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OF A COLLISIONAL ELECTRON PLASMA

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August 20, 1965

## LOW-FREQUENCY SPATIAL RESPONSE OF A COLLISIONAL ELECTRON PLASMA \*

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## ABSTRACT

A study is made of the linear spatial response of an electron plasma to a localized one-dimensional electric field, whose frequency  $\omega$  is low compared with the electron-collision frequency  $\nu$ . For a fully ionized plasma, both electron-electron and electron-ion collisions are included in the calculations, the ions being treated as fixed scatterers. It is shown that the neglect of ion dynamics is justified for a suitable choice of parameters. In the hydrodynamic regime, the response function decays exponentially, with decay length equal to a diffusion length  $a(\omega\nu)^{-1/2}$ , where  $a$  is electron thermal speed, and its amplitude is proportional to  $\omega^{3/2}$ . In the kinetic regime, the amplitude is proportional to  $\omega$ , and the decay is not exponential, with characteristic distance being the mean free path  $a/\nu$ . For a weakly ionized gas, only electron-neutral collisions are included; in the hydrodynamic region, the dependence of amplitude and decay length on  $\omega$  is the same as for the fully ionized gas, but the decay is no longer exponential.

## I. INTRODUCTION

The linear response of a thermal electron plasma with zero magnetic field was first studied by Landau,<sup>1</sup> with regard to both initial conditions and boundary conditions, under the assumption that collisions are negligible. In the present paper, we consider the boundary-value problem for the case in which collisions are important.

Interest in this problem was stimulated by the experiments of Wong, D'Angelo, and Motley,<sup>2</sup> who subjected the quiescent cesium plasma of a Q-machine<sup>3</sup> to a longitudinal field (of fixed frequency  $\omega$ ) produced by a plane grid, and observed the plasma response as a function of distance from the grid. Gould carried out a theoretical analysis<sup>4</sup> of the linear response in the collisionless approximation, replacing the single grid by a pair of closely spaced grids. He found that, for  $\omega \ll \omega_i$  (the ion plasma frequency), the response decayed spatially with three characteristic distances: (a) in the distance  $\lambda_D$  (the Debye length), Debye shielding occurred; (b) in the distance  $a_i/\omega$  ( $a_i \equiv$  ion thermal speed), ion Landau damping occurred; (c) in the distance  $a_e/\omega$  ( $a_e \equiv$  electron thermal speed), electron Landau damping occurred. The experiments indicated good agreement with theory in the region (b), the regions (a) and (c) being too short and somewhat too long for the experimental situation.

The experimental conditions of Wong et al. were such that the electron-collision frequency  $\nu_e$  greatly exceeded  $\omega$ . Clearly then, a study of the electron response must take electron collisions into account. In this paper we investigate the linear electron response,

treating the ions as fixed scatterers, and including both electron-ion and electron-electron collisions. (In Sec. VI we show that the neglect of ion dynamics is justifiable under appropriate conditions.) We shall in particular be interested in the low-frequency domain  $\omega \ll v_e$ .

Section II is devoted to a general formulation of the problem. The response function  $G(z, \omega)$ , defined by Eq. (2.2), is expressed in terms of the dielectric function  $K(k, \omega)$  in Eq. (2.5). The behavior of  $G$  as a function of  $z$  can thus be discussed in terms of the singularities of the analytic continuation of  $K^{-1}$  into the complex  $k$ -plane. Considering always<sup>5</sup> that  $\omega \ll v$ , we may divide the  $k$ -space into the three regions: (a)  $|k| \sim \kappa \equiv \lambda_D^{-1}$ , the Debye region; (b)  $|k| \sim \lambda_v^{-1} \equiv v/a \ll \kappa$ , the kinetic regime; and (c)  $|k| \ll v/a$ , the hydrodynamic regime.

There is no need to study the Debye region, since every plasma model, whether collisionless or collision-dominated, yields a zero of  $K$  at  $k = \pm i\kappa$ , and thus a contribution to  $G$  of  $\exp(-\kappa|z|)$ . We therefore always limit our discussion to  $|k| \ll \kappa$ , thereby allowing some algebraic simplifications.

The hydrodynamic regime allows a complete explicit solution, based on the equations of two-fluid hydrodynamics.<sup>6</sup> This solution is carried out in Sec. III, the results being given in Eqs. (3.16) and (3.15). The amplitude of the response is proportional to  $\omega^{3/2}$ , and the decay distance is a diffusion length  $\Delta \sim a/(\omega v)^{1/2}$ .

