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Distribution System Expansion Planning with Nonlinear Power Flow Models

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***Abstract*—We consider the electric distribution system expansion planning problem of selecting the least-cost set of upgrades to alleviate voltage magnitude violations. To this end, we use the full nonlinear power flow model to account for losses and represent meshed networks. The underlying problem is highly nonconvex due to the nonlinear power flow equations and the presence of discrete variables. Leveraging the efficacy of local search methods for handling continuous subproblems, a bilevel heuristic algorithm is proposed based on outer-approximation mixed-integer programming techniques. The proposed algorithm is shown to achieve the same optimal value as off-the-shelf mixed-integer nonlinear solvers with much less computation time on five benchmark IEEE distribution feeders.**

***Index Terms*—distribution systems, expansion planning, mixed-integer optimization**

I. Introduction

Growing electrification of personal vehicles and household appliances is primed to dramatically increase the load on electric distribution networks in the coming decades. Reactive compensation and voltage regulation will be necessary to prevent undervoltages from new electric loads with low power factors. Distribution lines and transformers will need to be reinforced to avoid exceeding their thermal limits. On the other hand, rooftop solar, vehicle-to-grid, and other distributed energy resources may cause large reverse power flows and local overvoltages. Electric utilities are tasked with bolstering the system against these changes at the least possible cost. The distribution system expansion problem (DSEP) considered here aims to find the least expensive portfolio of network upgrades to establish operational feasiblity for a power network.

Due to the discrete nature of investment decisions, DSEPs are mixed-integer programs, which are nonconvex and intractable in general. Existing work on DSEP has been largely restricted to radial networks, with many authors employing the DistFlow model [1]. Most existing literature uses a linearized version of the DistFlow model derived by neglecting losses, resulting in a mixed-integer linear program (MILP) [2]. For example, [3] and others use an iterative algorithm which alternates between a linear model with an affine losses term and a power flow to compute the losses given the last integer solution. However, as losses can be relatively high in distribution networks, handling them directly gives more reliable and low-cost solutions. [4] instead applies the popular second-order cone relaxation of the DistFlow equations, giving a mixed-integer conic program that can be solved globally. However, the second-order cone relaxation is only exact when the network satisfies certain sufficient conditions which may be violated for some DSEP problems. In particular, the relaxation is usually not exact when there are binding upper bounds on voltage magnitudes, a common occurrence in DSEP problems with overvoltages at the substation or at nodes with distributed generation.

Even when the second-order cone relaxation is exact, the DistFlow model can only represent radial networks, neglecting meshed distribution installations often present in downtown areas. DSEPs with (unrelaxed) nonlinear power flow models are underexplored in the literature due to the challenge of solving nonconvex mixed-integer nonlinear programs (MINLPs). Though all mixed-integer problems are nonconvex, we use "nonconvex" to refer to MINLPs for which the continuous subproblems with fixed integer variables are also nonconvex. Outer approximation algorithms are popular for *convex* MINLPs because they require solving only a limited number of continuous nonlinear subproblems associated with fixed integer variables [5]. Branch-and-bound algorithms, by contrast, solve many continuous subproblems with relaxed integrality constraints, which may converge prohibitively slowly to integer solutions. In this work, we exploit the reliable performance of local continuous nonlinear solvers for optimal power flow (OPF) to develop a bilevel optimization algorithm for the DSEP proglem. We adapt outer approximation methods to the nonconvex case by applying them to the outer subproblem. We demonstrate that the proposed algorithm solves the DSEP problem on five IEEE benchmark test cases, including one meshed network, in less time and with many fewer continuous nonlinear subproblems than off-the-shelf branch-and-bound solvers, which are the state-of-the-art heuristics for nonconvex MINLPs. We explore the experimental results to justify the application of the outer approximation algorithm.

II. Formulation

A. Model

In this section, we present a simple DSEP model featuring capacitor banks and voltage regulators installed to mitigate overvoltages and undervoltages. To avoid notational clutter, we consider a single future net load scenario and seek the least-cost set of upgrades necessary to satisfy voltage and thermal limits. However, the proposed technique can be easily extended to multiple scenarios. Every bus except the substation

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is considered to be a candidate for capacitor banks and every line is considered to be a candidate for a voltage regulator. When a given device is not installed, the associated variables of that device are simply constrained to be zero.

