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Spatial Reasoning in Elementary School Children’s Geometry Insight: A Neo-Piagetian Developmental Proposal

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Spatial Reasoning in Elementary School Children's Geometry Insight:

A Neo-Piagetian Developmental Proposal

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by

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ABSTRACT

Spatial Reasoning in Elementary School Children's Geometry Insight:

A Neo-Piagetian Developmental Proposal

by

David Allen Hallowell

Following Robbie Case's branch of neo-Piagetian theory, this dissertation proposed a central geometric structure for the domain of geometry. In consultation with a professor of mathematics skilled in early childhood mathematics education, a series of geometry investigations was designed, and manipulatives of special triangles were 3D printed for use in the study. Forty-eight children from 2 California charter schools participated in the study. Sixteen students from each of second-, fourth-, and sixth grade participated. Children completed the Figural Intersections Test to assess *M*-capacity (mental attention), the WISC-V block design subtest to assess spatial visualization, and the WISC-V visual span test to assess visuospatial working memory. Children completed an identical series of pretest and posttest spatial items to assess learning over the course of the sessions. Geometry investigations were video recorded, transcribed and analyzed for a subset of students. Two students from each grade were selected for analysis. The maximal variation method of purposeful sampling was used as the selection framework (Emmel, 2014). For each grade level, the student who showed the lowest initial performance and no improvement from pretest to posttest was contrasted with the student who showed the most improvement.

Children in the no-improvement group used fewer words to describe their activities when prompted, showed more constrained experimentation during the investigations, and exhibited fewer examples of hierarchical learning instances than their peers who did improvement from pretest to posttest. Theoretical and educational implications are discussed below, as well as limitations recommendations for future studies.

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Chapter 1: Introduction

Although it has traditionally been neglected as an important early competency, spatial reasoning ability represents a promising aspect of development for improving the educational experiences of children everywhere. There is a strong relation between persistence in STEM fields and spatial reasoning ability (Wai, Lubinski, & Benbow, 2009). Of special importance from an education perspective, spatial reasoning is not a fixed ability but one that can be improved through training, and those with the lowest initial spatial ability show the greatest gains in spatial reasoning ability when partaking in spatial interventions (Uttal et al, 2013). Spatial reasoning is especially important at the novice level, serving as a gatekeeper to STEM disciplines (Uttal & Cohen, 2012). Rigorous statistical analysis of empirical data suggests that the number of engineers in the United States could be doubled by improving spatial training early in education (Uttal & Cohen, 2012).

Many studies in recent years have found a strong link between spatial reasoning and mathematical abilities (e.g., Giofre, Mammarella, Ronconi, & Cornoldi, 2013; Mix et al., 2016; Simms, Clayton, Cragg, Gilmore, & Johnson, 2016), but most of the research is presented without a robust developmental theoretical framework. Without such a framework, observed developmental patterns and individual differences may be difficult to conceptualize. Concerning spatial reasoning in geometry, and building on the earlier influential work of van Hiele (1986), Clements, Wilson, and Sarama (2004) have suggested a research-based learning trajectory of geometric shape composition that is helpful in this regard. Children exhibit increasingly sophisticated levels of visualization abilities when performing on a concatenation task consisting of increasing levels of visualization (e.g., matching edges, rotating, reflecting, using mental images), but the trajectory is intended to be normative and descriptive for educators. It is not intended as a causal account of how children construct knowledge in this domain. For example, the trajectory does not account for the expansion of working memory capacity that occurs with biological maturation, nor does it propose to explain how children consolidate and reorganize their knowledge during important shifts in thinking.

In a masterful summary of the literature on spatial development in mathematics, Clements and Sarama (2007) subsequently proposed the notion of *hierarchic interactionalism*, whereby domain specific and domain general processes interact over development to usher mathematical understanding from intuition toward rich, self-reflective insight. These authors offer a compelling list of features and phenomena characteristic of cognitive development in the domain of mathematical thinking. This includes features such as developmental improvement (general and specific); hierarchic, syncretic integration of specific knowledge features; concrete-to-abstract tendency in general insights; the influence and limiting of prior knowledge on new knowledge construction; diversity in the timing and order of developmental apprehension; the strengthening of general and differentiated knowledge structures as metacognitive awareness develops in mathematical insight; the role of environment and culture in the development of mathematical insight; and finally the emergence of learning trajectories.

Clements and Sarama's (2007) work is discussed in greater detail below, but is sufficient to here to point out that the construct as proposed remains largely descriptive (see Pascual-Leone, 1987 for a distinction between descriptive and causal accounts of cognitive development). This framework does not account for the kind of underlying cognitional factors that would allow for predictions about the cognitive performance of individual children, such as working memory capacity. Without the presence of such hidden factors, it is difficult to explain the developmental asynchronicity observed between children that is implied by their postulate that children experience different developmental courses based on a child's individual characteristics and environmental milieu. For example, what accounts for the oft-observed pattern (e.g., Case, 1998) that under certain conditions an individual child may experience rapid learning and advanced performance, while in other contexts learning and performance gains of the same mathematical idea may take quite some time to accrue? Why do some children seem to pick up the significance of mathematical symbols and images quickly, while others languish to discern mathematical ideas in context? The answer to these questions would be highly valuable to educators, such as those who seek to assist children experiencing dyscalculia at various stages, but a robust account of human cognition would be required to begin to be able to effectively unravel these mysteries.

Another important characteristic of the literature on spatial reasoning and mathematics is the dearth of studies investigating development in the upper elementary school years. While number sense in the earliest school years is the strongest predictor of academic achievement across the elementary years (Duncan et al., 2007), and high school is a critical time for establishing a stable STEM identity (Wang, 2013), the upper elementary school years are a time when many students determine whether they are "a STEM person" (Maltese, Melki, & Wiebke, 2014). Despite this fact, much of the extant research focuses either on pre-Kindergarten through second grade, or on middle and high school mathematics. Third- through sixth-grade mathematics is often included in hypothesized learning trajectories, but less frequently researched directly. Numerous research summaries and books are available on early childhood mathematics, but a cursory search of "middle childhood mathematics" on

Google Scholar or Amazon yields the same early childhood, K-second resources one finds with the search term “early childhood mathematics.” Prior to Clements and colleagues’ work investigating early childhood, the majority of the research focused on middle school and older students’ geometric reasoning (Clements, Swaminathan, Hannibal, & Sarama, 1999).

While focusing on improving the mathematical experiences of children in early childhood has been a laudable and important movement, there may be particularities of development unique to middle childhood and beyond that constitute missed opportunities for supporting students’ development if we are to neglect this important phase of mathematical development. This dissertation posits that one such example resides in an area pointed out by Clements and Battista (2002). As children have mastered arithmetic and begin to exercise burgeoning working memory capacity to investigate the metacognitively-demanding insights of more advanced mathematical thinking, a pernicious confusion about what counts as knowing undermines many children’s development in mathematics (Clements & Battista, 2002). Mathematical insight is unique from common sense insight in that knowledge is often tentative, and the standards by which veridical judgments are arrived at often seem amorphous and elusive to students who are constructing new links between existing and unfamiliar insights for the first time. New knowledge reorganizes prior knowledge and opens new horizons of mathematical understanding (Lonergan, 1993), but this process is undermined by a lack of confidence in the fruits of earlier mathematical experiences (Clements & Battista, 1992). Additionally, what is developmentally appropriate as proof in elementary mathematics shifts as children proceed through the grade levels (Stylianides, 2007). The pedagogical exigencies are only intensified by the fact that professional mathematicians themselves hold diverse views about what constitutes proof in mathematics, and there may be differences in these standards along the dimension of applied versus pure mathematics practitioners (Inglis, Mejia-Ramos, Weber, & Alcock, 2013). Sense-making in mathematical thinking becomes an increasingly complicated affair as children advance through the school years.

While Clements and Sarama's (2007) presentation of hierarchic interactionalism contributes an adept evaluation of classical Piagetian theory in light of contemporary research findings, neo-Piagetian theory is not included in the discussion. This is disappointing considering neo-Piagetian theorists such as Pascual-Leone, Case, Okamoto, Morra and other collaborators have published important work that is relevant to children's developing mathematical knowledge (e.g., Okamoto & Case, 1996; Morra, 2012). Additionally, there is a striking degree of correspondence between Clements and Sarama's (2007) articulation of the hierarchic interactionalism perspective and Case's (1998) review of the neo-Piagetian perspective on intellectual development a decade earlier. Drawing on Piaget's notion of reflective abstraction (see Dubinsky, 2002), Case (1998) proposed the existence of *hierarchical learning loops* to account for knowledge consolidation and stage transition in cognitive development. Case's construct was grounded in a fully-articulated cognitional theory (see Pascual-Leone, Johnson, & Agostino, 2010), which allowed him to model how the interaction between specific and general knowledge interact to facilitate learning in low-exposure learning contexts. This model reconciles the empiricist assertion that knowledge is generally transferred from one context to another (associative learning), with the rationalist assertion that knowledge is abstracted from specific contexts into general, context-free insights (attentionally mediated learning; Case, 1998). It does so by proposing that changes in understanding flow bidirectionally in a learning loop, with new insights fed “upwards” into general understanding, and in turn, general insight facilitating the rate of specific insight in a “top-down” process. This coupling and restraining in learning takes place in a specified cognitive context that will be described in greater detail below. Suffice it to say that the neo-Piagetian account of intellectual development offers a rich context for understanding the development of mathematical thinking, and one that would bolster the theory that Clements and Sarama (2007) have provided.

This dissertation endeavored to contribute to what is understood regarding the developmental course of spatial and mathematical reasoning in geometry by conducting exploratory research with 48 children in second-, fourth-, and sixth grade. Through interdisciplinary collaboration we created a series of spatio-mathematical investigations using 3D-printed triangle manipulatives with properties and contexts aimed at eliciting key ideas in early geometric reasoning. Over the course of a week, each participant allowed us to video record their reasoning in cognitive clinical interview sessions (Ginsburg, 1997). One of these investigations was selected for analysis for this dissertation, and results from the Figural Intersections Test (FIT) were used to select a sub-sample of participants for analysis. Video recordings were transcribed and coded for exploratory analysis from a neo-Piagetian perspective.

Specifically, neo-Piagetian theory suggests several important cognitive factors that may offer a pedagogically fruitful lens through which cognitive development in this area may be understood. First, neo-Piagetian theory predicts that due to consolidations in *M*-capacity (i.e., working memory), 10-year-old children often exhibit evidence of qualitative shifts in thinking across various cognitive domains (Pascual-Leone, Johnson, & Agostino, 2010). In this study, the theory predicts that children of this age will start to demonstrate reorganization of geometric understanding to make qualitatively advanced geometric insights (e.g., noticing how angle measure factors into shape classification in a systematic fashion). Case and Okamoto (1996) predicted that children at age 10 are in transition from problem-solving using two instances of core knowledge in a domain, to being able to generalize core understanding to detect general principles of learning contexts for the given domain. Case (1998) proposed that this was achieved in part by the hierarchical learning loop, where new general insights reorganize the significance of specific insights. This study made special effort to identify potential examples of this process as children completed the geometric investigations we gave them.

As part of this exploratory study, several other cognitive dimensions that are theorized by neo-Piagetian theory to contribute to cognitive performance in spatial reasoning in geometry were administered, including intrinsic-dynamic spatial reasoning ability (i.e., spatial visualization), *M*-capacity (i.e., working memory capacity), visuospatial working memory, and field-dependency (i.e., cognitive style associated with attending to relevant features and inhibiting irrelevant perceptual features in problem solving).

A pre- and posttest set of concatenation tasks were administered to determine whether engaging in these investigations would lead to improvement in the kind of tasks they purport to facilitate. Additionally, analyses contrasted the student who experienced the greatest improvement from pretest to posttest with a student who did not improve for each grade level. While these examples are akin to case studies, considering the data this way would offer reflection points for future research aimed at helping students profit from these sorts of educational activities.

Several questions guided the research design and analysis: 1) What do the activities and reflections of children engaged in open-ended geometric inquiry tell us about core knowledge in the domain of geometry at different ages? 2) What might evidence of reflective abstraction and general versus specific insights look like in grade-school geometry? In other words, do we see evidence of specific insights that arise directly from the activities engaged in (e.g., this skinny shape does not make a complete table because it does not fit evenly around a point) get coupled with general insights (e.g., shape x always has these properties) to drive geometric understanding across investigations. A third, ancillary research question was whether the series of investigations conducted by children in this dissertation would facilitate performance on the pretest/posttest items that were designed to draw on insights nurtured by these investigations. I next review the existing research for a deeper engagement with what is known and unknown in children's burgeoning geometric reasoning.

Chapter 2: Literature Review

To achieve the aims stated in the introduction, I review extant research relevant to several pertinent questions: What does the literature say about the development of core knowledge in geometry, and how does spatial reasoning fit into this story? How might neo-Piagetian theory contribute to unresolved developmental questions in this domain? What does the literature say about the role of reflection in mathematical insight and the development of mathematical thinking? The relevant prior research is discussed below.

Spatial Reasoning and Geometry

Geometry is an important domain of primary mathematics education. It provides an explicitly spatial mode of entry into the wide universe of mathematical thinking that other areas in mathematics do not readily afford (Hilbert & Cohn-Vossen, 1999). The Cartesian plane opens up coordinate thinking in Algebra, with linear modeling on such planes constituting the basis of statistical analysis. Geometric thinking forms the basis of many STEM careers, such as chemistry, astrophysics, or just about anything else that can be quantized along a space-time vector. At the same time, educators in the U.S. are concerned about students' performance in geometry, especially when our national average is compared with the international average (*TIMSS*, 2012. Shape and space component, U.S. mean = 463, *SD* = 97; International mean = 490, *SD* = 98 (TIMSS data for 2015 not available at the time of this report); Retrieved from http://nces.ed.gov/surveys/pisa/idepisa/). Having reviewed a series of studies showing poor procedural and conceptual knowledge in geometry among high-achieving high school graduates, Clements and Battista (1992) asserted that what often underlies this undesirable outcome is a lack of authentic sense-making in geometric thinking. Students rarely make key generalizations across specific examples, and they often lack confidence about what constitutes truth in geometric reasoning. The result is that a flow of discreet, disconnected facts fails to converge into meaningful understanding, and are not encoded into long-term memory (Clements & Battista, 1992).

Additionally, Duval (2006) pointed out that geometry represents a special challenge pedagogically, since the geometric representations that students are exposed to never convey the idealized mathematical object in a single example. Instead, students must abstract essential mathematical properties across a series of examples, all the while ignoring accidental properties. Every representation is fraught with what Pascual-Leone (1979) called misleading cues. To be successful in geometry, a student must learn to actively inhibit misleading cues, while activating facilitating cues in the flow of attention. To complicate matters, Clements, Wilson, and Sarama (2004) have shown that what can be considered characteristic geometric insight of one developmental level differs in important ways at another (e.g., a square as its own shape and a square as a special case of rectangle). Clements and Battista (1992) argued that verification of truth, or what counts for knowledge in geometric sense-making, is critical for consolidating knowledge into retrievable schemes of understanding, and that developmentally appropriate criteria for truth in mathematics changes with age (Stylianides, 2007).

Clements et al. (2004) proposed an early learning trajectory that is domain-specific to geometry. These researchers elaborated the van Hiele model[1] to integrate shape composition

[1] van Hiele (1986) proposed a highly influential development of geometric thinking. Along this continuum, Level 1 children begin by making holistic judgments based on the visual features of geometric representations. In Level 2, children notice and describe certain parts of the representations, and how these relate spatially to one another. At Level 3 children start to make inferences about the presence of one property necessarily implying another. At Level 4 students can conduct axiomatic, formal proofs.

and decomposition actions into their model. They pointed out that work in this area continues to be hampered by a lack of consensus in the field for a commonly accepted learning trajectory. This may be due to the vexatious problem of articulating a developmental trajectory that is sufficiently differentiated for use by researchers, yet descriptively mundane enough to be of benefit to educators. Their 2004 article proposed such a developmental trajectory in geometry, based on their prior empirical research.

Clements, Wilson, and Sarama (2004) extended their earlier work with the proposal of the precomposer level, which occurs prior to van Hiele Level 1. They also enriched the original model by including shape composition and decomposition operations into the developmental scheme. These researchers administered several geometry measures to 72 children from preschool through second grade (ages 3-7 years-old), investigating the first 4 levels of their 7-level model. *Precomposers* represent the earliest of the first 4 stages, where children manipulate shapes but do not combine shapes to create a larger shape, and have trouble matching shapes to a frame. *Piece assemblers* are similar to prescomposers in their geometric activities, but they will concatenate to simple shapes and frames via a random "picking and discarding" strategy. *Picture Makers* effectively use sides to concatenate freely with rotation and flipping, but with little anticipation (i.e., trial and error) and without effective use of matching angles. The fourth stage is comprised of *shape composers*, who employ spatial visualization to anticipate matching both side lengths and angles profitably, flipping and rotating to concatenate in an anticipatory manner to fill frames and cover regions. That study found support for the first 4 levels of the developmental trajectory, with first graders either at the picture maker stage or in transition between picture maker and shape composer. In an earlier study, this research group found support for the latter 3 stages (*substitution composer*, where component shapes such as 2 trapezoids substitute a larger composite shape such as a hexagon; *shape composite iterator*, continuing a pattern to complete a covering; and *shape composer with superordinate units*, where children coordinate units of units of units [e.g., tiling a large rectangle with 4 2x2 square concatenations]; Clements et al., 1997). Children who have the relevant early spatial experiences may obtain all 7 levels by third grade, but many children advance beyond the third grade without getting the relevant spatial experience to fully develop this kind of thinking (Clements & Sarama, 2007).

As mentioned above, this learning trajectory is intentionally descriptive. It is intended to assist educators in assessing where a child is developmentally in shape composition and decomposition abilities. It is not intended as a causal account of mathematical cognition in geometry.

Newcombe and Shipley (2015) have proposed a 2 x 2 typology of spatial reasoning that theorizes the existence of two primary dimensions of spatial reasoning abilities that provides a useful framework for the kind of visualization acts geometry often requires of students. Along one dimension of spatial reasoning abilities are intrinsic versus extrinsic spatial contexts. Intrinsic spatial reasoning involves reasoning about the spatial properties of a single object (e.g., a desk is 1 meter long and three-quarters of a meter tall). Extrinsic spatial reasoning involves reasoning about an object or objects in relation to some frame of reference (e.g., the desk is adjacent to the window, behind the chair). Comprising the other dimension is static versus dynamic spatial reasoning. Static spatial reasoning involves situations where the spatial coordinates involved do not undergo any kind of spatial change, either intrinsically or extrinsically. In dynamic contexts, spatial information undergoes some kind of transformation, either intrinsically or extrinsically. This proposed classification system arose out of several prior meta-analyses of spatial reasoning, building on that prior work to extend the theoretical model to cover a broader range of studies on spatial reasoning. While the application of this model to educational contexts presents some particular challenges (see Davis & Spatial Reasoning Study Group, 2015 for a discussion), it represents an important contribution to the field of spatial reasoning, and a useful heuristic tool for educators to think about the various cognitive activities that children engage in in the classroom. This dissertation draws upon the distinction between static and dynamic spatial reasoning to classify observations made of participants engaging in geometric investigations.

In a later publication, Clements and Sarama (2007) presented their perspective of *hierarchic interactionalism*, which asserts that mathematical insight develops through the syncretic interaction of the child's existing knowledge bases and newly acquired knowledge. The authors summarized the perspective thusly: "Mathematical ideas are represented intuitively, then with language, then metacognitively, with the last indicating that the child possess an understanding of the topic and can access and operate on those understandings" (p. 464). This assertion leads to the question, how do the mathematical ideas in geometry emerge in development? Do children construct space, moving from egocentric spatial mapping (i.e., generating spatial maps in immediate, relative frames of reference) to allocentric constructions (i.e., distal spatial maps that include objects in space as related to each other), as Piaget proposed in the topological primary thesis (Piaget & Inhelder, 1967)? Reviewing the literature, Clements and Sarama (2007) presented research confirming that near space develops prior to far space, but that infants display more ability than Piaget conjectured and adults less. They pointed out that the topological primary thesis does not seem to be a useful construct for mathematics educators.

Taking a different route in their analysis, these researchers find that geometric reasoning is the result of combining knowledge from two separate subsystems: a *what* system (originating in the inferior temporal cortex ventral pathway) that makes use of property knowledge to determine what a shape is, and a *where* system (originating in the posterior parietal cortex ventral stream) that yields visuospatial information about the spatial features and location of an object. Their review of children's burgeoning notion of shape and spatial visualization in geometry contexts supports their hierarchical interactionalism perspective. They clarified that the literature supports the notion of *improvementive hierarchization* in geometric thinking, whereby children develop increasingly efficient and sophisticated knowledge networks across initially separate networks (Clements & Sarama, 2007).

In the case of geometry, children begin connecting visual, perceptual knowledge explicitly with verbal, semantic knowledge of shapes and their mathematical properties. These separate structures must be consolidated and reorganized into new, syncretic structures if children are to advance in their geometric thinking in middle childhood. Regarding shape analysis in early childhood, young children begin by comparing the shape under consideration to existing imagistic schemes that arise out of a small set of highly-regular prototypes (e.g., equilateral triangle with base parallel to the child's visual horizon) to more complex, mathematically-consistent prototypes (and corresponding sets of prototypes) as children mature in geometrically rich educational environments. For example, when learning to classify triangles, one preschool student held a stable visual scheme for a triangle, but developed a separate, unstable, semantic scheme for "a 3-sided shape," with these 2 schemes conflicting with one another until she was able to constrain and unify the two schemes (Spitler, Sarama, & Clements, 2003). These authors argued that Level 1 geometric thinking in the van Hiele model should be considered less a "visual" level (as it was originally proposed) and more of a "syncretic" level (as Clements (1992) proposed subsequently), since these types of knowledge structures coexist from early in development, and the emergence of effective spatial, geometric reasoning requires children link these knowledge networks.

This dissertation explored the geometric reasoning of second-, fourth-, and sixth-grade children as they undertook the same set of tasks in a cognitive interview setting (see Ginsberg, 1997 for the framework utilized here). As children begin to attend to richer analyses embedded in geometric contexts, the knowledge networks they construct are such that geometric:

> objects are 'neither words nor pictures' (Davis, 1984), but a synthesis of verbal declarative and rich imagistic knowledge, each interacting with and supporting the other. The question, therefore, should not be whether geometric thinking is visual or not visual but rather whether imagery is limited to unanalyzed, global visual patterns or includes flexible, dynamic, abstract, manipulable imagistic knowledge (Clements, Swaminathan, Hannibal, & Sarama, 1999). (Clements & Sarama, 2007, p. 506)

By fourth grade, many children should have achieved the highest of the composition/decomposition levels, shape composer with superordinate units. By age 10, children who have received consistent, high-quality experiences in geometry education should move on to what Clements and Sarama (2007) called the metacognitive phase of hierarchical interactionalism. Children should be capable of reflecting on their activities, as they typically have had important early experiences with shape composition and decomposition activities. It is the hypothesis of this dissertation that where this is not the case, the content of children's reflections during geometric activities should differ from the reflections of children who have reached the shape composer with superordinate units level of geometric thinking. Children who are not able to iterate with units of units were expected to understand the visual field differently than their more advanced counterparts.

Hsiu-Lan, De-Chih, Szu-Hsing and Der-Bang (2015) administered a measure of van Hiele levels that they designed and validated to 5,581 elementary school students in grades 1-6 in Taiwan in order to ascertain a distribution of van Hiele levels of thinking across that school system. Since Taiwan's elementary students are top-five performers globally on the geometry section of the *International Mathematics and Science Study* (TIMSS), their study results should be interpreted as developmentally advanced. They found that first and second graders assessed at Level 1, third and fourth graders assessed at Level 2, and only fifth and sixth graders performed at Level 3. These results suggest that fourth grade children's geometric thinking should be characterized predominately by van Hiele's Level 2 thinking, which is comprised of geometric insight that is largely descriptive in nature and highly embedded in the context at hand. It is possible that some Level 3 thinking, characterized by logical relations between features in Euclidian geometry, may be observed in advanced students who are in transitions. Age 10 was selected as the age of interest for this study, since both these data and data from neo-Piagetian theory suggest a qualitative shift in the quality of children's thinking at this age, and this study sought in part to investigate children's knowledge consolidation and reorganization in geometric sense-making.

Juan Pascual-Leone and Robbie Case's Neo-Piagetian Theories

How does the human brain generate insight? How is it that the human organism generates meaning from the complex environments we inhabit? How does one explain the learning paradox that human beings can generate novel responses to new situations for which an individual has no prior experience? Additionally, how does this capability unfurl over the developmental course of the human organism? Building on Baldwin's (2018) prior work, developmental psychologist Jean Piaget undertook several of these questions in presenting his theory of genetic epistemology.[2] While working in Piaget's developmental laboratory, a young Juan Pascual-Leone noticed that when children's cognitive activation capacity was considered, cognitive development actually corresponded with Piaget's substages (Cardellini & Pascual-Leone, 2004). "Le Patron," as Piaget was accustomed to being called, was not interested in discussing Pascual-Leone's observation. Pascual-Leone was subsequently banished to New York to complete his doctoral studies in psychology (he was already a medical doctor at that time) under famed learning-scientist Herman Witkin. Piaget signed off on Pascual-Leone's dissertation without any personal communication.

