Solution of Eigenproblems for Damped Structural Systems by the Lanczos Algorithm

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Abstract

A variant of the Lanczos algorithm is deduced to solve the eigenproblem arising in the analysis of viscously damped structural systems. Re-orthogonalization schemes are employed to restore the required orthogonality between the Lanczos vectors. A projection of the original eigenproblem onto the Krylov subspace spanned by the Lanczos vectors gives a standard tri-diagonal eigenproblem, of which the solutions are the Ritz approximations to the eigenpairs sought. An advantage of the algorithm is the fact that the Lanczos vectors are all real, even though the final solution is complex. A number of problems are solved by the algorithm and the results are very good.

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1. Introduction

In the analysis of dynamic response of linear structures, the equation of motion of a damped system can be expressed as

$$\mathbf{M}\,\ddot{\mathbf{q}}(t) + \mathbf{C}\,\dot{\mathbf{q}}(t) + \mathbf{K}\,\mathbf{q}(t) = \mathbf{f}(t) \tag{1.1}$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are, respectively, the $n \times n$ mass, damping, and stiffness matrices and $\ddot{\mathbf{q}}(t)$, $\dot{\mathbf{q}}(t)$, and $\mathbf{q}(t)$ are the $n \times 1$ acceleration, velocity, and displacement vectors. To find the free vibration solution of the system, we first solve eqn (1.1) for the homogeneous solution, which is of the form

$$\mathbf{q}(t) = e^{\lambda t}\,\mathbf{w} \tag{1.2}$$

Substituting this solution into eqn (1.1), we obtain the characteristic equation

$$(\lambda^2\,\mathbf{M} + \lambda\,\mathbf{C} + \mathbf{K})\,\mathbf{w} = 0 \tag{1.3}$$

where λ and $\mathbf{w}$ are the eigenvalue and eigenvector of the system and both are complex-valued in general. This quadratic characteristic problem can be reduced to a linear one by doubling the order of the system [1]. Accordingly,

$$\lambda \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \lambda\mathbf{w} \end{bmatrix} = \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \lambda\mathbf{w} \end{bmatrix} \tag{1.4}$$

which may be written as

$$\lambda\,\mathbf{A}\,\mathbf{z} = \mathbf{B}\,\mathbf{z} \tag{1.5}$$

with

$$\mathbf{A} = \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} \mathbf{w} \\ \lambda\mathbf{w} \end{bmatrix} \tag{1.6}$$

In this study, we are interested in nongyroscopic, damped systems. The associated $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ matrices are all symmetric. Hence the matrices $\mathbf{A}$ and $\mathbf{B}$ are also symmetric although neither is positive definite.

While eqn (1.5) is similar in form to the eigenproblem associated with an undamped system, i.e.,

$$\omega^2 \mathbf{M}\,\mathbf{w} = \mathbf{K}\,\mathbf{w} \tag{1.7}$$

there are important differences between the two problems. First, the pencil $(\mathbf{A}, \mathbf{B})$ is twice the size of the pencil $(\mathbf{M}, \mathbf{K})$. Second, both the $\mathbf{M}$ and the $\mathbf{K}$ matrices are positive definite, or at least positive semi-definite; whereas neither $\mathbf{A}$ or $\mathbf{B}$ is definite. Last, the solutions of $\lambda\,\mathbf{A}\,\mathbf{z} = \mathbf{B}\,\mathbf{z}$ are complex-valued while the solutions of $\omega^2\,\mathbf{M}\,\mathbf{w} = \mathbf{K}\,\mathbf{w}$ are real-valued. Due to these differences, the solution of $\lambda\,\mathbf{A}\,\mathbf{z} = \mathbf{B}\,\mathbf{z}$ is more complicated than the solution of $\omega^2\,\mathbf{M}\,\mathbf{w} = \mathbf{K}\,\mathbf{w}$ and therefore needs more computational effort and storage.

In this paper, we first employ an orthogonal projection method to reduce the original problem $\lambda\,\mathbf{A}\,\mathbf{z} = \mathbf{B}\,\mathbf{z}$ to a smaller one. To obtain an effective reduced problem, we choose a Krylov subspace generated from $\mathbf{B}^{-1}\mathbf{A}$ to construct the reduction. We deduce a variant of the Lanczos algorithm to account for the indefiniteness of the matrix pencil $(\mathbf{A}, \mathbf{B})$. The infamous loss of orthogonality between Lanczos vectors is discussed and schemes to restore the orthogonality or the semi-orthogonality are presented. The matrix pencil $(\mathbf{A}, \mathbf{B})$ is then projected onto the Krylov subspace spanned by the Lanczos vectors to obtain a reduced tri-diagonal system. The solutions of this reduced system are taken as the approximate eigensolutions of $(\mathbf{A}, \mathbf{B})$. The quality of this approximation can be measured by the residual vectors, which are readily obtained. In the end, the efficiency of the algorithm is shown by solving numerical examples.

2 Orthogonal Projection Method

In practical analysis, only the lower modes (i.e., those associated with the smallest $|\lambda|$) of a large dynamic system are required as they are the most important from a physical point of view. In addition, the mathematical idealization of any complex dynamic system also tends to be less reliable in predicting the higher modes. Consequently, the higher modes are of little practical interest. For these reasons, we are interested only in the lower mode solutions of the eigenproblem $\lambda\,\mathbf{A}\,\mathbf{z} = \mathbf{B}\,\mathbf{z}$. To find only the lower mode solutions, we employ an orthogonal projection method to obtain a reduced eigenproblem from the original eigenproblem. In practice, the order m of the reduced problem is considerably

smaller than the order n (or $2n$) of the original problem and hence a substantial computational effort can be eliminated by solving the reduced problem instead of the original problem. Nevertheless, some solutions of the reduced problem are very good approximations to the lower mode solutions of the original problem. In the following, we briefly describe how the orthogonal projection method can be used to reduce the eigenproblem $\lambda\ \mathbf{A}\ \mathbf{z} = \mathbf{B}\ \mathbf{z}$ to a smaller one.

Consider an order m subspace S of the solution space, referred to as the *admissible space*. The orthogonal projection method consists in using a vector $\mathbf{y}_j$ in subspace S to approximate an exact eigenvector $\mathbf{z}_j$ by requiring that the residual vector of $\mathbf{y}_j$ satisfy the Galerkin condition, i.e., the residual vector of $\mathbf{y}_j$ is orthogonal to the subspace S.

The orthogonal projection method applied to solve $\lambda\ \mathbf{A}\ \mathbf{z} = \mathbf{B}\ \mathbf{z}$ seeks approximate eigenvalues θ_j and their associated eigenvectors $\mathbf{y}_j$, which belong to the admissible subspace S, such that the following Galerkin condition is satisfied :

$$\theta_j\ \mathbf{A}\ \mathbf{y}_j - \mathbf{B}\ \mathbf{y}_j \ \mid\ S \tag{2.1}$$

Let $\mathbf{Q}$ be an arbitrary orthogonal basis of S; then we can replace the preceding equation by

$$\mathbf{Q}^T\ (\ \theta_j\ \mathbf{A}\ \mathbf{y}_j - \mathbf{B}\ \mathbf{y}_j\ ) = \mathbf{0} \tag{2.2}$$

The fact that $\mathbf{y}_j$ is in the subspace of S requires $\mathbf{y}_j = \mathbf{Q}\ \mathbf{s}_j$, where $\mathbf{s}_j$ are a set of free parameters. The choice of the free parameters in $\mathbf{s}_j$ is made to satisfy the Galerkin condition eqn (2.2), that is, $\mathbf{s}_j$ is the solution of

$$\mathbf{Q}^T\ (\ \theta_j\ \mathbf{A}\ \mathbf{Q} - \mathbf{B}\ \mathbf{Q}\ )\ \mathbf{s}_j = \mathbf{0} \tag{2.3}$$

This equation can be put in the following form :

$$\theta_j\ \mathbf{A}^*\ \mathbf{s}_j = \mathbf{B}^*\ \mathbf{s}_j \tag{2.4}$$

where $\mathbf{A}^*$ and $\mathbf{B}^*$ are the m by m projected matrices given by

$$\mathbf{A}^* = \mathbf{Q}^T\ \mathbf{A}\ \mathbf{Q} \qquad\qquad \mathbf{B}^* = \mathbf{Q}^T\ \mathbf{B}\ \mathbf{Q} \tag{2.5}$$

We note that eqn (2.4) actually represents an eigenproblem of the same form as eqn (1.5) but of reduced size. The solution of the reduced problem yields $\theta_1, \ldots, \theta_m$, which are

the Ritz values, and $s_1, \ldots, s_m$, which are the parameters used to form the Ritz vectors $\mathbf{Y} = [\, y_1, \ldots, y_m \,]$ by the relation $y_j = \mathbf{Q}\, s_j$. These Ritz pairs (θ_j, y_j) are the optimal approximations to the eigenpairs sought from the admissible subspace S in the sense that $\theta_j\, \mathbf{A}\, y_j - \mathbf{B}\, y_j$ is orthogonal to S.