The kinetic regime is discussed in Sec. IV. Here only a formal solution, Eq. (4.13), is possible without extensive numerical work. The

amplitude of the response is proportional to  $\omega$ , and the decay distance is the mean free path  $\lambda_v = a/v$ . The use of the Krook model, however, allows an explicit solution in terms of a quadrature, from which the asymptotic behavior for large and small  $z$  in the range  $z \sim \lambda_v$  is obtained [Eqs. (4.20 a,b)].

Section V is devoted to the Lorentz model, appropriate to a weakly ionized gas, in the hydrodynamic regime. The amplitude and decay length are the same as in Sec. III, but the decay is nonexponential. The asymptotic behavior is again found, this time in terms of the velocity dependence of the transport cross section.

## II. GENERAL CONSIDERATIONS

We consider a uniform electron gas in thermal equilibrium, neutralized by fixed positive ions. It is perturbed by an external electric field  $E_0(z,\omega)$ ; this field has fixed frequency  $\omega$ , has a component only in the  $z$ -direction, and varies with  $z$  only. Such a field is produced by a set of plane grids; we assume that they intercept a negligible fraction of the electrons crossing them, so that their only effect is electrical. We also assume that the field  $E_0$  extends over only a finite part  $\Delta z$  of the (theoretically infinite in extent) plasma, so that its spatial Fourier transform exists:

$$E_0(k,\omega) \equiv \int_{-\infty}^{\infty} dz e^{-ikz} E_0(z,\omega) \quad (2.1)$$



The plasma-response function is defined by the linear relation

$$E(z, \omega) = \int dz' G(z - z', \omega) E_0(z', \omega) \quad , \quad (2.2)$$

where  $E \equiv E_0 + E_e$  is the total Vlasov field;  $E_e$  is that due to the perturbed electron distribution, and is determined by

$$ik E_e(k, \omega) = 4\pi q \delta n(k, \omega) \quad , \quad (2.3)$$

where  $\delta n$  is the perturbed electron density, and  $q \equiv -e$  is the electron charge.

The object of this paper is the study of the response function  $G(z, \omega)$ . By Fourier transforming Eq. (2.2), we see that

$$G(k, \omega) = \frac{E(k, \omega)}{E_0(k, \omega)} \equiv [K(k, \omega)]^{-1} \quad , \quad (2.4)$$

where  $K(k, \omega)$  is the (longitudinal) dielectric function. The response function is thus given by

$$G(z, \omega) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikz} [K(k, \omega)]^{-1} \quad . \quad (2.5)$$

Since the unperturbed system is isotropic,  $K$  is even in  $k$ , and  $G$  is even in  $z$ . We may thus limit our attention to  $z > 0$ .

It is convenient to introduce the susceptibility  $\chi(k, \omega)$  :

$$\chi(k, \omega) \equiv K(k, \omega) - 1 = -E_e(k, \omega)/E(k, \omega) = -4\pi q \delta n(k, \omega)/ikE(k, \omega) \quad . \quad (2.6)$$

In the kinetic description, we use the electron phase-space density  $f(\underline{r}, \underline{v}; t)$ , and its perturbed part  $\delta f$  :

$$f(\underline{r}, \underline{v}; t) = n_0 g(v) + \delta f(\underline{r}, \underline{v}; t) , \quad (2.7)$$

where  $n_0$  is the unperturbed density, and

$$g(v) \equiv \pi^{-3/2} a^{-3} \exp(-v^2/a^2) \quad (2.8)$$

is the normalized three-dimensional Maxwell distribution. For  $\delta f$ , we introduce the dimensionless function

$$\psi(v, \mu; k, \omega) \equiv \theta_0 \delta f(k, \underline{v}; \omega) [n_0 q \phi(k, \omega) g(v)]^{-1} , \quad (2.9)$$

where  $\mu \equiv v_z/v$ , the unperturbed temperature is

$$\theta_0 \equiv \frac{1}{2} m a^2 , \quad (2.10)$$

and  $\phi$  is the total potential ( $E \equiv -ik\phi$ ). Equation (2.6) then becomes

$$\chi(k, \omega) = -(\kappa^2/k^2) \int d^3v g(v) \psi(v, \mu; k, \omega) , \quad (2.11)$$

where

$$\kappa^2 \equiv 4\pi n_0 q^2 \theta_0^{-1}$$

is the inverse square Debye length.

