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Note

A Note on Equivalent Thurstone Models

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Yellott (1978) has shown that there are Thurstone models with probability distributions of different types that are equivalent for complete experiments with three alternatives. This note generalizes and extends his findings by showing that for any number of alternatives n , there exists a pair of Thurstone models with probability distributions of different types that are equivalent for complete experiments with n alternatives, but which are not equivalent for complete experiments with $n + 1$ alternatives.

When testing choice theories experimentally it is reassuring to know that there is some experiment which will, at least theoretically, distinguish one model from another. In the case of Thurstone models (i.e., Thurstone, Case V models with some distribution F substituted for the normal) this is the case only if one assumes that the corresponding characteristic functions are nonvanishing.

Yellott (1977) has recently shown that two Thurstone models T_F and T_G are equivalent for n alternatives ($n \geq 3$) if and only if¹ the characteristic functions f and g , corresponding to the underlying probability distributions F and G , satisfy the functional equation

$$g\left(-\sum_{i=1}^{n-1} t_i\right) \prod_{i=1}^{n-1} g(t_i) = f\left(-a \sum_{i=1}^{n-1} t_i\right) \prod_{i=1}^{n-1} f(at_i) \quad (1)$$

for some constant $a > 0$ and all, $t_1, \dots, t_{n-1}$. Yellott (1977) has further shown that if f or g is nonvanishing then F and G must be distributions of the same type (i.e., $F(x) = G(ax + b)$, $a > 0$) but if f or g is allowed to vanish then there exists a pair of distributions of different types that produce Thurstone models that are equivalent for three alternatives (Yellott, 1978). This note generalizes and extends his later finding by demonstrating, for any n , a pair of Thurstone models with probability distributions of different types which are equivalent for n alternatives but are not equivalent for $n + 1$ alternatives.

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¹ See Appendix.

Consider the functions:

$$f(s) = \begin{cases} 1 - |s|, & s \in (-1, 1), \\ \frac{1}{2}(1 - |s - n|), & s \in (n - 1, n + 1), \\ \frac{1}{2}(1 - |s + n|), & s \in (-n - 1, -n + 1), \\ 0, & \text{elsewhere,} \end{cases}$$

$$g(s) = \begin{cases} 1 - |s|, & s \in (-1, 1), \\ -\frac{i}{2}(1 - |s - n|), & s \in (n - 1, n + 1), \\ \frac{i}{2}(1 - |s + n|), & s \in (-n - 1, -n + 1), \\ 0, & \text{elsewhere.} \end{cases}$$

If $n = 5$ these functions are those used by Yellott (1978, Eq. 5), for the 3-alternative case. In his 1978 paper, Yellott showed that if $n = 5$, the above functions, f and g , are characteristic functions and their probability density functions are

$$p_F(x) = \text{Sinc}^2(x)[1 + \cos 10\pi x],$$

$$p_G(x) = \text{Sinc}^2(x)[1 + \sin 10\pi x],$$

where $\text{Sinc}(x) = (\sin \pi x)/\pi x$. (A figure illustrating the graphs of these density functions is provided in that paper.)

The general case is similar. The graph of f now consists of three triangles centered at 0, n , and $-n$; each has width 2 and the heights are 1 at the origin and $\frac{1}{2}$ at $\pm n$. The graph of g consists of those same three triangles, but only the middle one lies in the real plane, while those centered at $\pm n$ lie in the imaginary plane, on opposite sides of the real axis. For a proof that f and g are characteristic functions for any n , with probability density functions

$$p_F(x) = \text{Sinc}^2(x)[1 + \cos 2\pi nx]$$

and

$$p_G(x) = \text{Sinc}^2(x)[1 + \sin 2\pi nx],$$

respectively, one can substitute n for 5 directly in Yellott's (1978) proof.

We show here that the corresponding Thurstone models, T_F , T_G , are equivalent for n alternatives, that is, that their characteristic functions satisfy (1).

For notational simplicity, let I_0 denote the interval $(-1, 1)$, I_n the interval $(n - 1, n + 1)$, I_{-n} the interval $(-n - 1, -n + 1)$, and let R denote $(I_0 \cup I_n \cup I_{-n})^c$.

