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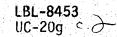
A MINICOURSE IN LIE PERTURBATIVE METHODS

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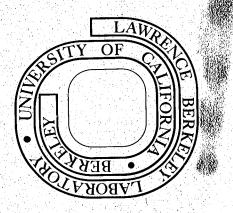
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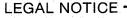
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### A Minicourse in Lie Perturbative Methods\*

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#### ABSTRACT

We give a skeletal presentation of the mechanics of Lie perturbative methods. The discussion is intended to enable the reader to begin to use the methods himself. The technique is illustrated by a specific example, in which we derive the ponderomotive Hamiltonian for a particle moving in a curl-free electric field.

<sup>\*</sup> This research was supported by the U. S. Department of Energy

Given any function G of the phase variables  $\underline{z} \equiv (\underline{q}, \underline{p})$ , we define the operator  $\check{G}$  by

$$\check{G} \equiv \{G, \} \equiv \partial G / \partial q \cdot \partial / \partial p - \partial G / \partial p \cdot \partial / \partial q.$$
(1)

One can show<sup>1</sup> that the operator  $T_G \equiv e^{\check{G}} \equiv \sum_{n=0}^{\infty} (1/n!)\check{G}^n$  induces a canonical transformation (C.T.).

Now if we have a Hamiltonian of the form

$$H^{0}(\underline{z}) \equiv H(\underline{z}) \equiv \sum_{m=0}^{\infty} \varepsilon^{m} H_{m}(\underline{z})$$
(2)

where  $H_0 = H_0(\underline{p})$  (so that the zero-order problem is exactly soluble). we can induce C.T.s of the type  $T_G$ , to remove the <u>q</u>-dependent part of H, to successively higher orders in the perturbation parameter  $\varepsilon$ , as follows. We write

$$H^{1} \equiv e^{\varepsilon G_{1}} H_{0} = \left(1 + \varepsilon \check{G}_{1} + \frac{1}{2} \varepsilon^{2} \check{G}_{1}^{2} + ...\right) \left(H_{0} + \varepsilon H_{1} + \varepsilon^{2} H_{2} + ...\right)$$
(3)  
$$= H_{0} + \varepsilon \left(\check{G}_{1} H_{0} + H_{1}\right) + \varepsilon^{2} \left(\frac{1}{2} \check{G}_{1}^{2} H_{0} + \check{G}_{1} H_{1} + H_{2}\right) + \mathcal{O}(\varepsilon^{3}).$$

Writing  $H^1 = \sum_{m=0}^{\infty} \varepsilon^m H_m^1$ , we read off the  $H_m^1$  from Eq.(3), to as high order as desired. Up to  $\mathcal{O}(\varepsilon^2)$ , one has

$$H_{0}^{1} \equiv H_{0}, \quad H_{1}^{1} \equiv \check{G}_{1}H_{0} + H_{1} = -\check{H}_{0}G_{1} + H_{1},$$

$$H_{2}^{1} \equiv \frac{1}{2}\check{G}_{1}^{2}H_{0} + \check{G}_{1}H_{1} + H_{2}.$$
(4)

Here we have used the property  $\check{G}F = -\check{F}G$ , which follows from (1), in obtaining the second form for  $H_1^1$ .

We now remove the <u>q</u> dependence to  $O(\epsilon)$ , and thereby determine the generator  $G_1$ , by stipulating that

$$H_{1}^{1} = -\check{H}_{0}G_{1} + H_{1} = \bar{H}_{1}, \qquad (5)$$

where  $\overline{H}_1 = \overline{H}_1(\underline{p})$  means an average over the zero-order orbit of  $H_1(\underline{q},\underline{p})$ . With this differential equation defining  $G_1$ , we have subtracted all the secularity out of  $G_1$  and the C.T. it induces (i.e.  $G_1$  oscillates about zero), in addition to removing the <u>q</u>-dependence in H up to  $(\epsilon^2)$ . To see the former property, we note from Eq. (1) that  $-H_0 = (d/dt)_0$ , the time derivative along the unperturbed trajectory, and so

$$G_1 = \int_{(0-\text{traj.})}^{t} dt' (\bar{H}_1 - H_1),$$
 (6)

where by definition of  $\bar{H}_1$ , the right-hand side is nonsecular.

Now we proceed analogously to second order.

$$H^{2} \equiv e^{\varepsilon^{2} \check{G}_{2}} H^{1} = (1 + \varepsilon^{2} \check{G}_{2} + ...) (H^{1}_{0} + \varepsilon H^{1}_{1} + \varepsilon^{2} H^{1}_{2} + ...)$$

$$\equiv \kappa^{1}(\underline{p}) + \varepsilon^{2} H^{2}_{2} + \mathcal{O}(\varepsilon^{3}),$$
(7)

where  $K^{1}(\underline{p}) \equiv H_{0}^{1} + \varepsilon H_{1}^{1}$  (the <u>q</u>-independent part of  $H^{1}$ ), and

$$H_2^2 \equiv -\tilde{H}_0 G_2 + H_2^1 = -\tilde{H}_0 G_2 + \frac{1}{2} \tilde{G}_1 (\tilde{H}_1 + H_1) + H_2.$$
(8)

The second form for  $H_2^2$  here comes from the third of Eqs.(4) and Eq. (5). Analogous to (5), we determine  $G_2$ , and remove the <u>q</u>-dependence of  $H^2$  to  $O(\epsilon^2)$ , by requiring

$$H_2^2 \equiv -\check{H}_0 G_2 + H_2^1 = \overline{H_2^1}$$
 (9)

Thus

$$G_{2} = \int_{(0-traj)}^{t} (\overline{H}_{2}^{1} - H_{2}^{1})$$
(10)

The extension of this process to arbitrary order in  $\ensuremath{\epsilon}$  should be clear.

