Until the Bitter End: On Prospect Theory in a Dynamic Context

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We provide a result on prospect theory decision makers who are naïve about the time-inconsistency induced by probability weighting. If a market offers a sufficiently rich set of investment strategies, investors postpone their trading decisions indefinitely due to a strong preference for skewness. We conclude that probability weighting in combination with naïvité leads to unrealistic predictions for a wide range of dynamic setups.
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Cumulative prospect theory (CPT, Tversky and Kahneman 1992) is arguably the most prominent alternative to expected utility theory (EUT, Bernoulli 1738/1954, von Neumann and Morgenstern 1944). EUT is well-studied in static and dynamic settings, ranging from game theory over investment problems to institutional economics. In contrast, CPT with probability weighting—the assumption that individuals overweight unlikely and extreme events—has mostly focused on the static case. This paper studies the dynamic investment and gambling behavior of CPT agents who are naïve, i.e., unaware of being time-inconsistent.

Our main result shows that naïve CPT agents never stop gambling when the set of gambling or investment opportunities is not too restrictive. This *never-stopping result* applies to highly unfavorable gambles and investments with arbitrarily large expected losses per time. It follows from a static result on skewness preference under CPT that we label *skewness preference in the small:* a CPT agent always wants to take a simple, small, lottery-like risk, even if it has negative expectation. At any point in time the naïve CPT investor reasons "If I lose just a little bit more, I will stop. And if I gain, I will continue." This simple strategy results in a right-skewed gambling experience that is attractive due to skewness preference in the small. Once a loss has occurred, however, a new skewed gambling strategy comes to the naïve CPT investor's mind, and—as long as such a strategy is feasible—he continues gambling.

No "malicious" third party is responsible for manipulating the CPT investor into this behavior. Never stopping arises naturally in numerous prominent economic and financial decision situations, thereby yielding predictions that are arguably too extreme. In a casino gambling model in the spirit of Barberis (2012), the naïve CPT gambler may gamble until the bitter end, i.e., until bankruptcy.

We also study the irreversible investment problem of Dixit and Pindyck (1994), and prove that naïve CPT agents never exercise American options even when it is profitable to exercise them immediately. Finally, our results imply that CPT cannot predict the disposition effect (Shefrin and Statman 1985) for naïve investors.

The results in this paper hold for a wide range of CPT specifications and are independent of the investor's reference point, which determines which outcomes are viewed as gains and losses. Our crucial assumption on CPT is that probability weighting is strong enough relative to loss aversion (the trait that losses feel worse than comparable gains feel good). This assumption, which ensures that individuals like skewness enough to bare the risk of potential losses, is fulfilled by all commonly employed CPT parametrizations that also received extensive empirical support.

We define CPT preferences precisely in Section I. In Section II, we present our static result that CPT implies skewness preference in the small. Section III presents the never-stopping result. Section IV discusses the implications for CPT models of casino gambling, real-option investment behavior, and the disposition effect. In Section V, we discuss options to evade the never-stopping result through relaxing our three main assumptions: probability weighting stronger than loss aversion, naïvité, and the availability of small and skewed gambles. Section VI summarizes our results. All proofs are in the appendix. Several additional results and illustrations are collected in a web appendix (Appendix W).

I. Prospect Theory with More Probability Weighting than Loss Aversion

We study CPT preferences over real-valued random variables X. In CPT, outcomes are evaluated by a *value function* (also called utility function) relative to a *reference point* that separates all outcomes into gains and losses. A *weighting function* distorts cumulative probabilities, as suggested by Quiggin (1982), rather than the probabilities of individual outcomes as in the original prospect theory of Kahneman and Tversky (1979). The idea of a reference point first appears in Markowitz's (1952) seminal paper. Common choices for the reference point are the status quo of current wealth or expected wealth.

For simplicity, first consider a binary risk $L(p, b, a)$ that yields outcome b with probability $p \in (0, 1)$, and $a < b$ otherwise. A prospect theory agent evaluates binary risks relative to the reference point $r \in \mathbb{R}$ as

$$(1) \qquad CPT(L(p, b, a)) = \begin{cases} (1 - w^+(p))U(a) + w^+(p)U(b), & \text{if } r \leq a \\ w^-(1 - p)U(a) + w^+(p)U(b), & \text{if } a < r \leq b \\ w^-(1 - p)U(a) + (1 - w^-(1 - p))U(b) & \text{if } b < r \end{cases}$$

with non-decreasing weighting functions $w^-, w^+ : [0, 1] \to [0, 1]$ with $w^+(0) = w^-(0) = 0$ and $w^+(1) = w^-(1) = 1$, and a continuous, strictly increasing value function $U : \mathbb{R} \to \mathbb{R}$ with $U(r) = 0$ that satisfy Assumptions 1 and 2 stated below. The CPT utility of general random variables X with possibly continuous outcomes can be defined as[1]

$$(2) \qquad CPT(X) = \int_{\mathbb{R}_+} w^+(\mathbb{P}(U(X) > y))\mathrm{d}y - \int_{\mathbb{R}_-} w^-(\mathbb{P}(U(X) < y))\mathrm{d}y,$$

but to understand the results of this paper it is sufficient to have formula (1) in mind. The following assumption on the value function means that any kinks it may have are not too extreme, which excludes infinite loss aversion.