In the ensuing years, the problems that Pascual-Leone had tried to warn his mentor about became apparent to the broader scientific community, with most developmental scientists moving onward to other theoretical perspectives. Many of Piaget's fundamental axioms about development were abandoned, such as the notion of universal stages in human cognitive development. However, despite his falling out with his mentor, Pascual-Leone believed there was much to be retained from Piaget's ideas about development. Instead of abandoning Piaget's theory, Pascual-Leone was the first to propose its revision. He

[2] He did not directly address the third question regarding the learning paradox. See Pascual-Leone for a discussion of this issue.

leveraged his medical background and concomitant knowledge of neuroscience to propose his *Theory of Constructive Operators* (TCO; Pascual-Leone, 1979). This branch of neo-Piagetian theory was presented to account not only for the kind of conscious, effortful learning that educators often focus on, but proposed a set of silent or hidden operators. These silent operators are so called because they are content-free cognitive factors that structure and pattern conscious, content-laden operations. What is proposed in the TCO forms a critical basis in offering a causal account of cognitive development and offers the mathematical cognition community a promising avenue for understanding intellectual development. The theory is worth exploring in depth, and it informed the core ideas of this dissertation.

Pascual-Leone (1987) viewed classical Piagetian theory (and indeed, many competing branches of neo-Piagetian theory) as engaging in a kind of neuropsychological reductionism, whereby psycho-logical structures are posited as the fundamental causal level of human cognition. But these logical structures give rise to a specific learning paradox: how is it that the human organism can generate novel, successful performance in misleading situations, if successful performance depends upon already-existing psycho-logical schemes for their generation? Pascual-Leone argued that classical Piagetian theory is not up to the task of explaining this learning paradox, instead turning to Marxist historical analysis for explanatory principles. From this perspective, developmental processes can be understood as a series of dialectical tensions that are in opposition to one another. As these processes interact, new phenomena emerge in a probabilistic fashion. He summarized the three laws of dialectics as follows: 1) quality and quantity flow back and forth in transformational processes; 2) opposing principles collide and combine in the formation of new entities; and

3) a chain of negations follows so that any subsystem may be subsumed by a higher system for that higher system's own functioning. Pascual-Leone's TCO is a dialectical theory, in that the subject and the metasubject account for the individual's performance on any given task. The subject is the individual with his or her set of existing schemes (or "repertoire" of prior learning) based on prior experiences, and the metasubject is the situation-free set of silent operators that acts upon and generates the schematic outputs in the subject. These silent operators are a set of individual factors that influence how schemes and sets of schemes are activated in the individual's conscious field, and they are expressed in enough detail to be operationalized in task analyses. Importantly, this theoretical approach draws upon dialectical principles to explicitly address the issue of multiple systemic levels (see Marr's [1982] famous analysis), such as the fact that the process occurs in a milieu of activated and inhibited neural networks to generate the subject's interaction with the environment.

One example of a task that Pascual-Leone and Goodman (1979) have characterized as misleading is the Embedded Figures Task (EFT), where the individual is shown a figure that he or she is to disambiguate from a more complex composite figure that is shown subsequently. Pascual-Leone provided a detailed analysis of the task demands on the metasubject, with particular attention to how the task creates a misleading situation via conflicting cues. While his analysis was impressively detailed, he did not provide empirical evidence aligning with his analysis in that paper. This task analysis was intended simply to demonstrate a clear case where a subject's existing psycho-Logical structures would be insufficient to explain successful task performance, and it is not difficult to imagine others like it, such as when 9 year-olds encounter fractions and must suppress the overwhelming temptations to use the magnitude of the denominator alone to determine the magnitude of the ratio (e.g., Siegler et al., 2010).

Pascual-Leone and Baillargeon (1994) created and validated the Figural Intersections Test (FIT) as a measure of a child's *M*-capacity in the presence of misleading cues. They modeled ten theoretical structural positions predicting outcomes of interactions between mental power of the child and mental demand characteristics of the task. The FIT is a scale that requires children to disambiguate the intersection of overlapping geometric figures. Items indicative of greater *M*-capacity feature greater geometric complexity and more misleading cues than items corresponding to lower *M*-capacity. FIT was given to 616 children ages 5-14 years old. Latent class analysis was conducted, and the model was shown to fit the data as predicted. The data matched a latent-distance model supporting the ten predictions, *M*-capacity of children approached interval scales when treated as classes, and performance was observed in the trade-off manner that the theory predicted.

Pascual-Leone and Goodman (1979) offer their solution to the learning paradox in the TCO, represented in Figure 1. It is beyond the scope of this literature review to provide a comprehensive account of Pascual-Leone's theory, but Figure 1 depicts the silent operators that give rise to the subject's performance. While some information processing models imply information bouncing along through some sort of factory machine, Pascual-Leone points out that this is not to be implied by his TCO model. Instead, activation patterns rapidly occur on the neural substrate as needed by the subject, not necessarily in a linear fashion (in many cases explicitly not so), and according to the various principles of the organism expressed in his theory. Especially important for the scope of this paper is the notion of *M*-power, which is a limited boosting capacity that can hold a given scheme or set of schemes (superordinate schemes) in the subject's conscious field at one time.

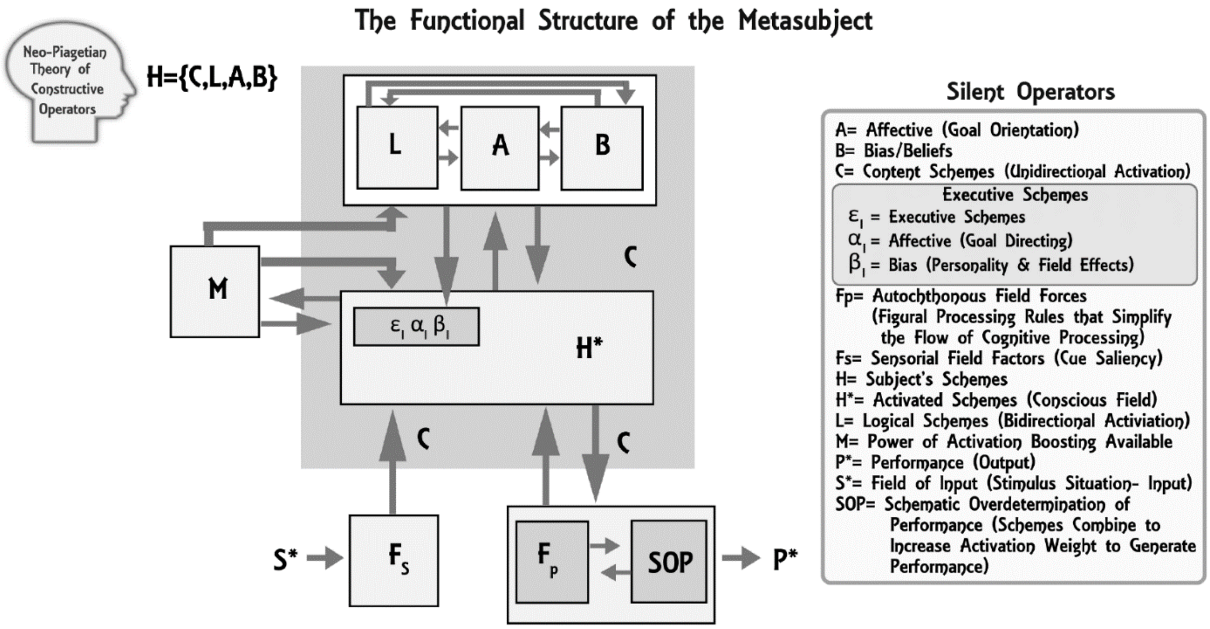

Figure 1. Silent Operators in Pascual-Leone's Theory of Constructive Operators. Adapted from Pascual-Leone, J., & Goodman, D. (1979). Intelligence and experience: A Neo-Piagetian approach. *Instructional Science*, 8(4), 301-367.

What is especially interesting from a developmental perspective is that M-power increases with biological maturation of the brain. When conducting task analyses in Piaget's laboratory, Pascual- Leone noticed that age-typical increases in the maximum number of schemes a child could hold active at once corresponded with Piaget's substages, as depicted in Figure 2 (Cardellini & Pascual- Leone, 2004). The TCO predicts that after emerging from the sensorimotor stage, children obtain a single increase in boosting capacity for every two years of maturation.

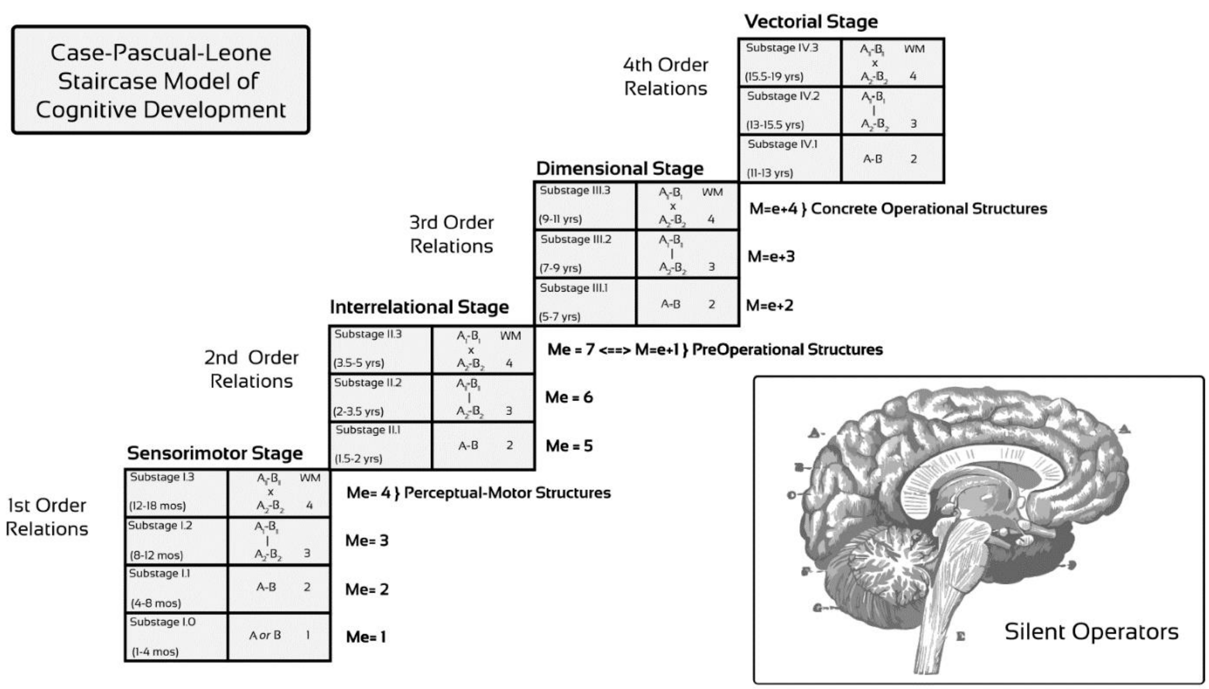

Figure 2. Case-Pascual-Leone Neo-Piagetian Model of Cognitive Development. Adapted from Pascual-Leone, J., Johnson, J., & Agostino, A. (2010). Mental attention, multiplicative structures, and the causal problems of cognitive development. In M. Ferrari & L. Vuletic (Eds.), *The developmental relations among mind, brain and education* (pp. 49-82). Amsterdam, Netherlands: Springer Netherlands.

Pascual-Leone (1987) used the Rho task to investigate whether this pattern held under experimental observation. For the Rho task, the child turns a crank mounted on a wooden box at varying intervals signaled by a light on the box, knocking down a metal bumper with the same hand that was doing the cranking when cued to do so. The task is named for the spatial pathway traversed by the participant during each trial. The dependent variable in the task is the speed at which the subject's hand moves across the required trajectory. From a neo-Piagetian perspective, there are four discrete components of the latency, two of which are tied directly to processes in executive function as intellective concrete-operational components. Reaction time (i.e., time from green light to starting to turn the crank) and circular time (i.e., circular motion of turning the crank) are motor performance factors. Pause time (i.e., time elapsed while overturning the crank to releasing the crank) and liner time (i.e., time elapsed while overturning the crank to hitting the paddle) because these portions of latency require fine adjustment in every trial to optimize performance time. A task analysis was performed to conceptualize the minimum number of schemas required for optimal performance for each variation of the trials. Testing was conducted at a middle-class public school in Toronto, with 10 children from each age group ages 7-12 participating. Latency times were found to decrease as a two-step function of age as the model of *M*-capacity growth predicted (Pascual-Leone, 1987).

While Pascual-Leone's theory offered a plausible model for how individuals construct insights in specific contexts across the developmental span, scholars in this tradition have not given intellectual development in educational contexts a priority of focus in subsequent work. It is important to note this, given Piaget's difficulty explaining horizontal decalage (i.e., "horizontal take-off"). This is the often-observed phenomenon where children master a form of thought in a given domain but fail to readily transfer the insight across to structurally similar problems in other domains and contexts (Case, 1985). While Pascual-Leone's theory offers a casual account of how new knowledge is created despite the misleading cues individuals often face in learning contexts, Case's (1998) theory provides an account of how knowledge develops within domains. This domain specificity is embodied in the construct of central conceptual structures, which are content-specific, core aspects of domain-specific knowledge that develop in a similar pattern due to the working memory changes that both Case and Pascual-Leone observe in their research (Case, 1996a). Prior to Case's untimely passing in 2001, three domains were mapped: spatial, numerical, and social/narrative. Citing Gardner's (1983) work on multiple intelligences, Case (1998) himself proposed that there are many domains of human knowing that may be mapped in in this fashion.

Case (1985) evaluated many developmental studies across domains, whereby he showed a clear pattern in intellectual development. He argued for the existence of executive control structures, which are operationalized as sets of problem situations (e.g., "interesting object disappears while moving downwards," Case, 1985, p.290), objectives (e.g., "Re-locate the moving object, in the center of one's visual field," p.290), and strategies (e.g., "move eyes in downward direction until object comes into view," p.290). The developmental pattern that Case showed seems to imply biologically determined "maximum executive processing loads" (EPLs), whose origin owes to Pascual-Leone's notion of *M*-power according to Case (1985, p.288). While the first example of scanning a visual field for an object of interest is relatively simple, these executive control structures are theorized to combine into superordinate structures in a reliable fashion. Namely, the two simpler executive structures from the prior stage of development are coordinated to form a new superordinate executive structure. It is expanding executive capacity which allows for this, just as in Pascual-Leone's approach. Case (1985) explained the effect on intellectual development with an analogy that is worth quoting at length:

> An analogy from the inorganic world may be useful at this point. Hydrogen and oxygen are two elements which share certain general formal properties in common, yet which are qualitatively distinct, both in terms of their specific properties, and in terms of their internal structure. Under certain conditions, however, they can combine into a superordinate structure. When this takes place, the internal structure of each is subtly altered. In addition, the new entity which results is qualitatively distinct from either of the components which went into it, and can itself function as a unit in a variety of further chemical reactions. What is being proposed with regard to children's thinking is that an analogous change can take place, as the basic mental elements that children consolidate at one stage of development are combined with each other, to form the new units which will be observed at the next. (p.87)

According to Case, it is this expanding capacity for schematic integration of control structures that leads to the qualitative differences in children's thinking that comprise the different stages. The EPL of each stage limits the maximal coordination of structures possible, and this capacity increases with biological maturation. Case specified four learning contexts that can lead to hierarchical integration. These include problem solving, exploration, imitation, and mutual regulation (Case, 1985). Case (1985) did not theorize that the learning context influences the EPL, as even in the context of imitation, a child still has to make an explicit link between two structures for himself or herself to coordinate them effectively. Additionally, the maximum EPL of each substage proceeds in the same formal fashion, with an increase in one capacity space along the developmental trajectory of each substage. What differs between stages is that after the conclusion of each fourth substage, the operations of the prior stage are consolidated into a single coordinated structure, freeing executive capacity for additional cognitive processing. This has the effect of “scaling up” the recursive developmental structure, increasing the number of executive control structures that can be coordinated at one time, and Case holds these consolidated structures to represent qualitatively distinct modes of executive functioning (Case, 1985).

The central conceptual structure of spatial reasoning was also investigated by Case and his colleagues (Case et al., 1996). Since the development of this structure unfolds on the same neural substrate as that of the numerical structure, its structure should look familiar. It is the content that differs. At age 4 precursor schemas exist, one for location on an array (e.g., up, down, left, right, etc.), and one for representing shape (e.g., round sun as a yellow circle). At age 6 these precursor schemas are consolidated into a single mental reference line (the name for the central spatial structure), where children’s representation of space involves figures lined up along a single spatial axis side-by-side. As before, age 8 sees this structure differentiate into two axes, where children will often have figures along one axis in the foreground (closer to the bottom of the page, with objects on this axis appearing larger than others), with a second axis depicting a skyline or a tree in the “background” (closer to the top of the page, with larger objects scaled smaller). Around age 10 children, children generalize these mental reference axes to represent vectors of space diminishing continuously from the viewer’s perspective.

Case et al. (1996) selected a battery of nine spatial measures, then created four levels of items corresponding with the hypothesized developmental levels. These were preaxial (age 4), uniaxial (age 6), biaxial (age 8), and integrated biaxial (age 10). In their first study, Case et al. validated the use of a spatial test battery by adapting 9 previously validated measures into 4 levels of difficulty that corresponded with their hypothesized developmental trajectory for each measure. The battery included seriation, a checkers task, a perspective-taking task requiring children to draw map, a wayfinding task on a playground using a map, a still-life drawing task from a physical display, a landscape drawing task from a verbal description, a handwriting task, a conservation task, and a judgment of lengths task, where sticks of varying lengths were offset from each other and the child was asked to judge the longer stick. Fifty-seven kindergarten and first-grade children completed the battery. Factor analysis revealed that all but two classical Piagetian measures, the conservation task and the length judgment task, loaded on the central spatial structure of interest.

In the second study, a single, 5-year-old male subject was compared to two control subjects to determine whether training in uniaxial thinking would transfer across tasks (Case, 1996). At pretest the participant tested into the preaxial substage. A training session used a drawing task designed to extend spatial representation from preaxial to uniaxial representation. The participant received eight 1-hour training sessions over 11 weeks. At posttest, the mean improvement the child experienced corresponded with 1.5 mental years. The two control children showed little change from pre- to posttest. These results suggested there is some degree of malleability for training the central spatial structure.

In a third study, these researchers administered the battery to older children to test whether the rate of development for the central spatial structure matched the rate of the central numeral and narrative structures that they had already validated (Case, 1996). Ninety-seven children ages 6-10 years old received the spatial battery to determine whether the predicted developmental pattern would hold. The observed developmental pattern matched the rates that had been observed for the other two structures.

Case and colleagues' analysis of the spatial domain raises some interesting questions. The approach to this analysis was to understand how children develop in their ability to represent 3-dimensional space projected onto a 2-dimensional plane. As mentioned above, spatial reasoning refers to a whole set of cognitive abilities. Case and colleagues' assertion that their mental reference line constitutes a core competency in this domain seems reasonable, but it is also difficult to consider how it might generalize to mental rotation of 3D objects, for example, or to predict the location of holes on an unfolded piece of paper after it was perforated in a folded state. These dynamic, stepwise spatial operations seem to increase the *M*-demand placed on the child, yet we know that even children in the uniaxial substage can successfully rotate a polyomino to match a target shape (e.g., see Casey et al., 2008). It seems some other spatial or analytic structure would underlie this competency. Since even infants can perform some mental rotation tasks, it seems that some sensorimotor executives may interact with the more advanced executive structure to facilitate these cognitive activities.

Case's work provided numerous entry points for understanding and modeling the developmental course of spatial reasoning, and his model offered no shortage of testable hypotheses. Additionally, Juan Pascual-Leone's perspective provided a robust model for understanding individual task performance. This is particularly important in a context such as spatial reasoning in geometry, where essentially every single mathematical diagram represents a misleading situation in relation to its ideal referent (Duval, 1999). Additionally, if students can be trained to use their executive schemas effectively in misleading situations as Pascual-Leone proposed, then special attention must be given to how this can be achieved. The combination of these approaches offers the potential for significant gains in understanding the development of children's thinking in cultures with formalized education.

Pascual-Leone, Johnson, and Agostino (2010) proposed a combination model that integrates the theory of constructive operators with Case's neo-Piagetian theory. This combination model is depicted in Figure 2 and has the advantage of providing a theoretical account of how novel performances are generated in real-time, while simultaneously accounting for developmental patterns observed within and across individuals. Pascual-Leone offered an updated version of his TCO, with some minor but important revisions to his notion of *M*-power. For example, while the precise magnitude of each boosting unit remains unknown, Pascual-Leone and colleagues elaborated on two sub-components of *M*-capacity. They posited that mental boosting units *k* have a stronger boosting magnitude than executive, sensorimotor units *e*. Sensorimotor units *e* are associated with lower-level schemas such as coordinating bodily movements in space, and are hypothesized to require less activation capacity than the kind of higher-level neo-cortical activities that *k* units are associated with. Mental boosting units *e* are associated with executive function in the frontal cortex and have a constant value of 6 units in older children. Units *k* emerge at about 3 years old and increase by 1 unit every 2 years until $M = e + 7$ around 15 years old. These authors argue that their model maps nicely onto Case's "staircase" conceptualization, proposing that TCO's predicted development of *M*-capacity aligns with Case's notion of multiplicative structure in formally educated children.

Under Pascual-Leone, Johnson, and Agostino (2010) conceptualization, Case's *sensorimotor* (0-18 mos.) and *interrelational* (1.5- 5 years) stages correspond to the development of M_e capacity. For each substage, an increase of 1 additional activation capacity develops until the maximum 6 is reached. Case's *dimensional* (5-11 years) and *vectorial* (11-19 years) stages produce the larger mental boosting units *k*, again by a single unit for each substage. While this paper is a theoretical proposal given without empirical studies to support it, it provides a testable model for researchers interested in continuing to investigate children's intellectual development from a neo-Piagetian perspective, especially in the context of education.

This dissertation attempted to combine the developmental trajectory in geometry outlined by Clements and Sarama (2007) with the neo-Piagetian cognitional theory of Pascual-Leone & Case (Case, 1998; Pascual-Leone, Johnson, & Agostino, 2010) to propose a new core conceptual structure (CCS) in the domain of geometry. At age 4, children are proposed to have two precursor schemas: one for imagistic geometric knowledge, and a second for verbal-declarative geometric knowledge. The *imagistic precursor schema* in this preaxial stage is largely figurative, holistic visuospatial awareness of shapes and reference frames. This precursor schema follows the trajectory specified by Clements and Sarama (2007), with shape knowledge based on highly standard visual prototypes of familiar shapes such as circles. The *verbal-declarative precursor schema* in the preaxial stage is proto-mathematical in that it is largely based on social knowledge of shapes and geometry. In its typical form, children label shapes discretely as familiar objects (e.g., a triangle is the "roof" of a house; a circle is a "ball"). Children who attend preschool or who have facilitating home environments may know shape names. Preaxial children are on the pathway to noticing the geometric nature of the world around them but working memory limitations prevent knowledge in these domains from being integrated and consolidated into a hierarchical structure such as that in observed in the syncretic phase.

By age 6, given exposure to the right learning contexts, children enter the uniaxial stage. As with the other CCSs proposed by Case (1996) and colleagues, it is posited here that the precursor structures become integrated through the syncretic thinking that Clements and Sarama (2007) have outlined in their theory of hierarchical interaction (the main axioms of this perspective are summarized in Table 1 below). Children's verbal-declarative knowledge shifts from holistic, social knowledge & analogies to early mathematical properties such as number and length of sides, points, roundness, flatness. Uniaxial children analyze the imagistic information they abstract from geometric diagrams and manipulatives to notice salient geometric properties, but these properties are tied to the context they occur in, and many geometric properties and relations may be virtually invisible to children at this stage when other salient features exist. For example, Hallowell, Okamoto, Romo, and La Joy (2015) showed 36 first-grade children (M_{age} = 6.96) a 2D image of a rectangle and asked them to identify matching faces of 4 plane- and solid-shape manipulatives in front of them, placed carefully to show all classes of faces to the children. One of the target matches, a triangular prism, had 3 rectangular faces and only 2 triangular faces. The rectangular faces comprised 2.5 times the surface area of the triangular prism compared with the triangular faces. Despite this, not 1 of the 36 children correctly identified the target as a match. When asked about the reason for excluding that shape, children most frequently cited "because it's a triangle" as the reason for the exclusion. Uniaxial children have integrated imagistic and verbal-declarative knowledge into a single, geometric core conceptual structure, but working memory limitations lead to errors of analysis as they can only operate on geometric contexts with a single instance of the CCS at a time.

