Title
Patterns and statistics on words

Author
Tiefenbruck, Mark

Publication Date
2012

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA, SAN DIEGO

Patterns and Statistics on Words

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Mark Tiefenbruck

Committee in charge:

Professor Jeff Remmel, Chair
Professor Adriano Garsia
Professor Ronald Graham
Professor Ramamohan Paturi
Professor Alexander Vardy
Professor Jacques Verstraete

2012

The dissertation of Mark Tiefenbruck is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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University of California, San Diego

2012

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ACKNOWLEDGEMENTS

A portion of Chapter 1 has been published in the Electronic Journal of Combinatorics. Rawlings, Don; Tiefenbruck, Mark. "Consecutive Patterns: From Permutations to Polyominoes and Back", Electronic Journal of Combinatorics, Volume 17, 2010. In addition, a portion of Chapter 1 has been submitted for publication in Pure Mathematics and Applications. Remmel, Jeff; Tiefenbruck, Mark. A portion of Chapter 1 is also currently being prepared for submission for publication of the material. Remmel, Jeff; Tiefenbruck, Mark. The dissertation author is an author for all of these papers.

Chapter 2, in part, has been published in the Electronic Journal of Combinatorics. Rawlings, Don; Tiefenbruck, Mark. "Consecutive Patterns: From Permutations to Polyominoes and Back", Electronic Journal of Combinatorics, Volume 17, 2010. The dissertation author was an author of this paper.

Chapter 3, in full, has been submitted for publication in Pure Mathematics and Applications. Remmel, Jeff; Tiefenbruck, Mark. The dissertation author is an author of this paper.

Chapter 4, in full, is currently being prepared for submission for publication of the material. Remmel, Jeff; Tiefenbruck, Mark. The dissertation author is an author of this material.

VITA

2004	Bachelor of Science, Northwestern University
2006-2007	Teaching Associate, California Polytechnic State University, San Luis Obispo
2007	Master of Science, California Polytechnic State University, San Luis Obispo
2008-2012	Graduate Teaching Assistant, University of California, San Diego
2012	Doctor of Philosophy, University of California, San Diego

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S. Kitaev, J. Remmel, and M. Tiefenbruck, Marked mesh patterns in 132-avoiding permutations I, *submitted to Pure Mathematics and Applications*

S. Kitaev, J. Remmel, and M. Tiefenbruck, Marked mesh patterns in 132-avoiding permutations II, *in preparation*

ABSTRACT OF THE DISSERTATION

Patterns and Statistics on Words

by

Mark Tiefenbruck

Doctor of Philosophy in Mathematics

University of California, San Diego, 2012

Professor Jeff Remmel, Chair

We study the enumeration of combinatorial objects by number of occurrences of patterns and other statistics. This work is broken into three main parts. In the first part, we enumerate permutations, compositions, column-convex polyominoes, and words by patterns relating consecutive entries. We show that there is a hierarchy of enumeration problems on these sets of objects, such that the problems in one set may be reformulated in terms of the higher sets, then solved using powerful techniques developed for those sets. We use this viewpoint to solve an open problem due to Kitaev and to produce many extensions of existing results and interesting new results. In the second part, we use the same viewpoint to generalize a theorem due to Garsia and Gessel on the major index statistic. We give many specializations and slight extensions of this result to apply it to a variety of combinatorial objects and variations of the statistic. In the third part, we present a general method for finding bijections between sets of objects that preserve various statistics. We use this method to solve problems posed by Claesson and Linusson and by Jones, and we also present several new results.

Chapter 1

Introduction

1.1 Basic definitions

1.1.1 Words and other common combinatorial objects

An *alphabet* is a set with distinguishable elements, called *letters*. A *word* on the alphabet X is a finite sequence of letters from X. The *length* of a word w, denoted $\ell(w)$, is the number of letters in w, including multiplicity. Every alphabet permits a single word of length zero, called the empty word and denoted by ϵ. The set of words of length n on the alphabet X is denoted by X^n. The set of all words on the alphabet X, including the empty word, is denoted by X^*, while the set of all non-empty words is denoted by X^+.

