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A Minicourse in Lie Perturbative Methods*

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ABSTRACT

We give a skeletal presentation of the mechanics of Lie perturbative methods. The discussion is intended to enable the reader to begin to use the methods himself. The technique is illustrated by a specific example, in which we derive the ponderomotive Hamiltonian for a particle moving in a curl-free electric field.

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Given any function G of the phase variables $\underline{z} \equiv (\underline{q},\underline{p})$, we define the operator $\check{G}$ by

$$\check{G} \equiv \{G,\ \} \equiv \partial G/\partial \underline{q}\cdot\partial/\partial \underline{p} - \partial G/\partial \underline{p}\cdot\partial/\partial \underline{q}. \tag{1}$$

One can show[1] that the operator $T_G \equiv e^{\check{G}} \equiv \sum_{n=0}^{\infty}(1/n!)\check{G}^n$ induces a canonical transformation (C.T.).

Now if we have a Hamiltonian of the form

$$H^0(\underline{z}) \equiv H(\underline{z}) \equiv \sum_{m=0}^{\infty} \varepsilon^m H_m(\underline{z}) \tag{2}$$

where $H_0 = H_0(\underline{p})$ (so that the zero-order problem is exactly soluble). we can induce C.T.s of the type T_G, to remove the $\underline{q}$-dependent part of H, to successively higher orders in the perturbation parameter ε, as follows. We write

$$\begin{aligned} H^1 &\equiv e^{\varepsilon\check{G}_1}H_0 = \left(1 + \varepsilon\check{G}_1 + \tfrac{1}{2}\varepsilon^2\check{G}_1^2 + \ldots\right)\left(H_0 + \varepsilon H_1 + \varepsilon^2 H_2 + \ldots\right) \\ &= H_0 + \varepsilon\left(\check{G}_1 H_0 + H_1\right) + \varepsilon^2\left(\tfrac{1}{2}\check{G}_1^2 H_0 + \check{G}_1 H_1 + H_2\right) + \mathcal{O}(\varepsilon^3). \end{aligned} \tag{3}$$

Writing $H^1 = \sum_{m=0}^{\infty} \varepsilon^m H_m^1$, we read off the H_m^1 from Eq.(3), to as high order as desired. Up to $\mathcal{O}(\varepsilon^2)$, one has

$$\begin{aligned} &H_0^1 \equiv H_0, \quad H_1^1 \equiv \check{G}_1 H_0 + H_1 = -\check{H}_0 G_1 + H_1, \\ &H_2^1 \equiv \frac{1}{2}\check{G}_1^2 H_0 + \check{G}_1 H_1 + H_2 . \end{aligned} \tag{4}$$

Here we have used the property $\check{G}F = -\check{F}G$, which follows from (1), in obtaining the second form for H_1^1.

We now remove the $\underline{q}$ dependence to $\mathcal{O}(\varepsilon)$, and thereby determine the generator G_1, by stipulating that

$$H_1^1 \equiv -\check{H}_0 G_1 + H_1 = \bar{H}_1, \tag{5}$$

where $\bar{H}_1 = \bar{H}_1(\underline{p})$ means an average over the zero-order orbit of $H_1(\underline{q},\underline{p})$. With this differential equation defining G_1, we have subtracted all the secularity out of G_1 and the C.T. it induces (i.e. G_1 oscillates about zero), in addition to removing the $\underline{q}$-dependence in H up to (ε^2). To see the former property, we note from Eq. (1) that $-\check{H}_0 = (d/dt)_0$, the time derivative along the unperturbed trajectory, and so

$$G_1 = \int^t_{(0\text{-traj.})} dt'(\bar{H}_1 - H_1), \tag{6}$$

where by definition of $\bar{H}_1$, the right-hand side is nonsecular.

Now we proceed analogously to second order.

$$\begin{aligned} H^2 &\equiv e^{\varepsilon^2 \check{G}_2} H^1 = (1 + \varepsilon^2 \check{G}_2 + \ldots)(H_0^1 + \varepsilon H_1^1 + \varepsilon^2 H_2^1 + \ldots) \\ &\equiv K^1(\underline{p}) + \varepsilon^2 H_2^2 + \mathcal{O}(\varepsilon^3), \end{aligned} \tag{7}$$

where $K^1(\underline{p}) \equiv H_0^1 + \varepsilon H_1^1$ (the $\underline{q}$-independent part of H^1), and

$$H_2^2 \equiv -\check{H}_0 G_2 + H_2^1 = -\check{H}_0 G_2 + \tfrac{1}{2}\check{G}_1(\bar{H}_1 + H_1) + H_2. \tag{8}$$

The second form for H_2^2 here comes from the third of Eqs.(4) and Eq. (5). Analogous to (5), we determine G_2, and remove the $\underset{\sim}{q}$-dependence of H^2 to $\mathcal{O}(\varepsilon^2)$, by requiring

$$H_2^2 \equiv -\check{H}_0 G_2 + H_2^1 = \overline{H_2^1} \ . \tag{9}$$

Thus

$$G_2 = \int_{(0\text{-traj})}^{t} (\overline{H_2^1} - H_2^1) \tag{10}$$

The extension of this process to arbitrary order in ε should be clear.

