}$ in (24) is UGAS for the hybrid system $\mathcal{H}_\theta$.

Note that the condition in (30) needs to be checked over a closed interval $[0, T_2]$, which is a difficult task. The following result relaxes this requirement.

Proposition 3.9. Let $T_2 > 0$ be given. The inequality in (30) holds for each $\nu \in [0, T_2]$ if there exist $\delta > 0$ and matrices $P, Q \in \mathbb{R}^{nN \times nN}$ satisfying $P = P^\top > 0$, $Q = Q^\top > 0$,

$$\begin{bmatrix} \mathrm{He}(A_\theta, P) & -P + A_\theta^\top \mathcal{K}^\top Q \\ \star & -\delta Q - \mathrm{He}(\mathcal{K}, Q)) \end{bmatrix} < 0, \tag{31}$$

$$\begin{bmatrix} \mathrm{He}(A_\theta, P) & -P + \exp(\delta T_2) A_\theta^\top \mathcal{K}^\top Q \\ \star & -\exp(\delta T_2)(\delta Q + \mathrm{He}(\mathcal{K}, Q)) \end{bmatrix} < 0. \tag{32}$$

Example 3.10. Consider the network estimation problem in Example 3.7. Using the same parameters $K_{11}, K_{12}, K_{21}, K_{22}$ and γ therein, it can be verified that (31) and (32) hold for $\delta = 10$ and a maximum value of $T_2 \approx 0.4$. As expected, replacing condition (30) by (31) and (32), which is a relaxation of (30), the largest T_2 allowed becomes smaller. △

Remark 3.11. Note that the matrices E_1 and E_2 involve the multiplication of $Q\mathcal{K}A_\theta$, which contains cross terms

[1] Note that the inequality in (25) is not linear. The tool developed in Fiala et al. (2013) provides a way to solve it.
[2] Code at `https://github.com/HybridSystemsLab/ObsSyncTimes2nd`.

involving Q, K_g, and γ. The presence of these terms makes the problem nonlinear and difficult to solve numerically. LMI conditions can be established following ideas in Ferrante et al. (2013). □

3.3 Robustness to measurement noise and channel noise

In this section, we investigate the scenario when the measurements y_i's are noisy, i.e., for each $i \in \mathcal{V}$, $\tilde{y}_i = y_i + m_i$, where $m_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^{p_i}$ denotes the measurement noise on the information received by agent i. Moreover, when agent i receives information (including $\hat{x}_k, y_k$) from agent k ($k \in \mathcal{N}(i)$), it is also affected by channel noise, i.e., $\tilde{x}_k^i = \hat{x}_k + c_i^x$ and $\tilde{y}_k^i = \tilde{y}_k + c_i^y$, where $c_i = (c_i^x, c_i^y) : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n+p_i}$ is the channel noise when agent i receives information from its neighbors. Then, according to the design in (11)-(12), the noise is injected through the update law of η_i, that is,

$$\dot{\eta}_i = 0 \tag{33}$$

when $\tau \in [0, T_2]$, and

$$\eta_i^+ = K_{ii} H_i e_i + \sum_{k \in \mathcal{V}} g_{ik}(K_{ik} H_k e_k + e_i - e_k) + \zeta_i \tag{34}$$

when $\tau = 0$, where

$$\zeta_i = -K_{ii} m_i + \sum_{k \in \mathcal{V}} g_{ik} K_{ik}(H_k c_i^x - c_i^y - m_k) - \gamma \sum_{k \in \mathcal{V}} g_{ik} c_i^x .$$

Then, the closed-loop system in (19) and (20) with added noise can be written in the following compact form:

$$\dot{\chi} = f_s(\chi) := (A_f \sigma, -1) \qquad \chi \in C_s, \tag{35}$$

$$\chi^+ \in G_s(\chi, \zeta) := (A_g \sigma + \zeta, [T_1, T_2]) \qquad \chi \in D_s, \tag{36}$$

where A_f and A_g are given in (21) and (22), respectively,

$$\zeta = (0, K_m m + K_c c),$$
$$K_m = -K_g * (I + \mathcal{G}),$$
$$K_c = [(K_g H_g - \gamma I) * \mathcal{G} \quad -K_g * \mathcal{G}],$$

$m = (m_1, m_2, \ldots, m_N)$, $c = (c^x, c^y)$, $c^x = (c_1^x, c_2^x, \ldots, c_N^x)$, and $c^y = (c_1^y, c_2^y, \ldots, c_N^y)$.[3] For this perturbed hybrid system, we have the following result.

