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Local existence and breakdown of scattering behavior for semilinear Schrödinger equations

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by

Gyu Eun Lee

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ABSTRACT OF THE DISSERTATION

Local existence and breakdown of scattering behavior for semilinear Schrödinger equations

by

Gyu Eun Lee
Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2021
Professor Rowan Brett Killip, Co-Chair
Professor Monica Vişan, Co-Chair

In this thesis, we study the behavior of solutions to some semilinear Schrödinger equations at short and long time scales. We first consider the nonlinear Schrödinger equations with power-type nonlinearity in three dimensions with periodic boundary conditions. We show that this equation is locally well-posed in critically scaling Sobolev spaces $H^s(\mathbb{T}^3)$. We then investigate the long-time asymptotic behavior of solutions to NLS in Euclidean space with defocusing, mass-subcritical power-type and Hartree nonlinearities. We discuss the divide between the wealth of results on the scattering theory for these equations in weighted L^2 spaces and the paucity of analogous results in $L^2(\mathbb{R}^d)$. To explain this, we show that the scattering problems for these equations are well-posed in weighted L^2 spaces in the sense that the scattering operators attain their natural and maximal regularity. Furthermore, we show that these scattering problems are ill-posed in L^2 in the sense that the scattering operators cannot be extended to all of L^2 without losing a positive (and, in the case of Hartree, infinite) amount of regularity.

The dissertation of Gyu Eun Lee is approved.

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2021

저희 형제를 위해 낯선 땅에서 고생하신 아버지와 어머니에게 이 논문을 바칩니다.

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CHAPTER 1

Introduction

1.1 Nonlinear Schrödinger equations

In this thesis we study aspects of the short-time and long-time behavior of solutions to certain *nonlinear Schrödinger equations* (NLS), which are a class of nonlinear partial differential evolution equations taking the form

$$i\partial_t u + \Delta u = F(u, t, x). \tag{1.1.1}$$

Here $t \in \mathbb{R}$, $x \in X$ for an appropriate choice of spatial domain X (typically Euclidean space $\mathbb{R}^d$ or the torus $\mathbb{T}^d$), and $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ complex spacetime scalar field. Such equations comprise a wide class of models for describing various wave phenomena.

The name derives from their close relationship with the *free/linear Schrödinger equation*

$$i\partial_t u + \Delta u = 0. \tag{1.1.2}$$

Solutions to this equation satisfying $u(t = 0) = u_0$ are denoted $e^{it\Delta}u_0$, where $e^{it\Delta}$ is known as the *free/linear propagator* for the Schrödinger equation.

The behavior of solutions to the free Schrödinger equation is characterized by *dispersion*, referring to the fact that different frequency components of a solution travel at different velocities, which manifests as a tendency for solutions to spread out in space while decaying in amplitude over time. Nonlinear Schrödinger equations tend to inherit significant aspects of their behavior from the properties of the free evolution. As such, they are classic examples of another large class of evolution equations known as *nonlinear dispersive PDEs*, which are equations characterized by a dispersive component to their behavior. In such equations, the analysis of solutions typically centers around understanding the interaction between the dispersive effect and the effects of the nonlinearity F.

Within this thesis we are interested in two cases of the nonlinearity F: the power-type nonlinearity $F(u) = \pm|u|^p u$, corresponding to the *power-type nonlinear Schrödinger equation* (pNLS)

$$i\partial_t u + \Delta u = \pm|u|^p u, \tag{1.1.3}$$

and the Hartree nonlinearity $F(u) = \pm(|x|^{-\gamma} * |u|^2)u$, corresponding to the *Hartree equation* (HNLS)

$$i\partial_t u + \Delta u = \pm(|x|^{-\gamma} * |u|^2)u. \tag{1.1.4}$$

These are among the most widely studied nonlinear Schrödinger equations, both as objects of mathematical interest and as physical models. pNLS, for instance, is a universal model of wave propagation in weakly nonlinear media, and arises in the study of water waves in deep ocean, light propagation in fiber optics, and Langmuir waves in hot plasmas, while both NLS and Hartree arise as limiting effective equations in the study of many-body quantum systems.

From the mathematical perspective, the primary objectives in the study of nonlinear Schrödinger equations are the following:

1. Local existence and well-posedness: the existence and continuous dependence of solutions upon initial data in an appropriately chosen function space, at least up to short times.

2. Global existence and well-posedness: the extension of the aforementioned local-in-time solutions to all times, or the demonstration of finite-time blowup.

3. Behavior: the detailed description of how solutions behave at various time scales, for instance as $t \to \pm\infty$.

Our results in this thesis are mainly related to the first and third of these objectives.

1.2 Local well-posedness for periodic NLS

We first turn our attention to the question of local existence and well-posedness. We focus in particular on the Cauchy problem for pNLS with periodic boundary conditions, i.e. with spatial domain $X = \mathbb{T}^d$, where $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ is the d-dimensional torus:

$$\begin{cases} i\partial_t u + \Delta u = \pm|u|^p u, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^d, \\ u(0,x) = u_0(x). \end{cases} \tag{1.2.1}$$

Here, we allow for the case of an irrational rectangular torus by embedding the irrationality into our choice of Riemannian metric, and denoting the corresponding Laplace-Beltrami operator by Δ.

This equation, when posed on $\mathbb{R}^d$, admits a scaling symmetry: if $u(t,x)$ is a solution to (1.2.1), then so is $u_\lambda(t,x) = \lambda^{-2/p} u(\lambda^{-2}t, \lambda^{-1}x)$. This yields the notion of *scaling-critical regularity* $s_c = \frac{d}{2} - \frac{2}{p}$, which is the regularity s at which the $\dot{H}^s(\mathbb{R}^d)$-norm is invariant under the scaling, i.e. $\|u_\lambda\|_{H^{s_c}(\mathbb{R}^d)} = \|u\|_{H^{s_c}(\mathbb{R}^d)}$. We carry this notion into the $\mathbb{T}^d$ setting as well.

The problem of local well-posedness for periodic NLS in critically scaling Sobolev spaces has a rich and storied history. Critical local well-posedness in the periodic setting has always been a greater challenge than in the Euclidean space setting. This is because the dispersive effect of the free Schrödinger evolution is weaker on compact manifolds since different frequency components can interact with each other repeatedly over long times, whereas in Euclidean space high-frequency components quickly move out to spatial infinity and have limited interactions with low-frequency components. This manifests through the *Strichartz estimates*, which are estimates on spacetime norms of the free Schrödinger propagator $e^{it\Delta}$. Though scaling-invariant Strichartz estimates are easily obtainable in Euclidean space, the corresponding estimates in the periodic setting are noticeably more challenging to come by.

Scaling-invariant Strichartz estimates are crucial to solving the periodic Cauchy problem in critically-scaling Sobolev spaces. The first such estimates for the Schrödinger equation on $\mathbb{T}^d$ are due to Bourgain [5], who obtained a range of scale-invariant Strichartz estimates on square tori and applied them to the local and small-data global theory for periodic NLS. In the same paper (and its companion work [6]), Bourgain introduced the idea of working in function spaces which are adapted to the dispersive symbol of the linear propagator, in this case the $X^{s,b}$ spaces.

Since then, progress on the scaling-critical Cauchy problem has progressed largely along these two ideas. Herr, Tataru, and Tzvetkov introduced the adapted function spaces X^s and Y^s in [30], and used them to prove local wellposedness for the H^1-critical pNLS on a partially irrational torus in $d = 4$. This built off earlier work by Hadac, Herr, and Koch [25], which introduced the atomic spaces U^p and V^p (precursors to X^s and Y^s), and by Herr, Tataru, and Tzvetkov [29], which established local wellposedness and small-data global wellposedness for H^1-critical pNLS on a square torus with $d = 3$. Ionescu and Pausader extended the result of [30] to obtain large-data global wellposedness H^1-critical defocusing pNLS on $\mathbb{T}^3$ [32]. In [24], Guo, Oh, and Wang proved a range of new scale-invariant Strichartz estimates for the linear Schrödinger evolution on irrational tori. As an application, they extended the result of [29] to a partially irrational torus with $d = 3$. They also established local wellposedness for Equation (1.2.1) for $d \in \{2, 3, 4\}$ and certain choices of $p = 2k$, $k \in \mathbb{N}$: $d = 2$ and $k \geq 6$; $d = 3$ and $k \geq 3$; and $d \geq 4$ and $k \geq 2$. Strunk extended this result to $d = 2, k \geq 3$ and $d = 3, k = 2$, using multilinear Strichartz estimates [54].

The problem has seen an explosion of progress in the last six years. This is due to the proof of the ℓ^2-decoupling conjecture by Bourgain and Demeter [8], which implies the full range of scale-invariant Strichartz estimates on square tori. Killip and Vişan extended these estimates to rectangular irrational tori, and used them to prove local wellposedness for H^1-critical NLS for such tori with $d = 3, 4$ [34]. Notably, they proved a bilinear version of the scale-invariant Strichartz estimates, which allowed for the proof to avoid the use of complicated multilinear estimates which were prevalent in earlier results.

The first of the results we will discuss in this thesis is a continuation of the above works:

Theorem 1.2.1 (Local well-posedness in $H^{s_c}(\mathbb{T}^3)$ [40])**.** *Let $p \geq 2$, and fix $d = 3$. Then Equation* (1.2.1) *is well-posed in $H^{s_c}(\mathbb{T}^3)$. In particular, for any $u_0 \in H^{s_c}(\mathbb{T}^3)$, there exists a time of existence $T = T(u_0)$ and a unique solution $u \in C_t([0,T); H^{s_c}(\mathbb{T}^3)) \cap X^{s_c}([0,T))$ to* (1.2.1)*.*

Chapter 2 is devoted to the proof of this result. Notably, this result recovers the $H^{\frac{1}{2}}$-critical NLS in three dimensions, corresponding to the cubic NLS $p = 2$. It also covers fractional-power nonlinearities, whereas all previous results were limited to even powers p in order to take advantage of the algebraic nature of the nonlinearity $|u|^p u$ for even p. Our proof relies primarily on the bilinear scale-invariant Strichartz estimates of [34]. To address the case of fractional p, this is combined with a paradifferential linearization introduced by Bony [4] in order to perform a frequency analysis of the nonlinearity.

1.3 Mass-subcritical scattering

Next we turn to the problem of describing the long-time behavior of solutions to nonlinear Schrödinger equations set in Euclidean space. Here the distinction between the signs in the nonlinearities $\pm|u|^p u$, $\pm(|x|^{-\gamma} * |u|^2)u$ becomes relevant. The $+$ sign is referred to as *defocusing*, while the $-$ sign is referred to as *focusing*. In the focusing case, the nonlinearity tends to fight the dispersive nature of the Laplacian, potentially leading to finite-time blowup and soliton formation. In the defocusing case, the nonlinearity tends to cooperate with the dispersion, and consequently dies out due to dispersive decay. This leads to long-time behavior which is strongly influenced by the free evolution.

In this thesis we are primarily concerned with the defocusing case. We restrict our attention to the *mass-subcritical* regimes for Equations (1.1.3) and (1.1.4) with defocusing nonlinearity, which are $0 < p < \frac{4}{d}$ for pNLS and $0 < \gamma < 2$ for HNLS. These equations obey conservation laws for *mass*

$$M(u(t)) = \int_{\mathbb{R}^d} |u(x)|^2 \, dx = M(u(0)) \tag{1.3.1}$$

and *energy*

$$E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(x)|^2 + Q(u(t)) = E(u(0)), \tag{1.3.2}$$

where the potential energy term $Q(u(t))$ is given by

$$Q(u(t)) = \frac{1}{p+2} \int_{\mathbb{R}^d} |u(x)|^{p+2} \, dx \tag{1.3.3}$$

in the case of pNLS and by

$$Q(u(t)) = \frac{1}{4} \int_{\mathbb{R}^d} (|x|^{-\gamma} * |u|^2)|u(x)|^2 \, dx \tag{1.3.4}$$

in the case of HNLS. The standard conjecture for the asymptotic long-time behavior of defocusing semilinear nonlinear Schrödinger equations like these is *scattering*, i.e. convergence to a free evolution.

For X a Banach space, we say that *asymptotic completeness* holds (in forward time) for a general nonlinear Schrödinger equation (1.1.1) if for all initial data $u_0 \in X$, there exists $u_+ \in X$ such that the global solution $u \in C_{t,\text{loc}}X$ to Equation (1.1.1) with Cauchy data $u(t=0) = u_0$ satisfies the asymptotic relation

$$\lim_{t \to +\infty} \|e^{-it\Delta} u(t) - u_+\|_X = 0.$$

When this occurs, we say that $u(t)$ *scatters* to u_+. In this case we may define the *initial-to-scattering-state operator*

$$\mathcal{S} : X \to X : \mathcal{S}(u_0) = u_+. \tag{1.3.5}$$

Similarly, we say that the (forward) wave operator exists on X for Equation (1.1.1) if for all final states $u_+ \in X$, there exists a unique global solution $u \in C_{t,\text{loc}}X$ to Equation (1.1.1) which scatters to u_+. When this holds, we may define the *wave operator*

$$\mathcal{W} : X \to X : \mathcal{W}(u_+) = u(t=0). \tag{1.3.6}$$

When the two operators exist, they are necessarily inverses of each other. Analogous definitions can be made in backwards time, i.e. as $t \to -\infty$. When asymptotic completeness holds and the wave operator exists on X (in both forward and backward time), we say that *scattering* holds for the equation on X. Scattering theory for these equations revolves around the question of the existence of these operators.

The existing scattering theory for defocusing pNLS and HNLS in the mass-subcritical regime is largely limited to data in weighted spaces $X = \Sigma^{\ell,m}$, which are Hilbert spaces defined by the norms

$$\|f\|_{\Sigma^{\ell,m}}^2 = \|f\|_{L^2}^2 + \||\nabla|^\ell f\|_{L^2}^2 + \||x|^m f\|_{L^2}^2. \tag{1.3.7}$$

The case $\Sigma^{1,1}$ is denoted by $\Sigma = H^1 \cap \mathcal{F}H^1$, and $\Sigma^{0,k}$ is also denoted $\mathcal{F}H^k$.

We briefly review the literature on the scattering theory of mass-subcritical pNLS and HNLS. We begin with pNLS: for the reader's convenience, these results are summarized in Table 1.1. The global theory for pNLS effectively begins with the work of Ginibre and Velo [18, 19], who proved who established local and global well-posedness, rigorously justified the conservation laws, and proved scattering in Σ for initial/final data in Σ, under the restriction $d \geq 2$, $\frac{4}{d} \leq p < \frac{4}{d-2}$. The first extension to mass-subcritical NLS is due to Tsutsumi [57], who extended the range of nonlinearities to $\alpha(d) < p < \frac{4}{d}$, where $\alpha(d) = \frac{2-d+\sqrt{(d-2)^2+16d}}{2d}$ is known as the *Strauss exponent*, a threshold exponent demarcating pNLS nonlinearities for which critically scaling global spacetime bounds for the solutions can be proved using the pseudoconformal energy law. Hayashi and Tsutsumi [27] gave another proof of scattering in the same range of p that avoided the use of the pseudoconformal transform, and also showed that the scattering operators are continuous on Σ. Cazenave and Weissler [11] improved the range to $\frac{4}{d+2} \leq p < \frac{4}{d}$ assuming small data, and recovered $p = \alpha(d)$ for large data, $d \geq 3$. Ginibre, Ozawa, and Velo [17] further improved the small-data range to $\frac{4}{d+2s} < p < \frac{4}{d}$ by working in $\Sigma^{s,s}$, $0 < s < 2$. Nakanishi and Ozawa [48] recovered $p = \frac{4}{d+2s} \leq p < \frac{4}{d}$ for small data in $\Sigma^{s,s}$, and $p = \alpha(d)$ for large data in Σ without restriction on dimension. Kita [36], Kita and Ozawa [37], and Masaki [42] investigated the asymptotic expansions of the scattering solutions near $t = \infty$. More results are available if one relaxes the topology of convergence to the free evolution. Tsutsumi and Yajima [58] showed that if $\frac{2}{d} < p < \frac{4}{d}$ (the *short-range regime*), then global solutions with initial data in Σ scatter in L^2 norm. Nakanishi [47] showed that under the same hypotheses, every final state in L^2 has a global solution which scatters to it in L^2, and Murphy [46] showed that such a global solution is almost surely unique in a probabilistic sense, establishing the almost-sure existence of wave operators in the L^2 topology. These results are optimal in light of the work of Glassey [21], Strauss [51], and Barab [2], which established that there is no nontrivial scattering theory for defocusing pNLS in the *long-range regime* $0 < p \leq \frac{2}{d}$: any solution which converges to a free evolution in the L^2 topology must be the zero solution. Instead, in the long-range regime the conjectured behavior is modified scattering, i.e. convergence to a free evolution modulated by a nonlinear phase.

The scattering theory of HNLS mirrors that of pNLS; these results are summarized in Table 1.2. The first positive results on scattering are due to Ginibre and Velo [20], who established local and global well-posedness and scattering in $\Sigma^{\ell,1}$ for $\ell \geq 1$ assuming $2 < \gamma < \min(4, d)$, which corresponds to a range of mass-*supercritical* nonlinearities. The first results in the mass-subcritical range were obtained by Strauss [52, 53], who proved existence of wave operators on $L^{\frac{4d}{2d+\gamma}} \cap L^2$ for $\frac{4}{3} < \gamma < \min(4, d)$. The exponent $\gamma = \frac{4}{3}$ plays a role for HNLS similar to that played by the Strauss exponent $\alpha(d)$ for pNLS. This was then brought into the setting of weighted spaces by Hayashi and Tsutsumi [28], who proved scattering in $\Sigma^{\ell,m}$, $\ell, m \geq 1$, for $d \geq 2$ and the same range of γ. Hayashi and Ozawa [26] were able to drop the regularity assumption on the scattering data, showing existence of the wave operators on $\mathcal{F}H^k$, $k \geq 1$, in $d \geq 3$, $\frac{4}{3} < \gamma < 2$. Nawa and Ozawa [49] improved this result to $d \geq 2$, $1 < \gamma < 2$. Masaki [43] showed global well-posedness and scattering in the critically scaling weighted L^2 spaces $\mathcal{F}\dot{H}^{s_c}$, where $s_c = 1 - \frac{\gamma}{2}$, $1 < \gamma < 2$, $d \geq 2$, assuming global spacetime bounds for the solution in $\mathcal{F}\dot{H}^{s_c}$. Again, relaxing the topology of convergence to the free evolution results in a greater range of results. Hayashi and Tsutsumi [28] proved that for

$d \geq 2$ and $1 < \gamma < \min(4, d)$, global solutions with initial data in Σ scatter in L^2. Hayashi and Ozawa [26] showed the analogous result for initial data in $\mathcal{F}H^1$ for $1 < \gamma < \min(2, d)$. As for the problem of the wave operators, Holmer and Tzirakis [31] showed that for $d = 2$ and $1 < \gamma < 2$, for any H^1 scattering state there exists a global H^1 solution u which scatters to it. However, because this global solution u is not known to be uniquely determined by u_+, this falls short of defining the wave operator. In all of these results, the lower threshold $\gamma = 1$ is sharp, playing a role analogous to $p = \frac{2}{d}$ for pNLS: for $\gamma \leq 1$, it was shown by Glassey [22] and Hayashi and Tsutsumi [28] that no nontrivial scattering theory with convergence in L^2-norm exists. Cho, Hwang, and Ozawa [12, 13] extended these results to potentials V with Fourier transform behaving like $|\xi|^{-(d-\gamma)}$ near the origin, $0 < \gamma \leq 1$. Thus the scattering problem for mass-subcritical HNLS also splits along the *short-range regime* $1 < \gamma < 2$, for which a nontrivial scattering theory is possible, and the *long-range regime* $0 < \gamma \leq 1$ for which no such theory exists. The conjectured asymptotic behavior in the long-range regime is modified scattering in the HNLS case as well.

Our work in this thesis addresses a defect in the scattering theory for short-range defocusing mass-subcritical pNLS and HNLS, which is the lack of a scattering theory on L^2. Indeed, we are in a rather curious position when it comes to the understanding of the long-time behavior of solutions to these equations in L^2. Both equations are well-known to be globally well-posed in L^2, a consequence of their mass-subcritical nature and the conservation of mass (1.3.1). However, all positive results on the short-range scattering theory impose a weighted condition on the initial or scattering data, even if (as in the case of [26, 28, 58]) the scattering behavior is only obtained in the weaker topology L^2. Therefore although we know that all L^2 solutions to our equations exist and are global, we have no understanding of their asymptotic behavior outside a small subset of solutions. Although Σ is not a totally unnatural space to consider Schrödinger equations (it emerges as the energy space after a Lens transform [55]), the situation in L^2 does leave something to be desired. Our work aims to explain this defect in the theory by showing that the scattering problem in L^2 is ill-posed.

Table 1.1: Summary of scattering theory for mass-subcritical pNLS

Type of result	Result
Scattering in Σ	Ginibre-Velo '79: $\frac{4}{d} \leq p < \frac{4}{d-2}$, $d \geq 2$: LWP, GWP, conservation laws, first scattering result in Σ Tsutsumi: '85: $\alpha(d) < p < \frac{4}{d}$: scattering in Σ Hayashi-Tsutsumi '87: $\alpha(d) < p < \frac{4}{d}$: $\mathcal{S}$ and $\mathcal{W}$ are continuous Cazenave-Weissler '92: $p \geq \frac{4}{d+2}$: small data scattering; $p = \alpha(d)$, $d \geq 3$: large data scattering Ginibre-Ozawa-Velo '94: $p > \frac{4}{d+2s}$: small data scattering in $\Sigma^{s,s}$ Nakanishi-Ozawa '01: $p = \frac{4}{d+2s}$: small data scattering in $\Sigma^{s,s}$); $p = \alpha(d)$: large data scattering in Σ Kita '03, Kita-Ozawa '05: $p > \alpha(d)$: obtained first term in asymptotic expansion of $u(t) - e^{it\Delta}u_+$ Masaki '09: $\alpha(d) \leq p < \frac{4}{d-2}$: Taylor expansion of $u(t) - e^{it\Delta}u_+$ near $t = \infty$ to maximal order allowed by nonlinearity
Asymptotic completeness	Tsutsumi-Yajima '84: $\frac{2}{d} < p < \frac{4}{d}$: scattering in L^2 for initial data in Σ
Final-state problem	Nakanishi '01: $\frac{2}{d} < p < \frac{4}{d}$: every L^2 scattering state has L^2 global solution scattering to it Murphy '19: $\frac{2}{d} < p < \frac{4}{d}$: final-state problem has unique solution almost surely
Negative results	Glassey '73, Strauss '73, Barab '84: $0 < p \leq \frac{2}{d}$: all global solutions in Σ scattering in L^2-norm are trivial

We now state the second result we will prove and discuss in this thesis, which addresses the case of pNLS.

Theorem 1.3.1 (Breakdown of regularity of the pNLS scattering operators [41])**.** *Let $d \geq 1$, and consider pNLS with $\alpha(d) < p < \frac{4}{d}$. Then:*

1. *The scattering operators $\mathcal{S}, \mathcal{W}$ for pNLS are well-defined as maps $\Sigma \to L^2$, and are maximally regular at $0 \in \Sigma$ in the sense that they are Hölder continuous of order $1+p$ at 0, but not of any higher order.*
2. *There exists $\beta = \beta(d,p) \in (0,p)$ such that $\mathcal{S}, \mathcal{W}$ admit no extensions to maps $L^2 \to L^2$ which are Hölder continuous of order $1+\beta$ on any ball $B \subset L^2$ containing the origin.*

In particular:

Corollary 1.3.2. *Assume $d \geq 1$ and $\alpha(d) < p < \frac{4}{d}$. Then:*

1. *Let $s > 1+p$, and let n be the integer part of s. Then $\mathcal{S}$ and $\mathcal{W}$, regarded as maps $\Sigma \to L^2$, cannot have an n-th Gateaux derivative defined about $0 \in \Sigma$ which is Hölder continuous of order $s-n$.*
2. *Let $s = 1+\beta$, where β is as in Theorem 1.3.1, and let n be the integer part of s. Then $\mathcal{S}$ and $\mathcal{W}$ cannot be extended to maps $L^2 \to L^2$ that admit an n-th Gateaux derivative defined about $0 \in L^2$ which is Hölder continuous of order $s-n$.*

The proof of this theorem is the focus of Chapter 3.

We interpret this result as a well-posedness result on the scattering problem for pNLS for initial/final data in Σ, and as an ill-posedness result for the scattering problem for data

Table 1.2: Summary of scattering theory for mass-subcritical HNLS

Type of result	Result
Scattering in Σ	Ginibre-Velo '80: $2 < \gamma < \min(4, d)$: LWP, GWP, conservation laws, first scattering result in Σ Strauss '81: $\frac{4}{3} < \gamma < \min(4, d)$: wave operators on $L^{\frac{4d}{2d+\gamma}} \cap L^2$ Hayashi-Tsutsumi '87: $\frac{4}{3} < \gamma < \min(4, d)$, $d \geq 2$: scattering in Σ Hayashi-Ozawa '88: $\frac{4}{3} < \gamma < 2$, $d \geq 3$: wave operators on $\mathcal{F}H^k$ Nawa-Ozawa '92: $1 < \gamma < 2$, $d \geq 2$: wave operators on $\mathcal{F}H^k$ Masaki '19: $1 < \gamma < 2$: GWP and scattering in $\mathcal{F}\dot{H}^{s_c}$ given global spacetime bounds
Asymptotic completeness	Hayashi-Tsutsumi '87: $1 < \gamma < \min(4, d)$, $d \geq 2$, scattering in L^2 for initial data in Σ Hayashi-Ozawa '88: $1 < \gamma < \min(2, d)$, scattering in L^2 for initial data in Σ
Final-state problem	Holmer-Tzirakis '10: $1 < \gamma < 2$, $d = 2$: every H^1 scattering state has H^1 global solution scattering to it
Negative results	Glassey '77, Hayashi-Tsutsumi '87: $0 < \gamma \leq 1$: all global solutions in Σ scattering in L^2-norm are trivial Cho-Hwang-Ozawa '16: $\mathcal{F}(V)(\xi) \sim \lvert\xi\rvert^{-(d-\gamma)}$ near 0, $0 < \gamma \leq 1$: nonexistence of L^2 scattering for H^s data

in L^2 in the sense of Bourgain [7]. It states that any hypothetical extension of the scattering theory of pNLS from Σ to L^2 must come at a cost. Such an extension of the theory would amount to an extension of the scattering operators $\mathcal{S}$ and $\mathcal{W}$ from $\Sigma \to L^2$ to $L^2 \to L^2$. Theorem 1.3.1 states that these hypothetical extensions of the scattering operators can only be defined at the cost of losing a positive amount of regularity through the extension. In particular, at the origin they fail to have the expected regularity C^{1+p} that one would expect from the smoothness of the pNLS nonlinearity $F(u) = |u|^p u$, which the scattering operators do enjoy as maps $\Sigma \to L^2$.

Finally, we state the third and last of the results we will prove and discuss in this thesis, which is a stronger version of Theorem 1.3.1 for the mass-subcritical HNLS.

Theorem 1.3.3 (Analyticity of the HNLS scattering operators [39]). *Let $d \geq 2$ and $\frac{4}{3} < \gamma < 1$. Let $\mathcal{T} \in \{\mathcal{S}, \mathcal{W}\}$. Then:*

1. *$\mathcal{T}$ is well-defined as a map $\Sigma \to \Sigma$, and is analytic in the sense that for all $u_0 \in \Sigma$ and $v \in \Sigma$, $\mathcal{T}$ admits the power series expansion*

$$\mathcal{T}(u_0 + \varepsilon v) = \mathcal{T}(u) + \sum_{k=1}^{\infty} \varepsilon^k w_k$$

   for all sufficiently small $\varepsilon > 0$, where $(w_k) \subset \Sigma$ and the series converges in Σ-norm.