Table 1**.** *Axioms of Clements and Sarama's (2007) Hierarchical Interactionalist Position*

Developmental progression	Content knowledge develops along progression levels that build on earlier concepts & processes that are naturally intuitive to children
Domain specific progression	Developmental progressions are primarily driven by insights within the specific domain under development, although metacognitive advances
Hierarchic development	New levels of thinking are characterized largely by reorganizations and incremental extensions of prior knowledge
Cyclic concretization	Development begins with sensory-concrete examples, and slowly builds through verbal generalizations into integrated-concrete mental models
Co-mutual development of concepts and skills	Concepts and skills influence each other, and effective instruction accounts for their bi-directional influence
Initial bootstraps	Children's early proto-mathematical insights emerge early in life and prepare children for learning in this domain
Different developmental courses	Individual and social differences give rise to different courses in development of mathematical thinking between individuals
Progressive hierarchization	Mathematical thinking develops within and across domains in such a way that mathematical insights become robust to misleading cues
Environment and culture	Environmental exposure to the vocabulary and activities of mathematical inquiry will affect the rate and breadth of mathematical development
Consistency of developmental progressions and instruction	Instruction should be aligned to children's natural developmental tendencies in order to optimize the development of mathematical thinking
Learning trajectories	Hypothesized learning trajectories are used to design instructional activities that support a range of age-appropriate developmental trajectories
Instantiation of hypothetical learning trajectories	Interactions between teachers as skilled facilitators and students as active inquirers are ultimately where insight and development take place

Consistent with Case's (1996) theory, it is proposed in this dissertation that children at 8 years old experience the biaxial stage, where the single dimension of the geometry CCS differentiates into 2 instances of the mental structure that may be held simultaneously in working memory while operating on a geometric context. For example, some children at this age can evaluate a complex frame in a concatenation task that they are trying to cover, and while at the same time mentally manipulating a composite shape to cover or iterate a covering before they even pick the shape up to place it (Clements, Wilson, & Sarama, 2004). This requires more than 1 instance of the geometry CCS as both the negative space in the frame and the potential composite shape are evaluated in an imagistic, verbal-declarative fashion. The area of the frame under consideration must be analyzed as a geometric entity, and this may yield clues as to which kind of composite shape candidate to search for (e.g., a triangle to fill a corner of a frame shaped like a bird). Once the candidate shape has been identified, a biaxial child who has strong spatial visualization ability, then has to imagine rotations, flips, angle matches, etc., to determine whether the optimal triangle has been selected. One must hold the negative space to be filled in the frame in mind to be able to do this.

At age 10, it is proposed here that children obtain the integrated biaxial stage, where working memory capacity has increased to such a degree that children can operate fluidly and efficiently with multiple instances of the geometry CCS, while simultaneously engaging in the kind of metacognitively rich reflection that allows for the eventual emergence of van Hiele's (1986) Level 3 geometric thinking. At this transition, children begin to notice properties that generalize beyond the specific contexts that they are operating in. While robust Level 3 thinking may not appear until fifth or sixth grade, 10-year-old children have enough working memory capacity to begin noticing the fruits of their geometric investigations. By sixth grade, it is hypothesized children who have had enough prior experience engaging in spatio-geometric activities should be able to fluently work across biaxial knowledge structures in this domain to systematically problem-solve geometric investigations. Figure 3 depicts the developmental trajectory of the geometric central conceptual structure proposed here. At age 4, children have the working memory capacity to attend to the visual or verbal aspects of a shape, but have difficulty unifying these precursor schemas into a robust structure for geometric reasoning. By age 6, these pre-cursor schemas have become integrated into a single mental reference axis that children can use to investigate a shape on a plane. At age 8, children have sufficient working memory capacity to reason with two instances of this mental reference axis, allowing for inferential reasoning and spatial visualization. By age 10, children are able to integrate two of these mental structures to engage in deep geometric reasoning from both visual and semantic dimensions of geometric contexts.

Figure 4 shows the geometric central conceptual structure that emerges in the uniaxial phase after the imagistic and verbal precursor structures become integrated. The visual-imagistic and verbal-declarative pre-cursor schemas are integrated in a single mental reference axis. During the uniaxial stage, language is descriptive and mathematically imprecise. Geometry as a domain is especially laden with misleading cues, whose presence may render less salient features virtually invisible especially for children with less experience encoding verbally imagistic properties.

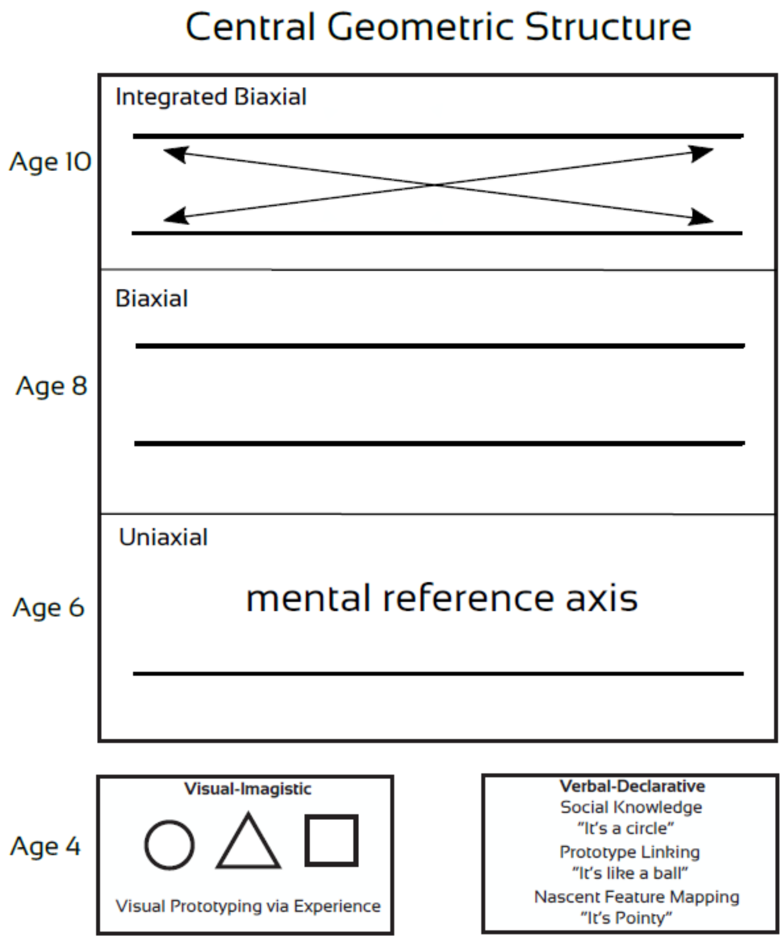

Figure 3. The developmental trajectory of the central geometric structure.

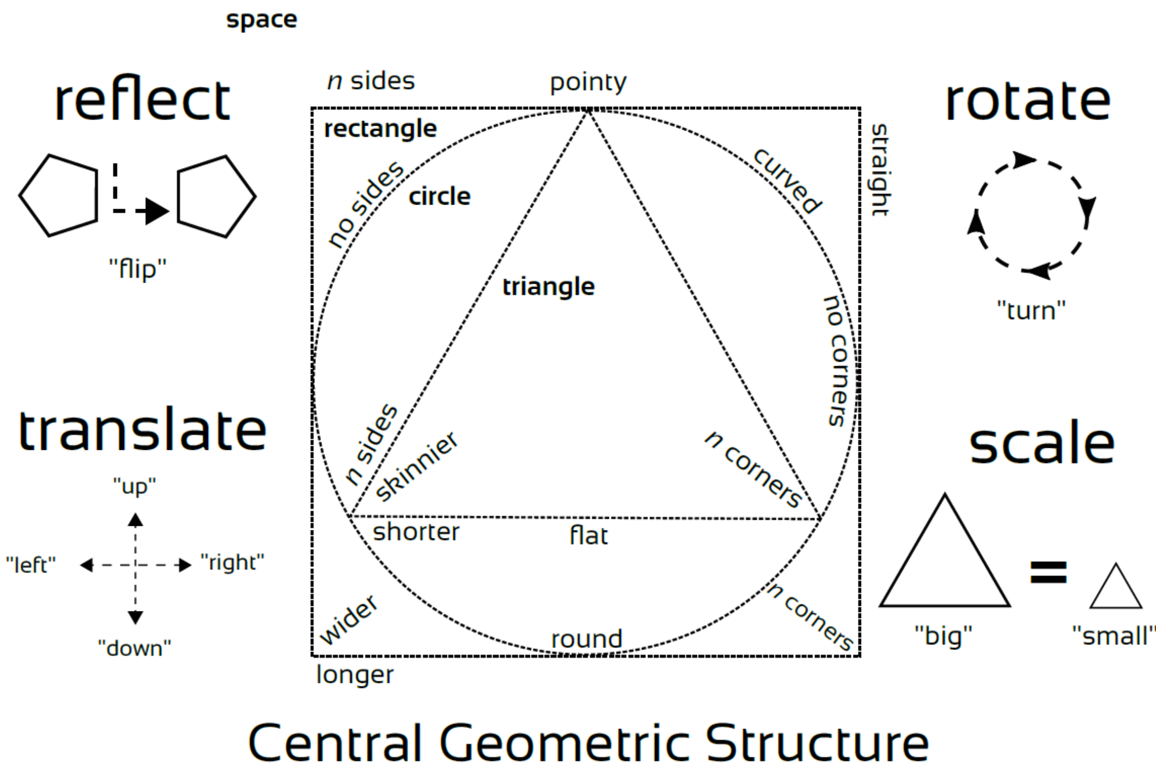

Figure 4. A single instance of the central geometric structure as present at age 6.

In summary, from this neo-Piagetian perspective, children of age 6 already have the ability to form a nascent mental structure that can be consolidated and expanded into powerfully effective geometric reasoning going into adolescence. This dissertation presents an exploratory analysis of geometric reasoning in second-, fourth-, and sixth-grade students to provide preliminary evidence and materials for further questions related to the central conceptual structure proposed above. Before turning to the present study though, a few prolegomena on the nature of mathematical knowledge and mathematical insight are necessary.

Reflection, Proof and Truth in Geometric Insight

Case's (1998) notion of a hierarchical learning loop (discussed above) involved the bidirectional influence of general insights upon specific insights and vice versa. He proposed this model of learning in the context of a discussion about stage transitions in neo-Piagetian theory. How is it that precursor schemas become integrated into a CCS? How does the CCS differentiate into multiple instances of itself, to become reintegrated to generalize insight? Case (1998) proposed that Piaget's notion of reflective abstraction was important to answer these questions. Where empirical abstraction in classical Piagetian theory involved deriving knowledge from the properties of objects, and pseudo-empirical abstraction involves knowledge derived from the placement of objects in context, reflective abstraction involves insights drawn from coordinated actions that are abstracted from their context and objectified in consciousness as higher-level insights (Dubinsky, 2002). A simple example of reflective abstraction occurs as children experiment with commutativity in addition. As children experiment with the order of addends, they come to represent internally the actions that become the insight that is the commutative property of addition (Dubinsky, 2002). This is why Clements and Sarama (2007) wrote:

> There is no understanding without reflection, and reflection is an activity students have to carry out themselves. No one else can do it for them. Yet a teacher who has some inkling as to where a particular student is in his or her conceptual development has a better chance of fostering a further reflective abstraction than one who merely follows the sequence of a preestablished curriculum (p. 466).

Yet, we know that students are confused about what counts as truth in geometric insight (Clements & Battista, 1992). Senk (1985) asked a large sample (n = 1520 students) of U.S. high school students from the Cognitive Development and Achievement in Secondary School Geometry Project to complete a proof-generation measure. These students had received formal instruction on proof writing over the year and included both honors and regular track students. Each student completed 6 items, which included filling in missing axioms for a proof, drawing a supporting geometric figure from a verbal proof, and writing 4 full proofs. There were 3 different forms of the test, resulting in 12 proofs across the full analysis. Participants were assigned a score of 0-4 based on the completeness of their responses, with '0' offering no supporting reasoning and '4' containing at most a single error. Responses were graded blindly by 2 teachers each, and the final score was averaged. If scores differed by more than a single point, a third grader was brought in. Senk determined that about 30% of students who took a full year of geometry reached a 75% mastery of proof. The overall accuracy rate exceeded 50% of students for only 3 of the 12 proofs scored. While this study was conducted some time ago, its methodology was impressive, and the findings were remarkable given that students had received explicit training in proof writing prior to administration of the instrument. Students are not sure about how to arrive at truth in geometry, even when they have been trained to do so.

Stylianides (2007) offered a definition of proof that she suggested is developmentally appropriate for elementary school. She suggested that "Proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim" with the added characteristics that argumentation is based on agreed-upon statements adopted by the classroom community, the modes of argumentation are accessible to the classroom community, and that arguments are presented in a developmentally appropriate manner (Stylianides, 2007, p. 291). However, Fosnot and Jacob (2009) took exception to the use of the term "argumentation" as synonym for proof. They point out that arguments are generally posited to decide which side has a stronger case, whereas in proof one relies on "rigorous reasoning from a set of clearly formulated hypotheses" (p.119). Fosnot and Jacob focused on the idea of establishing the validity of mathematical statements, with the act of re-examination and explanation of mathematical statements as essential. This dissertation adopted this perspective for the design, particularly as it is evocative of the notion of reflective abstraction.

Discussing the role of reflection in mathematics learning, Wheatley (1992) pointed out that reflective abstraction must be elicited in the proper context, problem-centered learning. The goal is to create a culture of inquiry, which is to be distinguished from "active learning," where students might bustle about a classroom working with manipulatives, etc. (Wheatley, 1992). In problem-centered learning, the teacher creates what van Hiele (1986) called "a crisis of thinking." The problem is carefully selected to elicit the big ideas that the teacher is aiming to facilitate and posed to students with minimal interference from the teacher. This is distinct from the "abstract-first" pedagogical method, where children are merely "shown" concrete examples of mathematical situations and asked to abstract important ideas. Instead, the activity should "confront" the student with his or her own actions as an object for reflection. There are often misleading cues designed into the problem-centered task to bring about such a reflection. Wheatley (1992) offered an example from his own work where a third-grader was given a string of mental arithmetic problems to perform:

> Interviewer: What is 21 take away 19?
>
> Jim: One…. No, TWO!
>
> I: What is 31 take away 28?
>
> J: 12.
>
> I: What is 31 take away 29?

J: 11

I: What is 31 take away 30?

J: [*Long pause*] One? (p.537).

It was not until Jim got to "31 take away 30" that the student realized the misconception in his earlier action. Wheatley followed up this interaction with MathLink cubes, and the student was given further opportunity to make sense of his composing and decomposing quantities in arithmetic.

While some years have passed since this interaction took place, the merits of inquiry-based activities continue to be debated as cognitive load theorists decry the cognitive processing demands placed on students when explicit guidance is not provided to students (Kirschner, Sweller, & Clark, 2006). Additionally, the results of the 2015 *PISA* show a negative association between "enquiry teaching" and science achievement on the PISA examination (OECD, 2016). Mathematics and science are not the same fields, but many advocates of direct instruction in mathematics have claimed a victory for their position based on these results. My intent was not to seek to enter this debate, but to suggest that it is difficult to ascertain the value of inquiry-based education when 1) researchers take an either/or perspective stance on problem-based instruction (it is possible that some optimal mix of guided and problem-based instruction exists, and that this ratio changes along the developmental course); 2) the assessments that are used as evidence of guided instruction's merits are in line with van Hiele's (1986) lower-order thinking (e.g., Kamii & Lewis (1991) interviewed 87 high-scoring second grade math students and found their mathematical understanding exemplified lower-order thinking in mathematics); and 3) highly-effective problem-based pedagogy such as the kind routinely found in top-achieving Japan requires a pervasive cultural shift in the way that teachers are trained and students are socialized, and it is clear that most countries in the world do not have such a robust teacher training system and school culture in place to support high-quality, inquiry-based education (see Hino, 2007 for a discussion of implementing Japanese problem-based culture in the United States).

Although I have designed a problem-centered geometry sequence for the purposes of investigating abstractive reflection in geometry, it is not to be understood that I was ignorant of the criticisms levied by empiricists against rationalists who advocate inquiry in education, nor did I believe that most of the important questions have yet been answered in properly placing inquiry learning's best role in the educational milieu. Rather, the activities designed for this dissertation align with insights from the best developmentally informed mathematics research available (Clements & Sarama, 2007), and with neo-Piagetian analysis of intellectual development across domains (Case, 1998).

The notion of high-level reflection facilitating intellectual development in mathematics finds some support in the research literature. Tan and Garces-Bacsal (2013) had 54 male secondary one students (age 13) in Singapore keep a math journal over a 6-week period. Journal-writing had specific prompts, such as "What are the steps in the 'algebraic method' of solving word problem?" (p.178). The intervention was aimed at spurring reflective abstraction through the venue of the journal. Twenty-seven students were assigned to the journal condition, while the other 27 were assigned to the control condition. Students in both conditions took a 35-minute pretest that assessed algebraic problem-solving (e.g., simplification, factoring, linear equations, substitution, and word problems). The post-test instrument was as similar as possible to the pretest, with some of the details altered. During the 6-week period, all students were assigned weekly worksheets, but only the journal-writing group received the journal prompt assignment. At pretest the experimental group and the control group scores did not differ significantly, but at posttest the experimental group outperformed the control group (experimental group $m = 18.37$, $sd = 4.71$ vs. control group $m = 16.31$ $sd = 4.71$, $p < 0.01$). Self-report measures suggest that about 90% of the students in the experimental condition perceived that the journal exercise was valuable in supporting mathematical insight. These students were all assessed as gifted and no females participated, so it is not clear that these findings would generalize beyond this unique group.

In a value-added assessment, Bicer, Capraro, and Capraro (2013) investigated whether 96 middle-school students assigned to either a 6-week writing condition on mathematical problem-solving or on homework and test preparation would show greater problem-solving ability gains between pre-test and posttest. They found that students who practiced interpreting information, writing mechanics, generating story problems, and solving each other's story problems, showed greater problem-solving ability between pre and posttest than students who focused on correcting exam and homework questions to increase procedural competence (Cohen's $d = 1.45$ across cognitive complexity and problem generation accuracy). The authors asserted that reflection made the students' mathematical acts themselves an object for reflection in a way that increased metacognitive awareness of their mathematical reasoning.

This dissertation made use of cognitive clinical interviews to capture active reflection as children engaged in geometric reasoning to further investigate the role of reflective abstraction in geometric insight. Transcripts of video-recorded sessions were the material for an exploratory analysis of children's geometric thinking in the middle and upper elementary years. Building on prior research, I extended the age range that studies on emerging spatial reasoning typically investigate (early childhood). Providing students with high quality mathematical experiences in domains such as geometry open the door for achievement that some students might not experience in less spatial domains. Additionally, this dissertation endeavored to extend neo-Piagetian theory by proposing a central conceptual structure in the domain of geometry. The research conducted for this dissertation elucidates some aspects of the proposed model, positioning these findings in a larger proposed research program.

Chapter 3: Methods

Participants

For recruiting purposes, local schools in a coastal community in Southern California were given the opportunity to participate and two charter schools accepted. At the time of the study, Charter School A had around 60% white enrollment, around 25% Hispanic/Latino enrollment, just under 10% identifying two or more ethnicities, and the remaining students identifying as Asian or African American (Ed-Data, www.ed-data.org, accessed May 14, 2019). Under 5% of students were classified as English language learners, and about 20% of students were eligible for free & reduced-price lunches. Charter School B had about 55% Hispanic/Latino enrollment, near 40% white, and the remaining students reporting mixed or Asian ethnicity. About 35% of students at Charter School B were English language learners at the time of the study, with nearly half of students qualifying for free & reduced-price lunches.

Teachers of second-, fourth-, and sixth-grade students were invited to participate in the study across the two schools. A third-grade teacher requested the opportunity to participate due to interest in the geometry activities. Those students were given the opportunity to participate but were excluded from analysis. All students in participating classrooms were given consent forms in English and Spanish and invited to participate. Many more students obtained permission to participate than could be accommodated by the researcher and an occasional assistant, so participants were run through the study based on the order that completed consent forms were returned to the researcher. Sixteen students completed the study from each of the 3 grade levels and all participants were fluent in English. Twenty-eight students identified as female, with participants ranging in age from 7 years, 4 months old to 12 years, 6 months old (Grade-level characteristics were as follows: second grade- 10 females, M_{age} = 8 years, 1 month, *SD* = 5.5 m months; fourth grade- 8 females, M_{age} = 9 years, 10 months, *SD* = 4.4 months; sixth grade- 10 females, M_{age} = 12 years, 0 months, *SD* = 2.7 months).

For this dissertation, transcripts generated by 6 participants from a single 30-minute session with each of them were coded and analyzed. The selection process and characteristics of these 6 individuals is described below.

Descriptive Statistics

This dissertation was designed with the clinical case study in mind the exploratory analysis undertaken in this chapter should be understood with this context in mind. It is hoped that these findings will be used as the basis for reasonable hypotheses that structure quantitatively rigorous studies that follow. Descriptive statistics are presented here to help the reader understand who participated in the study. Only a subset of these individuals was selected for presentation in this analysis in accordance with the procedure described below. The result of that process is that all students were divided into 2 groups based on whether or not students showed any increase in posttest item scores from pretest, then a single individual was selected from each group for each grade level for analysis.

Age. As described above, participants in this study were between the ages of 7 years, 4 months old and 12 years, 6 months old. Participants in second grade had a mean age of 8 years and 1 month old (*SD* = 5.5 months). The second-grade student selected for analysis for the no-improvement group was 7 years and 6 months old, and the improvement group student was 7 years and 4 months old at the start of the study dates. The mean age of the fourth-grade cohort was 9 years and 10 months (*SD* = 4.4 months). Both fourth-grade students selected for analysis were 10 years and 2 months old at the time of the study. The mean age for the sixth-grade cohort was 12 years and 0 months (*SD* = 2.7 months). The sixth-grade no-improvement group student was 11 years and 9 months old at the time of the study, and the improvement group student was 11 years and 11 months old. Figure 5 depicts the distribution of participants by age. Some data about the entire sample of participants who participated is provided here for context, but a subsample of 6 participants was selected for in-depth analysis in a procedure described below.

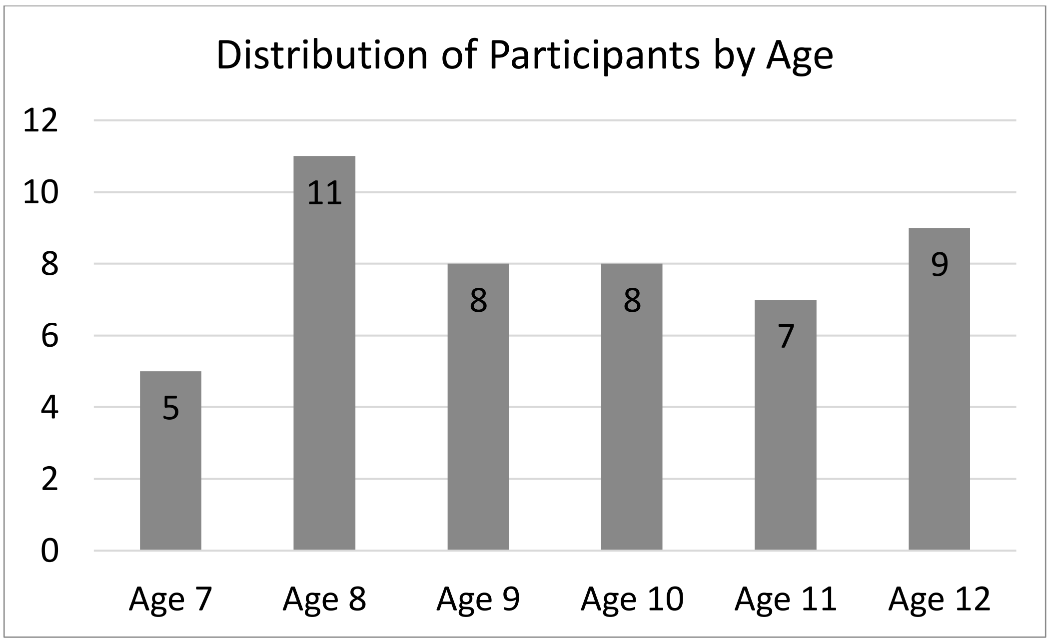

Figure 5. Distribution of Participants by Age

Selection Process and Selected Participant Characteristics. A subsample of participants was selected for analysis in accordance with a form of purposeful sampling called maximum variance (Patton, 1990). This method of selection entails selecting participants along a key variable of interest to emphasize the contrast of differences that may be masked by a small overall sample size. For this dissertation, the participant with the lowest and the highest pretest/posttest change score (described below) were selected for each grade level. In the case of a tie, the student with the lowest pretest score was selected. If candidates were tied on both metrics, the student who was the youngest at the time of participation was selected. The rationale for contrasting along the dimension of pretest-posttest change aligned with the goal of understanding children's geometric insight. It was reasoned that a child who improved his or her performance from pretest to posttest would likely have had helpful insights along the journey through the investigations and would have been able to share examples of his or thinking along the way. By contrast, a participant who did not exhibit improved performance may have exhibited traits the inhibited insight. The intention was to maximize this contrast. Participants who showed no improvement due to a ceiling effect were excluded from the selection process. The process also provided an unbiased way of selecting participants for analysis. This process resulted in the analysis sample depicted in Table 2 below. All students selected were within a standard deviation of the mean age for the respective grade. Participants with the low score formed the no-improvement group (pretest/posttest change of -1 or 0), and the participants with the high score formed the improvement group (pretest/posttest change of 1 or 2). Participants who showed no improvement due to a ceiling effect (i.e., perfect score on both pretest and posttest trials) were excluded from this selection process, given the rationale that we wanted to capture evidence of hierarchical learning in children who experienced improvement over the course of the study.

Table 2. *Summary Characteristics of Sample Selected for Analysis*

Group	**Gender**	**Age at Participation**	**FIT-1T Score**	**Pretest/Posttest Change**
Second Grade				
No Improvement	Female	7 years, 6 months	3	-1
Improvement	Male	7 years, 4 months	2	2
Fourth Grade				
No Improvement	Male	10 years, 2 months	5	0
Improvement	Male	10 years, 2 months	3	2
Sixth Grade				

No Improvement	Female	11 years, 9 months	No score	-1
Improvement	Female	11 years, 11 months	6	2

Materials and Procedures

Measures.