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[1]This expression indeed nests formula (1) for binary lotteries as well as the well-known definition of Tversky and Kahneman (1992) for general discrete prospects (cf. Kothiyal, Spinu, and Wakker 2011).

ASSUMPTION 1 (Value function): *The value function U has finite left and right derivatives, $\partial_- U(x)$ and $\partial_+ U(x)$, at every wealth level x. Further, $\lambda = \sup_{x \in \mathbb{R}} \frac{\partial_- U(x)}{\partial_+ U(x)} < \infty$ exists.*

In the well-studied case where U is S-shaped (i.e. convex over losses, concave over gains) and smooth everywhere except at the reference point, $\lambda = \frac{\partial_- U(r)}{\partial_+ U(r)}$ measures loss aversion; see Köbberling and Wakker (2005).[2] Note that Assumption 1 does not impose any restriction on r and thus the choice of the reference point is immaterial to our results. Moreover, many functional forms other than S-shape satisfy Assumption 1.

The final important feature of CPT is probability weighting, which is the driving force of this paper's results.[3] The assumptions we impose on the probability weighting functions are satisfied by the commonly used (inverse-S-shaped, i.e., first concave and then convex) probability weighting functions of Tversky and Kahneman (1992), Goldstein and Einhorn (1987), Prelec (1998), and the neo-additive weighting function (Wakker 2010, p. 208), for *all* parameter values.

ASSUMPTION 2 (Weighting Functions): *There exists at least one $p \in (0,1)$ such that*

1) $w^+(p) > \frac{\lambda p}{1-p+\lambda p}$ *and*

2) $w^-(1-p) < \frac{1-p}{1-p+\lambda p}$.

In Appendix W.1, we show that a sufficient condition for Assumption 2 is[4]

$$(3) \qquad\qquad \min\{w^{+'}(0), w^{-'}(1)\} > \lambda.$$

This condition says that extremely unlikely gains are overweighted and extremely likely losses are underweighted, both by more than the loss aversion parameter. Therefore, we refer to Assumption 2 as *probability weighting stronger than loss aversion.* Appendix W.1 provides a deeper examination of Assumption 2 and examples for when it may be violated.

II.   Skewness Preference in the Small

This paper starts out with a static result on prospect theory preferences over small, skewed risks. We say that a risk is *attractive* if the CPT utility of current wealth plus the risk is *strictly* higher than the CPT utility of current wealth.

THEOREM 1 (Prospect Theory's Skewness Preference in the Small): *Under Assumptions 1 and 2, for every wealth level there exists an attractive zero-mean binary lottery that is arbitrarily small.*

The proof in Appendix A explicitly constructs such a lottery. This binary lottery assigns the probability p that exists according to Assumption 2 to the larger outcome. Assumption 2 guarantees that this outcome is overweighted enough to overcome the individual's loss aversion. Since for inverse-S-shaped weighting functions this p is small, the larger outcome occurs with small probability, which is characteristic for a binary lottery being right-skewed (Ebert 2013). Thus Theorem 1 presents a rigorous result on CPT and skewness preference. Numerous articles find support for skewness preference (e.g., Kraus and Litzenberger (1976) for asset returns, Golec and Tamarkin (1998) for horse-race bets, and Ebert and Wiesen (2011) in a laboratory experiment). In many

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[2]S-shaped utility with exponential curvature as in de Giorgi and Hens (2006) satisfies Assumption 1. For a technical reason, S-shaped utility of the power form as in Kahneman and Tversky (1979) does not satisfy Assumption 1. We treat this important case explicitly in Appendix W.2 and obtain similar results.

[3]Some researchers (in particular in finance) also refer to "prospect theory" when an EUT investor has a loss-averse, S-shaped utility function. Therefore, we emphasize that our paper is on prospect theory *with* probability weighting.

[4]We thank an anonymous referee for suggesting this very intuitive sufficient condition.

situations, prospect theory may do a good job in explaining behavior because it implies skewness preference.

By continuity of CPT preferences we obtain the following corollary.