It is evident from the above description that any subspace can be used to construct a reduced eigenproblem. However, the key point is that the desired eigenvectors should be accurately approximated by vectors of the admissible subspace S. In the following, a Krylov subspace generated from the operator $\mathbf{D} = \mathbf{B}^{-1}\mathbf{A}$ and an arbitrary vector $\mathbf{q}$, that is, $span(\mathbf{q}, \mathbf{Dq}, \mathbf{D}^2\mathbf{q}, \ldots, \mathbf{D}^{m-1}\mathbf{q})$, is chosen as the admissible subspace for the reduction of a large eigenproblem. A variant of the Lanczos algorithm is deduced to generate an orthogonal basis for this Krylov subspace. A major advantage of using this basis, as shown in Section 5, is that the projected eigenproblem is a standard eigenproblem with a tri-diagonal coefficient matrix.

3 Lanczos Algorithm for Indefinite Matrix Pencil

The Lanczos algorithm was introduced in 1950 as an efficient method to extract some eigenvalues and associated eigenvectors of a symmetric standard eigenproblem. The algorithm may also be used to solve $\mathbf{K}\, \mathbf{w} = \omega^2\, \mathbf{M}\, \mathbf{w}$, since this generalized eigenproblem can be transformed into a standard one which is still symmetric. However, we can avoid this transformation by working directly with $\mathbf{K}_\sigma^{-1}\, \mathbf{M}\, \mathbf{w} = \dfrac{1}{\omega^2}\, \mathbf{w}$ where $\mathbf{K}_\sigma = \mathbf{K} - \sigma\, \mathbf{M}$ and σ is a shift. Although $\mathbf{K}_\sigma^{-1}\, \mathbf{M}$ is not symmetric, it is self-adjoint with respect to the M-weighted inner product defined by $(\mathbf{u}, \mathbf{v})_\mathbf{M} = \mathbf{u}^\mathrm{T}\, \mathbf{M}\, \mathbf{v}$ [2]. Given a starting vector, the Lanczos algorithm generates a sequence of vectors which are M-orthonormal to each other. These vectors, known as the Lanczos vectors, are used in the Rayleigh-Ritz procedure to reduce the original system into a smaller symmetric tridiagonal system. The solutions of the reduced tridiagonal system described above can be obtained easily and inexpensively. In addition, usually almost half of the solutions are very good approximates to the eigenpairs of the origin system. These advantages make the Lanczos algorithm an effective

method for the solution of large eigenproblems.

Both matrices of the pencil (**A**, **B**) associated with a damped system are indefinite. Therefore, the weighted inner-product of a vector is not necessarily positive. Except for this *improper* inner-product [4], the rest of the Lanczos algorithm for solving $\omega^2 \mathbf{M}\,\mathbf{w} = \mathbf{K}\,\mathbf{w}$ may be applied to solve $\lambda\,\mathbf{A}\,\mathbf{z} = \mathbf{B}\,\mathbf{z}$. Assuming that the first j Lanczos vectors ($\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_j$) have been found, we describe how to construct the next Lanczos vector $\mathbf{q}_{j+1}$. Here, we require that the $\mathbf{q}_{j+1}$ satisfy the condition $\mathbf{q}_{j+1}^T\,\mathbf{A}\,\mathbf{q}_i = 0$ for all i from 1 to j; that is, the new Lanczos vector is **A**-orthogonal to all the previous Lanczos vectors. To obtain $\mathbf{q}_{j+1}$, a preliminary vector $\bar{\mathbf{q}}_{j+1}$ is first calculated from the previous vector $\mathbf{q}_j$ as in the Krylov sequence.

$$\bar{\mathbf{q}}_{j+1} = \mathbf{B}^{-1}\,\mathbf{A}\,\mathbf{q}_j \tag{3.1}$$

In general, this preliminary vector can be expressed as a linear combination of all the previous Lanczos vectors and a residual vector; namely,

$$\bar{\mathbf{q}}_{j+1} = \hat{\mathbf{q}}_{j+1} + \alpha_j \mathbf{q}_j + \beta_{j-1}\mathbf{q}_{j-1} + \epsilon_{j-2}\mathbf{q}_{j-2} + \ldots \tag{3.2}$$

where $\hat{\mathbf{q}}_{j+1}$ is the residual vector, which is **A**-orthogonal to all previous Lanczos vectors, and α_j, β_{j-1}, ϵ_{j-2}, . . . are the components of $\bar{\mathbf{q}}_{j+1}$ in the directions of the previous Lanczos vectors. These component coefficients can be evaluated by imposing the condition of **A**-orthogonality among the Lanczos vectors. For example, pre-multiplying both sides of eqn (3.2) by $\mathbf{q}_j^T\,\mathbf{A}$, we obtain

$$\mathbf{q}_j^T\,\mathbf{A}\,\bar{\mathbf{q}}_{j+1} = \mathbf{q}_j^T\,\mathbf{A}\,\hat{\mathbf{q}}_{j+1} + \alpha_j \mathbf{q}_j^T\,\mathbf{A}\,\mathbf{q}_j + \tag{3.3}$$

$$\beta_{j-1}\mathbf{q}_j^T\,\mathbf{A}\,\mathbf{q}_{j-1} + \epsilon_{j-2}\mathbf{q}_j^T\,\mathbf{A}\,\mathbf{q}_{j-2} + \ldots$$

Here the first term on the right-hand side vanishes due to **A**-orthogonality, and all terms after the second vanish for the same reason. Hence, the component of $\bar{\mathbf{q}}_{j+1}$ along $\mathbf{q}_j$ can be readily obtained through

$$\alpha_j = \frac{\mathbf{q}_j^T\,\mathbf{A}\,\bar{\mathbf{q}}_{j+1}}{\mathbf{q}_j^T\,\mathbf{A}\,\mathbf{q}_j} \tag{3.4}$$

The component of $\bar{\mathbf{q}}_{j+1}$ along $\mathbf{q}_{j-1}$ may be found similarly by pre-multiplying eqn (3.2) by

$\mathbf{q}_{j-1}^{T}\mathbf{A}$. In this case all terms except the third vanish due to A-orthogonality, so we have

$$\beta_{j-1} = \frac{\mathbf{q}_{j-1}^{T}\,\mathbf{A}\,\bar{\mathbf{q}}_{j+1}}{\mathbf{q}_{j-1}^{T}\,\mathbf{A}\,\mathbf{q}_{j-1}} \tag{3.5}$$

Similarly, the component of $\mathbf{q}_{j-2}$ contained in $\bar{\mathbf{q}}_{j+1}$ is found to be

$$\epsilon_{j-2} = \frac{\mathbf{q}_{j-2}^{T}\,\mathbf{A}\,\bar{\mathbf{q}}_{j+1}}{\mathbf{q}_{j-2}^{T}\,\mathbf{A}\,\mathbf{q}_{j-2}} \tag{3.6}$$

Making use of eqn (3.1) and the fact that the transpose of a scalar and the scalar are identical, we obtain

$$\begin{aligned} \mathbf{q}_{j-2}^{T}\,\mathbf{A}\,\bar{\mathbf{q}}_{j+1} &= \mathbf{q}_{j-2}^{T}\,\mathbf{A}\,\mathbf{B}^{-1}\,\mathbf{A}\,\mathbf{q}_{j} \\ &= \mathbf{q}_{j}^{T}\,\mathbf{A}\,\mathbf{B}^{-1}\,\mathbf{A}\,\mathbf{q}_{j-2} \\ &= \mathbf{q}_{j}^{T}\,\mathbf{A}\,\bar{\mathbf{q}}_{j-1} \end{aligned} \tag{3.7}$$