2. *The same result holds with the space $\mathcal{F}H^1$ replacing Σ.*

Theorem 1.3.4 (Breakdown of analyticity of the HNLS scattering operators [39]). *Let $d \geq 2$ and $\frac{4}{3} < \gamma < 2$. Let $\mathcal{T} \in \{\mathcal{S}, \mathcal{W}\}$.*

1. *Let $s > \frac{5+5\gamma}{3+\gamma}$. Then $\mathcal{T} : \Sigma \to L^2$ admits no extension to a map $L^2 \to L^2$ which is Hölder continuous of order s on any ball B containing $0 \in L^2$.*

2. *Let $s > \frac{4+4\gamma}{2+\gamma}$. Then there exists $R > 0$ such that for any ball $B \subset B_R(0) \subset \Sigma$ (not necessarily containing the origin), $\mathcal{T} : B \to L^2$ admits no extension to a map $L^2 \to L^2$ which is Hölder continuous of order s at any point in $B \cap L^2$.*

These results are the focus of Chapter 4. We note that for $\frac{4}{3} < \gamma < 2$, $\frac{5+5\gamma}{3+\gamma} < \frac{4+4\gamma}{2+\gamma}$. Thus the breakdown of regularity we obtain is more severe at the origin than elsewhere. In particular, we see that $\mathcal{T} : \Sigma \to L^2$ has no C^3 extension to a map $L^2 \to L^2$. The lower range of s for which Hölder continuity fails is $s > \frac{5+5\gamma}{3+\gamma} > \frac{35}{13} \approx 2.69$ for part 1, and $s > \frac{4+4\gamma}{2+\gamma} > \frac{14}{5} = 2.8$ for part 2.

Theorems 1.3.3 and 1.3.4 are direct analogues of Theorem 1.3.1 for the Hartree equation. Theorem 1.3.3 states that the scattering problem for HNLS is analytically well-posed for initial/final data in Σ and $\mathcal{F}H^1$. The analyticity is consistent with the fact that the Hartree nonlinearity $F(u) = (|x|^{-\gamma} * |u|^2)u$ depends analytically on the solution u. We isolate Theorem 1.3.3 as a separate result from Theorem 1.3.4 because we consider it to be one of independent interest for the scattering theory of HNLS. Theorem 1.3.4 states that despite Theorem 1.3.3, which says that the scattering problem in Σ for HNLS is as well-posed as it can possibly be, the analogous problem in L^2 exhibits at best a finite amount of regularity with respect to the initial/final data.

Theorems 1.3.3 and 1.3.4 improve on Theorem 1.3.1 in the following ways:

1. They comprise an *infinite* loss of regularity between the scattering problems in Σ and in L^2, whereas in Theorem 1.3.1 the loss of regularity is finite in magnitude. This suggests that the smoothness of the nonlinearity does not play a significant role in the $\Sigma - L^2$ disparity in the scattering theory for mass-subcritical nonlinear Schrödinger equations. (However, we note that the gauge invariance of the nonlinearity does play an important role in all of our results.)

2. The expansion of $\mathcal{T}$ and the breakdown of regularity are proved at points $u_0 \neq 0$ as well. In Theorem 1.3.1 the analogous claims are only proved at the origin, due to technical difficulties in working with the fractional power $|u|^p u$ arising from its lack of smoothness.

A key component of the proof of Theorems 1.3.1 and 1.3.4 is the L_t^1-norm of the potential energy

$$\int_0^\infty Q(u(t))\ dt,$$

where Q is the potential energy functional 1.3.3, 1.3.4. This quantity can be thought of as measuring the rate of decay of $u(t)$. Boundedness equates to decay consistent with or faster than the dispersive decay of free evolutions, which is a necessary condition for scattering. This idea appeared in [11] in order to define a nation of "rapidly decaying" solution to pNLS and an associated scattering criterion. We obtain the breakdown of regularity in the scattering operators by relating this statement to the fact that by scaling, this energy-type functional cannot be bounded in L^2 for a scattering solution.

1.4 Notation

In this section we establish the notation we will employ for the remainder of this thesis.

Let X and Y be two quantities. We write $X \lesssim Y$ if there exists a constant $C > 0$ such that $X \leq CY$. If C depends on parameters $a_1, \ldots, a_n$, i.e. $C = C(a_1, \ldots, a_n)$ and we wish to indicate this dependence, then we will write $X \lesssim_{a_1,\ldots,a_n} Y$. If $X \lesssim Y$ and $Y \lesssim X$, we write $X \sim Y$. If the constant C is small, then we write $X \ll Y$. We also employ the asymptotic notation $\mathcal{O}(f)$ and $o(f)$ with their standard meanings.

We adopt the Japanese bracket notation $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$.

1.4.1 Notation for Chapter 2

We define the symmetric spacetime norms

$$\|u\|_{L^p_{t,x}([0,T)\times\mathbb{T}^3)} = \left(\int_0^T \int_{\mathbb{T}^3} |u(t,x)|^p\ dxdt \right)^{\frac{1}{p}}$$

where $1 \leq p \leq \infty$, with obvious changes if $p = \infty$. We often suppress the spacetime domain and write $\|u\|_{L^p_{t,x}}$.

Let $\phi : \mathbb{R}^d \to [0, \infty)$ be a smooth radial cutoff with $\phi(x) = 1$ for $|x| \leq 1$ and $\phi(x) = 0$ for $x \geq 2$. We take the convention $0 \in \mathbb{N}$. For $N \in 2^{\mathbb{N}}$ a dyadic integer, write $\phi_N(x) = \phi(x/N)$ and $\psi_N(x) = \phi_N(x) - \phi_{\frac{N}{2}}(x)$, with the convention $\psi_1(x) = \phi(x)$. For $f : \mathbb{T}^d \to \mathbb{C}$ with Fourier coefficients $\widehat{f}(\xi)$, $\xi \in \mathbb{Z}^d$, we define the Littlewood-Paley projections as Fourier multipliers:

$$\widehat{u_{\leq N}}(\xi) = \widehat{P_{\leq N} f}(\xi) = \phi_N(\xi)\widehat{f}(\xi), \ \widehat{u_N}(\xi) = \widehat{P_N f}(\xi) = \psi_N(\xi)\widehat{f}(\xi), \ \xi \in \mathbb{Z}^d.$$

In Chapter 2, integers denoted by N, N_0, N_1, and the like will be implicitly assumed to be dyadic integers.

1.4.2 Common notation and estimates for Chapters 3 and 4

We will be working with the mixed spacetime Lebesgue spaces $L_t^q L_x^r(I \times \mathbb{R}^d)$, with norms

$$\|u\|_{L_t^q L_x^r(I \times \mathbb{R}^d)} = \left(\int_I \left(\int_{\mathbb{R}^d} |u(t,x)|^r \ dx \right)^{\frac{q}{r}} \ dt \right)^{\frac{1}{q}}.$$

We will abbreviate the norm as $\|u\|_{L_t^q L_x^r(I \times \mathbb{R}^d)} = \|u\|_{L_t^q L_x^r(I)}$. When I is clear from context we will further abbreviate the norm as $\|u\|_{q,r}$. For purely spatial integration, we write $\|f\|_{L^r(\mathbb{R}^d)} = \|f\|_r$. For $1 \leq r \leq \infty$, we denote by r' the Hölder conjugate: $1 = \frac{1}{r} + \frac{1}{r'}$. We will also occasionally use the mixed Lorentz-Lebesgue spaces $L_t^{q,p} L_x^r(I \times \mathbb{R}^d)$, where for $1 \leq p < \infty$, $L_t^{q,p}(I)$ denotes the Lorentz space defined by the quasinorm

$$\|f\|_{L_t^{q,p}(I)} = q^{\frac{1}{p}} \left(\int_0^\infty t^p |\{s \in I : |f(s)| \geq t\}|^{\frac{p}{q}} \ \frac{dt}{t} \right)^{\frac{1}{q}},$$

and $L^{q,\infty}$ denotes weak L^q.

We recall the following fundamental estimates for the Schrödinger equation.

Proposition 1.4.1 (Dispersive estimate)**.** *Let $2 \leq r \leq \infty$. Then for all $t \neq 0$,*

$$\|e^{it\Delta}\phi\|_{L_x^r(\mathbb{R}^d)} \lesssim_{r,d} |t|^{\frac{d}{2} - \frac{d}{r}} \|\phi\|_{L^{r'}(\mathbb{R}^d)}.$$

Definition 1.4.1 (Admissible pair)**.** Let $d \geq 1$ and $2 \leq q, r \leq \infty$. We say that (q, r) is an *admissible pair* if it satisfies the scaling relation $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ and $(d, q, r) \neq (2, 2, \infty)$. We say that (α, β) is a *dual admissible pair* if (α', β') is an admissible pair.

Proposition 1.4.2 (Strichartz estimates). *Let $d \geq 1$, let (q, r) be an admissible pair, and let (α, β) be a dual admissible pair. Then for any interval $I \subset \mathbb{R}$,*

$$\|e^{it\Delta}\phi\|_{L_t^q L_x^r(I\times\mathbb{R}^d)} \lesssim \|\phi\|_{L^2(\mathbb{R}^d)},$$
$$\left\|\int_{\mathbb{R}} e^{-is\Delta}F(s)\ ds\right\|_{L^2(\mathbb{R}^d)} \lesssim \|F\|_{L_t^\alpha L_x^\beta(\mathbb{R}\times\mathbb{R}^d)},$$
$$\left\|\int_{s<t} e^{i(t-s)\Delta}F(s)\ ds\right\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|F\|_{L_t^\alpha L_x^\beta(\mathbb{R}\times\mathbb{R}^d)}.$$

Also, the following Lorentz space versions of these estimates hold for $2 < q < \infty$:

$$\|e^{it\Delta}\phi\|_{L_t^{q,2} L_x^r(I\times\mathbb{R}^d)} \lesssim \|\phi\|_{L^2(\mathbb{R}^d)},$$
$$\left\|\int_{s<t} e^{i(t-s)\Delta}F(s)\ ds\right\|_{L_t^{q,2} L_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|F\|_{L_t^{q',2} L_x^{r'}(\mathbb{R}\times\mathbb{R}^d)}.$$

Lastly, we define the notion of pointwise Hölder regularity.

Definition 1.4.2 (Pointwise Hölder space [1]). Let X and Y be Banach spaces. Let $x_0 \in X$ and U a convex open neighborhood of x_0. Fix $s > 0$, and let n be the integer part of s. For $s > 0$, we say that the map $G : X \to Y$ belongs to the *pointwise Hölder space* $C^s(x_0)$ if for all $h \in X$ with $\|h\|_X = 1$, there exist coefficients $\{a_j(x_0; h)\}_{j=0}^n \subset Y$ such that

$$\|G(x_0 + \varepsilon h) - G(x_0) - \sum_{j=1}^{n} \varepsilon^j a_j(x_0; h)\|_Y \lesssim \varepsilon^s$$

for all $\varepsilon > 0$ sufficiently small, with the implicit constant independent of the direction h.

Our main interest in $C^s(x_0)$ is that membership in $C^s(x_0)$ is a necessary, though not sufficient, condition for a stronger notion of regularity of order s:

Lemma 1.4.3. *Let X and Y be Banach spaces. Let $U \subset X$ be a convex neighborhood of $x_0 \in X$. Let $G : U \to Y$ be a map, and suppose $G \notin C^s(x_0)$ with $n < s < n + 1$. Then $d^nG(x; h)$ (the n-th Gateaux derivative of G), if it exists for $x \in U$, cannot be a Hölder continuous function of x of order $s - n$ with Hölder seminorm uniformly bounded in h.*

For the proof of Lemma 1.4.3, as well as the relationship between Definition 1.4.2 and other notions of regularity, we refer the reader to Appendix A.

CHAPTER 2

Local well-posedness for periodic NLS in scaling-critical Sobolev spaces in dimension three

In this chapter we prove Theorem 1.2.1, which we restate for the reader's convenience:

Theorem 2.0.1. *Let $p \geq 2$, and fix $d = 3$. Then Equation (1.2.1) is well-posed in $H^{s_c}(\mathbb{T}^3)$. In particular, for any $u_0 \in H^{s_c}(\mathbb{T}^3)$, there exists a time of existence $T = T(u_0)$ and a unique solution $u \in C_t([0,T); H^{s_c}(\mathbb{T}^3)) \cap X^{s_c}([0,T))$ to (1.2.1).*

Let us summarize the main ideas of the proof. The construction of the solution proceeds via a standard contraction mapping argument. The key tool in our analysis is the scale-invariant bilinear Strichartz estimate proved in [34], which straightforwardly handles the case of even values of p.

The main technical difficulty in this theorem arises when p is a non-even, possibly fractional power. In this case we lose access to some algebraic simplifications which would normally allow for an immediate decomposition of the nonlinearity along various frequency interactions. For instance, in the cubic case $p = 2$, we can write

$$|u|^2 u = \sum_{L,M,N} u_L \overline{u}_M u_N,$$

and the sum can then be further decomposed by classifying the types of these frequency interactions (high-high-high, high-high-low, etc.). However, when p is not an even integer this is not immediately possible.

We overcome this by iterating a paradifferential linearization technique of Bony, which allows us to isolate out various frequency interactions from the nonlinearity as long as we can continue to differentiate it. However, since the nonlinearity is differentiable only finitely many times, we eventually run into the limits of this technique. We overcome this last hurdle using the idea that a (well-behaved) nonlinear function of a low-frequency projection still lives at low frequencies, which manifests in the form of estimates such as the nonlinear Bernstein inequality. This allows us to obtain a sufficiently detailed decomposition of the nonlinearity into frequency interactions to close the contraction mapping estimate.

2.1 Preliminary definitions and estimates

We define the adapted function spaces U^p, V^p, X^s, and Y^s. We restrict ourselves to stating the definitions and basic properties. For a more complete reference, see [38]. Fix a finite time interval $[0, T)$. Let H be a separable Hilbert space over $\mathbb{C}$; for our purposes, this will be $\mathbb{C}$, $L^2(\mathbb{T}^3)$, or $H^{s_c}(\mathbb{T}^3)$. Let $\mathcal{Z}$ be the set of finite partitions $0 = t_0 < t_1 < \cdots < t_K \leq T$. We adopt the convention that $v(T) = 0$ for all functions $v : [0, T) \to H$.

Definition 2.1.1. Let $1 \leq p < \infty$. A U^p-*atom* is a function $a : [0, T) \to H$ of the form

$$a = \sum_{k=0}^{K-1} 1_{[t_k, t_k+1)} \phi_k,$$

where $\{t_k\} \in \mathcal{Z}$ and $\{\phi_k\} \subset H$ with $\sum_{k=0}^{K-1} \|\phi_k\|_H^p \leq 1$. The space $U^p([0, T); H)$ is the space of all functions $u : [0, T) \to H$ admitting a representation of the form

$$u = \sum_{j=1}^{\infty} \lambda_j a_j,$$

where a_j are U^p-atoms and $(\lambda_j) \in \ell^1(\mathbb{C})$. $U^p([0, T); H)$ is a Banach space under the norm

$$\|u\|_{U^p} = \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : u = \sum_{j=1}^{\infty} \lambda_j a_j \text{ with } (\lambda_j) \in \ell^1(\mathbb{C}) \text{ and } a_j \ U^p - \text{atoms} \right\}.$$

Definition 2.1.2. Let $1 \leq p < \infty$. $V^p([0, T); H)$ is the space of all functions $v : [0, T) \to H$ with finite V^p-seminorm $\|v\|_{V^p}$, where

$$\|v\|_{V^p} = \sup_{\{t_k\} \in \mathcal{Z}} \left(\sum_{k=1}^{K-1} \|v(t_k) - v(t_{k-1})\|_H^p \right)^{\frac{1}{p}}.$$

The space V^p_{rc} is the subspace of V^p consisting of right-continuous functions in V^p, normalized so that $\lim_{t\to 0^+} v(t) = 0$. The V^p-seminorm restricts to a norm on V^p_{rc}, and V^p_{rc} is a Banach space under this norm.

Definition 2.1.3. Let $s \in \mathbb{R}$, $d \geq 1$. We define $X^s([0,T))$ and $Y^s([0,T))$ to be the Banach spaces of all functions $u : [0,T) \to H^s(\mathbb{T}^d)$ such that for every $\xi \in \mathbb{Z}^d$, the map $t \mapsto \widehat{e^{-it\Delta}u(t)}(\xi)$ is in $U^2([0,T);\mathbb{C})$ and $V^2_{\mathrm{rc}}([0,T);\mathbb{C})$ respectively, with norms

$$\|u\|_{X^s([0,T))} = \left(\sum_{\xi\in\mathbb{Z}^d} \langle\xi\rangle^{2s} \|\widehat{e^{-it\Delta}u}(t)(\xi)\|^2_{U^2} \right)^{\frac{1}{2}},$$

$$\|u\|_{Y^s([0,T))} = \left(\sum_{\xi\in\mathbb{Z}^d} \langle\xi\rangle^{2s} \|\widehat{e^{-it\Delta}u}(t)(\xi)\|^2_{V^2} \right)^{\frac{1}{2}}.$$

Again, we will typically suppress the spacetime domain in our notation when it is obvious. X^s and Y^s have a dual pairing in the following sense:

Proposition 2.1.1 (X^s-Y^s duality; [29, Proposition 2.11])**.** *Let $s \geq 0$ and $T > 0$. For $f \in L^1([0,T); H^s(\mathbb{T}^d))$ we have*

$$\left\| \int_0^t e^{i(t-s)\Delta} f(s)\, ds \right\|_{X^s([0,T))} \leq \sup_{\|v\|_{Y^{-s}([0,T))}=1} \left| \int_0^T \int_{\mathbb{T}^3} f(t,x)\overline{v(t,x)}\, dxdt \right|.$$

Remark 2.1.1. We have a continuous embedding $X^s \hookrightarrow Y^s$. We also have

$$\|u\|_{L^\infty_t H^s_x} \lesssim \|u\|_{X^s},$$
$$\left\| \int_0^t e^{i(t-s)\Delta} F(s)\, ds \right\|_{X^s} \lesssim \|F\|_{L^1_t H^s_x}.$$

Remark 2.1.2. The spaces X^s and Y^s have the scaling of $L^\infty_t H^s_x$ and enjoy several of its Fourier-based properties: for instance, we have

$$\|P_N u\|_{Y^s} \sim N^s \|P_N u\|_{Y^0}$$

and

$$\|u\|_{Y^s} = \left(\sum_N \|P_N u\|^2_{Y^s} \right)^{\frac{1}{2}}.$$

We now state the main tools of our analysis.

Theorem 2.1.2 (Strichartz estimates [34])**.** *Fix $d \geq 1$, $1 \leq N \in 2^{\mathbb{N}}$, and $p > \frac{2(d+2)}{d}$. Then*

$$\|P_C u\|_{L^p_{t,x}([0,1]\times\mathbb{T}^d)} \lesssim N^{\frac{d}{2}-\frac{d+2}{p}} \|P_C u\|_{Y^0(\mathbb{T}^d)}$$

for all $p > \frac{2(d+2)}{d}$, where $C \subset \mathbb{R}^d$ is a cube of side length N and P_C denotes the Fourier projection to C.

As a direct consequence of Theorem 2.1.2, we have:

Lemma 2.1.3 (Bilinear Strichartz estimate [34, Lemma 3.1])**.** *Fix $d \geq 3$ and $T \leq 1$. Then for all $1 \leq N_2 \leq N_1$,*

$$\|u_{N_1} v_{N_2}\|_{L^2_{t,x}} \lesssim N_2^{\frac{d-2}{2}} \|u_{N_1}\|_{Y^0} \|v_{N_2}\|_{Y^0}.$$

The implicit constant does not depend on T.

Lastly, we state some useful fractional calculus estimates:

Proposition 2.1.4 (Fractional product rule)**.** *Let $d \geq 1$, $s > 0$, $1 < p < \infty$, and $1 < p_2, q_2 \leq \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}$. Then*

$$\||\nabla|^s (fg)\|_{L^p(\mathbb{T}^d)} \lesssim \||\nabla|^s f\|_{L^{p_1}(\mathbb{T}^d)} \|g\|_{L^{p_2}(\mathbb{T}^d)} + \||\nabla|^s g\|_{L^{q_1}(\mathbb{T}^d)} \|f\|_{L^{q_2}(\mathbb{T}^d)}.$$

Proposition 2.1.5 (Fractional chain rule)**.** *Suppose $F : \mathbb{C} \to \mathbb{C}$ satisfies $|F(u) - F(v)| \lesssim |u - v|(G(u) + G(v))$ for some $G : \mathbb{C} \to [0, \infty)$. Let $d \geq 1$, $0 < s < 1$, $1 < p < \infty$, and $1 < p_2 \leq \infty$, such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then*

$$\||\nabla|^s F(u)\|_{L^p(\mathbb{T}^d)} \lesssim \||\nabla|^s u\|_{L^{p_1}(\mathbb{T}^d)} \|G(u)\|_{L^{p_2}(\mathbb{T}^d)}.$$

Proposition 2.1.6 (Nonlinear Bernstein)**.** *Let $G : \mathbb{C} \to \mathbb{C}$ be Hölder continuous of order $\alpha \in (0, 1]$. Let $d \geq 1$ and $1 \leq p \leq \infty$. Then for $u : \mathbb{T}^d \to \mathbb{C}$ smooth and periodic, we have*

$$\|P_N G(u)\|_{L^{p/\alpha}(\mathbb{T}^d)} \lesssim N^{-\alpha} \|\nabla u\|^{\alpha}_{L^p(\mathbb{T}^d)}$$

for all $N > 1$.

In the Euclidean setting Propositions 2.1.4 and 2.1.5 are due to Christ and Weinstein [14]. Proposition 2.1.6 in the Euclidean setting appears in [33]. These results can be extended to the periodic setting via estimates on the periodic Littlewood-Paley convolution kernels.

2.2 Contraction mapping

Fix an initial datum $u_0 \in H^{s_c}(\mathbb{T}^3)$. Consider the Duhamel operator

$$\Phi(u)(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}F(u(s))\, ds,$$

where $F(u) = \pm|u|^p u$. As the choice of sign is irrelevant for everything that follows, we will take $F(u) = |u|^p u$ from here on.

The proof of Theorem 2.0.1 proceeds via a standard contraction mapping argument in the space $C_t([0,T); H^{s_c}(\mathbb{T}^3)) \cap X^{s_c}([0,T))$. For initial data u_0 it suffices to show that there exists a time T and a ball $B \subset C_t([0,T); H^{s_c}(\mathbb{T}^3)) \cap X^{s_c}([0,T))$ on which Φ is a self-map and contraction mapping. Then the Banach fixed-point theorem implies that Φ has a unique fixed point in B, which is the solution to (1.2.1) we seek. The goal of this section is the contraction mapping estimate:

Proposition 2.2.1. *Fix $p \geq 2$. Let $0 < T \leq 1$. Then*

$$\left\| \int_0^t e^{i(t-s)\Delta}[F(u+w)(s) - F(u)(s)]\, ds \right\|_{X^{s_c}([0,T))} \lesssim \|w\|_{X^{s_c}([0,T))}(\|u\|_{X^{s_c}([0,T))} + \|w\|_{X^{s_c}([0,T))})^p.$$

The implicit constant does not depend on T.

Proposition 2.2.1, together with Propositions 2.1.4 and 2.1.5, imply that Φ is indeed a self-map and contraction mapping on some ball in $X^{s_c}([0,T]) \cap C_t H^{s_c}_x([0,T] \times \mathbb{T}^3)$, provided that T is chosen sufficiently small. As just one example of this argument, we refer the reader to Section 4 of [34] for the details in the case of the H^1-critical cubic and quintic NLS.

Therefore, we focus our attention on the proof of Proposition 2.2.1 for the remainder of this paper.

First we make some reductions. By X^s-Y^s-duality (Proposition 2.1.1) and self-adjointness of Littlewood-Paley projections,

$$\begin{aligned}\left\| \int_0^t e^{i(t-s)\Delta} P_{\leq N}[F(u+w)(s) - F(u)(s)]\ ds \right\|_{X^{s_c}([0,T))} \\ &\leq \sup_{\|\tilde{v}\|_{Y^{-s_c}=1}} \left| \int_0^T \int_{\mathbb{T}^3} P_{\leq N}[F(u+w)(t) - F(u)(t)]\overline{\tilde{v}(t,x)}\ dxdt \right| \\ &= \sup_{\|\tilde{v}\|_{Y^{-s_c}=1}} \left| \int_0^T \int_{\mathbb{T}^3} [F(u+w)(t) - F(u)(t)]\overline{P_{\leq N}\tilde{v}(t,x)}\ dxdt \right|\end{aligned}$$

We will prove the following estimate:

$$\begin{aligned}\left| \int_0^T \int_{\mathbb{T}^3} v(t,x)[F(u+w)(t) - F(u)(t)]\ dxdt \right| \\ \lesssim \|v\|_{Y^{-s_c}} \|w\|_{Y^{s_c}} (\|u\|_{Y^{s_c}} + \|w\|_{Y^{s_c}})^p. \qquad (2.2.1)\end{aligned}$$

Taking $v = \overline{P_{\leq N}\tilde{v}}$ and letting $N \to \infty$ then gives us Proposition 2.2.1.

Let us give a brief outline of the proof of (2.2.1). This estimate is easiest to prove when p is an even integer. In this case, expanding u and w in terms of their Littlewood-Paley decompositions, we can write $F(u+w) - F(u)$ as a sum of products of frequency projections of $u, \overline{u}, w, \overline{w}$, and emulate earlier arguments such as those in [29, 32, 34]. We will provide an explicit argument for $p = 2$ in Section 2.2.1. This is also necessary, as some of our estimates for the case $p > 2$ do not extend down to the endpoint.

The remainder of the paper treats $p > 2$. When p is not an even integer, the above argument can no longer be carried out exactly. However, the nonlinearity can be rewritten in a more manageable form using a paradifferential calculus technique known as the Bony linearization formula: we write $F(u) = F(u_{\leq 1}) + \sum_{N \geq 2} [F(u_{\leq N}) - F(u_{\leq \frac{N}{2}})]$, and use the fundamental theorem of calculus to rewrite the differences, obtaining expressions essentially of the form $F(u) \sim \sum_{N \geq 1} u_N |u_{\leq N}|^p$. Iterating this sort of argument gives us an expression for $F(u+w) - F(u)$ that is essentially multilinear and treatable with known technology.

2.2.1 Proof of Theorem 2.0.1 for the cubic NLS

We first consider the $H^{\frac{1}{2}}$-critical cubic NLS, the case $p = 2$ of Theorem 2.0.1. This proof is very similar to the proof of the contraction estimate for the H^1-critical cubic NLS on $\mathbb{T}^4$ in [34]. By the previous discussion, to prove Theorem 2.0.1 for $p = 2$ it suffices to prove (2.2.1) for $p = 2$.

Proof of (2.2.1) *for* $p = 2$. Expanding u and w by their Littlewood-Paley decompositions and doing some combinatorics, (2.2.1) reduces to an estimate of the form

$$\sum_{N_0 \geq 1} \sum_{N_1 \geq N_2 \geq N_3 \geq 1} \left| \int_0^T \int_{\mathbb{T}^3} v_{N_0} u_{N_1}^{(1)} u_{N_2}^{(2)} u_{N_3}^{(3)} \, dxdt \right| \lesssim \|v\|_{Y^{-\frac{1}{2}}} \prod_{j=1}^{3} \|u^{(j)}\|_{X^{\frac{1}{2}}}, \tag{2.2.2}$$

where $u^{(j)}$ are chosen from $\{u, \overline{u}, w, \overline{w}\}$. In order for the integrals in (2.2.2) to be nonzero, the two highest frequencies must be comparable. Therefore the sum splits into two cases:

1. $N_0 \sim N_1 \geq N_2 \geq N_3$. We use the following idea which appeared in [29,34]. Let $\mathbb{Z}^3 = \bigcup_j C_j$ be a partition of frequency space $\mathbb{Z}^3$ into cubes C_j of side length N_2. We write $C_j \sim C_k$ if the sum set $C_j + C_k$ overlaps the Fourier support of $P_{\leq 2N_2}$. For a given C_k, there are a bounded number (independent of k and N_2) of C_j such that $C_j \sim C_k$. By Hölder,

Strichartz, and summing via Cauchy-Schwarz, we estimate:

$$\begin{aligned}
&\sum_{N_0\sim N_1\geq N_2\geq N_3\geq 1}\left|\int_0^T\int_{\mathbb{T}^3} v_{N_0}u^{(1)}_{N_1}u^{(2)}_{N_2}u^{(3)}_{N_3}\,dxdt\right|\\
&=\sum_{N_0\sim N_1\geq N_2\geq N_3}\sum_{C_j\sim C_k}\left|\int_0^T\int_{\mathbb{T}^3}(P_{C_j}v_{N_0})(P_{C_k}u^{(1)}_{N_1})u^{(2)}_{N_2}u^{(3)}_{N_3}\,dxdt\right|\\
&\leq\sum_{N_0\sim N_1\geq N_2\geq N_3}\sum_{C_j\sim C_k}\|P_{C_j}v_{N_0}\|_{L^{18/5}_{t,x}}\|P_{C_k}u^{(1)}_{N_1}\|_{L^{18/5}_{t,x}}\|u^{(2)}_{N_2}\|_{L^{18/5}_{t,x}}\|u^{(3)}_{N_3}\|_{L^6_{t,x}}\\
&\lesssim\sum_{N_0\sim N_1\geq N_2\geq N_3}\sum_{C_j\sim C_k}N_2^{\frac{1}{3}}N_3^{\frac{2}{3}}\|P_{C_j}v_{N_0}\|_{Y^0}\|P_{C_k}u^{(1)}_{N_1}\|_{Y^0}\|u^{(2)}_{N_2}\|_{Y^0}\|u^{(3)}_{N_3}\|_{Y^0}\\
&\lesssim\sum_{N_0\sim N_1\geq N_2\geq N_3}\sum_{C_j\sim C_k}\left(\frac{N_3}{N_2}\right)^{\frac{1}{6}}\|P_{C_j}v_{N_0}\|_{Y^{-\frac{1}{2}}}\|P_{C_k}u^{(1)}_{N_1}\|_{Y^{\frac{1}{2}}}\|u^{(2)}_{N_2}\|_{Y^{\frac{1}{2}}}\|u^{(3)}_{N_3}\|_{Y^{\frac{1}{2}}}\\
&\lesssim\|v\|_{Y^{-\frac{1}{2}}}\prod_{j=1}^{3}\|u^{(j)}\|_{Y^{\frac{1}{2}}}.
\end{aligned}$$

From here (2.2.2) follows from the embedding $X^s\hookrightarrow Y^s$.

