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Stability Analysis of Systems with Generalized Frequency Variables

Shinji Hara, Hideaki Tanaka, and Tetsuya Iwasaki

***Abstract*—A class of large-scal, multi-agent systems with decentralized information structures can be represented by a linear system with a generalized frequency variable. In this paper, we investigate fundamental properties of such systems, stability and $\mathcal{D}$-stability, exploiting the dynamical structure. Specifically, we first show that such system is stable if and only if the eigenvalues of the connectivity matrix lie in a region on the complex plane specified by the generalized frequency variable. The stability region is characterized in terms of polynomial inequalities, leading to an algebraic stability condition. We also show that the stability test can be reduced to a linear matrix inequality (LMI) feasibility problem involving generalized Lyapunov inequalities and that the LMI result can be extended for robust stability analysis of systems subject to uncertainties in the interconnection matrix. We then extend the result to $\mathcal{D}$-stability analysis to meet practical requirements, and provide a unified treatment of $\mathcal{D}$-stability conditions for ease of implementation. Finally, numerical examples illustrate utility of the stability conditions for the analysis of biological oscillators and for the design of cooperative stabilizers.**

I. Introduction

Due to the insatiable growth of computing power and the increasing demand for layers of networking, modern engineering systems have become more and more complex and are now subject to multitude of system dimensions. To cope with the challenges faced when dealing with large scale networked dynamical systems, many studies of different approaches have been made in a variety of areas. One of the important trends in these studies is the decentralized autonomous control of multi-agent systems (see e.g., [23] and the references therein). Numerous approaches have been proposed for the analysis and synthesis of such systems under some specific problem formulations, but very few results are currently available to provide a unifying framework for developing a theoretical paradigm broadly applicable to a general class of multi-agent systems in which agents exchange information each other and autonomously cooperate.

Recently, a linear system with a generalized frequency variable is proposed as a unifying framework for modeling multi-agent systems [10], [11]. Specifically, a collection of multiple identical SISO agents, of which the transfer function is expressed as $h(s)$, is described by the transfer function $\mathcal{G}(s) := G(\phi(s))$, where $G(s)$ is a proper rational function representing the interactions between agent and $\phi(s) := 1/h(s)$. Within this framework, $\mathcal{G}(s)$ is obtained by simply replacing s in $G(s)$ with $\phi(s)$, motivating us to call $\phi(s)$ the generalized frequency variable.

A similar class of systems was investigated earlier [25] in a purely theoretical context without any consideration of practical situations, which causes a difficulty of treating properties at $s = \infty$. In contrast, we argue that systems described by $G(\phi(s))$ with strictly proper rational transfer function $h(s)$ has the potential to provide a general theoretical foundation for analyzing and designing homogeneous large-scale dynamical network systems in a variety of areas. For example, the framework of the generalized frequency variable can be applied to the analysis and synthesis of central pattern generators (CPGs) [17], [3] and gene-protein regulatory networks [2], [26], as well as consensus and formation problems as surveyed in [23].

In this paper, we address a fundamental problem of assessing stability for linear systems with generalized frequency variables. There are three different types of stability test in control theory; graphical (Nyquist type), algebraic (Routh-Hurwitz type), and numeric (Lyapunov type) tests. Each condition has its own role in different situation with different purposes, and they provide foundations for, and play significant roles in, the progress of control theory. The graphical test is useful for understanding the stability condition intuitively. The algebraic test is completely algorithmic and is a powerful tool for parametric analysis and synthesis when combined with symbolic computation. The numerical test, often given in terms of linear matrix inequalities (LMIs), provides flexibility in the analysis to include other design specifications and/or robustness requirement. These standard methods can be directly used for stability analysis of linear systems with generalized frequency variable. However, our focus here is to exploit the structure of $G(\phi(s))$ to gain insights into how the network connectivity and individual dynamics interact to maintain stability. We also aim to simplify the analysis so as to make it applicable to very large-scale systems.

Some attempts have been made to exploit the internal structure for analysis of multi-agent systems in the literature [7], [10], [11], [25]. These attempts have lead to stability conditions that rely on graphical inspections. In contrast, the present paper is built upon the previous ideas and proposes an algebraic stability condition and an LMI stability condition, which can be checked systematically by using symbolic manipulation and numerical optimization, respectively. The LMI

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stability condition is extended to provide a sufficient condition for robust stability against uncertainties in the interconnection topology and strength.

To this end, we first point out that the poles of $G(s)$ coincide with the eigenvalues of the network connectivity matrix, and prove that the system is stable if and only if the eigenvalues are contained in a region on the complex plane specified by $\phi(s)$. The stability region is characterized through a Hurwitz type criterion [8] by a set of polynomial inequality conditions whose coefficients can be found systematically from $\phi(s)$. This provides us with a pure algebraic stability condition, and it is quite useful for parametric analysis and synthesis of multi-agent dynamical systems using symbolic computation. We then show, using a generalized Lyapunov theorem [15], that all the eigenvalues lie in the stability region if and only if a set of LMIs is feasible. The LMI formulation allows for an extension to a sufficient robust stability condition against a certain class of perturbations in the way subsystems interact each other.

We will further extend the result to deal with more practical requirements beyond stability. In particular, a design example of cooperative stabilization illustrates a case where the agent dynamics put a limitation on the achievable damping ratio regardless of the connectivity matrix. Such limitation cannot be detected by a standard stability analysis. As a possible approach to resolve such issues and to provide more flexibility in the analysis and design, we will consider the notion of $\mathcal{D}$-stability and derive conditions on the connectivity eigenvalues such that the poles of the multi-agent system lie in a prescribed half plane or a circular region. Finally, the utility of the results derived in this paper is demonstrated by an oscillation analysis for a nonlinear biological network and cooperative stabilization of networked inverted pendulums.

The idea of characterizing stability of multi-agent systems in terms of the connectivity matrix was presented earlier in the context of CPG analysis [17]. The present paper generalizes the idea and establishes a theoretical framework. In contrast with the previous results [10], [11], [25] derived in the frequency domain, our results are based on a state-space formulation and are possibly amenable to further developments in various directions including robust stability and performance analyses, control designs, and estimations.

The theoretical results on the stability analysis in this paper were reported with only outlines of the proofs in the authors' conference papers [27], [12]. The present paper provides complete proofs with several key lemmas and an algorithm for systematic stability assessment. Also new in this paper is an LMI condition for robust stability with a numerical example illustrating the trade-off between conservatism and computational efficiency. Two applications demonstrate effectiveness of the theoretical results. One is an oscillation analysis for nonlinear biological networks, and the other is on cooperative stabilization [14] of a system of multiple inverted pendulums, where the proposed $\mathcal{D}$-stability test is useful for performance analysis. An algebraic characterization of the stability region and its properties were presented in a paper by the authors written in Japanese [28].

This paper is organized as follows. We first define a class

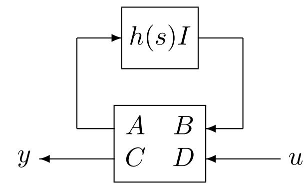

Fig. 1. Multi-agent representation of $\mathcal{G}(s)$

of linear systems with generalized frequency variables in Section II-A, and formulate a stability analysis problem in Section II-B. As main results, a polynomial characterization of the stability region and an algebraic stability test are presented in Section III-A, followed by an LMI condition for stability in Section III-B and an LMI robust stability condition in Section III-C. We then show some extensions to $\mathcal{D}$-stability analysis in Section IV. Section V illustrates two applications of the stability conditions to show the effectiveness in practice. Finally, conclusions are given in Section VI.

We use the following notation. The set of real, complex, and natural numbers are denoted by $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{N}$, respectively. The complex conjugate of $\lambda \in \mathbb{C}$ is denoted by $\bar{\lambda}$. For a matrix A, its transpose and complex conjugate transpose are denoted by A^{T} and A^*, respectively. For a square matrix A, the set of eigenvalues is denoted by $\sigma(A)$. The symbols $\mathbf{S}_n$ and $\mathbf{H}_n$ stand for the sets of $n \times n$ real symmetric and complex Hermitian matrices respectively, and $\mathbf{S}_n^+$ and $\mathbf{H}_n^+$ stand for their positive definite subsets. For matrices A and B, $A \otimes B$ means their Kronecker product. The open left-half complex plane and the closed right-half complex plane are denoted by $\mathbb{C}_-$ and $\mathbb{C}_+$, respectively.

II. Problem Formulation

A. Linear Systems with Generalized Frequency Variable

In this section, we define linear systems with generalized frequency variables and provide their dynamical equations in the frequency domain. Specifically, consider the linear time-invariant system described by the transfer function

$$\begin{aligned}\mathcal{G}(s) &= C\left(\frac{1}{h(s)}I_n - A\right)^{-1} B + D \\ &= \mathcal{F}_u\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, h(s)I_n\right),\end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, $h(s)$ is a single-input single-output, νth-order, strictly proper transfer function, and $\mathcal{F}_u$ denotes the upper linear fractional transformation as shown in Fig. 1. The system $\mathcal{G}(s)$ can thus be viewed as an interconnection of n identical agents, each of which has the internal dynamics $h(s)$, where the interconnection structure is specified by A, and the input-output structure for the whole system is specified by B, C, and D. Defining the transfer function

$$G(s) = C(sI_n - A)^{-1}B + D, \tag{2}$$

the system can also be described as

$$\mathcal{G}(s) = G(\phi(s)), \quad \phi(s) := 1/h(s). \tag{3}$$

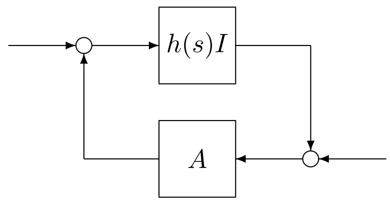

Fig. 2. Feedback system $\Sigma(h(s), A)$

Note that the variable s in (2) characterizes frequency properties of the transfer function $G(s)$ and that $\mathcal{G}(s)$ is generated by simply replacing s by $\phi(s)$ in G. Hence, we say that the system (1) is described by the transfer function G with the *generalized frequency variable* $\phi(s)$ [10].