The kinetic equation satisfied by  $f$  is

$$\left[ \frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z} + \frac{q}{m} E \frac{\partial}{\partial v_z} \right] f(z, v; t) = -C, \quad (2.12)$$

where  $-C$  represents the rate of change of  $f$  due to electron-ion and electron-electron collisions. For frequencies  $\omega$  much less than the plasma frequency  $\omega_e$ , and for  $k \ll \kappa$ , the Landau form of the Fokker-Planck equation may be used, with an accuracy of order  $(\ln \Lambda)^{-1}$ , i.e., about 10%. We linearize Eq. (2.12) about absolute equilibrium, Fourier transform it in space and time, and use the substitutions introduced above. We obtain as the equation for  $\psi(v, \mu; k, \omega)$  :

$$[-i(\omega - kv\mu) + v\mathcal{C}]\psi = -ikv\mu, \quad (2.13)$$

where  $v\mathcal{C}$  is the linear integro-differential collision operator obtained from the linearization of  $C$ . The factor  $v$  is an arbitrarily defined mean collision frequency; therefore  $\mathcal{C}$  is dimensionless.

The solution of Eq. (2.13) is to be substituted into Eq. (2.11) for  $\chi$ . We now show that the solution is unique, for real  $\omega, k$ . If it were not, the corresponding homogeneous equation

$$i(\omega - kv\mu)\psi = v\mathcal{C}\psi \quad (2.14)$$

would have a nonzero solution. Let us multiply both sides of (2.14) by  $g(v)\psi^*(v)$  and integrate over all  $v$  :

$$i \int d^3v g(v)(\omega - kv\mu)|\psi(v)|^2 = v \int d^3v g(v)\psi^*(v)\mathcal{C}\psi(v). \quad (2.15)$$

We use the fact<sup>7</sup> that  $\mathcal{C}$  is real and self-adjoint with respect to the weight function  $g$ . Hence the right side of (2.15) is real, but the left side is purely imaginary. Thus the only solution of (2.14) is identically zero, and (2.13) has a unique solution.

### III. HYDRODYNAMIC DESCRIPTION

For small  $k$  ( $k \ll v/a$ ) and small  $\omega$  ( $\omega \ll v$ ), the hydrodynamic approximation is appropriate. The complete set of two-fluid hydrodynamic equations has been derived by one of us,<sup>6</sup> by the generalization of the Chapman-Enskog method to unequal densities, temperatures, and flow velocities for the two fluids, electrons and ions. In these equations, we take the limit  $m_e/m_i \rightarrow 0$ , thereby neglecting the ion dynamics, the justification for which is deferred to Sec. VI. We then have a set of equations for the electron fluid with fixed ions, and the electron subscript is dropped. The equations are linearized about absolute thermal equilibrium, and are listed below:

(1) the equation of continuity:

$$i\omega\delta n(k,\omega) = ikn_0 u(k,\omega) \quad , \quad (3.1)$$

where  $u$  is the electron flow velocity;

(2) the momentum equation:

$$-i\omega m n_0 u(k,\omega) = n_0 q E(k,\omega) - ik\delta p(k,\omega) - \frac{4}{3} \eta k^2 u(k,\omega) + P(k,\omega) \quad , \quad (3.2)$$

where  $\delta p$  is the perturbed pressure,  $\eta$  is the electron viscosity, and  $P$  is the rate and density of momentum transfer from the ions to the electrons in collisions;

(3) the energy equation:

$$-i\omega \frac{3}{2} n_0 \delta\theta(k, \omega) = -p_0 i k u(k, \omega) - i k Q(k, \omega) \quad (3.3)$$

where  $\delta\theta$  is the perturbed temperature,  $p_0 \equiv n_0 \theta_0$  is the unperturbed pressure, and  $Q$  is the heat flow in the electron frame;

(4) the perturbed equation of state:

$$\delta p(k, \omega) = n_0 \delta\theta(k, \omega) + \theta_0 \delta n(k, \omega) \quad ; \quad (3.4)$$

(5) the generalized Ohm's law:

$$P(k, \omega) = -n_0 m \nu u(k, \omega) - c_1 i k n_0 \delta\theta(k, \omega) \quad , \quad (3.5)$$

where  $\nu$  is the effective momentum-transport collision frequency, calculated by Spitzer and Härm<sup>8</sup> to be

$$\nu = \frac{4}{3} \left( \frac{\pi}{2} \right)^{1/2} n_0 e^4 (\ln \Lambda)_m^{-1/2} \theta_0^{-3/2} \quad , \quad (3.6)$$

and  $c_1$  is the numerical thermoelectric coefficient, calculated<sup>8</sup> to equal 0.7 to 10% accuracy;