COROLLARY 1 (Unfair Attractive Gambles): *Under Assumptions 1 and 2, for every wealth level there exists an attractive, arbitrarily small binary lottery with negative mean.*

Recall that risk aversion is defined as every fair risk being unattractive. Skewness preference in the small thus implies that, at every wealth level, a CPT agent is *not* risk averse, because he does find fair risks attractive when they are sufficiently small and skewed.[5]

<p align="center">*Complex, tiny, and very skewed?*</p>

One may question the economic relevance of skewness preference in the small: are the potentially unfair risks that CPT agents find attractive unnatural in the sense that they are complex, extremely small, and extremely skewed?

First, note that the attractive risks we point out are simple binary risks. Gambling is not due to the construction of obscure St. Petersburg risks or the like.

Second, while we show that attractive risks *may* be small they do not have to be small. In general, the maximal size of an attractive gamble depends on further assumptions on the value function that we do not make in this paper. In Appendix W.2, we first show that results similar to those presented in this section also apply to the special case of a power-S-shaped value function even though it does not satisfy Assumption 1. Moreover, for this important value function and the reference point equal to current wealth, we can show that there are not only arbitrarily small, but also arbitrarily large attractive risks. This result on *skewness preference in the large* follows from a recent result due to Azevedo and Gottlieb (2012) when applied to skewness preference in the small.

Third, as regards the concern that attractive risks must be extremely skewed with a tiny gain probability, as condition (3) suggests, a closer examination of the weaker Assumption 2 in Appendix W.1 shows that this is not the case. There, we show that the right-hand sides of the conditions in Assumption 2 are weighting functions themselves, which we call the *benchmark weighting functions*. A comparison of the actual weighting functions with the benchmark weighting functions clarifies how skewed risks must be in order to be attractive. As an example, for the weighting function estimated in Tversky and Kahneman (1992) and a loss aversion parameter of $\lambda = 2.25$ the gain probability p may be as large as 7.2%.

III. On Prospect Theory in a Dynamic Context

In this section, we investigate the consequences of skewness preference in the small in a dynamic context. Consider a stochastic process that could reflect the accumulated returns of an investment project or the price development of an asset traded in the stock market. This process could likewise model an agent's wealth when gambling in a casino. Formally, we consider a Markov diffusion $X = (X_t)_{t \in \mathbb{R}_+}$ that satisfies

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t$$

where $(W_t)_{t \in \mathbb{R}_+}$ is a Brownian motion and the drift $\mu : \mathbb{R} \to \mathbb{R}$ and the volatility $\sigma : \mathbb{R} \to (0, \infty)$ are Lipschitz continuous. This definition covers the most frequently studied processes in economics and finance, arithmetic and geometric Brownian motion.

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[5]For early results on risk-aversion under a special case of CPT, see Chew, Karni, and Safra (1987) and Chateauneuf and Cohen (1994). Schmidt and Zank (2008) characterize strong risk-aversion (aversion to mean preserving spreads) in CPT while we provide sufficient conditions for weak risk-aversion (aversion to zero-mean risks) and skewness preference.

Investment or gambling strategies are modeled as uniformly integrable stopping times, which are plans when to sell an asset or when to stop gambling. Stopping times must be based only on past observations, i.e., all τ are adapted to the natural filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ of the process X. The prospect theory utility of a strategy τ given the information $\mathcal{F}_t$ at time t is given by $CPT(X_\tau, \mathcal{F}_t)$, the CPT utility of the risk X_τ that is generated by the strategy τ.

The probability weighting of prospect theory induces a time inconsistency (e.g. Machina 1989). Therefore, an initially optimal investment strategy τ may later on be dismissed for another strategy. A *naïve* investor is time-inconsistent and also unaware of this time-inconsistency. Therefore, he does not anticipate that later on he might deviate from his initial investment plan. At every point in time, the naïve CPT agent looks for a gambling or investment strategy τ that brings him higher CPT utility than stopping immediately. If such a strategy exists, he holds on to the investment—irrespective of his earlier plan. In the following, we always consider such a naïve agent.

Formally, the naïve agent stops at time t if and only if his prospect theory utility $CPT(X_\tau, \mathcal{F}_t)$ of any stopping strategy τ is less than or equal to $CPT(X_t, \mathcal{F}_t) = U(X_t)$, which is what he gets from stopping immediately:

$$U(X_t) \geq \sup_{\tau \geq t} CPT(X_\tau, \mathcal{F}_t).$$

The following theorem characterizes the gambling or investment behavior of a naïve CPT agent in continuous time, infinite horizon environments. In Section IV we study its implications for three selected dynamic decision problems. In Section V we explain why the result typically also applies in discrete and finite time. The agent's reference point remains arbitrary and may change over time.[6] The result applies to the stochastic processes above, even when the drift is negative. Moreover, the result is global in the sense that it applies at any time t, for every current wealth X_t, irrespectively of the gambling or investment history.