2. $N_0\leq N_1\sim N_2\geq N_3$. For this sum we can estimate with just Hölder, Strichartz, and Cauchy-Schwarz:

$$\begin{aligned}
&\sum_{N_0\leq N_1\sim N_2\geq N_3\geq 1}\left|\int_0^T\int_{\mathbb{T}^3} v_{N_0}u^{(1)}_{N_1}u^{(2)}_{N_2}u^{(3)}_{N_3}\,dxdt\right|\\
&\lesssim\sum_{N_0\leq N_1\sim N_2\geq N_3\geq 1}\|v_{N_0}\|_{L^{18/5}_{t,x}}\|u^{(1)}_{N_1}\|_{L^{18/5}_{t,x}}\|u^{(2)}_{N_2}\|_{L^{18/5}_{t,x}}\|u^{(3)}_{N_3}\|_6\\
&\lesssim\sum_{N_0\leq N_1\sim N_2\geq N_3\geq 1}N_0^{\frac{1}{9}}N_1^{\frac{1}{9}}N_2^{\frac{1}{9}}N_3^{\frac{2}{3}}\|v_{N_0}\|_{Y^0}\|u^{(1)}_{N_1}\|_{Y^0}\|u^{(2)}_{N_2}\|_{Y^0}\|u^{(3)}_{N_3}\|_{Y^0}\\
&\sim\sum_{N_0\leq N_1\sim N_2\geq N_3\geq 1}\frac{N_0^{\frac{11}{18}}N_3^{\frac{1}{6}}}{N_1^{\frac{7}{18}}N_2^{\frac{7}{18}}}\|v_{N_0}\|_{Y^{-\frac{1}{2}}}\|u^{(1)}_{N_1}\|_{Y^{\frac{1}{2}}}\|u^{(2)}_{N_2}\|_{Y^{\frac{1}{2}}}\|u^{(3)}_{N_3}\|_{Y^{\frac{1}{2}}}\\
&\lesssim\|v\|_{Y^{-\frac{1}{2}}}\|u\|_{Y^{\frac{1}{2}}}\sum_{N_1\sim N_2}\left(\frac{N_1}{N_2}\right)^{\frac{2}{9}}\|u^{(1)}_{N_1}\|_{Y^{\frac{1}{2}}}\|u^{(2)}_{N_2}\|_{Y^{\frac{1}{2}}}\\
&\lesssim\|v\|_{Y^{-\frac{1}{2}}}\prod_{j=1}^{3}\|u^{(j)}\|_{Y^{\frac{1}{2}}}.
\end{aligned}$$

□

Remark 2.2.1. Essentially the same proof will also give the contraction estimate for $p=4,6,8,\ldots$. We can even take the same Hölder exponents for the four highest frequencies; any remaining lower-frequency terms can be placed in $L^\infty_{t,x}$, and a similar argument as above will establish the claim.

2.2.2 Paradifferential linearization

We now extend the argument of the previous section to the case of general $p > 2$. Our main tool is the following paradifferential formula, which appeared in [4] with alternative hypotheses. See [56] for a textbook treatment of this subject.

Proposition 2.2.2 (Bony linearization formula)**.** *Let $g \in H^{s_c}(\mathbb{T}^d)$, $d \geq 3$, and let $F(z) = |z|^p z$ for $z \in \mathbb{C}$, $p \geq 0$. Then for all $1 \leq q < \frac{d}{2}$, we have*

$$F(g) = \lim_{N\to\infty} F(g_{\leq N}) = \lim_{N\to\infty} F(g_{\leq 1}) + \sum_{2\leq M\leq N} [F(g_{\leq M}) - F(g_{\leq \frac{M}{2}})], \tag{2.2.3}$$

where the limit is in the $L^q(\mathbb{T}^d)$-topology.

Proof. By the bound $|F(g) - F(h)| \lesssim |g - h|(|g|^p + |h|^p)$ and Sobolev embedding, we have

$$\begin{aligned}\|F(g) - F(g_{\leq N})\|_{L^q} &\lesssim \|g - g_{\leq N}\|_{L^{dq/(d-2q)}}(\|g\|^p_{L^{dp/2}} + \|g_{\leq N}\|^p_{L^{dp/2}}) \\ &\lesssim \|g - g_{\leq N}\|_{L^{dq/(d-2q)}}(\|g\|^p_{H^{s_c}} + \|g_{\leq N}\|^p_{H^{s_c}}).\end{aligned}$$

Under the given hypotheses, $1 < \frac{dq}{d-2q} < \infty$. Therefore $g_{\leq N} \to g$ in $L^{dq/(d-2q)}$. So in the limit, we obtain

$$\lim_{N\to\infty} \|F(g) - F(g_{\leq N})\|_{L^q} \lesssim \|g\|^p_{H^{s_c}} \lim_{N\to\infty} \|g - g_{\leq N}\|_{L^{dq/(d-2q)}} = 0. \qquad \square$$

We combine Proposition 2.2.2 and a linearization of $F(h_{\leq N}) - F(h_{\leq \frac{N}{2}})$. Using the identity

$$F(u + w) - F(u) = w \int_0^1 \partial_z F(u + \theta w)\, d\theta + \overline{w} \int_0^1 \partial_{\overline{z}} F(u + \theta w)\, d\theta, \tag{2.2.4}$$

we obtain

$$\begin{aligned}F(u_{\leq N}) - F(u_{\leq \frac{N}{2}}) = u_N &\int_0^1 \partial_z F((P_{\leq \frac{N}{2}} + \theta P_N)u)\, d\theta \\ &+ \overline{u_N} \int_0^1 \partial_{\overline{z}} F((P_{\leq \frac{N}{2}} + \theta P_N)u)\, d\theta. \end{aligned}\tag{2.2.5}$$

The above expression is essentially of the form $F(u_{\leq N})-F(u_{\leq \frac{N}{2}}) \sim u_N|u_{\leq N}|^p$ for the purpose of estimation, and thus (2.2.1) is morally equivalent to an estimate like

$$\sum_{N_0\geq 1}\sum_{N_1\geq 1}\int_0^T\int_{\mathbb{T}^3} v_{N_0}[(u_N+w_N)|u_{\leq N}+w_{\leq N}|^p - u_N|u_{\leq N}|^p]\ dxdt$$
$$\lesssim \|v\|_{Y^{-s_c}}\|w\|_{Y^{s_c}}(\|u\|_{Y^{s_c}}+\|w\|_{Y^{s_c}})^p.$$

We now make this precise. Under the convention $g_{\leq \frac{1}{2}} = 0$, Proposition 2.2.2 implies the weak convergence statement

$$\sum_{N\geq 1}\langle F(g_{\leq N}) - F(g_{\leq \frac{N}{2}}), \varphi\rangle_{L^2} = \langle F(g), \varphi\rangle_{L^2}$$

for all $\varphi \in C(\mathbb{T}^d)$. We now apply this to the integral appearing in (2.2.1). Noting that $v_{N_0}(t) \in C(\mathbb{T}^d)$ for all $v \in Y^{-s_c}([0,T))$ and $1 \leq N_0 \in 2^{\mathbb{Z}}$, we obtain

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^3} & v(t,x)[F(u+w)-F(u)](t,x)\ dxdt \\ &= \sum_{N_0\geq 1}\int_0^T\int_{\mathbb{T}^3} v_{N_0}(t,x)[F(u+w)-F(u)](t,x)\ dxdt \\ &= \sum_{N_0\geq 1}\sum_{N_1\geq 1}\int_0^T\int_{\mathbb{T}^3} v_{N_0}[(F(u_{\leq N_1}+w_{\leq N_1}) - F(u_{\leq \frac{N_1}{2}}+w_{\leq \frac{N_1}{2}}) \\ &\qquad - (F(u_{\leq N_1}) - F(u_{\leq \frac{N_1}{2}}))](t,x)\ dxdt.\end{aligned}$$

We split the inner sum into three regimes: $N_0 \gg N_1$, $N_0 \ll N_1$, and $N_0 \sim N_1$, so that (2.2.1) reduces to the following:

Proposition 2.2.3. *Fix $p \geq 2$. Let $0 < T \leq 1$. Then*

$$\begin{aligned}\Bigg|\sum_{N_0\geq 1}\sum_{N_1\gg N_0\geq 1}\int_0^T\int_{\mathbb{T}^3} & v_{N_0}[(F(u_{\leq N_1}+w_{\leq N_1}) - F(u_{\leq \frac{N_1}{2}}+w_{\leq \frac{N_1}{2}})) \\ &- (F(u_{\leq N_1}) - F(u_{\leq \frac{N_1}{2}}))](t,x)\ dxdt\Bigg| \\ &\lesssim \|v\|_{Y^{-s_c}}\|w\|_{Y^{s_c}}(\|u\|_{Y^{s_c}}+\|w\|_{Y^{s_c}})^p\end{aligned}$$

and

$$\Big|\sum_{N_0\geq 1}\sum_{1\leq N_1\ll N_0}\int_0^T\int_{\mathbb{T}^3} v_{N_0}[(F(u_{\leq N_1}+w_{\leq N_1})-F(u_{\leq \frac{N_1}{2}}+w_{\leq \frac{N_1}{2}}))$$
$$-(F(u_{\leq N_1})-F(u_{\leq \frac{N_1}{2}}))](t,x)\ dxdt\Big|$$
$$\lesssim \|v\|_{Y^{-s_c}}\|w\|_{Y^{s_c}}(\|u\|_{Y^{s_c}}+\|w\|_{Y^{s_c}})^p.$$

Proposition 2.2.4. *Fix $p\geq 2$. Let $0<T\leq 1$. Then*

$$\Big|\sum_{N_0\geq 1}\sum_{N_0\sim N_1\geq 1}\int_0^T\int_{\mathbb{T}^3} v_{N_0}[(F(u_{\leq N_1}+w_{\leq N_1})-F(u_{\leq \frac{N_1}{2}}+w_{\leq \frac{N_1}{2}}))$$
$$-(F(u_{\leq N_1})-F(u_{\leq \frac{N_1}{2}}))](t,x)\ dxdt\Big|$$
$$\lesssim \|v\|_{Y^{-s_c}}\|w\|_{Y^{s_c}}(\|u\|_{Y^{s_c}}+\|w\|_{Y^{s_c}})^p.$$

We treat each of these separately. The integrals in Proposition 2.2.4 can be controlled essentially as they are, while those in Proposition 2.2.3 are still difficult to estimate and require further linearization using Proposition 2.2.2 and Equation (2.2.5).

2.2.3 Controlling sums over incomparable frequencies

In this subsection we prove Proposition 2.2.3. We first express the integrands in Proposition 2.2.3 in a more manageable form. Linearizing via (2.2.5), we write

$$\begin{aligned}
&(F(u_{\leq N}+w_{\leq N})-F(u_{\leq \frac{N}{2}}+w_{\leq \frac{N}{2}}))-(F(u_{\leq N})-F(u_{\leq \frac{N}{2}}))\\
&=w_N\int_0^1 \partial_z F((P_{\leq \frac{N}{2}}+\theta P_N)(u+w)\ d\theta\\
&\quad+\overline{w_N}\int_0^1 \partial_{\bar z} F((P_{\leq \frac{N}{2}}+\theta P_N)(u+w))\ d\theta\\
&\quad+u_N\int_0^1 [\partial_z F((P_{\leq \frac{N}{2}}+\theta P_N)(u+w))-\partial_z F((P_{\leq \frac{N}{2}}+\theta P_N)u)]\ d\theta\\
&\quad+\overline{u_N}\int_0^1 [\partial_{\bar z} F((P_{\leq \frac{N}{2}}+\theta P_N)(u+w))-\partial_{\bar z} F((P_{\leq \frac{N}{2}}+\theta P_N)u)]\ d\theta.
\end{aligned}$$

In the last two terms, we may linearize the difference again using (2.2.4), obtaining:

$$
\begin{aligned}
&u_N \int_0^1 [\partial_z F((P_{\leq \frac{N}{2}} + \theta P_N)(u+w)) - \partial_z F((P_{\leq \frac{N}{2}} + \theta P_N)u)] \; d\theta \\
&= u_N \int_0^1 (P_{\leq \frac{N}{2}} + \theta P_N)w \int_0^1 \partial_z^2 F((P_{\leq \frac{N}{2}} + \theta P_N)(u+\eta w))] \; d\eta d\theta \\
&+ u_N \int_0^1 \overline{(P_{\leq \frac{N}{2}} + \theta P_N)w} \int_0^1 \partial_{\overline{z}} \partial_z F((P_{\leq \frac{N}{2}} + \theta P_N)(u+\eta w))] \; d\eta d\theta
\end{aligned}
$$

and similarly for the term involving $\overline{u_N}$. We summarize these calculations in the form

$$
\begin{aligned}
&(F(u_{\leq N_1} + w_{\leq N_1}) - F(u_{\leq \frac{N_1}{2}} + w_{\leq \frac{N_1}{2}})) - (F(u_{\leq N_1}) - F(u_{\leq \frac{N_1}{2}})) \\
&= w_{N_1} \int_0^1 \partial_z F((P_{\leq \frac{N_1}{2}} + \theta P_{N_1})(u+w) \; d\theta + \text{ similar terms} \\
&+ u_{N_1} \int_0^1 (P_{\leq \frac{N_1}{2}} + \theta P_N)w \int_0^1 \partial_z^2 F((P_{\leq \frac{N_1}{2}} + \theta P_{N_1})(u+\eta w))] \; d\eta d\theta \\
&+ \text{ similar terms.}
\end{aligned} \tag{2.2.6}
$$

where by "similar terms" we indicate the same expression up to complex conjugates and conjugate derivatives in the appropriate places.

We now recall our heuristic: this expression is morally of the form

$$
\begin{aligned}
&(F(u_{\leq N_1} + w_{\leq N_1}) - F(u_{\leq \frac{N_1}{2}} + w_{\leq \frac{N_1}{2}})) - (F(u_{\leq N_1}) - F(u_{\leq \frac{N_1}{2}})) \\
&\sim w_{N_1} \partial_z F(u_{\leq N_1} + w_{\leq N_1}) + u_{N_1} w_{\leq N_1} \partial_z^2 F(u_{\leq N_1} + w_{\leq N_1})
\end{aligned}
$$

Inspired by this, we claim that Proposition 2.2.4 follows from the following:

Proposition 2.2.5. *Fix $p \geq 2$. Let $0 < T \leq 1$. Then*

$$
\sum_{N_0 \gg N_1 \geq 1} \left| \int_0^T \int_{\mathbb{T}^3} v_{N_0} g_{N_1} D_{N_1} F(h_{\leq N_1}) \; dxdt \right| \lesssim \|v\|_{Y^{-s_c}} \|w\|_{Y^{s_c}} \max\{\|g\|_{Y^{s_c}}, \|h\|_{Y^{s_c}}\} \|h\|_{Y^{s_c}}^{p-1}
$$

and

$$
\sum_{N_0 \ll N_1 \geq 1} \left| \int_0^T \int_{\mathbb{T}^3} v_{N_0} g_{N_1} D_{N_1} F(h_{\leq N_1}) \; dxdt \right| \lesssim \|v\|_{Y^{-s_c}} \|w\|_{Y^{s_c}} \max\{\|g\|_{Y^{s_c}}, \|h\|_{Y^{s_c}}\} \|h\|_{Y^{s_c}}^{p-1}.
$$

where $g \in \{u, w\}$, $D_{N_1} \in \{\partial_z, \partial_{\overline{z}}\}$ *if* $g = w$, *and* $D_{N_1} \in \{w_{\leq N_1}\partial_z^2, w_{\leq N_1}\partial_z\partial_{\overline{z}}, w_{\leq N_1}\partial_{\overline{z}}^2\}$ *if* $g = u$.

First, let us see how this simplified estimate leads to Proposition 2.2.3.

Sketch of Proposition 2.2.3 assuming Proposition 2.2.5. By our work to this point, Proposition 2.2.3 can be proved by establishing the corresponding estimate for each term in Equation (2.2.6). Suppose, for instance, that we wish to prove the estimate

$$\Big|\sum_{N_0 \gg N_1} \int_0^T \int_{\mathbb{T}^3} v_{N_0} w_{N_1} \int_0^1 \partial_z F((P_{\leq \frac{N_1}{2}} + \theta P_{N_1})(u+w))\ d\theta dx dt\Big| \lesssim \|v\|_{Y^{-s_c}} \|w\|_{Y^{s_c}} (\|u\|_{Y^{s_c}} + \|w\|_{Y^{s_c}})^p.$$

Applying Fubini, it is sufficient to show that

$$\sum_{N_0 \gg N_1} \int_0^1 \left(\int_0^T \int_{\mathbb{T}^3} |v_{N_0} w_{N_1} \partial_z F((P_{\leq \frac{N_1}{2}} + \theta P_{N_1})(u+w))|\ dx dt \right) d\theta \lesssim \|v\|_{Y^{-s_c}} \|w\|_{Y^{s_c}} (\|u\|_{Y^{s_c}} + \|w\|_{Y^{s_c}})^p.$$

Take $g = w$, $D_{N_1} = \partial_z$, and $h = (P_{\leq \frac{N}{2}} + \theta P_N)(u+w)$. Then the first line in the above estimate is very nearly the expression that is estimated in Proposition 2.2.5. As we will point out after the proof of Proposition 2.2.5, the projection $P_{\leq \frac{N_1}{2}} + \theta P_{N_1}$ can be replaced by $P_{\leq N_1}$ by a simple argument, and consequently the integral in θ can be eliminated, leaving us with precisely the expression in Proposition 2.2.5. The point is that Proposition 2.2.5 is fairly lenient when it comes to choosing the functions g and h, in a way which will become apparent during its proof. See Remark 2.2.3. We thus obtain

$$\begin{aligned}\sum_{N_0 \gg N_1} \int_0^1 \left(\int_0^T \int_{\mathbb{T}^3} |v_{N_0} w_{N_1} \partial_z F((P_{\leq \frac{N_1}{2}} + \theta P_{N_1})(u+w))|\ dx dt \right) d\theta \\ \lesssim \|v\|_{Y^{-s_c}} \|w\|_{Y^{s_c}} \max\{\|w\|_{Y^{s_c}} \|u+w\|_{Y^{s_c}}^{p-1}, \|u+w\|_{Y^{s_c}}^p\} \\ \leq \|v\|_{Y^{-s_c}} \|w\|_{Y^{s_c}} (\|u\|_{Y^{s_c}} + \|w\|_{Y^{s_c}})^p.\end{aligned}$$

Lastly, to completely prove Proposition 2.2.3 we must obtain the analogous estimate for the remaining five terms in the expression (2.2.6), and also the corresponding estimates for the sum over $N_0 \ll N_1$. A similar argument as above shows how to go from Proposition 2.2.5 to the required estimate in each case. □

Before we prove Proposition 2.2.5, let us outline the proof. The point of isolating the regimes $N_0 \gg N_1$ and $N_0 \ll N_1$ is that in these cases, we can further restrict the nonlinear factor $D_{N_1}F(h_{\leq N_1})$ to the highest frequencies; e.g. if $N_0 \gg N_1$, we can replace $D_{N_1}F(h_{\leq N_1})$ with $P_{\sim N_0}(D_{N_1}F(h_{\leq N_1}))$ inside the integral. We may then differentiate this term to obtain some extra decay in the highest frequency. When $2 < p < 4$, the first and second derivatives of $F(z) = |z|^p z$ admit one full derivative, while when $p \geq 4$ they admit two full derivatives. This produces enough decay to defeat the critical regularity: $s_c < 1$ for $2 < p < 4$, and $s_c < 2$ for $p \geq 4$. We will therefore be able to obtain an estimate for

$$\sum_{N_0 \gg N_1 \geq 1} \left| \int_0^T \int_{\mathbb{T}^3} v_{N_0} g_{N_1} D_{N_1} F(h_{\leq N_1})\, dxdt \right|$$

which we will be able to sum much like as in the proof of Proposition 2.2.1 for the cubic NLS.

We record some useful estimates before proceeding.

Lemma 2.2.6 (Scaling-critical Strichartz estimate)**.** *For $p > \frac{4}{3}$,*

$$\|u\|_{L^{5p/2}_{t,x}([0,T)\times\mathbb{T}^3)} \lesssim \|u\|_{Y^{s_c}}. \tag{2.2.7}$$

Also, if $r > \frac{5p}{2}$ with $p > \frac{4}{3}$, then

$$\|u_{\leq N}\|_{L^r_{t,x}([0,T)\times\mathbb{T}^3)} \lesssim N^{\frac{2}{p}-\frac{5}{r}} \|u_{\leq N}\|_{Y^{s_c}}. \tag{2.2.8}$$

Proof. By the square-function estimate,

$$\|u\|_{L^{5p/2}_{t,x}} \sim \|(\sum_N |u_N|^2)^{\frac{1}{2}}\|_{L^{5p/2}_{t,x}} \leq (\sum_N \|u_N\|^2_{L^{5p/2}_{t,x}})^{\frac{1}{2}}.$$

Here we have used $p > \frac{4}{3} > \frac{4}{5}$, which ensures that $\|\cdot\|_{L^{5p/4}_{t,x}}$ is a norm and not merely a quasinorm. The condition $p > \frac{4}{3}$ also ensures that $\frac{5p}{2} > \frac{10}{3}$. Therefore we may apply the

Strichartz estimate and obtain

$$(\sum_N \|u_N\|^2_{L^{5p/2}_{t,x}})^{\frac{1}{2}} \lesssim (\sum_N N^{2s_c}\|u_N\|^2_{Y^0})^{\frac{1}{2}} \sim \|u\|_{Y^{s_c}}.$$

This proves (2.2.7). We proceed similarly for (2.2.8): by the square-function estimate, Strichartz, and Cauchy-Schwarz, we obtain

$$\begin{aligned}\|u_{\leq N}\|_{L^r_{t,x}} &\lesssim (\sum_{M\leq N} \|u_M\|^2_{L^r_{t,x}})^{\frac{1}{2}} \lesssim (\sum_{M\leq N} (M^{\frac{2}{p}-\frac{5}{r}})^2\|u_M\|^2_{Y^{s_c}})^{\frac{1}{2}} \\ &\lesssim N^{\frac{2}{p}-\frac{5}{r}}\|u_{\leq N}\|_{Y^{s_c}}.\end{aligned}$$

□

Lemma 2.2.7. *Let $p > 2$ and $0 < T \leq 1$. Then*

$$\|\nabla u_{\leq N}\|_{L^{\frac{10p}{p+4}}_{t,x}([0,T)\times\mathbb{T}^3)} \lesssim N^{\frac{1}{2}}\|u_{\leq N}\|_{Y^{s_c}([0,T))}, \tag{2.2.9}$$

$$\|\nabla u_{\leq N}\|_{L^{\frac{20p}{p+8}}_{t,x}} \lesssim N^{\frac{3}{4}}\|u_{\leq N}\|_{Y^{s_c}}, \tag{2.2.10}$$

and

$$\|\Delta u_{\leq N}\|_{L^{\frac{10p}{p+4}}_{t,x}} \lesssim N^{\frac{3}{2}}\|u_{\leq N}\|_{Y^{s_c}}. \tag{2.2.11}$$

Proof. Note that $\frac{10p}{p+4} > \frac{10}{3}$ for $p > 2$. Applying Bernstein, Strichartz, and Cauchy-Schwarz,

$$\begin{aligned}\|\nabla u_{\leq N}\|_{L^{\frac{10p}{p+4}}_{t,x}} &\leq \sum_{M\leq N}\|\nabla u_M\|_{L^{\frac{10p}{p+4}}_{t,x}} \sim \sum_{M\leq N} M\|u_M\|_{L^{\frac{10p}{p+4}}_{t,x}} \lesssim \sum_{M\leq N} M^{\frac{1}{2}}\|u_M\|_{Y^{s_c}} \\ &\lesssim N^{\frac{1}{2}}\|u_{\leq N}\|_{Y^{s_c}}.\end{aligned}$$

The other two estimates are proved similarly. □

Remark 2.2.2. Since the Strichartz estimate at the $L^{10/3}_{t,x}$ endpoint only holds with a derivative loss, (2.2.9) and (2.2.11) do not hold for $p = 2$. This is one reason why we have provided a separate argument for the cubic NLS.

Lemma 2.2.8. *Let $u \in C^1(\mathbb{T}^d)$. If $p \geq 2$ and $G \in \{\partial_z F, \partial_{\bar{z}} F\}$, then*

$$|\nabla(G(u(x)))| \lesssim_p |u(x)|^{p-1}|\nabla u(x)|.$$

and

$$|\Delta G(u(x))| \lesssim_p |u(x)|^{p-1}|\Delta u(x)| + |u(x)|^{p-2}|\nabla u(x)|^2.$$

If $p \geq 3$ *and* $G \in \{\partial_z^2 F, \partial_z \partial_{\bar{z}} F, \partial_{\bar{z}}^2 F\}$*, then*

$$|\nabla(G(u(x)))| \lesssim_p |u(x)|^{p-2}|\nabla u(x)|$$

and

$$|\Delta(G(u(x)))| \lesssim_p |g(x)|^{p-2}|\Delta u(x)| + |u(x)|^{p-3}|\nabla u(x)|^2.$$

The proof of Lemma 2.2.8 is a straightforward calculus exercise that we omit.

Proof of Proposition 2.2.5. Our proof splits along the cases $2 < p < 4$ and $p \geq 4$.