B. Stability Analysis Problem

Let $h(s)$ have a minimal realization $h(s) \sim (A_h, b_h, c_h, 0)$ where $A_h \in \mathbb{R}^{\nu\times\nu}$, $b_h \in \mathbb{R}^{\nu}$, $c_h \in \mathbb{R}^{1\times\nu}$. It can be shown [10] that a realization of $\mathcal{G}(s)$ is given by $\mathcal{G}(s) \sim (\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ where

$$\begin{aligned} &\mathcal{A} = I_n \otimes A_h + A \otimes (b_h c_h) \in \mathbb{R}^{n\nu\times n\nu}, \\ &\mathcal{B} = B \otimes b_h \in \mathbb{R}^{n\nu\times m}, \quad \mathcal{C} = C \otimes c_h \in \mathbb{R}^{p\times n\nu}, \\ &\mathcal{D} = D \in \mathbb{R}^{p\times m}. \end{aligned} \tag{4}$$

If (A, B, C) and (A_h, b_h, c_h) are both minimal realizations, then $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is minimal [10]. Hence, stability of $\mathcal{G}(s)$ can be checked by computing the eigenvalues of $\mathcal{A}$.

Our objective is to develop a stability analysis method for the system $\mathcal{G}(s)$, exploiting its particular structure to gain insights into interconnected dynamical systems and to reduce the computational burden associated with the analysis. Hence, it is not our approach to directly deal with the realization of $\mathcal{G}(s)$ ignoring the structural information, but rather, we will aim to characterize the stability property of $\mathcal{G}(s)$ in terms of $G(s)$ and $h(s)$.

The linear time-invariant system with the generalized frequency variable $\mathcal{G}(s)$ given by (3) is stable (all the poles of $\mathcal{G}(s)$ are in $\mathbb{C}_-$), if and only if the feedback system $\Sigma(h(s), A)$ shown in Fig. 2 is internally stable. This condition is in turn equivalent to stability of

$$\mathcal{H}(s) := \left(\frac{1}{h(s)} I - A\right)^{-1} = (\phi(s) I - A)^{-1}, \tag{5}$$

i.e., the property that $\mathcal{H}(s)$ is proper and analytic in the closed right half complex plane. The problem is to find a necessary and sufficient condition for stability of the linear time-invariant system (5) in terms of the generalized frequency variable $\phi(s) := 1/h(s)$ and the interconnection matrix A.

III. Stability Region and LMI Condition

In this section, we show that the system (5) is stable if and only if the eigenvalues of the interconnection matrix A are in a particular region specified by the generalized frequency variable $\phi(s)$. We will first provide a characterization of the stability region, then give an LMI condition for the eigenvalues to lie in the region.

A. Characterization of the Stability Region

Let us first state a lemma that characterizes a necessary and sufficient condition for $\mathcal{H}(s)$ to be stable, explicitly expressed in terms of the generalized frequency variable $\phi(s) = 1/h(s)$ and the interconnection matrix A.

Lemma 1: Let a matrix $A \in \mathbb{R}^{n\times n}$, and a strictly proper rational function $h(s)$ be given and consider the system $\mathcal{H}(s)$ in (5). Define the polynomial $p(\lambda, s)$ for $\lambda \in \mathbb{C}$ by

$$p(\lambda, s) := d(s) - \lambda n(s), \tag{6}$$

where $n(s)$ and $d(s)$ are coprime polynomials such that $h(s) = n(s)/d(s)$. The following statements are equivalent.

(i) $\mathcal{H}(s)$ is stable.

(ii) $\sigma(A) \subset \Lambda(h) := \{ \lambda \in \mathbb{C} \mid p(\lambda, s) \text{ is Hurwitz}^1 \}$.

Proof: The system $\mathcal{H}(s)$ is stable if and only if $\det(I/h(s) - A) \neq 0$ for all $s \in \mathbb{C}_+$, i.e., $\lambda h(s) \neq 1$ for all $s \in \mathbb{C}_+$ and $\lambda \in \sigma(A)$ [10], [25]. This condition is equivalent to the property that $p(\lambda, s)$ is Hurwitz for all $\lambda \in \sigma(A)$, which is further equivalent to $\sigma(A) \subset \Lambda(h)$. The proof is now complete. ∎

As stated in Lemma 1, the system $\mathcal{H}(s)$ is stable if and only if the eigenvalues of the interconnection matrix A are all in the stability region $\Lambda(h)$ defined for given subsystem dynamics $h(s)$. The following result gives a characterization of $\Lambda(h)$ in terms of polynomial inequalities.

Theorem 1: Let a real rational, strictly proper, νth order transfer function $h(s) = n(s)/d(s)$ be given with coprime polynomials $n(s)$ and $d(s)$. Define $p(\lambda, s)$ and $\Lambda(h)$ as in Lemma 1. Then there exist positive integers $\ell_k \le k+1$ and $\Phi_k \in \mathbf{S}_{\ell_k}$ for $k = 1, 2, \ldots, \nu$ such that

$$\Lambda(h) = \bigcap_{k=1}^{\nu} \Sigma_k, \quad \Sigma_k := \{ \lambda \in \mathbb{C} \mid l_{\ell_k}(\lambda)^* \Phi_k l_{\ell_k}(\lambda) > 0 \} \tag{7}$$

where

$$l_\ell(\lambda) := \begin{bmatrix} 1 & \lambda & \lambda^2 & \cdots & \lambda^{\ell-1} \end{bmatrix}^{\mathsf{T}}.$$

Moreover, the system $\mathcal{H}(s)$ in (5) is stable if and only if $\sigma(A) \subset \Lambda(h)$.

A proof of this result is given in Appendix B. Below, we briefly outline the idea behind the proof by indicating the steps to construct Φ_k such that (7) holds.

1) Let $h(s)$ be given as

$$h(s) = \frac{n(s)}{d(s)} = \frac{b_1 s^{\nu-1} + \cdots + b_\nu}{s^\nu + a_1 s^{\nu-1} + \cdots + a_\nu} \tag{8}$$

and define $p(\lambda, s)$ by (6).

2) Use an extended Routh-Hurwitz criterion (Lemma 2 in Appendix A) to obtain a necessary and sufficient condition for $p(\lambda, s)$ to have all its roots s in the open left half plane. The condition is given as $\Delta_k(\lambda, \bar{\lambda}) > 0$ for $k = 1, 2, \ldots, \nu$, where Δ_k is a polynomial of λ and $\bar{\lambda}$ with its value in $\mathbb{R}$.

[1] The polynomial $p(\lambda, s)$ is called Hurwitz if all its roots s are in the open left half complex plane.

3) Obtain the maximum degree $\ell_k - 1$ of λ in $\Delta_k(\lambda, \bar{\lambda})$ and the coefficient matrix Φ_k of $\Delta_k(\lambda, \bar{\lambda})$ for $k = 1, 2, \ldots, \nu$, where the (i, j) entry of Φ_k is the coefficient of $\lambda^{i-1}\bar{\lambda}^{j-1}$ in $\Delta_k(\lambda, \bar{\lambda})$.

The following example illustrates how to derive a form of Φ_k in a parametric way.

Example 1: Consider the general second order ($\nu = 2$) strictly proper transfer function

$$h(s) = (cs + d)/(s^2 + as + b). \qquad (9)$$

Suppose that $n(s) = cs+d$ and $d(s) = s^2+as+b$ are coprime. The constructive steps for generating Φ_k proceeds as follows:

1) Define the polynomial

$$\begin{aligned} p(\lambda, s) &= s^2 + as + b - \lambda(cs + d) \\ &= s^2 + (a - cx - jcy)s + b - dx - jdy, \end{aligned}$$

where x and y are the real and imaginary part of λ, respectively.

2) Applying Lemma 2 to $p(\lambda, s)$, we obtain the following condition

$$\Delta_1(\lambda, \bar{\lambda}) = a - cx = a - \frac{c}{2}\lambda - \frac{c}{2}\bar{\lambda} > 0,$$

$$\begin{aligned} \Delta_2(\lambda, \bar{\lambda}) &= \begin{vmatrix} a - cx & 0 & dy \\ 1 & b - dx & cy \\ 0 & -dy & a - cx \end{vmatrix} \\ &= -\frac{1}{2}c^2 d(\lambda^2\bar{\lambda} + \lambda\bar{\lambda}^2) + \frac{1}{2}(3acd + bc^2 - d^2)\lambda\bar{\lambda} \\ &+ \frac{1}{4}\left(adc + bc^2 + d^2\right)(\lambda^2 + \bar{\lambda}^2) \\ &- \frac{1}{2}(2abc + a^2 d)(\lambda + \bar{\lambda}) + a^2 b > 0. \end{aligned}$$

3) Find the maximum degree $\ell_k - 1$ of λ in $\Delta_k(\lambda, \bar{\lambda})$ and the coefficient matrix Φ_k of $\Delta_k(\lambda, \bar{\lambda})$ for $k = 1, 2$:

$$\ell_1 = 2,\ \ell_2 = 3,\ \ \Phi_1 = \begin{bmatrix} 2a & -c \\ -c & 0 \end{bmatrix},$$

$$\Phi_2 = \frac{1}{4}\begin{bmatrix} 4a^2 b & -2a^2 d - 4abc & acd + bc^2 + d^2 \\ -2a^2 d - 4abc & 6acd + 2bc^2 - 2d^2 & -2c^2 d \\ acd + bc^2 + d^2 & -2c^2 d & 0 \end{bmatrix}.$$

As illustrated above, the process of constructing Φ_k is systematic and can be automated for the general νth-order $h(s)$. A general algorithm is given in Appendix C, which can be effectively utilized not only for the parametric analysis of low order systems with small number of parameters by symbolic computation as explained through the above example but also for numerical computation of Φ_k for fairly high order systems appearing in many practical applications.

The stability region $\Lambda(h)$ is now characterized by polynomial inequalities, and hence it can be easily visualized on the complex plane as the intersection of the subregions Σ_k as seen in the next example.