THEOREM 2 (Main Result): *Under Assumptions 1 and 2, the naïve CPT agent never stops.*

The idea of the proof is to construct a gambling strategy that results in a right-skewed binary risk which the agent prefers to stopping immediately, due to Theorem 1. We show that a simple two-threshold strategy, where the agent stops when the process falls a little bit, and continues until it has risen significantly, meets these requirements. Whenever the process reaches either threshold, a new two-threshold strategy comes to the naïve investor's mind that makes him hold on to his investment.

Comparison with Expected Utility

It is insightful to contrast the prediction made by Theorem 2 with EUT. Is probability weighting really needed to generate never stopping? While there exists *no* process the naïve prospect theory agent will ever stop, a risk-averse expected utility maximizer stops all fair and unfair processes immediately. We now show that never stopping is not even obtained under EUT when assuming an extreme degree of risk-lovingness. In particular, we consider an EUT agent whose utility function u is of exponential growth at least at some wealth level $\hat{x}$, i.e., u is not more convex at $\hat{x}$ than all possible exponential functions.[7]

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[6]To avoid confusion: the dynamic results in this paper also assume a "classical" prospect theory reference point. However, this reference point may change over time and evolve according to any $\mathcal{F}_t$-adapted stochastic reference point process $(r_t)_{t \in \mathbb{R}_+}$ as long as the reference point is viewed as constant by the agent when the stopping decision has to be made. As such, the reference point may depend on the past investment evolution and past behavior, but not on future behavior as is the case for the stochastic, expectations-based reference points considered by Kőszegi and Rabin (2006, 2007).

[7]Every power and exponential utility function (concave or convex) is of exponential growth everywhere. If u is infinitely risk-loving at x, i.e., kinked and gain-seeking ($\partial_+ u(x) > \partial_- u(x)$), then the function u is not of exponential growth at x.

PROPOSITION 1: *An EUT maximizer with continuous and strictly increasing utility stops all Brownian motions with sufficiently negative drift and small variance at every wealth level where his utility function is of exponential growth.*

Therefore, while for a naïve CPT agent never stopping occurs as a consequence of natural and commonly made assumptions on the CPT value and weighting functions, in EUT, never stopping does not even emerge under most extreme and unnatural assumptions on the utility function. Never stopping will be the result of a time-dependent utility model when utility is gain-seeking at a reference point that always equals current wealth. But this assumption of predictably changing preferences seems even more unrealistic as it predicts infinite risk-lovingness for all small risks always (rather than respecting loss aversion and distinguishing between right-skewed and left-skewed risks as does CPT).

IV. Applications

A. Casino Gambling

Our first example is the continuous, infinite time horizon analogue to the discrete, finite time casino gambling model of Barberis (2012). Barberis studies the behavior of prospect theory agents who gamble 50-50 bets in a casino for up to five periods. Because no analytical solution is available, the author investigates planned and actual behavior by computing the CPT utilities of all possible gambling strategies that can be generated by a five-period, 50-50 binomial tree, for more than 8000 parameter combinations of the CPT parametrization of Tversky and Kahneman (1992). The reference point is constant and assumed to equal initial wealth when entering the casino. An advantage of this approach is that it also yields results on the behavior of sophisticated agents with and without commitment. For naïvité, in this setting, the simulation results show that gamblers typically plan to follow a stop-loss strategy when entering the casino, but end up playing a gain-exit strategy (i.e., continue gambling when losing and stop gambling when winning). We now first give the analytical solution to the continuous, infinite time analogue of the casino gambling model (which applies to our general version of CPT). We will compare with Barberis (2012) in Section V when we discuss finite and discrete time.

Let X be an arithmetic Brownian motion with negative drift $\mu(x) = \mu < 0$ and constant volatility $\sigma(x) = \sigma > 0$, i.e.,

$$\mathrm{d}X_t = \mu \mathrm{d}t + \sigma \mathrm{d}W_t.$$

Due to the negative drift the agent loses money in expectation if he does not stop. Further assume that the process absorbs at zero because then the agent goes bankrupt. From Theorem 2 it follows that the naïve agent gambles until the bitter end, i.e., he will continue gambling until forced to stop due to bankruptcy. From standard results in probability theory we know that this will happen almost surely.

B. Exercising an American Option

Let X be a geometric Brownian motion with drift $\mu < 0$ and variance $\sigma > 0$, i.e.,

$$\mathrm{d}X_t = X_t(\mu \mathrm{d}t + \sigma \mathrm{d}W_t).$$

However, almost everywhere this cannot be the case. Since u is monotonic it is differentiable almost everywhere and thus without risk-loving kinks at almost all wealth levels.