Consider a network with a set of buses $\mathcal{N}_+ = \mathcal{N} \cup \{0\}$, where 0 is the substation bus and $\mathcal{N}$ denotes the set of PQ buses. Let the set of lines be shown as $\mathcal{L} \subseteq \mathcal{N}_+ \times \mathcal{N}_+$.

The power flows from bus i to bus j on a line $(i, j) \in \mathcal{L}$ are given by

$$\begin{aligned} P_{ij} &= V_i^2 g_{ij} \\ &\quad - V_i \tilde{V}_{ij} \left(g_{ij} \cos(\theta_i - \theta_j) + b_{ij} \sin(\theta_i - \theta_j) \right) \end{aligned} \tag{1a}$$

$$\begin{aligned} Q_{ij} &= -V_i^2 b_{ij} \\ &\quad - V_i \tilde{V}_{ij} \left(g_{ij} \sin(\theta_i - \theta_j) - b_{ij} \cos(\theta_i - \theta_j) \right) \end{aligned} \tag{1b}$$

where $P, Q \in \mathbb{R}^{|\mathcal{L}|}$ are the real and reactive sending-end power flows, respectively; $V \in \mathbb{R}^{|\mathcal{N}|}$ are the voltage magnitudes; $\theta \in \mathbb{R}^{|\mathcal{N}|}$ are the voltage angles; $\tilde{V} \in \mathbb{R}^{|\mathcal{L}|}$ are intermediate voltage magnitudes associated with voltage regulators (attached at the recieving end by convention); and $b, g \in \mathbb{R}^{|\mathcal{L}|}$ are the line series conductances and susceptances, respectively. The power balance equations at a node $i \in \mathcal{N}$ are

$$p_i = \sum_{i \to j} P_{ij}, \quad q_i = \sum_{i \to j} Q_{ij} \tag{2}$$

where $p, q \in \mathbb{R}^{|\mathcal{N}|}$ are the real and reactive power injections, respectively. For simplicity, we neglect shunt admittances. The injections at PQ buses $i \in \mathcal{N}^+$ are given by

$$p_i = -p_{d,i}, \quad q_i = -q_{d,i} + q_{\text{cap},i} \tag{3}$$

where $p_d, q_d \in \mathbb{R}^{|\mathcal{N}^+|}$ are the real and reactive future net loads and $q_{\text{cap}} \in \mathbb{R}^{|\mathcal{N}^+|}$ is the reactive power output of the capacitor banks. Capacitor banks are modeled as reactive power sources with a continuous range of output, as opposed to discrete switching levels:

$$0 \le q_{\text{cap},i} \le z_{\text{cap},j} \Delta \overline{q}_{\text{cap},j} \quad \forall j \in \mathcal{N}. \tag{4}$$

where $z_{\text{cap}} \in \mathbb{Z}^{|\mathcal{N}^+|}$ is the discrete capacitor investment decision and $\Delta \overline{q}_{\text{cap}} \in \mathbb{R}^{|\mathcal{N}^+|}$ is the capacitor sizing increment. Similarly, the voltage regulator tap model is continuous. Furthermore, we approximate the voltage regulation as additive rather than proportional, a reasonable approximation since voltage magnitudes are constrained to be close to their nominal values of 1 (assuming a per-unit system). For an edge $(i, j) \in \mathcal{L}$:

$$|\tau_{ij}| \le z_{\text{reg},ij} \Delta \overline{\tau}_{ij} \tag{5a}$$

$$\tilde{V}_{ij} = V_j + \tau_{ij} \tag{5b}$$

where $\tau \in \mathbb{R}^{|\mathcal{L}|}$ is the regulator tap, $z_{\text{reg}} \in \mathbb{Z}^{|\mathcal{L}|}$ is the binary regulator investment decision and $\Delta\tau \in \mathbb{R}^{|\mathcal{L}|}$ is the one-sided regulator tap limit. Voltage magnitudes are subject to operational limits:

$$\underline{v}_j \le V_i \le \overline{v}_j \quad \forall i \in \mathcal{N} \tag{6}$$

where $\underline{v}, \overline{v} \in \mathbb{R}^{|\mathcal{N}|}$ are the upper and lower voltage magnitude limits. Finally, there is an upper bound on the discrete upgrades:

$$z_{\text{cap},j} \le \overline{z}_{\text{cap},j} \qquad \forall j \in \mathcal{N} \tag{7a}$$

$$z_{\text{reg},e} \le 1 \qquad \forall j \in \mathcal{L} \tag{7b}$$

The cost of the upgrades is given by

$$c(z) = \sum_{j \in \mathcal{PQ}} C_{\text{cap},j} z_{\text{cap},j} + \sum_{e \in \mathcal{L}} C_{\text{reg},e} z_{\text{reg},e} \tag{8}$$