(6) the generalized Fourier's Law :

$$Q(k, \omega) = -\frac{\theta_0}{mv} (c_2 ik n_0 \delta\theta + c_1 P) \quad (3.7)$$

where  $c_2$  is the numerical thermal conductivity, equal<sup>8</sup> to 4.0 (to 10%), and  $c_1$  is the same as in (3.5), by Onsager's relation; and

(7) an estimate for the viscosity:

$$\eta \sim p_0/v \quad (3.8)$$

From the inequality  $\omega \ll v$  and Eq. (3.5), we see that the inertial term of Eq. (3.2) may be dropped; likewise from  $k \ll v/a$  and (3.8), we may drop the viscous term from (3.2). The remaining terms are all comparable in our  $k, \omega$  range.

The solution of the set of equations is most conveniently expressed in terms of the generalized conductivity:

$$\sigma(k, \omega) \equiv \frac{n_0 q u(k, \omega)}{E(k, \omega)} \quad (3.9)$$

which is related to the susceptibility by

$$\chi(k, \omega) = (4\pi/-i\omega)\sigma(k, \omega) \quad (3.10)$$

The solution is straightforwardly found to be

$$\frac{\sigma(k, \omega)}{\sigma_0} = \frac{1 - c_3 k^2 a^2 (3i\omega v)^{-1}}{1 - c_4 k^2 a^2 (3i\omega v)^{-1} + \frac{3}{2} c_3 [k^2 a^2 (3i\omega v)^{-1}]^2} \quad (3.11)$$

where

$$\sigma_0 \equiv n_0 q^2 m^{-1} v^{-1} \equiv 4\pi\omega_e^2 v^{-1} \quad (3.12)$$

is the dc conductivity, being the limit<sup>9</sup> of  $\sigma(k,\omega)$  as  $k^2 a^2 (\omega v)^{-1} \rightarrow 0$  ;

and

$$c_3 \equiv c_2 - c_1^2 \quad ,$$

$$c_4 \equiv \frac{5}{2} + c_2 + 2c_1 \quad .$$

(Spitzer and Härm's values yield  $c_3 = 3.5$  ,  $c_4 = 7.9$ .)

The response function  $G(z,\omega)$  is

$$G(z,\omega) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikz} [1 + (4\pi/-i\omega)\sigma(k,\omega)]^{-1} \quad . \quad (3.13)$$

Since  $\sigma(k,\omega)$  is a rational function of  $k$  , the only singularities of the integrand are the roots of the equation

$$\sigma(k,\omega) = i\omega/4\pi \quad ; \quad (3.14)$$

but since  $\omega \ll v \ll \sigma_0$  , the roots are simply the zeroes of  $\sigma(k,\omega)$  , namely

$$k_{\pm} \equiv \pm (3i\omega v)^{1/2} (c_3 a^2)^{-1/2} \quad . \quad (3.15)$$

Calculating the residue of the integrand, we finally obtain

$$G(z, \omega) = \frac{(1 + c_1)^2}{2c_3} \frac{\omega v}{\omega_e^2} k_+ \exp ik_+ |z| \quad (3.16)$$

We note that the existence of a singularity in the  $k$ -plane, or of a zero of  $\sigma(k, \omega)$ , requires  $c_3$  to be nonzero. Since  $c_3$  represents the thermal conductivity and the thermoelectric effect, it is evident that a crude model based on the standard Ohm's law ( $c_1 = 0$ ) and the adiabatic equation of state ( $Q = 0$ ) leads to no hydrodynamic response.

The response decays as a pure (complex) exponential in the hydrodynamic domain. The decay length is

$$\Delta \sim a / (\omega v)^{1/2} \quad (3.17)$$

which is characteristic of diffusion phenomena. The amplitude of the response is proportional to  $\omega^{3/2}$ .