The agent holds an American option that pays $X_t - K$ if exercised at time t. Here $K \in \mathbb{R}_+$ represents the costs of investment. The American option could be interpreted as an investment opportunity, i.e., a real option (compare Dixit and Pindyck 1994). The agent is allowed to exercise his option at every point in time $t \geq 0$. If the agent does not exercise the option, he receives a payoff of zero.

From Theorem 2 it follows that the agent will never exercise his option and hence the naïve prospect theory agent gets a payoff of zero even though he could get a strictly positive payoff by exercising the option immediately whenever $X_0 > K$.

C. Prospect Theory Predicts no Disposition Effect for naïve Investors

The disposition effect (Shefrin and Statman 1985) refers to the empirical observation that individual investors are more inclined to sell stocks that have gained in value rather than stocks that have declined in value. Several papers have investigated whether prospect theory can explain the disposition effect. However, all of them seem to have done so without the consideration of probability weighting. Barberis (2012) notes that the binomial tree in his paper, which models a casino, may likewise represent the evolution of a stock price over time. Then, naïve investors may exhibit a disposition effect, even though they plan to do the opposite of the disposition effect. Our result can be related to stock trading in the same spirit, but implies that naïve CPT agents *never* sell. In contrast, Henderson (2012) shows that a prospect theory model similar to ours, but without probability distortion, makes reasonable predictions for trading behavior and can explain the disposition effect.

V. Discussion

Continuous time price processes such as geometric Brownian motion fit particularly well for financial market models. However, never stopping will be the prediction for a wide range of investment and gambling situations in continuous or discrete, finite or infinite time. A global result like Theorem 2 requires that at any time—and no matter how wealth has changed by that time—at least one gambling or investment strategy is available that results in an attractive (skewed) gamble. This insight requires some elaboration.

Consider again casino gambling over 50-50 bets for up to five periods. In that case, there is no skewness in the first place: the basic one-shot gamble is 50-50. The agent only gambles because he can generate a skewed payoff distribution by means of a stop-loss strategy. However, this generation of skewness takes time and is thus only feasible in the beginning. Therefore, the *combination* of symmetric one-shot gambles and finite (very short) time horizon ensures that the "casino dries out of skewness." At some exogenous point in time, the casino does not offer skewed gambling experiences any more, and thus the agent stops gambling.

With this in mind it is immediate that, typically, we also have never stopping for a finite time space. To this means, the casino (or the financial market) must offer an attractive (sufficiently small and skewed) gamble in a single period, i.e., in the final period. The commonly made assumption of complete markets (which says that securities with any payoff structure are available) yields our extreme prediction of never stopping for any time horizon.

In Appendix W.3, we illustrate this point through a numerical example. There we assume the original finite, discrete time setting introduced by Barberis (2012), and only change the probability of an up-movement in the binomial tree from $1/2$ to $1/37$. In the casino paradigm, this corresponds to the assumption that a casino offers bets on a single number in French Roulette. We show that an agent with CPT preferences, as estimated in Tversky and Kahneman (1992), never stops gambling for any finite or infinite time horizon. Another example in Appendix W.3 presents a discrete-time, infinite horizon model with only symmetric one-shot gambles available. This model features never

stopping because skewness can be generated through a sequence of bets with different stake sizes. An environment where our never-stopping result would not readily apply is one where bets are rather symmetric and large. This may be the case with indivisible and expensive investments, illiquid investments whose prices jump discontinuously, or high-stake casino gambles. For example, Barberis' (2012) result seems reasonable if one is playing blackjack, where the winning probability is close to 50% and the hands are $10,000. Whether a particular real-life environment offers sufficiently small and skewed gambles is ultimately an empirical question.

To evade the never-stopping result, one could also dispense with one of the other two main assumptions of this paper. First, one may abstract from probability weighting as many papers do. Prospect theory without the probability weighting component (thus focusing on the impact of a loss-averse, S-shaped utility function) is indeed extensively applied, and successful in explaining various empirical phenomena such as individual trading behavior (Henderson 2012) or life insurance decisions (Gottlieb 2013).

Second, naïvité may be questioned as a suitable concept to deal with time-inconsistency. Machina (1989) provides a discussion of time-inconsistency in generalized EUT models that received renewed attention through the casino model of Barberis (2012). Xu and Zhou (2013), for example, study the investment behavior of a CPT agent who can commit to her initial plan.

Finally, a pure prospect theory model may be regarded as a straw man, as it is bound to produce strange results when it moves to the realm of small probabilities, even in a static setting.[8] While the ideas of prospect theory help us understand many behavioral phenomena much better, their formalization may have potential for improvement.