Next, expanding $\bar{\mathbf{q}}_{j-1}$ in terms of the preceding Lanczos vectors and the residual vector $\hat{\mathbf{q}}_{j-1}$ as in eqn (3.2), we obtain

$$\mathbf{q}_{j}^{T}\,\mathbf{A}\,\bar{\mathbf{q}}_{j-1} = \mathbf{q}_{j}^{T}\,\mathbf{A}\,(\hat{\mathbf{q}}_{j-1} + \alpha_{j-2}\mathbf{q}_{j-2} + \beta_{j-3}\mathbf{q}_{j-3} + \epsilon_{j-4}\mathbf{q}_{j-4} + \ldots) \tag{3.8}$$

Since all terms on the right hand side vanish due to the A-orthogonality, we achieve the anticipated result $\epsilon_{j-2} = 0$. A similar manipulation could be applied to eqn (3.2) to demonstrate that all further terms in the expansion of $\bar{\mathbf{q}}_{j+1}$ vanish. In other words, the $\bar{\mathbf{q}}_{j+1}$ can expressed as the combination of only the previous two Lanczos vectors and the residual vector. Therefore, we can combine eqn (3.1) and eqn (3.2) to give the recurrence formula for deriving the residual vector $\hat{\mathbf{q}}_{j+1}$ as

$$\hat{\mathbf{q}}_{j+1} = \mathbf{B}^{-1}\,\mathbf{A}\,\mathbf{q}_{j} - \alpha_{j}\,\mathbf{q}_{j} - \beta_{j-1}\,\mathbf{q}_{j-1} \tag{3.9}$$

with α_j and β_{j-1} given in eqn (3.4) and (3.5) respectively. The new Lanczos vector is then obtained simply by scaling the residual vector $\hat{\mathbf{q}}_{j+1}$, i.e.,

$$\mathbf{q}_{j+1} = \frac{\hat{\mathbf{q}}_{j+1}}{\gamma_{j+1}} \tag{3.10}$$

where γ_{j+1} is the *pseudo length* of $\hat{\mathbf{q}}_{j+1}$ and is defined as

$$\gamma_{j+1} = (\delta_{j+1}\, \hat{\mathbf{q}}_{j+1}^{\mathrm{T}}\, \mathbf{A}\, \hat{\mathbf{q}}_{j+1})^{1/2} \tag{3.11}$$

with $\delta_{j+1} = sgn\ (\hat{\mathbf{q}}_{j+1}^{\mathrm{T}}\, \mathbf{A}\, \hat{\mathbf{q}}_{j+1})$. Here we use an extra array $\boldsymbol{\delta}$ to store the pseudo length of the Lanczos vectors, which are normalized to be 1 or -1.

In the above derivation, we assumed that γ_{j+1} is not equal to zero. Although in practice it is highly improbable that a zero γ_{j+1} is encountered; we still include the discussion of this situation for the completeness of the proposed algorithm. For the indefinite matrix pencil under consideration, there are two possible alternatives when γ_{j+1} is equal to zero : (1) $\hat{\mathbf{q}}_{j+1}$ is equal to zero, or (2) $\hat{\mathbf{q}}_{j+1}$ is not equal to zero. If the first case occurs, which means that we have captured an *invariant subspace* [2], every Ritz pair of the invariant subspace is an exact eigenpair of the original system. If the second case occurs, which implies that unfortunately we have chosen a defective starting vector, we can start the procedure over by choosing a different starting vector.

Frequently, we encounter problems in which the stiffness matrix **K** is singular. In this case, we perform a spectral shift σ to transform eqn (1.4) into

$$(\lambda-\sigma) \begin{bmatrix} (\mathbf{C}+2\sigma\mathbf{M}) & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ (\lambda-\sigma)\mathbf{w} \end{bmatrix} = \tag{3.12}$$

$$\begin{bmatrix} (-\mathbf{K}-\sigma\mathbf{C}-\sigma^2\mathbf{M}) & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ (\lambda-\sigma)\mathbf{w} \end{bmatrix}$$

where the shifted **A** and **B** matrices remain symmetric. In this formulation, we only need to factor the shifted stiffness matrix $\mathbf{K} + \sigma\, \mathbf{C} + \sigma^2\, \mathbf{M}$, which is still in banded form. An alternative is to form $(\mathbf{B} - \sigma\mathbf{A})\, \mathbf{z} = (\lambda - \sigma)\, \mathbf{A}\, \mathbf{z}$; however it is necessary to factor the entire $(\mathbf{B} - \sigma\mathbf{A})$, which is much more expensive and hence should be avoided.

An algorithm for computing the Lanczos vectors is summarized in Box 1, showing that only $14n + 2\mu(\mathbf{M}) + \mu(\mathbf{C}) + \nu(\mathbf{K})$ multiplications are required to generate each new Lanczos vector, where $\mu(\mathbf{M})$, $\mu(\mathbf{C})$ represents the number of operations to compute $\mathbf{M}\,\mathbf{x}$, $\mathbf{C}\,\mathbf{x}$, respectively, and $\nu(\mathbf{K})$ represents the number of operations to solve $\mathbf{K}\,\mathbf{y} = \mathbf{x}$ for typical n-vectors $\mathbf{x}$, $\mathbf{y}$. To compute m Lanczos vectors, the algorithm only needs the storage

for m $2n$-vectors and 4 m-vectors to keep all the essential information. The algorithm does not need to explicitly form matrices **A** and **B**; moreover, it can take full advantage of the symmetry and sparsity of the matrices **M**, **C**, and **K**.

4 Orthogonality between Lanczos Vectors

The algorithm presented above involves orthogonalization against only the two preceding vectors at each step. In finite precision arithmetic, inevitable rounding errors in computation will generate vectors which are not orthogonal to each other. To be precise about the loss of orthogonality between Lanczos vectors, we measure orthogonality between $\mathbf{q}_j$ and $\mathbf{q}_k$ by

$$\eta_{j\,k} = \mathbf{q}_j^{\mathrm{T}}\,\mathbf{A}\,\mathbf{q}_k \tag{4.1}$$

The $\eta_{j\,k}$ are zero for all $k < j$ and $|\eta_{j\,j}|$ is equal to 1 if the $\mathbf{q}$'s are orthonormal. In finite precision arithmetic, we might expect these $|\eta_{j\,k}|$ to be at ϵ level, where ϵ is the machine round-off error unit. The output from a simple Lanczos solution, however, shows that the computed η's are much larger than ϵ and can even be of the order 1. This loss of orthogonality is unfortunately widespread. To illustrate the loss in orthogonality, we have solved a test problem using the simple Lanczos algorithm given in Box 1. The beam structure shown in Figure 2 is divided into five equal segments and has 10 degrees of freedom. The simple Lanczos algorithm is used to generate 20 Lanczos vectors which is the order of (**A**, **B**). We show in Figure 1 the quantities $\log_{10}(|\eta_{i\,j} - \delta_{ij}\,\delta_j|/\epsilon)$ rounded to the next integer, with δ_{ij} being the Kronecker delta function. In exact arithmetic, all these quantities should be zero.

Figure 1 shows that the orthogonality relation starts to fail at an early stage of the calculation and the growth of η's appears to follow a regular pattern. We may simulate this pattern in the following way. To include the effect induced by rounding errors, we change the three-term recurrence formula eqn (3.9) into

$$\gamma_{j+1}\,\mathbf{q}_{j+1} = \mathbf{B}^{-1}\,\mathbf{A}\,\mathbf{q}_j - \alpha_j\,\mathbf{q}_j - \beta_{j-1}\,\mathbf{q}_{j-1} + \mathbf{f}_{j+1} \tag{4.2}$$

