Title
CALCULATION OF GEOMETRICAL EFFICIENCY OF COUNTERS

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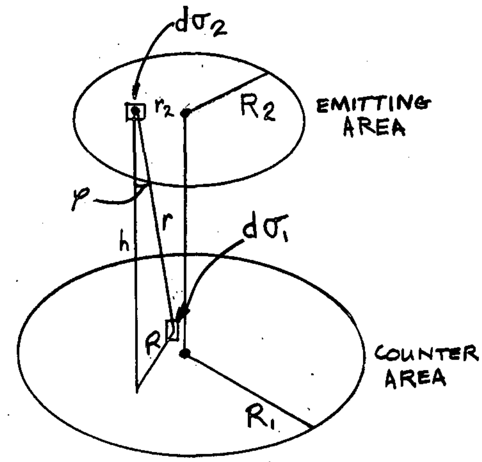


Let F be the flux of radiation going from a sample of radius R_2 to a counter of radius R_1 then Φ the emitted intensity of radiation per unit solid angle. Then the flux between two elements of area $d\sigma_1$ and $d\sigma_2$ will be given by

$$dF = \Phi \frac{\cos \phi}{r^2} d\sigma_1 \, d\sigma_2 \quad . \tag{1}$$

Since

$$\cos \phi = \frac{h}{r} = \frac{h}{\sqrt{h^2 + R^2}} \quad , \tag{2}$$

and

$$dF = \Phi \frac{h}{r^3} d\sigma_1 \, d\sigma_2 \quad . \tag{3}$$

a. Point Source ($R_1 \geq R_2$)

If there is only one element of area emitting ($d\sigma_2$) then the total emitted flux is $4\pi \Phi \, d\sigma_2$ and the geometrical counting efficiency will be given by

$$dG = \frac{dF}{4\pi\Phi\, d\sigma_2} = \frac{h}{r^3}\, d\sigma_1 \quad . \tag{4}$$

To integrate over σ_1 we set

$$d\sigma_1 = R\, dR\, d\alpha \quad . \tag{5}$$

Then by symmetry we can limit $0 \leq \alpha \leq \pi$ and put in a factor 2 so that in this case

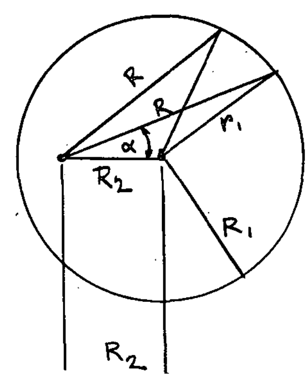
$$dG = \frac{h}{2\pi r^3}\, R\, dR\, d\alpha \quad . \tag{6}$$

Then integrate first over α .

For $R_2 + R \leq R_1$, $0 \leq \alpha \leq \pi$;

and for $R_2 + R \geq R_1$

the upper limit on α is determined by



$$R_1^{\,2} = R^2 + R_2^{\,2} - 2\, R_2\, R \cos\alpha \ , \tag{7}$$

or

$$\alpha = \cos^{-1}\left[\frac{R^2 + R_2^{\,2} - R_1^{\,2}}{2\, R_2\, R}\right] , \tag{8}$$

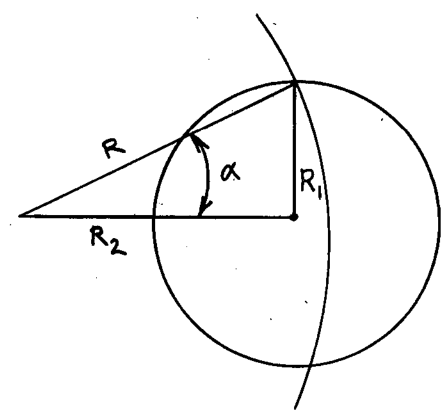
so

$$G = \frac{h}{2} \int_0^{R_1-R_2} \frac{R\,dR}{r^3} + \frac{h}{2\pi} \int_{R_1-R_2}^{R_1+R_2} \cos^{-1}\left\{\frac{R^2 + R_2^2 - R_1^2}{2\,R_2\,R}\right\} \frac{R\,dR}{r^3} \tag{9}$$