Example 2: Let us consider a specific $h(s)$ to clearly show the advantage of the algebraic stability condition given in Theorem 1 from the view point of parametric analysis. The system comprises the first order low pass filter (time constant T) and a time delay L with the first order Pade approximation, and is expressed as

$$h(s) = \frac{1}{1 + Ts} \cdot \frac{1 - \frac{L}{2}s}{1 + \frac{L}{2}} = \frac{-\frac{1}{T}s + \frac{2}{LT}}{s^2 + (\frac{2}{L} + \frac{1}{T})s + \frac{2}{LT}}, \qquad (10)$$

which is a special case of (9) with

$$a = \frac{L + 2T}{LT}, \qquad b = d = \frac{2}{LT}, \qquad c = -\frac{1}{T}.$$

We can see from pure algebraic computations based on the formula of $\Delta_1 > 0$ and $\Delta_2 > 0$ in Example 1 that the stability region is characterized by a single parameter $\rho := L/T$, or the ratio of the delay time L and the time constant T as

$$\Delta_1 > 0 \quad \Leftrightarrow \quad x > -(1 + 2/\rho) \qquad (11)$$

$$\Delta_2 > 0 \quad \Leftrightarrow \quad y^2 < \frac{(1 - x)(\rho x + \rho + 2)^2}{\rho(\rho x + \rho + 4)}, \qquad (12)$$

where x and y are the real and imaginary part of λ, respectively.

Clearly $x < 1$ should hold to satisfy the second inequality $\Delta_2 > 0$ under $\Delta_1 > 0$, and the stability region on the real axis is given by $(-(1 + 2/\rho), 1)$. It is readily seen that the stability regions for two extreme cases $\rho \to 0$ (or $L \to 0$) and $\rho \to \infty$ (or $T \to 0$) can be represented by $y^2 < 4(1 - x)$ and $x^2 + y^2 < 1$, respectively. Note that the former condition is exactly the one for first order systems. Note also that the latter condition, or the unit disc condition, is coincident with the one for discrete-time systems as expected, since the first order Pade approximation of the time delay is a Tustin transformation. We can also confirm by symbolic computation such as QE (Quantifier Elimination) that the stability region for ρ_1 contains that for ρ_2 when $\rho_2 > \rho_1 > 0$.

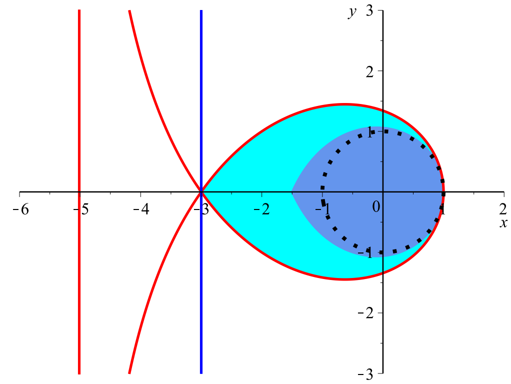

Fig. 3. Stability region for $h(s)$ given by (10)

We now illustrate how inequalities $\Delta_k > 0$ ($k = 1, \ldots, \nu$) specify the stability region. The shaded region in Fig. 3 shows the stability region for $\rho = 1$, which is the intersection of two regions determined by $\Delta_1 > 0$ (the half plane to the right of the blue line $x = -3$) and $\Delta_2 > 0$ (the two regions enclosed by the red curve). The red curve except the vertical line $x = -5$ is in fact the Nyquist plot of $\phi(s)$. We see that the stability region is a collection of points around which the number of encirclements by the Nyquist plot is zero, and thus the result is essentially coincident with the graphical test in [7]. It should be emphasized that $\Delta_1 > 0$, or equivalently $x := \text{Re}\{\lambda\} > -(1+2/\rho)$, systematically removes the left part of the region corresponding to $\Delta_2 > 0$, allowing for characterization of the stability region without counting the number of encirclements. This is one of the advantages of the proposed algebraic result in comparison with the graphical result in [7]. Figure 3 also illustrates how the stability region shrink as ρ becomes larger, where the dark shaded region corresponds to the case of $\rho = 4$ and it will converge to the unit disk when ρ tends to ∞.

B. LMI Stability Condition

In this section, we show stability conditions for the linear time-invariant system with the generalized frequency variable in terms of LMIs. The following is our main result for stability analysis, derived as a simple consequence of the developments in the previous sections.

Theorem 2: Let a matrix $A \in \mathbb{R}^{n\times n}$, and a strictly proper rational function $h(s) = n(s)/d(s)$ be given and define $\mathcal{H}(s)$ and $p(\lambda, s)$ by (5) and (6), respectively. Suppose that $n(s)$ and $d(s)$ are coprime. Let $\ell_k \in \mathbb{N}$ and $\Phi_k \in \mathbf{S}_{\ell_k}$ for $k = 1, 2, \ldots, \nu$ be specified as in Theorem 1. Then $\mathcal{H}(s)$ is stable if and only if, for each $k = 1, 2, \ldots, \nu$, there exists $X_k \in \mathbf{S}_n^+$ such that

$$L_{\ell_k}(A)^\mathsf{T}(\Phi_k \otimes X_k)L_{\ell_k}(A) > 0, \tag{13}$$

where

$$L_\ell(A) := \begin{bmatrix} I \\ A \\ \vdots \\ A^{\ell-1} \end{bmatrix}. \tag{14}$$

Proof: From Theorem 1, a necessary and sufficient condition for stability of $\mathcal{H}(s)$ is given by $\sigma(A) \subset \Lambda(h)$. The equivalence of this condition and (13) follows from Lemma 3 in Appendix A. ■

Theorems 1 and 2 can systematically check the stability of the linear time-invariant system with the generalized frequency variable, given the generalized frequency variable $\phi(s) = 1/h(s)$ and the interconnection matrix A. We have two methods for checking stability: one is to compute the eigenvalues of A and check if $\sigma(A) \subset \Lambda(h)$, and the other is to solve the LMIs[2] in (13). While the eigenvalue method would be more computationally efficient, the LMI method has multiple advantages for developing system theories for interconnected dynamics.

Since the LMIs are expressed explicitly in terms of the property of $h(s)$ representing the common subsystem dynamics and A characterizing the information exchange between the subsystems, these LMIs may lead to a further understanding of these large-scale systems. Therefore, this theorem would provide a foundation for further performance and robustness analyses of linear time-invariant systems with generalized frequency variables. The following section provides an extension to robust stability analysis.

[2]The stability condition is given in terms of ν independent LMIs of dimension $n \times n$ with ν variables $X_k \in \mathbf{S}_n$. If the standard Lyapunov inequality $\mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^\mathsf{T} < 0$ is used for a state space realization of $\mathcal{G}(s)$, we have a single LMI of dimension $n\nu \times n\nu$ with one variable $\mathcal{X} \in \mathbf{S}_{n\nu}$.

C. LMI Robust Stability Condition

This section provides an extension of the nominal stability result in the previous section to the case where the interconnections are subject to structured parametric uncertainties. The basic idea is to consider the stability condition in Theorem 2 with A replaced by an uncertain matrix A_Δ, and then apply the quadratic separator [18]. The resulting robust stability condition is given as follows.

Theorem 3: Consider the uncertain system described by

$$\mathcal{H}_\Delta(s) := (I/h(s) - A_\Delta)^{-1}, \quad A_\Delta := A + B\Delta C,$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$ are given matrices, $h(s)$ is a strictly proper, real rational, νth order transfer function, and Δ is an uncertain matrix belonging to a compact subset of real matrices $\mathbf{\Delta}$. Let $\Lambda(h)$, $\ell_k \in \mathbb{N}$, and $\Phi_k \in \mathbf{S}_{\ell_k}$ for $k = 1, 2, \ldots, \nu$ be specified as in Theorem 1. Then $\mathcal{H}_\Delta(s)$ is robustly stable for all $\Delta \in \mathbf{\Delta}$ if one of the following equivalent conditions holds for each $k = 1, 2, \ldots, \nu$:

(i) There exists $X_k \in \mathbf{S}_n^+$ such that

$$L_{\ell_k}(A_\Delta)^\mathsf{T}(\Phi_k \otimes X_k)L_{\ell_k}(A_\Delta) > 0, \quad \forall \Delta \in \mathbf{\Delta}. \tag{15}$$

(ii) There exist $X_k \in \mathbf{S}_n^+$ and $\Theta_k \in \mathbf{S}_{n_k}$ such that

$$F^\mathsf{T}(\Phi_k \otimes X_k)F > H^\mathsf{T}\Theta_k H,$$

$$\begin{bmatrix} I \\ \Lambda \end{bmatrix}^\mathsf{T} \Theta_k \begin{bmatrix} I \\ \Lambda \end{bmatrix} \geq 0, \quad \forall\, \Lambda \in \mathbf{\Lambda}_{\ell_k}$$

where $n_k := (p+m)(\ell_k - 1)$ and

$$\begin{aligned} &F := \mathsf{A}_{\ell_k}\mathsf{B}_{\ell_k}, \\ &H := \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}, \quad H_1 := \begin{bmatrix} \mathsf{C}_{\ell_k}\mathsf{A}_{\ell_k-1}\mathsf{B}_{\ell_k-1} & 0 \end{bmatrix} \\ &\qquad\qquad\qquad\ \ H_2 := \begin{bmatrix} 0 & I \end{bmatrix}, \\ &\mathsf{B}_\ell := \text{diag}(I, B, \ldots, B), \\ &\mathsf{C}_\ell := \text{diag}(C, \ldots, C), \\ &\mathbf{\Lambda}_\ell := \text{diag}(\mathbf{\Delta}, \ldots, \mathbf{\Delta}), \end{aligned}$$

$$\mathsf{A}_\ell := \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ A & I & 0 & \ddots & \vdots \\ A^2 & A & I & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ A^{\ell-1} & \cdots & A^2 & A & I \end{bmatrix},$$

where B, C, and $\mathbf{\Delta}$ repeat $\ell - 1$ times in B_ℓ, C_ℓ, and $\mathbf{\Lambda}_\ell$, respectively.