Theorem 3.12. Let $0 < T_1 \leq T_2$ be given. Suppose there exist matrices $K_g \in \mathbb{R}^{nN \times p}$ and $P \in \mathbb{R}^{2nN \times 2nN}$ such that $P = P^\top > 0$ and condition (25) holds. Then, the set $\mathcal{A}$ is ISS with respect to measurement noise and communication noise, i.e., each $\phi \in \mathcal{S}_{\mathcal{H}_s}$ satisfies, for any $(t, j) \in \operatorname{dom} \phi$,

$$|\phi(t,j)|_{\mathcal{A}} \leq \max\left\{\sqrt{2\tfrac{\alpha_2}{\alpha_1}}\lambda_d^{j/2}|\phi(0,0)|_{\mathcal{A}}, \widetilde{\gamma}_m(j)|m|_\infty, \widetilde{\gamma}_c(j)|c|_\infty\right\},$$

where $\lambda_d = 1 - \frac{\mu}{\alpha_2} + \epsilon \in (0,1)$, $\widetilde{\gamma}_m(j) = 2\sqrt{\frac{\rho}{\alpha_1}\frac{1-\lambda_d^j}{1-\lambda_d}}|K_m|$, $\widetilde{\gamma}_c(j) = 2\sqrt{\frac{\rho}{\alpha_1}\frac{1-\lambda_d^j}{1-\lambda_d}}|K_c|$, $\rho = \frac{1}{\epsilon\alpha_1}w_1 + w_2$, $\epsilon \in \left(0, \frac{\mu}{\alpha_2}\right)$, $\mu \in \left(0, \min\left\{\alpha_2, -\hat{\beta}\right\}\right)$,

$$\alpha_1 = \min_{\nu \in [0,T_2]} \underline{\lambda}\left(\exp(A_f^\top \nu) P \exp(A_f \nu)\right),$$

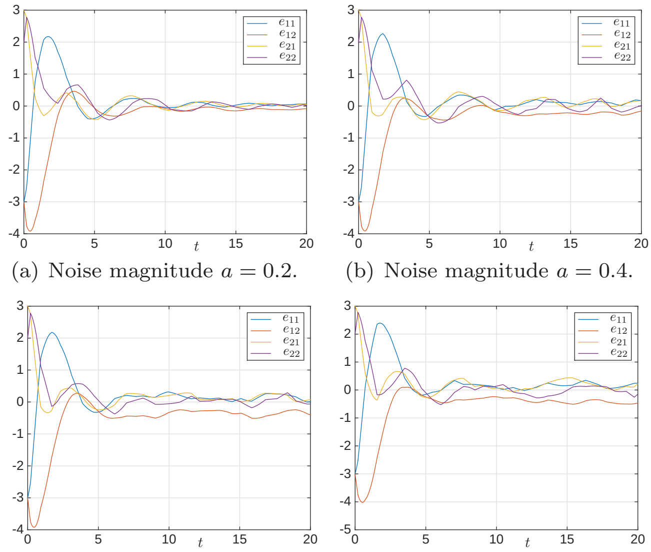
$$\alpha_2 = \max_{\nu \in [0,T_2]} \overline{\lambda}\left(\exp(A_f^\top \nu) P \exp(A_f \nu)\right),$$

$$w_1 = \max_{\nu \in [T_1,T_2]} |\exp(A_f^\top \nu) P \exp(A_f \nu) A_g|^2,$$

$$w_2 = \max_{\nu \in [T_1,T_2]} |\exp(A_f^\top \nu) P \exp(A_f \nu)|,$$

$$\hat{\beta} = \max_{\nu \in [T_1,T_2]} \overline{\lambda}(\exp(\delta\nu) A_g^\top \exp(A_f^\top \nu) P \exp(A_f \nu) A_g - P).$$

[3] Note that the Khatri-Rao product $-K_g * (I + \mathcal{G})$ is such that the (i,k)-th entry K_{ik} of K_g is multiplied by the (i,k)-th scalar entry of the matrix $I + \mathcal{G}$ for all $i, k \in \mathcal{V}$.



(a) Noise magnitude $a = 0.2$. (b) Noise magnitude $a = 0.4$.

(c) Noise magnitude $a = 0.6$. (d) Noise magnitude $a = 0.8$.

Fig. 3. Effect of measurement noise and communication noise for the system in Example 3.13.