If $w \in X^n$, then for $1 \leq i \leq n$, w_i will denote the i-th letter of w, and w may be written as $w = w_1 w_2 \cdots w_n$. Given two words $u, v \in X^*$, the product (or concatenation) of u and v, written uv, is the word consisting of the letters of u followed by the letters of v. That is, if $w = uv$, then $\ell(w) = \ell(u) + \ell(v)$, $w_i = u_i$ for $1 \leq i \leq \ell(u)$, and $w_j = v_{j-\ell(u)}$ for $\ell(u)+1 \leq j \leq \ell(w)$. Given two sets of words $U, V \subseteq X^*$, $UV = \{uv : u \in U, v \in V\}$. Since UV is a set, it does not contain multiple copies of words that can be formed in multiple ways by different choices of u and v.

In this work, we study many combinatorial objects that can be expressed intuitively as words on some alphabet. We will make heavy use of the natural numbers, *i.e.* $\mathbb{N} = \{0, 1, 2, \dots\}$, the positive integers, *i.e.* $\mathbb{P} = \{1, 2, 3, \dots\}$, and the set $[n] = \{1, 2, \dots, n\}$ as alphabets. Other alphabets will be formed from Cartesian products of these, or we will introduce them separately.

One important combinatorial object is the permutation. A *permutation* of $[n]$ is a bijection mapping $[n]$ to itself. The set of all permutations of $[n]$ is denoted by S_n. A permutation $\sigma \in S_n$ may naturally be thought of as a word on the alphabet $[n]$ by letting $\sigma_i = \sigma(i)$. From that point of view, which we will adopt for the entirety of this work, $S_n \subseteq [n]^n$.

Another important combinatorial object is the composition. A *composition* of m into n *parts* is a sequence of n positive integers that sum to m. Thus, $\mathbb{P}^n$ is the set of all compositions with n parts. A *weak composition* is similar to a composition, except that the parts are taken from the natural numbers. Given a (weak) composition w of length n, we define $\text{sum}(w) = \sum_{i=1}^{n} w_i$, so that w is a (weak) composition of $\text{sum}(w)$ into n parts.

Related to compositions are partitions. A *partition* of m into n parts is a collection of n positive integers that sum to m, where two partitions are considered distinct only if they have a different number of parts of a given size, regardless of which order we write them in. For example, 21 and 12 are distinct as compositions but the same as partitions. The set of partitions with n parts is denoted by Λ_n. For convenience, we will also let $\Lambda = \bigcup_{n \geq 0} \Lambda_n$. We commonly give a canonical representation to partitions by sorting their entries, usually in decreasing order. Thus, if $\lambda \in \Lambda_n$, then $\lambda = \lambda_1 \lambda_2 \cdots \lambda_n$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. We will define a *weak partition* in the obvious way, as well as $\text{sum}(\lambda)$.

It is common to display a partition graphically using a *Ferrers diagram*. In such a diagram, each part λ_i of the partition λ is represented by a row of λ_i square cells with shared edges. These rows are then stacked and left-aligned. In French notation, the rows are sorted with the largest row on the bottom, as in the Ferrers diagram for $\lambda = 5331$ in Figure 1.1 (in English notation, they are sorted with the largest row on top). To retrieve the partition λ from this diagram, we read the lengths of the rows from bottom to top. However, it is clear that reading the heights of the columns from left to right gives another partition of the same

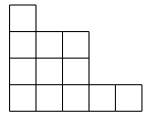


Figure 1.1: The Ferrers diagram for $\lambda = 5331$ in French notation

integer, the *conjugate* of λ, denoted by λ'. Thus, the conjugate of 5331 is 43311. In general, λ'_i is the number of parts of size i or larger in λ.