VI. Conclusion

This paper derives results on prospect theory when probability weighting is strong enough relative to loss aversion. We first prove that probability weighting implies skewness preference in the small. At any wealth level, a CPT agent wants to take a sufficiently right-skewed binary risk that is arbitrarily small, even if it has negative expectation.

Skewness preference in the small has consequences for CPT in a dynamic context. We investigate the predictions of prospect theory for a naïve agent who is unaware of his time-inconsistency. When small and skewed gambling experiences are possible, naïve CPT agents never stop gambling. The implications of this result are extreme, as we illustrate along the lines of casino gambling, option exercise, and stock trading.

As the time-inconsistency of CPT arises naturally as a consequence of probability weighting, the question of how this time-inconsistency should be dealt with needs to be addressed. This paper analyzed the predictions of a dynamic version of prospect theory and their sensitivity to probability weighting and the richness of the investment opportunity set. We hope that this advances our understanding of some of the mechanisms at work in models that take prospect theory to the dynamic context.

PROOFS

A1. Proof of Theorem 1

Let x be the agent's wealth. We split the proof into three cases $x > r$, $x < r$, and $x = r$. We prove that for all $x \in \mathbb{R}$ and every $\epsilon > 0$ there exists a binary lottery $L = L(p, b, a)$ with mean x and $a, b \in (x - \epsilon, x + \epsilon)$ such that $CPT(L) > CPT(x)$. The arbitrarily small zero-mean risk mentioned

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[8]We thank an anonymous referee for this perspective.

in the statement is thus given by $L - x$. Note that L having mean x yields $a < x < b$ and

$$x = (1-p)a + pb \iff p = \frac{x-a}{b-a}.$$

Proof of case 1 $(x > r)$. Choose $a > r$ such that both a and b are gains. Then lottery L gives the agent a utility of $CPT(L) = w^+(p)U(b) + (1-w^+(p))U(a)$. Therefore, the agent prefers L over x if there exist $a < x$ and $b > x$ such that

$$0 < \left(1 - w^+\left(\frac{x-a}{b-a}\right)\right)U(a) + w^+\left(\frac{x-a}{b-a}\right)U(b) - U(x)$$

$$= (U(b) - U(a))\left(w^+\left(\frac{x-a}{b-a}\right) - \frac{U(x)-U(a)}{U(b)-U(a)}\right)$$

(A1)
$$= \underbrace{p\,(U(b) - U(a))}_{>0}\left(\frac{w^+(p)}{p} - \frac{\frac{U(x)-U(a)}{x-a}}{\frac{U(b)-U(a)}{b-a}}\right).$$

Consider sequences $(a_n, b_n)_{n\in\mathbb{N}}$ with $a_n = x - \frac{p}{n}$ and $b_n = x + \frac{1-p}{n}$. Note that by construction

$$\frac{U(b_n) - U(a_n)}{b_n - a_n} = \frac{U(b_n) - U(x)}{b_n - x}\frac{b_n - x}{b_n - a_n} + \frac{U(x) - U(a_n)}{x - a_n}\frac{x - a_n}{b_n - a_n}$$

$$= \frac{U(b_n) - U(x)}{b_n - x}(1 - p) + \frac{U(x) - U(a_n)}{x - a_n}p.$$

Therefore, according to equation (A1), the agent prefers lottery $L(p, b_n, a_n)$ over x if

$$0 < \frac{w^+(p)}{p} - \frac{\frac{U(x)-U(a_n)}{x-a_n}}{\frac{U(b_n)-U(x)}{b_n-x}(1 - p) + \frac{U(x)-U(a_n)}{x-a_n}p}.$$

Note that

$$\lim_{n\to\infty} \frac{\frac{U(x)-U(a_n)}{x-a_n}}{\frac{U(b_n)-U(x)}{b_n-x}(1 - p) + \frac{U(x)-U(a_n)}{x-a_n}p} = \frac{\partial_-U(x)}{\partial_+U(x)(1-p) + \partial_-U(x)p} = \frac{\frac{\partial_-U(x)}{\partial_+U(x)}}{1 - p + \frac{\partial_-U(x)}{\partial_+U(x)}p} \leq \frac{\lambda}{1 - p + p\lambda}.$$

Therefore, for n sufficiently large the agent finds lottery $L(p, b_n, a_n)$ attractive if

(A2)
$$0 < \frac{w^+(p)}{p} - \frac{\lambda}{1 - p + p\lambda} \iff w^+(p) > \frac{\lambda p}{1 - p + p\lambda},$$

and condition 1 of Assumption 2 ensures that there exists at least one such p.