Proof: Fix $\Delta \in \mathbf{\Delta}$ and $M := \Phi_k \otimes X_k$. Condition (15) holds if and only if $\eta^\mathsf{T} M \eta > 0$ for all $\eta \in \mathbb{L}$, where

$$\mathbb{L} := \{L_{\ell_k}(A_\Delta)x : \ x \in \mathbb{R}^n \}.$$

For $i = 1, 2, \ldots, \ell_k - 1$, define

$$\eta_{i+1} = A_\Delta \eta_i, \quad \xi_{i+1} = \Delta C \eta_i, \quad \xi_1 = \eta_1 = x.$$

Then we have

$$\mathbb{L} = \{F\xi : \ \begin{bmatrix} \Lambda & -I \end{bmatrix} H\xi = 0 \}, \quad \Lambda := \text{diag}(\Delta, \ldots, \Delta).$$

The equivalence (i) ⇔ (ii) then follows from Lemma 10 in [18]. ■

The robust stability condition in statement (ii) is potentially conservative due to the assumption that X_k is independent of Δ, and can be made computationally tractable through reduction to a finite number of LMIs using the vertex separator or the DG-scaling [18].

Example 3: Consider the system described by the feedback connection of $h(s)I$ and $A_\Delta = A + \delta B$ where $h(s)$ is given by (10) with $L = T = 1$ and

$$A = \frac{1}{3}\begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 2 & 1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

We have analyzed the robust stability of the above system by setting $\Delta = \delta I$ and $C = I$ in Theorem 3 with the vertex separator. In combination with the bisection search, we estimated the maximum value of γ for which the stability is guaranteed for all perturbations satisfying $|\delta| \leq \gamma$. The computational algorithm implemented on Matlab converged after 4.33 s of the CPU time, and the result indicated that the system remains stable when $\gamma = 1.50$. This stability margin turned out to be exact as shown in Fig. 4 where the eigenvalues of A_Δ are plotted (red dots) on the complex plane when δ is varied in the interval $-1.5 \leq \delta \leq 1.5$. The root locus remains within the stability region for all δ in this interval, except when $\delta = 1.5$ at which one of the eigenvalues moves to -3, exactly on the boundary.

For comparison, the same robust stability condition was applied to another representation of the same system, namely, the feedback connection of $(1/s)I$ and $\mathcal{A} + \delta\mathcal{B}$ with $\mathcal{A} := I \otimes A_h + A \otimes (b_h c_h)$ and $\mathcal{B} := B \otimes (b_h c_h)$. The algorithm converged in 1.83 s to $\gamma = 1.03$. This example indicates that there are cases where a representation with the generalized frequency variable can be advantageous in reducing the conservatism of the robust stability condition, at the expense of more computational time.

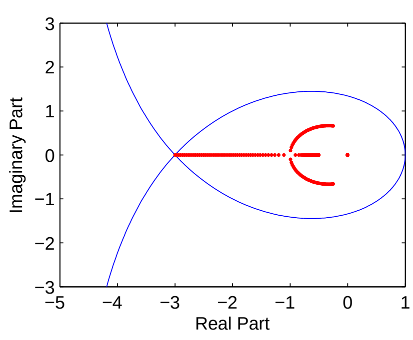

Fig. 4. Eigenvalues of A_Δ for $|\delta| \leq 1.5$

IV. Extensions to $\mathcal{D}$-Stability Analysis

This section is devoted to $\mathcal{D}$-stability analysis for dealing with practical situations, where we specify stability degrees such as the damping ratio and convergence rate rather than just guaranteeing stability of the system. Our $\mathcal{D}$-stability analysis relates to the discussions on time responses and robustness in Example 2 of [7], where the information flow filter is suggested to modify $h(s)$ to adjust the stability region relative to the eigenvalues of the interconnection matrix. The $\mathcal{D}$-stability result presented below can be used to directly characterize the eigenvalue region for specific damping and stability margin evaluated by distance from the stability boundary.

Given a region $\mathcal{D}$ on the complex plane, the system $\mathcal{H}(s)$ is said to be $\mathcal{D}$-stable if all the poles of $\mathcal{H}(s)$ lie in $\mathcal{D}$. The $\mathcal{D}$-stability region is defined to be the set of points on the complex plane such that $\mathcal{H}(s)$ is $\mathcal{D}$-stable if and only if all the eigenvalues of A lie in the set. The objective of this section is to extend the characterization of the stability region developed in Section III to the $\mathcal{D}$-stability region.

A. *Half plane and disk*

One of typical $\mathcal{D}$-stability regions useful for a variety of practical applications is illustrated in Fig. 5. Using this region, we can analyze stability degree corresponding to parameters α, θ and r that define region $\mathcal{D}$. Specifically, α, θ and r correspond to convergence rate, oscillation damping / degree of overshoot, and robustness against unmodeled dynamics in the high frequency range, respectively.

Region $\mathcal{D}$ is constructed as the intersection of three half planes $\mathcal{C}_{\alpha,\theta}$ (Fig. 6) defined from

$$\mathcal{C}_{\alpha,\theta} := \{x + jy \in \mathbb{C} : y \tan\theta < \alpha - x\}, \tag{16}$$

for $\alpha \in \mathbb{R}$ and $|\theta| < \pi/2$, and a disk $\mathcal{D}_{c,r}$ (Fig. 7) defined by

$$\mathcal{D}_{c,r} := \{z \in \mathbb{C} : |z - c| < r\} \tag{17}$$

for $c \in \mathbb{C}$ and $r > 0$. In other words, we have

$$\mathcal{D} = \mathcal{C}_{0,\theta} \cap \mathcal{C}_{0,-\theta} \cap \mathcal{C}_{\alpha,0} \cap \mathcal{D}_{0,r},$$

and hence we only need to investigate two independent necessary and sufficient $\mathcal{D}$-stability conditions for for all poles of $\mathcal{H}(s)$ to lie in $\mathcal{C}_{\alpha,\theta}$ or $\mathcal{D}_{c,r}$.

1) Condition for half plane: Since $\mathcal{C}_{\alpha,\theta}$ is a region derived by rotating $\mathbb{C}_-$ by θ counterclockwise around the origin and shifting the resulting region by α in the direction of the real axis, we can get a $\mathcal{D}$-stability condition for the half plane from the stability condition for continuous-time systems by transforming the variable "s" into "$e^{j\theta}s + \alpha$." Specifically, all the poles of $\mathcal{H}(s)$ belong to $\mathcal{C}_{\alpha,\theta}$ if and only if

$$\sigma(A) \subset \Lambda(h)^{\mathcal{C}}(\alpha,\theta) := \bigcap_{k=1}^{\nu} \Lambda(h)_k^{\mathcal{C}}(\alpha,\theta)$$
$$\Lambda(h)_k^{\mathcal{C}}(\alpha,\theta) := \{\lambda \in \mathbb{C} \mid= l_{\ell_k^{\mathcal{C}}}(\lambda)^* \Phi_k^{\mathcal{C}}(\alpha,\theta) l_{\ell_k^{\mathcal{C}}}(\lambda) > 0\}, \tag{18}$$

where $\ell_k^{\mathcal{C}} \in \mathbb{N}$ and $\Phi_k^{\mathcal{C}}(\alpha,\theta) \in \mathbf{H}_{\ell_k^{\mathcal{C}}}$ are specified by applying extended Routh-Hurwitz criterion [8] to $p(\lambda, e^{j\theta}s + \alpha)$ given by (6).

2) Condition for disk: There are several ways to get the stability region for a disk $\mathcal{D}_{c,r}$. The most natural way is based on the stability condition for discrete-time systems (z-domain) or Schur-Cohn criterion [22]. Since $\mathcal{D}_{c,r}$ is a region obtained by enlarging the unit circle with factor r and shifting the center to c, we can derive a condition for the disk by applying Schur-Cohn criterion to $p(\lambda, rs + c)$ given by (6).

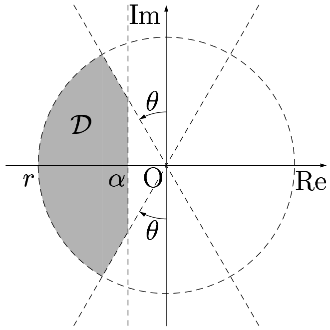

Fig. 5. Region $\mathcal{D}$

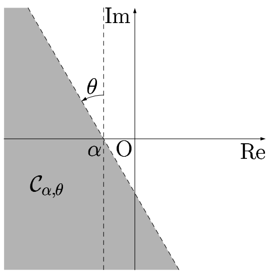

Fig. 6. Half plane $\mathcal{C}_{\alpha,\theta}$

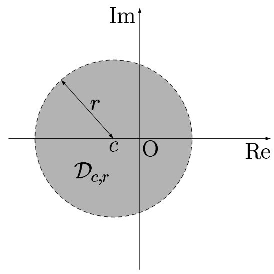

Fig. 7. Disk $\mathcal{D}_{c,r}$

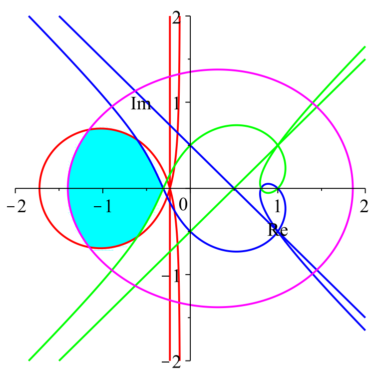

Fig. 8. $\mathcal{D}$-stability region for $h(s)$ in Example 2

Example 4: Let us consider $h(s)$ defined by

$$h(s) = (2s+1)/(s^2+s+1)$$

and set the parameters to specify the desired region $\mathcal{D}$ as $\alpha = -11/15$, $\theta = \pi/4$, and $r = 3$. We then apply the proposed algorithm to the intersection of $\mathcal{C}_{-11/15,0}$, $\mathcal{C}_{0,\pi/4}$, $\mathcal{C}_{0,-\pi/4}$, and $\mathcal{D}_{0,3}$. The result is shown as the shaded region in Fig. 8, which is the exact region where all the eigenvalues of A should lie to guarantee the $\mathcal{D}$-stability of the interconnected system. Note that the region does not include the origin of the complex plane, since the target agent $h(s)$ considered is stable but not $\mathcal{D}$-stable. We should emphasize that our proposed algorithm systematically gives the exact $\mathcal{D}$-stability region even if it is very complicated as seen in Fig. 8.