Aside from the discrete variables, the feasible set defined by (1), (2), (3), (4), (5), (6), (7) is nonconvex because of the nonlinear power flow equations (1).

B. DSEP Problem

We collect the operational variables into a vector as

$$x := \begin{bmatrix} V^T & \tilde{V}^T & \theta^T & p_0 & q_0 & q_{\text{cap}}^T & \tau^T \end{bmatrix}^T$$

The DSEP problem can then be written compactly as

$$\min_{\substack{z \in \mathbb{Z}^{n_z} \\ x \in \mathbb{R}^{n_x}}} \; c(z) \tag{9a}$$

$$\text{s.t. } f(x) = 0 \tag{9b}$$

$$h_1(x, z) \le 0 \tag{9c}$$

$$h_2(x, z) \le 0 \tag{9d}$$

$$z \in \mathcal{Z} \tag{9e}$$

where $f : \mathbb{R}^{n_x} \to \mathbb{R}^{n_f}$ represents the power flow equations (1), (2), (3), (5b), $h_1 : \mathbb{R}^{n_x} \times \mathbb{Z}^{n_z} \to \mathbb{R}^{n_{h_1}}$ collects *operational constraints* (4), (5a), and $h_2 : \mathbb{R}^{n_x} \times \mathbb{Z}^{n_z} \to \mathbb{R}^{n_{h_2}}$ collects the voltage limits (6). The feasible set of upgrades defined in (7) is represented by $\mathcal{Z}$.

Though we introduce a specific model in Section II-A, the compact form (9) can represent many power system expansion problems, including transmission system expansion problems featuring flexible AC transmission system (FACTS) devices or utility-owned storage. (9d) may also include thermal line flow limits, which are not considered here because they are not included in benchmark IEEE distribution networks in our experiments (see Section V). The proposed algorithm can be applied to any model of the form (9), but since it is a heuristic, its performance may not generalize to more complex problems where, for example, h_1 and h_2 may be nonconvex.

We will assume that there exists a set of upgrades allowing for feasibility; otherwise, the DSEP problem would be ill-posed.

Assumption 1. *The optimization problem* (9) *is feasible.*

The problem (9) is an MINLP because the power flow function f is nonlinear and z is an integer variable. Moreover, since f is an *equality-constrained* nonlinear function, (9) is a nonconvex MINLP. Given the difficulty of solving nonconvex MINLPs, we propose a heuristic algorithm to solve (9) by exploiting the structural properties of the problem.

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III. Approach

First, we modify the problem by introducing slack variables to the voltage limits (9d) and penalizing them in the objective function. For some $\alpha \in [0,1)$, define the penalized approximate DSEP problem

$$\min_{\substack{z\in\mathbb{Z}^{n_z}\\ x\in\mathbb{R}^{n_x}\\ \delta\in\mathbb{R}^{n_{h_2}}}} \quad (1-\alpha)c(z) + \alpha\|\delta\| \tag{10a}$$

$$\text{s.t. } f(x,z) = 0 \tag{10b}$$

$$h_1(x,z) \le 0 \tag{10c}$$

$$h_2(x,z) \le \delta \tag{10d}$$

$$z \in \mathcal{Z} \tag{10e}$$

Here, α controls the tradeoff between cost and safety constraint violations. Define the *value function*

$$\begin{aligned} \phi(z) = \min_{\substack{x\in\mathbb{R}^{n_x}\\ \delta\in\mathbb{R}^{n_{h_2}}}} \quad & \|\delta\| \\ \text{s.t. } & f(x,z) = 0 \\ & h_1(x,z) \le 0 \\ & h_2(x,z) \le \delta \end{aligned} \tag{11}$$

The following corollary is a consequence of Assumption 1.