Define $t_0 = -\sum_{i=1}^{n-1} t_i$ and rewrite (1) as

$$\prod_{i=0}^{n-1} g(t_i) = \prod_{i=0}^{n-1} f(at_i). \quad (1')$$

To prove that (1') is satisfied, consider the different values that t_i may take. Note first that the value of t_0 is determined by the values of the other t_i 's.

If $t_i \in R$, $f(t_i) = g(t_i) = 0$ and both sides of (1') are zero, so assume $t_i \notin R$, $i = 1, \dots, n-1$. There are then four cases to consider. Letting $C_n = \text{Card}\{t_i \in I_n \mid i = 1, \dots, n-1\}$ and $C_{-n} = \text{Card}\{t_i \in I_{-n} \mid i = 1, \dots, n-1\}$, the cases are:

- (1) $C_n = C_{-n} = 0$
- (2) $C_n = C_{-n} \neq 0$
- (3) $|C_n - C_{-n}| = 1$
- (4) $|C_n - C_{-n}| = k > 1$.

Case 1. $C_n = C_{-n} = 0$. In this case, $t_i \in I_0$, $i = 1, \dots, n-1$, so $f(t_i) = g(t_i)$, $i = 1, \dots, n-1$. But $t_0 = -\sum_{i=1}^{n-1} t_i \in (-(n-1), n-1)$ and $(-(n-1), n-1) \cap (I_n \cup I_{-n}) = \emptyset$ so $t_0 \in I_0 \cup R$. If $t_0 \in I_0$, $f(t_0) = g(t_0)$ and we have $\prod_{i=0}^{n-1} f(t_i) = \prod_{i=0}^{n-1} g(t_i)$. If $t_0 \in R$, both sides of (1') are zero.

Case 2. $C_n = C_{-n} \neq 0$. Pair each $t_i \in I_n$ with a $t_j \in I_{-n}$ ($i, j \neq 0$), then for any such pair

$$g(t_i)g(t_j) = -if(t_i)if(t_j) = f(t_i)f(t_j)$$

hence

$$\prod_{\substack{t_i \in I_n \\ t_j \in I_{-n} \\ i, j \neq 0}} g(t_i)g(t_j) = \prod_{\substack{t_i \in I_n \\ t_j \in I_{-n} \\ i, j \neq 0}} f(t_i)f(t_j).$$

We also know that $\forall t_i \in I_0$, $f(t_i) = g(t_i)$, so we only have to consider t_0 .

$$-t_0 = \sum_{i=1}^{n-1} t_i = \sum_{\substack{t_i \in I_n \\ t_j \in I_{-n} \\ i, j \neq 0}} (t_i + t_j) + \sum_{\substack{t_i \in I_0 \\ i \neq 0}} t_i.$$

Now, for any pair $t_i \in I_n$, $t_j \in I_{-n}$, $t_i + t_j \in (-2, 2)$ so

$$\sum_{\substack{t_i \in I_n \\ t_j \in I_{-n} \\ i, j \neq 0}} (t_i + t_j) \in (-2C_n, 2C_n)$$

and

$$\sum_{\substack{t_i \in I_0 \\ i \neq 0}} t_i \in (-(n-1-2C_n), n-1-2C_n)$$

since $\text{Card}\{t_i \in I_0 \mid i = 1, \dots, n-1\} = n-1-2C_n$. Adding the endpoints of the two intervals we find that:

$$-t_0 \in (-(n-1), n-1) \quad \text{or} \quad t_0 \in (-n+1, n-1).$$

But $(-n+1, n-1) \cap (I_{-n} \cup I_n) = \emptyset$ so either $t_0 \in R$ and both sides of (1') are zero, or $t_0 \in I_0$ and $f(t_0) = g(t_0)$ which, with the above, yields

$$\begin{aligned} \prod_{i=0}^{n-1} f(t_i) &= \prod_{\substack{t_i \in I_n \\ t_j \in I_{-n}}} f(t_i) f(t_j) \prod_{t_i \in I_0} f(t_i) \\ &= \prod_{\substack{t_i \in I_n \\ t_j \in I_{-n}}} g(t_i) g(t_j) \prod_{t_i \in I_0} g(t_i) \\ &= \prod_{i=0}^{n-1} g(t_i). \end{aligned}$$