WISC-V Block Design subtest. The Wechsler Intelligence Scale for Children- V (WISC-V) Block Design subtest is a validated measure of intrinsic-dynamic spatial reasoning validated for use with children ages 6-16 years-old (Pearson Efficacy, 2016; Uttal et al., 2013). Assessment equivalency between paper and electronic versions has been empirically validated (Daniel, Wahlstrom, & Zhang, 2014). The electronic version is fun for children and is administered on an iPad, which eliminates scoring errors due to researcher error. Children were administered the measure in accordance with the provider's guidelines. Participants completed the task with the physical blocks provided by Pearson.

WISC-V Picture Span subtest. A new subtest on the WISC-V is the Picture Span subtest, which assesses visuospatial working memory by presenting a series of simple figures (e.g., plant-gift-ladybug, etc.). This subtest has undergone the same validation procedures as cited above, and the rationale for administering the electronic version is the same.

WISC-V Scores. The block design and picture span subtests from WISC-V were administered and scored for 27 participants. Due to limited time and resources, the fifth session of each week was used as a makeup session for one of the geometry investigations whenever a child missed an investigation. Additionally, the sixth-grade class had numerous extracurricular activities that limited possible session days, resulting in very few test completions from that grade. Due to the low testing rate overall, these items were excluded from this analysis.

Figural Intersections Test (FIT). Pascual-Leone and Baillargeon (1994) validated the FIT to detect M-capacity (neural activation boosting capacity, functionally similar to working memory) in the context of misleading cues. Performance on FIT provides the best diagnostic tool available to determine at what stage in Pascual-Leone's neo-Piagetian model a child has matured to, since difficult items require a combination of schema activation and inhibition for successful performance.

FIT Scores. The Figural Intersections Test (FIT) is a measure of working memory capacity and is described in detail below (Pascual-Leone and Baillargeon, 1994). All scales of the FIT scores for 45 of the 48 participants were assessed and recorded. Two fourth grade participants and one sixth-grade participant were absent on the day of testing and scheduling a makeup test date for each student was not possible due to limited time and resources. Since FIT-1T scores are scaled according to the theoretical M-capacity described above in Chapter 2, those scores are presented here to give the reader insight into the sample that the participants were selected from for the qualitative analysis described below. Figure 6 shows the distribution of FIT-1T scores for the entire sample. Figure 7 shows the distribution of scores by age, and Figure 8 shows the overall distribution by grade.

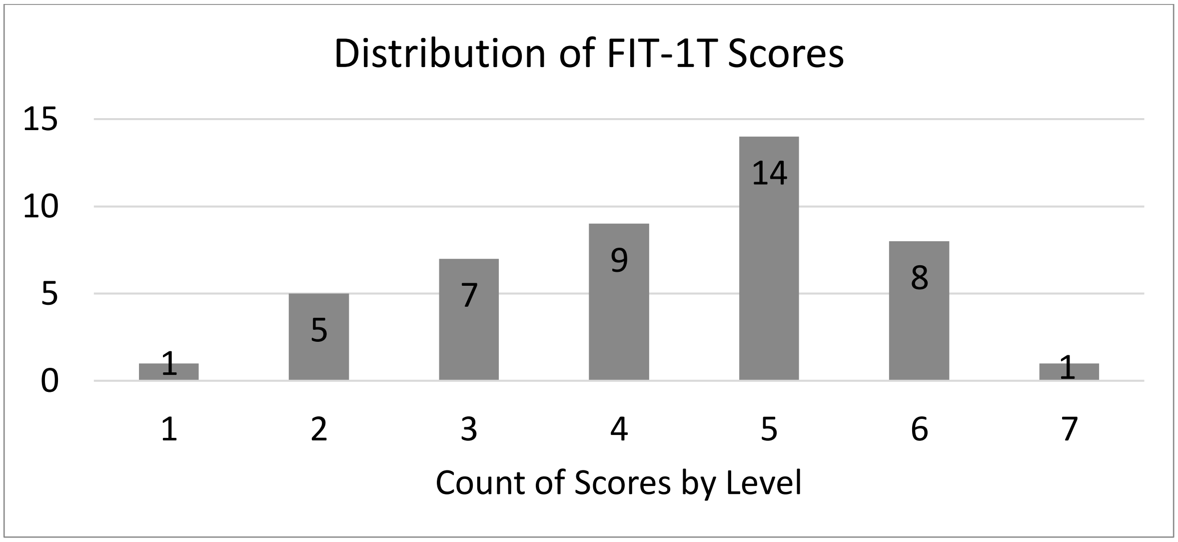

Figure 6. Distribution of FIT-1T scores

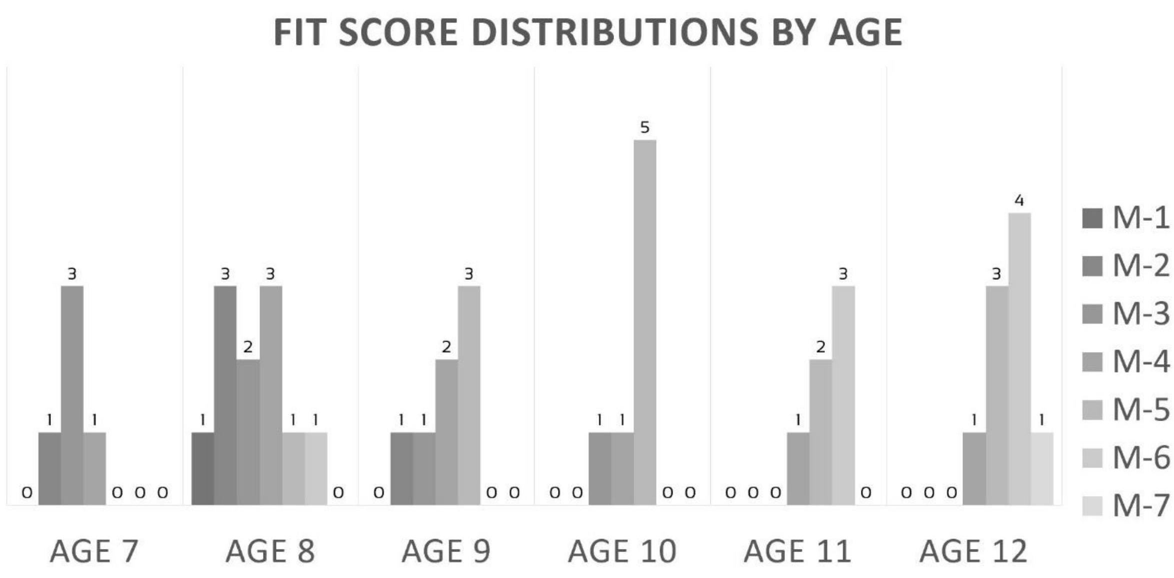

Figure 7. FIT Score Distributions by Age

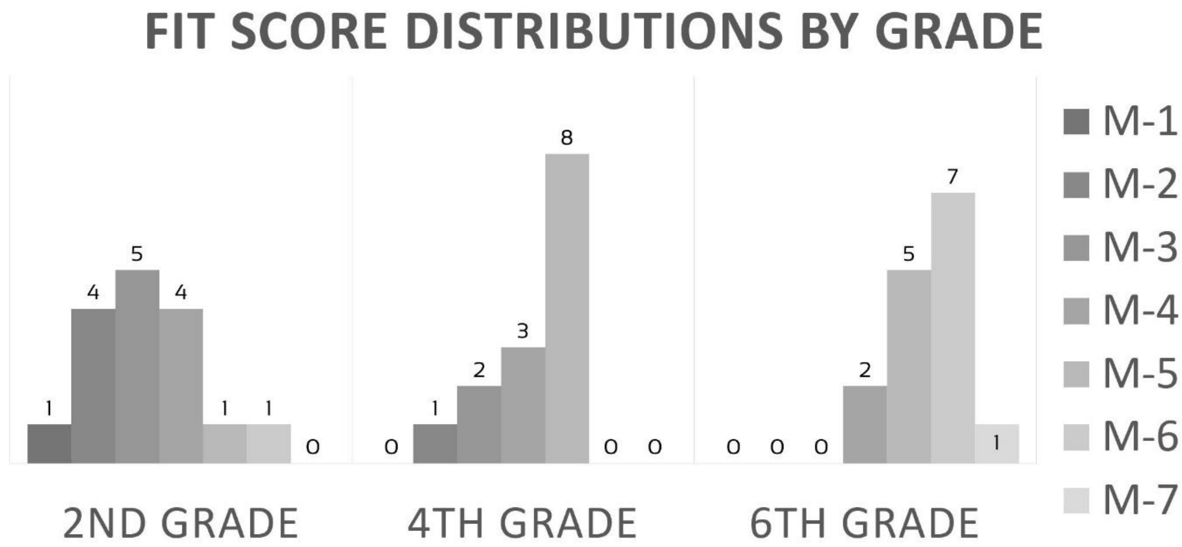

Figure 8. FIT Score Distribution by Grade

Spatial Composition/Decomposition Pretest/Posttest Items. Three spatial composition items were utilized as a pretest/posttest measure. These were administered on an individual basis, with the pretest session conducted prior to beginning the first geometry investigation, and the posttest administered after completion of the third investigation. One geometry achievement item was developed by Jacob and Mendoza's (personal communication, 2017) for their *Young Mathematicians at Work* spatial curriculum. This item involved growing a trapezoid. Students were directed to attend to the edges of a trapezoidal pattern block, then given additional trapezoids to "grow" the shape, "doubling" the trapezoid. Two additional frame-filling items comprised the other two items in the pretest-posttest measure. The first involved filling a kite frame and was a star-shaped frame. Figure 9 depicts the three solutions to these items, but students did not see the internal lines during testing. For the first two items children were given the correct pieces to cover the frame. The third item was designed to be the most difficult, and students were given a distractor set of triangles in addition to the two correct triangles that were necessary to cover the frame. The distractor triangle had the same dimensions as the target triangle, but swapped along the hypotenuse and legs (i.e., the side length of the target hypotenuse was the same as the distractor legs and vice versa). Participants were provided an empty frame to cover for the second and third items. Children were given as much time as they desired to work on each item but were told they could move on to the next item when they wished. No assistance or instruction was given for these items.

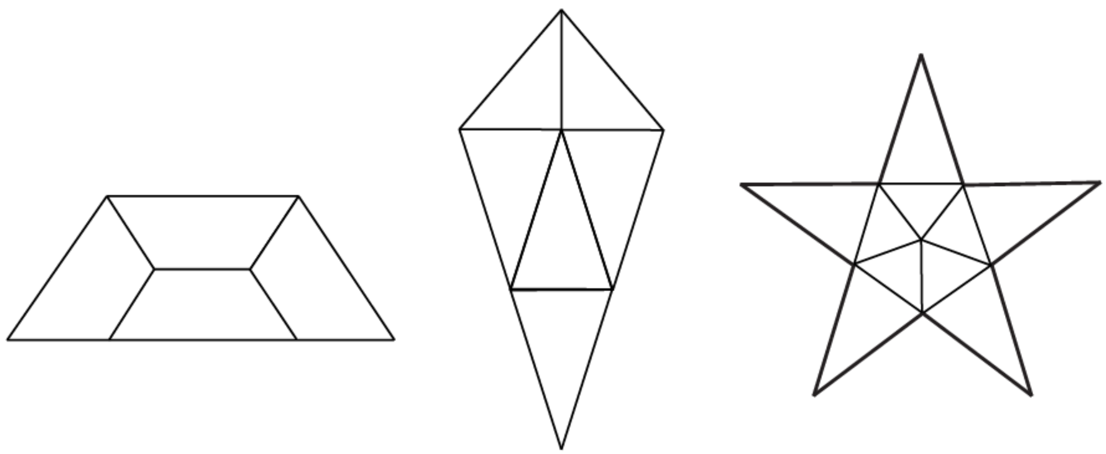

Figure 9. The three pretest-posttest items with lines superimposed on the frames to depict the solutions. Participants were given an empty frame with only the external boundaries to cover for the kite and the star items.

Pretest/Posttest Item Scores. Since pretest and posttest sessions were conducted on days where participants engaged in the geometry investigations, all participants completed the 3 pretest and posttest items. While the Solomon four-group design would have been preferable to address issues related to internal and external validity with this design (Braver & Braver, 1988), limited resources did not allow for this possibility. The pretest/posttest items were designed so that successful performance required the kind of spatial and geometric insights students would generate while conducting the geometry participants completed during the session in this study. Items were scored on the basis of completion (1 point) or no completion (0 points). To receive a score for completing an item, the participant needed to independently double the trapezoid or cover the frames respectively, indicating to the researcher that the task was complete. No partial score was awarded for partial completion. Figure 10 depicts the item completion rates by grade for both pretest and posttest trials. Overall completion rates increased across all three items. The sixth-grade participants saw a single unit decline in the grade-level completion-rate for the trapezoid item, but this was the only pre-post item drop. Scores increased in all other cases for grade-level and item category.

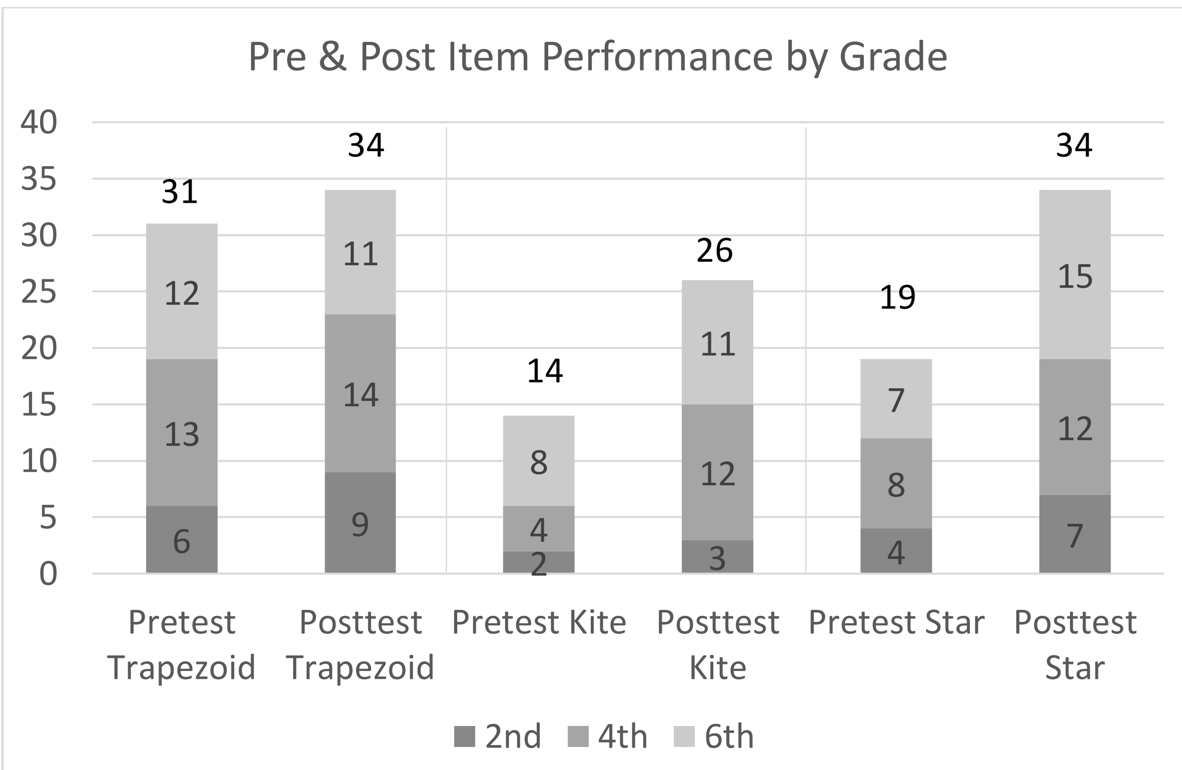

Figure 10. Pre & post item performance by grade

Figure 11 shows the overall distribution of pretest/posttest change. More than half of the children exhibited an increase in their performance on these items, with cumulative change skewed positively. Individual change scores for participants included in this analysis are reported above in Table 2. While causation cannot be inferred from these data, the performance gains children showed here suggest that a study with the resources to implement the Solomon four-group design might be worthwhile in the future.

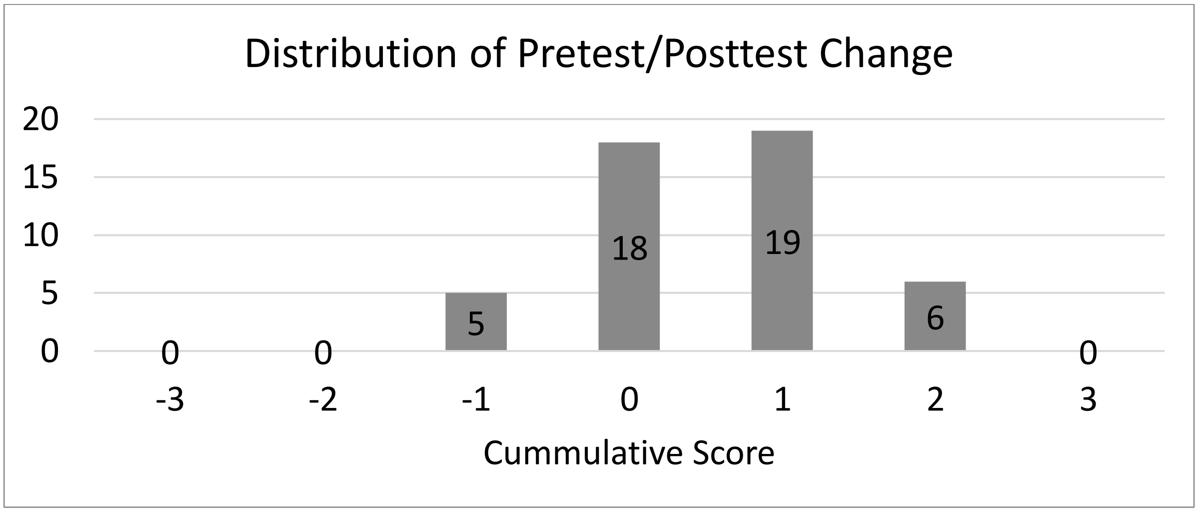

Figure 11. Distribution of pretest/posttest change

Materials.

Triangle manipulatives. Triangle design prototypes were specified after extensive collaboration with Jacob and Mendoza from the University of California, Santa Barbara Department of Mathematics. Manipulatives of special triangles were created to elicit specific geometric ideas in the context of the investigations presented below. Manipulatives were modeled in AutoDesk 123D° Design 2.2.14. A LulzBot TAZ 5 3D printer was used to print the triangle manipulatives utilized in the geometry investigations. Several colors of ColorFabb's nGen filament (2.85 mm) were used to print the triangles, with each species of triangle printed in a single unique color. A single letter was printed into each vertex measure (e.g., A = 30°) and used consistently across triangles for that measure to give curious students a consistent reference point. Triangle manipulative blueprints and model files for 3D printing will be provided on request.

Geometry Investigations. All 3 geometry investigations were based on the pedagogical framework proposed by Fosnot and Jacob (2001), which was further expanded in the area of geometry by Jacob and Mendoza (personal communication, 2017). That framework is an inquiry-based model, where children are provided with specially chosen contexts and problems that naturally lead to the development of "mathematizing." Mathematizing is deep mathematical reasoning the child engages in that has much deeper roots than correct execution of a standard algorithm. While such proficiency is important, mathematizing is a way that children can appropriate their own mathematical understanding to build profound conceptual understanding in concert with their burgeoning procedural understanding. These geometric investigations made use of 3D-printed special triangles to draw out key notions in early geometric reasoning (e.g., 3-4-5, 3-5-5, and 3-3-3$\sqrt{2}$ triangles). A key feature of these investigations is that they are designed to be tools for thinking that students of very diverse developmental milestones can generate insight from. Students in second- fourth- and sixth grade were administered the same set of investigations to learn what geometric reasoning might look like at various ages with the same opportunities for insight.

Investigation 1: Area & Units- Doubling and Tripling with Pattern Blocks & Special Triangles. Participants completed a series of doubling and tripling exercises with standard pattern blocks. The first exercise was with a square rectangle. Participants were provided 14 square rectangle pattern blocks and asked to "grow" a single square to a 2 x 2 and then 3 x 3 specimen. Participants were told, "This is a square-rectangle. Can you double this square rectangle? Can you triple this square rectangle?" The researcher clarified the ambiguity of doubling or tippling units as side length versus area units whenever this was unclear. It was hypothesized that this would be easy for most children even in second grade, and it was found this to be true. The primary purpose of this exercise was to build confidence and start with a familiar context known to the children. This same procedure was repeated with the wide rhombi and the equilateral triangles. Figure 12 depicts the intended outputs of the first 3 activities.

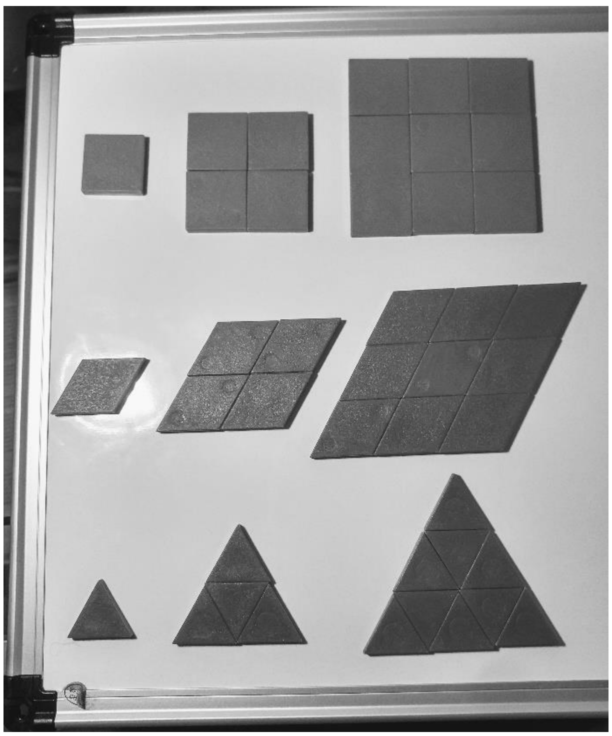

Figure 12. Doubling and tripling shapes with pattern blocks.

After completing the three "shape growing" exercises with the pattern blocks, participants repeated these exercises with three special triangles. First, participants were given 14 of the green triangles, which featured the proportions of a 3-3-3$\sqrt{2}$ right triangle. This triangle was selected for use in this series of investigations for a variety of reasons. Its symmetry from the midpoint of the hypotenuse to the right-angle vertex makes it an easier triangle to work with than the 3-4-5 triangle since it is the same when reflected. However, it differs from the standard equilateral triangle in that two of them joined along the hypotenuses form a square rectangle. We made use of this feature especially in Investigation 2, but we hypothesized it would be a manageable starting point for children to grow these triangles, and that proved to be the case.

The second special triangle used in this investigation was the 3-5-5 purple triangle. This triangle appeared strikingly different from the 3-3-3√2 green triangle, but for the purposes of the shape growing exercise, they had identical requirements for growing. Students with more advanced geometric insight completed this exercise immediately after figuring out the first task. Many other participants spent some time experimenting with trial and error and were surprised and delighted to discover that the strategy to solve the task was the same as with the first one.

The final shape growing activity for this investigation required students to work with the 3-4-5 gold triangle. This triangle proved challenging for most of the students, with younger students asking for a hint in many cases. The researcher was careful to place the 14 triangles down in different orientations, so that the student needed to work out that the lack of symmetry required all of the gold triangles to be in the same orientation to grow (“face-up” or “face down”). This investigation brought up interesting insights about side lengths, symmetry, and reflection (i.e., “flipping”). For some students, this marked a turning point in attending to some subtler aspects of the properties of shapes, especially in the context of concatenation and shape iteration. Some students discussed area and units, but many did not. Figure 13 depicts the shape growing activity with the special triangles.