Case 1.1: ($2 < p < 4$, $N_0 \gg N_1 \geq 1$). For fixed N_0, we may write

$$\left| \int_0^T \int_{\mathbb{T}^3} v_{N_0} g_{N_1} D_{N_1} F(h_{\leq N_1}) \, dxdt \right| = \left| \int_0^T \int_{\mathbb{T}^3} v_{N_0} g_{N_1} P_{\sim N_0}(D_{N_1} F(h_{\leq N_1})) \, dxdt \right|.$$

If $g = w$, then $D_{N_1} F = G \in \{\partial_z F, \partial_{\bar{z}} F\}$. Therefore by Bernstein, Lemma 2.2.8, Hölder, Lemma 2.2.6, and Lemma 2.2.7, we have

$$\begin{aligned}
\|P_{\sim N_0}(D_{N_1} F(h_{\leq N_1}))\|_{L^2_{t,x}} &\lesssim N_0^{-1} \|\nabla G(h_{\leq N_1})\|_{L^2_{t,x}} \\
&\lesssim N_0^{-1} \| |h_{\leq N_1}|^{p-1} \nabla h_{\leq N_1}\|_{L^2_{t,x}} \\
&\leq N_0^{-1} \|h_{\leq N_1}\|^{p-1}_{L^{5p/2}_{t,x}} \|\nabla h_{\leq N_1}\|_{L^{\frac{10p}{p+4}}_{t,x}} \\
&\lesssim N_0^{-1} N_1^{\frac{1}{2}} \|h_{\leq N_1}\|^p_{Y^{s_c}}.
\end{aligned}$$

If $g = u$, then $D_{N_1} F = w_{\leq N_1} G$ where $G \in \{\partial_z^2 F, \partial_z \partial_{\bar{z}} F, \partial_{\bar{z}}^2 F\}$. In this case $P_{\sim N_0}(D_{N_1} F(h_{\leq N_1})) =$

$w_{\le N_1} P_{\sim N_0}(G(h_{\le N_1}))$, and similarly to above we have

$$\begin{aligned}
\|P_{\sim N_0}(D_{N_1} F(h_{\le N_1}))\|_{L^2_{t,x}} &= \|w_{\le N_1} P_{\sim N_0}(G(h_{\le N_1}))\|_{L^2_{t,x}} \\
&\le \|w_{\le N_1}\|_{L^{5p/2}_{t,x}} \|P_{\sim N_0}(G(h_{\le N_1}))\|_{L^{\frac{10p}{5p-4}}_{t,x}} \\
&\lesssim N_0^{-1} \|w_{\le N_1}\|_{L^{5p/2}_{t,x}} \|\nabla(G(h_{\le N_1}))\|_{L^{\frac{10p}{5p-4}}_{t,x}} \\
&\lesssim N_0^{-1} \|w_{\le N_1}\|_{L^{5p/2}_{t,x}} \||h_{\le N_1}|^{p-2} \nabla h_{\le N_1}\|_{L^{\frac{10p}{5p-4}}_{t,x}} \\
&\lesssim N_0^{-1} \|w_{\le N_1}\|_{L^{5p/2}_{t,x}} \||h_{\le N_1}\|^{p-2}_{L^{5p/2}_{t,x}} \|\nabla h_{\le N_1}\|_{L^{\frac{10p}{p+4}}_{t,x}} \\
&\lesssim N_0^{-1} N_1^{\frac{1}{2}} \|w_{\le N_1}\|_{Y^{s_c}} \|h_{\le N_1}\|^{p-1}_{Y^{s_c}}.
\end{aligned}$$

We now estimate using Hölder, the bilinear Strichartz estimate (Lemma 2.1.3), and the above estimates. When $g = w$ we obtain

$$\begin{aligned}
&\left| \int_0^T \int_{\mathbb{T}^3} v_{N_0} g_{N_1} P_{\sim N_0}(D_{N_1} F(h_{\le N_1}))\ dxdt \right| \\
&\lesssim \|v_{N_0} g_{N_1}\|_{L^2_{t,x}} \|P_{\sim N_0}(D_{N_1} F(h_{\le N_1}))\|_{L^2_{t,x}} \\
&\lesssim \frac{N_1}{N_0} \|v_{N_0}\|_{Y^0} \|w_{N_1}\|_{Y^0} \|h_{\le N_1}\|^p_{Y^{s_c}} \\
&\lesssim \left(\frac{N_1}{N_0}\right)^{1-s_c} \|v_{N_0}\|_{Y^{-s_c}} \|w_{N_1}\|_{Y^{s_c}} \|h_{\le N_1}\|^p_{Y^{s_c}},
\end{aligned}$$

while when $g = u$ we similarly obtain

$$\begin{aligned}
&\left| \int_0^T \int_{\mathbb{T}^3} v_{N_0} g_{N_1} P_{\sim N_0}(D_{N_1} F(h_{\le N_1}))\ dxdt \right| \\
&\lesssim \left(\frac{N_1}{N_0}\right)^{1-s_c} \|v_{N_0}\|_{Y^{-s_c}} \|u_{N_1}\|_{Y^{s_c}} \|w_{\le N_1}\|_{Y^{s_c}} \|h_{\le N_1}\|^{p-1}_{Y^{s_c}}.
\end{aligned}$$

Since $2 < p < 4$, we have $1 - s_c > 0$. Therefore the above estimate is summable over $N_0 \gg N_1$ using Cauchy-Schwarz similarly to the proof of (2.2.1) for $p = 2$ from Section 3.1. Performing the sum, this part of Proposition 2.2.5 follows.

Case 1.2: ($2 < p < 4$, $1 \le N_0 \ll N_1$). This case is similar to the previous case. Since the highest frequency is N_1, we now have

$$\left| \int_0^T \int_{\mathbb{T}^3} v_{N_0} g_{N_1} D_{N_1} F(h_{\le N_1})\ dxdt \right| = \left| \int_0^T \int_{\mathbb{T}^3} v_{N_0} g_{N_1} P_{\sim N_1}(D_{N_1} F(h_{\le N_1}))\ dxdt \right|.$$

Estimating as before, when $g = w$ we have

$$\|P_{\sim N_1}(D_{N_1}F(h_{\le N_1}))\|_{L^2_{t,x}} \lesssim N_1^{-\frac{1}{2}}\|h_{\le N_1}\|^p_{Y^{s_c}}$$

while when $g = u$ we have

$$\|P_{\sim N_1}(D_{N_1}F(h_{\le N_1}))\|_{L^2_{t,x}} \lesssim N_1^{-\frac{1}{2}}\|w_{\le N_1}\|_{Y^{s_c}}\|h_{\le N_1}\|^{p-1}_{Y^{s_c}}.$$

Applying Hölder, bilinear Strichartz, and the above estimates, when $g = w$ we obtain

$$\begin{aligned}
&\left|\int_0^T\int_{\mathbb{T}^3} v_{N_0}g_{N_1}P_{\sim N_0}(D_{N_1}F(h_{\le N_1}))\ dxdt\right| \\
&\lesssim \|v_{N_0}g_{N_1}\|_{L^2_{t,x}}\|P_{\sim N_0}(D_{N_1}F(h_{\le N_1}))\|_{L^2_{t,x}} \\
&\lesssim \left(\frac{N_0}{N_1}\right)^{\frac{1}{2}}\|v_{N_0}\|_{Y^0}\|w_{N_1}\|_{Y^0}\|h_{\le N_1}\|^p_{Y^{s_c}} \\
&\lesssim \left(\frac{N_0}{N_1}\right)^{\frac{1}{2}+s_c}\|v_{N_0}\|_{Y^{-s_c}}\|w_{N_1}\|_{Y^{s_c}}\|h_{\le N_1}\|^p_{Y^{s_c}},
\end{aligned}$$

while when $g = u$ we obtain

$$\begin{aligned}
&\left|\int_0^T\int_{\mathbb{T}^3} v_{N_0}g_{N_1}P_{\sim N_0}(D_{N_1}F(h_{\le N_1}))\ dxdt\right| \\
&\lesssim \left(\frac{N_0}{N_1}\right)^{\frac{1}{2}+s_c}\|v_{N_0}\|_{Y^{-s_c}}\|u_{N_1}\|_{Y^{s_c}}\|w_{\le N_1}\|_{Y^{s_c}}\|h_{\le N_1}\|^{p-1}_{Y^{s_c}}.
\end{aligned}$$

Again, $\frac{1}{2} + s_c > 0$, so this is summable over $N_0 \ll N_1$ and this case is proved.

Case 2.1: ($p \ge 4$, $N_0 \gg N_1 \ge 1$). This proof proceeds similarly to that of **Case 1.1**, except that instead of a gradient we take a Laplacian. Since the highest frequency is N_0, we localize the nonlinear factor to frequencies $\sim N_0$. When $g = w$, we have (with $G \in \{\partial_z F, \partial_{\bar{z}} F\}$)

$$\begin{aligned}
&\|P_{\sim N_0}(D_{N_1}F(h_{\le N_1}))\|_{L^2_{t,x}} \\
&\qquad \lesssim N_0^{-2}\|\Delta G(h_{\le N_1})\|_{L^2_{t,x}} \\
&\qquad \lesssim N_0^{-2}\||h_{\le N_1}|^{p-1}|\Delta h_{\le N_1}| + |h_{\le N_1}|^{p-2}|\nabla h_{\le N_1}|^2\|_{L^2_{t,x}} \\
&\qquad \le N_0^{-2}(\|h_{\le N_1}\|^{p-1}_{L^{5p/2}_{t,x}}\|\Delta h_{\le N_1}\|_{L^{\frac{10p}{p+4}}_{t,x}} + \|h_{\le N_1}\|^{p-2}_{L^{5p/2}_{t,x}}\|\nabla h_{\le N_1}\|^2_{L^{\frac{20p}{p+8}}_{t,x}}) \\
&\qquad \lesssim N_0^{-2}N_1^{\frac{3}{2}}\|h_{\le N_1}\|^p_{Y^{s_c}},
\end{aligned}$$

while when $g = u$ we have (with $G \in \{\partial_z^2 F, \partial_z \partial_{\overline{z}} F, \partial_{\overline{z}}^2 F\}$)

$$\begin{aligned}
&\|P_{\sim N_0}(D_{N_1} F(h_{\le N_1}))\|_{L^2_{t,x}} \\
&\qquad = \|w_{\le N_1} P_{\sim N_0}(G(h_{\le N_1}))\|_{L^2_{t,x}} \\
&\qquad \lesssim N_0^{-2} \|w_{\le N_1}\|_{L^{5p/2}_{t,x}} \|\Delta G(h_{\le N_1})\|_{L^{\frac{10p}{5p-4}}_{t,x}} \\
&\qquad \lesssim N_0^{-2} \|w_{\le N_1}\|_{Y^{s_c}} \| |h_{\le N_1}|^{p-2} |\Delta h_{\le N_1}| + |h_{\le N_1}|^{p-3} |\nabla h_{\le N_1}|^2 \|_{L^2_{t,x}} \\
&\qquad \le N_0^{-2} \|w_{\le N_1}\|_{Y^{s_c}} (\|h_{\le N_1}\|^{p-2}_{L^{5p/2}_{t,x}} \|\Delta h_{\le N_1}\|_{L^{\frac{10p}{p+4}}_{t,x}} + \|h_{\le N_1}\|^{p-3}_{L^{5p/2}_{t,x}} \|\nabla h_{\le N_1}\|^2_{L^{\frac{20p}{p+8}}_{t,x}}) \\
&\qquad \lesssim N_0^{-2} N_1^{\frac{3}{2}} \|w_{\le N_1}\|_{Y^{s_c}} \|h_{\le N_1}\|^{p-1}_{Y^{s_c}}.
\end{aligned}$$

By Hölder, bilinear Strichartz, and the above estimate, when $g = w$ we obtain

$$\begin{aligned}
&\left| \int_0^T \int_{\mathbb{T}^3} v_{N_0} g_{N_1} P_{\sim N_0}(D_{N_1} F(h_{\le N_1}))\ dxdt \right| \\
&\lesssim \|v_{N_0} g_{N_1}\|_{L^2_{t,x}} \| \|P_{\sim N_0}(D_{N_1} F(h_{\le N_1}))\|_{L^2_{t,x}} \\
&\lesssim \left(\frac{N_1}{N_0}\right)^2 \|v_{N_0}\|_{Y^0} \|w_{N_1}\|_{Y^0} \|h_{\le N_1}\|^p_{Y^{s_c}} \\
&\lesssim \left(\frac{N_1}{N_0}\right)^{2-s_c} \|v_{N_0}\|_{Y^{-s_c}} \|w_{N_1}\|_{Y^{s_c}} \|h_{\le N_1}\|^p_{Y^{s_c}}
\end{aligned}$$

while when $g = u$ we obtain

$$\begin{aligned}
&\left| \int_0^T \int_{\mathbb{T}^3} v_{N_0} g_{N_1} P_{\sim N_0}(D_{N_1} F(h_{\le N_1}))\ dxdt \right| \\
&\lesssim \|v_{N_0} g_{N_1}\|_{L^2_{t,x}} \| \|P_{\sim N_0}(D_{N_1} F(h_{\le N_1}))\|_{L^2_{t,x}} \\
&\lesssim \left(\frac{N_1}{N_0}\right)^2 \|v_{N_0}\|_{Y^0} \|u_{N_1}\|_{Y^0} \|w_{\le N_1}\|_{Y^{s_c}} \|h_{\le N_1}\|^{p-1}_{Y^{s_c}} \\
&\lesssim \left(\frac{N_1}{N_0}\right)^{2-s_c} \|v_{N_0}\|_{Y^{-s_c}} \|u_{N_1}\|_{Y^{s_c}} \|w_{\le N_1}\|_{Y^{s_c}} \|h_{\le N_1}\|^{p-1}_{Y^{s_c}}.
\end{aligned}$$

Since $p \ge 4$, $2 - s_c > 0$ and thus this is summable over $N_0 \gg N_1$.

Case 2.2: ($p \ge 4$, $1 \le N_0 \ll N_1$). This is covered by the proof of **Case 1.2**, since $\frac{1}{2} + s_c > 0$ for all $p > 2$. □

Remark 2.2.3. As we have alluded to earlier, these estimates are rather lenient with respect to the precise functions inside the integrals. For instance, take $g = w$. If in **Case 1.1** we estimate $\|P_{\sim N_0}(D_{N_1}F(h_{\leq \frac{N_1}{2}} + \theta h_N))\|_{L^2_{t,x}}$ instead of $\|P_{\sim N_0}(D_{N_1}F(h_{\leq N_1})\|^2_{L_{t,x}}$, then since $h_{\leq \frac{N_1}{2}} + \theta h_N = P_{\leq 2N}(h_{\leq \frac{N_1}{2}} + \theta h_N)$ we would find via the same proof that

$$\begin{aligned}\|P_{\sim N_0}(D_{N_1}F(h_{\leq \frac{N_1}{2}} + \theta h_N))\|^2_{L_{t,x}} &\lesssim N_0^{-1}N_1^{\frac{1}{2}}\|h_{\leq \frac{N_1}{2}} + \theta h_N\|^p_{Y^{s_c}} \\ &\lesssim N_0^{-1}N_1^{\frac{1}{2}}(\|h_{\leq \frac{N_1}{2}}\|_{Y^{s_c}} + \|h_{\leq N}\|_{Y^{s_c}})^p \\ &\lesssim N_0^{-1}N_1^{\frac{1}{2}}\|h_{\leq N}\|^p_{Y^{s_c}}.\end{aligned}$$

This sort of argument can be used to fill in the remaining gaps in our sketch of the proof of Proposition 2.2.3 from Proposition 2.2.5.

2.2.4 Controlling sums over comparable frequencies

In this section we prove Proposition 2.2.4. In the regime of comparable frequencies $N_0 \sim N_1$, the methods in the previous section cannot be used because there is no way to restrict the nonlinear factor to a specific frequency. Instead, we aim to recreate the case of $p = 2$ as best as possible by iterating the paradifferential linearization technique used earlier. The precise details of how this is done differ between the cases $p \geq 3$ and $2 < p < 3$, and we treat them separately. The difference arises from the regularity of $F(z) = |z|^p z$, which determines how many times we may iterate the linearization.

2.2.4.1 The case $p \geq 3$

Let $p \geq 3$. In this case, $F(z) = |z|^p z$ admits four derivatives, and hence we may iterate our paradifferential linearization process four times. We begin with the formal expression arising from Proposition 2.2.2:

$$F(u+w)-F(u)=\sum_{N\geq 1}[F(u_{\leq N}+w_{\leq N})-F(u_{\leq \frac{N}{2}}+w_{\leq \frac{N}{2}})]$$
$$-\sum_{N\geq 1}[F(u_{\leq N})-F(u_{\leq \frac{N}{2}})]$$

where we interpret this equality in terms of convergence in L^q, $1\leq q<\frac{d}{2}$, and in particular weak convergence against continuous functions. Fixing N_0, we throw away the summands with $N\gg N_0$ and $N\ll N_0$, and apply (2.2.5) to obtain

$$\sum_{N_0\sim N_1\geq 1}(u_{N_1}+w_{N_1})\int_0^1\partial_z F((P_{\leq \frac{N_1}{2}}+\theta P_{N_1})(u+w))\ d\theta+\text{similar terms}$$
$$-\sum_{N_0\sim N_1\geq 1}u_{N_1}\int_0^1\partial_z F((P_{\leq \frac{N_1}{2}}+\theta P_{N_1})u)\ d\theta+\text{similar terms.}$$

We now apply Proposition 2.2.2 and (2.2.4) again to the terms inside the integral: that is, we write

$$\begin{aligned}\partial_z F((P_{\leq \frac{N_1}{2}}+\theta P_N)u)&=\sum_{N\geq 1}[\partial_z F(P_{\leq N}(P_{\leq \frac{N_1}{2}}+\theta P_N)u)-\partial_z F(P_{\leq \frac{N}{2}}(P_{\leq \frac{N_1}{2}}+\theta P_N)u]\\&=\sum_{N\geq 1}P_N(P_{\leq \frac{N_1}{2}}+\theta P_N)u\int_0^1\partial_z^2 F((P_{\leq \frac{N}{2}}+\eta P_N)(P_{\leq \frac{N_1}{2}}+\theta P_N)u)\ d\eta\\&\quad+\text{simliar terms,}\end{aligned}$$

and substitute these inside the integrals. Note that $P_N(P_{\leq \frac{N_1}{2}}+\theta P_{N_1})=0$ for $N>2N_1$, and $P_N(P_{\leq \frac{N_1}{2}}+\theta P_{N_1})$ is equivalent to P_N for $N\leq 2N_1$ for the purpose of estimates in the manner that we have explained in our sketch of Proposition 2.2.3 and Remark 2.2.3. Similarly, we may treat $(P_{\leq \frac{N}{2}}+\eta P_N)(P_{\leq \frac{N_1}{2}}+\theta P_{N_1})$ as $P_{\leq N}$ for $N\leq 2N_1$ for the purpose of proving Proposition 2.2.4.

We summarize these ideas and calculations in the notation $\sim$. For two expressions A and B, we say $A\sim B$ if A and B are related by collapsing Littlewood-Paley projections (e.g. $P_N(P_{\leq \frac{N_1}{2}}+\theta P_{N_1})\sim P_N$ for $N\leq 2N_1$), removing integrals over $[0,1]$, and conjugating factors or derivatives (as is encapsulated in the phrase "similar terms" as we have used up to this point). If $A \sim B$, then A and B should admit the same type of estimates in the manner we have described in the sketch of Proposition 2.2.3 and Remark 2.2.3. Thus we may succinctly express our first two iterations of the linearization in the following way:

$$
\begin{aligned}
&\sum_{N_0 \sim N_1 \geq 1} [F(u_{\leq N_1} + w_{\leq N_1}) - F(u_{\leq \frac{N_1}{2}} + w_{\leq \frac{N_1}{2}})] - \sum_{N_0 \sim N_1 \geq 1} [F(u_{\leq N_1}) - F(u_{\leq \frac{N_1}{2}})] \\
&\sim \sum_{N_0 \sim N_1 \geq 1} (u_{N_1} + w_{N_1}) \partial_z F(u_{\leq N_1} + w_{\leq N_1}) - \sum_{N_0 \sim N_1 \geq 1} u_{N_1} \partial_z F(u_{\leq N_1}) \\
&\sim \sum_{N_0 \sim N_1 \gtrsim N_2 \geq 1} (u_{N_1} + w_{N_1})(u_{N_2} + w_{N_2}) \partial_z^2 F(u_{\leq N_2} + w_{\leq N_2}) - \sum_{N_0 \sim N_1 \gtrsim N_2 \geq 1} u_{N_1} u_{N_2} \partial_z^2 F(u_{\leq N_2}).
\end{aligned}
$$

From here on we will use the symbol $\sim$ to summarize all calculations involving Proposition 2.2.2, (2.2.4), and (2.2.5). We now linearize with Proposition 2.2.2 and (2.2.5) once more, then shift terms and linearize with (2.2.4) one last time to obtain:

$$
\begin{aligned}
&\sum_{N_0 \sim N_1 \gtrsim N_2 \geq 1} (u_{N_1} + w_{N_1})(u_{N_2} + w_{N_2}) \partial_z^2 F(u_{\leq N_2} + w_{\leq N_2}) - \sum_{N_0 \sim N_1 \gtrsim N_2 \geq 1} u_{N_1} u_{N_2} \partial_z^2 F(u_{\leq N_2}) \\
&\sim \sum_{N_0 \sim N_1 \gtrsim N_2 \gtrsim N_3 \geq 1} (u_{N_1} + w_{N_1})(u_{N_2} + w_{N_2})(u_{N_3} + w_{N_3}) \partial_z^3 F(u_{\leq N_3} + w_{\leq N_3}) \\
&\qquad - \sum_{N_0 \sim N_1 \gtrsim N_2 \gtrsim N_3 \geq 1} u_{N_1} u_{N_2} u_{N_3} \partial_z^3 F(u_{\leq N_3}) \\
&\sim \sum_{N_0 \sim N_1 \gtrsim N_2 \gtrsim N_3 \geq 1} u_{N_1}^{(1)} u_{N_2}^{(2)} u_{N_3}^{(3)} \partial_z^3 F(u_{\leq N_3} + w_{\leq N_3}) \\
&\qquad - \sum_{N_0 \sim N_1 \gtrsim N_2 \gtrsim N_3 \geq 1} u_{N_1} u_{N_2} u_{N_3} [\partial_z^3 F(u_{\leq N_3} + w_{\leq N_3}) - \partial_z^3 F(u_{\leq N_3})] \\
&\sim \sum_{N_0 \sim N_1 \gtrsim N_2 \gtrsim N_3 \geq 1} u_{N_1}^{(1)} u_{N_2}^{(2)} u_{N_3}^{(3)} \partial_z^3 F(u_{\leq N_3} + w_{\leq N_3}) \\
&\qquad - \sum_{N_0 \sim N_1 \gtrsim N_2 \gtrsim N_3 \geq 1} u_{N_1} u_{N_2} u_{N_3} w_{\leq N_3} \partial_z^4 F(u_{\leq N_3} + w_{\leq N_3}).
\end{aligned}
$$

Here $u^{(j)} \in \{u, w\}$ with at least one $u^{(j)} = w$. Note that in the last step of this linearization, we require $p \geq 3$ so that the fourth-order derivatives of F in z and $\overline{z}$ are well-defined. Therefore to establish Proposition 2.2.4 for $p \geq 3$, it suffices to prove:

Proposition 2.2.9. *Fix $p \geq 3$. Let $0 < T \leq 1$. Then*

$$\sum_{N_0 \sim N_1 \gtrsim N_2 \gtrsim N_3 \geq 1} \left| \int_0^T \int_{\mathbb{T}^3} v_{N_0} u^{(1)}_{N_1} u^{(2)}_{N_2} u^{(3)}_{N_3} D_{N_3} F(h_{\leq N_3}) \ dxdt \right|$$
$$\lesssim \|v\|_{Y^{-s_c}} \|u^{(1)}\|_{Y^{s_c}} \|u^{(2)}\|_{Y^{s_c}} \|u^{(3)}\|_{Y^{s_c}} \max\{\|w\|_{Y^{s_c}}, \|h\|_{Y^{s_c}}\} \|h\|^{p-3}_{Y^{s_c}},$$

where

$$D_{N_3} \in \{\partial_z^3, \partial_z^2 \partial_{\bar{z}}, \partial_z \partial_{\bar{z}}^2, \partial_{\bar{z}}^3\} \text{ if } u^{(j)} = w \text{ for some } j = 1, 2, 3,$$

and

$$D_{N_3} \in \{w_{\leq N_3} \partial_z^4, w_{\leq N_3} \partial_z^3 \partial_{\bar{z}}, w_{\leq N_3} \partial_z^2 \partial_{\bar{z}}^2, , w_{\leq N_3} \partial_z \partial_{\bar{z}}^3, w_{\leq N_3} \partial_{\bar{z}}^4\} \text{ if } u^{(j)} \neq w \text{ for all } j = 1, 2, 3.$$

Remark 2.2.4. The condition $p \geq 3$ is only required for the estimates where $u^{(j)} \neq w$ for $j = 1, 2, 3$, since these require the fourth-order derivative of F to be defined. For the estimates where some $u^{(j)} = w$, we need only $p \geq 2$. This will be useful in the next section.

Proof. For any integer $0 \leq k < p + 1$, and any k-th order derivative G of $F(z) = |z|^p z$, we have $|G(z)| \lesssim |z|^{p-k}$. Therefore

$$|D_{N_3} F(h_{\leq N_3})| \lesssim \max\{|w_{\leq N_3}|, |h_{\leq N_3}|\} |h_{\leq N_3}|^{p-3}.$$

Therefore by Lemma 2.2.6 we have

$$\|D_{N_3} F(h_{\leq N_3})\|_{L^\infty_{t,x}} \lesssim N^{\frac{2(p-2)}{p}} \max\{\|w_{\leq N_3}\|_{Y^{s_c}}, \|h_{\leq N_3}\|_{Y^{s_c}}\} \|h_{\leq N_3}\|^{p-3}_{Y^{s_c}}.$$

Estimating the integral by Hölder, bilinear Strichartz, and Strichartz, we have

$$\begin{aligned} &\left| \int_0^T \int_{\mathbb{T}^3} v_{N_0} u^{(1)}_{N_1} u^{(2)}_{N_2} u^{(3)}_{N_3} D_{N_3} F(h_{\leq N_3}) \ dxdt \right| \\ &\lesssim \|v_{N_0} u^{(2)}_{N_2}\|_{L^2_{t,x}} \|u^{(1)}_{N_1} u^{(1)}_{N_3}\|_{L^2_{t,x}} \|D_{N_3} F(h_{\leq N_3})\|_{L^\infty_{t,x}} \\ &\lesssim \left(\frac{N_0}{N_1}\right)^{s_c} \left(\frac{N_3}{N_2}\right)^{s_c - \frac{1}{2}} \|v_{N_0}\|_{Y^{-s_c}} \|u^{(1)}_{N_1}\|_{Y^{s_c}} \|u^{(2)}_{N_2}\|_{Y^{s_c}} \|u^{(3)}_{N_3}\|_{Y^{s_c}} \\ &\qquad \cdot \max\{\|w_{\leq N_3}\|_{Y^{s_c}}, \|h_{\leq N_3}\|_{Y^{s_c}}\} \|h_{\leq N_3}\|^{p-3}_{Y^{s_c}}. \end{aligned}$$

Summing with Cauchy-Schwarz first over $N_2 \gtrsim N_3$, then over $N_0 \sim N_1$ establishes the desired estimate. ☐

2.2.4.2 The case $2 < p < 3$

The last case to consider is $2 < p < 3$. In this case, F does not admit four derivatives, so we cannot obtain the linearized expression we used for the case $p \geq 3$. However, F admits three derivatives and the third derivatives of F are Hölder continuous, which we can "differentiate" to obtain sufficient decay to sum.

First we obtain the linearization of the nonlinearity. Employing the notation $\sim$ as in the previous section, we obtain:

$$
\begin{aligned}
&\sum_{N_0 \sim N_1 \geq 1} [F(u_{\leq N_1} + w_{\leq N_1}) - F(u_{\leq \frac{N_1}{2}} + w_{\leq \frac{N_1}{2}})] - \sum_{N_0 \sim N_1 \geq 1} [F(u_{\leq N_1}) - F(u_{\leq \frac{N_1}{2}})] \\
&\sim \sum_{N_0 \sim N_1 \geq 1} (u_{N_1} + w_{N_1}) \partial_z F(u_{\leq N_1} + w_{\leq N_1}) - \sum_{N_0 \sim N_1 \geq 1} u_{N_1} \partial_z F(u_{\leq N_1}) \\
&\sim \sum_{N_0 \sim N_1 \gtrsim N_2 \geq 1} (u_{N_1} + w_{N_1})(u_{N_2} + w_{N_2}) \partial_z^2 F(u_{\leq N_2} + w_{\leq N_2}) \\
&\qquad - \sum_{N_0 \sim N_1 \gtrsim N_2 \geq 1} u_{N_1} u_{N_2} \partial_z^2 F(u_{\leq N_2}) \\
&\sim \sum_{N_0 \sim N_1 \gtrsim N_2 \geq 1} u_{N_1}^{(1)} u_{N_2}^{(2)} \partial_z^2 F(u_{\leq N_2} + w_{\leq N_2}) \\
&\qquad + \sum_{N_0 \sim N_1 \gtrsim N_2 \geq 1} u_{N_1} u_{N_2} [\partial_z^2 F(u_{\leq N_2} + w_{\leq N_2}) - \partial_z^2 F(u_{\leq N_2})] \\
&\sim \sum_{N_0 \sim N_1 \gtrsim N_2 \gtrsim N_3 \geq 1} u_{N_1}^{(1)} u_{N_2}^{(2)} (u_{N_3} + w_{N_3}) \partial_z^3 F(u_{\leq N_3} + w_{\leq N_3}) \\
&\qquad + \sum_{N_0 \sim N_1 \gtrsim N_2 \geq 1} u_{N_1} u_{N_2} w_{\leq N_2} \partial_z^3 F(u_{\leq N_2} + w_{\leq N_2}).
\end{aligned}
$$

The first summation in the last line is precisely of a form that is controlled by Proposition 2.2.9; see Remark 2.2.4. Writing $w_{\leq N_2} = \sum_{N_2 \geq N_3} w_{N_3}$, to establish Proposition 2.2.4 for $2 < p < 3$, it suffices to prove the following estimate:

Proposition 2.2.10. *Fix $2 < p < 3$. Let $0 < T \leq 1$. Then*

$$\sum_{N_0 \sim N_1 \gtrsim N_2 \geq N_3 \geq 1} \left| \int_0^T \int_{\mathbb{T}^3} v_{N_0} u_{N_1} u_{N_2} w_{N_3} G(h_{\leq N_2}) \ dxdt \right| \lesssim \|v\|_{Y^{-s_c}} \|u\|_{Y^{s_c}}^2 \|w\|_{Y^{s_c}} \|h\|_{Y^{s_c}}^{p-2}$$

where $G \in \{\partial_z^3 F, \partial_z^2 \partial_{\bar{z}} F, \partial_z \partial_{\bar{z}}^2 F, \partial_{\bar{z}}^3 F\}$.