Proof of case 2 $(x < r)$. Choose $b < r$ such that both a and b are losses. In that case, lottery $L = L(p, b, a)$ secures the agent a prospect theory utility of

$$CPT(L) = (1 - w^-(1-p))U(b) + w^-(1-p)U(a)$$

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with $1 - p = \frac{b-x}{b-a}$. Therefore, the agent continues gambling if there exist $a < x$ and $b > x$ such that

$$0 < \left(1 - w^- \left(\frac{b-x}{b-a}\right)\right) U(b) + w^- \left(\frac{b-x}{b-a}\right) U(a) - U(x)$$

$$= U(b) - U(a) + U(a) - U(x) - w^- \left(\frac{b-x}{b-a}\right)(U(b) - U(a))$$

$$= (U(b) - U(a))\left(1 - w^- \left(\frac{b-x}{b-a}\right) + \frac{U(a) - U(x)}{U(b) - U(a)}\right)$$

$$\text{(A3)} \qquad = \underbrace{p\left(U(b) - U(a)\right)}_{>0} \left(\frac{1 - w^-(1-p)}{p} - \frac{\frac{U(x)-U(a)}{x-a}}{\frac{U(b)-U(a)}{b-a}}\right)$$

which is the analogue to inequality (A1). The proof continues similarly to that of case 1.

Proof of case 3 $(x = r)$. When $x = r$, a is a loss and b is a gain. Therefore,

$$CPT(L) = w^-(1-p)U(a) + w^+(p)U(b).$$

Note that, since $x = r$ by definition $U(x) = U(r) = 0$. Therefore, the agent chooses L over x if there exist $a < x$ and $b > x$ such that

$$0 < w^-(1-p)U(a) + w^+(p)U(b) - U(x)$$

$$= w^+(p)\left(U(b) - U(a)\right) + (U(a) - U(x))\left(w^-(1-p) + w^+(p)\right)$$

$$= (U(b) - U(a))\left(w^+(p) - \frac{U(x) - U(a)}{U(b) - U(a)}\left(w^-(1-p) + w^+(p)\right)\right)$$

$$\text{(A4)} \qquad = \underbrace{p\left(U(b) - U(a)\right)}_{>0} \left(\frac{w^+(p)}{p} - \frac{\frac{U(x)-U(a)}{x-a}}{\frac{U(b)-U(a)}{b-a}}\left(w^-(1-p) + w^+(p)\right)\right).$$

Similarly to before, it can be shown that the agent prefers lottery $L(p, b_n, a_n)$ over x for large enough n if

$$\text{(A5)} \qquad 0 < \frac{w^+(p)}{p} - \frac{\lambda}{1-p+p\lambda}\left(w^-(1-p) + w^+(p)\right),$$

which is the analogue to what inequality (A2) is for case 1 of the proof. We conclude the proof by verifying

$$\text{(A6)} \qquad w^+(p) > \lambda p \frac{w^-(1-p)}{1-p},$$

which is equivalent to inequality (A5). The first (second) inequality below follows from condition 1 (condition 2) of Assumption 2.

$$w^+(p) > \frac{\lambda p}{1-p+p\lambda} = \frac{\lambda p}{1-p} \cdot \frac{1-p}{1-p+\lambda p} > \frac{\lambda p}{1-p} w^-(1-p),$$

which is inequality (A6). □

10

Suppose the agent arrives at wealth x at time t, i.e., $X_t = \bar{x}$. She continues gambling if there exists a gambling strategy τ such that $CPT(X_\tau, \mathcal{F}_t) > U(\bar{x})$ where

$$CPT(X_\tau, \mathcal{F}_t) = \int_{\mathbb{R}_+} w^+ \left(\mathbb{P}(u(X_\tau - r) > y \,|\, \mathcal{F}_t)\right)\mathrm{d}y - \int_{\mathbb{R}_-} w^- \left(\mathbb{P}(u(X_\tau - r) < y \,|\, \mathcal{F}_t)\right)\mathrm{d}y \,.$$

We consider strategies $\tau_{a,b}$ with two absorbing endpoints $a < \bar{x} < b$ which stop if the process X leaves the interval (a, b), i.e.,

$$\tau_{a,b} = \inf\{s \geq t : X_s \notin (a,b)\} \,.$$

Denote with $p = \mathbb{P}(X_{\tau_{a,b}} = b)$ the probability that with strategy $\tau_{a,b}$ the agent will stop at b. Note that strategy $\tau_{a,b}$ results in a binary lottery for the agent.