#### IV. KINETIC DESCRIPTION

For  $k \sim v/a$ , we must use the full kinetic equation (2.13). Since we are interested in  $\omega \ll v$ , we expand the solution in powers of  $\omega/v$ :

$$\psi(v, \mu; k, \omega) = \psi^{(0)}(v, \mu; k) + (\omega/v)\psi^{(1)}(v, \mu; k) + O(\omega/v)^2 \quad (4.1)$$

Equation (2.13), to zero order in  $\omega/v$ , is



$$(ikv\mu + v\mathcal{E})\psi^{(0)}(v, \mu; k) = -ikv\mu \quad (4.2)$$

Since the operation of  $\mathcal{E}$  on a constant yields zero, representing conservation of particles, a solution of (4.2) is

$$\psi^{(0)}(v, \mu; k) = -1 \quad ; \quad (4.3)$$

and by the uniqueness theorem proved at the end of Sec. II, it is the only solution. From Eq. (2.11), we find the static susceptibility

$$\chi(k, \omega = 0) = \kappa^2/k^2 \quad , \quad (4.4)$$

which is well known.

The first-order equation is

$$(ikv\mu + v\mathcal{E})\psi^{(1)}(v, \mu; k) = -iv \quad . \quad (4.5)$$

We introduce dimensionless variables

$$v' \equiv v/a \quad , \quad (4.6)$$

$$k' \equiv ka/v \quad , \quad (4.7)$$

and rewrite (4.5) as

$$(-k'v'\mu + i\mathcal{E})\psi^{(1)}(v', \mu; k') = 1 \quad . \quad (4.8)$$

From its solution, which must be found numerically for all  $k'$ ,

we form

$$\bar{\psi}(k') \equiv \int d^3v g(v) \psi^{(1)}(v', \mu; k') \quad (4.9)$$

The susceptibility is then, to first order,

$$\chi(k, \omega) = (\kappa^2/k^2)[1 - (\omega/v)\bar{\psi}(k') + O(\omega/v)^2] \quad (4.10)$$

Since  $k'$  is of order unity,  $\bar{\psi}$  will be of order unity, and  $\chi$  has no zeroes on the real  $k$ -axis. Further, since  $\kappa \gg k$ , we have  $\chi \gg 1$ , so that

$$\begin{aligned} G(z, \omega) &= \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikz} [\chi(k, \omega)]^{-1} \\ &= -\kappa^{-2} \frac{d^2}{dz^2} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikz} [1 + (\omega/v)\bar{\psi}(k')] \dots \quad (4.11) \end{aligned}$$

to first order in  $\omega/v$ . The first term of the integrand yields a  $\delta$ -function, which must be dropped, since we are in the kinetic domain  $z \sim \lambda_v$ . (A more careful treatment yields Debye shielding for  $z \ll \lambda_v$ .) We thus obtain

$$G(z, \omega) = -\frac{\omega}{v} \kappa^{-2} \frac{d^2}{dz^2} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikz} \bar{\psi}(k') \quad (4.12)$$

in terms of the dimensionless distance  $z' \equiv z/\lambda_v$ , this is

$$G(z, \omega) = -\frac{\omega v^2}{2a\omega_e^2} \frac{d^2}{dz'^2} \int_{-\infty}^{+\infty} \frac{dk'}{2\pi} \Psi(k') \exp(ik'z') \quad (4.13)$$

The form (4.13) indicates that the amplitude of the response function is proportional to  $\omega$ , and its shape is a function of  $z'$ , independent of  $\omega$ . We have not undertaken the extensive numerical work necessary to evaluate this function.

A more explicit expression for  $G$  in the kinetic regime may be obtained by using a model for the collision operator  $\mathcal{C}$ . The Krook model<sup>10</sup> conserves particles and energy, and provides for momentum transfer:

$$\mathcal{C} \psi(\underline{v}) \equiv \psi(\underline{v}) - \int d^3u g(u) \psi(\underline{u}) - \left( \frac{2}{3} \frac{v^2}{a^2} - 1 \right) \int d^3u \left( \frac{u^2}{a^2} - \frac{3}{2} \right) g(u) \psi(\underline{u}) \quad (4.14)$$