Case 3. $|C_n - C_{-n}| = 1$. Assume $C_n = C_{-n} + 1$ (the other case is identical). Let t_i be the variable with value in I_n which is not paired with a variable with value in I_{-n} (see Case 2). Then we can write

$$-t_0 = \sum_{\substack{t_i \in I_n \\ i \neq l \\ t_j \in I_{-n} \\ i, j \neq 0}} (t_i + t_j) + \sum_{\substack{t_i \in I_0 \\ i \neq 0}} t_i + t_l.$$

Since there are C_{-n} pairs t_i, t_j ,

$$\sum_{\substack{t_i \in I_n \\ i \neq l \\ t_j \in I_{-n} \\ i, j \neq 0}} (t_i + t_j) \in (-2C_{-n}, 2C_{-n}) \quad (\text{using Case 2})$$

and

$$\sum_{\substack{t_i \in I_0 \\ i \neq 0}} t_i \in (-(n-1-2C_{-n}-1), n-1-2C_{-n}-1)$$

which with $t_l \in (n-1, n+1)$ gives

$$-t_0 \in (-2C_{-n} - (n-1-2C_{-n}-1) + n-1, 2C_{-n} + n-1-2C_{-n}-1 + n+1)$$

or

$$t_0 \in (-2n+1, -1).$$

But

$$(-2n+1, -1) \cap (I_0 \cup I_n) = \emptyset$$

so either $t_0 \in R$ and the equation is satisfied, or $t_0 \in I_{-n}$ providing a match for t_l so that we may write

$$\prod_{i=0}^{n-1} g(t_i) = \prod_{\substack{t_i \in I_n \\ t_j \in I_{-n}}} g(t_i) g(t_j) \prod_{t_i \in I_0} g(t_i)$$

$$\begin{aligned}
 &= \prod_{\substack{t_i \in I_n \\ t_j \in I_{-n}}} f(t_i) f(t_j) \prod_{t_i \in I_0} f(t_i) \\
 &= \prod_{i=0}^{n-1} f(t_i).
 \end{aligned}$$

Case 4. $|C_n - C_{-n}| = k > 1$. Assume $C_n = C_{-n} + k$. Since $k > 1$ we cannot match every $t_i \in I_n$ with a $t_j \in I_{-n}$ so we want to show that whenever the above condition holds, $t_0 \in R$.

Let $t_{l_1}, \dots, t_{l_k}$ be the unpaired variables with values in I_n , then

$$-t_0 = \sum_{\substack{t_i \in I_n \\ i \neq l_1, \dots, l_k \\ t_j \in I_{-n} \\ i, j \neq 0}} (t_i + t_j) + \sum_{\substack{t_i \in I_0 \\ i \neq 0}} t_i + \sum_{i=1}^k t_{l_i}.$$

As before,

$$\sum_{\substack{t_i \in I_n \\ i \neq l_1, \dots, l_k \\ t_j \in I_{-n} \\ i, j \neq 0}} (t_i + t_j) \in (-2C_{-n}, 2C_{-n})$$

and therefore,

$$\sum_{\substack{t_i \in I_0 \\ i \neq 0}} t_i \in (-(n-1-2C_{-n}-k), n-1-2C_{-n}-k),$$

while $t_{l_i} \in (n-1, n+1)$ implies $\sum_{i=1}^k t_{l_i} \in (k(n-1), k(n+1))$ thus $-t_0 \in ((k-1)n+1, (k+1)n-1)$ or $t_0 \in (-(k+1)n+1, -(k-1)n-1)$. But for $k > 1$, $(-(k+1)n+1, -(k-1)n-1) \cap (I_n \cup I_{-n} \cup I_0) = \emptyset$ so $t_0 \in R$ as required.