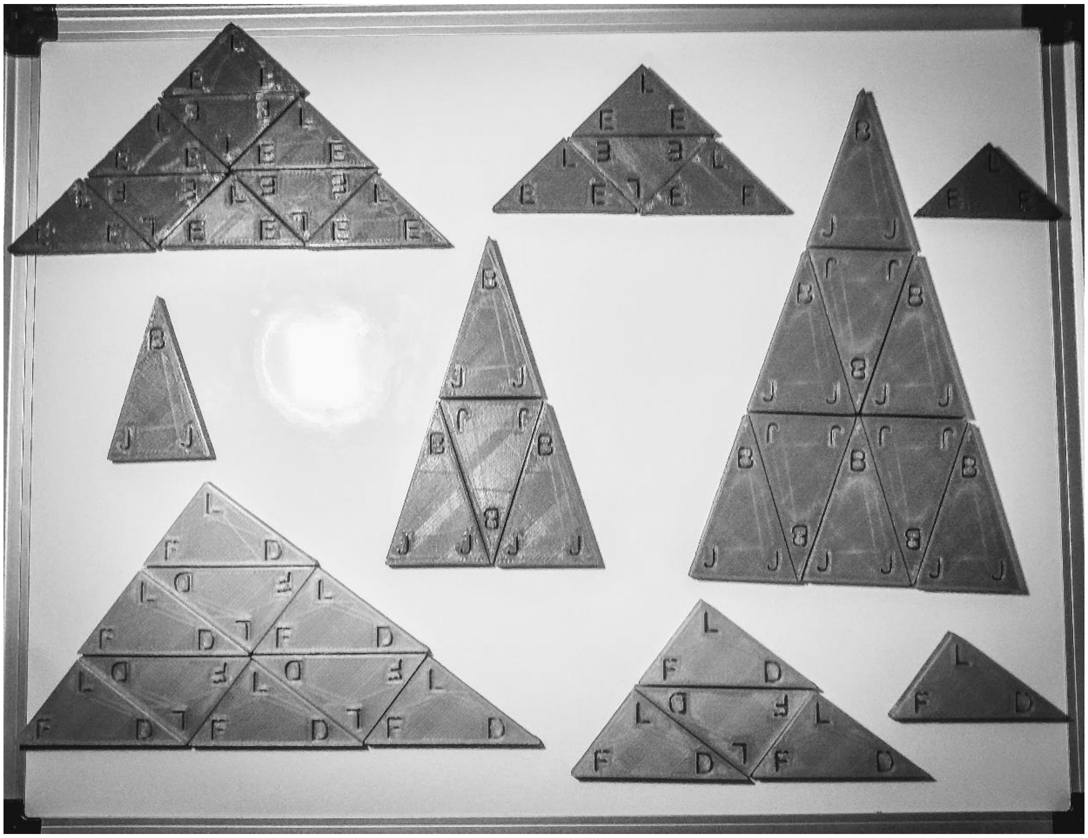

Figure 13. Growing shapes with special triangles. The top row shows the progression of the 3-3-3√2 green triangles, the middle row shows the 3-5-5 purple triangles, and the bottom row shows the 3-4-5 gold triangles.

Investigation 2: Fun with Frames. Investigation 2 involved covering frames. Participants were given the frames described in this section, printed on yellow paper with thick black lines and laminated for ease of use. The first frame was a small kite, depicted with target orientations indicated in Figure 14. Participants were given the frame, two of the 3-3-3√2 green triangles, and 2 of the 3-4-5 gold triangles. The researcher asked, “Do you think you could cover this frame with these triangles?” Participants were given as much time as they pleased and could ask for hints as needed. The researcher endeavored to give the fewest hints possible to facilitate performance (e.g., place one triangle in the target location or suggest flipping a single gold triangle). As above, students were very reluctant to ask for hints, and only did so after a long period of unfruitful trial-and-error. This frame presented two requirements that were especially challenging potentially. First, children had to rotate the green triangle to an orientation that was instinctively non-canonical for many of them (e.g., hypotenuse aligned to top edges of the kite instead of the right angle nested in the apex). Second, it required children to use the asymmetrical 3-4-5 gold triangle in a reflected orientation, drawing on the experience from the previous investigation that the reflected orientation matters. This and other aspects of the investigations described below were designed intentionally to provide opportunities for related geometric insight across geometric contexts and to elicit reflections suggestive of hierarchical learning loops where they may present themselves in the verbalizations of participants.

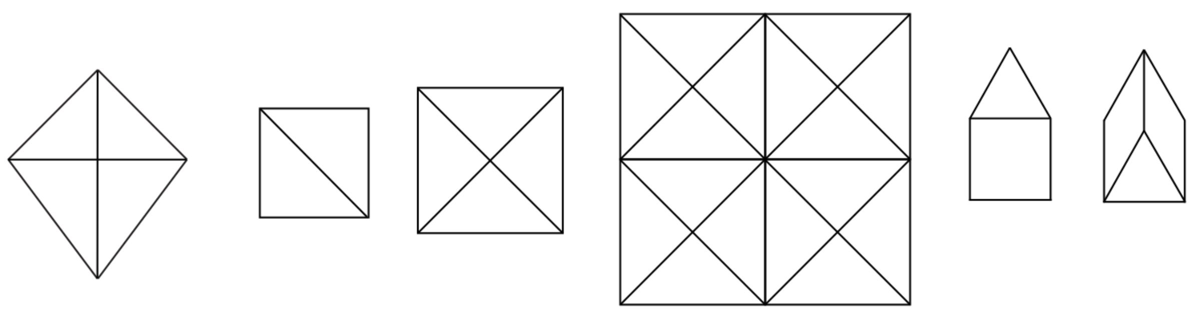

Figure 14. Frames from Investigation 2: Fun with Frames with target orientations indicated. The small and large square-rectangle frames have more than one possible solution.

The second part of the Fun with Frames investigation presented children with a series of square-rectangle frames, also depicted in Figure 14. These frames were also printed on yellow paper with thick black lines comprising the outside edge and laminated. The first frame (i.e., "small frame") required only two of the 3-3-3√2 green triangles concatenated along their hypotenuses to cover the frame. This activity was designed to elicit the idea of rotation and side-matching, since the green triangle would not fit in canonical orientation (i.e., right angle at the apex, hypotenuse at the base). The second frame (i.e., “medium frame”) required 4 green triangles to cover and was designed to build on ideas from the first frame. Rotation could be used as a strategy to notice that in this case matching the hypotenuse to the exterior frame boundary would yield a side match. Students might immediately iterate on this insight and match triangles in the same relative orientation to the subsequent 3 sides. Finally, students were presented with a large square-rectangle frame that could be covered by 16 green triangles in one of two different target orientations. Participants who were extremely effective at spatial visualization might see this frame as a tiling of 4 of the medium frames, but it was expected that this insight might arise later in the activity after building along the perimeter using the side-matching strategy, then filling the interior through trial and error rotation.

The final part of the Fun with Frames investigation involved covering a simple house frame 2 different ways with standard pattern blocks, then doubling the standard house construction (i.e., equilateral triangle pattern block concatenated to the “top” edge of a square rectangle). The researcher provided the participant with a supply of numerous pieces that included every kind of standard pattern block. The researcher then introduced a single house frame and asked the participant, “Do you think it’s possible to cover this frame using some of these pattern blocks?” It was anticipated (and proved to be the case) that children would quickly cover the frame using a square-rectangle and equilateral triangle for the standard house construction described above. The researcher then produced a second identical copy of the house frame and place it next to the first. The researcher then said, “I wonder, would it be possible to cover the frame using a different combination of pieces?” If participants were unclear about the meaning of the question, the researcher answered any questions the child had. Once a child discovered that the same frame could be covered by two skinny rhombi concatenated at the apex of the house and an equilateral triangle in the space at the bottom, the researcher then asked, “Looking at the square rectangle versus the 2 skinny rhombi put together [points with finger to shapes named], what do you think covers more area, this 1 square or these 2 skinny rhombi together?” This was done in order to give the child an opportunity to reflect upon and discuss the notion of conservation. It was hypothesized (and found to be the case) that some children might infer deductively that the square and rhombi covered the same space since in both cases an identical equilateral triangle covered the remaining space. These are depicted in Figure 13. Finally, participants were asked if it was possible to double the standard house construction. This was done to build on the pattern block growing activities from Investigation 1.

Investigation 3: Knights of the Polygonal Table. The third investigation was based on Jacob, Katzburg, and Mendoza’s (2015) Markus’ Table Designs, which is an open-ended concatenation exercise undertaken with standard pattern blocks. Students experiment with building tables under various geometric constraints that set the context for mathematizing about parallel lines, angles, and composing space generally. For this dissertation, we iterated on this curriculum specifically for special triangles. Participants were shown the images in Figure 15 and given the instruction, “This is Smith. Smith is a knight who likes to build tables for his guests. Sometimes Smith has 4 knights visit, other times Smith has 12 visitors. One knight can sit at the edge of each piece, like so [indicates image]. Smith only has one rule: all of the corners have to meet at 1 point in the center like that [points to image example]. Do you see this table [indicates counterexample at left]? That looks like you could use it, but Smith wouldn't allow it. Can you tell why?" The researcher waited for the student to generate the reason, and further discussion ensued until each child understood the constraint of the activity. Then the researcher said, "I wonder how many different tables we can make from these triangles?" A sequence of triangles was given. Participants were given unlimited time to explore the different tables that they could construct, and many students asked the researcher to meticulously document their discoveries. First, children were given equilateral triangles from standard pattern blocks. Next, they were given the 3-3-3√2 green triangles to construct with. The third set of triangles were 54° x 54° x 72° white triangles. The fourth set of triangles were 72° x 72° x 36° pink triangles. The final set of triangles to be presented for this exercise were 75° x 75° x 30° yellow triangles. As participants noticed that the yellow triangles were narrower at the vertex than the pink triangles, the researcher asked questions such as, "Do you think the yellow table we seat greater or fewer number of knights"? Participants were also asked about the space filled around the center point to probe for a conversation about circles or angle measure.

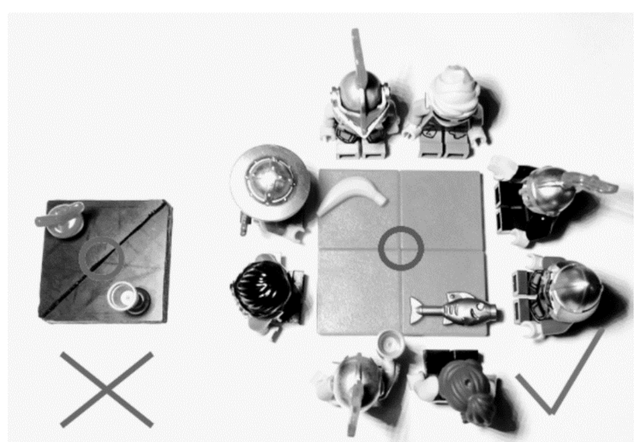

Figure 15. Instructional images for the Knights of the Polygonal Tables illustrating the single rule that all tables must have a common center point where all the corners meet.

Session materials. A double-sided magnetic 15 x12 inch whiteboard with a robust metal frame was provided for the participants to work on. This provided a high-contrast rectangular visual field for the children to work on and facilitated video transcriptions after the sessions. A small unobtrusive tabletop tripod was used to position a Canon Vixia HF S21 over the active frame for a birds-eye-view of the participant's activities. A backup voice recording was made on a Google Pixel smartphone whenever students felt comfortable wearing a clip-on microphone. Two Apple iPads were used to administer the WISC-V measures via the Q-Interactive app.

Procedure.

This study was comprised of 5 sessions spread out over a typical school week. The first day was spent together as a group completing the FIT. The second, third, and fourth day were spent in individual sessions with participants completing the geometry investigations described above. Participants completed the posttest items on day 4 after completion of Investigation 3. The fifth day was dedicated to completing the WISC assessments on an iPad in individual sessions. All participants were interviewed by the researcher in the method of clinical cognitive interviews throughout each investigation (Ginsburg, 1997). All sessions were video and audio recorded, transcribed for analysis, and coded. Throughout these investigations, participants were given as much time as the session would allow to arrive at a solution. They were instructed from the beginning that they may ask for a hint at any time, and the researcher would give a single tip or two to support the student's insights but avoided didactic exposition. In the vast majority of cases, participants applied heroic efforts to solve these investigations for themselves without assistance. While the purpose of these investigations was pedagogical in nature as opposed to evaluative, it was important to make space for children in the study to own their mathematical discoveries. Children were deeply engaged, and many expressed disappointment at the conclusion of the last day despite already having met for five sessions.

Day 1. On the first day of the study all students who had a completed consent form were given the opportunity to provide assent in accordance with IRB guidance. Students were gathered together as a group in their regular classroom setting for administration of the Figural Intersections Test (FIT; Pascual-Leone and Johnson, 2001). Each student was provided their own copy of the test and was invited to complete the practice questions in accordance with the measure instruction manual. Every student was provided a red pen to complete the measure. Students were provided the board demonstration as instructed in the manual on a portable white board or on the classroom whiteboard. All of the primary concepts were discussed in an interactive manner, and students shared their responses after completing the practice questions. Students were shown incorrect responses as directed in the manual and asked to correct the researcher to comply with the rules. Participants were encouraged to ask any questions about the instructions before and during the test. Students were instructed to mark a single dot in each of the figures on the right, then place a single dot at the shapes intersection on the left. No assistance was provided for solving items. Most students completed the measure in 45 minutes or less.

Day 2. The remaining days were conducted individually with the participant, either in a quiet corner in the participants regular classroom, or just outside the classroom at a lunch table. Care was taken to shade students from excessive sun and to ensure optimal temperature comfort. An unobtrusive tabletop tripod was set up and the camera pointed straight down at the whiteboard working surface, which filled the frame. Participants who felt comfortable wearing a small clip on microphone during the session (all but a handful of students) wore a microphone and audio was recorded directly onto a mobile phone. Audio files were held locally on the device until transferred to a password-protected hard drive and the local files on the mobile device were deleted. Participants were reminded that assent could be withdrawn at any time and participation was completely voluntary. Participants completed the pretest items described above to begin the session on the second day, prior to beginning Investigation 1: Area & Units- Doubling and Tripling with Pattern Blocks & Special Triangles. After completing the pretest items, participants completed Investigation 1. Video files were also transferred to a password-protected hard drive and the local files deleted.

Day 3. Participants completed Investigation 2: Fun with Frames on the third day.

Day 4. The fourth day was spent engaged in Investigation 3: Knights of the Polygonal Tables and the completion of the posttest items.

Day 5. The final session was spent on the WISC subtests for block design and picture span. These were administered in accordance with Pearson guidelines using 2 iPads and the block manipulatives included with the block design subtest. Anonymized data was sent to Pearson's Q-Interactive cloud-based scoring software interface. These data were used for exploratory analysis and not used to provide any clinical assessment to participants.

At the conclusion of the study, all participants were invited to choose a designer pencil and 2 stickers featuring rainforest animals as gratuity for participation. After conclusion of the study, remaining students in each of the participating classrooms were also provided the reward regardless of participation status.

Transcription & Coding Method. This author transcribed and coded all videos using Inqscribe 2.2.4.262. This coding process was exploratory in nature and its iterative nature was informed by grounded theory analysis (Saldaña, 2015). This researcher transcribed and coded all 3 investigations from a single participant in order to establish a coding scheme based on the behaviors the participant was observed engaging in during the investigations. Once this coding scheme was established, Investigation 2 was transcribed and coded by this researcher. This resulted in 1,881 lines transcribed. Transcripts were annotated with the coding scheme, which was iterated on through multiple passes to increase the specificity of the scheme to differentiate behaviors. The final categories of observations are presented in Table 3, with exhaustive lists of observations associated with the categories. While study resources were too meager to support the multiple coders required to establish interrater reliability, this analysis can be understood as observational notes from this experienced author who has conducted published studies on children's spatial reasoning in early geometry elsewhere (e.g., Hallowell, Okamoto, Romo, & La Joy, 2015). This table is provided as a means to orienting the reader to how the video recordings were analyzed.

Table 3. *Coding scheme categories and associated observations*

Category	Definition/explanation
Inhibition	**Participant inhibits a strategy or starts over** • All manipulatives are cleared from the target field in order to get a fresh start • Abandons current strategy
Explanation	**Participant offers some form of explanation for an action or sequence of actions taken** • A **verbal explanation** is offered for an insight or successful performance • A **visual explanation** is offered for an insight or successful performance (i.e., shows instead of tells) • Child explicitly cites a **prior experience** as a source of insight for problem solving • Cites **geometric property**, such as side length, angle, etc. • **Recites a sequence of events** describing how a geometric challenge was resolved that is either historical (i.e., matches observed behavior) or is non-historical (i.e., might make sense but contradicts observed events)
Static Spatial Reasoning	**Exhibits actions that are associated with filling a target area without reorienting the visual field (Newcombe and Shipley, 2015)**

	• Triangle is initially placed with base parallel to the “bottom” of the visual frame and the apex at the “top.” This is called the “canonical start” because it was the most observed way to begin an investigation. • Tries to force piece into non-fitting region • A manipulative fills a space a space between at least two other manipulative edges and is accepted or rejected as a fit (i.e., the child correctly accepts a fitting manipulative or incorrectly rejects a fitting manipulative) • A manipulative does not fill a space a space between two other manipulative edges and is accepted or rejected as a fit (i.e., child correctly rejects a non-fitting manipulative or incorrectly accepts it) • A manipulative is tested for fit along the horizontal axis of the frame • A manipulative is tested for fit along the vertical axis of the frame • Manipulative hangs over boundaries, but continues construction • Attempts to fill a corner with the right angle of a triangle • Begins investigation by placing right angle into apex of kite • Strategies: o Fills a layer of a frame, then builds on top or underneath the layer using the same technique o Explicitly references using filled boundaries or frame edge to visualize shape in an unfilled area o Uses side matching as a strategy to identify target orientation
Dynamic Spatial Reasoning	**Exhibits actions that are associated with some type of reorienting the visual field or manipulatives orientation to the field beyond simple translation (Newcombe and Shipley, 2015)** • A triangle is flipped onto its opposite face • Rotates the entire frame • Repeats an effective strategy • Repeats an ineffective strategy • Repeats a concatenation of 2+ pieces to cover a space • Rotates a manipulative to assess for fit • Reorients piece in transit for placement on frame in target orientation
Hint	**Participant asks the researcher for a hint** • Participant asks for a hint. A visual affordance is offered (e.g., rotates a single triangle into target alignment for the child) • Participant asks for a hint. A verbal affordance is offered (e.g., “try the other edge”) • Actions immediately following a hint indicate that the hint facilitated or did not facilitate performance
Experimentation	**Participant exhibits a novel strategy while attempting to complete a task** • A strategy or behavior is observed that is rare or unique
Conservation	**For the small house frame item, participant states that the 2 skinny rhombi cover the same space as the square, and provides a coherent justification (e.g., “because if you take away the triangle, they both cover the remaining space”)** • Equivalence of coverage is identified, and a reasonable justification is provided

Chapter 4: Results

This dissertation collected 54 hours, 54 minutes, and 18 seconds of video evidence across the 48 participants and the third-grade students who requested the investigations. It was not possible with the limited time and budget of a single graduate student to analyze the full data set in all its richness, so a subset of the data was selected for qualitative analysis as described in the Methods chapter above. The primary aim of this dissertation was to provide an exploratory analysis of elementary school children's mathematizing in geometry contexts in light of neo-Piagetian theory. Two primary research questions informed this exploration: 1) What do the activities and reflections of children engaged in open-ended geometric inquiry tell us about core knowledge in the domain of geometry at different ages? 2) What might evidence of reflective abstraction and general versus specific insights look like in grade-school geometry? In other words, do we see evidence of specific insights that arise directly from the activities engaged in (e.g., this skinny shape does not make a complete table because it does not fit evenly around a point) get coupled with general insights (e.g., shape *x* always has these properties) to drive geometric understanding across investigations. A third, ancillary research question was whether the series of investigations conducted by children in this dissertation would facilitate performance on the pretest/posttest items that were designed to draw on insights nurtured by these investigations.

For this dissertation, video data were analyzed, and transcripts were generated for 6 participants working through Investigation 2: Fun with Frames that was described above in Chapter 3. The second Investigation was selected for analysis for two primary reasons. First, it increased the potential to capture examples of hierarchical learning, since children were likely to have relevant insights while completing the first Investigation. Second, the third

Investigation entailed activities that were more open-ended, making comparison across participants less tenable. These exploratory analyses were conducted with several aims in mind. Firstly, I wanted to know what children's geometric mathematizing in the upper grammar school years might entail. Given the same investigation in second-, fourth-, and sixth grade, how might children's actions, insights, and reflections differ developmentally? To that end, cognitive clinical interviews (Ginsburg, 1997) were conducted during the investigations and form the basis of the reflections upon activities that the children engaged in during these mathematizing opportunities. Secondly, given the central geometric structure proposed above and the hierarchical learning processes hypothesized in the supporting theoretical perspectives, this analysis presents exemplars of these phenomena at work in the geometric reasoning of children. These were identified when children explicitly referred to insights from prior activities or contexts in their verbal explanations. Thirdly, half of the sessions presented for analysis here were with children who did not exhibit improvement in the number of pretest/posttest items they were able to complete unaided from pretest to posttest. The other half were children who did experience performance improvement from pretest to posttest. These cases are presented in tandem in order to highlight the maximal variation between participants as stated above (Patton, 1990). The goal of this approach is to identify patterns that are substantive differences between groups along a key variable. In this case I wanted to know, why were some students able to learn from the investigations while others had difficulty doing so?

Second Grade.

Second grade overview. The second-grade participant selected for the no improvement group was a female age 7 years and 6 months at the time of participation. The improvement counterpart for this grade was a male age 7 years and 4 months. An overview of their respective observation tallies is submitted for review in Table 4. The key differences between the second-grade participants in this study are shown primarily in two ways. The first revealed itself in the sheer volume of activity of the participant in the improvement cohort versus his counterpart. The second-grade participant from the improvement group exhibited spatial activities on the order of magnitude of 3x that of the other child. This child also seemed to fearlessly pursue experimentation in his mathematizing, stopping and beginning anew via 8 inhibition behaviors compared with 1 inhibition observation for the second-grade child from the no-improvement group. This dissertation does not claim to answer whether this is merely an individual difference from a single case study, but the contrast is stark. It is difficult to imagine that a child who is engaging in prolific experimentation would not have a greater frequency of geometric insight. In this case, the student seems to have benefited from these labors with a fruitful performance gain.

The second primary difference to note between the two second grade students is an interesting pattern in the explanations category. Where explanations with a significant verbal component were concerned, the 2 participants were fairly similar to one another. For second grade children in particular, this might be particularly challenging due to working memory limitations and the nascent nature of geometric property knowledge as outlined previously in Chapter 2. However, the second-grade participant from the improvement group offered a four-fold frequency of visual explanations in comparison with the child who did not experience improvement gains. While both children used words sparingly in their explanations, the improvement participant provided numerous visual demonstrations of his geometric reasoning. Future research might attend to the developmental opportunities and nuances of supporting children's developing spatial reasoning, particularly regarding how to encourage children to offer more visual explanations for their thinking, and when they are ready, how to start to support children's integration of visual and verbal sense-making.

Table 4. *Observation Tallies for Second-Grade Participants*

	Participant	
	second- No Improvement	**second- Improvement**
Inhibition	1	8
Explanation	19	43
Verbal Explanation	4	6
Visual Explanation	8	30
Cites Prior Experience	4	3
Cites Geometric Property	2	3
Accurate Sequence Recollection	0	1
Inaccurate Sequence Recollection	1	0
Static Spatial Reasoning	22	60
Dynamic Spatial Reasoning	17	45
Hint	4	5
Hint Facilitated Performance	2	1
Experimentation	0	1
Conservation	Yes	Yes

Second grade, no improvement cohort. While this analysis presents 2 sets of transcripts grouped by whether a participant made improvement from pretest to posttest or not, an exchange between the researcher and the second-grade participant in the no-improvement group illustrates the point that children in both groups exhibited effective mathematical insight:

Researcher (R): So, I'm wondering, can we do the same house using a different combination of pieces? [places second house frame in front of student]

Participant (P): [Places equilateral triangle along bottom edge. Quickly covers remaining space with two skinny rhombi.]

R: Wow. Have you done it using those small skinny pieces before?

P: No.

R: That was pretty cool! Have you ever seen that before?

P: Well, I've done something like that. I've done these [picks up skinny rhombus] in a circle and then fill it in with triangles.

R: I bet that was really pretty... so you already knew, you've seen that before in a different pattern, so you knew, 'hey, I could probably do this if I needed to.' That's pretty cool. Um, I 'm wondering, do you think these two put together [indicates two skinny rhombi in second frame] cover more area, or this orange one [square piece in first frame] covers more area? Which one covers more space?

P: I think they cover the same.

R: Whoa, that's pretty good! How did you know that?

P: Because, if you like shrink this one into a square [indicates rhombus] and add it to this one [indicates second rhombus] it's sort like the same as that [indicates orange square piece].

R: Oh... what if your friend was like, 'I don't believe you, you're not right.' What would you say?

P: I would say... because it like has four edges [two rhombi] and that has four edges [indicates orange square] but they're [rhombi] smaller than this [square], so if you add them together...

In this exchange the child explains that she noticed the similarity between the two concatenated skinny rhombi with the equilateral triangle to another concatenation activity she had done previously. This prior knowledge allowed the her to quickly visualize an alternative way of filling the house frame. While the child’s explanation for why the two skinny rhombi fill the same area as the square piece is a bit fuzzy, the child grasps something about the side lengths and the relative differences. She is on the verge of insight, even if she cannot quite put it into rigorous terms.

Earlier in the session, this participant was given the kite frame to fill. The child started by placing the green 3-3-3√2 with the right angle into the apex of the kite. Across all participants in this study, this was the most common way to start. After some experimentation, the child correctly placed a gold 3-4-5 triangle in target orientation in the bottom right region of the kite. Without hesitation, the child flipped the second gold 3-4-5 triangle in the air and placed it immediately in the target orientation, filling the lower region of the kite frame. This orientation gave many second-grade students trouble, since it required a student to remember the insight from the triangle-growing investigation that face-up or face down orientation mattered for the gold triangle, or at least to rediscover it. This child was able to anticipate the fit based on the opposing orientation of the first gold triangle. This shows that at least on an instinctual level, she has carried over her experience with the asymmetrical 3-4-5 gold triangle from the growing triangles exercise in Investigation 1 to the kite frame activity in Investigation 2. Over the course of these two investigations, the child has used the same spatial insight, that flipping an asymmetrical triangle can fill a space in different ways depending on orientation, in two different contexts (growing a shape versus covering a frame).