B. Unified formulation

Although the result presented in the previous subsection is effective as seen in Example 4, the method requires two completely separate algorithms for half planes and disks. This section develops a unified condition that allows for characterization of half planes and disks by a single algorithm with different parameter values. The idea for the unification is to use δ operator which nicely links the continuous-time result (or the left half plane) to one for the discretized system with small sampling period T (or disk with center at $-1/T$ and radius T). We, therefore, try to get a unified $\mathcal{D}$-stability condition based on δ operator ($\delta := (z-1)/T$) or by applying δ-Schur Cohn criterion [6]. Note that region $\mathcal{D}_{c,r}$ is derived from

$$\mathbb{D}_T := \{\delta \in \mathbb{C} : |\delta + 1/T| < 1/T\},$$

which is the stability region of δ-domain with sampling period $T > 0$, by shifting it by $1/T$ in the positive direction of the real axis, enlarging the result by factor rT, and shifting the center by c. Hence, we can obtain a condition for the roots s of a polynomial to lie in $\mathcal{D}_{c,r}$ from a stability condition for δ-domain by transforming the variable δ into $s := rT\delta + r + c$. Specifically, all the poles of $\mathcal{H}(s)$ lie in $\mathcal{D}_{c,r}$ if and only if

$$\sigma(A) \subset \Lambda(h)^{\mathcal{D}}(c,r,T) := \bigcap_{k=1}^{\nu} \Lambda(h)_k^{\mathcal{D}}(c,r,T)$$
$$\Lambda(h)_k^{\mathcal{D}}(c,r,T) := \{\lambda \in \mathbb{C} \mid l_{\ell_k^{\mathcal{D}}}(\lambda)^* \Phi_k^{\mathcal{D}}(c,r,T) l_{\ell_k^{\mathcal{D}}}(\lambda) > 0\} \tag{19}$$

where $\ell_k^{\mathcal{D}} \in \mathbb{N}$ and $\Phi_k^{\mathcal{D}}(c,r,T) \in \ell_k^{\mathcal{D}}$ are specified by applying δ-Schur Cohn criterion [6] to $p(\lambda, rT\delta + r + c)$ given by (6).

One can also derive the condition for $\mathcal{C}_{\alpha,\theta}$ from this condition. Let $\mathcal{D}_T$ be defined by

$$\mathcal{D}_T := \mathcal{D}_{-\frac{1}{T}e^{j\theta}+\alpha, \frac{1}{T}}.$$

Then, as illustrated in Fig. 9, we have

$$\lim_{T\to 0} \mathcal{D}_T = \mathcal{C}_{\alpha,\theta}.$$

Hence, the following theorem holds.

Theorem 4: Let a νth order and strictly proper transfer function $h(s) = n(s)/d(s)$, regions $\mathcal{C}_{\alpha,\theta}$ and $\mathcal{D}_{c,r}$ be given. Define $p(\lambda, s)$ by (6). Suppose that $n(s)$ and $d(s)$ are coprime. the positive integer $\ell_k^{\mathcal{C}} \in \mathbb{N}$ and $\Phi_k^{\mathcal{C}}(\alpha,\theta) \in \mathbf{H}_{\ell_k^{\mathcal{C}}}$ for $k = 1, 2, \ldots, \nu$ are specified by applying extended Routh-Hurwitz criterion [8] for $p(\lambda, e^{j\theta}s + \alpha)$, and the positive integer $\ell_k^{\mathcal{D}} \in \mathbb{N}$ and $\Phi_k^{\mathcal{D}}(c,r,T) \in \mathbf{H}_{\ell_k^{\mathcal{D}}}$ for $k = 1, 2, \ldots, \nu$ are specified by applying δ-Schur-Cohn criterion [6] for $p(\lambda, rT\delta + r + c)$. Define $\Lambda(h)_k^{\mathcal{C}}(\alpha,\theta)$ and $\Lambda(h)_k^{\mathcal{D}}(c,r,T)$ for $k = 1, 2, \ldots, \nu$ by (18) and (19), respectively. Then, the following equation holds for $k = 1, 2, \ldots, \nu$

$$\lim_{T\to 0} \Lambda(h)_k^{\mathcal{D}}\left(-\frac{1}{T}e^{j\theta} + \alpha, \frac{1}{T}, T\right) = \Lambda(h)_k^{\mathcal{C}}(\alpha,\theta).$$

Theorem 4 leads to the following algorithm, which provides unifying $\mathcal{D}$-stability conditions for linear time-invariant systems with generalized frequency variables.

Algorithm: Let $h(s)$ be given

1) Write $h(s) = n(s)/d(s)$ with $n(s)$ and $d(s)$ coprime. Define the polynomial $p(\lambda, s)$ by (6).

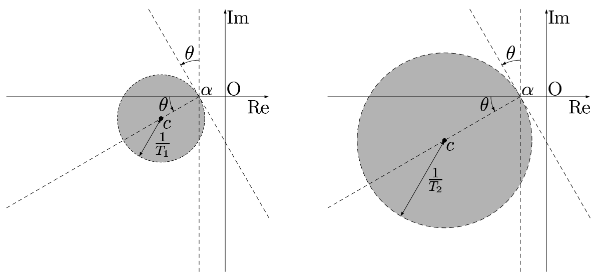

Fig. 9. Disk $\mathcal{D}_T$: $T_1 > T_2$

2) Obtain $\Phi_k^{\mathcal{D}}(c, r, T)$ by applying δ-Schur-Cohn criterion for $p(\lambda, rT\delta + r + c)$.
3) • Disk region $\mathcal{D}_{c',r'}$:
   Substitute $c = c'$, $r = r'$ and $T = 1$ into $\Phi_k^{\mathcal{D}}(c, r, T)$.
   • Half plane region $\mathcal{C}_{\alpha,\theta}$:
   Substitute $c = -\frac{1}{T}e^{j\theta} + \alpha$, $r = \frac{1}{T}$ into $\Phi_k^{\mathcal{D}}(c, r, T)$, and set $T = 0$.

The main advantage of the unified condition is that we need only one algorithm which can be applied to both continuous-time systems and discrete-time systems / sampled-data systems, and symbolic computations are quite useful to derive the corresponding conditions in parametric ways.

V. Applications

This section provides two applications, namely oscillation analysis for nonlinear biological networks and cooperative stabilization [14] for a system of multiple inverted pendulums. The target agent in the former case is stable and instability of linearized feedback system around an equilibrium point is investigated. The latter application is completely opposite, i.e., stabilizability for an unstable agent is discussed.

A. Oscillation Analysis for Nonlinear Biological Networks

A model of biological circuits for central pattern generators [17] is given by

$$x = h(s)A\varphi(x)$$

where $x(t) \in \mathbb{R}^n$ is the vector of cell membrane potentials, $h(s)$ is a scalar transfer function representing the linear dynamics of a neuron, φ is a sigmoid function capturing the threshold property, and $A \in \mathbb{R}^{n\times n}$ is the neuronal interconnection matrix. This model could also represent gene regulatory networks [16]. We consider a recurrent cyclic inhibition oscillator [30] with an additional self-feedback as in [29], and the model is given by

$$h(s) := \frac{2}{(s+1)^2}, \quad \varphi(x) := \tanh(x), \qquad A := \begin{bmatrix} k & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix},$$

where φ acts on x elementwise, and $k \in \mathbb{R}$ is the parameter dictating the strength and the excitatory/inhibitory nature of the self-feedback.

An oscillation in the sense of Yakubovich exists if all the equilibrium points are hyperbolically unstable due to ultimate boundedness of all the trajectories [24], [17]. We vary k and

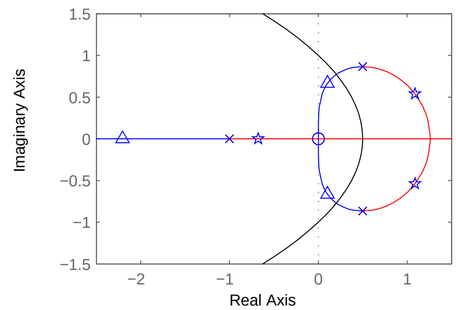

Fig. 10. Stability region and root loci

examine stability of the equilibrium at the origin through linearization $\varphi(x) \simeq x$. By Example 1 and Theorem 1, the origin is stable when $\Delta_1 = 2 > 0$ and $\Delta_2 = 4(1 - 2\lambda_R - \lambda_I^2) > 0$, where $\lambda = \lambda_R + j\lambda_I$ is an arbitrary eigenvalue of A. This stability region of λ on the complex plane can be visualized as the region to the left of the black curve in Fig. 10. Thus the origin is stable if all the eigenvalues of A are in this region, or otherwise, we expect to see a stable limit cycle.

The eigenvalues of A are also plotted in Fig. 10, where the root loci are indicated by blue ($k < 0$) and red ($k > 0$) curves. Without the self-feedback ($k = 0$, marked by "x" in Fig. 10), two of the eigenvalues are outside of the stability region, and a simulation shows a stable limit cycle (Fig. 11). With an excitatory self-feedback ($k = 1.5$, "$\star$"), the origin is unstable, and a limit cycle is continued to be observed (Fig. 12). Note that the amplitudes of the fundamental component as well as the phase relationship between neurons remain roughly the same, but the period of oscillation is substantially increased by the positive self-feedback, consistently with the prediction in [17], [16], [29]. With an inhibitory self-feedback ($k = -2$, "$\triangle$"), however, all the eigenvalues are in the stability region, and the origin is indeed (at least locally) stable (Fig. 13).