Time Dependent H

In the event that $H = H(\underset{\sim}{z},t)$, we reduce the problem to the situation already treated by extending our phase space to have one additional degree of freedom, with new coordinate t, conjugate momentum E, and Hamiltonian

$$\mathcal{H}(\underset{\sim}{q},t;\underset{\sim}{p},E) \equiv H(\underset{\sim}{q},\underset{\sim}{p},t) + E. \tag{11}$$

The system develops in a new time variable θ (so $\dot{f} \equiv df/d\theta$). The equations of motion are then

$$\dot{t} = \partial\mathcal{H}/\partial E = 1 \qquad \text{(hence } t = \theta), \tag{12a}$$

$$\dot{E} = -\partial\mathcal{H}/\partial t = -\partial H/\partial t, \quad \text{and} \tag{12b}$$

$$\dot{\underset{\sim}{q}} = \partial H/\partial \underset{\sim}{p}, \quad \dot{\underset{\sim}{p}} = -\partial H/\partial \underset{\sim}{q}, \tag{12c}$$

as before. [We may think of the term E in $\mathcal{H}$ as a reservoir, feeding energy in and out of the term H representing our original system, at just the rate needed to give H its time dependence. Thus Eq. (12b) is to be expected.] We now treat this θ-independent Hamiltonian, with exactly soluble zero order part $\mathcal{H}_0(\underline{p},E) \equiv H_0(\underline{p}) + E$, just as described above.

Example

We consider the motion of a particle, moving freely except for the perturbing influence of any number of electrostatic plane waves, each having wave vector $\underline{k}_\ell$ and frequency ω_ℓ. The Hamiltonian is then

$$H(\underline{z},t) = H_0(\underline{p}) + \varepsilon H_1(\underline{q},t), \tag{13}$$

where $H_0(p) \equiv p^2/(2m)$, $H_1(\underline{q},t) \equiv \sum_\ell V_\ell \exp i(\underline{k}_\ell \cdot \underline{q} - \omega_\ell t)$, and $V_{-\ell} = V_\ell^*$, and $(\underline{k}_{-\ell},\omega_{-\ell}) = -(\underline{k}_\ell,\omega_\ell)$. The unperturbed trajectories of $\mathcal{H} = H + E$ are given by

$$\begin{aligned} \underline{p}(\theta) &= \underline{p} = \text{constant}, \quad E = \text{constant}, \\ \underline{q}(\theta) &= q(0) + \theta\underline{v}, \qquad t = \theta \ , \end{aligned} \tag{14}$$

where $\underline{v} \equiv \partial H/\partial\underline{p} = \underline{p}/m$. We thus have $\bar{\mathcal{H}}_1 = \bar{H}_1 = 0$, and so Eq. (5) reads

$$0 = \mathcal{H}_1^1 \equiv H_1 + \{G_1, \mathcal{H}_0\}$$

$$= H_1 + \partial G_1/\partial t + (\partial G_1/\partial \underline{q})\cdot(\partial G_1/\partial \underline{p}) - (\partial G_1/\partial \underline{p})\cdot(\partial G_1/\partial \underline{q}) \tag{15}$$

$$\equiv H_1 + dG_1/d\theta)_0.$$

Thus

$$G_1(\underline{z},t) = -\int_{(0\ \mathrm{traj.})}^{\theta} d\theta' \sum_{\ell} V_\ell \exp i[\underline{k}_\ell\cdot(q(0) + \theta'\underline{v}) - \omega_\ell\theta']$$

$$= -i\sum_{\ell} [V_\ell \exp i(\underline{k}_\ell\cdot\underline{q} - \omega_\ell t)](\omega_\ell - \underline{k}_\ell\cdot\underline{v})^{-1} \tag{16}$$

From Eq. (8), we have, finally

$$\mathcal{H}_2^2 = \overline{\mathcal{H}_2^1} = \frac{1}{2}\,\overline{\check{G}_1 H_1} \equiv \frac{1}{2}\overline{\{G_1, H_1\}}\ . \tag{17}$$

Eq. (17) is the standard expression for the ponderomotive Hamiltonian. Using (16) in (17), one readily obtains the more explicit form

$$\mathcal{H}_2^2 = (2m)^{-1} \sum_{\ell} |V_\ell \underline{k}_\ell|^2 (\omega_\ell - \underline{k}_\ell\cdot\underline{v})^{-2}. \tag{18}$$

We now see at a calculational level how to obtain this result using Lie methods, as well as how to proceed in other problems. The interested reader is referred to the bibliography for a fuller development of the mathematical theory,[1,2] as well as for applications to problems in plasma physics, such as magnetic field self-generation[3], expressions for the ponderomotive force in magnetized plasmas,[3] mode coupling in inhomogeneous magnetized plasmas,[4] and particle motion and guiding center theory.[5,6]

References

1) A. J. Dragt and J. M. Finn, J. Math Phys. 17, 2215 (1976).

2) J. R. Cary, LBL preprint #6350 (6/78).

3) J. R. Cary, Ph.D. dissertation, University of California at Berkeley (1978-1979).

4) S. Johnston and A. N. Kaufman, Phys. Rev. Letters 40, 1266 (1978).

5) H. E. Mynick, Ph.D. dissertation, University of California at Berkeley (1978-1979).

6) A. J. Dragt and J. M. Finn, J. Geophysical Research 81, 2327 (1976).