There is now no need to assume  $\omega \ll v$ ; an explicit solution of Eq. (2.13) is found, yielding the susceptibility

$$\chi(k, \omega) = (\kappa^2/k^2)(1 + \zeta T)(1 + \gamma T)^{-1} \quad (4.15)$$

where

$$\zeta \equiv (\omega + i\nu)/ka \quad (4.16)$$

$$\gamma \equiv i\nu/ka \quad (4.17)$$

$$T \equiv \left\{ \gamma(z^2 - \zeta^2) + \left[ \frac{3}{2} - \gamma\zeta(\zeta^2 - \frac{1}{2}) \right] z \right\} \\ \times \left[ \left( \zeta^4 - \zeta^2 + \frac{5}{4} \right) \gamma z + \frac{3}{2} + \gamma\zeta(\zeta^2 - \frac{1}{2}) \right]^{-1}, \quad (4.18)$$

and

$$Z \equiv Z(\zeta) \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dx \frac{e^{-x^2}}{x - \zeta} \quad (4.19)$$

is the plasma dispersion function.<sup>11</sup> In the hydrodynamic domain  $\omega \ll v$  and  $k \ll v/a$ , the susceptibility reduces to the same form (3.10)-(3.12) as in the exact treatment of Sec. III, with values  $c_3 = 2.5$  and  $c_4 = 5$ .

In the kinetic regime  $k \sim v/a$  and  $\omega \ll v$ , we expand (4.15) to first order in  $\omega/v$ , and obtain the form (4.13), with  $\bar{\Psi}(k')$  a lengthy analytic expression involving  $Z(\gamma)$ . The function  $\bar{\Psi}$  therefore has a branch cut along the imaginary  $k'$ -axis. The contour of integration can be moved up to this cut, the integral then taking the form (for  $z > 0$ )

$$\int_0^{\infty} dk'' F(k'') \exp(-k''z')$$

The function  $F$  is too complicated for analytic quadrature. Rather than carrying out the numerical quadrature for this model, we content ourselves with evaluating the asymptotic forms for large and small  $z'$ :

$$G(z, \omega) = \frac{8}{25\sqrt{3}} \frac{i\omega v^2}{a\omega_e^2} \left(\frac{z'}{2}\right)^{5/3} \exp\left[-3\left(\frac{z'}{2}\right)^{2/3}\right] \text{ for } z' \gg 1, \quad (4.20a)$$

$$G(z, \omega) = \frac{2}{\sqrt{\pi}} \frac{i\omega v^2}{a\omega_e^2} (z')^{-2} \text{ for } z' \ll 1. \quad (4.20b)$$

[The inequalities in (4.20) must still satisfy  $(v/\omega_e) \ll z' \ll (v/\omega)^{1/2}$ .]

#### V. LORENTZ MODEL

For a weakly ionized plasma, the Lorentz model is appropriate, wherein we neglect electron-electron collisions and treat only the collisions of electrons with the neutral atoms. We assume that the temperature is so low that the collisions are elastic, and we ignore the recoil of the neutrals. Thus the neutrals are considered as fixed scatterers, with a differential scattering cross section  $\sigma(v, \theta)$ .

We shall content ourselves with studying the hydrodynamic limit, i.e.,  $\omega \ll v$  and  $k \ll v/a$ . However, the equations of Sec. III do not apply here, since in the Lorentz model there is no relaxation of speed toward a local Maxwellian, the speed of an electron being unaltered in a collision. The electron gas thus behaves not like a single fluid, but somewhat like a superposition of nearly isotropic and monoenergetic fluids.

We return to the kinetic equations (2.12) and (2.13). The Lorentz model collision operator  $C$  is already linear, and is given by

$$C \equiv n_s v \int d^2\Omega' \sigma(v, \theta) [f(z, v, \underline{\Omega}; t) - f(z, v, \underline{\Omega}'; t)] \quad (5.1)$$

where  $\underline{\Omega} \equiv \underline{v}/v$  is the direction of  $\underline{v}$ ,  $\theta$  is the scattering angle between  $\underline{\Omega}$  and  $\underline{\Omega}'$ , and  $n_s$  is the density of scatterers. Since  $C$  is linear,  $v e^\psi$  is given by the same form as (5.1), with  $f$  replaced by  $\psi(v, \mu; k, \omega)$ .

It is now convenient to expand  $\psi$  in Legendre polynomials:

$$\psi(v, \mu; k, \omega) \equiv \sum_{\ell} P_{\ell}(\mu) \psi_{\ell}(v; k, \omega) \quad (5.2)$$

Using the Legendre addition theorem, we then obtain

$$v e^\psi \equiv \sum_{\ell} v_{\ell}(v) P_{\ell}(\mu) \psi_{\ell}(v; k, \omega) \quad (5.3)$$

where

$$v_{\ell}(v) \equiv n_s v \int d^2\Omega \sigma(v, \theta) [1 - P_{\ell}(\cos \theta)] \quad (5.4)$$

We note that  $v_0 \equiv 0$ ,  $v_{\ell}(v) > 0$  for  $\ell > 0$ , and that  $v_1(v)$  is the conventional momentum-transport collision frequency.