or

$$G = \frac{1}{2}\left[1 - \frac{h}{\sqrt{(R_1 - R_2)^2 + h^2}}\right] + \frac{h}{2\pi} \int_{R_1-R_2}^{R_1+R_2} \cos^{-1}\left\{\frac{R^2 + R_2^2 - R_1^2}{2\,R_2\,R}\right\} \frac{R\,dR}{r^3} . \tag{10}$$

b. Point Source $(R_2 \geq R_1)$



In this case α is determined as in Eq. (8). The limits on integration are different so that here we have

$$G = \frac{h}{2\pi} \int_{R_2-R_1}^{R_2+R_1} \cos^{-1}\left\{\frac{R^2 + R_2^2 - R_1^2}{2\,R_2\,R}\right\} \frac{R\,dR}{r^3} . \tag{11}$$

This agrees with the stated expression in Calvin, "Isotopic Carbon", App. IV, for $R_2 > R_1$.

b. Circular Source $R_1 \geq R_2$

In this case the total flux is $4\pi\Phi(\pi R_2^2)$ so that

$$dG = \frac{dF}{4\pi^2 R_2^2 \Phi} = \frac{h}{4\pi^2 R_2^2} \frac{d\sigma_1 \, d\sigma_2}{r^3} \quad . \tag{12}$$

Let

$$d\sigma_2 = r_2 \, dr_2 \, d\theta \quad , \tag{13}$$

$$d\sigma_1 = R \, dR \, d\alpha \quad . \tag{14}$$

We can integrate readily over θ and restrict $0 \leq \alpha \leq \pi$ so that we get

$$dG(r_2, R, \alpha) = \frac{h}{\pi R_2^2} \frac{r_2 \, dr_2 \, R \, dR \, d\alpha}{r^3} \quad . \tag{15}$$

Now integrate over α

$$dG(r_2, R) = \left(\frac{h}{\pi R_2^2} \frac{R \, dR}{r^3} \right)(\alpha \; r_2 \, dr_2) \quad . \tag{16}$$

Now either we shall have

$$\alpha = \pi$$

or

$$\alpha = \cos^{-1} \left\{ \frac{R^2 + r_2^2 - R_1^2}{2 \, r_2 \, R} \right\} , \tag{17}$$

depending on the circumstances.

We now want to find out over what ranges the integration of R and r_2 must be carried for different values of α. If we take a fixed r_2 we can draw as follows the ranges of integration for R.