Despite the impressive visualization of the gold triangle, the student returned to the single green 3-3-3√2 triangle in the top region. The child tried concatenating it to the 2 gold triangles now properly covering the bottom region of the frame in canonical orientation (i.e., right angle pointing towards top of visual frame, hypotenuse parallel to bottom), but noticed the gap left and opted to nest the triangle in the top angle of the kite. She realized that the frame was not covered, stating "this is the farthest I could go." Compared to other participants, the 65 seconds spent on this part of the investigation was on the shorter side. Students typically spent several minutes in trial-and-error trying to cover this frame. When the researcher offered a visual hint by placing one of the green 3-3-3√2 triangles in target orientation, the participant quickly placed the final green triangle in the reflected target orientation to the hint triangle. When the researcher inquired about what the participant noticed in order to complete the frame, the child had difficulty recounting what happened:

> P: Um, that like, I tried that way [points to top portion of frame] and it didn't work, so I tried that way [points to bottom right] and it fit, so I did that one [points to 3-3-3√2 triangle above, but pauses realizing this was not the actual sequence of events]… well... I did it right there [indicates bottom left] and then.... [voice trails off].

The participant effectively fills the small and medium square frames without much trouble, but again encounters difficulty while working on covering the large square frame. The child's first approach is to employ a strategy that can often be effective: to iterate along the bottom edge of the frame a layer of squares like the one that covered the small frame. The result is the entire bottom edge is covered. The layer hangs over the left and right sides by just a half centimeter. The child iterates again to cover the top, and starts to cover the middle of the frame, but runs out of pieces. The child sees that it does not fit and clears the frame, starting over. The child repeats this again, this time tinkering with the overall fit of the pieces. Maybe if the pieces are concatenated tightly and aligned carefully to the edges, this tiling will fit. The participant realizes it does not and asks for another hint. The researcher offers a verbal and visual hint:

> R: Okay, let's look at it like this... watch this... So before we were trying the shorter sides on the outside... Let's try it with the longer side along the outside and see what that does for us [places a single green 3-3-3√2triangle with hypotenuse aligned to bottom right side of the frame].

From this single hint, the child quickly regenerated the medium-frame square construction (i.e., right angles at center, hypotenuses comprising the four sides), then iterated this to fill the large square frame without hesitation. Throughout Investigation 2, this child seemed confident in their prior knowledge, and was very effective at drawing on key strategies like iterating patterns. However, the participant did not show much evidence of openness to experimentation or discovery throughout the investigation, relying on the researcher for hints. It may be that the child was not socialized to math as a creative, generative endeavor where new insights are fair game, or perhaps the complexity of the challenges were enough and striking out new insights was too much to add to an already burdened working memory bank. Future studies might investigate both these factors.

Here the child was observed drawing upon prior knowledge when working with skinny rhombi to fill a space in a central angle. She was seen effectively utilizing an asymmetrical 3-4-5 gold triangle manipulative, visualizing how to fill a space in the bottom of the kite frame she was attempting to cover. At the same time, she got stuck on the kite frame and appealed for a hint relatively early in her investigation. Again, she was stymied while attempting to cover the medium square frame, asking the researcher for a hint. This child exhibited numerous moments of geometric brilliance, but did not show any particular bent towards exploration or experimentation and relied on the researcher as a source of knowledge relatively quickly when stuck.

Second grade, improvement cohort. While attempting to cover the kite frame, the second-grade participant in the improvement group displayed a great deal more trial-and-error actions, as well as a greater variety of strategies and actions in comparison to the second-grade peer discussed above. The researcher observed 5 actions for the other second-grade participant versus 25 actions for this student prior to asking for a hint. When this student was provided a hint, he immediately completed the frame covering. When asked how the hint helped, he explained, "you just do the same thing on the other side." The child had a grasp of the significance of symmetry.

When observing the participant cover the medium-square frame, the researcher noticed that the child covered that frame very quickly compared to most children in the study. The researcher asked the child to explain how he figured it out. The child explained:

> P: So, what I did is I turned into the triangle shape again [aligns right angle of green 3-3-3√2 triangle in the top right corner of medium square frame] and did all that [demonstrates rotating, frame side-length match-testing in medium square frame]. So, I first started like that [shows process of getting green 3-3-3√2 triangle matched to left side-length of medium square frame] then like that [demonstrates process of trial-and-error strategy for concatenating the second triangle to complete the top-left half of the medium square frame]. Then it was like the same thing as this one [grabs kite frame and holds alongside medium square frame with concatenated triangles that match the construction that fills the top portion of the

kite frame]. So, it was just um, it was just smaller {means narrower} on the bottom {portion of the kite frame}.

Figure 16 depicts the participant's explanation that he repeated the construction from the top portion of the kite frame to fill half of the medium-square frame. Given the child's grasp of symmetry discussed above, it makes sense that he was able to cover the medium frame so quickly.

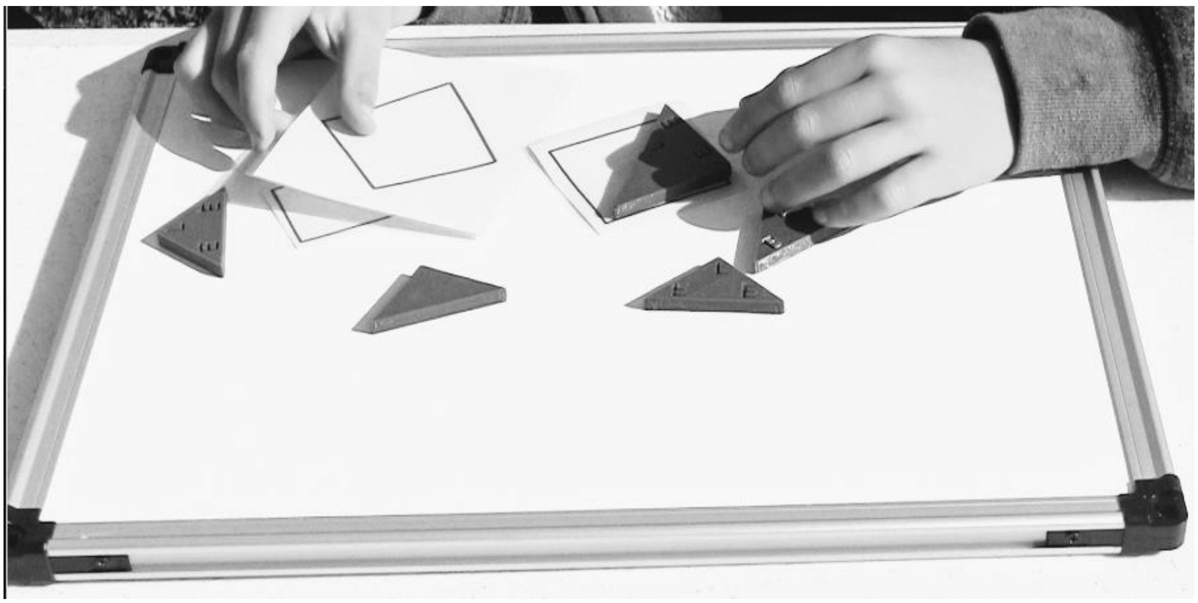

Figure 16. Second grade participant explaining that half of the medium-square construction along the diagonal line of symmetry requires an identical construction to the top portion of the kite frame.

While it is tempting to interpret the child's explanation about iterating the construction from the top of the kite, it is unclear whether the child made this connection prior to resolving the medium frame. When the child is given the large frame to fill, the child struggles to fill for the frame for some time. There is no evidence that the child is able to iterate the medium square construction they seemed to masterfully produce moments ago. The participant asks for a hint after some time spent experimenting. The researcher indicates that the medium-frame construction may hold a key insight. The child does not make the link and covers the frame through trial-and-error after some time has transpired.

After covering the house frames two different ways, the participant seems to reason his way into conservation:

> R: So another question I have for you... looking at these two houses, do you think this shape right here covers more space [indicates square of first construction on frame], the orange one? Or do you think that these 2 cover more space [indicates both skinny rhombi on second frame]?
>
> P: Well... These 2... No... Well... They cover the same amount of space.
>
> R: Oh! Wow. How did you know that?
>
> P: It's because if you use the same one [indicates equilateral triangle] and you can make the same one fit, then it would just be the same size.
>
> R: So, could we say, each one of these [indicates one of the skinny rhombi] is something about a square?
>
> P: Um, well, like... well... [holds up skinny rhombus] this is not a square. This is a square [holding square pattern block in other hand]. So.... so what would you do if you had this [shows a skinny rhombus in his hand] and then you had this [holds up square]?

This child's explanation for why the 2 rhombi cover the same space as the square is one of the most robust generated by any of the students in the study. Many participants made topological statements that were geometrically sound but based on visual estimation (e.g., this part that sticks up could be flattened to make it like a square). Using the negative space occupied by the triangle to make an inference about the remaining area approached a kind of inferential rigor that a robust verbal if-then statement could inject into proof. The researcher was surprised to encounter this from a second-grade student. When the researcher pushed a bit further to see if the child might make another inference about a single rhombus covering half the area of the square, the child evaded an answer and the conversation meandered out of the realm of geometric rigor.

Fourth Grade.

Fourth grade overview.

Both fourth grade participants selected for analysis were male. The student from the no improvement cohort was 10 years and 2 months old at the time of participation. His peer from the improvement cohort was the same age at the time of the study. Table 5 summarizes their respective observation tallies. These 2 participants differed along some important dimensions. The participant from the no-improvement cohort showed many more instances of static and dynamic spatial reasoning and had many more inhibition instances than his improvement counterpart. This was because the student engaged in much more trial-and-error sequences than did the participant from the improvement cohort. Whereas the second-grade no-improvement child demonstrated a reluctance to experiment and perhaps an overreliance on the researcher for mathematical insight, the fourth grade no-improvement participant did not show this pattern. Instead, he was willing to spend long periods of time engaged in problem solving and experimented with novel approaches to filling the frames. It is important to note that the novel approach the child experimented with was attempting to fill the frame by starting to construct at the line of symmetry. This turned out not be adaptive in most cases and may have indicated some non-typical gestalt way of perceiving the visual field.

Another interesting contrast shows itself along the lines of verbal explanations. Eight of the 25 verbal explanations that the fourth-grade participant from the improvement cohort referenced geometric properties (e.g., "a rectangle has four sides"), whereas the participant from the no-improvement cohort did not provide any direct references to geometric properties. This is in line with the working hypothesis of this dissertation, and with both Clements' and Case's positions: that imagistic and verbal schemas support optimal development in geometric thinking & insight. The idea is that as children engage in the kind of geometric mathematizing activities provided to participants in this study, they should have a richer framework from which to gather and organize their insights, and then to generalize these mental models to novel problem spaces in the domain. It is hoped that future research will investigate the nature of this using experimental methods.

Table 5. *Observation Tallies for Fourth-Grade Participants*

	Participant	
	fourth- No Improvement	**fourth- Improvement**
Inhibition	14	3
Explanation	39	59
Verbal Explanation	15	25
Visual Explanation	21	20
Cites Prior Experience	2	3
Cites Geometric Property	0	8
Accurate Sequence Recollection	1	2
Inaccurate Sequence Recollection	0	1
Static Spatial Reasoning	76	38
Dynamic Spatial Reasoning	37	26
Hint	2	2
Hint Facilitated Performance	1	2
Experimentation	4	0
Conservation	Yes	No

Fourth grade, no improvement cohort. The fourth-grade participant in the no improvement group also struggled with the kite frame for a bit, unfruitfully rotating and flipping triangles in trial-and-error before asking for a hint. As with the participant above, the researcher placed a single green 3-3-3√2 triangle at top left as a hint upon request. The participant then concatenated a gold 3-4-5 in the target orientation in the bottom portion of the kite, covering the left half of the frame. The participant then tried to fit the second gold 3-4-5 triangle to the first by matching the side 5 of the triangle to side 4 of the first triangle running along the line of symmetry. The second gold 3-4-5 triangle hung well over the side of the frame in a poor fit, so the child removed it and tried the second green 3-3-3√2 triangle with no better result. The child appeared to have difficulty assessing the empty space and the shape of the kind of triangle needed to fill that space.

The green 3-3-3√2 was not a close fit for the bottom right region of the kite, but the child seemed to genuinely struggle with whether the gold or green triangle should go there. While many of the children 2 grades below did not have difficulty quickly completing a frame already covered on one half of a symmetrical frame, this participant appeared to glean little value from achieving this milestone. After more trial-and-error, the child discovered the target orientation of the second green triangle. Lastly, the child discovered the target orientation of the remaining gold triangle after more trial-and-error where 360 degrees of rotation resulted in the insight that a reflection was needed.

As was the case for the second-grade participant in the example above, the child did not experience excessive difficulty covering the small frame. When filling the medium frame though, some subtle but perhaps pernicious glitches were observed:

> P: [Begins by concatenating along hypotenuses, recreating small square. Leaves one green 3-3-3√2 triangle in top left corner, right-angle in. Rotates second green 3-3-3√2 triangle in trial-and-error strategy, reaches target orientation along bottom,

but not matched to frame side. It seems he does not notice that the triangle is in the target orientation because instead of joining it to the side of the frame and constructing off its placement, he removes it and rotates it. The triangle is placed in target orientation in relation to right side, but again not matched to the side. Removes and rotates. Gets a second triangle in target orientation along bottom, concatenates to piece along right side but doesn't match to sides. Pulls apart, tries another going RA in at bottom right. Removes. Tries concatenating 2 green 3-3-3√2 triangles so the hypotenuses meet at vertical line of symmetry. Removes, aligns single green 3-3-3√2 triangle so hypotenuse is aligned to horizontal line of symmetry. Recreates "small square" just underneath hypotenuse]

I need a hint.

R: You want a hint? Here, let me give you a hint. What do you notice... [aligns green 3-3-3√2 triangle in target orientation along right edge]. What do you notice there?

P: Here?

R: Right here [indicates triangle aligned to right side of medium frame]. What do you think?

P: [concatenates second green 3-3-3√2 triangle to first triangle along right side]

Oh!

[Places third triangle in target orientation along left side. Quickly fills in remaining space to complete medium frame]

R: What did you notice there? What happened?

P: Well I kind of already knew this I just forgot it, I guess.

R: Yeah, that's okay! What did you notice though? What did you know?

P: Um... instead usually you'd wanna do this [demonstrates matching two green 3-3-3√2 triangles to top corners, right angles in] ... but, instead you can maybe go like that [demonstrates 2 green 3-3-3√2 triangles aligned along right and left sides]

R: I see.

P: And then, that [fills in top triangle] because I was looking for something that formed the shape of this triangle.

Here the participant uses the vertical line of symmetry as the focal point of the visual field. Multiple times the hypotenuse of a green triangle is parallel to the side it needs to be matched to in order to fill the frame, but the child does not notice this. There is a gap between the hypotenuse and the matching side of the frame, and even when the researcher explicitly aligns the hypotenuse to the side as a hint, it takes a moment for the participant to have the insight. The child's attention was naturally focused on the line of symmetry, and the child ignored the spoils of the frame side lengths until it was impossible to ignore.

The child found the large square frame troublesome when approaching the frame with the same strategy. Very few children in this study did not use a perimeter-first strategy to fill the outside edges of the large square frame prior to filling in the middle area last. In this case, the child iterated the medium square construction up the line of symmetry as a starting strategy (see Figure 17). When the researcher asked the child about their strategy, the child was silent and did not provide a response. Despite having just seen the side-matching strategy in the medium frame exercise, the participant's instinct to construct up the vertical line of symmetry was robust.

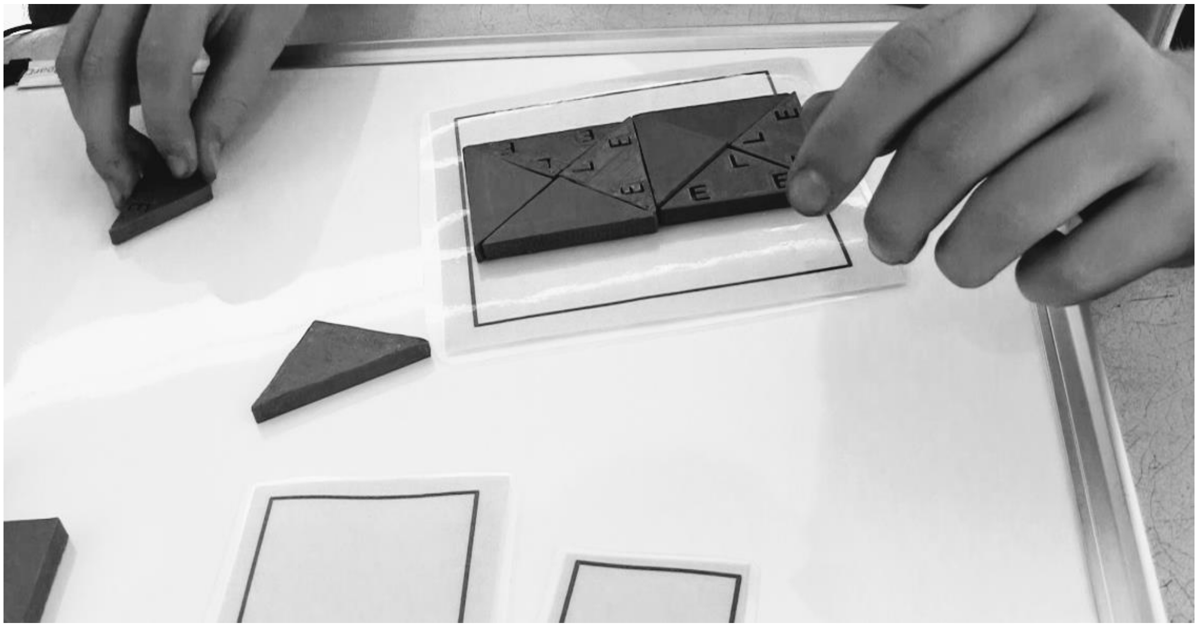

Figure 17. Fourth grade participant from the no improvement group using the line of symmetry to fill the large square frame

For a child who can effectively visualize the frame as a reflected figure along the line of symmetry, constructing the 2 medium square constructions centered over the line of symmetry might be the visual cue that unlocks the solution. Simply shift the squares up or down to the edge, repeat, and complete. Many children in the study exhibited such behavior in similar contexts. This child attempted to concatenate a single layer of green triangles on one edge before realizing there was not enough space. Eventually the child did shift the construction down to the bottom edge, but instead of iterating the same construction, added to it in a trial-and-error fashion until it was completed. The possibility of dividing the space into sub-regions where effective patterns could be repeated seemed invisible to the child.

Due to a scheduling conflict, the child needed to stop for the day. The following day the child was very eager to talk about their insights from the prior day. Asking for the frame and triangles, the researcher inquired:

R: So this is what you figured out yesterday when we were doing our activity. So you were thinking about this?

P: Yeah.

R: For like, a long time.

P: Yeah.

R: Interesting. Did you practice at home with anything, or....

P: No.

R: No. You were just thinking about it in your mind?

P: Yeah, it just came into my head.

R: Cool! I love it when that happens. Some of the most creative people think like that.

P: [Constructs the perimeter from left side, then top, then bottom, then tiles the interior. Instead of concatenating two medium squares, completes interior with two green 3-3-3√2 triangles reflected along vertical line of symmetry (hypotenuses concatenated)]

And then, I guess it's just double the size of this.

R: Ahhhh.... but that's a different way, right?

P: Yeah.

R: So you figured out both ways.

P: Uh huh.

R: That's really good. Um, actually, I haven't had anybody figure out both ways yet. You're the only one. Nice job.

While this participant may have had difficulty with visualizing space as units for iterating, the child was engaged and invested in the activity. The participant went home and spent enough time trying to visualize the possibilities that they were able to return to school the next morning and share their geometric insight with the researcher. This time the construction was not the product of a trial-and-error process, but instead was the child's construction. The child had put in the effort to visualize the frame and to consider the pieces and was able to produce the construction quickly without the aid of the frame.

This child seemed to understand the visual field differently from the other children. With half the kite frame covered along the line of symmetry, he did not perceive the second half as merely a kind of iteration of the covered part. It was an alien landscape, still awaiting its own solution. This problem presented itself again in the context of the sequence of square frames. I wondered, had this child a set of verbal syllogisms about symmetry and iteration, would that have improved his capacity for learning from each of his trial-and-error endeavors? Ironically, although he had difficulty seeing how covering 1 half of a symmetrical figure made covering the second half a simpler affair, he was also overly focused on starting his constructions along the line of symmetry. He ignored the external boundaries of the frame as tools for side matching. I wondered how some verbal hints that the child never requested might have helped him to inhibit his excessive attention on the line of symmetry. An astute teacher might ask such a student covering the large square frame, "How many of these [hypotenuse] side lengths match to one side? Does it change from side to side?" It was not the child's instinct to reflect on these things, and they never came up during the interview.

Fourth grade, improvement cohort. The fourth-grade participant in the improvement group also began the kite frame exercise with bold experimentation. Instead of trying for the top portion of the kite with a green 3-3-3√2 triangle in canonical orientation as was commonly observed, this child immediately saw that the long hypotenuse of the gold 3-4-5 triangle was needed to match the long side length of the bottom left side of the kite frame. He proclaimed "Easy!" and then struggled to ascertain why the second gold triangle would not fit no matter how many times rotated. The participant did not seem to remember the insight about flipping the gold triangles from the shape growing exercise the day before. The child removed the gold triangle and tried a green triangle, then tried gold again, then cleared the whole frame and started over. The number and variety of actions observed to this point far exceeded that of his fourth-grade peer in this analysis. The child cleared the frame multiple times to start over with another permutation of triangles (e.g., gold to green to gold just in case something changed to 2 green, etc.).

After a while of this, the child asked for a hint. When the researcher placed a single green 3-3-3√2 triangle at the top left region of the kite frame, the child immediately concatenated a reflected green triangle to fill the top portion, then quickly filled the bottom portion with 2 gold triangles in target orientation. The researcher asked the child how he might help a peer struggling with this investigation:

> R: What would you say to somebody else, if you're trying to help your friend, what advice would you give them about this one? What's important to notice?
>
> P: So... if it was hard for them, I'd put it right there too [places green 3-3-3√2

triangle at top left, just as the researcher had done for them]. Well if I was helping a little kid then I would give them a hint [places gold 3-4-5 triangle in target orientation at bottom left] and then...

R: That's exactly what I do with the little kids!

P: [fills top of kite with second green 3-3-3√2 triangle] And then, I would tell them, 'put the same on the other side.'

R: I see. 'Put the same on the other side.'

This participant suggested a combination of visual and verbal clues to help a struggling peer. The child was comfortable utilizing both modalities in geometrically oriented mathematizing contexts.

Moving on to the sequence of square frames, the participant's activities and reflections provide some interesting characteristics:

R: Now I'm wondering, using the green triangles [places small square frame and green 3-3-3√2 triangles in front of student], can we cover this frame? Is that possible?

P: [Grabs a green 3-3-3√2 triangle, aligns to top left in target orientation. Quickly rotates second green 3-3-3√2 triangle to place in target orientation, filling the frame.]

R: Wow, that was way too easy for you. So how did you know to do that?

P: I just went like that [places single green 3-3-3√2 triangle so vertices extend beyond opposite sides] and saw that it was too big. But then I just put it at a random spot [aligns to corner in target orientation] and I was like, 'whoa! Look at that.' And then I saw the other half [demonstrates completing the fill with the second triangle] and it was an equal shape.

For some children, it is difficult enough to construct a narrative that makes sense for how they arrived at a solution in the realm of geometric reasoning. Reporting a historically accurate sequence of events with commentary is just too far a stretch for some children, but this child can keep track of his own process. This allowed him to rule out ineffective actions he had already attempted so he could spend his energy on new experiments. I was struck with the sense that this was one key to his success.

R: Nicely done! So is it possible to do this one? [places medium square frame in front of student].

P: Uh huh. [Grabs green 3-3-3√2 triangle, begins trial-and-error strategy of fitting piece to target orientation. First, tries the small square frame construction at top left, sees that it doesn't fit. Tries a single green 3-3-3√2 triangle with the right angle in toward the bottom left. Repeats with a second green 3-3-3√2 triangle at top right. Abandons, tries other novel orientations. After multiple trials, realizes that hypotenuse matches medium square frame side length. Leaves first green 3-3-3√2 triangle in target orientation at left side. Takes second green 3-3-3√2 triangle, flips, rotates into target orientation at bottom edge. Rotates third green 3-3-3√2 triangle into target orientation along right side. Quickly completes construction at top].

The participant's capacity for tracking and experimentation was very clear from his actions. The child did not try the same thing twice and was systematic about where he conducted his geometric investigations. Something else was on the child's mind that could not be directly observed in his behavior:

R: Hmmm, interesting. Nicely done! Good work. What did you notice there? What was the secret?

P: [clears frame]. I just... I was like playing around with it [demonstrates trial-and-error strategy] and then I just went like... so I'll line this up [indicates hypotenuse to side match] and then I saw another... I saw a negative shape right here [replicates placement of green (3-3-rt3) triangle at bottom of frame].

R: Ah, a negative shape!

P: I saw 2 more of the negative shapes [aligns along right side, then completes top]. This one, you fit it right there [triangle in top].

I: So, where did you learn about negative shapes? How'd you know about that?

P: Well, I just like... in art, when I used to do art here, we went to art class. We did this thing with foam and then those were positive... our teacher said like, there is like positive shapes and there are negative shapes.

R: Oh.... that's fascinating!