Using (5.3) in Eq. (2.13), and projecting the latter onto  $P_{\ell}$ , we obtain

$$[-i\omega + v_{\ell}(v)] \psi_{\ell} + ikv \left[ \frac{\ell}{2\ell - 1} \psi_{\ell-1} + \frac{\ell + 1}{2\ell + 3} \psi_{\ell+1} \right] = -ikv \delta_{\ell 1} \quad (5.5)$$

In particular, for  $l = 0$  and  $l = 1$ , we have

$$\omega\psi_0 = \frac{1}{3}kv\psi_1 \quad (5.6a)$$

$$[-i\omega + v_1(v)]\psi_1 + ikv[\psi_0 + \frac{2}{5}\psi_2] = -ikv \quad (5.6b)$$

For  $l > 1$ , we can conclude from Eq. (5.5) that either  $\psi_l/\psi_{l-1} = O(kv/v_l) \ll 1$  or  $\psi_l/\psi_{l+1} = O(kv/v_l) \ll 1$ . The former choice leads to a convergent series for (5.3), and from the uniqueness theorem of Sec. II, is then the only solution. In (5.6b) we thus drop  $\psi_2$ , and also  $\omega$ , and then solve the set (5.6) for  $\psi_0$ :

$$\psi_0(v; k, \omega) = \frac{k^2 v^2}{3i\omega v_1(v) - k^2 v^2} \quad (5.7)$$

The susceptibility is thus

$$\chi(k, \omega) = k^2 \int d^3v g(v) v^2 [k^2 v^2 - 3i\omega v_1(v)]^{-1} \quad (5.8)$$

In contrast to the result of Sec. III,  $\chi$  is now no longer a rational function of  $k$ , but has a branch cut in the  $k$ -plane along the line rotated  $\pi/4$  from the real axis (see Fig. 1). For the evaluation of the response function

$$G(z, \omega) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikz} [1 + \chi(k, \omega)]^{-1}, \quad (5.9)$$

we may again neglect unity compared with  $\chi$ , and then deform the contour from  $C_1$  to  $C_2$  (for  $z > 0$ ; recall that  $G$  is even in  $z$ ). No poles are swept over in this deformation, as is easily shown by the Nyquist method.

For the integration along  $C_2$ , we set  $k^2 = i(\rho^2 \pm i\epsilon)$ , and obtain

$$G(z, \omega) = \kappa^{-2} e^{i\pi/4} \int_0^{\infty} \frac{d\rho}{\pi} \frac{F(\rho)}{F^2(\rho) + P^2(\rho)} \exp(e^{3i\pi/4} \rho z), \quad (5.10)$$

where

$$F(\rho) \equiv \pi \int d^3v g(v) \delta[\rho^2 - 3\omega v_1(v)v^{-2}], \quad (5.11)$$

and

$$P(\rho) \equiv P \int d^3v \frac{g(v)}{\rho^2 - 3\omega v_1(v)v^{-2}}. \quad (5.12)$$

It is clear from these formulas that the characteristic decay distance is of order  $a(3\omega\bar{v})^{-1/2}$ , where  $\bar{v}$  is a mean collision frequency; i.e., the behavior is a diffusion process and is characterized by a



diffusion length. However, the decay is not exponential, as in Sec. III, because here  $\chi$  has a branch cut rather than a pole.

Explicit evaluation requires numerical quadrature, given  $v_1(v)$  for the neutral gas of interest. However, we can obtain the asymptotic behavior by assuming a power law for  $v_1(v)$  as  $v \rightarrow 0$  and  $\infty$ . We suppose that

$$v_1(v) \xrightarrow{v \rightarrow 0} v_0 \left( \frac{a}{v} \right)^{s_0-2}, \quad (5.13a)$$

$$v_1(v) \xrightarrow{v \rightarrow \infty} v_\infty \left( \frac{a}{v} \right)^{s_\infty-2}; \quad (5.13b)$$

we note that the classical interaction of an electron with a polarizable molecule yields  $s = 2$  for all  $v$ , i.e.,  $v_1$  independent of  $v$ .