Time Dependent H

In the event that  $H = H(\underline{z},t)$ , we reduce the problem to the situation already treated by extending our phase space to have one additional degree of freedom, with new coordinate t, conjugate momentum E, and Hamiltonian

$$H(q,t;p,E) \equiv H(q,p,t) + E.$$
(11)

The system develops in a new time variable  $\theta$  (so  $f \equiv df/d\theta$ ). The equations of motion are then

$$t = \partial H/\partial E = 1$$
 (hence  $t = \theta$ ), (12a)

$$\dot{\mathbf{E}} = -\partial H/\partial t = -\partial H/\partial t$$
, and (12b)

$$\dot{\mathbf{q}} = \partial \mathbf{H} / \partial \mathbf{p}, \quad \dot{\mathbf{p}} = -\partial \mathbf{H} / \partial \mathbf{q}, \quad (12c)$$

as before. [We may think of the term E in H as a reservoir, feeding energy in and out of the term H representing our original system, at just the rate needed to give H its time dependence. Thus Eq. (12b) is to be expected.] We now treat this  $\theta$ -independent Hamiltonian, with exactly soluble zero order part  $H_0(\underline{p}, E) \equiv H_0(\underline{p}) + E$ , just as described above.

#### Example

We consider the motion of a particle, moving freely except for the perturbing influence of any number of electrostatic plane waves, each having wave vector  $\underline{k}_{g}$  and frequency  $\omega_{g}$ . The Hamiltonian is then

$$H(\underline{z},t) = H_0(\underline{p}) + \varepsilon H_1(\underline{q},t), \qquad (13)$$

where  $H_0(\underline{p}) \equiv p^2/(2m)$ ,  $H_1(\underline{q},t) \equiv \sum_{\ell} V_{\ell} \exp i(\underline{k}_{\ell} \cdot \underline{q} - \omega_{\ell} t)$ , and  $V_{-\ell} = V_{\ell}^*$ , and  $(\underline{k}_{-\ell}, \omega_{-\ell}) = -(\underline{k}_{\ell}, \omega_{\ell})$ . The unperturbed trajectories of H = H + E are given by

 $\underline{p}(\theta) = \underline{p} = \text{constant}, \quad \mathbf{E} = \text{constant},$ 

(14)

$$q(\theta) = q(0) + \theta v, \quad t = \theta,$$

where  $\underline{v} \equiv \partial H/\partial \underline{p} = \underline{p}/m$ . We thus have  $\overline{H}_1 = \overline{H}_1 = 0$ , and so Eq. (5) reads

$$0 = H_1^1 \equiv H_1 + \{G_1, H_0\}$$
  
=  $H_1 + \partial G_1 / \partial t + (\partial G_1 / \partial g) \cdot (\partial G_1 / \partial p) - (\partial G_1 / \partial p) \cdot (\partial G_1 / \partial g)$  (15)  
=  $H_1 + dG_1 / d\theta_0$ .

Thus

$$G_{1}(\underline{z},t) = -\int_{(0 \text{ traj.})}^{\theta} d\theta' \sum_{\ell} V_{\ell} \exp i[\underline{k}_{\ell} \cdot (q(0) + \theta' \underline{v}) - \omega_{\ell} \theta']$$

$$= -i \sum_{\ell} [V_{\ell} \exp i(\underline{k}_{\ell} \cdot \underline{q} - \omega_{\ell} t)] (\omega_{\ell} - \underline{k}_{\ell} \cdot \underline{v})^{-1}$$
(16)

From Eq. (8), we have, finally

$$H_2^2 = \overline{H_2^1} = \frac{1}{2} \,\overline{\breve{G}_1 H_1} \equiv \frac{1}{2} \,\overline{(G_1, H_1)} \,. \tag{17}$$

Eq. (17) is the standard expression for the ponderomotive Hamiltonian. Using (16) in (17), one readily obtains the more explicit form

$$H_2^2 = (2m)^{-1} \sum_{\mathcal{Q}} |\mathbf{v}_{\mathcal{Q}} \mathbf{k}_{\mathcal{Q}}|^2 (\omega_{\mathcal{Q}} - \mathbf{k}_{\mathcal{Q}} \cdot \mathbf{y})^{-2}.$$
(18)

We now see at a calculational level how to obtain this result using Lie methods, as well as how to proceed in other problems. The interested reader is referred to the bibliography for a fuller development of the mathematical theory,<sup>1,2</sup> as well as for applications to problems in plasma physics, such as magnetic field self-generation<sup>3</sup>, expressions for the ponderomotive force in magnetized plasmas,  $^3$  mode coupling in inhomogeneous magnetized plasmas,  $^4$  and particle motion and guiding center theory.  $^{5,6}$ 

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