Example 3.13. To show the effect of communication noise and measurement noise, we revisit Example 3.7. Moreover, we consider the communication noise and measurement noise is generated randomly and with values in $[0, a]$, where $a \in \mathbb{R}_{\geq 0}$. With the same set of parameters, four simulations are shown in Figure 3. It can be seen that the steady-state estimation error increases with the size of the measurement and communication noises. △

Remark 3.14. Since the hybrid system $\mathcal{H}_s$ satisfies the hybrid basic conditions and, under the conditions in Theorem 3.5, the compact set $\mathcal{A}$ is UGAS for $\mathcal{H}_s$, it follows from (Goebel et al., 2012, Lemma 7.20) that the stability of $\mathcal{A}$ is robust to general small perturbations. □

4. ASYNCHRONOUS EVENT TIMES

In this section, we study the scenario where each agent receives information asynchronously. To model such mechanism, instead of having one timer for all agents, each agent has a timer τ_i that triggers the events following the dynamics in (5), i.e.,

$$\dot{\tau}_i = -1 \qquad \tau_i \in [0, T_2], \tag{37a}$$

$$\tau_i^+ \in [T_1, T_2] \qquad \tau_i = 0, \tag{37b}$$

for each $i \in \mathcal{V}$. Moreover, for each $i \in \mathcal{V}$, the resulting flow dynamics of η_i is given by (14) when $\tau_i \in [0, T_2]$, and the jump dynamics of η_i is given by (16) when $\tau_i = 0$. Let $e_i = \hat{x}_i - x$ and $e = (e_1, \cdots, e_N)$, $\theta = (\theta_1, \cdots, \theta_N)$, $\tau = (\tau_1, \cdots, \tau_N)$, and

$$\theta_i = K_{ii} y_i^e + \sum_{k \in \mathcal{V}} g_{ik} K_{ik} y_k^e + \gamma \sum_{k \in \mathcal{V}} g_{ik}(e_i - e_k) - \eta_i. \tag{38}$$

Then, the interconnection between (1), (6), (37), and (38) leads to a hybrid system $\mathcal{H}_a = (C_a, f_a, D_a, G_a)$ with state $\chi_a = (\sigma_\theta, \tau) \in \mathcal{X}_a := \mathbb{R}^{nN} \times \mathbb{R}^{nN} \times [0, T_2]^N$, $\sigma_\theta = (e, \theta)$ and data given by

$$f_a(\chi_a) := (A_{f\theta}\sigma_\theta, -\mathbf{1}_N)$$

for each $\chi_a \in C_a = \mathcal{X}_a$, and

$$G_a(\chi_a) := \{G_i(\chi_a) : \chi_a \in D_i, i \in \mathcal{V}\}$$

when $\chi_a \in D_a = \bigcup_{i \in \mathcal{V}} D_i$, $D_i = \{\chi_a \in C_a : \tau_i = 0\}$,

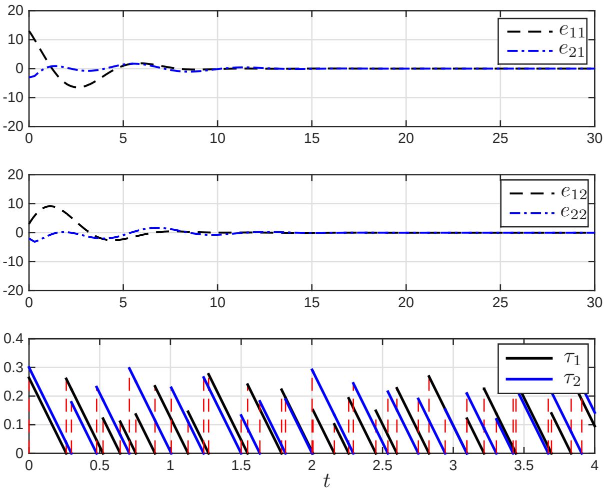


Fig. 4. Estimation errors e_1 and e_2 for the network system in Example 4.3, where $x = (x_1, x_2)$, $\hat{x}_i = (\hat{x}_{i1}, \hat{x}_{i2})$, $e_{i1} = \hat{x}_{i1} - x_1$, $e_{i2} = \hat{x}_{i2} - x_2$, for $i \in \{1, 2\}$. Initial conditions are $x(0,0) = (2,2)$, $\hat{x}_1(0,0) = (5,5)$, $\hat{x}_2(0,0) = (-1,0)$, $\eta_1(0,0) = (1,1)$, $\eta_2(0,0) = (-1,-1)$, $\tau(0,0) = 0.2$.

$$G_i(\chi_a) = \begin{bmatrix} e \\ (\theta_1, \theta_2, \cdots, \theta_{i-1}, 0, \theta_{i+1}, \cdots, \theta_N) \\ (\tau_1, \tau_2, \cdots, \tau_{i-1}, [T_1, T_2], \tau_{i+1}, \cdots, \tau_N) \end{bmatrix},$$

where $A_{f\theta}$ is given in (29). In this case, the set of interest is $\mathcal{A}_a = \{0_{nN}\} \times \{0_{nN}\} \times [0, T_2]^N$.