The limit  $v \rightarrow 0$  corresponds to  $\rho \rightarrow \infty$  (for  $s_0 > 0$ ), and thus to  $z \rightarrow 0$  [by which we mean, of course,  $a/\bar{v} \ll z \ll a/(\omega\bar{v})^{1/2}$ ]; likewise  $v \rightarrow \infty$  corresponds to  $z \rightarrow \infty$ . The asymptotic evaluation of (5.13) is straightforward, and we find

$$G(z, \omega) \xrightarrow{z \rightarrow 0} -\frac{4}{\pi^{1/2} s_0} \left( 2 - \frac{6}{s_0} \right) i \exp \left( -\frac{3\pi i}{2s_0} \right) \kappa^{-2} \Delta_0^{-3} \left( \frac{|z|}{\Delta_0} \right)^{-3+(6/s_0)}, \quad (5.14a)$$

$$G(z, \omega) \xrightarrow{z \rightarrow \infty} \frac{\pi}{4\sqrt{2}} \frac{s_\infty}{(s_\infty + 4)^{1/2}} \left[ \left( \frac{s_\infty + 1}{2} \right) ! \right]^{-2} \kappa^{-2} \Delta_\infty^{-4} |z|$$

$$\times \exp \left\{ -(s_\infty + 4) \left[ (|z|/4\Delta_\infty)^4 s_\infty^{-s_\infty} e^{-\pi i} \right]^{1/(s_\infty+4)} \right\}, \quad (5.14b)$$

where

$$\Delta_0 \equiv a(3\omega v_0)^{-1/2} ,$$

$$\Delta_\infty \equiv a(3\omega v_\infty)^{-1/2} .$$

## VI. ION DYNAMICS

For an electron-ion plasma, the total response is determined by the total susceptibility

$$\chi = \chi_e + \chi_i , \quad (6.1)$$

where  $\chi_e$  and  $\chi_i$  represent the contributions of electrons and ions respectively. For the applicability of our results, it is necessary to justify the neglect of the ion response, and therefore to show that there exists a parameter range of  $k, \omega$  where

$$|\chi_i| \ll |\chi_e| \quad (6.2)$$

When, on the other hand,  $|\chi_i| \sim |\chi_e|$ , the responses are comparable; this is the range of the quasi-neutral ion-acoustic waves. They occur at  $k \sim \omega/a_i$  for  $\omega \ll \omega_i$ , and for all values of the ratio  $\omega/v_i$ . They have been studied by Gould<sup>4</sup> for  $v_i = v_e = 0$ , by Kulsrud and Shen<sup>12</sup> for  $\omega \gg v_i \neq 0$ , and by Kivelson and Du Bois<sup>13</sup> for  $\omega \ll v_i$ . (The subscript i refers to purely ion quantities; in particular,  $v_i$  is the ion-ion collision frequency, and  $\omega_i$  is the ion plasma frequency.)

FOOTNOTES AND REFERENCES

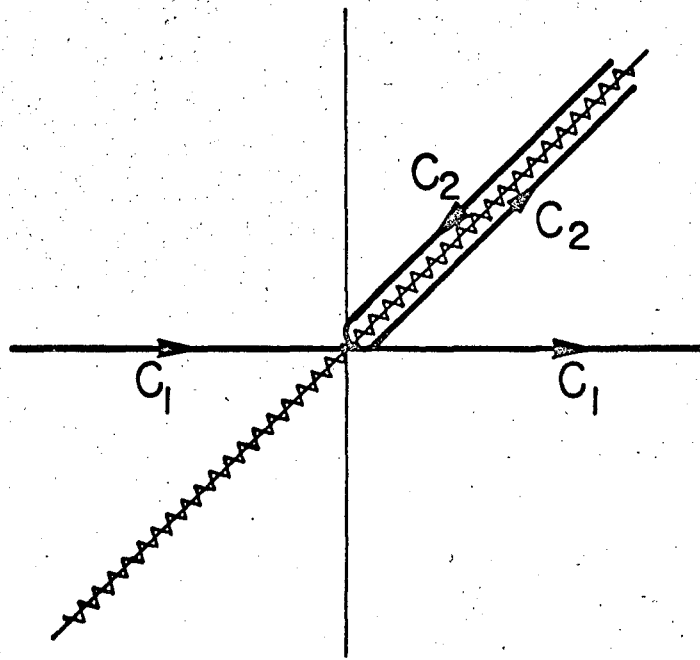
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FIGURE CAPTION

Fig. 1 Branch cut and contours in the  $K$ -plane for  $\chi$  and  $G$ , in the Lorentz model.



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