In each case above, if $n+1$ alternatives are used, there are values of t_i that cause (1) not to hold. For example, in Case 1, where all of the t_i are in I_0 , if $t_i = 1, i = 1, \dots, n$, then $t_0 = -\sum_{i=1}^n t_i = -n \in I_{-n}$, giving us

$$f(t_0) \sum_{i=1}^n f(t_i) \neq g(t_0) \sum_{i=1}^n g(t_i),$$

since $f(t_0) \neq g(t_0)$ when $t_0 \in I_{-n}$.

This completes the proof. Several things should be noted about the result. (1) If a pair of nonequivalent Thurstone models is indistinguishable with n alternatives, then it is indistinguishable with less than n alternatives. (2) For each n , there is a distinct pair of nonequivalent Thurstone models that is indistinguishable with at most n -alternatives (the pair defined in the proof above). (3) As a consequence of (1) and (2), for each n there is an infinite number of pairs of nonequivalent Thurstone models that are indistinguishable with n alternatives, namely, those that are indistinguishable for at most m alternatives for some $m > n$.

APPENDIX

Yellott (1977) proved that if two Thurstone models T_F and T_G are equivalent for complete experiments with n objects ($n \geq 3$) then the characteristic functions f and g of F and G satisfy Eq. (1). Following is a proof of the converse, which was inadvertently omitted in the earlier paper.

Suppose the characteristic functions of F and G satisfy

$$g(t_1)g(t_2)\cdots g(t_{n-1})g\left(-\sum_{i=1}^{n-1}t_i\right)=f(at_1)f(at_2)\cdots f(at_{n-1})f\left(-\sum_{i=1}^{n-1}at_i\right), \quad (1)$$

where a is a constant such that $D_F(x) = D_G(ax)$ (where D_F and D_G are the associated difference distributions), then T_F and T_G are equivalent for complete experiments with n objects.

Proof. Consider the special case $n = 3$. The reasoning is the same for arbitrary n . Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ denote three independent identically distributed random variables having distribution F and let $\mathbf{x}'_1, \mathbf{x}'_2, \mathbf{x}'_3$ denote three i.i.d. random variables having distribution G . Then suppose for a given system of choice probabilities the scale values for model T_F are u_1, u_2, u_3 . Then

$$\begin{aligned} p(i, j) &= P(u_i + \mathbf{x}_i > u_j + \mathbf{x}_j) \\ &= P(\mathbf{x}_j - \mathbf{x}_i < u_i - u_j) \\ &= D_F(u_i - u_j), \end{aligned}$$

where D_F is the distribution function of $\mathbf{x}_i - \mathbf{x}_j$. Now from (1) (setting $n = 3, t_2 = 0$) we know that for some $a > 0$, $g(t)g(-t) = f(at)f(-at)$, and since these are the respective characteristic functions of $\mathbf{x}'_j - \mathbf{x}'_i$ and $a(\mathbf{x}_j - \mathbf{x}_i)$ it follows that $D_F(y) = D_G(ay)$, and consequently model T_G yields the same pair comparison probabilities as T_F provided we use the scale values $\{au_i\}$ in T_G and $\{u_i\}$ in T_F .

Now let $p(i)$ denote the probability that alternative i is selected when all three alternatives are available. For model T_F we have

$$p(1) = P(\mathbf{x}_2 - \mathbf{x}_1 < u_1 - u_2, \mathbf{x}_3 - \mathbf{x}_1 < u_1 - u_3).$$

Let $J_F(x, y) = P(\mathbf{x}_2 - \mathbf{x}_1 \leq x, \mathbf{x}_3 - \mathbf{x}_1 \leq y)$ so that $p(1) = J_F(u_1 - u_2, u_1 - u_3)$ under model T_F . Similarly $p(1) = J_G(a(u_1 - u_2), a(u_1 - u_3))$ under model T_G . Then the characteristic function of $J_F(x, y)$ is $f(t_1)f(t_2)f(-t_1 - t_2)$ and the c.f. of $J_G(ax, ay)$ is $g(t_1/a)g(t_2/a)g(-t_1/a - t_2/a)$, and so it follows from (1) that $J_G(ax, ay) = J_F(x, y)$; consequently $p(1)$ is the same under models T_F and T_G . The same argument holds for all the $p(i)$, and so T_F and T_G agree on all choice probabilities provided their scale values are $\{u_i\}$ and $\{au_i\}$ respectively.

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