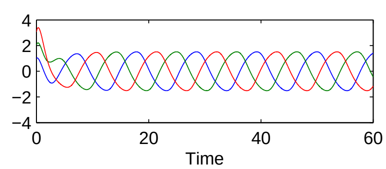

Fig. 11. Simulation with $k = 0$ (marked by x)

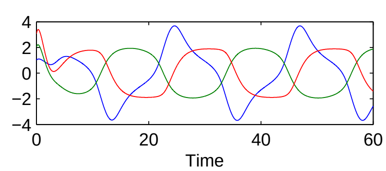

Fig. 12. Simulation with $k = 1.5$ (marked by $\star$)

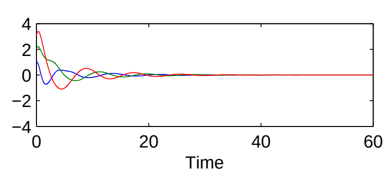

Fig. 13. Simulation with $k = -2$ (marked by $\triangle$)

B. Cooperative Stabilization

This subsection is concerned with a synthesis type question named *cooperative stabilizability* proposed in [14]. We say that $h(s)$ is *cooperatively stabilizable* if there exists a matrix $A \in \mathbb{R}^{n\times n}$ such that all the poles of $\mathcal{H}(s)$ are stable. Clearly, $h(s)$ is cooperatively stabilizable if and only if the associated stability region is nonempty. We are particularly interested in the situation where the target agent alone cannot be stabilized by a constant output feedback but it is cooperatively stabilizable, because this situation clearly shows an advantage of cooperation from the view point of control.

To illustrate such situation, we consider unstable agents for which the stability regions tend to be small or even empty. In particular, we give an example where the stability region has no intersection with the real axis, in which case the connectivity matrix A cannot have any real eigenvalue, requiring two or more (even number of) agents for stabilization. This is quite different from the case for stable agents as seen in the previous subsection, where the stability region is always nonempty and includes a region around origin.

The target agent or subsystem in this application example is an inverted pendulum system illustrated in Fig. 14, where an inverted pendulum is mounted on a motor-driven cart. Here, M, m, and 2ℓ denote the mass of the cart, the mass of the pendulum, and the length of the pendulum, respectively. We assume that the pendulum moves only within the vertical plane and that it has a uniform density so that its moment of inertia is given by $I = \frac{1}{3}m\ell^2$. The friction coefficient between the track and the cart is denoted by $\mu_{\rm t}$ and that between the pendulum and the cart is denoted by $\mu_{\rm p}$.

The dynamical relationships between the control input u, or the force applied to the cart, and the resulting position of the cart x and the angle of the pendulum θ are represented by

$$\begin{aligned} &\tfrac{4}{3}m\ell^2\ddot{\theta} + \mu_{\rm p}\dot{\theta} - mg\ell\theta = -m\ell\ddot{x}\ , \\ &(M+m)\ddot{x} + \mu_{\rm t}\dot{x} + m\ell\ddot{\theta} = u \end{aligned}$$

under the assumption that the angle θ is small. Taking the Laplace transform of the system equations yields

$$\begin{aligned} &\tfrac{4}{3}m\ell^2\Theta(s)s^2 + \mu_{\rm p}\Theta(s)s - mg\ell\Theta(s) = -m\ell X(s)s^2, \\ &(M+m)X(s)s^2 + \mu_{\rm t}X(s)s + m\ell\Theta(s)s^2 = U(s)\ . \end{aligned}$$

Then, the transfer functions from u to θ (denoted by P_θ) and from u to x (denoted by P_x) are, respectively, expressed as

$$P_\theta(s) = \frac{-m\ell s}{D(s)}\ ,\ \ P_x(s) = \frac{\frac{4}{3}m\ell^2s^2 + \mu_{\rm p}s - mg\ell}{sD(s)}\ ,$$

where

$$D(s) = a_3s^3 + a_2s^2 + a_1s + a_0$$

$$\begin{aligned} a_0 &:= -\mu_{\rm t}mg\ell, \\ a_1 &:= -(M+m)mg\ell + \mu_{\rm p}\mu_{\rm t}, \\ a_2 &:= (M+m)\mu_{\rm p} + \tfrac{4}{3}\mu_{\rm t}m\ell^2, \\ a_3 &:= \tfrac{1}{3}(4M+m)m\ell^2. \end{aligned}$$

Since [14] proved that both $P_\theta(s)$ with any $\mu_{\rm t} > 0$ and $P_x(s)$ with sufficiently small $\mu_{\rm p} > 0$ are not cooperatively

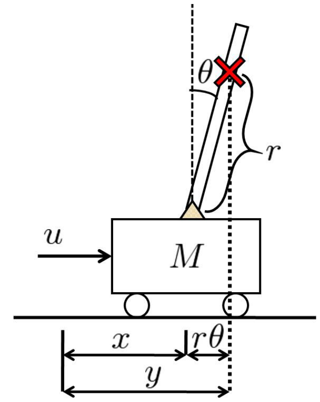

Fig. 14. Inverted pendulum

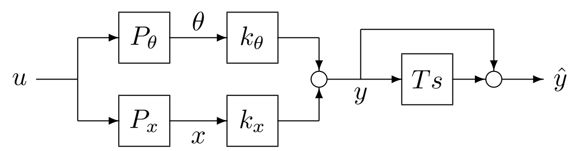

Fig. 15. PD control setting

stabilizable, we consider a different situation, where the measured output $y(t)$ is a linear combination of the two outputs of $\theta(t)$ and $x(t)$:

$$y(t) := k_\theta\theta(t) + k_x x(t) \qquad (20)$$

which generalizes the case illustrated in Fig. 14.

The transfer function from $u(t)$ to $y(t)$ in Fig. 14 can be expressed as

$$h_{\rm P}(s) = k_\theta P_\theta(s) + k_x P_x(s) = \frac{b_2s^2 + b_1s + b_0}{sD(s)}\ , \qquad (21)$$

where

$$\begin{aligned} b_0 &:= -mg\ell k_x, \\ b_1 &:= \mu_{\rm p}k_x, \\ b_2 &:= \tfrac{4}{3}m\ell^2k_x - m\ell k_\theta\ . \end{aligned}$$

Our further modification is to use a PD controller rather than just a P controller. in order to guarantee the possibility of cooperative stabilization. In other words, our target system here is given by

$$h(s) \quad = \quad \frac{(Ts+1)(b_2s + b_1s + b_0)}{sD(s)}\ ,$$

where $T > 0$ denotes the ratio of P-gain and D-gain as depicted in Fig. 15. Setting the physical parameters of the inverted pendulum

$$\begin{aligned} &m = 5.99M, \quad \ell = 2.94, \quad \mu_{\rm p} = 0.164M, \\ &\mu_{\rm t} = 9.98M, \quad k_\theta = -26.8M, \quad k_x = -2.09M\ , \end{aligned}$$

where the mass M is arbitrary, we have

$$h(s) = \frac{(Ts+1)(\frac{19}{10}s^2 - \frac{1}{500}s + \frac{21}{10})}{s(s-2)(s+1)(s+5)}\ . \qquad (22)$$

An application of Theorem 1 to $h(s)$ with $T = 0$ show that the stability region is empty, which implies that there

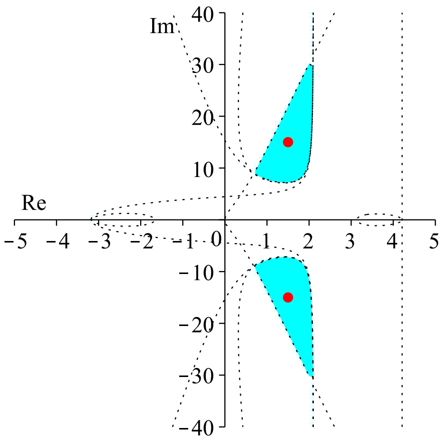

Fig. 16. Stability region for $h(s)$ given by (22) with $T = 1/2$

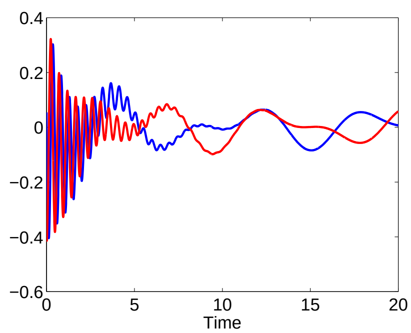

Fig. 17. Free response of the interacting two-pendulum system

exists no P control that stabilizes the system even if multiple agents collaborate each other. Fortunately, however, we can show that $h(s)$ with $T = 1/2$ is cooperatively stabilizable with the nonempty stability region illustrated in Fig. 16. It should be noted that the stability region does not intersect with the real axis. This means that each $h(s)$ cannot be stabilized by the PD control without cooperation since no real diagonal matrix A can be designed to have eigenvalues in the stability region.

Let us consider the case with two inverted pendulum systems that interact with each other under the following interconnection matrix:

$$A = \begin{bmatrix} 1.5 & -15 \\ 15 & 1.5 \end{bmatrix} .$$

The red points in Fig. 16 represent the eigenvalues of A which are $-1.5 \pm j15$. We can see from Fig. 16 that all the eigenvalues of A lie in the shaded region, or $\sigma(A) \subset \Lambda(h)$ holds, which guarantees that $\mathcal{H}(s)$ is stable. Figure 17 shows free responses of the outputs $y(t)$ of the two pendulums with a certain nonzero initial condition. The responses are very oscillatory, although the system is stable. This is because the poles of $\mathcal{H}(s)$ are $-0.5601 \pm j13.3444$, $-1.9596 \pm j0.3407$, $-0.0508 \pm j1.4097$, $-0.0046 \pm j0.8448$, and there exists a pair of poles near the imaginary axis.