Figure 1 $\log_{10}\left(\dfrac{|\eta_{i\,j} - \delta_{ij}\,\delta_j|}{\epsilon}\right)$

row/col	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*
1	*0*	*1*	*0*	*1*	*3*	*4*	*7*	*8*	*11*	*13*	*15*	*16*	*15*	*12*	*10*	*12*	*13*	*14*	*15*	*15*
2		*0*	*1*	*0*	*2*	*3*	*5*	*7*	*9*	*11*	*14*	*15*	*14*	*11*	*11*	*13*	*14*	*15*	*17*	*17*
3			*0*	*1*	*2*	*2*	*4*	*6*	*8*	*10*	*13*	*14*	*13*	*10*	*11*	*13*	*15*	*16*	*17*	*16*
4				*0*	*1*	*1*	*3*	*4*	*7*	*9*	*11*	*12*	*11*	*11*	*12*	*14*	*15*	*16*	*16*	*16*
5					*0*	*1*	*2*	*3*	*5*	*7*	*10*	*11*	*9*	*10*	*11*	*13*	*14*	*15*	*15*	*15*
6						*0*	*1*	*1*	*4*	*6*	*8*	*9*	*8*	*10*	*11*	*13*	*14*	*15*	*15*	*15*
7							*0*	*1*	*2*	*4*	*6*	*7*	*6*	*8*	*9*	*12*	*13*	*14*	*14*	*14*
8								*0*	*2*	*1*	*5*	*5*	*6*	*8*	*9*	*11*	*12*	*13*	*14*	*14*
9									*0*	*1*	*2*	*3*	*4*	*6*	*7*	*9*	*11*	*12*	*12*	*12*
10										*0*	*2*	*1*	*3*	*5*	*6*	*8*	*10*	*11*	*12*	*12*
11											*0*	*0*	*1*	*3*	*4*	*7*	*8*	*10*	*13*	*13*
12												*0*	*0*	*1*	*4*	*7*	*9*	*11*	*14*	*14*
13													*0*	*2*	*2*	*6*	*8*	*10*	*13*	*13*
14														*0*	*1*	*2*	*5*	*7*	*10*	*10*
15			*S*	*y*	*m*	*m*	*e*	*t*	*r*	*i*	*c*				*0*	*2*	*2*	*5*	*8*	*8*
16																*0*	*1*	*1*	*5*	*4*
17																	*0*	*1*	*3*	*4*
18																		*1*	*3*	*3*
19																			*1*	*2*
20																				*1*

where the n-vector $\mathbf{f}_{j+1}$ accounts for rounding errors introduced during the step and α_j, β_{j-1}, γ_{j+1} and $\mathbf{q}'s$ denote the quantities computed according to Box 1. Pre-multiplying eqn (4.2) by $\mathbf{q}_k^{\mathrm{T}}\,\mathbf{A}$ and using the definition in eqn (4.1), we obtain

$$\gamma_{j+1}\,\eta_{j+1\,k} = \mathbf{q}_k^{\mathrm{T}}\,\mathbf{A}\,\mathbf{B}^{-1}\,\mathbf{A}\,\mathbf{q}_j - \alpha_j\,\eta_{j\,k} - \beta_{j-1}\,\eta_{j-1\,k} + \mathbf{q}_k^{\mathrm{T}}\,\mathbf{A}\,\mathbf{f}_{j+1} \tag{4.3}$$

Interchanging the index j and k of eqn (4.3) we obtain a similar expansion :

$$\gamma_{k+1}\,\eta_{k+1\,j} = \mathbf{q}_j^{\mathrm{T}}\,\mathbf{A}\,\mathbf{B}^{-1}\,\mathbf{A}\,\mathbf{q}_k - \alpha_k\,\eta_{k\,j} - \beta_{k-1}\,\eta_{k-1\,j} + \mathbf{q}_j^{\mathrm{T}}\,\mathbf{A}\,\mathbf{f}_{k+1} \tag{4.4}$$

From the symmetry of $\mathbf{A}\,\mathbf{B}^{-1}\,\mathbf{A}$, the first term on the right-hand side of eqn (4.3) and (4.4) are equal and therefore can be eliminated by subtraction, resulting in

$$\gamma_{j+1}\,\eta_{j+1\,k} = \gamma_{k+1}\eta_{j\,k+1} + (\alpha_k - \alpha_j)\eta_{j\,k} + \tag{4.5}$$

$$\beta_{k-1}\eta_{j\,k-1} - \beta_{j-1}\eta_{j-1\,k} + \mathbf{q}_k^T \mathbf{A}\, \mathbf{f}_{j+1} - \mathbf{q}_j^T \mathbf{A}\, \mathbf{f}_{k+1}$$

The last two terms of eqn (4.5) are due to unknown local rounding errors which are assumed to be at round-off level. We can denote them simply by a number $\psi_{j+1\,k}$. Thus, we achieve a recurrence that governs the evolution of the $\eta_{j+1\,k}$

$$\gamma_{j+1}\,\eta_{j+1\,k} = \gamma_{k+1}\,\eta_{j\,k+1} + (\alpha_k - \alpha_j)\,\eta_{j\,k} + \tag{4.6}$$

$$\beta_{k-1}\eta_{j\,k-1} - \beta_{j-1}\eta_{j-1\,k} + \psi_{j+1\,k}$$

This recurrence can also be expressed compactly in vector form as

$$\gamma_{j+1}\,\boldsymbol{\eta}_{j+1} = \mathbf{T}\,\boldsymbol{\eta}_j - \alpha_j\boldsymbol{\eta}_j - \beta_{j-1}\boldsymbol{\eta}_{j-1} + \boldsymbol{\psi}_{j+1} \tag{4.7}$$

where $\eta_{j+1\,k}$ and $\psi_{j+1\,k}$ are the elements of $\boldsymbol{\eta}_{j+1}$ and $\boldsymbol{\psi}_{j+1}$, respectively, and $\mathbf{T}$ is the tri-diagonal matrix defined in eqn (5.2). This formula states that $\boldsymbol{\eta}_{j+1}$ is some combination of $\boldsymbol{\eta}_j$ and $\boldsymbol{\eta}_{j-1}$ plus the contamination from round-off $\boldsymbol{\psi}_{j+1}$ occurring at this step. The loss of orthogonality therefore can be viewed as the result of an amplification of each local error after its introduction into the computation. This statement is consistent with the phenomenon observed in Figure 1.

Full re-orthogonalization. To maintain all the $\eta_{j+1\,k}$'s at round-off level, a full re-orthogonalization (FRO) scheme can be adopted which performs the explicit orthogonalization of $\mathbf{q}_{j+1}$ against all the previous $\mathbf{q}'s$. To this end, we add the following Gram-Schmidt orthogonalization step after the three-term recurrence step to force $\hat{\mathbf{q}}_{j+1}$ to be orthogonal to $\mathbf{q}_1, \ldots, \mathbf{q}_j$ up to the round-off level

$$\mathbf{p}_{j+1} = \mathbf{A}\,\hat{\mathbf{q}}_{j+1}$$

$$\hat{\mathbf{q}}_{j+1} \leftarrow \hat{\mathbf{q}}_{j+1} - \sum_{i=1}^{j} \delta_i\,\mathbf{q}_i\,(\mathbf{q}_i^T\,\mathbf{p}_{j+1}) \tag{4.8}$$

where $\delta_i = \mathbf{q}_i^T\,\mathbf{A}\,\mathbf{q}_i$ is 1 or -1. During the summation, the vector $\mathbf{p}_{j+1}$ is not changed to avoid extra $2\mu(\mathbf{M}) + \mu(\mathbf{C})$ multiplication of $\hat{\mathbf{q}}_{j+1}$ by $\mathbf{A}$. This modification can bring all the η's to the round-off level, however an extra $4jn$ multiplication is added to the original cost. This additional cost is not small compared to the cost of performing the three-term recurrence. Indeed, this cost will become dominant after some steps depending on the

costs $\mu(\mathbf{M})$, $\mu(\mathbf{C})$ and $\nu(\mathbf{K})$ relative to n.

Partial re-orthogonalization. The FRO scheme just discussed aims at keeping all the η's at ϵ level, where ϵ is the machine round-off error unit. However, recent research [2] and [3] shows that semi-orthogonality; i.e., maintaining all the η's at $\epsilon^{1/2}$ level, between Lanczos vectors generated from the operator $\mathbf{K}^{-1}\,\mathbf{M}$ is sufficient to achieve the eigensolutions of $(\mathbf{M}, \mathbf{K})$ within the desired accuracy. Following this approach, we examine whether semi-orthogonality between the Lanczos vectors generated from the operator $\mathbf{B}^{-1}\,\mathbf{A}$ is enough to achieve satisfactory eigensolutions of $(\mathbf{A}, \mathbf{B})$. To this end, we monitor the components in $\boldsymbol{\eta}_{j+1}$ when computing a new vector $\mathbf{q}_{j+1}$. We purge the $\mathbf{q}_{j+1}$ against only those $\mathbf{q}_k$ with $\eta_{j+1\,k}$ greater than $\epsilon^{1/2}$ to restore the required semi-orthogonality. In this way, re-orthogonalization is performed against a portion, rather than all, of the previous Lanczos vectors. Therefore, this scheme is called partial re-orthogonalization (PRO). To determine whether re-orthogonalization is required, we need to know all the elements in $\boldsymbol{\eta}_{j+1}$. Forming $\boldsymbol{\eta}_{j+1}$ explicitly requires $2jn$ multiplications, which is actually half the cost of the re-orthogonalization process. Thus, for economic reasons, we use the estimated $\boldsymbol{\eta}_{j+1}$, represented by eqn (4.7), to determine whether re-orthogonalization is required. The unknown vector $\boldsymbol{\psi}_{j+1}$ in eqn (4.7) can be replaced by appropriately chosen random numbers, which are based on a statistical study to reflect the effect of round-off, as detailed in [5]. Accordingly, we use the following recurrence in the algorithm to estimate the level of orthogonality.

$$\boldsymbol{\eta}_{j+1} = \frac{1}{\gamma_{j+1}} \left[\mathbf{T}\,\boldsymbol{\eta}_j - \alpha_j \boldsymbol{\eta}_j - \beta_{j-1}\boldsymbol{\eta}_{j-1} + \boldsymbol{\psi}_{j+1} \right] \tag{4.9}$$

This formula holds for $j+1 \geq 3$ and starts by assuming that $\eta_{1\,1} = \delta_1$, $\eta_{2\,1} = \psi_{2\,1}$ and $\eta_{2\,2} = \delta_2$.