Let $r_2 \leq R_1$ then

for $r_2 + R \leq R_1$; $\alpha = \pi$; and $0 \leq R \leq R_1 - r_2$

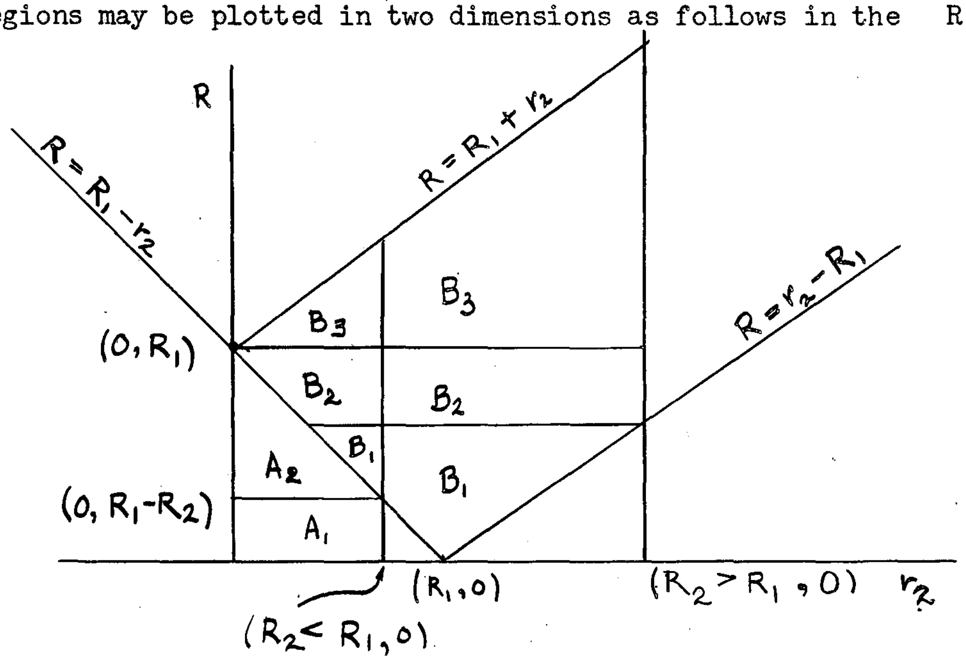
$r_2 + R \geq R_1$; $\alpha = \cos^{-1} \left\{ \dfrac{R^2 + r_2^2 - R_1^2}{2\, r_2\, R} \right\}$; and

$$R_1 - r_2 \leq R \leq R_1 + r_2 .$$

While for $r_2 \geq R_1$; $\alpha = \cos^{-1} \left\{ \dfrac{R^2 + r_2^2 - R_1^2}{2\, r_2\, R} \right\}$; and

$$r_2 - R_1 \leq R \leq r_2 + R_1 .$$

These regions may be plotted in two dimensions as follows in the R, r_2 plane



If we change the order of integration to an integration over r_2 first, we have various regions to consider.

b1 $(R_1 > R_2)$

Region	Limits	α
A_1	$0 \leq r_2 \leq R_2$; $0 \leq R \leq R_1 - R_2$	π
A_2	$0 \leq r_2 \leq R_1 - R$; $R - R_2 \leq R \leq R_1$	π
$B_1 + B_2$	$R_1 - R \leq r_2 \leq R_2$; $R_1 - R_2 \leq R \leq R_1$	Eq. (17)
B_3	$R - R_1 \leq r_2 \leq R_2$; $R_1 \leq R \leq R_1 + R_2$	Eq. (17).

b2 $(R_1 < R_2)$

Region	Limits	α
A	$0 \leq r_2 \leq R_1 - R$; $0 \leq R \leq R_1$	π
B_1	$R_1 - R \leq r_2 \leq R + R_1$; $0 \leq R \leq R_2 - R_1$	Eq. (17)
B_2	$R_1 - R \leq r_2 \leq R_2$; $R_2 - R_1 \leq R \leq R_1$	Eq. (17)
B_3	$R - R_1 \leq r_2 \leq R_2$; $R_1 \leq R \leq R_1 + R_2$	Eq. (17)

Now to evaluate the expressions for G in the two cases we have to integrate expressions of the type

$$\int \alpha \, r_2 \, dr_2$$

where $\alpha = \pi$ or $\alpha = \cos^{-1}\left[\frac{R^2 + r_2^2 - R_1^2}{2 R r_2}\right]$

for

$\alpha = \pi$
$$\int_a^b \alpha \, r_2 \, dr_2 = \frac{\pi}{2} r_2^2 \Big|_a^b . \tag{18}$$

For the other case

$$\int_a^b \left\{ \cos^{-1}\left[\frac{R^2 - R_1^2 + r_2^2}{2 R r_2}\right] r_2 \right\} dr_2$$

$$= \left\{ \frac{r_2^2}{2} \cos^{-1}\left[\frac{R^2 - R_1^2 + r_2^2}{2 R r_2}\right] - \frac{R_1^2}{2} \sin^{-1} \frac{R^2 + R_1^2 - r_2^2}{2 R R_1} - \frac{1}{4}\sqrt{4 r_2^2 R^2 - (R^2 - R_1^2 + r_2^2)^2} \right\}_a^b$$