In a similar manner, we can obtain all possible poles of $\mathcal{H}(s)$ resulting from admissible choices of A whose eigenvalues lie in the stability region. For each point λ in the stability

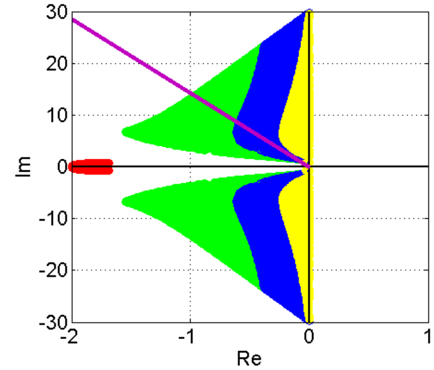

Fig. 18. Poles of $\mathcal{H}(s)$ given by (22) with $T = 1/2$

region shown in Fig. 16, the four characteristic roots of the polynomial $p(\lambda, s)$ in (6) are found and plotted in Fig. 18 with different colors. There may be some overlap of the colored regions, but a pair of closed-loop poles are always contained in the yellow region for any stabilizing choices of A. The purple line is drawn so that the upper half of the yellow region is entirely above the line. The angle between the imaginary axis and this line is $\pi/45$. This implies that there always exists a pole of $\mathcal{H}(s)$ of which the damping ratio is less than $\zeta := \sin(\pi/45) \cong 0.07$ for any A that stabilizes $\mathcal{H}(s)$. Hence, system responses are oscillatory for $h(s)$ given by (22) with $T = 1/2$ and any A which stabilizes $\mathcal{H}(s)$.

This phenomena can be confirmed through $\mathcal{D}$-stability analysis shown in the previous section. Set $\mathcal{D}$-stability region parameters as $\alpha = 0$, $\theta = \pi/180$, and $r = \infty$. In other words, the $\mathcal{D}$-stability region is given by an intersection of $\mathcal{C}_{0,\pi/180}$ and $\mathcal{C}_{0,-\pi/180}$. The $\mathcal{D}$-stability analysis leads to the corresponding $\mathcal{D}$-stability region, which is the shaded region in Fig. 19. Since both eigenvalues of A marked as red points in the figure are located outside the $\mathcal{D}$-stability region, we see that the closed-loop system is not $\mathcal{D}$-stable. This implies that the smallest damping ratio of the closed-loop system is less than $\sin(\pi/180)$. We can also see that $\mathcal{D}$-stability region becomes empty by setting $\theta = \pi/45$. This is consistent with our investigation based on Fig. 18 .

It should be noticed that no simple Hurwitz stability analysis cannot examine these properties. This is one of our motivations to investigate $\mathcal{D}$-stability conditions by which we can analyze the stability degree such as the damping and convergence rates.

VI. Conclusion

In this paper, we have considered linear time-invariant systems with generalized frequency variables $\phi(s)$, described as $C(\phi(s)I - A)^{-1}B + D$. Such systems arise from interconnections of multiple identical subsystems, where $h(s) := 1/\phi(s)$ is the common subsystem dynamics, and A is the connectivity matrix characterizing the information exchange among subsystems.

We have shown that the interconnected system is stable if and only if the eigenvalues of the interconnection matrix A are in a particular region specified by the generalized frequency variable $\phi(s)$. We provided a characterization of the stability

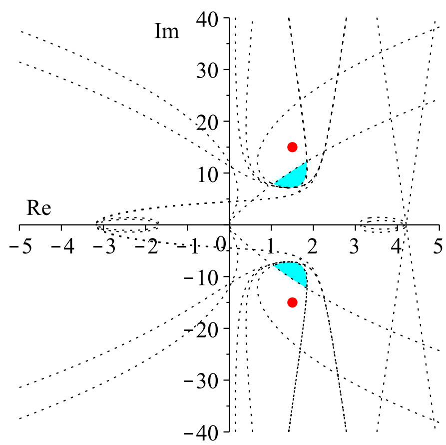

Fig. 19. $\mathcal{D}$-Stability region for $h(s)$ given by (22) with $T = 1/2$

region and a necessary and sufficient LMI condition for the eigenvalues to lie in the region. The LMI result was extended for robust stability analysis of systems subject to uncertainties in the interconnection matrix. We have then proposed a unified treatment of D-stability region and shown its complete characterization. Finally, two numerical examples were provided to illustrate applications of the stability condition.

The future research directions along this line include robust stability analysis for heterogeneous multi-agent dynamical systems [12], where the idea from [19] may be useful. Control performance analysis such as $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm computations [13] would be another direction.

Acknowledgment: The authors would like to thank Mr. S. Fukamachi for his help to complete examples in Section III-A.

Appendix A
Preliminary Lemmas

This section is devoted to several lemmas which are instrumental to the proof of the main stability theorem.

Since the coefficients of $p(\lambda, s)$ are complex numbers, the standard Routh-Hurwitz criterion is not applicable. However, the following result [8] provides a stability condition for polynomials with complex coefficients, and will be useful for characterizing the set of λ for which $p(\lambda, s)$ is Hurwitz.

Lemma 2 ([8]): The polynomial with complex coefficients

$$p(s) = s^\nu + \alpha_1 s^{\nu-1} + \alpha_2 s^{\nu-2} + \cdots + \alpha_\nu,$$
$$\alpha_i = p_i + jq_i \ (i = 1, 2, \ldots, \nu)$$

is Hurwitz if and only if the determinants

$$\Delta_1 = p_1,$$
$$\Delta_k = \det \begin{bmatrix} F(\mathsf{p}_k) & -F(\mathsf{q}_k)R \\ UF(\mathsf{q}_k) & F(\mathsf{p}_{k-1}) \end{bmatrix} \ (k = 2, 3, \cdots, \nu)$$

are all positive, where

$$F(x) := \begin{bmatrix} x_2 & x_4 & x_6 & \cdots & x_{2k} \\ x_1 & x_3 & x_5 & \ddots & \vdots \\ 0 & x_2 & x_4 & \ddots & x_{k+3} \\ \vdots & \ddots & \ddots & \ddots & x_{k+2} \\ 0 & \cdots & 0 & x_{k-1} & x_{k+1} \end{bmatrix},$$
$$\mathsf{p}_k := \begin{bmatrix} 1 & p_1 & p_2 & \cdots & p_{2k-1} \end{bmatrix}^\mathsf{T},$$
$$\mathsf{q}_k := \begin{bmatrix} 0 & 0 & q_1 & q_2 & \cdots & q_{2k-2} \end{bmatrix}^\mathsf{T},$$
$$p_i = q_i = 0 \ (i > \nu),$$
$$R := \begin{bmatrix} 0 \\ I \end{bmatrix} \in \mathbb{R}^{k\times(k-1)}, \ U := \begin{bmatrix} I & 0 \end{bmatrix} \in \mathbb{R}^{(k-1)\times k}.$$

Using this lemma, the set of $\lambda \in \mathbb{C}$, for which $p(\lambda, s)$ is a Hurwitz polynomial in s, can be expressed in the form of (23) since the determinant of a matrix is a polynomial of its entries.

The following lemma provides a necessary and sufficient condition for the eigenvalues of a matrix to lie in a region in the complex plane specified by a polynomial inequality. The condition is given in terms of an LMI feasibility problem.

Lemma 3: Let positive integers $\ell, n \in \mathbb{N}$, and matrices $\Phi \in \mathbf{S}_\ell$ and $A \in \mathbb{R}^{n\times n}$ be given and define Σ, $l_\ell(\lambda)$ and $L_\ell(A)$ by

$$\Sigma := \{ \lambda \in \mathbb{C} \mid l_\ell(\lambda)^* \Phi l_\ell(\lambda) > 0 \}, \tag{23}$$

The following statements are equivalent.

(i) $\sigma(A) \in \Sigma$.
(ii) There exists $X \in \mathbf{S}_n^+$ such that

$$L_\ell(A)^*(\Phi \otimes X)L_\ell(A) > 0. \tag{24}$$

Moreover, the result holds when the matrices are complex, i.e., $\Phi \in \mathbf{H}_\ell$, $A \in \mathbb{C}^{n\times n}$, and $X \in \mathbf{H}_n^+$.

Proof: (i) $\Rightarrow$ (ii): Let J be the Jordan form of A. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that

$$P^{-1}AP = J = \Lambda + U,$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ ($\lambda_i \in \sigma(A)$) is diagonal and all the elements of U are either 0 or 1, with all nonzero elements located on the diagonal above the main diagonal. For a small $\varepsilon > 0$, defining the nonsingular matrix $D(\varepsilon) = \mathrm{diag}(1, \varepsilon, \ldots, \varepsilon^{n-1})$, we obtain

$$A(\varepsilon) := D(\varepsilon)^{-1}JD(\varepsilon) = \Lambda + \varepsilon U.$$

For $k \in \mathbb{N}$, the k-th power of $A(\varepsilon)$ can be represented in the form

$$A(\varepsilon)^k = \Lambda^k + \varepsilon U(k, \varepsilon)$$

where $U(k, \varepsilon)$ is bounded as $\varepsilon \to 0$ for all $k \in \mathbb{N}$. Since $\sigma(A) \subset \Sigma$, we have $l_\ell(\lambda_i)^* \Phi l_\ell(\lambda_i) > 0$. Choosing

$$Y = \mathrm{diag}\left(\frac{1}{l_\ell(\lambda_1)^* \Phi l_\ell(\lambda_1)}, \ldots, \frac{1}{l_\ell(\lambda_n)^* \Phi l_\ell(\lambda_n)}\right) > 0,$$

we obtain

$$
\begin{aligned}
&L_\ell(A(\varepsilon))^*(\Phi\otimes Y)L_\ell(A(\varepsilon))\\
&= L_\ell(\Lambda)^*(\Phi\otimes Y)L_\ell(\Lambda) + Q(\varepsilon) = I_n + Q(\varepsilon).
\end{aligned}
$$

where $Q(\varepsilon)$ is a Hermitian matrix such that $\lim_{\varepsilon\to 0} Q(\varepsilon) = 0$. Therefore,

$$L_\ell(A(\varepsilon))^*(\Phi\otimes Y)L_\ell(A(\varepsilon)) > 0$$

holds for a sufficiently small $\varepsilon > 0$. Then, note that

$$
\begin{aligned}
&L_\ell(A(\varepsilon))^*(\Phi\otimes Y)L_\ell(A(\varepsilon))\\
&= D(\varepsilon)^*P^*L_\ell(A)^*(\Phi\otimes X)L_\ell(A)PD(\varepsilon) > 0\\
&\Leftrightarrow L_\ell(A)^*(\Phi\otimes X)L_\ell(A) > 0,
\end{aligned}
$$

where $X := (P^{-1})^*(D(\varepsilon)^{-1})^*YD(\varepsilon)^{-1}P^{-1} > 0$. Hence, there exists $X > 0$ such that (24) is satisfied.
(ii) $\Rightarrow$ (i): Suppose $X > 0$ satisfies (24). For an arbitrary $\lambda\in\sigma(A)$, let $v\in\mathbb{C}^n$ be an associated eigenvector so that $Av = \lambda v$. Then

$$v^*L_\ell(A)^*(\Phi\otimes X)L_\ell(A)v = l_\ell(\lambda)^*\Phi l_\ell(\lambda)v^*Xv > 0.$$