5 Reduction to Tri-diagonal System

After m steps, we have the Lanczos vectors $\mathbf{Q} = [\mathbf{q}_1, \ldots, \mathbf{q}_m]$ satisfying the matrix form of three-term recurrence formula :

$$\gamma_{m+1}\, q_{m+1}\, e_m^T = B^{-1} A Q - Q T \tag{5.1}$$

where $e_m^T = (0\ 0\ \ldots\ 0\ 1)$ and T is a tri-diagonal matrix of the form

$$T = \begin{bmatrix} \alpha_1 & \beta_1 & & & & \\ \gamma_2 & \alpha_2 & \beta_2 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \gamma_{m-1} & \alpha_{m-1} & \beta_{m-1} \\ & & & & \gamma_m & \alpha_m \end{bmatrix} \tag{5.2}$$

These Lanczos vectors are A-orthogonal; i.e., they satisfy

$$Q^T A Q = \Delta \tag{5.3}$$

where Δ is the diagonal matrix with diagonal elements δ_j (i.e., 1 or -1). After pre-multiplying eqn (5.1) by $Q^T A$ and using eqn (5.3), we obtain

$$Q^T A B^{-1} A Q = \Delta T \tag{5.4}$$

Now, we choose the span of the set of Lanczos vectors, Q, as the admissible space and apply the projection method described in section 2 to obtain a reduced eigenproblem. To this end, we first premultiply eqn (1.5) by AB^{-1} to obtain

$$\lambda A B^{-1} A z = A z \tag{5.5}$$

Using this equation, we obtain the reduced problem :

$$\theta_j \Delta T s_j = \Delta s_j \tag{5.6}$$

or equivalently

$$T s_j = s_j \theta_j^{-1} \tag{5.7}$$

which is a standard eigenproblem. We solve this reduced eigenproblem by a QR algorithm, which is readily available in the EISPACK library [6]. After the solution of this reduced eigenproblem $\theta_1, \ldots, \theta_m$ are the Ritz values and the Ritz vectors $y_1, \ldots, y_m$ are given by

$$y_j = Q s_j \tag{5.8}$$

To make the Ritz vectors satisfy $| y_j^T A y_j | = 1$, we can scale the s_j such that $| s_j^T \Delta s_j | = 1$ for $j = 1, \ldots, m$. These Ritz pairs are the approximates to the least

dominant eigenpairs of the original system (**A**, **B**). To measure the quality of this approximation, we compute the residual vectors defined by

$$\mathbf{r}_j = \mathbf{B}^{-1}\,\mathbf{A}\,\mathbf{y}_j - \mathbf{y}_j\,\theta_j^{-1} \tag{5.9}$$

The three-term recurrence formula can be used to simplify the computation of these residual vectors. Post-multiplying eqn (5.1) by $\mathbf{s}_j$ leads to

$$\gamma_{m+1}\,\mathbf{q}_{m+1}\,\mathbf{e}_m^{\mathrm{T}}\,\mathbf{s}_j = \mathbf{B}^{-1}\,\mathbf{A}\,\mathbf{Q}\,\mathbf{s}_j - \mathbf{Q}\,\mathbf{T}\,\mathbf{s}_j \tag{5.10}$$

Making use of $\mathbf{T}\,\mathbf{s}_j = \mathbf{s}_j\,\theta_j^{-1}$ and $\mathbf{Q}\,\mathbf{s}_j = \mathbf{y}_j$, we obtain

$$\mathbf{B}^{-1}\,\mathbf{A}\,\mathbf{y}_j - \mathbf{y}_j\,\theta_j^{-1} = \gamma_{m+1}\,\mathbf{q}_{m+1}\,\mathbf{s}_j(m) \tag{5.11}$$

where $\mathbf{s}_j(m)$ represents the m^{th} element of the vector $\mathbf{s}_j$. That is, the residual vector can be obtained simply from

$$\mathbf{r}_j = \gamma_{m+1}\,\mathbf{q}_{m+1}\,\mathbf{s}_j(m) \tag{5.12}$$

and its pseudo length can accordingly be obtained from

$$|\,\mathbf{r}_j^{\mathrm{T}}\,\mathbf{A}\,\mathbf{r}_j\,|^{1/2} = \gamma_{m+1}\,\mathbf{s}_j(m) \tag{5.13}$$

where in general $\mathbf{s}_j(m)$ is a complex number. Note that γ_{m+1} and $\mathbf{s}_j(m)$ are readily available from the Lanczos algorithm and therefore no extra computational effort is required to form the quantity $|\,\mathbf{r}_j^{\mathrm{T}}\,\mathbf{A}\,\mathbf{r}_j\,|^{1/2}$. Since $\mathbf{r}_j = \mathbf{0}$ corresponds to an exact solution, we can assess the quality of the approximate solution by examining the magnitude of the components of $\mathbf{r}_j$. It is convenient to use the scalar quantity $|\,\mathbf{r}_j^{\mathrm{T}}\,\mathbf{A}\,\mathbf{r}_j\,|^{1/2}$ instead. However, as already stated in section 3, it is possible for $|\,\mathbf{r}_j^{\mathrm{T}}\,\mathbf{A}\,\mathbf{r}_j\,|^{1/2}$ to be zero with a nonzero $\mathbf{r}_j$. Therefore, one has to be cautious about the use of $|\,\mathbf{r}_j^{\mathrm{T}}\,\mathbf{A}\,\mathbf{r}_j\,|^{1/2}$ as a measure of how good the approximation is. We suggest that one compute the Euclidean length $|\,\mathbf{r}_j^{\mathrm{T}}\,\mathbf{r}_j\,|^{1/2}$ in addition to the pseudo length $|\,\mathbf{r}_j^{\mathrm{T}}\,\mathbf{A}\,\mathbf{r}_j\,|^{1/2}$ to make sure that the components of $\mathbf{r}_j$ are indeed small.

6 Numerical Examples

In this section, we use several Test Problems to assess the performance of the proposed algorithm to extract the eigenpairs of damped dynamic systems. The solution algorithm proposed is implemented in the research version of FEAP, a "Finite Element Analysis Program," (see Chapter 24 in [7] for a description of a simplified version of this program). The results reported herein are obtained using a VAX Station II/GPX computer system using the Ultrix 1.2 operating system and the f77 Fortran compiler.

Test problem 1 : The structure is modeled as a cantilever beam with a lumped translational viscous-damper attached at the tip. The beam is modeled using the elementary beam theory where the geometrical configuration and physical properties are shown in Figure 2. The consistent mass is used to define **M**. The damping matrix **C** has only one nonzero element representing the magnitude c of the lumped damper. The cantilever beam is divided into 20 equal elements and has 40 degrees of freedom. The order of the associated (**A**, **B**) is 80.

We use the Lanczos algorithm with the FRO scheme to solve this problem. Figure 3 summarizes the results of 8 experiments. Here, we call a Ritz pair good if the pseudo length of its associated residual vector is less than 10^{-8}. This criterion ensures that a good Ritz pair approximates the eigenpair sought with high accuracy. From the results in Figure 3, we see that the first few eigenpairs can be extracted at a fairly low cost compared to the other eigenpairs. This is because the re-orthogonalization cost is greater at later steps in the Lanczos algorithm.

In this problem, We have run the algorithm up to the size of the problem to test the robustness of the computer program developed. However we emphasize that the algorithm is intended only for partial solution of a large eigenproblem. After the 80 steps, we see that the pseudo length of the 81^{th} Lanczos vector is 0.9×10^{-15}, which is at the round-off level, implying that the computed Lanczos vectors have spanned the whole solution space as it should in exact arithmetic. This desirable result will ensure that all the Ritz pairs

obtained from the solution of the reduced tri-diagonal system are good and hence are accurate eigenpairs of the system.