Proof. Proposition 2.2.10 follows from the following two estimates:

$$\sum_{N_0 \sim N_1 \geq N_2 \geq N_3} \left| \int_0^T \int_{\mathbb{T}^3} v_{N_0} u_{N_1} u_{N_2} w_{N_3} P_{\leq N_2}(G(h_{\leq N_2})) \ dxdt \right| \lesssim \|v\|_{Y^{-s_c}} \|u\|_{Y^{s_c}}^2 \|w\|_{Y^{s_c}} \|h\|_{Y^{s_c}}^{p-2}, \tag{2.2.12}$$

$$\sum_{N_0 \sim N_1 \geq N_2 \geq N_3} \sum_{N > N_2} \left| \int_0^T \int_{\mathbb{T}^3} v_{N_0} u_{N_1} u_{N_2} w_{N_3} P_N(G(h_{\leq N_2})) \ dxdt \right| \lesssim \|v\|_{Y^{-s_c}} \|u\|_{Y^{s_c}}^2 \|w\|_{Y^{s_c}} \|h\|_{Y^{s_c}}^{p-2}. \tag{2.2.13}$$

Proof of (2.2.12)**:** For a given N_2, let $\mathbb{Z}^3 = \bigcup_j C_j$ be a partition of frequency space into cubes C_j of side length N_2. We write $C_j \sim C_k$ if the sum set $C_j + C_k$ intersects the Fourier support of $P_{\leq 3N_2}$. For a given C_j, there are finitely many C_k with $C_j \sim C_k$, and the number of such C_k is uniformly bounded independently of N_2.

To prove (2.2.12) it then suffices to prove

$$\sum_{N_0 \sim N_1 \geq N_2 \geq N_3} \sum_{C_j \sim C_k} \left| \int_0^T \int_{\mathbb{T}^3} (P_{C_j} v_{N_0})(P_{C_k} u_{N_1}) u_{N_2} w_{N_3} P_{\leq N_2}(G(h_{\leq N_2})) \ dxdt \right| \lesssim \|v\|_{Y^{-s_c}} \|u\|_{Y^{s_c}}^2 \|w\|_{Y^{s_c}} \|h\|_{Y^{s_c}}^{p-2}.$$

First let us proceed formally. Note that $|G(z)| \lesssim |z|^{p-2}$. By Hölder and Strichartz, we have

$$\begin{aligned} &\left| \int_0^T \int_{\mathbb{T}^3} (P_{C_j} v_{N_0})(P_{C_k} u_{N_1}) u_{N_2} w_{N_3} P_{\leq N_2}(G(h_{\leq N_2})) \ dxdt \right| \\ &\leq \|P_{C_j} v_{N_0}\|_{L^{r_0}_{t,x}} \|P_{C_k} u_{N_1}\|_{L^{r_0}_{t,x}} \|u_{N_2}\|_{L^{r_0}_{t,x}} \|w_{N_3}\|_{L^{r_1}_{t,x}} \|P_{\leq N_2}(G(h_{\leq N_2}))\|_{L^{\infty}_{t,x}} \\ &\lesssim \left(\frac{N_0}{N_1} \right)^{s_c} \left(\frac{N_3}{N_2} \right)^{\frac{3}{2} - \frac{5}{r_1} - s_c} \|P_{C_j} v_{N_0}\|_{Y^{-s_c}} \|P_{C_k} u_{N_1}\|_{Y^{s_c}} \|u_{N_2}\|_{Y^{s_c}} \|w_{N_3}\|_{Y^{s_c}} \|h_{\leq N_2}\|_{Y^{s_c}}^{p-2}, \end{aligned}$$

provided that r_j are Hölder exponents with $r_j > \frac{10}{3}$, $j = 0, \dots, 4$. The lowest frequency is summable if $r_1 > \frac{5p}{2}$. We take $r_0 = \frac{15p}{5p-2(1-\varepsilon)}$ and $r_1 = \frac{5p}{2(1-\varepsilon)}$. Then for $\varepsilon = \varepsilon(p)$ sufficiently small, we have $r_0, r_1 > \frac{10}{3}$. Summing using Cauchy-Schwarz, (2.2.12) follows.

Proof of (2.2.13)**:** As before, let $\mathbb{Z}^3 = \bigcup_j C_j$ be a partition of frequency space into cubes, except that the C_j now have side length N and $C_j \sim C_k$ if $C_j + C_k$ intersects the Fourier support of $P_{\leq 3N}$. Then like the previous proof, (2.2.13) follows from

$$\sum_{N_0 \sim N_1 \geq N_2 \geq N_3} \sum_{N > N_2} \sum_{C_j \sim C_k} \left| \int_0^T \int_{\mathbb{T}^3} (P_{C_j} v_{N_0})(P_{C_k} u_{N_1}) u_{N_2} w_{N_3} P_N(|h_{\leq N_2}|^{p-2})\, dxdt \right|$$
$$\lesssim \|v\|_{Y^{-s_c}} \|u\|_{Y^{s_c}}^2 \|w\|_{Y^{s_c}} \|h\|_{Y^{s_c}}^{p-2}.$$

The new ingredient relative to the preceding is the following estimate: for $2 < p < 3$ and $r > \frac{10}{3}$,

$$\|P_N(G(h_{\leq N_2}))\|_{L_{t,x}^{r/(p-2)}} \lesssim N^{-(p-2)} N_2^{(\frac{5}{2} - \frac{5}{r_4} - s_c)(p-2)} \|u_{\leq N_2}\|_{Y^{s_c}}^{p-2}. \tag{2.2.14}$$

This estimate follows from Lemma 2.1.6 and the Littlewood-Paley square-function estimate. Choosing $r_0 = r_1 = \frac{20p}{(1-\varepsilon)p^2 + (1+5\varepsilon)p + 4\varepsilon}$, $r_2 = \frac{10p}{2p^2 - 4 - 3(1-\frac{\varepsilon}{3})p(p-2)}$, $r_3 = \frac{5p}{2(1-\varepsilon)}$, and $r_4 = \frac{10}{3(1-\varepsilon)}$, and taking $\varepsilon = \varepsilon(p) > 0$ small, we find that $r_j > \frac{10}{3}$ for $j = 0, 1, 2, 3, 4$. Proceeding as above, by Hölder, Bernstein, and (2.2.14) we obtain

$$\left| \int_0^T \int_{\mathbb{T}^3} (P_{C_j} v_{N_0})(P_{C_k} u_{N_1}) u_{N_2} w_{N_3} P_N(G(h_{\leq N_2}))\, dxdt \right|$$
$$\lesssim \|P_{C_j} v_{N_0}\|_{L_{t,x}^{r_0}} \|P_{C_k} u_{N_1}\|_{L_{t,x}^{r_1}} \|u_{N_2}\|_{L_{t,x}^{r_2}} \|w_{N_3}\|_{L_{t,x}^{r_3}} \|P_N(G(h_{\leq N_2}))\|_{L_{t,x}^{r_4/(p-2)}}$$
$$\lesssim \left(\frac{N_0}{N_1} \right)^{s_c} N^{5(\frac{1}{r_2} + \frac{1}{r_3} + \frac{p-2}{r_4}) - p} N_2^{-5(\frac{1}{r_2} + \frac{p-2}{r_4}) + p - \frac{2}{p}} N_3^{\frac{2}{p} - \frac{5}{r_3}}$$
$$\cdot \|P_{C_j} v_{N_0}\|_{Y^{-s_c}} \|P_{C_k} u_{N_1}\|_{Y^{s_c}} \|u_{N_2}\|_{Y^{s_c}} \|w_{N_3}\|_{Y^{s_c}} \|h_{\leq N_2}\|_{Y^{s_c}}^{p-2}.$$

Our choices of r_j also ensure that this is summable over $N_0 \sim N_1 \gtrsim N_2 \gtrsim N_3$, $N > N_2$. Performing the summation establishes (2.2.13). □

CHAPTER 3

Breakdown of regularity for the scattering operators of the defocusing mass-subcritical NLS

In this chapter we prove Theorem 1.3.1, which we restate for the reader's convenience:

Theorem 3.0.1. *Let $d \geq 1$, and consider pNLS with $\alpha(d) < p < \frac{4}{d}$. Then:*

1. *The scattering operators $\mathcal{S}, \mathcal{W}$ for pNLS are well-defined as maps $\Sigma \to L^2$, and are maximally regular at $0 \in \Sigma$ in the sense that they are Hölder continuous of order $1+p$ at 0, but not of any higher order.*

2. *There exists $\beta = \beta(d,p) \in (0,p)$ such that $\mathcal{S}, \mathcal{W}$ admit no extensions to maps $L^2 \to L^2$ which are Hölder continuous of order $1+\beta$ on any ball $B \subset L^2$ containing the origin.*

Here, the notion of Hölder continuity of order s at a point x_0 is defined as membership in the class $C^s(x_0)$; see Definition 1.4.2.

We now summarize the main ideas in the proof of this theorem. This discussion also applies to the proof of part (1) of Theorem 1.3.4. The basic strategy is one that has been applied by various authors (e.g. [3, 15, 35]) to obtain ill-posedness results such as norm inflation for nonlinear dispersive equations in low-regularity spaces. The first step is to decompose the solution operator into a main term, which we expect to exhibit ill-posedness properties, as well as an error term. This decomposition is generally performed in a stronger topology which contains the low-regularity topology as a dense subspace. The next step is to prove that the main term exhibits the desired ill-posedness properties when considered in the low-regularity topology. The final step is to show that the error term is subdominant to the main term in the regime where the main term exhibits ill-posedness, thus establishing that the full solution operator inherits this behavior.

For us, the stronger topology is Σ, and the weaker topology where we expect ill-posedness is L^2. We begin by obtaining an asymptotic expansion of the scattering operator $\mathcal{T}$ with respect to the initial/scattering data which holds in Σ. Such expansions for pNLS and HNLS scattering operators have been considered previously in the work of Kita [36], Kita and Ozawa [37], Carles and Ozawa [10], Carles and Gallagher [9], Masaki [42], and Miao, Wu, and Zhang [44]. This expansion take the form

$$\mathcal{T}(\phi) = \phi \pm i \int_0^\infty e^{-is\Delta} F(e^{is\Delta}\phi)\ ds + e(\phi),$$

where F denotes the nonlinearity and $e(\phi)$ is an error term. Here ϕ is the first derivative term in the expansion of $\mathcal{T} : \Sigma \to L^2$.

The key idea is that the term

$$i \int_0^\infty e^{-is\Delta} F(e^{is\Delta}\phi)\ ds, \tag{3.0.1}$$

which corresponds to a certain order of derivative in the Taylor expansion of $\mathcal{T}$, is well-behaved for $\phi \in \Sigma$ but poorly behaved on L^2 in the sense that (3.0.1) cannot be bounded in terms of $\|\phi\|_{L^2}$. The reason for this misbehavior is the failure of "nonlinear free energy" estimates of the form

$$\int_0^\infty Q(e^{it\Delta}\phi)\ dt \lesssim \|\phi\|_2^\alpha,\ \alpha > 0, \tag{3.0.2}$$

which is entirely due to scaling; here we recall that $Q(u)$ is the potential energy functional (1.3.3), (1.3.4). The rest of the argument is to control the L^2-norm of the error $e(\phi)$, in order to ensure that the bad behavior in (3.0.1) is the dominant behavior. By doing so, we can show that no estimates of the form

$$\frac{\|\mathcal{T}(\phi) - \phi\|_{L^2}}{\|\phi\|_{L^2}^s} = \mathcal{O}(1) \text{ as } \|\phi\|_{L^2} \to 0$$

can hold for certain values of s, which implies the desired breakdown of regularity of $\mathcal{T}$ at least at the origin.

3.1 Preliminary definitions and estimates

We define the energy functional

$$E(v) = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla v|^2 + \frac{1}{p+2}|v|^{p+2}\ dx.$$

We will need the following effective version of the pseudoconformal energy law:

Lemma 3.1.1 (Pseudoconformal energy estimate)**.** *Let* $0 < p < \frac{4}{d}$ *and* $\phi \in \Sigma$. *Let* u *be the global solution to* (1.1.3) *with initial data* ϕ. *Then for all* $t \geq 1$,

$$\|u(t)\|_{L_x^{p+2}}^{p+2} \lesssim_{d,p} t^{-\frac{dp}{2}}(\|u(1)\|_\Sigma^2 + \|u(1)\|_\Sigma^{p+2}). \tag{3.1.1}$$

Moreover, if $\varepsilon = \varepsilon(d,p) > 0$ *is sufficiently small and* $\|\phi\|_\Sigma < \varepsilon$, *then for all* $t \geq 0$

$$\|u(t)\|_{L_x^{p+2}}^{p+2} \lesssim_{d,p} \langle t\rangle^{-\frac{dp}{2}}\|\phi\|_\Sigma^2. \tag{3.1.2}$$

Lemma 3.1.1 has a well-known formal proof for regular solutions via the virial identity, though it is usually stated without the explicit dependence on $u(1)$ or the small-data statement. We reproduce it here for the reader's convenience; our proof follows the presentation in [45]. A proof that recovers Lemma 3.1.1 for rougher solutions can be found in [16].

Proof. Let $J(t) = x + it\nabla$. (3.1.1) follows from the more general inequality

$$\|J(t)u(t)\|_{L_x^2}^2 + \|u(t)\|_{L_x^{p+2}}^{p+2} \lesssim_{d,p} t^{-\frac{dp}{2}}(\|u(1)\|_\Sigma^2 + \|u(1)\|_\Sigma^{p+2}).$$

Expanding,

$$\|J(t)u(t)\|_{L_x^2}^2 = \int |x|^2|u|^2 - 2t\mathrm{Im}(\overline{u}\nabla u \cdot 2x) + 4t^2|\nabla u|^2\ dx.$$

Next, we invoke the virial identity for solutions to (1.1.3):

$$\frac{d^2}{dt^2}\int |x|^2|u|^2\ dx = \frac{d}{dt}2\mathrm{Im}\int \overline{u}\nabla u \cdot 2x\ dx = \int \frac{4dp}{p+2}|u|^{p+2} + 8|\nabla u|^2\ dx.$$

It follows that

$$\frac{d}{dt}\int |x|^2|u|^2 - 2t\mathrm{Im}\overline{u}\nabla u \cdot 2x\ dx = -\int \frac{4dpt}{p+2}|u|^{p+2} + 8t|\nabla u|^2\ dx.$$

By conservation of energy,

$$4t^2 \frac{d}{dt} \int |\nabla u|^2 \, dx = -8t^2 \frac{d}{dt} \int \frac{1}{p+2} |u|^{p+2} \, dx.$$

Combining, we obtain

$$\frac{d}{dt} \int |J(t)u(t)|^2 \, dx = -\int \frac{4dpt}{p+2} |u|^{p+2} - 8t^2 \frac{d}{dt} \int \frac{1}{p+2} |u|^{p+2} \, dx.$$

Defining

$$e(t) = \int |J(t)u(t)|^2 + \frac{8t^2}{p+2} |u|^{p+2} \, dx,$$

we find that

$$\dot{e}(t) = \frac{4t(4-dp)}{p+2} \int |u|^{p+2} \, dx = \frac{2 - \frac{dp}{2}}{t} \frac{8t^2}{p+2} \int |u|^{p+2} \, dx.$$

With

$$U(t) = \frac{8t^2}{p+2} \int |u|^{p+2} \, dx,$$

it follows that

$$U(t) \leq e(t) = e(1) + \int_1^t \dot{e}(s) \, ds = e(1) + \int_1^t \frac{2 - \frac{dp}{2}}{s} U(s) \, ds.$$

By Gronwall's inequality and Sobolev embedding, we conclude that

$$U(t) \leq e(1) \exp\left(\int_1^t \frac{2 - \frac{dp}{2}}{s} \, ds \right) \lesssim (\|u(1)\|_\Sigma^2 + \|u(1)\|_\Sigma^{p+2}) t^{2 - \frac{dp}{2}}$$

which gives the claim. The small-data statement now follows from the local well-posedness theory for NLS in Σ developed in [18]. □

3.2 Small-data expansion of the scattering operators

In this section we perform the small-data expansion of the wave operator and the initial-to-scattering-state map.

Henceforth we take $q = \frac{4(p+2)}{dp}$; then $(q, p+2)$ is an admissible pair. We write $\mathcal{T}_- = \mathcal{S}$, $\mathcal{T}_+ = \mathcal{W}$, regarding them as maps $\mathcal{T}_\pm : \Sigma \to L^2$.

Proposition 3.2.1 (Small-data expansion)**.** *Let $\alpha(d) < p < \frac{4}{d}$. Then there exists $\varepsilon = \varepsilon(d,p) > 0$ small so that if $\|\phi\|_\Sigma < \varepsilon$, then*

$$\mathcal{T}_\pm(\phi) = \phi \pm i \int_0^\infty e^{-is\Delta} F(e^{is\Delta}\phi)\ ds + e_\pm(\phi)$$

where the error term $e_\pm(\phi)$ satisfies

$$\|e_\pm(\phi)\|_2 \lesssim_{d,p} \|\phi\|_\Sigma^{\frac{2(2p+1)}{p+2}}. \tag{3.2.1}$$

The proof of Proposition 3.2.1 is based on the proof of scattering in Σ above the Strauss exponent established in [18, 19], combined with the effective pseudoconformal energy law 3.1.1. It follows from the following two estimates:

Lemma 3.2.2. *Let $d \geq 1$, $\alpha(d) < p < \frac{4}{d}$, and $\phi \in \Sigma$. Then there exists $\varepsilon = \varepsilon(d,p) > 0$ small so that if $\|\phi\|_\Sigma < \varepsilon$, then*

$$\|u\|_{\frac{pq}{q-2},p+2} + \|u\|_{q,p+2} \lesssim_{d,p} \|\phi\|_\Sigma^{\frac{2}{p+2}}.$$

Proof. By Lemma 3.1.1,

$$\begin{aligned}\|u\|_{\frac{pq}{q-2},p+2} &\lesssim \left(\int_0^\infty (\langle t\rangle^{-\frac{dp}{2(p+2)}} \|\phi\|_\Sigma^{\frac{2}{p+2}})^{\frac{pq}{q-2}}\ dt\right)^{\frac{q-2}{pq}} \\ &= \|\phi\|_\Sigma^{\frac{2}{p+2}} \left(\int_0^\infty \langle t\rangle^{-\frac{2p}{q-2}}\ dt\right)^{\frac{q-2}{pq}}.\end{aligned}$$

The integral in time is finite provided $\frac{2p}{q-2} > 1$, which is true whenever $p > \alpha(d)$. A similar argument shows that

$$\|u\|_{q,p+2} \lesssim \|\phi\|_\Sigma^{\frac{2}{p+2}} \left(\int_0^\infty \langle t\rangle^{-2}\ dt\right)^{\frac{1}{q}} \lesssim \|\phi\|_\Sigma^{\frac{2}{p+2}}.$$

□

Lemma 3.2.3. *Let $d \geq 1$, $\alpha(d) < p < \frac{4}{d}$, and $\phi \in \Sigma$. Then*

$$\|e^{it\Delta}\phi\|_{\frac{pq}{q-2},p+2} \lesssim_{d,p} \|\phi\|_\Sigma.$$

Proof. The dispersive estimate (Proposition 1.4.1), combined with the Gagliardo-Nirenberg inequality and the embedding $\Sigma \hookrightarrow L^q$ ($\frac{2d}{d+2} < q \leq 2$), gives us the decay estimate

$$\|e^{it\Delta}\phi\|_{p+2} \lesssim \langle t\rangle^{-\frac{dp}{2(p+2)}}\|\phi\|_\Sigma.$$

From here the proof is nearly identical to that of Lemma 3.2.2, where we invoke the above decay estimate in place of the pseudoconformal energy estimate. □

Proof of Proposition 3.2.1. By the construction of the wave operator given in [11], for a given final state $\phi \in \Sigma$ the global solution u to (1.1.3) with final state ϕ satisfies

$$\begin{aligned} u(t) &= e^{it\Delta}\phi + \int_t^\infty e^{i(t-s)\Delta}F(u)(s)\ ds \\ &= e^{it\Delta}\phi + \int_t^\infty e^{i(t-s)\Delta}F(e^{is\Delta}\phi)(s)\ ds + r_+(\phi)(t), \end{aligned}$$

where

$$r_+(\phi)(t) = u(t) - e^{it\Delta}\phi - \int_t^\infty e^{i(t-s)\Delta}F(e^{is\Delta}\phi)(s)\ ds.$$

Sending $t \to 0$, we obtain

$$\mathcal{W}(\phi) = u(0) = \phi + \int_0^\infty e^{-is\Delta}F(e^{is\Delta}\phi)(s)\ ds + e_+(\phi)$$

where $e_+(\phi) = r_+(\phi)(0)$. A similar expression holds for $\mathcal{S}(\phi)$: we have

$$\mathcal{S}(\phi) = [\mathcal{S}(\phi) - e^{-it\Delta}u(t)] + \left[\phi - i\int_0^t e^{-is\Delta}F(e^{is\Delta}\phi)\ ds\right] + e^{-it\Delta}r_-(\phi)(t), \tag{3.2.2}$$

where

$$r_-(\phi)(t) = u(t) - e^{it\Delta}\phi + i\int_0^t e^{i(t-s)\Delta}F(e^{is\Delta}\phi)\ ds.$$

By the definition of $\mathcal{S}(\phi)$, we have $\|\mathcal{S}(\phi) - e^{-it\Delta}u(\phi)(t)\|_2 \to 0$ as $t \to \infty$. Sending $t \to \infty$ in (3.2.2), we obtain

$$\mathcal{S}(\phi) = \phi - i\int_0^\infty e^{-is\Delta}F(e^{is\Delta}\phi)\ ds + e_-(\phi),$$

where $e_-(\phi) = \lim_{t\to\infty} e^{-it\Delta}r_-(\phi)(t)$. Therefore we have

$$\|e_\pm(\phi)\|_2 \leq \|r_\pm(\phi)\|_{\infty,2}.$$

Since u satisfies the integral equation, we may write

$$r_{\pm}(\phi)(t) = \pm i \int_0^t e^{i(t-s)\Delta}[F(u(\phi)(s)) - F(e^{is\Delta}\phi)]\ ds.$$

By repeated applications of the Strichartz inequality (Proposition 1.4.2) and Hölder, we obtain

$$\begin{aligned}\|r_{\pm}(\phi)\|_{\infty,2} &= \left\| \int_0^t e^{i(t-s)\Delta}[F(u(\phi)(s)) - F(e^{is\Delta}\phi)]\ ds \right\|_{\infty,2} \\ &\lesssim_{d,p} (\|u\|_{\frac{pq}{q-2},p+2} + \|e^{it\Delta}\phi\|_{\frac{pq}{q-2},p+2})\|u - e^{it\Delta}\phi\|_{q,p+2} \\ &\lesssim_{d,p} (\|u\|^p_{\frac{pq}{q-2},p+2} + \|e^{it\Delta}\phi\|^p_{\frac{pq}{q-2},p+2})\|u\|^p_{\frac{pq}{q-2},p+2}\|u\|_{q,p+2}.\end{aligned}$$

Using Lemmas 3.2.2 and 3.2.3 to control the terms in the last line, and noting that $\|\phi\|_\Sigma^{\frac{2}{p+2}} \gg \|\phi\|_\Sigma$ for $\|\phi\|_\Sigma$ small, we obtain Proposition 3.2.1. □

3.3 Breakdown of regularity

We are now ready to proceed with the proof of Theorem 3.0.1.

3.3.1 Proof in high dimensions

First we restrict to the case $d \geq 4$. This choice is entirely for expository reasons: it simplifies some technical details, while preserving the essence of the proof.

Proof of Theorem 3.0.1, $d \geq 4$. Our first goal is part (1) of Theorem 3.0.1. First we show that $\mathcal{T}_\pm : \Sigma \to L^2$ is of class $C^s(0)$ for all $0 < s \leq 1 + p$. Applying Strichartz and arguing as in the proof of Lemma 3.2.3, we have the estimate

$$\left\| \int_0^\infty e^{-is\Delta} F(e^{is\Delta}\phi)\ ds \right\|_2 \lesssim \|\phi\|_\Sigma^{1+p}.$$

Therefore Proposition 3.2.1 gives us

$$\mathcal{T}_\pm(\phi) - \phi = \mathcal{O}_{L^2}(\|\phi\|_\Sigma^{1+p}) + \mathcal{O}_{L^2}(\|\phi\|_\Sigma^{\frac{2(2p+1)}{p+2}})$$

whenever $\|\phi\|_\Sigma$ is small. Noting that $\frac{2(2p+1)}{p+2} > 1+p$ (the condition is equivalent to $p < 1$, which holds for mass-subcritical NLS whenever $d \geq 4$), we find that

$$\mathcal{T}_\pm(\phi) - \phi = \mathcal{O}_{L^2}(\|\phi\|_\Sigma^{1+p}).$$

From this we conclude that $\mathcal{T}_\pm : \Sigma \to L^2$ belongs to the class $C^s(0)$ for all $0 < s \leq 1+p$. Moreover, this identifies the first variation of $\mathcal{T}_\pm$ at 0 as $d\mathcal{T}_\pm(0)(\phi) = \phi$.

Next we show that $\mathcal{T}_\pm : \Sigma \to L^2$ fails to be of class $C^s(0)$ whenever $s > 1+p$. It suffices to show that

$$\mathcal{T}_\pm(\phi) - \phi \neq \mathcal{O}_{L^2}(\|\phi\|_\Sigma^s) \tag{3.3.1}$$

as $\|\phi\|_\Sigma \to 0$ for any $s > 1+p$; see Lemma A.0.3.

By Proposition 3.2.1, L^2 duality, and the unitarity of the linear propagator, we have

$$\begin{aligned}
\|\mathcal{T}_\pm(\phi) - \phi\|_2 &\geq \left\| \int_0^\infty e^{-is\Delta} F(e^{is\Delta}\phi)\, ds \right\|_2 - \|e_\pm(\phi)\|_2 \\
&\geq \frac{1}{\|\phi\|_2} \left| \int_0^\infty \langle e^{-is\Delta} F(e^{is\Delta}\phi), \phi \rangle_{L_x^2}\, ds \right| - \|e_\pm(\phi)\|_2 \\
&= \frac{\|e^{it\Delta}\phi\|_{p+2,p+2}^{p+2}}{\|\phi\|_2} - \|e_\pm(\phi)\|_2.
\end{aligned}$$

Therefore (3.3.1) is proved if we exhibit a sequence $(\phi_n) \subset \Sigma$ with $\|\phi_n\|_\Sigma \to 0$ and

$$\frac{\|e^{it\Delta}\phi_n\|_{p+2,p+2}^{p+2}}{\|\phi_n\|_2 \|\phi_n\|_\Sigma^s} - \frac{\|e_\pm(\phi_n)\|_\Sigma}{\|\phi_n\|_\Sigma^s} \to \infty.$$

Let $\phi \in \Sigma$ with $\|\phi\|_2 = 1$, and for $\varepsilon, \sigma > 0$ define

$$\phi_{\varepsilon,\sigma}(x) = \frac{\varepsilon}{\sigma^{\frac{d}{2}}} \phi\left(\frac{x}{\sigma}\right).$$

Then $\phi_{\varepsilon,\sigma}$ satisfies the following scalings:

$$\|\phi_{\varepsilon,\sigma}\|_2 = \varepsilon,\ \|\nabla \phi_{\varepsilon,\sigma}\|_2 \sim \frac{\varepsilon}{\sigma},\ \|\phi_{\varepsilon,\sigma}\|_\Sigma \sim \varepsilon(1 + \frac{1}{\sigma} + \sigma),$$

and

$$\|e^{it\Delta}\phi_{\varepsilon,\sigma}\|_{p+2,p+2}^{p+2} \sim \varepsilon^{p+2}\sigma^{2-\frac{dp}{2}},$$

where for the last expression we have used the parabolic scaling symmetry of the linear Schrödinger equation.