We first prove that the agent never stops if X is a martingale. For every stopping time $\tau_{a,b}$ consider the sequence of bounded stopping times $\min\{\tau_{a,b}, n\}$ for $n \in \mathbb{N}$. By Doob's optional sampling theorem (Revuz and Yor 1999, p. 70), $\mathbb{E}(X_{\min\{\tau_{a,b},n\}}) = X_t = \bar{x}$. By the theorem of dominated convergence it follows that

$$\mathbb{E}\left(X_{\tau_{a,b}}\right) = \mathbb{E}\left(\lim_{n \to \infty} X_{\min\{\tau_{a,b},n\}}\right) = \lim_{n \to \infty} \mathbb{E}\left(X_{\min\{\tau_{a,b},n\}}\right) = \bar{x} \,.$$

Hence, $X_{\tau_{a,b}}$ implements the binary lottery $L(p, a, b)$ with expectation $\bar{x}$. From Theorem 1 it follows that there exist $a < b$ such that the agent prefers the binary lottery $L(p, a, b)$ induced by the strategy $\tau_{a,b}$ over the certain outcome $\bar{x}$.

In the last step we prove that the naïve agent never stops even if X is not a martingale. Define the strictly increasing scale function $S : \mathbb{R} \to \mathbb{R}$ by

$$S(x) = \int_0^x \exp\left(-\int_0^y \frac{2\mu(z)}{\sigma^2(z)}\mathrm{d}z\right)\mathrm{d}y \,.$$

Define a new process $\hat{X}_t = S(X_t)$ and a new value function $\hat{U}(\hat{x}) = (U \circ S^{-1})(\hat{x})$. Note that the index λ from Assumption 1 of the original value function U equals that of $\hat{U}$ as for all $\hat{x} = S(x)$

$$\frac{\partial_- \hat{U}(\hat{x})}{\partial_+ \hat{U}(\hat{x})} = \frac{\frac{\partial_- U(x)}{S'(x)}}{\frac{\partial_+ U(x)}{S'(x)}} = \frac{\partial_- U(x)}{\partial_+ U(x)} \,.$$

The process $\hat{X}$ satisfies (Revuz and Yor 1999, p. 303 ff)

$$\mathbb{P}\left(\hat{X}_{\tau_{a,b}} = S(b)\right) = \frac{S(\bar{x}) - S(a)}{S(b) - S(a)} = \mathbb{P}\left(X_{\tau_{a,b}} = b\right) = p \,.$$

Therefore, a CPT agent with the value function $\hat{U}$ facing the process $\hat{X}$ evaluates all stopping times exactly as a CPT agent with value function U who faces X and the proof follows from the martingale case. $\qquad\square$

Consider a wealth level $\hat{x}$ where u is of exponential growth, i.e., is not more concave than all exponential functions. Formally, there exist $\beta \in \mathbb{R}, \alpha \in \mathbb{R}_+$ such that $[u(x) - \beta] \leq [u(\hat{x}) -$

$\beta]\exp(\alpha(x - \hat{x}))$ for all x. Since EUT preferences are invariant under quasi-linear transformations, without loss of generality $\beta = 0$. We will show that at $\hat{x}$ the EUT maximizer will stop all Brownian motions with negative drift $(\mu < 0)$ and small enough variance $(\sigma < \sqrt{\left|\frac{2\mu}{\alpha}\right|})$.

Consider the Brownian motion X with $X_t = \hat{x} + \mu t + \sigma W_t$ starting in $X_0 = \hat{x}$ with negative drift $\mu < 0$ and variance $\sigma < \sqrt{\left|\frac{2\mu}{\alpha}\right|}$. For every stopping time τ such that $\mathbb{P}(\tau > 0) > 0$

$$\mathbb{E}\left[u(X_\tau)\right] \leq u(\hat{x})\mathbb{E}\left[\exp(\alpha(X_\tau - \hat{x}))\right] = u(X_0)\mathbb{E}\left[\exp(\alpha\mu\tau + \alpha\sigma W_\tau)\right]$$
$$= u(X_0)\mathbb{E}\left[1 + \int_0^\tau \left(\alpha\mu + \frac{1}{2}\alpha^2\sigma^2\right)\exp(\alpha\mu s + \alpha\sigma W_s)\mathrm{d}s + \int_0^\tau \alpha\sigma\exp(\alpha\mu s + \alpha\sigma W_s)\mathrm{d}W_s\right]$$
$$= u(X_0)\mathbb{E}\left[1 + \int_0^\tau \left(\alpha\mu + \frac{1}{2}\alpha^2\sigma^2\right)\exp(\alpha\mu s + \alpha\sigma W_s)\mathrm{d}s\right] < u(X_0)\,.$$

The second equality follows from Itô's Lemma. The last equality follows from Doob's optional sampling theorem. Hence, the expected utility from stopping X immediately at $\hat{x}$ is strictly higher than the expected utility from any other stopping strategy. □

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