Theorem 4.1. Let $0 < T_1 \le T_2$ be given. Suppose N agents are connected via a digraph $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{G})$. Moreover, suppose there exist $\delta > 0$ and matrices $K_g \in \mathbb{R}^{nN \times p}$, $P_i, Q_i \in \mathbb{R}^{n \times N}$ satisfying $P_i = P_i^\top > 0$, $Q_i = Q_i^\top > 0$ for all $i \in \mathcal{V}$, and

$$\begin{bmatrix} \mathrm{He}(A_\theta, P) & -P + A_\theta^\top \mathcal{K}^\top \widetilde{Q}(\nu) \\ \star & -\delta \widetilde{Q}(\nu) - \mathrm{He}(\mathcal{K}, \widetilde{Q}(\nu)) \end{bmatrix} < 0 \qquad (39)$$

for all $\nu = (\nu_1, \nu_2, \ldots, \nu_N) \in [0, T_2]^N$, where $P = \mathrm{diag}(P_1, P_2, \cdots, P_N)$,

$$\widetilde{Q}(\nu) = \mathrm{diag}\left(\widetilde{Q}_1(\nu_1), \widetilde{Q}_2(\nu_2), \cdots, \widetilde{Q}_N(\nu_N)\right)$$

and $\widetilde{Q}_i(\nu_i) = \exp(\delta \nu_i) Q_i$ for each $i \in \mathcal{V}$. Then, the set $\mathcal{A}_a$ is UGAS for the hybrid system $\mathcal{H}_a$.

Condition (39) needs to be checked over a closed set $[0, T_2]^N$, which might be a difficult task. The following result relaxes this requirement.

Proposition 4.2. Let $T_2 > 0$ be given. The inequality in (39) holds if there exists $\delta > 0$ and matrices $K_g \in \mathbb{R}^{nN \times p}$, $P_i, Q_i \in \mathbb{R}^{n \times N}$ satisfying $P_i = P_i^\top > 0$, $Q_i = Q_i^\top > 0$ for all $i \in \mathcal{V}$ such that

$$\begin{bmatrix} \mathrm{He}(A_\theta, P) & -P + A_\theta^\top \mathcal{K}^\top Q \\ \star & -\delta Q - \mathrm{He}(\mathcal{K}, Q) \end{bmatrix} < 0, \qquad (40)$$

$$\begin{bmatrix} \mathrm{He}(A_\theta, P) & -P + \exp(\delta T_2) A_\theta^\top \mathcal{K}^\top Q \\ \star & -\exp(\delta T_2)(\delta Q + \mathrm{He}(\mathcal{K}, Q)) \end{bmatrix} < 0, \qquad (41)$$

where $P = \mathrm{diag}(P_1, \ldots, P_N)$ and $Q = \mathrm{diag}(Q_1, \ldots, Q_N)$.

Example 4.3. Consider the network system in Example 3.7. Using the same parameters K_{11}, K_{12}, K_{21}, K_{22}, and γ therein, it can be verified that (40) and (41) hold for $\delta = 10$ and a maximum value of $T_2 \approx 0.3$. As expected, for the case of asynchronous event times, the maximum value of T_2 allowed is smaller than the one obtained for synchronous event times ($T_2 = 1$). A simulation [4] is shown in Figure 4 with $T_1 = 0.1$ and $T_2 = 0.3$. △

5. CONCLUSION

In this paper, a distributed state observer under intermittent information communication is proposed. In contrast to classic observers for linear time-invariant systems, a node with enough information from its neighbors can estimate the plant state even without detectability or even taking measurements of the plant output. Sufficient conditions that guarantee UGAS of the convergence of estimation error to zero are presented. Though not explored in detail, these conditions are expected to also guarantee global exponential stability. Furthermore, robustness of the stability under certain conditions with respect to communication and measurement noises are studied in terms of ISS.

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[4] Code at `https://github.com/HybridSystemsLab/ObsSyncTimes2nd`.