Test Problem 2 : The system consists of two beams connected by a hinge with a rotational viscous-damper. The geometrical configuration and physical properties of the system are shown in Figure 4. The consistent mass matrix is used for **M**. The damping matrix **C** has only four nonzero elements, which are due to the lumped rotational damper. The system is divided into 40 equal elements and has 83 degrees of freedom. The associated (**A**, **B**) is of order 166. Note that the system is unrestrained and hence has rigid body modes. We use the shifted (**A**, **B**) defined by eqn (3.12) to compute the eigenpairs of this unrestrained system. The Lanczos algorithm with FRO scheme is used to solve this problem. Figure 5 summarizes the results of 9 experiments. Similar conclusions as in the first test problem can be inferred from Figure 5. The pseudo length of the 167^{th} Lanczos vector is 0.1×10^{-14}, which again exhibits the robustness of the computer program developed.

In general, the starting vector for the Lanczos algorithm may be chosen arbitrarily. However, if the starting vector is orthogonal to any of the eigenvectors of (**A**, **B**), all the Lanczos vectors will also be orthogonal to these eigenvectors. In practice, round-off errors eventually will introduce components along these eigenvectors; however, round-off enters slowly and the convergence to these eigenvectors is deferred. Therefore, we need to avoid the possibility of the starting vector being orthogonal to the wanted eigenvectors of the system. Since the structural system in this test problem is symmetric, there are anti-symmetric modes as well as symmetric modes. If a symmetric starting vector is used, such as (1, 1, . . . , 1), all the Lanczos vectors will be symmetric. Accordingly, all the anti-symmetric modes of the structure will be suppressed by this biased starting vector. To obtain all the required lower modes, we cannot choose a symmetric or anti-symmetric vector as the starting vector. This undesirable situation can usually be avoided by using a random vector as the starting vector.

Test Problem 3 : This problem is a three dimensional space truss system. There are 44 nodes and the 4 end nodes are fully restrained, as shown in Figure 6. Thus, there are

120 degrees of freedom and the associated (**A**, **B**) is of order 240. All truss bars have the same density and Young's modulus but different damping, as shown in Figure 6, resulting in a nonproportionally damped system. We use the Lanczos algorithm with the FRO scheme to generate 60 Lanczos vectors. We also use the Lanczos algorithm with the proposed PRO scheme to generate 60 Lanczos vectors. The results from the two schemes are compared in Table 2. The PRO scheme is shown to be adequate to compute the desired solution.

Table 2 *Results of Test Problems 3 and 4*

item problem	*Test Problem 3*		*Test Problem 4*	
	FRO	*PRO*	*FRO*	*PRO*
number of Lanczos vectors generated	*60*	*60*	*80*	*80*
number of re-orthogonalization	*1770*	*602*	*3159*	*1246*
CPU time spent on generating Lanczos vectors	*40.3*	*32.6*	*536.6*	*473.5*
CPU time spent on solving reduced eigenproblem	*50.8*	*51.7*	*100.9*	*101.2*
total CPU time spent on solving the whole problem	*113.9*	*106.7*	*893.2*	*830.9*
number of good Ritz pairs obtained	*28*	*28*	*40*	*40*

Test Problem 4 : This problem is a larger three dimensional space truss system. There are 300 nodes and the 4 end nodes are fully restrained, as shown in Figure 7. A typical cell is the same as the typical cell in the test problem 3. There are 888 degrees of freedom and the order of the associated (**A**, **B**) is 1776. We use the Lanczos algorithm with the FRO scheme to generate 80 Lanczos vectors. We also use the Lanczos algorithm with the PRO scheme to generate 80 Lanczos vectors. The results from the two schemes

are also compared in Table 2.

From Table 2, we see that the 60 Ritz pairs obtained provide 28 good eigenpairs for test problem 3 and the 80 Ritz pairs obtained provide 40 good eigenpairs for test problem 4 for both FRO and PRO cases. That is, approximately two Lanczos vectors, on the average, are required to capture a new eigenvector for these two large problems. This implies that the Krylov subspace generated by $\mathbf{B}^{-1}\mathbf{A}$ and a random vector is very effective in approximating the least dominant eigenvectors of the damped dynamic systems considered.

By maintaining semi-orthogonality between the Lanczos vectors with the PRO scheme, the resulting Ritz values are as accurate as those obtained with the FRO scheme, as shown in Table 2. But a great part of the re-orthogonalization steps can be eliminated by using the PRO scheme instead of the FRO scheme. That is, we can eliminate some re-orthogonalization effort without sacrificing accuracy of the final solution when solving $\lambda\,\mathbf{A}\,\mathbf{z} = \mathbf{B}\,\mathbf{z}$ with the PRO scheme. This is in agreement with the case of solving $\omega^2\,\mathbf{M}\,\mathbf{w} = \mathbf{K}\,\mathbf{w}$ by standard Lanczos method, as shown by [5].

To assess the efficiency of the Lanczos algorithm, the lower mode solutions of the above four test problems are also computed using a subspace iteration algorithm. The subspace iteration algorithm reported in [8] is used for this purpose. The subspace dimension is determined by $\min\{\,2n\,,\,n+8\,\}$, where n is the number of wanted eigenpairs. Table 3 compares the cost of the Lanczos algorithm with the cost of the subspace iteration algorithm. It is apparent that the Lanczos algorithm is considerably more efficient than the subspace iteration algorithm for the examples considered.

7 Conclusions

A variant of the standard Lanczos algorithm is presented which yields an efficient solution of the eigenproblem arising in the analysis of damped linear dynamic systems. The algorithm takes full advantage of the sparsity and symmetry of the associated matrices. From the results shown, the proposed Lanczos algorithm is considerably more economical

Table 3 *Results from different algorithms*

Test Problem	Lanczos algorithm		subspace iteration algorithm	
	No. of good Ritz pairs	CPU time (second)	No. of good Ritz pairs	CPU time (second)
1	8	8.5	8	40.2
2	20	41.7	16	214.6
3	28	113.9	24	1012.9
4	40	893.2	40	20992.8

than the subspace iteration algorithm for extracting lower mode eigenpairs of damped dynamic systems.

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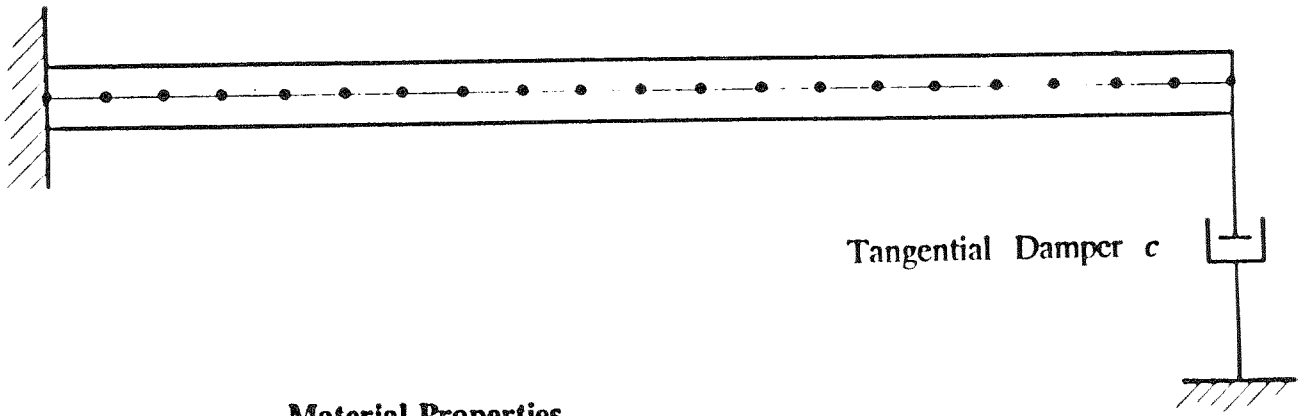
Box 1 *Simple Lanczos Algorithm*