One more combinatorial object that we will study is the column-convex polyomino (CCP). A *column-convex polyomino* is constructed by successively gluing a finite sequence of columns, each consisting of a finite number of unit square cells, together in the xy-plane so that each pair of adjacent columns shares an edge of positive integer length. CCPs with the same sequence of column heights and overlaps are considered equivalent, so we give a canonical representation by saying that the bottom left corner of the left-most column is at $(0, 0)$. Figure 1.2 gives a diagram of a CCP. Let CCP be the set of column-convex polyominoes, which

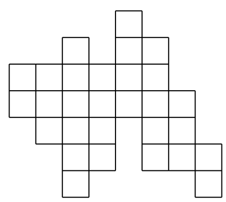


Figure 1.2: A column-convex polyomino

we will regard as a set of words whose letters are columns. The area of $Q \in CCP$ is denoted by area(Q), which is also the sum of the heights of the columns. The perimeter of Q is denoted by per(Q), and the height of the column Q_i in Q will be denoted by $|Q_i|$. The CCP in Figure 1.2 has area$(Q) = 29$, per$(Q) = 38$, and $\ell(Q) = 8$.

1.1.2 Permutation patterns and statistics

A *statistic* on a set of words W is a function mapping W to the set of integers, *i.e.* $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$, although most statistics commonly of interest map to $\mathbb{N}$. For example, the length, sum, and area functions discussed earlier are statistics. We will define many more.

Permutation patterns give rise to a common type of statistic on permutations. Given a sequence $a = a_1, a_2, \ldots, a_m$ of distinct numbers, define the *reduction* of a, denoted by $\mathrm{red}(a)$, to be the permutation $\tau \in S_m$ whose letters have the same relative order as the elements of a. That is, for all $i, j \in [m]$, $\tau_i < \tau_j$ if and only if $a_i < a_j$. For example, $\mathrm{red}(2853) = 1432$. Then, given $\sigma \in S_n$ and $\tau \in S_m$, an *occurrence* of the pattern τ in σ is a subsequence of the letters in σ whose reduction is τ. If the subsequence consists of consecutive letters of σ, then it is called a *match*, a *consecutive occurrence*, or an occurrence of the consecutive pattern τ. If σ has no occurrences of a (consecutive) pattern, then σ *avoids* that (consecutive) pattern. The set of permutations in S_n that avoid τ is denoted by $S_n(\tau)$. For example, if $\sigma = 31452$, then 314 and 315 are occurrences of 213 in σ, the first of which is consecutive, and σ avoids 321, so $\sigma \in S_5(321)$.

For $\tau \in S_m$, we define the statistics $\tau, \tau\text{-mch} : \bigcup_{n \geq 0} S_n \to \mathbb{N}$ such that $\tau(\sigma)$ is the number of occurrences of τ in σ and $\tau\text{-mch}(\sigma)$ is the number of consecutive occurrences. For example, if $\sigma = 31452$, then $213(\sigma) = 2$, $213\text{-mch}(\sigma) = 1$, and $321(\sigma) = 0$. A consecutive occurrence of 12 is frequently called an *ascent* or *rise*, and a consecutive occurrence of 21 is frequently called a *descent*. We will thus define $\mathrm{asc}(\sigma) = 12\text{-mch}(\sigma)$ and $\mathrm{des}(\sigma) = 21\text{-mch}(\sigma)$. An occurrence of 21 is frequently called an *inversion*, and an occurrence of 12 is frequently called a *co-inversion*. We will thus define $\mathrm{inv}(\sigma) = 21(\sigma)$ and $\mathrm{coinv}(\sigma) = 12(\sigma)$.

We will also frequently consider sets of patterns. For example, a *peak* in σ is a subsequence $\sigma_i\sigma_{i+1}\sigma_{i+2}$ such that $\sigma_i < \sigma_{i+1}$ and $\sigma_{i+1} > \sigma_{i+2}$. Thus, we see that a peak is a consecutive occurrence of either 132 or 231. We will define the statistic $\mathrm{peak}(\sigma)$ to be the number of peaks in σ. Similarly, a *valley* is a consecutive occurrence of either 213 or 312, and we define $\mathrm{val}(\sigma)$ to be the number of valleys in σ. In general, if P is a set of patterns, then $P(\sigma)$ will be defined as $\sum_{\tau \in P} \tau(\sigma)$, and $P\text{-mch}(\sigma)$ will be defined as $\sum_{\tau \in P} \tau\text{-mch}(\sigma)$.