We will work in the regime $\varepsilon \ll 1$, $\sigma \gg 1$, and $\varepsilon\sigma \ll 1$. These together imply that $\|\phi_{\varepsilon,\sigma}\|_\Sigma \sim \varepsilon\sigma \ll 1$, and therefore we are in the small-data regime of Proposition 3.2.1.

By the error estimate (3.2.1) of Proposition 3.2.1, we have

$$\|e_\pm(\phi_{\varepsilon,\sigma})\|_2 \lesssim (\varepsilon\sigma)^{\frac{2(2p+1)}{p+2}}.$$

Next we assume that $\varepsilon = \sigma^{-j}$ with $j > 1$. Since $\sigma \gg 1$, under this assumption that we still have $\varepsilon\sigma \ll 1$. We now compute:

$$\begin{aligned}\frac{\|e^{it\Delta}\phi_{\varepsilon,\sigma}\|_{p+2,p+2}^{p+2}}{\|\phi_{\varepsilon,\sigma}\|_2} - \|e_\pm(\phi_{\varepsilon,\sigma})\|_2 &\gtrsim \varepsilon^{p+1}\sigma^{2-\frac{dp}{2}} - (\varepsilon\sigma)^{\frac{2(2p+1)}{p+2}} \\ &= \sigma^{-j(p+1)+2-\frac{dp}{2}} - \sigma^{\frac{2(2p+1)}{p+2}(1-j)}.\end{aligned}$$

We wish for the main term to dominate the error term in the regime $\sigma \gg 1$. Since we are free to take j arbitrarily large, the main term will dominate provided $p+1 < \frac{2(2p+1)}{p+2}$; as we have already observed, this is automatically satisfied whenever $d \geq 4$.

Therefore we have

$$\|\mathcal{T}_\pm(\phi_{\varepsilon,\sigma}) - \phi_{\varepsilon,\sigma}\|_2 \gtrsim \sigma^{-j(p+1)+2-\frac{dp}{2}} \tag{3.3.2}$$

and thus

$$\frac{\|\mathcal{T}_\pm(\phi_{\varepsilon,\sigma}) - \phi_{\varepsilon,\sigma}\|_2}{\|\phi_{\varepsilon,\sigma}\|_\Sigma^s} \gtrsim \sigma^{j[s-(p+1)]+2-\frac{dp}{2}-s}.$$

Since $s > 1+p$, for j sufficiently large we have $j[s-(p+1)] + 2 - \frac{dp}{2} - s > 0$. Taking j large to guarantee this inequality and that the main term dominates the error, then taking $\sigma \to \infty$, we find that

$$\frac{\|\mathcal{T}_\pm(\phi_{\varepsilon,\sigma}) - \phi_{\varepsilon,\sigma}\|_2}{\|\phi_{\varepsilon,\sigma}\|_\Sigma^s} \to \infty.$$

We thus conclude, as desired, that $\mathcal{T}_\pm$ is not of class $C^s(0)$ as a map $\Sigma \to L^2$ whenever $s > 1+p$.

We proceed to part (2) of Theorem 3.0.1. It suffices to show there exists $0 < \beta < p$ so that

$$\mathcal{T}_{\pm}(\phi) - \phi \neq \mathcal{O}_{L^2}(\|\phi\|_2^{1+\beta}) \tag{3.3.3}$$

as $\|\phi\|_{L^2} \to 0$. For suppose $\mathcal{T}_{\pm} \in C^{1+\beta}(0)$. Then $\mathcal{T}_{\pm}(\varepsilon\phi) - \varepsilon a(\phi) = \mathcal{O}_{L^2}(\varepsilon^{1+\beta})$ for $\|\phi\|_2 = 1$ and $\varepsilon > 0$ small. Dividing through by ε, noting $\mathcal{T}_{\pm}(0) = 0$, and letting $\varepsilon \to 0$, we find that $a(\phi) = d\mathcal{T}_{\pm}(0)(\phi)$, the first variation of $\mathcal{T}_{\pm}$ at 0 in the direction ϕ; but we already know that $d\mathcal{T}_{\pm}(0)(\phi) = \phi$ when $\mathcal{T}_{\pm}$ is regarded as a map $\Sigma \to L^2$, and by density this would be preserved if $\mathcal{T}_{\pm}$ were to admit an extension to L^2. Therefore $\mathcal{T}_{\pm}(\phi) - \phi$ is the only expression that has any hope of satisfying the $\mathcal{O}(\|\phi\|_2^{1+\beta})$ bound; showing that this fails proves that $\mathcal{T}_{\pm} \notin C^{1+\beta}(0)$ as a map $L^2 \to L^2$.

From here the proof is similar to the proof we gave for (3.3.1). Arguing identically as before, (3.3.3) is proved if we exhibit a sequence $(\phi_n) \subset \Sigma$ with $\|\phi_n\|_{\Sigma} \to 0$ and

$$\frac{\|e^{it\Delta}\phi_n\|_{p+2,p+2}^{p+2}}{\|\phi_n\|_2^{2+\beta}} - \frac{\|e_{\pm}(\phi_n)\|_2}{\|\phi_n\|^{1+\beta}} \to \infty.$$

We take $\sigma \gg 1$, $\varepsilon = \sigma^{-j}$, and j sufficiently large. Starting from (3.3.2) and dividing through by $\|\phi_{\varepsilon,\sigma}\|_2^{1+\beta}$, we obtain

$$\frac{\|\mathcal{T}_{\pm}(\phi_{\varepsilon,\sigma}) - \phi_{\varepsilon,\sigma}\|_2}{\|\phi_{\varepsilon,\sigma}\|_2^{1+\beta}} \gtrsim \sigma^{j(\beta-p)+2-\frac{dp}{2}}$$

For this to be large in the regime $\sigma \gg 1$, we require $j(\beta - p) + 2 - \frac{dp}{2} > 0$, or equivalently $\beta > p - \frac{1}{j}(2 - \frac{dp}{2})$. This shows that if j is sufficiently large and this inequality for β holds, then $\mathcal{T}_{\pm}$ fails to extend to a map $L^2 \to L^2$ of class $C^{1+\beta}(0)$. Since the constraint on β is an open condition, we can optimize by taking the smallest admissible value of j, which depends only on p and d. Therefore we have found $j = j(d, p)$ so that if $\beta > p - \frac{1}{j}(2 - \frac{dp}{2})$, then u_+ fails to extend to a map $L^2 \to L^2$ of class $C^{1+\beta}(0)$, which completes the proof. □

3.3.2 Proof in low dimensions

Here we outline the proof of Theorem 3.0.1 in $d = 1, 2, 3$. There is no truly serious obstruction to be overcome to obtain the result in low dimensions; the choice to break up the proof is entirely for expository purposes, as the proof for $d \geq 4$ is particularly clean and encompasses all of the main ideas.

The main reason why the previous proof does not extend to lower dimensions is that the error estimate

$$\|e_{\pm}(\phi)\|_2 \lesssim \|\phi\|_{\Sigma}^{\frac{2(2p+1)}{p+2}}$$

is no longer strong enough for the main term to dominate the error when $d \leq 3$ and $\alpha(d) < p < \frac{4}{d}$. Therefore the main task is to sharpen this estimate until the error is once again dominated by the main term.

The inefficiency in the above estimate arises from the use of the pseudoconformal energy estimate (Lemma 3.1.1), which is obviously not scaling-invariant and thus leads to losses every time it is invoked. However, noting that there is some slack in the integrability conditions for the time integrals in the proofs of Lemmas 3.2.2 and 3.2.2, we can reduce the total degree to which we rely on the pseudoconformal energy estimate.

As before, we write $q = \frac{4(p+2)}{dp}$, so that $(q, p+2)$ is an admissible pair. We now state the sharpened version of (3.2.1):

Proposition 3.3.1. *Let $d \geq 1$, $\alpha(d) < p < \frac{4}{d}$. Define $e_{\pm}(\phi)$ as before. Let $\frac{q-2}{2p} < \eta \leq 1$ and $\frac{1}{2} < \nu \leq 1$. Then there exists $\varepsilon = \varepsilon(d,p) > 0$ small so that if $\|\phi\|_{\Sigma} < \varepsilon$, then*

$$\|e_{\pm}(\phi)\|_2 \lesssim_{d,p,\eta,\nu} \|\phi\|_{\Sigma}^{Q(d,p,\eta,\nu)} \tag{3.3.4}$$

where

$$Q(d,p,\eta,\nu) = 2p(1-\eta) + (1-\nu) + \frac{2}{p+2}(2\eta p + \nu).$$

We begin the proof. Write $\theta = 1 - \frac{dp}{2(p+2)}$. First, we have the following sharpened forms of Lemmas 3.2.2 and 3.2.3:

Lemma 3.3.2. *Let $d \geq 1$, $\alpha(d) < p < \frac{4}{d}$, and $\phi \in \Sigma$. Then there exists $\varepsilon = \varepsilon(d,p) > 0$ small so that if $\|\phi\|_\Sigma < \varepsilon$, then for $\frac{q-2}{2p} < \eta \leq 1$, we have*

$$\|u(\phi)\|_{\frac{pq}{q-2},p+2} \lesssim_{d,p,\eta} (\|\phi\|_2^\theta (E(\phi)^{\frac{1}{2}})^{1-\theta})^{1-\eta} \|\phi\|_\Sigma^{\frac{2\eta}{p+2}},$$

and for $\frac{1}{2} < \nu \leq 1$, we have

$$\|u(\phi)\|_{q,p+2} \lesssim_{d,p,\eta} (\|\phi\|_2^\theta (E(\phi)^{\frac{1}{2}})^{1-\theta})^{1-\nu} \|\phi\|_\Sigma^{\frac{2\nu}{p+2}},$$

Proof. We seek to control

$$\left(\int_0^\infty \|u(\phi)(t)\|_{L_x^{p+2}}^{\frac{pq}{q-2}}\, dt\right)^{\frac{q-2}{pq}}.$$

Let $\eta \in [0,1]$. We factor the integrand into powers $\|u(\phi)(t)\|_{L_x^{p+2}}^{\frac{pq}{q-2}(1-\eta)} \|u(\phi)(t)\|_{L_x^{p+2}}^{\frac{pq}{q-2}\eta}$. We estimate the first piece using Gagliardo-Nirenberg, and the second using the pseudoconformal energy estimate. We obtain

$$\begin{aligned}\left(\int_0^\infty \|u(\phi)(t)\|_{L_x^{p+2}}^{\frac{pq}{q-2}}\, dt\right)^{\frac{q-2}{pq}} &= \left(\int_0^\infty \|u(\phi)(t)\|_{L_x^{p+2}}^{\frac{pq}{q-2}(1-\eta)} \|u(\phi)(t)\|_{L_x^{p+2}}^{\frac{pq}{q-2}\frac{2\eta}{p+2}}\, dt\right)^{\frac{q-2}{pq}} \\ &\leq (\|\phi\|_2^\theta (E(\phi)^{\frac{1}{2}})^{1-\theta})^{(1-\eta)} \|\phi\|_\Sigma^{\frac{2\eta}{p+2}} \left(\int_0^\infty \langle t\rangle^{-\frac{2p\eta}{q-2}}\, dt\right)^{\frac{q-2}{pq}}.\end{aligned}$$

The last integral is finite assuming $\eta > \frac{q-2}{2p}$. This establishes the first estimate in Lemma 3.3.2. The second estimate for $\|u(\phi)\|_{q,p+2}$ is proved in exactly the same way: we split $\|u(\phi)(t)\|_{L_x^{p+2}} = \|u(\phi)(t)\|_{L_x^{p+2}}^{(1-\nu)} \|u(\phi)(t)\|_{L_x^{p+2}}^{\nu}$, estimate the first piece using Gagliardo-Nirenberg, and the second by the pseudoconformal energy estimate. The condition $\nu > \frac{1}{2}$ is required to make the final integral in time finite. We leave the details to the reader. □

Lemma 3.3.3. *Let $d \geq 1$, $\alpha(d) < p < \frac{4}{d}$, and $\phi \in \Sigma$. Then for $\frac{q-2}{2p} < \eta \leq 1$,*

$$\|e^{it\Delta}\phi\|_{\frac{pq}{q-2},p+2} \lesssim_{d,p,\eta} (\|\phi\|_2^\theta \|\nabla\phi\|_2^{1-\theta})^{1-\eta} \|\phi\|_\Sigma^\eta.$$

Proof. The proof proceeds almost identically to that of the first part of Lemma 3.3.2. As earlier, we factor $\|e^{it\Delta}\phi\|_{p+2} = \|e^{it\Delta}\phi\|_{p+2}^{1-\eta} \|e^{it\Delta}\phi\|_{p+2}^\eta$. The first factor can be controlled using Gagliardo-Nirenberg and the conservation of $\dot{H}^s$ norms under the linear Schrödinger flow.

The second factor is controlled near time 0 by Gagliardo-Nirenberg, and at large times by the dispersive estimate and the embedding $\Sigma \hookrightarrow L^q$ for all $\frac{2d}{d+2} < q \leq 2$. The condition $\eta > \frac{q-2}{2p}$ ensures that the time integral that remains is finite. We leave the details to the reader. $\square$

Proof of Proposition 3.3.1. We argue as in the proof the error bound in of Proposition 3.2.1, but using Lemmas 3.3.2 and 3.3.3. Doing so, we arrive at an estimate of the form

$$\|e_\pm(\phi)\|_2 \lesssim_{d,p,\eta,\nu} \|\phi\|_2^\alpha (E(\phi)^{\frac{1}{2}})^{\beta+\delta} \|\phi\|_\Sigma^\gamma + \|\phi\|_2^\alpha (E(\phi)^{\frac{1}{2}})^\beta \|\nabla\phi\|_2^\delta \|\phi\|_\Sigma^{\gamma + \frac{p}{p+2}\eta p},$$

where:

$$\begin{aligned}
\alpha &= \theta(2p(1-\eta) + (1-\nu)); \\
\beta &= (1-\theta)(p(1-\eta) + (1-\nu)); \\
\gamma &= \frac{2}{p+2}(2\eta p + \nu); \\
\delta &= (1-\theta)p(1-\eta).
\end{aligned}$$

Note that $Q(d,p,\eta,\nu) = \alpha + \beta + \delta + \gamma$. By Sobolev embedding, $E(\phi)$ is controlled by $\|\phi\|_\Sigma^2 + \|\phi\|_\Sigma^{p+2}$, and by the assumption $\|\phi\|_\Sigma \ll 1$ the second term is negligible. Therefore every norm and each $(E(\phi))^{\frac{1}{2}}$ is majorized by $\|\phi\|_\Sigma$, and we have:

$$\|e_\pm(\phi)\|_2 \lesssim_{d,p,\eta,\nu} \|\phi\|_\Sigma^{Q(d,p,\eta,\nu)} + \|\phi\|_\Sigma^{Q(d,p,\eta,\nu) + \frac{p}{p+2}\eta p} \lesssim \|\phi\|_\Sigma^{Q(d,p,\eta,\nu)}. \qquad \square$$

We are finally equipped to prove Theorem 3.0.1 in full generality.

Proof of Theorem 3.0.1, $d = 1,2,3$. We mention only the necessary changes relative to the proof in dimensions $d \geq 4$.

The first step is to prove that $\mathcal{T}_\pm : \Sigma \to L^2$ is of class $C^s(0)$ for all $0 < s < 1 + p$. It suffices as before to show that

$$\mathcal{T}_\pm(\phi) - \phi = \mathcal{O}(\|\phi\|_\Sigma^{1+p})$$

whenever $\|\phi\|_\Sigma$ is small. To ensure this we must show that $e_+(\phi)$ is of higher order in $\|\phi\|_\Sigma$ than the main term, i.e. $1+p < Q(d,p,\eta,\nu)$ for some admissible choice of η and ν. Since η can be arbitrarily close to $\frac{q-2}{2p}$ and ν can be arbitrarily close to $\frac{1}{2}$, it suffices to show that

$$1+p < Q\left(d, p, \frac{q-2}{2p}, \frac{1}{2}\right).$$

This is equivalent to the condition

$$2dp^2 + (11d-8)p + (8d-16) > 0.$$

When $d \geq 2$, this is automatically satisfied for $p > 0$ because the coefficients are nonnegative. When $d = 1$, the positive root of this polynomial is smaller than $\frac{3}{2}$, and thus this is satisfied for $p > 2 = \frac{2}{d}$.

Next we show that $\mathcal{T}_\pm : \Sigma \to L^2$ is not of class $C^s(0)$ for $s > 1+p$, and does not extend to a map $L^2 \to L^2$ of class $C^{1+\beta}(0)$ for some $0 < \beta < p$. As before it suffices to show that

$$\mathcal{T}_\pm(\phi) - \phi \neq \mathcal{O}_{L^2}(\|\phi\|_\Sigma^s) \tag{3.3.5}$$

in the first case, and

$$\mathcal{T}_\pm(\phi) - \phi \neq \mathcal{O}_{L^2}(\|\phi\|_{L^2}^{1+\beta}). \tag{3.3.6}$$

in the latter case. Examining the proof in $d \geq 4$, we observe that the only way in which the size of $e_\pm(\phi)$ enters into either argument is to show that there exists a regime $\varepsilon \ll 1$, $\sigma \gg 1$ and $\varepsilon\sigma \ll 1$ so that $\|\phi_{\varepsilon,\sigma}\|_2$ (where $\phi_{\varepsilon,\sigma}$ is defined as before) is dominated by the main term $\|\phi_{\varepsilon,\sigma}\|_2^{-1}\|e^{it\Delta}\phi_{\varepsilon,\sigma}\|_{p+2,p+2}^{p+2}$. Taking $\varepsilon = \sigma^{-j}$ with $j > 1$ to be determined, the main term is still of size

$$\frac{\|e^{it\Delta}\phi_{\varepsilon,\sigma}\|_{p+2,p+2}^{p+2}}{\|\phi_{\varepsilon,\sigma}\|_2} \sim \sigma^{-j(p+1)+2-\frac{dp}{2}}.$$

We use (3.3.4) to control the error by

$$\|e_\pm(\phi_{\varepsilon,\sigma})\|_2 \lesssim_{d,p,\eta,\nu} \sigma^{(1-j)Q(d,p,\eta,\nu)}.$$

Noting as before that $Q(d,p,\eta,\nu) > p+1$ for a judicious choice of η and ν, we see that $\|e_\pm(\phi_{\varepsilon,\sigma})\|_\Sigma$ is negligible relative to the main term for j sufficiently large and $\sigma \gg 1$. From here the proof of (3.3.5) and (3.3.6) proceeds exactly as when $d \geq 4$. □

CHAPTER 4

Analyticity and infinite breakdown of regularity for the mass-subcritical Hartree scattering problem

In the final chapter we prove Theorems 1.3.3 and 1.3.4, which we restate for the reader's convenience:

Theorem 4.0.1. *Let $d \geq 2$ and $\frac{4}{3} < \gamma < 1$. Let $\mathcal{T} \in \{\mathcal{S}, \mathcal{W}\}$. Then:*

1. *$\mathcal{T}$ is well-defined as a map $\Sigma \to \Sigma$, and is analytic in the sense that for all $u_0 \in \Sigma$ and $v \in \Sigma$, $\mathcal{T}$ admits the power series expansion*

$$\mathcal{T}(u_0 + \varepsilon v) = \mathcal{T}(u) + \sum_{k=1}^{\infty} \varepsilon^k w_k$$

   for all sufficiently small $\varepsilon > 0$, where $(w_k) \subset \Sigma$ and the series converges in Σ-norm.

2. *The same result holds with the space $\mathcal{F}H^1$ replacing Σ.*

Theorem 4.0.2. *Let $d \geq 2$ and $\frac{4}{3} < \gamma < 2$. Let $\mathcal{T} \in \{\mathcal{S}, \mathcal{W}\}$.*

1. *Let $s > \frac{5+5\gamma}{3+\gamma}$. Then $\mathcal{T} : \Sigma \to L^2$ admits no extension to a map $L^2 \to L^2$ which is Hölder continuous of order s on any ball B containing $0 \in L^2$.*

2. *Let $s > \frac{4+4\gamma}{2+\gamma}$. Then there exists $R > 0$ such that for any ball $B \subset B_R(0) \subset \Sigma$ (not necessarily containing the origin), $\mathcal{T} : B \to L^2$ admits no extension to a map $L^2 \to L^2$ which is Hölder continuous of order s at any point in $B \cap L^2$.*

We now summarize the main ideas of the proofs of these results. For convenience, we restrict our discussion to the proof of the result for $\mathcal{S}$, but the discussion adapts to the case of $\mathcal{W}$ easily. The proofs of Theorem 4.0.1 and part 1 of Theorem 4.0.2 have already been outlined at the beginning of Chapter 3, so we are left to outline the proof of part 2.

Recall that when $u_0 = 0$, the key idea of the proof of breakdown of regularity is to identify ill-posedness behavior in the term

$$i \int_0^\infty e^{-is\Delta} F(e^{is\Delta} v)\; ds, \tag{4.0.1}$$

which manifests as the failure to control the nonlinear free energy

$$\int_0^\infty Q(e^{it\Delta} v)\; dt$$

in terms of $\|v\|_{L^2}$. The same strategy holds in the case $u_0 \neq 0$. In this case, we look to identify the breakdown of regularity in w_3^+, the third derivative term in the expansion

$$\mathcal{S}(u_0 + v) = \mathcal{S}(u_0) + \sum_{k \geq 1} w_k^+.$$

When $u_= \neq 0$, w_3^+ is plays a role analogous to (4.0.1), and we must show it cannot be controlled in terms of $\|v\|_{L^2}$. The basic reason is, once again, the failure of the nonlinear free energy estimates, with w_1^+ (the first derivative term in the expansion) taking the place of v (the first derivative term when $u_0 = 0$).

However, two complications arise when $u_0 \neq 0$. The first is that w_1^+ is no longer merely the limit of a free evolution: it arises as a limit $\lim_{t\to\infty} e^{-it\Delta} w_1(t)$, where $w_1(t)$ satisfies a linear Schrödinger equation with nontrivial potential if $u_0 \neq 0$. This means that we cannot apply the earlier scaling argument to the nonlinear free energy functional right away. We get around this issue by using the fact that although $w_1(t)$ is not itself a free evolution, it does converge asymptotically in time to one. We therefore decompose the integral

$$\int_0^\infty Q(w_1(t))\; dt \tag{4.0.2}$$

into two regions, one consisting of large times where $w_1(t)$ is effectively a free evolution, and the other consisting of short times. We then rescale in spacetime, which replaces w_1 with a rescaled version which we can effectively treat as a free evolution for all times outside of an arbitrarily small time interval, allowing us to use the earlier scaling argument for the nonlinear free energy.

The second complication is that w_3^+ also contains additional terms which are cubic in v, which we regard as error terms and must show are subdominant to the main term (4.0.2). Since the main term is also cubic in v, this requires a more efficient analysis than we need for the case $u_0 = 0$, for which the main term is the only cubic term and all errors are strictly higher order. For this we employ a Lorentz space refinement of the scattering theory built in [26, 28, 48] in order to obtain essentially optimal estimates on the cubic error terms.

4.1 Preliminary definitions and estimates

We will make use of the vector field $J(t) = x + 2it\nabla$, which is standard in the scattering theory of Schrödinger equations. J obeys the identity

$$J(t) = M(t)(2it\nabla)M(-t) = e^{it\Delta}xe^{-it\Delta} \tag{4.1.1}$$

where $M(t) = e^{i|x|^2/4t}$; it measures the evolution of the center of mass for free evolutions. It is associated to the following decay estimate:

Lemma 4.1.1 ([49]). *For $2 \leq r < \frac{2d}{d-2}$ and $t \neq 0$, we have*

$$\|u(t)\|_r \lesssim_{d,r} |t|^{-\theta(d,r)}\|u(t)\|_2^{1-\theta(d,r)}\|J(t)u(t)\|_2^{\theta(d,r)},$$

where $\theta(d,r) = \frac{d(r-2)}{2r}$.

Proof. By the decomposition $J(t) = M(t)(2it\nabla)M(-t)$ (4.1.1) and the Gagliardo-Nirenberg inequality,

$$\begin{aligned}\|u(t)\|_r = \|M(-t)u(t)\|_r &\lesssim \|u(t)\|_2^{1-\theta(d,r)} \|\nabla M(-t)u(t)\|_2^{\theta(d,r)} \\ &= |t|^{-\theta(d,r)} \|u(t)\|_2^{1-\theta(d,r)} \|J(t)u(t)\|_2^{\theta(d,r)}. \qquad \square\end{aligned}$$

We note that for the special case $r = \frac{4d}{2d-\gamma}$, $\theta = \frac{\gamma}{4}$; from this point on we fix this as the value of θ.

Lastly, we will need some preliminary estimates on the nonlinearity. Define

$$T(u, v, w) = (|x|^{-\gamma} * (u\bar{v}))w. \tag{4.1.2}$$

Note that $F(u) = (|x|^{-\gamma} * |u|^2)u = T(u, u, u)$. Applications of Hölder's inequality and the Hardy-Littlewood-Sobolev inequality yield the following multilinear estimates:

Lemma 4.1.2 (Hartree nonlinearity estimates [28])**.** *Let $0 < \gamma < d$ and $r = \frac{4d}{2d-\gamma}$. Then for all $u, v, w \in L^r(\mathbb{R}^d)$,*

$$\begin{aligned} Q(u) = \int_{\mathbb{R}^d} (|x|^{-\gamma} * |u|^2)|u(x)|^2 dx &\lesssim \|u\|_r^4, \\ \|T(u, v, w)\|_{r'} &\lesssim \|u\|_r \|v\|_r \|w\|_r, \\ \|\nabla T(u_1, u_2, u_3)\|_{r'} &\lesssim \sum_{i=1}^{3} \|\nabla u_i\|_r \prod_{j\neq i} \|u_j\|_r, \\ \|J(t)T(u_1, u_2, u_3)\|_{r'} &\lesssim \sum_{i=1}^{3} \|J(t)u_i\|_r \prod_{j\neq i} \|u_j\|_r. \end{aligned}$$

4.2 Analyticity of the Hartree scattering operators

For the remainder of this paper, we assume $d \geq 2$ and $\frac{4}{3} < \gamma < 2$.

Our goal in this section is to prove Theorem 4.0.1. As we have mentioned in the introduction, this proceeds largely along the lines of the framework set out in [9], adapted to the estimates we have for the mass-subcritical Hartree equation.

We first address the analyticity of the wave operator. Let $u_+ \in \Sigma$ be a scattering state, and let $v \in \Sigma$ with $\|v\|_\Sigma = 1$ be arbitrary. By [28], under our current assumptions there exists a unique global solution $u \in C_t\Sigma(\mathbb{R})$ to Equation (1.1.4) which scatters to u_+, and for each $\varepsilon > 0$ there exists a unique global solution $u^\varepsilon \in C_t\Sigma$ which scatters to $u_+ + \varepsilon v$. Moreover, the wave operator $\mathcal{W} : \Sigma \to \Sigma$ is well-defined.

Write $u^\varepsilon = u + w^\varepsilon$. Our goal is to show that for $\|u_+\|_\Sigma$ sufficiently small, w^ε admits the norm-convergent expansion

$$w^\varepsilon(t) = \sum_{k=1}^{\infty} \varepsilon^k w_k(t) \text{ as } \varepsilon \to 0,$$

where (w_k) are elements of an appropriate function space determined by contraction mapping. This argument consists of three parts:

1. determining the hierarchy of equations satisfied by the sequence (w_k);
2. showing that (w_k) is sufficiently strongly bounded in a global spacetime norm, so that the series for u^ε is norm convergent;
3. showing that the series for w^ε does actually converge to w^ε.

It will emerge as a consequence that $\mathcal{W}$ admits the norm-convergent expansion

$$\mathcal{W}(u_+ + \varepsilon v) = \sum_{j=0}^{\infty} \varepsilon^k v_k \text{ as } \varepsilon \to 0$$

where $(v_k) \subset \Sigma$.