Step 1 :	Operation Count
pick a random vector $\mathbf{r}$	
$\mathbf{p} = \mathbf{A}\,\mathbf{r}$	$2\mu(\mathbf{M}) + \mu(\mathbf{C})$
solve $\mathbf{B}\,\mathbf{q} = \mathbf{p}$	$\nu(\mathbf{K})$
$\mathbf{p} = \mathbf{A}\,\mathbf{q}$	$2\mu(\mathbf{M}) + \mu(\mathbf{C})$
$\gamma_1 = sqrt\,[\,abs\,(\mathbf{q}^T\,\mathbf{p})\,]$	$2n$
$\delta_1 = sgn\,(\mathbf{q}^T\,\mathbf{p})$	
$\mathbf{p} \leftarrow \mathbf{p} / \gamma_1$	$2n$
$\mathbf{q} \leftarrow \mathbf{q} / \gamma_1$	$2n$
solve $\mathbf{B}\,\mathbf{r} = \mathbf{p}$	$\nu(\mathbf{K})$
$\alpha_1 = (\mathbf{r}^T\,\mathbf{p}) \cdot \delta_1$	$2n$
$\mathbf{r} \leftarrow \mathbf{r} - \alpha_1 \cdot \mathbf{q}$	$2n$
$\mathbf{oldp} = \mathbf{A}\,\mathbf{r}$	$2\mu(\mathbf{M}) + \mu(\mathbf{C})$
$\gamma_2 = sqrt\,[abs\,(\mathbf{r}^T\,\mathbf{oldp})]$	$2n$
$\delta_2 = sgn\,(\mathbf{r}^T\,\mathbf{oldp})$	
store $\mathbf{q}$ as $\mathbf{q}_1$	

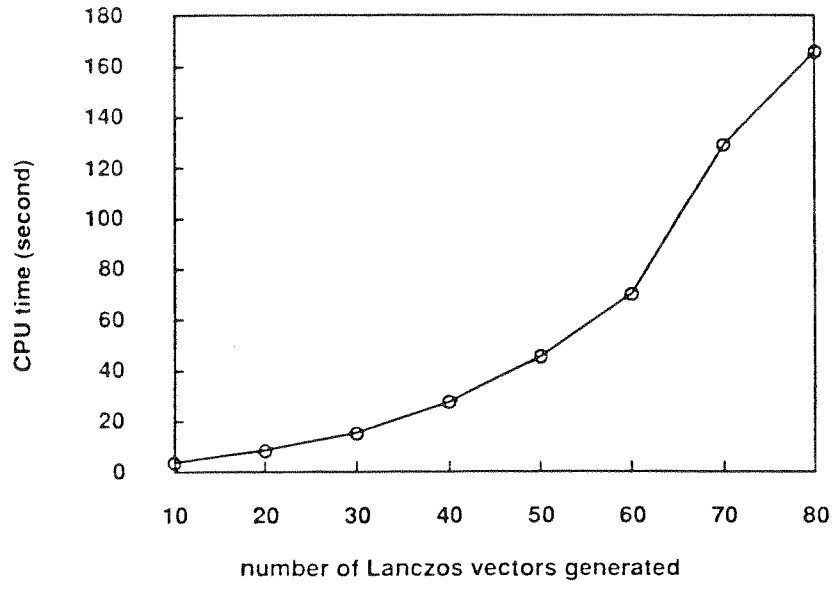
Loop : For j = 2,3...	Operation Count
$\mathbf{oldq} \leftarrow \mathbf{q}$	
$\mathbf{oldp} \leftrightarrow \mathbf{p}$	
$\mathbf{q} = \mathbf{r} / \gamma_j$	$2n$
$\mathbf{p} \leftarrow \mathbf{p} / \gamma_j$	$2n$
solve $\mathbf{B}\,\mathbf{r} = \mathbf{p}$	$\nu(\mathbf{K})$
$\alpha_j = (\mathbf{r}^T\,\mathbf{p}) \cdot \delta_j$	$2n$
$\beta_{j-1} = (\mathbf{r}^T\,\mathbf{oldp}) \cdot \delta_{j-1}$	$2n$
$\mathbf{r} \leftarrow \mathbf{r} - \alpha_j \cdot \mathbf{q}$	$2n$
$\mathbf{r} \leftarrow \mathbf{r} - \beta_{j-1} \cdot \mathbf{oldq}$	$2n$
$\mathbf{oldp} = \mathbf{A}\,\mathbf{r}$	$2\mu(\mathbf{M}) + \mu(\mathbf{C})$
$\gamma_{j+1} = sqrt\,[\,abs\,(\mathbf{r}^T\,\mathbf{oldp})\,]$	$2n$
$\delta_{j+1} = sgn\,(\mathbf{r}^T\,\mathbf{oldp})$	
store $\mathbf{q}$ as $\mathbf{q}_j$	



Material Properties

modulus	1000
length	5
density	1
inertia	1
area	1

Figure 2 Test Problem 1



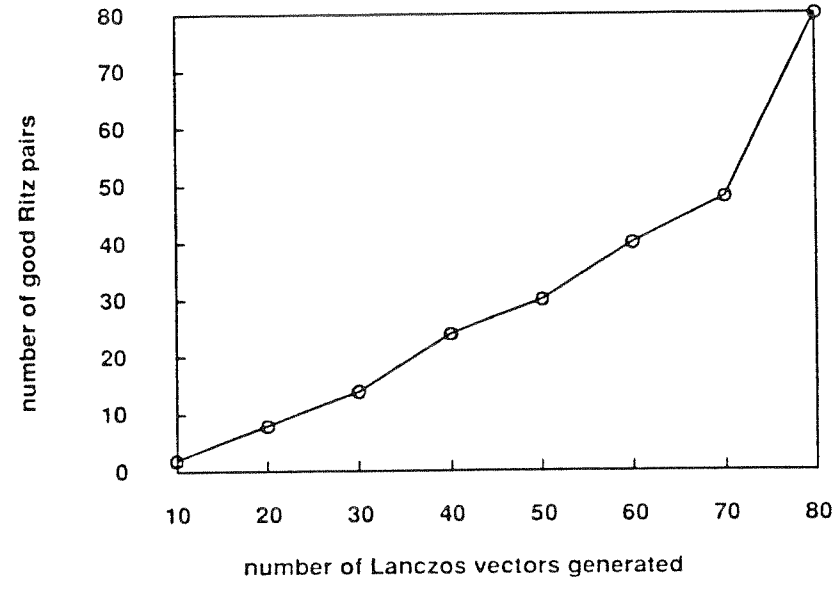
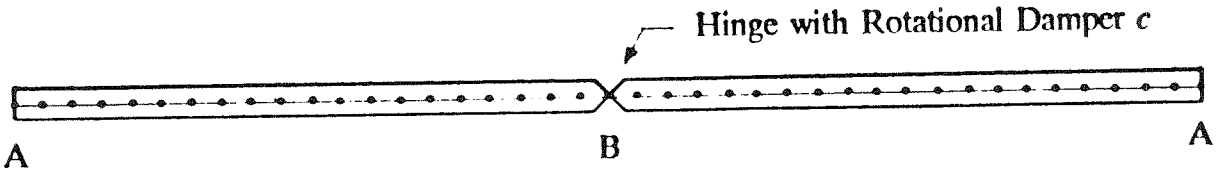
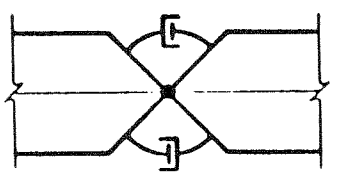