Another common permutation statistic is the *major index*. Roughly speaking, the major index of a permutation $\sigma \in S_n$ is the sum of the positions of its descents. More precisely, if $\mathrm{Des}(\sigma) = \{i \in [n-1] : \sigma_i > \sigma_{i+1}\}$, then the major index of σ, denoted $\mathrm{maj}(\sigma)$, is given by $\sum_{i\in\mathrm{Des}(\sigma)} i$. MacMahon [44] showed that inv and maj have the same distribution on S_n. We will also define the co-major index, denoted by $\mathrm{comaj}(\sigma)$, to be the sum of the positions of the ascents in σ.

1.1.3 Formal series

Consider the sequence $a = a_0, a_1, a_2, \ldots$. The *ordinary generating function* for a is given by the function $f(x) = \sum_{n\geq 0} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$. We will use the notation $f(x)|_{x^n}$ to refer to the coefficient of x^n in a generating function; in this case, $f(x)|_{x^n} = a_n$. Given a sequence we wish to study, we will typically try to find an algebraic expression for its generating function. For example, the generating function for the Fibonacci numbers, defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$, is given by $f(x) = \frac{x}{1-x-x^2}$. Once an algebraic expression for the generating function has been obtained, we can use standard techniques to calculate elements of the sequence quickly or to find good approximations for them. To a combinatorist, a generating function is nearly as good as (and sometimes better than) an explicit formula.

A common variant on the ordinary generating function is the *exponential generating function*. The exponential generating function for the sequence $a = a_0, a_1, a_2, \ldots$ is given by the function $f(x) = \sum_{n\geq 0} a_n x^n/n!$. In other words, it is the ordinary generating function for the sequence b, where $b_n = a_n/n!$. Exponential generating functions are useful in some cases when the ordinary generating function cannot be written as a simple algebraic expression. For example, if $a_n = n!$, there is no simple expression for the ordinary generating function, but the exponential generating function is simply $1 + x + x^2 + \cdots = (1-x)^{-1}$. Given an exponential generating function, it is a simple exercise to retrieve the original sequence a, making it a useful variant. Clearly, many other variants are possible, and we will often need to choose one that suits the problem.

We will treat generating functions as *formal power series*, meaning that x is merely a symbol that is not meant to stand in for an unknown quantity and that the sum will never be evaluated. Instead, the function notation and sum are used for notational convenience, indicating that the algebraic properties of formal power series, *e.g.* their sum and multiplication rules, are the same as for ordinary power series. In this way, we avoid questions of convergence that, while giving information about the asymptotic behavior of the sequence, will not be considered in this work. We will also consider formal power series in several variables.

The sequence $a = a_0, a_1, a_2, \ldots$ can be thought of as a function a mapping $\mathbb{N}$ to some other set, such that $a(i) = a_i$. However, as suggested in Subsection 1.1.2, we will frequently consider functions whose domains are sets of words. We would thus like to extend the notion of generating functions to encompass these cases. If $W \subseteq X^*$ is a set of words and h is a function whose domain is W, then we can define the generating function for h to be $f = \sum_{w \in W} h(w)w$. Again, we will define $f|_w$ to be the coefficient of w in f, so that $f|_w = h(w)$. As there is no conventional definition for a sum of words, f is clearly considered a formal series, with algebraic properties defined naturally: multiplication distributes over sums, and if c_1, c_2 are coefficients and u, v are words, then $(c_1u)(c_2v) = c_1c_2uv$ and $c_1u + c_2u = (c_1 + c_2)u$. For example, if $X = \{a, b\}$, then $(3a + ab)(2\epsilon + 4b) = 6a\epsilon + 12ab + 2ab\epsilon + 4abb = 6a + 14ab + 4abb$. Note that, since $uv \neq vu$ in general for $u, v \in X^*$, the order of the terms in a product is important. It is never necessary to explicitly write ϵ, so we will refrain from doing so in the future.