The participant took a valuable notion learned in an art setting and applied it effectively here in this geometry context. He noticed important aspects of the context and made links to prior knowledge. The child showed similar intentionality when completing the large square frame, although the participant needed a hint to complete the middle region of the large square frame. When the researcher asked the child to explain how he solved the challenge, the participant's explanation exhibited an interesting form of regression:

R: Wow, that's really good. Um, and then what was hard about the middle [indicates rectangular space in center of construction]?

P: Because the first time I did this earlier today, I kinda forgot.

R: Uh huh.

P: So, like... I went like that. [Removes triangles from center region, attempts to demonstrate his earlier process but inadvertently places green 3-3-3√2 triangle in target orientation at left side, then bottom] It didn't really... [notices that what he is demonstrating adequately fills the region] well, it could have worked like that but... I went like that [takes one triangle to show ineffective strategy of concatenating the hypotenuse to the first triangle, which remains in target orientation]. And that didn't work [stops and quietly reflects on the incomplete construction].

Several things have happened here for the child. For the first time in the session, the child's explanation does not match the actual sequence of events that transpired. He seemed to have lost track of the experiments that were conducted. Perhaps most interestingly, this is the first time the child provides solely visual explanations for what occurred, showing rather than telling what transpired.

Moving on to the conservation exercise with the 2 house frames, the child is not sure whether the 2 rhombi are equal or different from the square. When challenged, the child does not have clear reason why they would be the same but estimates it might be the case. When the researcher and the participant discuss doubling the house, things get interesting:

R: Okay. What if your friend did something like this [extends rectangle by an additional layer laterally resulting in 2x3 rectangle, moves 2x2 triangle at top of house over the center], 'There you go, it's doubled!'

P: I'd say like, well, that's not equal because the edge [indicates open edge of square at top right between the 2x2 triangle and the corner]. It has to be like, equal [removes extra column of squares, moves 2x2 triangle so corner aligns to 2x2 square corner]. It has to be like touching the edges.

R: Ah, it has to line up.

P: And that wasn't touching the edges [demonstrates previous 2x3 construction]. It [indicates 2x2 triangle] was in the middle like that.

To this point, the conversation has been somewhat circular. The criterion for whether the house construction has been doubled or not has to do with the square sides lining up with the triangle corners.

R: Oh, okay. Alright. Very cool. And what if... Let me ask you this, what if they did this [moves 2x2 triangle to left side of 2x3 rectangle] and then they made... they finished building this part here. Let's do it real fast [completes 3x3 triangle atop 2x3 rectangle].

P: Can I try to triple this?

R: You can but really quick... Actually, I'm getting at that. So, I want to ask you, what if your friend did this [completes 3x3 triangle atop 2x3 construction] and said, 'It's doubled!' What's wrong with that?

P: It's NOT doubled because this [indicates 2x3 rectangle] is a rectangle not a square.

At this point in the conversation, the participant has started to appeal to properties, like shape categories. The child was not careful to make the distinction that a square is in fact a special kind of rectangle. The child was too busy making geometric distinctions.

R: Nice!

P: If you put three extra down here [indicates bottom edge of 2x3 rectangle] that'd make it a square.

R: So do you wanna fix it?

P: [takes 3 squares and completes bottom layer]

R: Nice, how did you know that?

P: I remember all this from yesterday.

R: Wow. That's so cool. That is a really cool house. I bet you, you could make it five, six, seven times bigger if I gave you enough pieces. I bet you could figure out how to do that.

P: Yeah. It's crazy how [Disassembles tripled house] it starts out from this [indicates first house construction] all the way to this [reconstructs tripled house]. It's kind of like multiplication, but with shapes. So you do it three times. So..

At this point, the child has started to think about the shape components as units, and the child has noticed that units can be subjected to multiplication. The mathematizing wheels are turning.

R: Here, do you want a pen? Do you want to write down numbers?

P: Yeah.

R: Okay, let me give you a pen. I want to hear more about this, because this is fascinating. So tell me about multiplication with shapes. What's going on here?

P: This [draws first house construction on whiteboard, followed by multiplication symbol and numeral three, equal sign] equals [erases equation with finger]

R: Oh, let me get ya a little towel so you don't have to do that with your finger.

P: Tighten this up a little bit [realigns tripled house for good fit]. So... [redraws equation so that tripled house appears on the other end of the equals symbol].

Figure 18 depicts the participant’s multiplication of houses schema and notation. This investigation became a playground of ideas for the child during this segment.

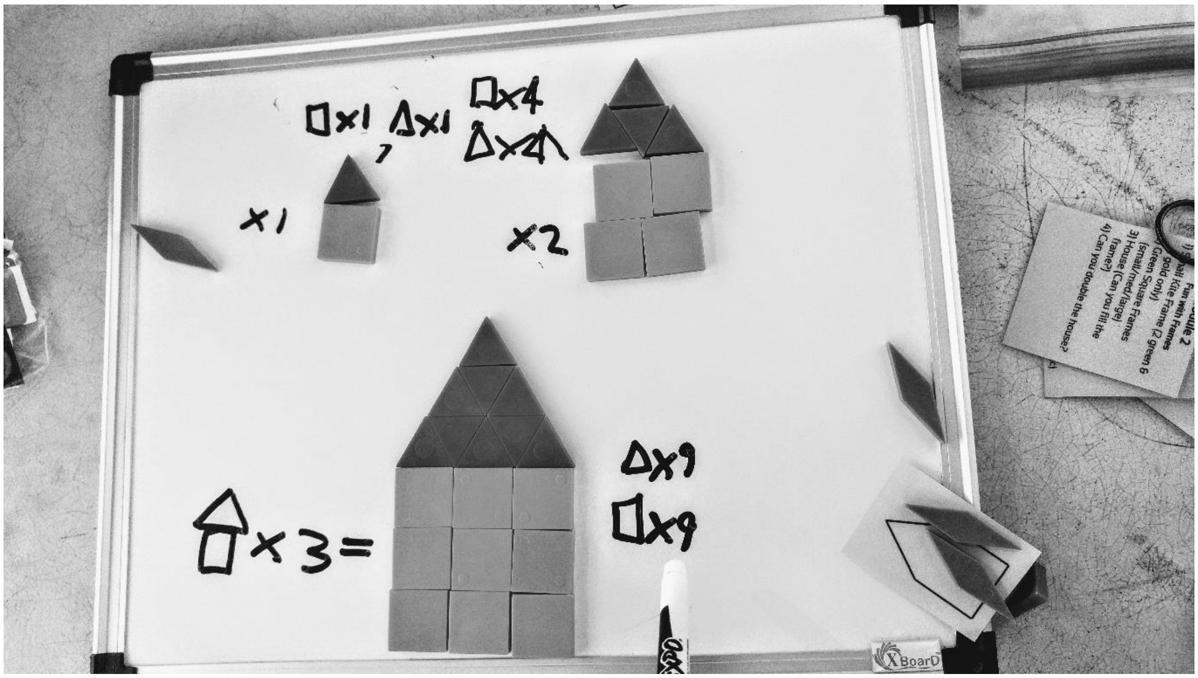

Figure 18. Multiplying shapes to make houses of different scales.

R: I think that's brilliant. Okay. I really like this. Multiplication with shapes... that's a really cool way of thinking about it. So I wonder if... I wonder, when we're multiplying [begins placing additional squares and triangle in front of student] we start off with this house [constructs first construction]. How many squares and triangles do we have?

P: You have [touching each with pen] 1 square and 1triangle.

R: Okay. So that's times one, right?

P: Yeah. So, I'll write [writes "x1" next to construction]

R: Okay. And then when we do this [constructs doubled house]...

P: Times two [writes "x2" adjacent to doubled house]

R: Times two. How many squares do we have?

P: So, 2 squares.

R: We have two squares here [indicates 2x2 square construction of doubled house]?

P: Oh yeah. No. I mean... [writes "1 square" above single house] 1 square. I'll just do symbol shape [erases shape name, replaces with geometric symbol]. So there's...

The notation for this mathematizing activity is relatively open. There is no adult at the front of the room to tell the child what to write. This experiment is for the child, and the child is leading the inquiry.

R: How many squares do we have here [indicates doubled house]?

P: [Counts four squares with pen] 4 squares.

R: Okay, 4 squares. how many triangles do we have?

P: [Counts 4 triangles]. 1, 2, 3, 4.

R: Interesting. This is really great, by the way.

P: [Carefully writes shape symbols and quantities adjacent to doubled house]

R: And then how many squares do we have here [indicates tripled house]?

P: So... one, two, three [counting aloud with pen, by column from left to right].. 1, 2, 3... 9.

R: Nice! You used an array there, huh? You didn't even count them.

P: [Anticipating next question, counts 3x3 triangle quantity at top. Counts each shape individually with pen, from left to right and bottom to top] 1, 2, 3, 4, 5, 6, 7, 8, 9. [Completes annotation adjacent to tripled triangle]. One thing that I realized is they have the same amount of shapes. So this one [indicates first house construction] has 1triangle, 1 square. This one [indicates doubled house] has 4 triangles, 4 squares. And then this one [indicates tripled house] has 4... this one has 9 triangles and 9 squares.

After having created a manageable taxonomy of shape multiplication examples, the participant starts to extrapolate properties. The child is looking across the examples for new layers of insight, and the child finds them.

R: You're right! What's goin' on here?

P: It's like multiplication.

R: It's like multiplication! This is fascinating...

P: But it's not with numbers, really.

This fourth-grade participant from the improvement cohort exhibited a remarkable degree of coordination between the visual and verbal systems, to the point of articulating an analogy of geometry as multiplication. When the participant was pushed to the limits of this with the large square frame, the child’s words faltered in conjunction with his performance. The child’s enthusiasm returned when he began to make logical-mathematical statements around the manipulatives he was working with, as depicted in Figure 17 above. There seemed to be a correlation between the child’s general ability to accurately recall the sequence of his processes, and his efficient, intentional experimentation. Watching this child engage in these activities gave the impression a strong coupling of the imagistic and verbal systems in geometry provided a robust framework for effective insight making.

Sixth Grade.

Sixth grade overview. The selection process described above yielded 2 female students for the sixth-grade analysis. The sixth-grade participant representing the no-improvement cohort was age 11 years and 9 months at the time of the study. Her improvement counterpart was age 11 years and 11 months at the time of participation. A summary of their observation tallies is presented in Table 6. In this case, the 2 participants are similar in many respects, with some important distinctions. The biggest difference is the frequency of verbal explanations for geometric insights that the researcher asked for. The participant from the improvement cohort gave 14 verbal explanations for a geometric insight that the researcher inquired about, compared to only 4 for the no-improvement cohort child. Of those 14 verbal explanations, 12 referenced some specific geometric property. The no-improvement participant did not provide any of these. Additionally, the no-improvement participant asked for 5 hints, but none of the provided hints seemed to lead to facilitated performance. Immediately after the researcher provided the hint, the participant continued to struggle as before. In this case, the child from the improvement cohort did not request any hints. The child from the no-improvement cohort exhibited more actions associated with static spatial reasoning, mostly in the course of extended trial-and-error sequences.

Table 6. *Observation Tallies for Sixth-Grade Participants*

	Participant	
	Sixth- No Improvement	**Sixth- Improvement**
Inhibition	7	7
Explanation	16	31
Verbal Explanation	4	14
Visual Explanation	8	5
Cites Prior Experience	3	0
Cites Geometric Property	0	12
Accurate Sequence Recollection	1	0
Inaccurate Sequence Recollection	0	0
Static Spatial Reasoning	57	35
Dynamic Spatial Reasoning	38	37
Hint	5	0
Hint Facilitated Performance	0	0
Experimentation	1	0
Conservation	No	Yes

Sixth grade, no improvement cohort. The sixth-grade participant in the no improvement group also had trouble with the kite frame. The participant started off with a few of the common strategies: trying the green 3-3-3√2 triangle in canonical orientation in the apex of the kite, trial-and-error rotation, triangle swapping in various corners to test fit. A minute into the activity, the child rotated a gold 3-4-5 triangle into the target orientation in the bottom left region of the kite. While many of the sixth-grade children in this study quickly resolved the remaining placements on this frame once they identified the target orientation of one of the gold triangles, this participant paused, noticed the gap between the green 3-3-3√2 triangle resting in canonical position at the apex, then removed the gold triangle entirely and tried a green triangle in this position. Figure 18 shows the moment of decision when the child was evaluating the construction for a match. Perhaps the child felt confident that the green triangle at the top was correctly placed, so the gold triangle's side match was not salient or convincing. It is also possible that the gold triangle's fit was too tenuous since it was only supported by a single frame side, whereas the green triangle had 2 frame sides to support its case for correctness.

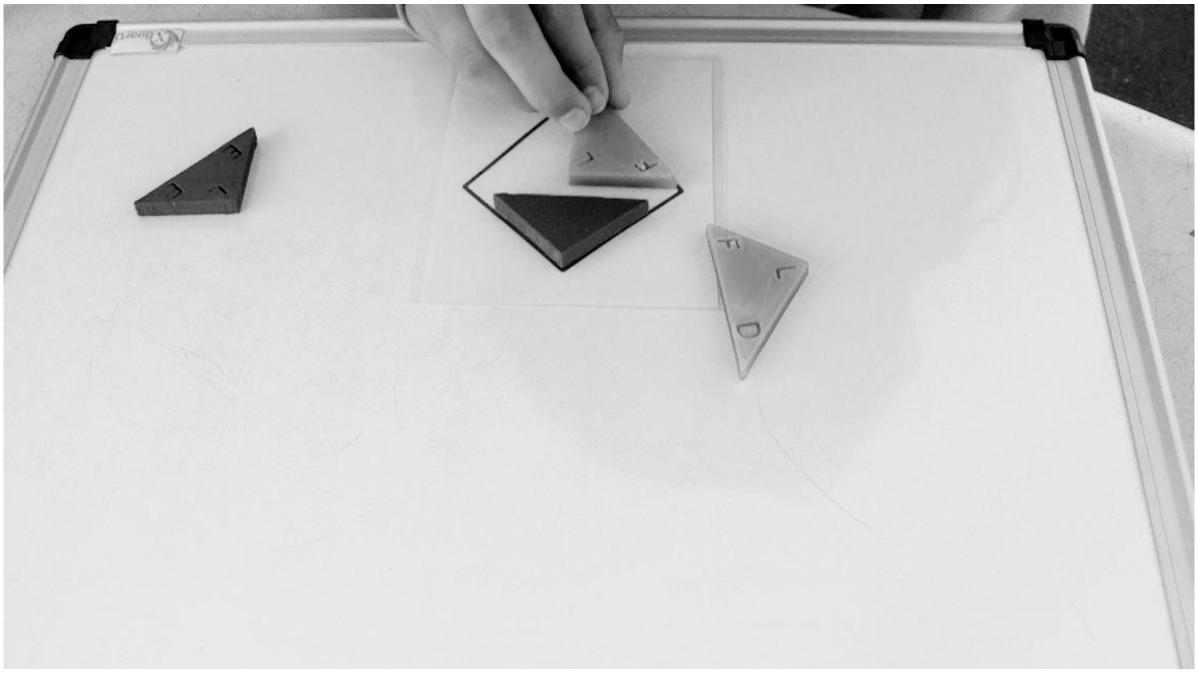

Figure 19. Participant ponders accepting the gold 3-4-5 triangle in target orientation, ultimately deciding against it

The child requests a hint following this sequence of events, but the researcher encouraged the child to experiment a bit further before taking the hint. The participant continues the investigation:

R: I actually think you can solve this without my help.

P: [Engages in trial-and-error strategy with 2 gold 3-4-5 triangles, rotating and flipping, but focused on aligning to the side vertices. Repeats trial-and-error strategy that results in inverted kite negative space. Tries concatenating 2 gold 3-4-5 triangle along hypotenuses at top left, forming a rectangle that doesn't fill the top left sector. Grabs a green 3-3-3√2 triangle and tries to fit it into negative space that is much too narrow for any possible orientation. Disassembles. Tries gold 3-

4-5 triangle nested in right vertex and green 3-3-3√2 triangle nested in left vertex. Attempts to fit gold 3-4-5 triangle in between the 2 triangles, pushing gold 3-4-5 triangle at left vertex out of the frame. Tries rotating and aligning green 3-3-3√2 triangle at bottom vertex. Places green 3-3-3√2 triangle so hypotenuse is aligned to vertical line of symmetry in bottom vertex. Attempts trial-and-error strategy with additional green 3-3-3√2 triangle in top right quadrant, rotating but not finding a workable alignment. Disassembles.]

At this point, the child's activities appear somewhat random to the researcher. The child moves from vertex to vertex. In one moment, the child tries 2 different triangles in the left and right vertices, indicating that either they are unaware of the implications of symmetry in the context, or the child is having difficulty keeping it all in mind during the activity. There is no direct evidence that any kind of insight is carrying over from trial to trial. The researcher intervenes:

R: So I'll give you a hint, you had the gold one in the right spot earlier.

P: This one? [places gold 3-4-5 triangle at right vertex, s3 toward bottom].

R: Um, when it was down below.

P: Right here [concatenates to second gold 3-4-5 triangle along hypotenuses to create rectangle at top left]?

R: Nope, the other way. Down in the bottom.

P: [Isn't sure what to do. Holds gold 3-4-5 triangle in bottom half, flipping and rotating without placement]. Oh.

R: You had the long part down and the flat top up.

P: Like that [places gold 3-4-5 triangle in bottom vertex, right side, hypotenuse along line of symmetry]?

R: [Removes second gold 3-4-5 triangle. Places first gold 3-4-5 triangle in target orientation in bottom left region]. So you did that.

As the researcher attempts to provide the hint, it becomes clear that the child did not notice the moment the gold triangle was in the target orientation. The child has great difficulty receiving the hint as anything other than direct placement of the manipulative by the researcher in the target orientation. The verbal hints are completely unhelpful to the child. The verbal and the visual are not deeply connected for this participant.

P: Oh, there you go [concatenates green 3-3-3√2 triangle to top of gold 3-4-5 triangle so that bottom left vertex is nested in left side vertex of kite frame. Picks up second gold 3-4-5 triangle, flips, concatenates to first in target orientation. Translates green 3-3-3√2 triangle so that it is concatenated to gold triangles, and its line of symmetry is aligned to the kite frame line of symmetry. Looks expectantly at the researcher]. Yeah?

The child is ready to accept an approximation. Perhaps this exercise has been exhausting for the participant and the child is ready to move on. The researcher prompts to see if the participant can go a little further:

R: Let's see if we can make it fit with both green ones.

P: [Discards the first green triangle. Grabs the second, tries placing it in canonical orientation into angle at top of kite. Picks up two green 3-3-3√2 triangles and sees that one is rotated in a potential matching orientation. Places at top left in target orientation, then removes. Holds two green 3-3-3√2 triangles in canonical orientations. Recreates prior rotation, places at top left in target orientation. Rotates second triangle and places at top right to complete construction]. There you go.

With the bottom region of the kite covered by the 2 gold triangles and the stipulation that the top region of the kite be covered with the 2 green 3-3-3$\sqrt{2}$ triangles, the participant completed the frame in relatively little time. She completed the small and medium square constructions without excessive difficulty, but responded "You know, I really don't know" when asked how she solved the medium square frame covering. The child moved on to the large square frame and after persisting through some difficulties getting the covering to fit to the frame, found success. The researcher inquired:

R: Nicely done! That's kinda hard, how did you figure that out?

P: Oh, I just copied this one [Indicates medium square construction]

R: Nice! Okay. But you didn't copy it... you didn't just... [Moves medium square construction adjacent to large square construction] some kids just straight copy it. You can do it this way.

P: Yeah.

R: If we wanted to make it a perfect copy, what would we do to this [indicates large square construction] to make it a perfect copy? Can you see it?

P: Yeah, you would just do the same thing [points to center construction], over again [points to four corners].

R: Yeah you'd have to take this [indicates interior triangles] and turn it so the line went that way [indicates perpendicular concatenation line], and then you'd have four. You'd take this out, turn it [demonstrates action with hand over pieces], take this out, turn it. Take this out, turn it. So, that's really cool. Nicely done. Um, if you were giving hints to someone who is having a hard time, what would you say?

P: Um, I would say, like... try a different strategy. You should flip over the triangles, like what you said. You shouldn't just look at it, just holding it... just look at it another way.

R: Oh, I see.

P: Try putting them in different places [Demonstrates trial-and-error placement strategy].

R: Ah, I see. So reorient it on the frame. Well done! Okay.

The participant seemed to have trouble finding words to attach to the images. She could see the differences and similarities between some constructions, such as the medium-square construction and its twin embedded in the large-square construction. This knowledge was useful in some contexts but made systematic geometric insight difficult and limited. This participant went on to construct both house constructions without a moment's hesitation. The child had worked with the skinny rhombi in an earlier grade level, and very quickly carried the experience over. When asked the conservation question, the child confidently asserted that the square pattern block covered a greater area than the 2 skinny rhombi, since it was "bigger." The researcher probed about the remaining triangle pattern block in both cases, but this did not prompt insight as it did with other children in this study. This participant did not have the tools of inference that a peer with a stronger verbal grasp of geometric reasoning might have.

Sixth grade, improvement cohort. The sixth-grade participant in the improvement group completes the kite and the small square frame in below average time. In both cases, the child reports thinking of the spaces to be filled in terms of composite shapes. For the kite, the bottom and the top regions each require 1 large triangle (i.e., composite triangle of 2 gold or green triangles concatenated into target position). There is not much problem solving to observe because the child is an effective visualizer. Through some effective side-matching & visual estimation, the child rapidly completes the medium square frame item. When asked how she figured that one out, the she reported, "Um, they kind of fit together like a pie or something…like a square pie."

For the large square frame, the participant starts off attempting what worked out with the medium square frame. She puts together a small square to see how it combines with the medium square construction, but immediately sees that this would not work and finished the frame in the expected manner. When asked how they knew the way to do so, the participant reported:

> R: So what was your... why was your instinct to try that first?
>
> P: Um... I guess I thought that it makes sense if it were just like one of these ones [indicates their medium square construction on the medium square frame] but multiple [indicates 3 remaining empty quadrants in large square frame] ... multiple squares.

With each new challenge, the child assessed the context, looked for a promising avenue to pursue for a solution, effectively pursues that avenue, and finds the effective way to achieve the goal in relatively little time. The participant reported not enjoying geometry very much, despite having good spatial instincts.

When the researcher asked about the large square frame, the participant pulled out the medium frame and pointed out that the large frame is just a tiling of 4 medium frames. The child then showed that when divided along the diagonal line of symmetry, the medium square construction is just 2 bigger triangles.

The only other point of interest for this participant was when she attempted the 2 house frame constructions, she was frustrated for a few moments while trying to figure the out the construction. The researcher observed the child rotating the frame about 45° to get a different perspective. While this did not directly bear fruit, it was interesting to see the fearlessness with which this child pursued an insight.

Overall, this participant’s transcripts were the briefest. The child was so effective at strategically breaking down problems into manageable units, she never stayed stuck for very long. She was highly proficient in both the imagistic and verbal tracks of geometric reasoning, an effective visualizer who could quickly and elegantly translate her visual insights into verbal analogies. Across frames, she was able to do this in different ways, such as when she saw the medium frame as 2 triangles but the large frame as 4 medium squares. She was not anxious about “getting it right,” but seemed confident to look at the spatial puzzles before her from multiple vantage points.

Summary. Across these case studies, it was difficult to identify a single factor that held across grade levels between cohorts. Overall summary tallies contrasting the 2 cohorts are presented in Table 6, but the reader is encouraged to keep in mind the grade-level tallies presented above, which provide important nuance to these numbers. The single factor that held across grade levels between cohorts supported the central hypothesis of this dissertation. The frequency of verbal explanations tendered strongly trended towards the improvement group, and the difference was especially striking for those verbal explanations that cited a geometric property in the verbal content. The content of these verbalizations was almost never particularly advanced. Instead they often cited a class of shapes or some simple feature like number of sides. This group of children also provided more visual explanations, which entailed showing the researcher how they did something instead of telling (they received a tally for each when they showed and told simultaneously).

Grade-wise, the 2 participants did not differ widely on the verbal explanations, although the participant in the improvement cohort provided far greater visual explanations (30 observations vs. 6). This might be explained by working memory limitations and might suggest that the verbal channel becomes especially important in the months separating second and fourth grade. This question cannot be answered from single case studies but offers an interesting possibility for future research.

Contrary to expectations, fourth-grade participants in this study showed more advanced geometric reasoning than the sixth-grade participants on average. Among these case studies, the fourth-grade participant from the improvement cohort demonstrated his imaginative geometric multiplication system. In another case shared below in the Discussion chapter, a fourth-grade child used algebra to solve angle measures of triangles. Almost none of the sixth-grade participants even mentioned the word "angle" in their interviews. That fourth-grade student did not qualify for selection due to a ceiling effect across pretest/posttest items, which were too easy for him. When the researcher asked about how he had become so strong in the domain of geometry, he mentioned that he and some others from that class attended an afterschool math club. In line with Case's (1996) theory, experience plays a very big role in the development of proficiency in any domain, and that certainly proved to be this case for this study. Further implications are discussed in the next chapter.