4.2.1 Hierarchy equations

The coefficients (w_k) of the series for u^ε formally satisfy a hierarchy of coupled PDEs. We express u^ε in integral form, then match like powers of ε to obtain the coefficients. Let us write

$$\mathcal{N}(u, v, w)(t) = i \int_t^\infty e^{i(t-s)\Delta} T(u, v, w) \, ds$$

where T is the trilinear form defined in (4.1.2). Matching zero-th order terms in ε yields

$$w_0(t) = u(t) = e^{it\Delta}u_+ + \mathcal{N}(u,u,u)(t)$$

Matching first order terms in ε yields

$$w_1(t) = e^{it\Delta}v + \mathcal{N}(u,u,w_1)(t) + \mathcal{N}(u,w_1,u)(t) + \mathcal{N}(w_1,u,u)(t).$$

Higher-order terms behave similarly, involving symmetric sums of the trilinear operators T with arguments in $\{u, w_1, w_2, \ldots\}$. To simplify notation, we introduce the symmetric sum operator S which sums over all distinct permutations of the ordered triple (u, v, w). For example,

$$S\mathcal{N}(u,u,w_1) = \mathcal{N}(u,u,w_1) + \mathcal{N}(u,w_1,u) + \mathcal{N}(w_1,u,u).$$

Such a symmetric sum has either one, three, or six summands. With this notation, the full hierarchy of equations for the coefficients takes the following form:

$$w_0(t) = e^{it\Delta}u_+ + \mathcal{N}(u,u,u)(t), \tag{4.2.1}$$

$$w_1(t) = e^{it\Delta}v + S\mathcal{N}(u,u,w_1)(t), \tag{4.2.2}$$

$$w_N(t) = \sum_{j+k+\ell=N} \mathcal{N}(w_j, w_k, w_\ell)(t), \;\; N \geq 2. \tag{4.2.3}$$

4.2.2 Coefficient estimates

Fix $r = \frac{4d}{2d-\gamma}$ and $q = \frac{8}{\gamma}$; then (q,r) is a Schrödinger-admissible pair. Also fix $\alpha = \frac{8}{4-\gamma}$. With these choices we have $\frac{1}{q'} = \frac{1}{q} + \frac{2}{\alpha}$.

For a time interval I, we define the space $Y(I)$ via its norm

$$\|f\|_{Y(I)} = \|f\|_{L_t^\infty L_x^2(I)} + \|f\|_{L_t^q L_x^r(I)}.$$

We define the space $X(I)$ by the norm

$$\|f\|_{X(I)} = \|f\|_{Y(I)} + \|J(t)f\|_{Y(I)} + \|\nabla f\|_{Y(I)}.$$

Remark 4.2.1. $X(I)$ is adapted to the Σ-norm and is thus used to prove part (1) of Theorem 4.0.1. The results of this section can also be proved in the $\mathcal{F}H^1$-adapted space $Z(I)$ defined by the norm

$$\|f\|_{Z(I)} = \|f\|_{Y(I)} + \|J(t)f\|_{Y(I)}.$$

We leave it to the reader to verify that all of the relevant estimates hold with $Z(I)$ replacing $X(I)$, thus obtaining part (2) of Theorem 4.0.1 as well.

The results of this section also hold in the Lorentz-modified space $X^*(I)$, defined analogously to $X(I)$ but replacing $Y(I)$ by $Y^*(I)$, where

$$\|f\|_{Y^*(I)} = \|f\|_{L_t^\infty L_x^2(I)} + \|f\|_{L_t^{q,2} L_x^r(I)}.$$

Since $L_t^{q,2}$ is normable for our choice of q, these are still Banach spaces. Again, we leave it to the reader to check that all of the results we prove in this section can be adapted to the X^* setting as well: the key estimate is

$$\|T(u,v,w)\|_{L_t^{q',2} L_x^{r'}} \lesssim \|u\|_{L_t^{\alpha,\infty} L_x^r} \|v\|_{L_t^{\alpha,\infty} L_x^r} \|w\|_{L_t^{q,\infty} L_x^r}$$

and the analogous estimates for $J(t)T(u,v,w)$ and $\nabla T(u,v,w)$, which follow from Lemma 4.1.2 and Hölder's inequality for Lorentz spaces. The Lorentz space refinement will become relevant in Section 4.3.

We construct the power series expansions of the wave and scattering operators by constructing the coefficients (w_k) on the interval $[0,\infty)$ and then taking the appropriate limits in t. For notational convenience we construct (w_k) first on $[1,\infty)$. Composing with the time-translation symmetry of HNLS then gives us the coefficients on $[0,\infty)$.

Proposition 4.2.1. *For any $u_+ \in \Sigma$ and any $v \in \Sigma$ with $\|v\|_\Sigma = 1$, there exists a constant $\Lambda = \Lambda(R,d,\gamma) > 0$ such that for all $k \geq 1$,*

$$\|w_k\|_{X([1,\infty))} \leq a_k \Lambda^k,$$

where (a_k) *is a sequence of positive numbers satisfying*

$$a_k \lesssim (C_0 a_1)^k$$

for some positive constant C_0*.*

Corollary 4.2.2. *Under the same hypotheses, the series*

$$\sum_{k=1}^{\infty} \varepsilon^k w_k$$

converges in the norm topology of $X(\mathbb{R})$ *for all sufficiently small* $\varepsilon > 0$*.*

Lemma 4.2.3 ([3, 35])**.** *Let* (a_j) *be a sequence of positive numbers satisfying*

$$a_N \le C \sum_{\substack{j+k+\ell=N \\ j,k,\ell \neq N}} a_j a_k a_\ell, \;\; N \ge 2.$$

Then there exist constants $C_0, C_1 > 0$ *such that*

$$a_N \le C_1 (C_0 a_1)^N$$

for all $N \ge 1$*.*

Proof. We claim the stronger inequality

$$\langle N \rangle^2 a_N \le C_1 (C_0 a_1)^N. \tag{4.2.4}$$

We proceed by induction on N.

First we assume $C_1 C_0 \ge 1$. Under this assumption, the base case $N = 1$ is trivial.

Now assume (4.2.4) holds for $1, \dots, N-1$. We estimate:

$$\begin{aligned}
\langle N\rangle^2 a_N &\leq C \sum_{\substack{j+k+\ell=N \\ j,k,\ell\neq N}} a_j a_k a_\ell \langle j+k+\ell\rangle^2 \\
&\leq CC_1^3 (C_0 a_1)^N \sum_{\substack{j+k+\ell=N \\ j,k,\ell\neq N}} \frac{\langle j+k+\ell\rangle^2}{\langle j\rangle^2\langle k\rangle^2\langle \ell\rangle^2} \\
&\leq 3CC_1^3 (C_0 a_1)^N \sum_{\substack{j+k+\ell=N \\ j,k,\ell\neq N}} \frac{\langle j\rangle^2+\langle j\rangle^2+\langle j\rangle^2}{\langle j\rangle^2\langle k\rangle^2\langle \ell\rangle^2} \\
&\leq 9CC_1^3 (C_0 a_1)^N \sum_{\substack{j+k+\ell=N \\ j,k,\ell\neq N}} \frac{\langle j\rangle^2}{\langle j\rangle^2\langle k\rangle^2\langle \ell\rangle^2}.
\end{aligned}$$

The remaining sum we bound as follows:

$$\sum_{\substack{j+k+\ell=N \\ j,k,\ell\neq N}} \frac{\langle j\rangle^2}{\langle j\rangle^2\langle k\rangle^2\langle \ell\rangle^2} \leq \sum_{\ell=0}^{N}\sum_{k=0}^{N-\ell} \frac{1}{\langle k\rangle^2\langle \ell\rangle^2} \leq \left(\sum_{k=0}^{\infty}\langle k\rangle^{-2}\right)^2 = C_2^2.$$

Therefore

$$\langle N\rangle^2 a_N \leq (9CC_1^2C_2^2)C_1(C_0a_1)^N.$$

The claim then follows by choosing $C_1 = (9CC_2^2)^{-\frac{1}{2}}$.

Finally, we note that once C_1 is fixed as above, we are free to choose C_0 as large as we like in (4.2.4). Thus we can always assume $C_1C_0 \geq 1$, justifying our earlier assumption. □

Proof of Proposition 4.2.1. We proceed by induction on k.

Take $k = 1$. Let I be a time interval. By Strichartz, Lemma 4.1.2, and Hölder in time, we find that

$$\begin{aligned}
\|1_{t\in I} w_1\|_{\infty,2} &\leq \|1_{t\in I} e^{it\Delta} v\|_{\infty,2} + C(d,\gamma)\|S\mathcal{N}(u,u,w_1)\|_{\infty,2} \\
&\leq \|1_{t\in I} e^{it\Delta} v\|_{\infty,2} + C(d,\gamma)\|1_{t\in I} u\|_{\alpha,r}^2 \|1_{t\in I} w_1\|_{q,r}.
\end{aligned}$$

By Lemma 4.1.1 and our choice of r and α,

$$\begin{aligned}\|1_{t\in I}u\|_{\alpha,r} &\lesssim \|1_{t\in I}u\|_{\infty,2}^{1-\theta}\|1_{t\in I}J(t)u\|_{\infty,2}^{\theta}\left(\int_I |t|^{-2\gamma/(4-\gamma)}\,dt\right)^{1/\alpha}\\ &\leq \|1_{t\in I}u\|_{X([1,\infty))}\left(\int_I |t|^{-2\gamma/(4-\gamma)}\,dt\right)^{1/\alpha}.\end{aligned}$$

Since $\gamma > \frac{4}{3}$, $\frac{2\gamma}{4-\gamma} > 1$. Since $u \in X([1,\infty))$, we can decompose $[1,\infty)$ into a union of finitely many disjoint intervals I_k such that

$$\|1_{t\in I_k}w_1\|_{\infty,2} \leq \|1_{t\in I_k}e^{it\Delta}v\|_{\infty,2} + \frac{1}{12}\|1_{t\in I_k}w_1\|_{X([1,\infty))}.$$

Arguing similarly for the remaining parts of the $X(I)$ norm, we find that

$$\|1_{t\in I_k}w_1\|_{X([1,\infty))} \leq \|1_{t\in I_k}e^{it\Delta}v\|_{X([1,\infty))} + \frac{1}{2}\|1_{t\in I_k}w_1\|_{X([1,\infty))}.$$

Since $\|1_{t\in A\cup B}f\|_{X([1,\infty))} \sim \|1_{t\in A}f\|_{X([1,\infty))} + \|1_{t\in B}f\|_{X([1,\infty))}$ whenever A and B are disjoint, we conclude by Strichartz and (4.1.1) that

$$\|w_1\|_{X([1,\infty))} \lesssim_{d,\gamma} \|e^{it\Delta}v\|_{X([1,\infty))} \lesssim \|v\|_\Sigma = 1.$$

Let $C(d,\gamma)$ be the implicit constant in this estimate, and set $\Lambda = C(d,\gamma)$. This establishes the case $k = 1$.

Now define the sequence (a_N) by $a_1 = C(d,\gamma)$ from above, and

$$a_N = C'(d,\gamma)\sum_{\substack{j+k+\ell=N\\ j,k,\ell\neq N}} a_j a_k a_\ell,$$

where $C'(d,\gamma)$ is a constant to be determined. Assume the bound

$$\|w_j\|_{X([1,\infty))} \leq a_j\Lambda^j$$

for $j = 1,\ldots,N-1$, and consider w_N. Working as in the previous case, we find that

$$\begin{aligned}\|1_{t\in I}w_N\|_{X([1,\infty))} &\lesssim \sum_{\substack{j+k+\ell=N\\ j,k,\ell\neq N}} \|1_{t\in I}S\mathcal{N}(w_j,w_k,w_\ell)\|_{X([1,\infty))}\\ &\quad + C(I)^2\|1_{t\in I}w_0\|_{X([1,\infty))}^2\|1_{t\in I}w_N\|_{X([1,\infty))},\end{aligned}$$

where

$$C(I) = \left(\int_I |t|^{-2\gamma/(4-\gamma)} \, dt \right)^{1/\alpha}.$$

Once again, this implies that

$$\|w_N\|_{X([1,\infty))} \lesssim \sum_{\substack{j+k+\ell=N \\ j,k,\ell \neq N}} \|S\mathcal{N}(w_j, w_k, w_\ell)\|_{X([1,\infty))}.$$

By Strichartz, Lemma 4.1.2, Hölder, Lemma 4.1.1, and invoking the induction hypothesis, we find that

$$\|w_N\|_{X[1,\infty)} \lesssim \Lambda^N \sum_{\substack{j+k+\ell=N \\ j,k,\ell \neq N}} a_j a_k a_\ell.$$

The implicit constant in this estimate can be defined independently of N. Thus if we set $C'(d,\gamma)$ to be this constant, then we arrive at

$$\|w_N\|_{X([1,\infty))} \lesssim a_N \Lambda^N.$$

Invoking Lemma 4.2.3 to control the growth of (a_N) completes the proof. □

4.2.3 Convergence

We have shown that the series

$$\sum_{k \geq 1} \varepsilon^k w_k$$

is norm convergent in the space $X([1,\infty))$ for all sufficiently small $\varepsilon > 0$. Our next goal is to show that it converges to the correct object.

Proposition 4.2.4. *Let $\varepsilon > 0$ be such that $\sum_{k \geq 1} \varepsilon^k w_k$ converges in $X([1,\infty))$. Then*

$$\left\| u^\varepsilon - u - \sum_{k=1}^{N-1} \varepsilon^k w_k \right\|_{X([1,\infty))} \to 0 \text{ as } N \to \infty.$$

Proof. Let $W_{\geq N} = u^\varepsilon - u - \sum_{k=1}^{N-1} \varepsilon^k w_k = u^\varepsilon - u - W_{<N}$. Then for $N \geq 2$, $W_{\geq N}$ satisfies the

equation

$$\begin{aligned} W_{\geq N}(t) &= \mathcal{N}(u^\varepsilon, u^\varepsilon, u^\varepsilon) - \mathcal{N}(u,u,u) - \sum_{M=1}^{N-1} \varepsilon^M \sum_{j+k+\ell=M} S\mathcal{N}(w_j, w_k, w_\ell) \\ &= i\int_t^\infty e^{i(t-s)\Delta} G(u, W_{<N}, W_{\geq N})\ ds + \mathcal{N}(W_{<N}, W_{<N}, W_{<N}) \\ &\qquad - \sum_{M=1}^{N-1} \varepsilon^M \sum_{j+k+\ell=M} S\mathcal{N}(w_j, w_k, w_\ell) \end{aligned}$$

where $G(u, W_{<N}, W_{\geq N})$ consists of all the terms $T(a,b,c)$ with at least one argument equal to $W_{\geq N}$. Arguing as in the proof of Proposition 4.2.1, we can decompose $[1,\infty)$ into a finite collection of disjoint intervals I_k such that

$$\left\| 1_{t\in I_k} \int_t^\infty e^{i(t-s)\Delta} G(u, W_{<N}, W_{\geq N})\ ds \right\|_{X([1,\infty))} \leq \frac{1}{2} \| 1_{t\in I_k} W_{\geq N} \|_{X([1,\infty))}.$$

For the remaining terms, we write

$$\begin{aligned} \mathcal{N}(W_{<N}, W_{<N}, W_{<N}) &= \sum_{M=1}^{N-1} \varepsilon^M \sum_{j+k+\ell=M} S\mathcal{N}(w_j, w_k, w_\ell) \\ &\quad + \sum_{\substack{1\leq j,k\leq N-1 \\ j+k\geq N}} \varepsilon^{j+k} S\mathcal{N}(u, w_j, w_k) \\ &\quad + \sum_{\substack{1\leq j,k,\ell\leq N-1 \\ j+k+\ell\geq N}} \varepsilon^{j+k+\ell} S\mathcal{N}(w_j, w_k, w_\ell). \end{aligned}$$

The first term cancels exactly with the remaining terms in the previous expression for $W_{\geq N}$. For the latter two terms, by Proposition 4.2.1 we have

$$\left\| \sum_{\substack{1\leq j,k\leq N-1 \\ j+k\geq N}} \varepsilon^{j+k} S\mathcal{N}(u, w_j, w_k)\ ds \right\|_{X([1,\infty))} \lesssim (\varepsilon\Lambda)^N$$

as long as $\varepsilon\Lambda < 1$, and similarly

$$\left\| \sum_{\substack{1\leq j,k,\ell\leq N-1 \\ j+k+\ell\geq N}} \varepsilon^{j+k+\ell} S\mathcal{N}(w_j, w_k, w_\ell)\ ds \right\|_{X([1,\infty))} \lesssim (\varepsilon\Lambda)^N.$$

Thus we conclude that

$$\|W_{\geq N}\|_{X([1,\infty))} \lesssim (\varepsilon\Lambda)^N,$$

and sending $N \to \infty$ proves the desired claim. □

Thus we have shown that the map $\Sigma \to X([1,\infty))$ that sends the scattering data u_+ to the solution $u : [1,\infty) \times \mathbb{R}^d \to \mathbb{C}$ depends analytically on u_+, and we can compute the coefficients of the power series expansion around any point u_+ inductively. Consequently, the same holds for the map $\Sigma \to \Sigma : u_+ \mapsto u(t=1)$.

A similar argument applies to proving the analyticity of the initial-to-scattering state operator $\mathcal{S} : \Sigma \to \Sigma$. For initial data $u \in \Sigma$ and $v \in \Sigma$ we define $u \in X([0,\infty))$ to be the global solution to HNLS with $u(0) = u_0$, and $u^\varepsilon(t)$ to the global solution to HNLS with $u^\varepsilon(0) = u_0 + \varepsilon v$; these are well-defined by the existing scattering theory in Σ. Then arguments similar to those of Propositions 4.2.1 and 4.2.4 show that u^ε admits the expansion

$$u^\varepsilon = u + \sum_{k\geq 1} \varepsilon^k w_k$$

with $\|w_k\|_{X([0,\infty))} \lesssim \Lambda^k$ for some fixed Λ and small ε. To finish the proof of Theorem 4.0.1 it remains to show:

Proposition 4.2.5. *For all $k \geq 1$, the limits*

$$w_k^+ = \lim_{t\to\infty} e^{-it\Delta} w_k(t)$$

exist in Σ, and

$$\mathcal{S}(u_0 + \varepsilon v) = u_+ + \sum_{k\geq 1} \varepsilon^k w_k^+$$

with the latter series converging in Σ.

Proof. We claim that for each $N \geq 1$, $(e^{-it\Delta} w_N(t))_{t\geq 0}$ is Cauchy in Σ as $t \to \infty$. We have

$$e^{-it_1\Delta} w_N(t_1) - e^{-it_2\Delta} w_N(t_2) = \sum_{j+k+\ell=N} -i \int_{t_1}^{t_2} e^{-is\Delta} ST(w_j, w_k, w_\ell)\, ds$$

By 4.1.1, Lemma 4.1.2, Strichartz, and Hölder as before,

$$\|e^{-it_1\Delta}w_N(t_1) - e^{-it_2\Delta}w_N(t_2)\|_\Sigma \lesssim I(t_1,t_2) \sum_{j+k+\ell=N} \|w_j\|_{X([1,\infty))}\|w_k\|_{X([1,\infty))}\|w_\ell\|_{X([1,\infty))}$$

where

$$I(t_1,t_2) = \left(\int_{t_2}^{t_1} |t|^{-2\gamma/(4-\gamma)}\ dt\right)^{2/\alpha}.$$

Since the integral tends to 0 as $t_1, t_2 \to 0$, the claim follows. Therefore the sequence $(w_N^+) \subset \Sigma$ is well-defined, and for $N \geq 2$ we may write

$$w_N^+ = \sum_{j+k+\ell=N} -i\int_0^\infty e^{-is\Delta} ST(w_j, w_k, w_\ell)\ ds$$

(with appropriate changes for $N = 1$). Working as in the proof of Proposition 4.2.1, we can show that there exists a constant $\Lambda > 0$ such that

$$\|w_N^+\|_\Sigma \lesssim \Lambda^N.$$

Therefore the series $\sum_k \varepsilon^k w_k^+$ converges in Σ for $\varepsilon > 0$ sufficiently small, and a similar argument to Proposition 4.2.4 shows that

$$\mathcal{S}(u_0 + \varepsilon v) = u_+ + \sum_{k=1}^\infty \varepsilon^k w_k^+$$

in the sense of convergence of the series in Σ to the LHS. □

4.3 Breakdown of analyticity

We now turn to the proof of Theorem 4.0.2.

4.3.1 Breakdown at the origin

We first consider part (1) of Theorem 4.0.2. From here on, we abuse notation and redefine

$$\mathcal{N}(u,v,w) = i\int_0^\infty e^{-is\Delta} T(u,v,w)\ ds.$$

Let $\mathcal{T} \in \{\mathcal{S}, \mathcal{W}\}$, regarding it as a map $\Sigma \to L^2$, and consider its power series expansion $\mathcal{T}(v) = \sum_{k\geq 1} w_k$ at $0 \in \Sigma$ (for small $\|v\|_\Sigma$). A careful accounting of the hierarchy of equations governing the coefficients shows that all even-indexed terms vanish, so

$$\mathcal{T}(v) = v + \mathcal{N}(e^{it\Delta}v, e^{it\Delta}v, e^{it\Delta}v) + \sum_{\substack{k\geq 5 \\ k \text{ odd}}} w_k.$$

To establish part (1) of Theorem 4.0.2 it is enough to show:

Proposition 4.3.1. *For any* $s > \frac{5+5\gamma}{3+\gamma}$, *we have*

$$\|\mathcal{T}(v) - v\|_2 \neq \mathcal{O}_{L^2}(\|v\|_2^s).$$

Proof. Let $v \in \Sigma$ be sufficiently small so that the expansion $\mathcal{T}(v) = \sum_k w_k$ holds. By L^2-duality, Fubini, and unitarity of the free propagator $e^{it\Delta}$ we have

$$\begin{aligned}
\|\mathcal{T}(v) - v\|_2 &\geq \frac{1}{\|v\|_2} |\langle \mathcal{N}(e^{it\Delta}v, e^{it\Delta}v, e^{it\Delta}v), v\rangle| \|e(v)\|_2 \\
&= \frac{1}{\|v\|_2} \int_0^\infty Q(e^{is\Delta}v)\, ds - \|e(v)\|_2
\end{aligned}$$

where

$$e(v) = \sum_{\substack{k\geq 5 \\ k \text{ odd}}} w_k.$$

For fixed nonzero $v \in \Sigma$ and $\varepsilon, \sigma > 0$, we define

$$v_{\varepsilon,\sigma}(x) = \frac{\varepsilon}{\sigma^{\frac{d}{2}}} v\left(\frac{x}{\sigma}\right).$$

We will show the existence of a sequence of parameters (ε, σ) such that:

1. $\|v_{\varepsilon,\sigma}\|_\Sigma \ll 1$ (so that the series expansion holds for $\mathcal{T}(v_{\varepsilon,\sigma})$);

2. taking the limit along the sequence (ε, σ), we have

$$\lim_{(\varepsilon,\sigma)} \frac{1}{\|v_{\varepsilon,\sigma}\|_2^s} \left(\frac{1}{\|v_{\varepsilon,\sigma}\|_2} \int_0^\infty Q(e^{is\Delta}v_{\varepsilon,\sigma})\, ds - \|e(v_{\varepsilon,\sigma})\|_2 \right) \to \infty.$$

Since $(v_{\varepsilon,\sigma})$ is an L^2-bounded sequence, the claim immediately follows.

We will take $\varepsilon \ll 1$ and $\sigma \gg 1$ with $\varepsilon\sigma \ll 1$. The last condition keeps us in the regime of small $\|v_{\varepsilon,\sigma}\|_\Sigma$, so that the power series expansion continues to hold. Then the family $(v_{\varepsilon\sigma})$ obeys the following scalings:

$$\|v_{\varepsilon,\sigma}\|_2 \sim \varepsilon, \ \|v_{\varepsilon,\sigma}\|_\Sigma \sim \varepsilon\sigma, \ \|\nabla v_{\varepsilon,\sigma}\|_2 \sim \varepsilon\sigma^{-1}.$$

Moreover, by the parabolic scaling symmetry $e^{it\Delta}v(x) \leftrightarrow e^{i\sigma^{-2}t\Delta}v(\sigma^{-1}x)$ of the free Schrödinger flow,

$$\int_0^\infty Q(e^{is\Delta}v_{\varepsilon,\sigma})\ ds = \varepsilon^4\sigma^{2-\gamma}\int_0^\infty Q(e^{is\Delta}v)\ ds \sim_{\|v\|_\Sigma} \varepsilon^4\sigma^{2-\gamma}.$$

Here, the finiteness of the integral follows from Lemmas 4.1.1, 4.1.2, and the Gagliardo-Nirenberg inequality to control $\|e^{is\Delta}v\|_r$ near $t=0$.

By Proposition 4.2.1, the error term obeys the estimate

$$\|e(v_{\varepsilon,\sigma})\|_2 \lesssim \|v_{\varepsilon,\sigma}\|_2^5 \sim \varepsilon^5\sigma^5.$$

Therefore

$$\|\mathcal{T}(v_{\varepsilon,\sigma}) - v_{\varepsilon,\sigma}\|_2 \gtrsim \varepsilon^3\sigma^{2-\gamma} - \varepsilon^5\sigma^5.$$

We now take $\varepsilon = \sigma^{-j}$ for some $j > 1$ we will choose momentarily; this guarantees that $\|v_{\varepsilon,\sigma}\|_\Sigma \sim \varepsilon\sigma \ll 1$ as $\sigma \gg 1$. Then

$$\varepsilon^3\sigma^{2-\gamma} - \varepsilon^5\sigma^5 = \sigma^{-3j+2-\gamma} - \sigma^{-5j+5} \sim \sigma^{-3j+2-\gamma}$$

as $\sigma \to \infty$ so long as $-3j+2-\gamma > -5j+5$, which is equivalent to the condition $j > \frac{3+\gamma}{2}$. For such j and $\sigma \gg 1$, we have

$$\frac{1}{\|v_{\varepsilon,\sigma}\|_2^s}\|\mathcal{T}(v_{\varepsilon,\sigma}) - v_{\varepsilon,\sigma}\|_2 \gtrsim \sigma^{(s-3)j+2-\gamma}.$$

The RHS tends to ∞ as $\sigma \to \infty$ provided that $(s-3)j + (2-\gamma) > 0$. When $s \geq 3$, this is automatically satisfied since $\gamma < 2$; when $s < 3$, it is equivalent to the condition $j < \frac{2-\gamma}{3-s}$. Therefore it suffices to find a j satisfying

$$\frac{3+\gamma}{2} < j < \frac{2-\gamma}{3-s}.$$

Such a j exists whenever $\frac{3+\gamma}{2} < \frac{2-\gamma}{3-s}$, which is equivalent to $\frac{5+5\gamma}{3+\gamma} < s$. □

Remark 4.3.1. This proof can be done essentially without change for the power series expansion in $\mathcal{F}H^1$ as well.

4.3.2 Breakdown away from the origin

We now move to part (2) of Theorem 4.0.2. We adopt the following abuse of notation: w_1 refers both to the function in $X([0,\infty))$ defined by the results of Section 4.2, and also to the map

$$v \mapsto w_1(v) = v - S\mathcal{N}(u, u, w_1).$$

We will use a similar convention for w_1^+.

To keep things concrete, let us work specifically with $\mathcal{T} = \mathcal{S}$; the discussion adapts easily to $\mathcal{W}$. We recall the notation

$$\mathcal{S}(u_0 + v) = \mathcal{S}(u_0) + \sum_{k=1}^{\infty} w_k^+.$$

Our goal here is the following:

Proposition 4.3.2. *There exists $R = R(d,\gamma) > 0$ such that for all $u_0 \in \Sigma$ satisfying $\|u_0\|_\Sigma < R$, all $\|v\|_\Sigma$ small, and all $s > \frac{4+4\gamma}{2+\gamma}$, we have*

$$\|\mathcal{S}(u_0 + v) - \mathcal{S}(u_0) - w_1^+ - w_2^+\|_2 \neq \mathcal{O}(\|v\|_2^s).$$

This will emerge as a consequence of the following estimate on the third derivative term:

Proposition 4.3.3. *There exists $R = R(d,\gamma) > 0$ such that for all $\|u_0\|_\Sigma < R$ and for $\varepsilon \ll 1$, $\varepsilon\sigma \ll 1$, $\sigma \gg 1$, we have*

$$\|w_3^+(v_{\varepsilon,\sigma})\|_2 \gtrsim_{d,\gamma} \varepsilon^3 \sigma^{2-\gamma}.$$

The crux of the proof is to identify the source of the breakdown of regularity in w_3^+, which has the form

$$w_3^+ = S\mathcal{N}(u, u, w_3) + S\mathcal{N}(u, w_1, w_2) + \mathcal{N}(w_1, w_1, w_1).$$

The bad behavior we seek arises from the term $\mathcal{N}(w_1, w_1, w_1)$, which we expect to be dominant as it is essentially a resonant interaction. We will first establish that this resonant term has the optimal scaling $\varepsilon^3\sigma^{2-\gamma}$ as was the case at $u_0 = 0$, and then show that the remaining cubic terms are subdominant.