Since $X > 0$ we conclude that $l_\ell(\lambda)^*\Phi l_\ell(\lambda) > 0$ and hence $\sigma(A)\subset\Sigma$. ■

Lemma 3 converts the stability condition given in terms of the generalized Lyapunov equation [15] to the inequality condition. The matrix Φ is required to have a single negative eigenvalue in the equality theorem, but this requirement is eliminated in the inequality counterpart. An important implication of Lemma 3 in our context is that, if the set $\Lambda(h)$ in statement (ii) of Lemma 1 can be expressed in the form of (23), then the necessary and sufficient condition for $\mathcal{H}(s)$ to be stable can be described as an LMI feasibility problem.

Appendix B
Proof of Theorem 1

In view of Lemma 1, it suffices to show that the set $\Lambda(h)$ is characterized by (7). As outlined right after the theorem, the Hurwitz property of the polynomial $p(\lambda,s)$ in (6) can be characterized as $\Delta_k(\lambda,\bar\lambda) > 0$ using Lemma 2, where the coefficients p_i and q_i are defined as

$$p_i = a_i - b_i\frac{\lambda+\bar\lambda}{2},\quad q_i = -b_i\frac{\lambda-\bar\lambda}{2j}. \tag{25}$$

Let us show that the coefficients of the polynomial $\Delta(\lambda,\bar\lambda)$ are all real. To see this, let $\hat{\mathrm{q}}_k := \mathrm{q}_k/j$ and note that p_k and $\hat{\mathrm{q}}_k$ are affine functions of λ and $\bar\lambda$ with real coefficients. Since F is a linear function, we have

$$
\begin{aligned}
\Delta_k &= \det\begin{bmatrix} F(\mathrm{p}_k) & -jF(\hat{\mathrm{q}}_k)R\\ jUF(\hat{\mathrm{q}}_k) & F(\mathrm{p}_{k-1})\end{bmatrix}\\
&= \det\begin{bmatrix} F(\mathrm{p}_k) & F(\hat{\mathrm{q}}_k)R\\ UF(\hat{\mathrm{q}}_k) & F(\mathrm{p}_{k-1})\end{bmatrix}
\end{aligned}
$$

where we noted the identity $\det(A) = \det(J^*AJ)$ with $J = \mathrm{diag}(I, jI)$. Therefore, there exists a real symmetric matrix Φ_k such that $\Delta_k = l_{\ell_k}(\lambda)^*\Phi_k l_{\ell_k}(\lambda)$.

It remains to show that Δ_k is a polynomial of λ and $\bar\lambda$, of order at most k each. Define

$$T := \begin{bmatrix} 1 & 0 & 0\\ 0 & I & jI\\ 0 & jI & I\end{bmatrix},\quad M_k := \begin{bmatrix} p^\mathsf{T} & -q^\mathsf{T}\\ P & -\hat{Q}\\ Q & \hat{P}\end{bmatrix},$$

$$
\begin{aligned}
&p\in\mathbb{R}^k, && P,Q\in\mathbb{R}^{(k-1)\times k},\\
&q\in\mathbb{R}^{k-1}, && \hat{P},\hat{Q}\in\mathbb{R}^{(k-1)\times(k-1)}.
\end{aligned}
$$

where

$$
\begin{aligned}
p^\mathsf{T} &:= \begin{bmatrix} p_1 & p_3 & p_5 & \cdots & p_{2k-1}\end{bmatrix}\\
q^\mathsf{T} &:= \begin{bmatrix} q_2 & q_4 & \cdots & q_{2k-2}\end{bmatrix}\\
P &:= \begin{bmatrix} 1 & p_2 & p_4 & \cdots & p_{2k-2}\\ 0 & p_1 & p_3 & \cdots & p_{2k-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & \cdots & p_{k-2} & p_k\end{bmatrix}\\
Q &:= \begin{bmatrix} 0 & q_2 & q_4 & \cdots & q_{2k-2}\\ 0 & q_1 & q_3 & \cdots & q_{2k-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & \cdots & q_{k-2} & q_k\end{bmatrix}\\
\hat{P} &:= \begin{bmatrix} p_1 & p_3 & \cdots & p_{2k-3}\\ 1 & p_2 & \cdots & p_{2k-4}\\ \vdots & \vdots & \ddots & \vdots\\ \cdots & \cdots & p_{k-3} & p_{k-1}\end{bmatrix}\\
\hat{Q} &:= \begin{bmatrix} q_1 & q_3 & \cdots & q_{2k-3}\\ 0 & q_2 & \cdots & q_{2k-4}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & q_{k-3} & q_{k-1}\end{bmatrix}
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
2^{k-1}\Delta_k &= \det(TM_k)\\
&= \det\begin{bmatrix} p^\mathsf{T} & q^\mathsf{T}\\ P + jQ & j(\hat{P}+j\hat{Q})\\ j(P-jQ) & \hat{P}-j\hat{Q}\end{bmatrix}\\
&= \det\begin{bmatrix} a^\mathsf{T} - (\lambda+\bar\lambda)b^\mathsf{T}/2 & -(\lambda-\bar\lambda)\hat{b}^\mathsf{T}/(2j)\\ A - \lambda B & j(\hat{A}-\lambda\hat{B})\\ j(A-\bar\lambda B) & \hat{A}-\bar\lambda\hat{B}\end{bmatrix}
\end{aligned}
$$

for some real coefficient vectors a, b, and $\hat{b}$ and matrices A, B, $\hat{A}$, and $\hat{B}$, where we noted that

$$p_i + jq_i = a_i - \lambda b_i$$

with a_i and b_i being the real coefficients of $d(s)$ and $n(s)$, respectively. By the Leibniz formula, the determinant Δ_k is a polynomial of λ and $\bar\lambda$ with degree at most k since the matrices B and $\hat{B}$ have $k-1$ rows.

Appendix C
Algorithm for Computing Φ_k

Here we provide an algorithm to compute Φ_k ($k = 1,\ldots,\nu$) systematically based on the coefficients of numerator and denominator of $h(s)$ for characterizing the stability region and checking the stability even in a parametric way.

Algorithm h2Phi($h(s)$) :

Input : $h(s) = \dfrac{b_1 s^{\nu-1} + \cdots + b_\nu}{s^\nu + a_1 s^{\nu-1} + \cdots + a_\nu}$

Output : ℓ_k and Φ_k

1) $p_0 \leftarrow 1$, $q_0 \leftarrow 0$
   for $i \leftarrow 1$ **until** $2\nu - 1$ **do**
   if $i \leq \nu$ **then**
   $p_i \leftarrow a_i - b_i x$, $q_i \leftarrow -b_i y$
   else
   $p_i \leftarrow 0$, $q_i \leftarrow 0$
2) $\Delta_1 \leftarrow p_1$
   for $k \leftarrow 2$ **until** 2ν **do**
   $M \leftarrow O^{(2k-1)\times(2k-1)}$
   for $i \leftarrow 1$ **until** k **do**
   if i is odd **then**
   for $m \leftarrow 1$ **until** $k - (i-1)/2$ **do**
   $M(i, k-m+1) \leftarrow p_{2k-i-2(m-1)}$
   for $m \leftarrow 1$ **until** $k - (i-1)/2 - 1$ **do**
   $M(i, 2k-m) \leftarrow -q_{2k-i-1-2(m-1)}$
   else
   for $m \leftarrow 1$ **until** $k - (i-2)/2$ **do**
   $M(i, k-m+1) \leftarrow p_{2k-i-2(m-1)}$
   for $m \leftarrow 1$ **until** $k - (i-2)/2 - 1$ **do**
   $M(i, 2k-m) \leftarrow p_{2k-i-1-2(m-1)}$
   $M^{k+1,k+2,\ldots,2k-1}_{2,3,\ldots,k} \leftarrow -M^{1,2,\ldots,k-1}_{k+1,k+2,\ldots,2k-1}$
   $M^{k+1,k+2,\ldots,2k-1}_{k+1,k+2,\ldots,2k-1} \leftarrow M^{1,2,\ldots,k-1}_{1,2,\ldots,k-1}$
   $\Delta_k(x,y) \leftarrow |M|$
3) **for** $k \leftarrow 1$ **until** ν **do**
   $\Delta'_k(\lambda, \bar{\lambda}) \leftarrow \Delta_k((\lambda+\bar{\lambda})/2, (\lambda-\bar{\lambda})/2j)$
4) **for** $k \leftarrow 1$ **until** ν **do**
   $\ell_k \leftarrow$ the maximum of the degree of λ in $\Delta'_k(\lambda, \bar{\lambda})$ -1
   $\Phi_k \leftarrow O^{\ell_k \times \ell_k}$
   for $m \leftarrow 0$ **until** $\ell_k - 1$ **do**
   for $l \leftarrow 0$ **until** $\ell_k - 1$ **do**
   $\Phi_k(m+1, l+1)$
   $\leftarrow$ the coefficient of $\lambda^m \bar{\lambda}^l$ in $\Delta'_k(\lambda, \bar{\lambda})$
5) **return** ℓ_k and Φ_k.

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