Table 6. *Summative Observation Tallies for No Improvement versus Improvement Cohort*

	Participant	
	No Improvement Cohort	**Improvement Cohort**
Inhibition	22	27
Explanation	74	133
Verbal Explanation	23	45
Visual Explanation	37	55
Cites Prior Experience	9	6
Cites Geometric Property	2	23
Accurate Sequence Recollection	2	3
Inaccurate Sequence Recollection	1	1
Static Spatial Reasoning	155	133
Dynamic Spatial Reasoning	92	108
Hint	11	7
Hint Facilitated Performance	3	3
Experimentation	5	1
Conservation	2 Yes, 1 No	2 Yes, 1 No

Chapter 5: Discussion

The high-level research aims of this dissertation were stated above at the conclusion of Chapter 1. The geometry investigations and data collection were structured in such a fashion as to provide rich material akin to clinical case studies. The idea was to remove the abstraction of computer screens and static images by putting tangible manipulatives of special triangles into the hands of elementary school children. We observed them using these special triangles and pattern blocks to engage in rich geometric mathematizing. The broadest aim was to provide sufficient rich qualitative data to support the informed design of future research on this kind of educational theory and intervention. The theoretical orientation of neo-Piagetian theory already has a very robust framework to draw upon to make sense of the data presented in this dissertation, and the participants' activities and reflections in turn have contributed to a further development of neo-Piagetian theory with the central geometric structure that has been proposed.

The research was conducted with several specific questions in mind, in addition to the general aims just described. This study was designed and conducted to gain insight into what the activities and reflections of children engaged in open-ended geometric inquiry might tell us about core knowledge in the domain of geometry at different ages. This was especially salient for the fourth- and sixth-grade participants, since neo-Piagetian theory predicts a two-fold increase in the central conceptual structure that working memory can support in problem solving. This study also aimed to uncover evidence of reflective abstraction and general versus specific insights might look like in grade-school geometry. I also sought to identify whether the series of investigations conducted by children in this dissertation would facilitate performance on the pretest/posttest items that were designed to draw on insights nurtured by these investigations. In this chapter I revisit these questions and discuss new questions that arose as a result of the data presented in the analysis.

Implications: Neo-Piagetian Theory

Verbal & Imagistic Precursor Schemas.

This dissertation proposed a new central geometric structure to add to the canon of domain-specific knowledge structures proposed by neo-Piagetian theorists working in Robbie Case's framework. It was proposed that at age 4, children have 2 precursor schemas to reason with in this domain: the verbal and the imagistic precursor schema (see Figure 3). By age 6, children gain enough working memory capacity to unify these precursor schemas into a single instance of the central geometric structure proposed in Figure 4. In the analysis section, we saw multiple instances where not having words to assign to images seemed to inhibit performance. All 3 children in the no-improvement groups exhibit a paucity of verbal labels for actions in key moments. The second-grade child had difficulty describing how she arrived at the covering for the kite frame. The fourth-grade student had trouble finding words to describe resolving the medium frame. The participant completely ignored the researcher's prompt for a verbal reflection when he was working on the large frame. The sixth-grade participant experienced great difficulty interpreting the verbal hint given by the researcher for the placement of the gold 3-4-5 triangle in the bottom of the kite frame. The participant's response to the researcher's asking how she figured out the medium frame covering was, "You know, I don't know."

With some exceptions, the participants in the improvement group exhibited moments of greater faculty with geometric language. When asked about his covering the medium square frame, the second-grade participant gave a verbally accurate description of his sequence of events and used some geometric language in the explanations, such as “smaller” and “on the bottom.” While these words are not precise mathematical notions, they indicate an encoding that goes beyond the visual channel. The fourth-grade participant in the improvement group actually generated a verbal hint to give to a struggling peer in a researcher prompted hypothetical. Most of the hints for peers that children suggested were imagistic nudges, such as placing a triangle in a certain orientation. Drawing from his effective grasp of symmetry, the fourth-grade participant from the improvement cohort suggested, “put the same on the other side.” There was no line on the frame indicating the left versus right side of the kite, but the line of symmetry was a salient entity on the frame for this child, and the child expected words could make this real for a struggling peer.

The fourth-grade participant from the improvement cohort showed a strong capacity for spatial experimentation. The child was observed trying numerous strategies in a systematic fashion. The child was silent during this process, but given the child’s abundance of words when prompted, it is difficult to imagine that the child was not keeping some kind of silent verbal tally on the attempted strategies. Additionally, the child comported prior knowledge from art class to utilize the notion of “negative shapes” to describe the ghosted outlines of missing members when a frame was covered around an unresolved region.

Especially interesting was when this same child faced the challenge of the large square frame. The complexity of the challenge required the child to ask for a hint when he had difficulty covering the center of the construction. After the hint helped the child finish the frame, for the first time he was without words to explain how the challenge was surmounted. Uncharacteristically, the child could not recall the correct order of events, and the child did not use intelligible statements to describe his process. This is the only time during the entire session that the child seems to lose the plot on his own mathematizing. When doubling houses, the child's verbal powers were restored, and the researcher was told that the roof needed to be "touching the edges," and was not doing so when it was "in the middle." The researcher was also told "It's NOT doubled because this is a rectangle not a square." This child then conducted a stunning experiment about "multiplication but with shapes," drawing inferences across observations and using mathematical notation (the most precise form of 'verbal' articulation in this context) to track observations.

The sixth-grade participant in the improvement group showed a deep grasp of both imagistic and verbal reasoning. The child used images in such a pregnant fashion, that words were used sparingly. The child iterated with composite shapes, and used words to describe visual similes (e.g., "like a square pie"). There was an elegant terseness to the child's experiments and insights, which were layered with imagistic insight. Although the child's effectiveness with images was orders of complexity deep, the child was capable of providing succinct, clear, and accurate verbal descriptions of events and reasoning when prompted.

From a neo-Piagetian perspective, a child's ability to engage the visual and verbal cognitive systems in geometric insight is critical to effective mathematizing in the grade-school years, and will form the basis of optimal development as the child progresses through the years and attempts to transition from the dimensional central geometric structure proposed in Figure 4, to the vectorial version depicted in Figure 20. This also aligns with Duval's (2006) work, which showed that the verbal and visual systems are both required to parse geometric diagrams in advanced geometry. Proficiency with reasoning in both channels is required to comprehend all the mathematical information depicted in a geometric diagram, and children's optimal cognitive development in spatial mathematics requires attention and nurturing to reach full fruition.

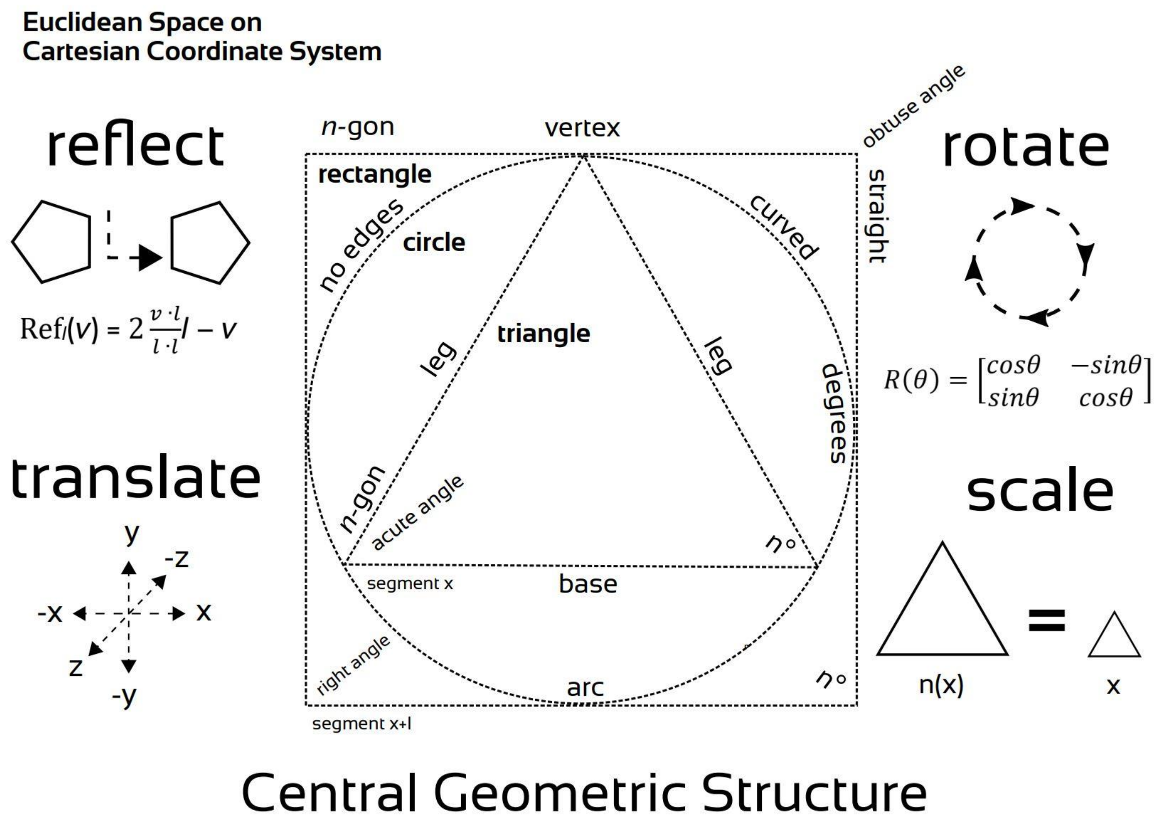

Figure 20. Hypothesized central geometric structure for the vectorial stage

Prior Experience.

Neo-Piagetian theory does not take development for granted. In line with developmental research across fields for the last 50 years, Case's theory recognizes the role of nurture upon nature. Central conceptual structures are conceptualized and offered as models of optimal development limited by the constraints of neurological maturation and working memory capacity (Case, 1996). Children need the right kind of learning opportunities at the right moments to develop the robust mental structures that comprise core knowledge in a particular domain. While only a select sub-group of participants was presented in this analysis, it was very clear that the fourth-grade cohort in this study exhibited the most advanced mathematically advanced reasoning during the investigations. The most advanced student in the study was a fourth grader who was in the no-improvement cohort due to the ceiling effect (successfully completed all items at both pretest and posttest). Surprisingly, almost none of the children asked about the letters printed on the triangles. For the few that did, most of them abandoned the topic once the researcher mentioned the notion of angle measure. Most of them were not ready to engage this topic. For this mathematically talented fourth-grade child, the answer to the question was an invitation. Figure 21 shows one example of his work. The child used the triangles to set up multiple scenarios where he could use algebra or division to discover the precise angle measure. The researcher did not explain that the sum of the angles was 360˚, nor was this information necessary. The context became a playground for this child who brought numerous mathematical gifts to the context. This level of insight was not observed among any of the sixth-grade cohort, and there were several other fourth-grade participants who also showed exceptional ability. When the researcher inquired where this student had picked up such a great ability in mathematics, the child reported that he was part of an after-school math club. This child was clearly comfortable with open mathematical inquiry and was very effective at coordinating insights across contexts.

Another interesting phenomenon observed in these results is that for the 2 grade levels that had FIT scores for both participants (i.e., second and fourth grade), the child in the no improvement cohort had the higher FIT score. Presumably these children should have had greater cognitive capacity to capture materials for mathematical insight. It would make sense that these children would have been the ones to experience greater improvement, since theoretically their spatial reasoning should come more easily than for their lower scoring counterparts. A couple of possible explanations here are relevant to future research. First, prior experience may play a bigger role than cognitive capacity when it comes to putting these experiences together into insight. It may be that the children in the improvement group had more of the right kind of prior spatial experiences to set them up for achievement in this domain. Secondly, achievement on FIT involves disambiguating complex geometric figures. It does not directly assess information processing in the verbal channel. In contrast to this, the findings reported above suggest that integration of spatial and verbal information may have undergirded the participants' improvement on these items. This is inline with the primary thesis posited in this dissertation: that integration of imagistic and verbal schemas into a central conceptual structure for geometry are critical for bearing pedagogical fruit in this domain.

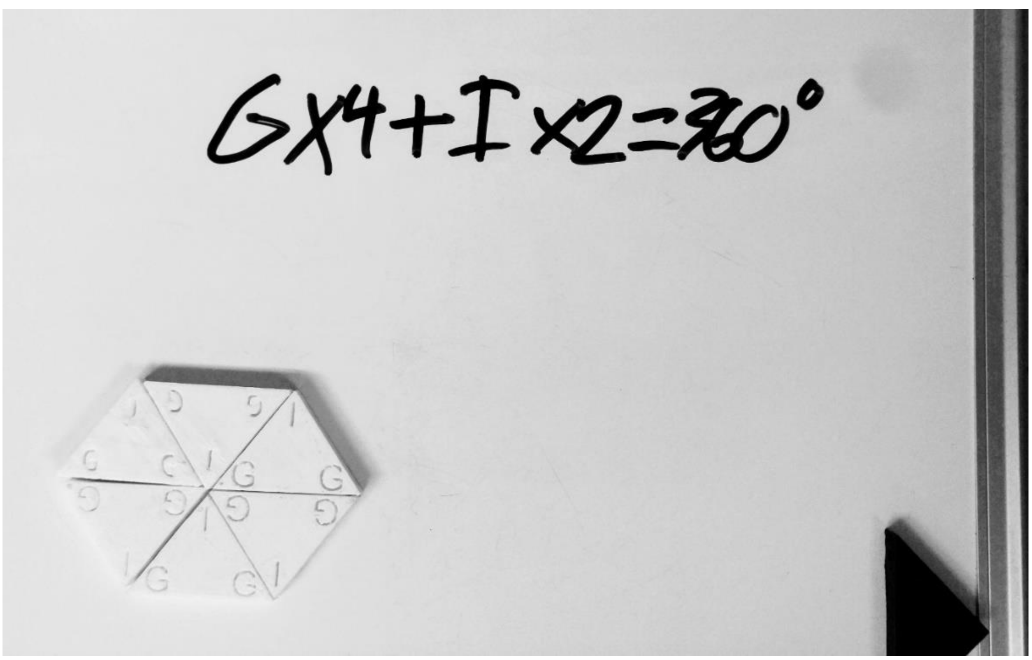

Figure 21. Fourth grade participant using algebra to solve specific angle measures during the Knights of the Polygonal Tables investigation

Hierarchical learning loops: Transfer vs Hierarchical Learning.

How do educators provide a pathway from the dimensional version of the central geometric structure to the vectorial mental model? There has been no small amount of attention by psychologists to the notions of near and far transfer in learning (e.g., Kassai et al., 2019). One motivation for the large-scale abandonment of classical Piagetian theory was the failure of modern statistical analyses of research data to support Piaget's claims of domain general knowledge transfer (Case, 1985). Pascual-Leone had been working in Piaget's lab and had tried to warn him about a need to revise his claims based on the data, but Piaget did not receive Pascual-Leone's feedback (Cardellini & Pascual-Leone, 2004). Most psychologists completely abandoned Piaget's framework, viewing most claims about knowledge transfer with skepticism and adopting an empiricist perspective on learning. Case innovated in this area by specifying how learning works from a rationalist perspective. The child is conceived of as a problem solver, and hierarchical learning loops are iterative feedback loops between associative learning and attentionally mediated learning (Case, 1996). Associative learning is the kind of learning that happens when 2 stimuli become linked through repeated exposure. An example of this is when a child sees the label "Italy" associated with a boot-like outline over various geographic representations. Eventually, an unlabeled map with the shape of Italy will automatically generate the insight "Italy" after enough exposures. Attentionally mediated learning happens when a learner is focusing his or her attention on a map of Europe to memorize the relative locations and names of all the countries on a map. A neurotypical individual probably does not pick up a robust mental model of all the countries of Europe through associative learning, but every school year hundreds of thousands of school children learn this information well enough to pass a geography test through the kind of attentional activation boost afforded by attentionally mediated learning.

Case (1996) asserted that in every learning situation there is some degree of associative learning at play whereby the learner brings general insights to the context. Then there is the kind of specific insight that comes from problem solving through attentionally mediated learning. The hierarchical learning loop occurs when a bidirectional feedback loop happens. Figure 22 is adapted from Case (1996) and depicts the interplay between these two kinds of learning. In this study, a child may have the general insight that covering a frame means an object fits snugly over it, and a specific insight that matching the side of a triangle to the side of a frame along different side lengths can be an effective way to establish the way that a piece constitutes a snug fit. Perhaps, as was observed in the sessions presented in this analysis, a child noticed that side matching was a key to covering the medium square frame, so a child will try it again on the large frame. Often the child would try to match a leg rather than the base to the side, but children who had the insight that one needed to try both side lengths would also try the hypotenuse. Some fortunate children might associate their preexisting grasp of symmetry with the insight that a single hypotenuse for the green 3-3-$3\sqrt{2}$ covered exactly one half the large square frame side length.

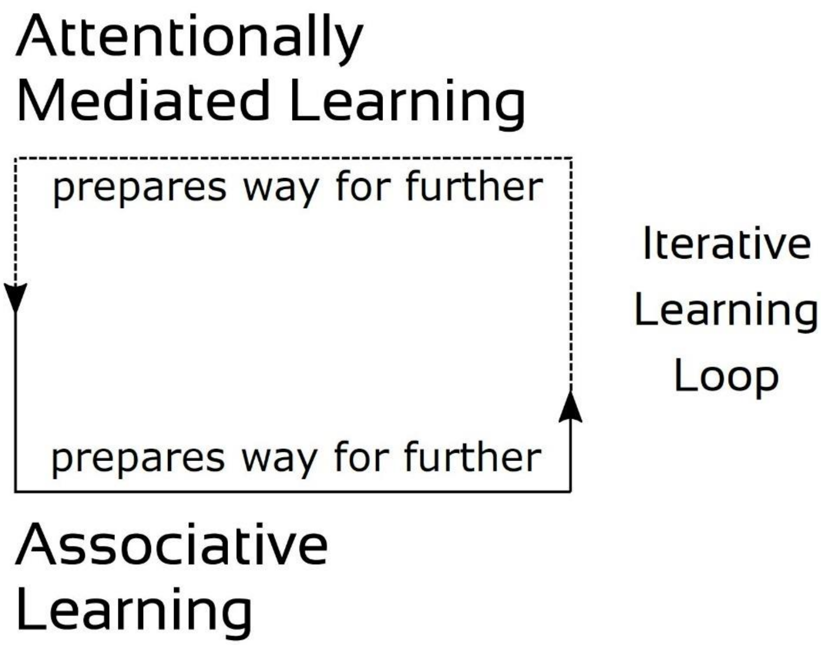

Figure 22. Iterative learning loop adapted from Case (1996)

From the perspective of this analysis, children will expand and consolidate existing insight into new insights when they are given lots of opportunities to build on what they know with new experiences and new connections. The sixth-grade participant in the improvement cohort was especially ready to make new connections because she could visualize composite shapes as tiling the plane across investigations. The child was able to take a specific insight about the top region of the kite and carry it over in a new fashion for the medium and large square frames. There is a subtle but important distinction from the learning theorists' notion of transfer. What we are discussing here is not simply comporting a logical structure from one context to another, dressing the same logical structure in a different outfit of detail. This is prior insight being applied in an innovative fashion to solve a new kind of problem. A good example of this is the fourth-grade student mentioned above who applied algebra to the polygonal tables problem to find the angle measures. The child makes effective use of general insights (e.g., when I want to solve for unknowns, I use algebraic equations) to support specific insights (e.g., I solved for angle measure "G" which is 36˚, so I need 10 of these triangles to build a table). From this perspective, the goal of education should not be just about training specific skill sets (although that's not a bad thing to do), but it should be about supporting the child's noticing their own noticing. The child should be given opportunities to problem solve in contexts that are sufficiently open to allow children at various stages of learning to experiment with what they know and what they are able to discover. For children to make the journey from the dimensional central geometric structure to the vectorial structure, their verbal knowledge need to deepen to the semantic purity of mathematical semiotics, and to extend to inferential logic. Their visual knowledge will have to grow to see the plane as an infinite and continuous space for geometric possibility, and then extend to other kinds of planes that do not follow the neat constraints of a Newtonian world. To meet this invitation fruitfully, children need more than near or far transfer. Children need to be able to link robust core knowledge structures across domains in ways that make sense for the problem at hand.

Implications: Educational Contexts, Pedagogy, and Learners.

This analysis presented in this dissertation has several important implications for educational contexts, pedagogy, and learners. Firstly, individual differences are important and should not be ignored. The opportunity cost to a student who is mismatched to their environment might be devastating. It was noted in this analysis that a single simple visual hint was often all a child needed to have a key insight about the investigation. While some educators may have a very strict stance towards giving this kind of hint on the grounds that it robs the child of a learning opportunity, some children in this study actually showed accelerated insight when given a subtle facilitating cue. Importantly though, it was observed that some participants did not benefit from hints. The child continued to struggle with trial-and-error until he or she happened upon the solution, and a reflection question from the researcher revealed that the child could not articulate an insight about the solution. It seems that where this is the case, children may be especially under-developed in their ability to link the visual and the verbal. It is possible that these children may benefit from focused interventions aimed at increasing their capacity to verbally encode visual states and actions. This is a question for future research.

Another key issue identified here is the issue of experimentation. Mathematics may be presented as a static body of knowledge, where the teacher has all the answers, and the student is to absorb those methods that are computationally fruitful in generating the letter 'A.' The second-grade participant in the no-improvement group was very effective at utilizing prior knowledge, such as when she identified that the skinny rhombi could be used to construct the second house construction but seemed quite reluctant to try something new. The student looked to the researcher to serve as the knowledge arbiter in the investigation. In an older but important paper, Clements and Battista (1992) wrote about the notion of verification in geometry. Who decides what counts as knowledge? If children are socialized in the classroom to be compliant absorbers of standard algorithms and worksheet masters, they will miss out on the development of the kind of confidence they need to be able build the rich knowledge structures discussed above. Due to personality differences, some children may be especially susceptible to this kind of socialization. Additionally, a student who experiences stereotype threat (e.g., see Spencer & Steele, 1999) in a mathematics classroom environment may also regress to an overly passive role as a strategy for dealing with excessive math anxiety. It is important the educators identify these triggers early and provide supportive environments where all children feel safe to explore their mathematical insights without fear of failure. If a child fails to build confidence in this domain early on, it is unlikely that they will achieve the central geometric structure characteristic of the vectorial child's potential.

Another major pedagogical issue unique to the domain of geometry is the profound hazard of misleading cues that is intrinsic to the content of this domain. This issue presented itself for the fourth-grade participant from the no-improvement group who was fixated on the line of symmetry as a starting point for his constructions. For the medium square frame, the child had several green 3-3-3√2 triangles within half a centimeter of target position, already in target orientation, but the child could not decouple from the line of symmetry to notice the side matches available. The sides seemed to be invisible to the child. For the purposes of this study, the researcher did not attempt to intervene, but it might be pedagogically valuable to gently guide a student to attend to the right cues. Additionally, pointing out mislead cues and making them explicit conversation points may provide children with robust tools for geometric reasoning as they encounter geometric figures of increasing complexity.

Finally, it was noted earlier that one aim of the study was identify grade-level specific patterns in children's early geometric mathematizing. While the distribution of FIT scores across all the participants matched normed expectations, it was strikingly clear that grade level was a poor proxy for geometric sophistication among the participants in this study. The fourth-grade class as a group showed the most advanced geometric reasoning and the broadest application of general insight to the specific problems given in the investigations. Of course, the second-grade students indicated a lower capacity to deal with complexity in general, but some of the precocious children in that cohort approached the level of performance seen in some of the upper grades. A child's prior experience seemed to the researcher to play a much larger role than age in this study. This may be related to the fact that children rarely get time to engage in open inquiry in a geometry learning context like the one I offered to children in this study. It is possible that a future study conducted in a country or region with highly effective geometry curriculum may find discernable differences between grade levels, since in that case most children are being provided with an optimal learning environment and biological maturation becomes more of a factor.

Limitations & Future Studies.

This dissertation presented an exploratory analysis of qualitative data of students at 2 California charter schools. The aim was to provide case studies of upper grade-school aged children's mathematizing activities. There were many limitations of this study design that may be addressed by future research. Children in the study comprised a convenience sample, and all of the caveats associated with that method should be considered when interpreting these results. These findings are not generalizable in any kind of inferential sense, although the children in the study were not atypical in any kind of obvious way. The coding schema presented in the analysis was the result of a single researcher's observations and notes and was intended to offer a starting point for future research studies or analyses. Interrater reliability should be established before using such a schema. As mentioned above, the pretest/posttest design did not have a control condition, nor the other 2 conditions called for in the Solomon four-group design (Braver & Braver, 1988). These items were designed under the guidance of a mathematics professor skilled in early-childhood math education, but they were not validated and were not scored with enough variance to be used in statistical analysis. Future studies should be conducted to address these methodological shortcomings.

This dissertation proposed a new central geometric structure for the domain of geometry. I offered a dimensional model and a vectorial model of this structure to provide hypothesized developmental endpoints for the pre-college years. This study did not submit an instrument to validate the existence of either one of these central conceptual structures. While the activities, insights, and difficulties of children in this study collectively informed the proposed structures, the study design of this dissertation prevents us from answering questions of normativity. In the future, a battery of measures similar to the one developed by Case et al. (1996) to assess spatial development should be developed and validated to evaluate these structures.

Several additional research questions for future studies were discussed in this chapter. These include: Do students with richer spatial vocabulary learn geometry at a faster rate than their less-verbal peers? Do students with a fixed mindset about mathematical knowledge gain accrete mathematical insights more slowly than students with an inquiry-oriented mindset? Do teachers socialize their students to be one way or the other depending on pedagogical approach? Are there children who are more susceptible to misleading cues in geometry, and can they be taught to identify and inhibit misleading cues? There are many more questions than the current dissertation was able to provide answers for. However, this is a promising step forward in understanding how children develop core knowledge in geometry.

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