As an illustration, we will derive the generating function for $h : \Lambda \to \mathbb{N}$ such that $h(\lambda) = 1$ for all λ. In order to avoid confusion, we will let the letters of λ be represented by $x_1, x_2, \ldots$ rather than $1, 2, \ldots$. Also, we will sort the letters in increasing order. It should be clear that any partition can then be decomposed uniquely into some number (possibly zero) of x_1s, followed by some number of x_2s, and so on. Also, any word composed of some number of x_1s, followed by some number of x_2s, and so on, corresponds to a partition. Thus, we see that $(1 + x_1 + x_1x_1 + \cdots)(1 + x_2 + x_2x_2 + \cdots) \cdots = \prod_{i \geq 1}(1 + x_i + x_ix_i + \cdots)$ expands to the generating function we seek.

Next, we claim that $(1 - x_i)^{-1} = 1 + x_i + x_ix_i + \cdots$. Indeed, if we multiply the right-hand side by $1 - x_i$, we find that 1 is the only term that doesn't cancel out. Two formal series f and g are considered equal if they have the same coefficients, *i.e.* if $f|_w = g|_w$ for all w. Thus, $(1 + x_i + x_i x_i + \cdots)(1 - x_i) = 1$ as formal series. By definition, then, $1 + x_i + x_i x_i + \cdots = (1 - x_i)^{-1}$. Therefore, we see that

$$\sum_{\lambda \in \Lambda} \lambda = (1 - x_1)^{-1}(1 - x_2)^{-1} \cdots = \prod_{i \geq 1} (1 - x_i)^{-1}. \tag{1.1}$$

Such a generating function can give us a wealth of information by substituting different quantities for the x_is. For example, if we replace $x_i = x^i$, then we get the generating function for $p(m)$, the number of partitions of m:

$$\sum_{m \geq 0} p(m) x^m = \prod_{i \geq 1} (1 - x^i)^{-1}. \tag{1.2}$$

Similarly, by substituting $x_i = x^i z$, we get the bivariate generating function for the number of partitions of m into n parts:

$$\sum_{\lambda \in \Lambda} x^{\mathrm{sum}(\lambda)} z^{\ell(\lambda)} = \prod_{i \geq 1} (1 - x^i z)^{-1}. \tag{1.3}$$

Thus, Equation (1.1) generalizes both of these results and many more.

A notable feature of Equation (1.3) is the form of its left-hand side. We see that the coefficient of $x^m z^n$ in this generating function is the number of partitions with sum m and length n. This form is a common paradigm, and we will say that this is the generating function for partitions by sum and length. Equation (1.2) is the generating function for partitions by sum.

Define the q-shifted factorial of n by $(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$. Then, we may rewrite the right-hand side of Equation (1.2) as $(x; x)_\infty^{-1}$ and the right-hand side of Equation (1.3) as $(xz; x)_\infty^{-1}$. The q-shifted factorial will commonly play a role in our generating functions, behaving similarly to $n!$. For example, we define the q-exponential function, the q-sine function, and the q-cosine function as follows:

$$\mathrm{e}_q(z) = \sum_{n \geq 0} \frac{z^n}{(q; q)_n} = 1 + \frac{z}{(q; q)_1} + \frac{z^2}{(q; q)_2} + \cdots,$$

$$\sin_q(z) = \sum_{n \geq 0} \frac{(-1)^n z^{2n+1}}{(q; q)_{2n+1}} = \frac{z}{(q; q)_1} - \frac{z^3}{(q; q)_3} + \frac{z^5}{(q; q)_5} - + \cdots, \text{ and}$$

$$\cos_q(z) = \sum_{n \geq 0} \frac{(-1)^n z^{2n}}{(q; q)_{2n}} = 1 - \frac{z^2}{(q; q)_2} + \frac{z^4}{(q; q)_4} - + \cdots.$$

1.2 Synopsis

1.2.1 Chapter 2: Consecutive Patterns

Define an up-down permutation to be a permutation $\sigma \in S_n$ such that if $i \in [n-1]$, then $\sigma_i < \sigma_{i+1}$ if i is odd and $\sigma_i > \sigma_{i+1}$ if i is even. In 1881, André [1] derived the exponential generating function for up-down permutations by length. That is, if $\mathrm{UD}S_n$ is the set of up-down permutations of length n, then André showed that

$$\sum_{n\geq 0}\sum_{\sigma\in \mathrm{UD}S_n} \frac{z^n}{n!} = \sec(z) + \tan(z). \tag{1.4}$$