(19)

Evaluating this $\int$ at the various limits needed, we find that

$$\int^{R_2} \cos^{-1}\left[\frac{R^2 - R_1^{\,2} + r_2^{\,2}}{2\,R\,r_2}\right] r_2\,dr_2$$

$$= \frac{R_2^{\,2}}{2}\cos^{-1}\left[\frac{R^2 - R_1^{\,2} + R_2^{\,2}}{2\,R\,R_2}\right] - \frac{R_1^{\,2}}{2}\sin^{-1}\left[\frac{R^2 + R_1^{\,2} - R_2^{\,2}}{2\,R\,R_1}\right]$$

$$- \frac{1}{4}\sqrt{4\,R_2^{\,2}\,R - (R^2 - R_1^{\,2} + R_2^{\,2})^2} + C$$

$$= \frac{\pi R_2^{\,2}}{4} - \frac{\pi R_1^{\,2}}{4} + f(R_2) + C \quad , \tag{20}$$

where

$$f(R_2) = \frac{R_1^{\,2}}{2}\cos^{-1}\left[\frac{R^2 + R_1^{\,2} - R_2^{\,2}}{2\,R\,R_1}\right] - \frac{R_2^{\,2}}{2}\sin^{-1}\left[\frac{R^2 + R_2^{\,2} - R_1^{\,2}}{2\,R\,R_2}\right]$$

$$- \frac{1}{4}\sqrt{4\,R^2\,R_1^{\,2} - (R^2 + R_1^{\,2} - R_2^{\,2})^2} \quad . \tag{21}$$

$$\int^{R_1 - R} \cos^{-1}\left[\frac{R^2 - R_1^{\,2} + r_2^{\,2}}{2\,R\,r_2}\right] r_2\,dr_2 = \frac{\pi}{2}\,(R_1 - R)^2 - \frac{\pi}{4}\,R_1^{\,2} + C \tag{20a}$$

$$\int^{R - R_1} \cos^{-1}\left[\frac{R^2 - R_1^{\,2} + r_2^{\,2}}{2\,R\,r_2}\right] r_2\,dr_2 = -\frac{\pi}{4} R_1^{\,2} + C \tag{20b}$$

$$\int^{R_1 + R} \cos^{-1}\left[\frac{R^2 - R_1^{\,2} + r_2^{\,2}}{2\,R\,r_2}\right] r_2\,dr_2 = \frac{\pi R_1^{\,2}}{4} + C \tag{20c}$$

$$\int^{R_2} \cos^{-1}\left[\frac{R^2 - R_1^{\,2} + r_2^{\,2}}{2\,R\,r_2}\right] r_2\,dr_2 = \frac{\pi R_2^{\,2}}{4} - \frac{\pi R_1^{\,2}}{4} + f(R_2) + C\,. \tag{20d}$$

We now substitute the limits in the expression for G.

Case b_1. $(R_1 > R_2)$

$$G = \frac{h}{\pi R_2^{\,2}}\left\{ \int_0^{R_1 - R_2} \frac{R\,dR}{r^3}\left[\frac{\pi}{2} R_2^{\,2}\right] \int_{R_1 - R_2}^{R_1} \frac{R\,dR}{r^3}\ \frac{\pi}{2}(R_1 - R)^2 \right.$$

$$+ \int_{R_1 - R_2}^{R_1} \frac{R\,dR}{r^3}\left[\frac{\pi R_2^{\,2}}{4} - \frac{\pi R_1^{\,2}}{4} + f(R_2) - \frac{\pi}{2}(R_1 - R)^2 + \frac{\pi}{4} R_1^{\,2}\right]$$

$$\left. + \int_{R_1}^{R_1 + R_2} \frac{R\,dR}{r^3}\left[\frac{\pi R_2^{\,2}}{4} - \frac{\pi R_1^{\,2}}{4} + f(R_2) + \frac{\pi}{4} R_1^{\,2}\right] \right\}$$