Corollary 1.1. *There exists a vector $z_0 \in \mathcal{Z} \cap \mathbb{Z}^{n_z}$ such that $\phi(z_0) = 0$.*

(10) can be equivalently expressed as

$$\begin{aligned} \min_{z\in\mathbb{Z}^{n_z}} \quad & (1-\alpha)c(z) + \alpha\phi(z) \\ \text{s.t. } & z \in \mathcal{Z} \end{aligned} \tag{12}$$

Proposition 2. *If z^* solves* (12) *and $\phi(z^*) = 0$, then there exists a vector x^* such that (z^*, x^*) solves* (9).

Proof. Let x^* solve (11) when $z = z^*$. Since $\phi(z^*) = 0$, (z^*, x^*) along with $\delta^* = 0$ solves (10). (10) may then be simplified by replacing δ with 0 everywhere it appears, recovering (9) except with a factor of $(1-\alpha)$ in the objective. Since $\alpha < 1$ and $(1-\alpha) > 0$, the scaling factor does not change the solution and (z^*, x^*) solves (9). $\square$

Let z_α denote an arbitrary solution to (12) for a given α.

Proposition 3. *There exists a threshold $\underline{\alpha} \in [0,1)$ such that $\phi(z_\alpha) = 0$ for all $\alpha \in (\underline{\alpha}, 1]$.*

Proof. Define $\overline{c} = \max_{z\in\mathbb{Z}^{n_z}}\{c(z) : z \in \mathcal{Z}\}$ and $\underline{p} = \min_{z\in\mathbb{Z}_+^{n_z}}\{\phi(z) : z \in \mathcal{Z}, \phi(z) > 0\}$. Note that the latter is well-defined since z, and by extension $\phi(z)$, can take only discrete values and $\phi(0) > 0$ (otherwise (9) is trivial). Consider a candidate $\tilde{z} \in \mathcal{Z} \cap \mathbb{Z}_+^{n_z}$ such that $\phi(\tilde{z}) > 0$. Choose $\underline{\alpha} = \frac{\overline{c}}{\overline{c}+\underline{p}}$. We can lower-bound the objective value evaluated at $\tilde{z}$ for all $\alpha \in (\underline{\alpha}, 1]$:

$$\begin{aligned} (1-\alpha)c(\tilde{z}) + \alpha\phi(\tilde{z}) &\ge \alpha\phi(\tilde{z}) \\ &\ge \alpha\underline{p} \\ &> (1-\alpha)\overline{c} \\ &\ge (1-\alpha)c(z_0) \\ &= (1-\alpha)c(z_0) + \alpha\phi(z_0) \end{aligned}$$

The first inequality comes from the nonnegativity of $c(\tilde{z})$. The second follows from the definition of $\underline{p}$. The third follows from $\alpha > \underline{\alpha}$ and the choice of $\underline{\alpha}$. The fourth holds by construction of $\overline{c}$. The equality follows from Corollary 1.1. The resulting lower bound implies $\tilde{z}$ is not a solution to (12) because z_0 is feasible by construction and achieves a strictly lower objective. It follows that at optimality we must have $\phi(z_\alpha) = 0$. $\square$

Put simply, Proposition 2 says that if (12) is solved and the slack variables are zero at optimality, then (9) is also solved. Proposition 3 guarantees that there is a sufficiently large penalty $\underline{\alpha} < 1$ which will drive the slack variables to zero. Taken together, this suggests a general algorithm where (12) is solved for a given α, the solution z^* is returned if $\phi(z^*) = 0$, and otherwise α is increased.

IV. Algorithm

In general, the value function ϕ is not convex because the minimization in (11) is nonconvex [6]. However, we will assume that ϕ is well-approximated by a convex function. This assumption is motivated and empirically validated in Section V. Define the first-order approximation of ϕ around z_0 as

$$\phi_{z_0}(z) := \phi(z_0) + \nabla\phi(z_0)^T(z - \tilde{z}). \tag{13}$$

Algorithm 1 Outer Approximation (OA)

```
α ← 0
S ← ∅
b ← 0
while true do

    z ← arg min_{z∈Z_+^{n_z}} (1 − α)c(z) + α max_{z̃∈S} {φ_z̃(z) − b_z̃}
        s.t. z ∈ 𝒵

    if z ∈ S then
        if φ(z) = 0 then
            return z
        else
            α = (α+1)/2
        end if
    else
        S ← S ∪ {z}
        for (z, z̃) ∈ S × S do
            b_z̃ ← max{b_z̃, φ_z̃(z) − φ(z)}
        end for
    end if
end while
```

$$z \leftarrow \arg\min_{z\in\mathbb{Z}_+^{n_z}} \; (1-\alpha)c(z) + \alpha\max_{\tilde{z}\in\mathcal{S}}\{\phi_{\tilde{z}}(z) - b_{\tilde{z}}\} \quad \text{s.t. } z \in \mathcal{Z}$$