Figure 3 Results of Test Problem 1



$M = c\ \Delta\dot{\theta}$

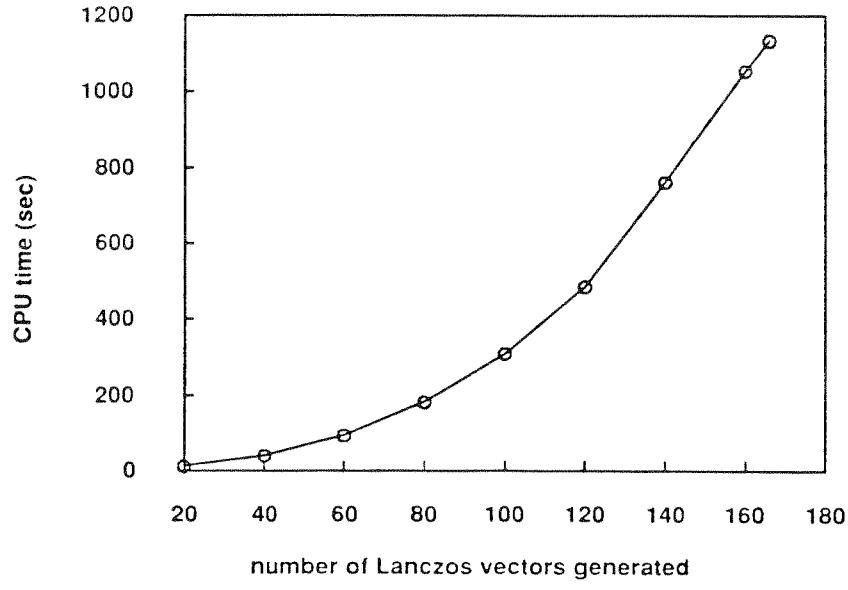


Detail for Rotational Damper

Material Properties

beam	1	2
modulus	1000	1000
length	20	20
density	1	1
inertia	1	1
area	1	1

Figure 4 Test Problem 2



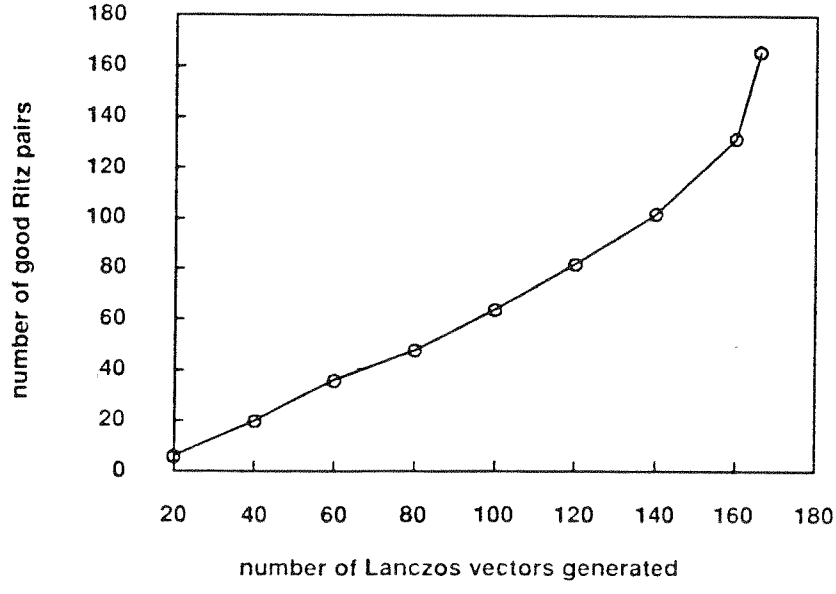
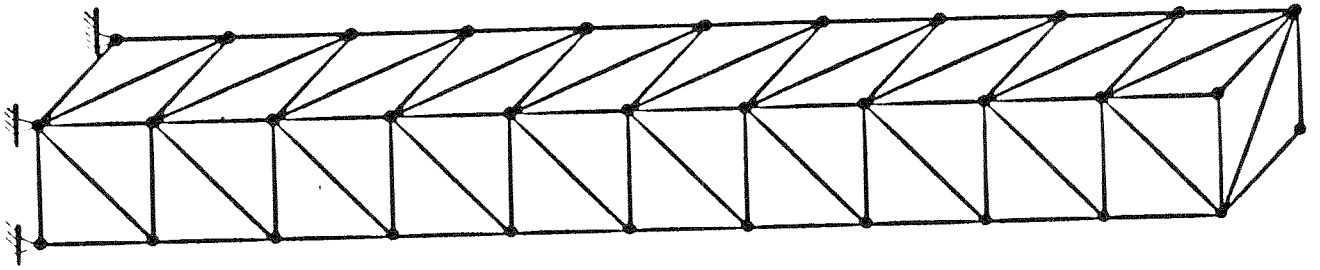
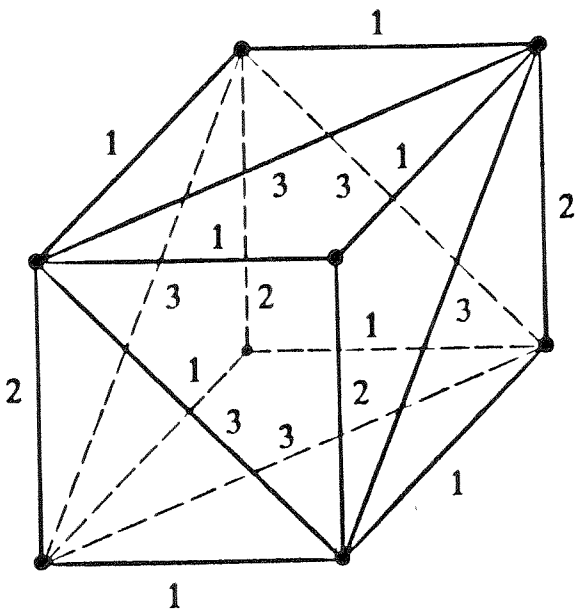


Figure 5 Results of Test Problem 2





Typical Cell

Material Properties

member	1	2	3
modulus	1	1	1
density	1	1	1
inertia	1	1	1
area	1	1	1
damping	0.5	1.0	2.0

Figure 6 Test Problem 3

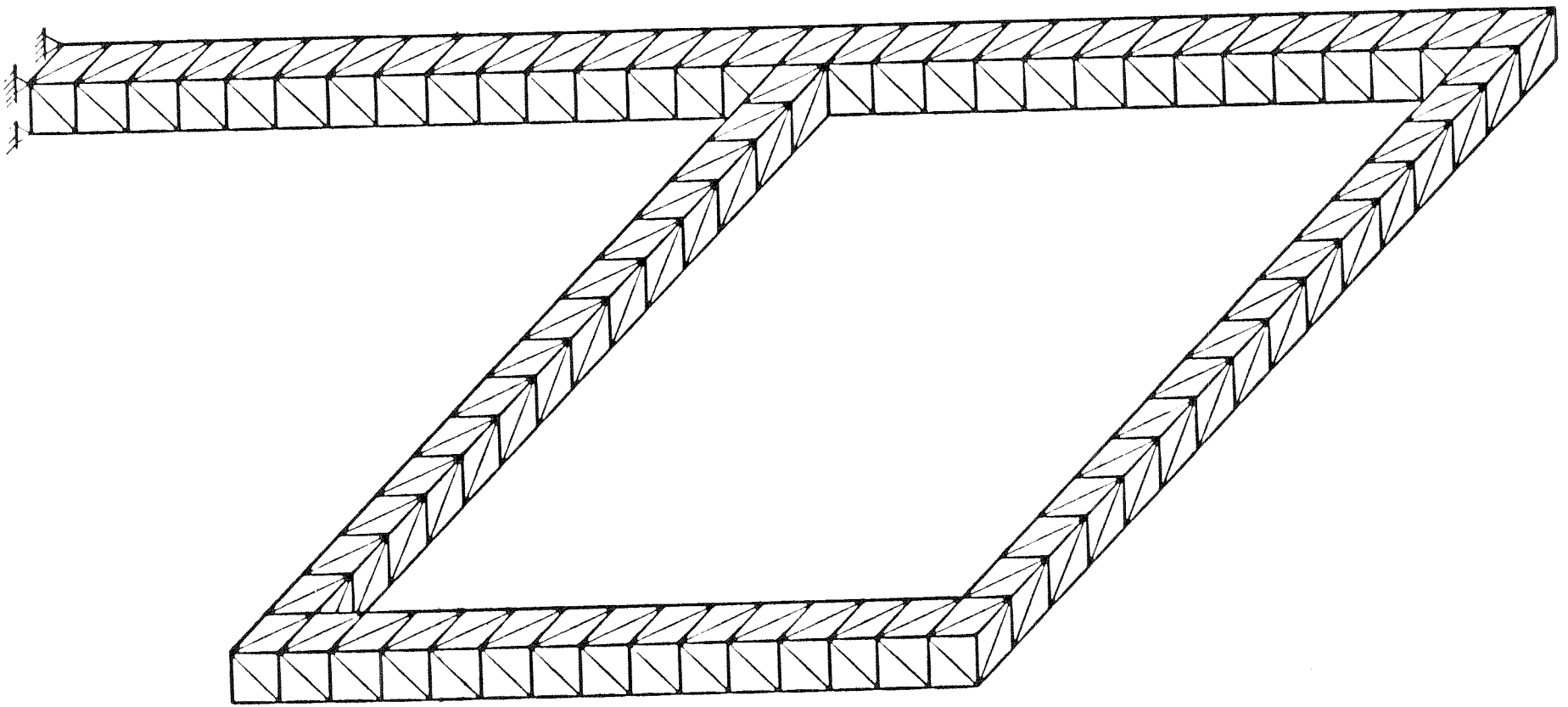


Figure 7 Test Problem 4