(22) Cont.

$$= \frac{h}{2} \int_{0}^{R_1 - R_2} \frac{R\,dR}{r^3} + \frac{h}{4} \int_{R_1 - R_2}^{R_1 + R_2} \frac{R\,dR}{r^3} + \frac{h}{R_2^{\,2}} \int_{R_1 - R_2}^{R_1 + R_2} f(R_2)\, \frac{R\,dR}{r^3}$$

$$= \frac{1}{2} \left[1 - \frac{h}{\sqrt{h^2 + (R_1 - R_2)^2}} \right] + \frac{1}{4} \left[\frac{h}{\sqrt{h^2 + (R_1 - R_2)^2}} - \frac{h}{\sqrt{h^2 + (R_1 + R_2)^2}} \right]$$

$$+ \frac{h}{\pi R_2^{\,2}} \int_{R_1 - R_2}^{R_1 + R_2} f(R_2)\, \frac{R\,dR}{r^3}$$

$$= \frac{1}{2} \left[1 - \frac{1}{2} \quad \frac{h}{\sqrt{h^2 + (R_1 - R_2)^2}} + \frac{h}{\sqrt{h^2 + (R_1 \quad R_2)^2}} \right]$$

$$+ \frac{h}{\pi R_2^{\,2}} \int_{R_1 - R_2}^{R_1 + R_2} f(R_2)\, \frac{R\,dR}{r^3} \,, \tag{22}$$

which is formula (4) in Calvin, "Isotopic Carbon", App. IV.

Case b_2 $(R_1 < R_2)$:

$$G = \frac{h}{\pi R_2^2}\left\{ \int_0^{R_1} \frac{R\,dR}{r^3}\left[\frac{\pi}{2}(R_1 - R)^2\right] + \int_0^{R_2-R_1} \frac{R\,dR}{r^3}\left[\frac{\pi R_1^2}{4} - \frac{\pi}{2}(R_1 - R)^2 + \frac{\pi}{4}R_1^2\right] \right.$$

$$+ \int_{R_2-R_1}^{R_1} \frac{R\,dR}{r^3}\left[\frac{\pi R_2^2}{4} - \frac{\pi R_1^2}{4} + f(R_2) - \frac{\pi}{2}(R_1 - R)^2 + \frac{\pi R_1^2}{4}\right]$$

$$\left. + \int_{R_1}^{R_1+R_2} \frac{R\,dR}{r^3}\left[\frac{\pi R_2^2}{4} - \frac{\pi R_1^2}{4} + f(R_2) + \frac{\pi R_1^2}{4}\right] \right\}$$

$$= \frac{h}{\pi R_2^2}\left\{ \frac{\pi R_1^2}{2}\int_0^{R_2-R_1} \frac{R\,dR}{r^3} + \frac{\pi R_2^2}{4}\int_{R_2-R_1}^{R_1+R_2} \frac{R\,dR}{r^3} + \int_{R_2-R}^{R_1+R_2} f(R_2)\frac{R\,dR}{r^3} \right\}$$

$$= \frac{1}{2}\left\{ \frac{R_1^2}{R_2^2}\left[1 - \frac{h}{\sqrt{h^2 + (R_2 - R_1)^2}}\right] + \frac{1}{2}\left[\frac{h}{\sqrt{h^2 + (R_2 - R_1)^2}} - \frac{h}{\sqrt{h^2 + (R_1 + R_2)^2}}\right] \right\}$$

$$+ \frac{h}{\pi R_2^2}\int_{R_2-R_1}^{R_2+R_1} f(R_2)\,\frac{R\,dR}{r^3}\,, \qquad (23)$$

which is formula (5) in Calvin.