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Algorithm 1 solves (12) by approximating $\phi(z)$ as the pointwise maximum of adjusted first-order approximations of ϕ evaluated at a set of points S. The adjustments b, set in Step 5, ensure that the approximation underestimates every visited point. If the solution in Step 5 is in the set of fixed points S, (12) is considered solved for the current α. Otherwise, the solution is added to S to tighten the approximation. The α update is a simple bisection causing α to asymptotically approach 1.

Notice that the cuts added for previous α remain in $\mathcal{S}$ for future α and the approximation of p grows tighter the longer the algorithm runs.

The optimization in Step 5 is a mixed-integer linear program (MILP) with its dimension equal to the dimension of z. While we take every line and bus to be a candidate for upgrades in the model presented here, practical networks will typically have a limited number of candidate sites for investment. That is, the dimension of the MILP may grow much slower than the size of the network, allowing it to be solved efficiently even for large cases. Additionally, a well-known advantage of outer approximation algorithms is that the MILPs can often rapidly reoptimize for each new cut by warm-starting at the previous solution.

Computing (13) in Algorithm 1 requires evaluating ϕ and computing the gradient $\nabla\phi$. We will discuss the latter, which is simpler, first. Assuming (11) can be solved, the gradients are the sensitivity of (11) with respect to z. Assuming the solution to (11) is a regular point satisfying the second-order sufficient optimality conditions, the sensitivities can be computed from the Lagrange multipliers [6], which are returned along with the (local or global) optimizaton solution by off-the-shelf NLP solvers.

Evaluating the primal function ϕ requires solving an OPF problem, which is a *continuous* nonconvex nonlinear program (NLP). Due to the availability of good warm starts, nonconvex OPF problems are solved efficiently and reliably in practice using local search solvers such as the interior-point solver IPOPT [7].

Algorithm 1 depends on the assumption that, for a given z, (11) can be solved globally by local search algorithms and moreover that its solution x^* is a regular point satisfying the second-order sufficient conditions.

V. Experiments

A. Performance

We demonstrate the proposed algorithm on five benchmark IEEE distribution feeders: 18-bus, 22-bus, 33-bus, 69-bus, and 141-bus. We make the following changes to the networks:

1) All inactive lines activated. The only case with default-inactive lines is the 33-bus network, which becomes meshed after activation.
2) Voltage magnitude bounds are tightened to ± 0.05 p.u. from the original ± 0.1 p.u.
3) All active and reactive loads are iteratively increased by 100% of their original values until a power flow with unit substation voltage magnitude results in a maximum voltage deviation between buses of at least 0.2 p.u.

TABLE I
PERFORMANCE OF OA AND BNB ON BENCHMARK IEEE DISTRIBUTION NETWORKS

Network	Method	Cost	Upgrades	NLP solves	Time (s)
18	OA	**2.757**	20C/0R	**10**	**0.84**
	BnB	**2.757**	20C/0R	332	11.52
22	OA	**2**	0C/2R	**16**	**1.40**
	BnB	**2**	0C/2R	116	4.94
33	OA	**2.767**	20C/1R	**16**	**2.89**
	BnB	**2.767**	19C/1R	12320	862.13
69	OA	**1.810**	9C/1R	**24**	**5.44**
	BnB	**1.810**	9C/1R	389	46.04
141	OA	**3**	0C/3R	**56**	**35.48**
	BnB	**3**	0C/3R	1358	349.0

Capacitor bank investment sizes are equal to the reactive loads at each bus, $\Delta\overline{q}_{\text{cap}} = -q_d$. Voltage regulator taps can change voltages by up to 5%, $\Delta\overline{\tau} = .05$. Capacitor costs are set proportional to their upgrade sizes and such that the most expensive capacitor on the network costs one unit: $c_{\text{cap}} = \Delta\overline{q}_{\text{cap}} / \max_{n\in\mathcal{N}}\{\Delta\overline{q}_{\text{cap},n}\}$. Regulator costs are set to one unit: $c_{\text{reg}} = 1$. Up to five upgrade levels are permitted for every capacitor bank: $\overline{z}_{\text{cap}} = 5$. We define ϕ using the 1-norm.