4.3.2.1 The main term

We continue to work in the regime $\varepsilon \ll 1, \sigma \gg 1, \varepsilon\sigma \ll 1$. Our goal is to establish the following:

Proposition 4.3.4. *There exists $R = R(d, \gamma) > 0$ such that if $\|u_0\|_\Sigma < R$, then*

$$\|\mathcal{N}(w_1(v_{\varepsilon,\sigma}), w_1(v_{\varepsilon,\sigma}), w_1(v_{\varepsilon,\sigma}))\|_2 \gtrsim \varepsilon^3\sigma^{2-\gamma}.$$

for all σ sufficiently large.

When $u_0 \neq 0$, the fact that w_1 is no longer purely a free evolution complicates the proof of this relationship; we cannot use the scaling argument for the potential energy

$$\int_0^\infty Q(e^{is\Delta}v)\ ds$$

right away. We get around this issue by using the fact that w_1 *does* behave like a free evolution at large times, and rescaling in time so that the rescaled version of w_1 behaves like a free evolution at almost all times.

Lemma 4.3.5. *There exists $R = R(d, \gamma) > 0$ such that if $\|u_0\|_\Sigma < R$, then*

$$\|w_1\|_{Y([0,\infty))} \sim \|w_1^+\|_2 \sim \|v\|_2,$$

$$\|w_1\|_{X([0,\infty))} \sim \|w_1^+\|_\Sigma \sim \|v\|_\Sigma,$$

$$\|\nabla w_1\|_{\infty,2} \lesssim \|v\|_{H^1}.$$

Proof. Writing $w_1^+ = v + S\mathcal{N}(u,u,w_1)$ and arguing as usual,

$$\|w_1^+\|_2 \leq \|w_1\|_{Y([0,\infty))} \lesssim \|v\|_2 + \|u\|_{X([0,\infty))}^2 \|w_1\|_{Y([0,\infty))}.$$

Taking $\|u_0\|_\Sigma$ small, we can make $\|u\|_{X([0,\infty))}$ arbitrarily small. Doing so, we find that

$$\|w_1^+\|_2 \leq \|w_1\|_{Y([0,\infty))} \lesssim \|v\|_2.$$

Similarly, we have

$$\|v\|_2 \leq \|w_1^+\|_2 + \|u\|_{X([0,\infty))}^2 \|w_1\|_{Y([0,\infty))} \lesssim \|w_1^+\|_2 + \|u\|_{X([0,\infty))}^2 \|v\|_2,$$

and thus taking $\|u_0\|_\Sigma$ small we obtain

$$\|v\|_2 \lesssim \|w_1^+\|_2.$$

The other estimates follow similarly. $\square$

In particular, we have the scaling

$$\|w_1(v_{\varepsilon,\sigma})\|_{Y([0,\infty))} \sim \|w_1^+(v_{\varepsilon,\sigma})\|_2 \sim \|(w_1^+)_{\varepsilon,\sigma}\|_2 \sim \varepsilon,$$

$$\|w_1(v_{\varepsilon,\sigma})\|_{X([0,\infty))} \sim \|w_1^+(v_{\varepsilon,\sigma})\|_\Sigma \sim \|x(w_1^+)_{\varepsilon,\sigma}\|_2 \sim \varepsilon\sigma;$$

this will be useful because most quantities we estimate henceforth depend more directly on w_1 and w_1^+ than on v.

Proof of Proposition 4.3.4. Fix $v \neq 0$. For any $\tau > 0$, L^2-duality, Fubini, and unitarity of the free propagator we have

$$\|\mathcal{N}(w_1,w_1,w_1)\|_2 \geq \frac{1}{\|w_1^+\|_2} \left| \int_\tau^\infty \langle T(w_1,w_1,w_1), e^{is\Delta} w_1^+ \rangle_{L_x^2} \, ds \right| - \left\| \int_0^\tau e^{-is\Delta} T(w_1,w_1,w_1) \, ds \right\|_2 .$$

We write

$$\int_\tau^\infty \langle T(w_1,w_1,w_1), e^{is\Delta}w_1^+\rangle_{L_x^2}\, ds = \int_\tau^\infty Q(e^{is\Delta}w_1^+)\, ds - \int_\tau^\infty \langle ST(w_1,w_1,e^{is\Delta}w_1^+ - w_1), e^{is\Delta}w_1^+\rangle_{L_x^2}\, ds.$$

Since $\|e^{is\Delta}w_1^+ - w_1\|_{Y([\tau,\infty))} \to 0$ as $\tau\to\infty$, it follows that for sufficiently large $\tau = \tau(v)$ we have

$$\int_\tau^\infty \langle T(w_1,w_1,w_1), e^{is\Delta}w_1^+\rangle_{L_x^2}\, ds \sim \int_\tau^\infty Q(e^{is\Delta}w_1^+)\, ds$$

We now rescale $w_1^+ \mapsto (w_1^+)_{\varepsilon,\sigma}$, which (at least on L^2 and Σ) is essentially equivalent to rescaling $v\mapsto v_{\varepsilon,\sigma}$. Under this rescaling, the parabolic scaling symmetry of the free Schrödinger flow yields

$$\int_\tau^\infty Q(e^{is\Delta}(w_1^+)_{\varepsilon,\sigma})\, ds = \varepsilon^4\sigma^{2-\gamma}\int_{\tau/\sigma^2}^\infty Q(e^{is\Delta}w_1^+)\, ds.$$

Thinking of σ being arbitrarily large, we estimate this by the integral over all of $[0,\infty)$. Using Lemma 4.1.2 and the Gagliardo-Nirenberg inequality, this incurs an error of size

$$\int_0^{\tau/\sigma^2} Q(e^{is\Delta}w_1^+)\, ds \lesssim \frac{\tau}{\sigma^2}(\|w_1^+\|_2^{1-\theta}\|\nabla w_1^+\|_2^\theta)^4 \lesssim \frac{\tau}{\sigma^2}\|v\|_\Sigma^4 \ll 1.$$

This establishes the $\varepsilon^3\sigma^{2-\gamma}$ scaling on the main term. By Strichartz, Gagliardo-Nirenberg, and Lemma 4.3.5 we find that the remainder satisfies

$$\begin{aligned}\left\|\int_0^\tau e^{-is\Delta}T(w_1,w_1,w_1)\, ds\right\|_2 &\lesssim \|w_1\|^2_{L_t^\alpha L_x^r([0,T])}\|w_1\|_{Y([0,T])}\\ &\lesssim \left(\int_0^\tau (\|w_1\|_{\infty,2}^{1-\theta}\|\nabla w_1\|_{\infty,2}^\theta)^\alpha\right)^{2/\alpha}\|v\|_2\\ &\lesssim \tau^{2/\alpha}\|v\|_{H^1}^3.\end{aligned}$$

Since we are working in the regime $\sigma\gg 1$, $\|v_{\varepsilon,\sigma}\|_{H^1}\sim\varepsilon$. Therefore rescaling yields

$$\left\|\int_0^\tau e^{-is\Delta}T(w_1(v_{\varepsilon,\sigma}),w_1(v_{\varepsilon,\sigma}),w_1(v_{\varepsilon,\sigma}))\, ds\right\|_2 \lesssim \tau^{2/\alpha}\varepsilon^3.$$

Since τ depends only on v and $2-\gamma>0$, this term is subdominant to $\varepsilon^3\sigma^{2-\gamma}$ for $\sigma\gg 1$. $\square$

4.3.2.2 Nonresonant cubic terms

We are left to control the nonresonant cubic error terms $\|S\mathcal{N}(u,w_1,w_2)\|_2$ and $\|S\mathcal{N}(u,u,w_3)\|_2$.

Proposition 4.3.6. *There exists $R = R(d,\gamma) > 0$ such that for all $\|u_0\|_\Sigma < R$ and for $\varepsilon \ll 1$, $\varepsilon\sigma \ll 1$, $\sigma \gg 1$ we have*

$$\|S\mathcal{N}(u,w_1,w_2)\|_2 + \|S\mathcal{N}(u,u,w_3)\|_2 \lesssim R^2\varepsilon^3\sigma^{2-\gamma}.$$

Taking R small will ensure that the main term continues to dominate the errors. Together with Proposition 4.3.4 this immediately implies Proposition 4.3.3.

The idea of this proof is that the estimate $\|w_1\|_{\alpha,r} \lesssim \|v\|_\Sigma$, which holds due to the fact that $|t|^{-\frac{\alpha\gamma}{4}}$ is integrable near ∞, is slack. Naïvely using it to control $\|S\mathcal{N}(u,w_1,w_2)\|_2$ and $\|S\mathcal{N}(u,u,w_3)\|_2$ only yields an estimate of $\varepsilon^3\sigma^3$, which is *not* subdominant to $\varepsilon^3\sigma^{2-\gamma}$. For small data, the estimate can be improved to one of the form $\|w_1\|_{\alpha,r} \lesssim \|v\|_\Sigma^a\|v\|_2^b\|\nabla v\|_2^c$ by relying less on Lemma 4.1.1. This is done analogously to how we sharpened the error estimate in Proposition 3.2.1 to the one given in Proposition 3.3.1. The Lorentz space refinement is used to recover an endpoint, which is necessary as the scaling $\varepsilon^3\sigma^{2-\gamma}$ is sharp.

Lemma 4.3.7. *There exists $R = R(d,\gamma) > 0$ such that if $\|u_0\|_\Sigma < R$, then*

$$\|w_1\|_{L_t^{\alpha,\infty}L_x^r([0,\infty))} \lesssim \|v\|_2^{1-\gamma/4}\|v\|_\Sigma^{(4-\gamma)/8}\|\nabla w_1\|_{\infty,2}^{(3\gamma-4)/8}.$$

Proof. For any $A \in [0,1]$, by Lemma 4.1.1 and the Gagliardo-Nirenberg inequality we have

$$\|w_1(t)\|_r \lesssim |t|^{-A\gamma/4}\|w_1(t)\|_2^{1-\gamma/4}\|J(t)w_1(t)\|_2^{A\gamma/4}\|\nabla w_1(t)\|_2^{(1-A)\gamma/4}.$$

Choose $A = \frac{4}{\alpha\gamma} = \frac{4-\gamma}{2\gamma}$. Note that for $\frac{4}{3} < \gamma < 2$, $\frac{1}{2} < \frac{4-\gamma}{2\gamma} < 1$. Since $|t|^{-A\gamma/4} \in L_t^{\alpha,\infty}([0,\infty))$ for this choice of A, we find that

$$\|w_1\|_{L_t^{\alpha,\infty}L_x^r([0,\infty))} \lesssim \|w_1(t)\|_{\infty,2}^{1-\gamma/4}\|J(t)w_1(t)\|_{\infty,2}^{(4-\gamma)/8}\|\nabla w_1(t)\|_{\infty,2}^{(3\gamma-4)/8}.$$

The claim now follows from the Lorentz space version of Lemma 4.3.5; we leave the details to the reader. $\square$

Lemma 4.3.8. *There exist $R = R(d,\gamma) > 0$ and $L = L(d,\gamma) > 0$ such that for all $\|u_0\|_\Sigma < R$ and $v \in \Sigma$ with $\|v\|_\Sigma \leq L$, we have*

$$\|\nabla w_1^+\|_2 + \|\nabla w_1\|_{Y([0,\infty))} \lesssim \|\nabla v\|_2.$$

Proof. By the usual estimates and Lemma 4.3.7, for all $\tau > 0$ we have

$$\begin{aligned}\|\nabla w_1\|_{Y([0,\tau))} &\lesssim_{d,\gamma} \|\nabla v\|_2 + \|u\|_{X([0,\tau))}^2 (\|w_1\|_{L_t^{\alpha,\infty} L_x^r([0,\tau))} + \|\nabla w_1\|_{Y([0,\tau))}) \\ &\lesssim_{d,\gamma} \|\nabla v\|_2 + R^2 (\|v\|_2^{1-\gamma/4} \|v\|_\Sigma^{(4-\gamma)/8} \|\nabla w_1\|_{Y([0,\tau))}^{(3\gamma-4)/8} + \|\nabla w_1\|_{Y([0,\tau))}) \\ &\leq \|\nabla v\|_2 + \delta_1 \|\nabla w_1\|_{Y([0,\tau))}^{(3\gamma-4)/8} + \delta_2 \|\nabla w_1\|_{Y([0,\tau))},\end{aligned}$$

where $\delta_1 = \delta_1(R,L)$ and $\delta_2 = \delta_2(R)$ can be made arbitrarily small by an appropriate choice of R and L. The estimate for ∇w_1 then follows via a bootstrap argument. The estimate for ∇w_1^+ is then immediate from the definition of w_1^+. □

Proof of Proposition 4.3.6. By the Lorentz space adaptations of our earlier estimates, we have

$$\begin{aligned}\|S\mathcal{N}(u, w_1, w_2)\|_2 &\lesssim \|u\|_{X^*(I)} \|w_1\|_{L_t^{\alpha,\infty} L_x^r} \|w_2\|_{L_t^{q,2} L_x^r} \\ &\lesssim \|u\|_{X^*(I)}^2 \|w_1\|_{L_t^{\alpha,\infty} L_x^r}^2 \|w_1\|_{L_t^{q,2} L_x^r}\end{aligned}$$

and

$$\begin{aligned}\|S\mathcal{N}(u,u,w_3)\|_2 &\lesssim \|u\|_{X^*(I)}^2 \|w_3\|_{L_t^{q,2} L_x^r} \\ &\lesssim \|u\|_{X^*(I)}^2 (\|u\|_{X^*(I)} \|w_1\|_{L_t^{\alpha,\infty} L_x^r} \|w_2\|_{L_t^{q,2} L_x^r} + \|w_1\|_{L_t^{\alpha,\infty} L_x^r}^2 \|w_1\|_{L_t^{q,2} L_x^r}) \\ &\lesssim \|u\|_{X^*(I)}^2 (\|u\|_{X^*(I)}^2 \|w_1\|_{L_t^{\alpha,\infty} L_x^r}^2 \|w_1\|_{L_t^{q,2} L_x^r} + \|w_1\|_{L_t^{\alpha,\infty} L_x^r}^2 \|w_1\|_{L_t^{q,2} L_x^r}) \\ &= \|u\|_{X^*(I)}^2 (1 + \|u\|_{X^*(I)}^2) \|w_1\|_{L_t^{\alpha,\infty} L_x^r}^2 \|w_1\|_{L_t^{q,2} L_x^r}.\end{aligned}$$

The claim now follows by taking $R = R(d,\gamma)$ small, $\|u_0\|_\Sigma < R$, replacing w_1 with $w_1(v_{\varepsilon,\sigma})$, and applying Lemmas 4.3.5, 4.3.7, and 4.3.8. □

4.3.2.3 Conclusion of the proof

At last, we are ready to proceed with the proof of Proposition 4.3.2, thereby completing the proof of Theorem 4.0.2.

Proof. We define $v_{\varepsilon,\sigma}$ as before and work in the regime $\varepsilon \ll 1, \sigma \gg 1, \varepsilon\sigma \ll 1$. We write

$$\mathcal{S}(u_0 + v) - \mathcal{S}(u_0) - w_1^+ - w_2^+ = w_3^+(v) + e(v),$$

where

$$e(v) = \sum_{k \geq 4} w_k^+.$$

By Propositions 4.2.1 and 4.3.3, for sufficiently small $\|u_0\|_\Sigma$ we have the lower bound

$$\begin{aligned}\|\mathcal{S}(u_0 + v) - \mathcal{S}(u_0) - w_1^+ - w_2^+\|_2 &\geq \|w_3^+(v_{\varepsilon,\sigma})\|_2 - \|e(v_{\varepsilon,\sigma}\|_2 \\ &\gtrsim \varepsilon^3\sigma^{2-\gamma} - \varepsilon^4\sigma^4.\end{aligned}$$

We take $\varepsilon = \sigma^{-j}$ with $j > 1$ to be determined: then $\varepsilon\sigma \ll 1$ for $\sigma \gg 1$. For the main term to dominate the quartic error we require $-3j+2-\gamma > -4j+4$, which is equivalent to $j > 2+\gamma$ and supersedes the condition $j > 1$. Assuming this condition, we have

$$\frac{1}{\|v_{\varepsilon,\sigma}\|^2}\|\mathcal{S}(u_0 + v_{\varepsilon,\sigma}) - \mathcal{S}(u_0) - w_1^+ - w_2^+\|_2 \gtrsim \sigma^{(s-3)j+2-\gamma}.$$

The RHS is unbounded as $\sigma \to \infty$ as long as $(s-3)j + 2 - \gamma > 0$. This condition is automatically met if $s \geq 3$ since $\gamma < 2$, while if $s < 3$ then it is equivalent to $j < \frac{2-\gamma}{3-s}$. Thus an appropriate value of j can be found provided that $2+\gamma < \frac{2-\gamma}{3-s}$, which is equivalent to the condition $s > \frac{4+4\gamma}{2+\gamma}$. □

APPENDIX A

Pointwise Hölder spaces and Gateaux derivatives

In this appendix we relate the notion of pointwise Hölder regularity given in Definition 1.4.2 to more familiar notions. For convenience we reproduce the definition here:

Definition A.0.1 (Pointwise Hölder space [1])**.** Let X and Y be Banach spaces. Let $x_0 \in X$ and U a convex open neighborhood of x_0. Fix $s > 0$, and let n be the integer part of s. For $s > 0$, we say that the map $G : X \to Y$ belongs to the *pointwise Hölder space* $C^s(x_0)$ if for all $h \in X$ with $\|h\|_X = 1$, there exist coefficients $\{a_j(x_0; h)\}_{j=0}^n \subset Y$ such that

$$\|G(x_0 + \varepsilon h) - G(x_0) - \sum_{j=1}^{n} \varepsilon^j a_j(x_0; h)\|_Y \lesssim \varepsilon^s$$

for all $\varepsilon > 0$ sufficiently small, with the implicit constant independent of the direction h.

This is related to two notions: the Peano derivative (also known as the de la Vallée-Poussin derivative), and the Gateaux derivative.

Definition A.0.2 (Peano, de la Vallée-Poussin derivative)**.** Let X and Y be Banach spaces. Let $x_0 \in X$, let U be a convex open neighborhood of x_0, and let $h \in X$ with $\|h\|_X = 1$. For $n \geq 1$, we say that a map $G : U \to Y$ has an n-th *Peano derivative*, or *de la Vallée-Poussin derivative*, at x_0 in the direction h if there exist $\{a_j(x_0; h)\}_{j=1}^n \subset Y$ such that

$$\|G(x_0 + \varepsilon h) - G(x_0) - \sum_{j=1}^{n} \frac{1}{j!} \varepsilon^j a_j(x_0; h)\|_Y = o(\varepsilon^n; h)$$

as $\varepsilon \to 0$.

Therefore if $G \in C^s(x_0)$ with $s \geq n$, then G automatically has an n-th Peano derivative, with an asymptotic bound as $\varepsilon \to 0$ which is uniform in h; moreover, if $s > n$, then the asymptotic bound is stronger.

Definition A.0.3 (Gateaux derivative [23])**.** Let X and Y be Banach spaces. Let $x_0 \in X$ and $U \subset X$ a convex neighborhood of x_0. We say that the map $G : U \to Y$ is *Gateaux differentiable* at x_0 in the direction $h \in X$ if the limit

$$dG(x_0; h) = \lim_{\varepsilon \to 0^+} \frac{G(x_0 + \varepsilon h) - G(x_0)}{\varepsilon} = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} G(x_0 + \varepsilon h)$$

exists in Y. In that case, we call $dG(x_0; h)$ the *Gateaux derivative*, or *first variation*, of G at u in the direction v. If $dG(x_0; h)$ exists for all $h \in X$, we say that G is Gateaux differentiable at x_0. Similarly, we define the Gateaux derivative of order n, or n-th variation, by

$$d^n G(x_0; h) = \left.\frac{d^n}{d\varepsilon^n}\right|_{\varepsilon=0} G(x_0 + \varepsilon h).$$

Gateaux derivatives are homogeneous in their second argument: $d^j G(x_0; \varepsilon h) = \varepsilon^j d^j G(x_0; h)$ for all $\varepsilon \in \mathbb{R}$ ([23], Lemma 1.2).

It is clear that if $n \geq 1$ and $G : U \to Y$ has an n-th Peano derivative $a_n(x_0; h)$ at x_0 in the direction h, then it also has j-th Peano derivatives $a_j(x_0; h)$ at x_0 in the direction h for $j = 1, \ldots, n-1$; moreover, G is Gateaux differentiable at x_0 in the direction h with first variation $dG(x_0; h) = a_1(x_0; h)$. It is not, however, true that G has variations of any higher order, even in the real-valued case: a counterexample is $f(x) = x^3 \sin(1/x)$ for $x \neq 0$, $f(0) = 0$, for which the second Peano derivative exists at 0, but not $f''(0)$ [50]. For this reason, $C^s(x_0)$ is not exactly a replacement for the space of n-times Gateaux differentiable maps with $d^n G(x_0; h)$ Hölder continuous of order $s - n$ in x_0. When $s > 2$, we are not even able to detect from the definition whether a map in $C^s(x_0)$ has a second variation at x_0. However, $C^s(x_0)$ is still a useful notion for detecting when a map *fails* to have a certain level of Gateaux regularity, which is what is relevant for the breakdown of regularity statements in Corollary 1.3.2. This arises through the generalization of Taylor's theorem with remainder for Banach space valued functions.

Theorem A.0.1 (Taylor's theorem with remainder; [23], Theorem 5)**.** *Let X and Y be Banach spaces. Let $U \subset X$ be a convex neighborhood of $u \in X$. Let $G : U \to Y$ be n-times Gateaux differentiable on U, and let $x_0 \in X$ be such that $d^n G(x_0 + s\varepsilon h; h)$ is Riemann integrable (defined in* [23]*) over $s \in (0,1)$ whenever $\varepsilon > 0$ is sufficiently small. Then for all $h \in X$ with $\|h\|_X = 1$ and $\varepsilon > 0$ small,*

$$G(x_0 + \varepsilon h) = G(x_0) + \sum_{j=1}^{n} \frac{\varepsilon^j}{j!} d^j G(x_0; h) + \varepsilon^{n+1} R_{n+1}(x_0, h, \varepsilon)$$

where

$$R_{n+1}(x_0, h, \varepsilon) = \frac{1}{n!} \int_0^1 (1-s)^n d^{n+1} G(x_0 + s\varepsilon h; h)\ ds.$$

We now arrive at the main statement of interest. It states that for $n < s < n+1$, membership in $C^s(x_0)$ is necessary for a map G to be n times Gateaux differentiable with $d^n G(x; h)$ Hölder continuous of order $s - n$. This gives us a way of detecting whether G admits s derivatives in this latter sense.

Lemma A.0.2. *Let X and Y be Banach spaces. Let $U \subset X$ be a convex neighborhood of $x_0 \in X$. Let $G : U \to Y$ be a map, and suppose $G \notin C^s(x_0)$ with $n < s < n+1$. Then $d^n G(x; h)$, if it exists for $x \in U$, cannot be a Hölder continuous function of x of order $s - n$ with Hölder seminorm uniformly bounded in h.*

Proof. Suppose for contradiction that $d^n G(x; h)$ exists on U and is Hölder continuous of order $s - n$ in x, with Hölder seminorm uniformly bounded in h. Then all lower order Gateaux derivatives must also exist. This implies that G satisfies the conditions of Theorem A.0.1, and hence admits the expansion

$$G(x_0 + \varepsilon h) = G(x_0) + \sum_{j=1}^{n-1} \frac{\varepsilon^j}{j!} d^j G(x_0; h) + \varepsilon^n R_n(x_0, h, \varepsilon)$$

as $\varepsilon \to 0$, where R_n is given as in Theorem A.0.1. By the Hölder continuity assumption, we have

$$
\begin{aligned}
\|R_n(x_0,h,\varepsilon) - \frac{1}{n!}d^nG(x_0;h)\|_Y \\
&= \left\| \frac{1}{(n-1)!}\int_0^1 (1-r)^{n-1}[d^nG(x_0+r\varepsilon h;h) - d^nG(x_0;h)]\ dr \right\|_Y \\
&\leq \frac{1}{(n-1)!}\int_0^1 (1-r)^{n-1}\|d^nG(x_0+r\varepsilon h;h) - d^nG(x_0;h)\|_Y\ dr \\
&\lesssim \varepsilon^{s-n}\int_0^1 (1-r)^{n-1}r^s\ dr \leq \varepsilon^{s-n}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
G(x_0+\varepsilon h) &= G(x_0) + \sum_{j=1}^{n-1}\frac{\varepsilon^j}{j!}d^jG(x_0;h) + \varepsilon^n R_n(x_0,h,\varepsilon) \\
&= G(x_0) + \sum_{j=1}^{n}\frac{\varepsilon^j}{j!}d^jG(x_0;h) + \varepsilon^n[R_n(x_0,h,\varepsilon) - \frac{1}{n!}d^nG(x_0;h)] \\
&= G(x_0) + \sum_{j=1}^{n}\frac{\varepsilon^j}{j!}d^jG(x_0;h) + \mathcal{O}_Y(\varepsilon^s).
\end{aligned}
$$

But then $G \in C^s(x_0)$, contradiction. □

Lastly, we need a way of checking that a given G does not belong to the class $C^s(x_0)$.

Lemma A.0.3. *Let n be a positive integer, and let $n < s < s+\delta < n+1$. Assume $G \in C^s(x_0)$ with Peano derivatives $\{a_j(x_0;h)\}_{j=1}^n$, so that*

$$
\|G(x_0+\varepsilon h) - G(x_0) - \sum_{j=1}^{n}\varepsilon^j a_j(x_0;h)\|_Y \lesssim \varepsilon^s.
$$

Suppose also that

$$
\|G(x_0+\varepsilon h) - G(x_0) - \sum_{j=1}^{n}\varepsilon^j a_j(x_0;h)\|_Y \not\lesssim \varepsilon^{s+\delta}.
$$

Then $G \notin C^{s+\delta}(x_0)$.

The proof is based on the following uniqueness statement for the Peano derivatives:

Theorem A.0.4 ([23], Theorem 6)**.** *Let X and Y be Banach spaces. Let $U \subset X$ be a convex neighborhood of $x_0 \in X$. Let $G : U \to Y$ be a map. Then for each positive integer n, there exists at most one expansion of the form*

$$G(x_0 + h) = G(x_0) + \sum_{j=1}^{n} a_j(x_0; h) + R_{n+1}(x_0, h)$$

satisfying $a_j(x_0; sh) = s^j a_j(x_0; h)$ and $R_{n+1}(x_0, h) = o(\|h\|_Y^n)$ as $h \to 0$.

Proof of Lemma A.0.3. Suppose to the contrary that $G \in C^{s+\delta}(x_0)$. Then there are coefficients $\{b_j(x_0; h)\}_{j=1}^n$ such that

$$\|G(x_0 + \varepsilon h) - G(x_0) - \sum_{j=1}^{n} \varepsilon^j b_j(x_0; h)\|_Y \lesssim \varepsilon^{s+\delta}.$$

Then we have two polynomial expansions for $G(x_0 + h)$ around x_0 of degree n with $o(\|h\|_Y^n)$ remainder as $h \to 0$. By Theorem A.0.4, it follows that $b_j = a_j$. But this contradicts the assumption that the the error in the expansion $G(x_0 + \varepsilon h) \sim G(x_0) + \sum_{j=1}^{n} \varepsilon^j a_j(x_0; h)$ is not $\mathcal{O}(\varepsilon^{s+\delta})$. □

The utility of Lemma A.0.3 is that so long as we can verify one asymptotically valid polynomial approximation of $G(x_0 + \varepsilon h)$, the same polynomial approximation can be used to check the membership of G in $C^s(x_0)$, as long as there is no need to add a higher-order derivative term to the expansion.

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