For comparison, we also solve (9) using the off-the-shelf MINLP solver BONMIN, which implements the branch-and-bound algorithm from [8]. Such branch-and-bound algorithms with local NLP subproblem solvers are the state-of-the-art heuristic for nonconvex MINLPs [5] and are also implemented in the open-source solvers Minotaur and Juniper as well as the commercial solver Knitro. We will refer to our method as OA (outer approximation), technically a misnomer because the approximation in Step 5 is not necessarily a lower bound. We refer to to the branch-and-bound algorithm as BnB. In Algorithm 1, the inner NLP subproblems are solved using IPOPT and the outer MILP subproblems are solved using Gurobi. Simulations are run on on a 6-core 2.60 GHz laptop computer running Windows 11. Network data is taken from MATPOWER case files. The performance of OA and BnB are compared in Table I. “C” and “R” in the “Upgrades” column indicate the number of capacitors and regulators upgrades, respectively.

For all five networks, $b_z = 0$ for all $z \in \mathcal{S}$ when Algorithm 1 terminated. This means that, for each upgrade z for which $\phi(z)$ was evaluated, the first-order approximation ϕ_z *always* lower-bounded all other evaluated upgrades in $\mathcal{S}$. Moreover, the optimal cost of OA and BnB are equal for all cases as seen in the cost column of Table I. As long as the NLP subproblems are solved globally, BnB will give the global optimum, meaning that OA did not return suboptimal solutions despite the theoretical possibility that the “outer” approximation in Step 5 of Algorithm 1 is not a universal lower bound. Taken together, these observations empirically support the claim that ϕ has approximately convex structure and that an outer-approximation algorithm is appropriate.

For the same reason that outer approximation algorithms

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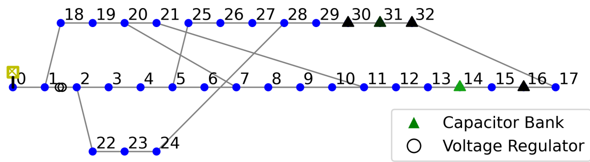

Fig. 1. Upgrades made on the 33-bus network by the proposed algorithm.

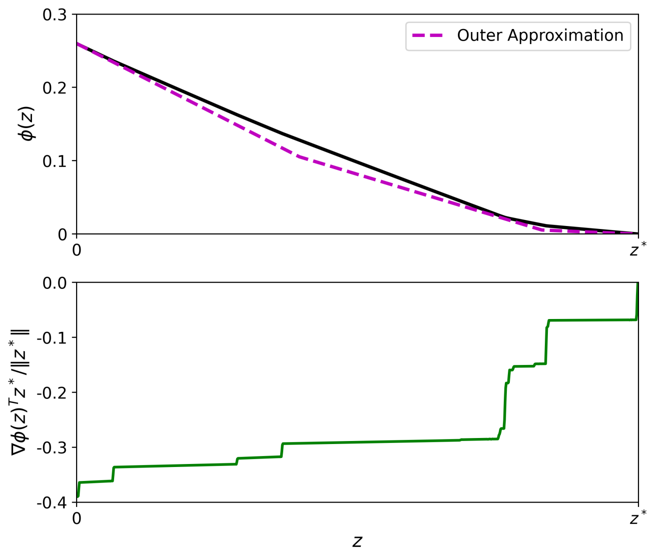

Fig. 2. The value function and its directional derivative along the one-dimensional subspace in the direction of the OA solution z^*.

are often faster than BnB for convex MINLPs, they perform well here too. That is, the results indicate that the NLP solves are the computational bottleneck: OA, even though it has to solve MILP subproblems, runs much faster than BnB due to having to solve an order of magnitude fewer NLPs. Notice that the performance gap was particularly large for the 33-bus network, which is the only meshed network of the five.

B. Illustrative demonstration

Recall that the foundational assumption of the OA algorithm is that the value function ϕ is well-approximated by a convex function. To build intuition and further support this hypothesis, we include an illustrative demonstration of OA on the 33-bus network [9], which is the only meshed network of the five evaluated in Section V-A. Figure 1 shows the upgrades made by OA in the experiments from Section V-A. Darker triangles indicate larger z_{cap}. Figure 2 plots the one-dimensional cross section of ϕ (upper subplot) and its directional derivative (lower subplot) from the zero upgrade vector to the solution z^* depicted in Figure 1. Note that while z is constrained to discrete values in the DSEP problem, Figure 2 shows non-integer investments for the sake of illustration. The upper subplot also includes $\max_{\tilde{z}\in\mathcal{S}}\{\phi_{\tilde{z}}(z) - b_{\tilde{z}}\}$, which for this cross-section is an underapproximation as desired.

Figure 2 gives some geometric intuition behind the reliable performance of OA in the experiments from Section V-A. Between 0 and z^*, there are a series of jump discontinuities in the directional derivative. Though not evident from the figure, further analysis reveals that these jumps occur when the set of active voltage violations in (9d) changes. In the regions between the jumps, ϕ changes relatively little. In other words, in experiments so far, any concave curvature in ϕ induced by the nonlinear power flow equations has been dominated by the comparatively large convex effect of relieving voltage violations, meaning that the first-order approximation of ϕ around a given z is likely to underapproximate almost all other points. This observation is consistent across test cases and provides some empirical justification for application of Algorithm 1.

VI. Conclusion and Future Work

We propose a new algorithm for the distribution system expansion planning problem using nonlinear power flow models, which can represent meshed networks and networks with non-negligible losses. The algorithm is based on an outer approximation heuristic which reformulates the full MINLP as a bilevel optimization problem. The inner problem solves a continuous OPF problem to generate piecewise-linear convex approximations of the DSEP objective function, which in turn constitute the MILP outer problem. We demonstrate strong computational performance relative to an off-the-shelf solver on a number of IEEE test systems. The empirical effectiveness of the proposed technique motivates a theoretical exploration of the structural properties of the value function. In particular, it may be possible to derive conditions under which ϕ is globally convex, e.g., by choice of a particular norm in (11).

References

[1] S. H. Low, "Convex Relaxation of Optimal Power Flow—Part II: Exactness," *IEEE Transactions on Control of Network Systems*, vol. 1, no. 2, pp. 177–189, Jun. 2014.

[2] A. Tabares, J. F. Franco, M. Lavorato, and M. J. Rider, "Multistage Long-Term Expansion Planning of Electrical Distribution Systems Considering Multiple Alternatives," *IEEE Transactions on Power Systems*, vol. 31, no. 3, pp. 1900–1914, May 2016.

[3] A. Moreira, M. Heleno, A. Valenzuela, J. H. Eto, J. Ortega, and C. Botero, "A Scalable Approach to Large Scale Risk-Averse Distribution Grid Expansion Planning," *IEEE Transactions on Power Systems*, vol. 39, no. 1, pp. 2115–2128, Jan. 2024.

[4] N. Amjady, A. Attarha, S. Dehghan, and A. J. Conejo, "Adaptive Robust Expansion Planning for a Distribution Network With DERs," *IEEE Transactions on Power Systems*, vol. 33, no. 2, pp. 1698–1715, Mar. 2018.

[5] P. Belotti, C. Kirches, S. Leyffer, J. Linderoth, J. Luedtke, and A. Mahajan, "Mixed-integer nonlinear optimization," *Acta Numerica*, vol. 22, pp. 1–131, May 2013.

[6] D. P. Bertsekas, *Nonlinear programming*, 3rd ed. Belmont, Mass: Athena scientific, 2016.

[7] A. Wächter and L. T. Biegler, "On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming," *Mathematical Programming*, vol. 106, no. 1, pp. 25–57, Mar. 2006.

[8] O. K. Gupta and A. Ravindran, "Branch and Bound Experiments in Convex Nonlinear Integer Programming," *Management Science*, vol. 31, no. 12, pp. 1533–1546, Dec. 1985.

[9] M. Baran and F. Wu, "Network reconfiguration in distribution systems for loss reduction and load balancing," *IEEE Transactions on Power Delivery*, vol. 4, no. 2, pp. 1401–1407, Apr. 1989.

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