Later, Gessel [27] generalized this classic result to include inversions, showing that

$$\sum_{n\geq 0}\sum_{\sigma\in \mathrm{UD}S_n} \frac{q^{\mathrm{inv}(\sigma)} z^n}{(q;q)_n} = \frac{1}{\cos_q(z)} + \frac{\sin_q(z)}{\cos_q(z)}. \tag{1.5}$$

Define an up-down composition to be a composition $w \in \mathbb{P}^n$ such that if $i \in [n-1]$, then $w_i \leq w_{i+1}$ if i is odd and $w_i > w_{i+1}$ if i is even. If $\mathrm{UD}\mathbb{P}^n$ is the set of up-down compositions, then we show that the generating function for up-down compositions by length and sum is given by

$$\sum_{n\geq 0}\sum_{w\in \mathrm{UD}\mathbb{P}^n} q^{\mathrm{sum}(w)}(z/q)^n = \frac{1}{\cos_q(z)} + \frac{\sin_q(z)}{\cos_q(z)}. \tag{1.6}$$

Comparing this to Equation (1.5), one may suspect that these problems are related. In fact, there is a natural correspondence between certain enumeration problems on permutations and compositions. The bijection ∇_n, defined in the following paragraph, gives the connection.

Let $w \in \mathbb{P}^n$ be a composition, and let $\lambda \in \Lambda_n$ be a partition whose letters are sorted in increasing order. For $\sigma \in S_n$, define $\mathrm{inv}_i(\sigma) = |\{j : j > i, \sigma_j < \sigma_i\}|$, *i.e.* the number of inversions starting at position i. We define the function $\nabla_n : S_n \times \Lambda_n \to \mathbb{P}^n$ by the following rule: if $\nabla_n(\sigma, \lambda) = w$, then $w_i = \mathrm{inv}_i(\sigma) + \lambda_{\sigma_i}$. For example, if $\sigma = 2431$ and $\lambda = 1134$, then $w_1 = 1 + 1 = 2$, $w_2 = 2 + 4 = 6$, $w_3 = 1+3 = 4$, and $w_4 = 0+1 = 1$, so $w = 2641$. In private communication, Foata showed that ∇_n is the inverse of Fédou's [20] insertion-shift bijection. In Subsection 2.3.2, we will prove that ∇_n is a bijection and show some of its properties.

One important property of ∇_n is that $\sigma_i < \sigma_{i+1}$ if and only if $w_i \leq w_{i+1}$. Therefore, occurrences of consecutive patterns in permutations correspond roughly to similar subsequences in compositions. For example, if $\sigma_i\sigma_{i+1}\sigma_{i+2}$ is a peak, then $w_i \leq w_{i+1}$ and $w_{i+1} > w_{i+2}$. We use this viewpoint to define consecutive patterns in compositions. That is, if $\nabla_n(\sigma, \lambda) = w$, then given $\tau \in S_m$, we say that $w_i w_{i+1} \cdots w_{i+m-1}$ is a consecutive occurrence of τ in w if $\sigma_i\sigma_{i+1}\cdots\sigma_{i+m-1}$ is a consecutive occurrence of τ in σ. Equivalently, if $w_i w_{i+1} \cdots w_{i+m-1}$ is a consecutive occurrence of τ in w, then there exists $\mu \in \Lambda_m$ such that $\nabla_m(\tau, \mu) = w_i w_{i+1} \cdots w_{i+m-1}$. Naturally, we define the statistic τ-mch(w) to be the number of consecutive occurrences of τ in w. Theorem 2.3, reproduced below in a simpler form, allows us to obtain the so-called q-exponential generating function for permutations by inversions and other statistics from the corresponding ordinary generating function for compositions by sum. Indeed, if we use weak compositions and partitions in Fédou's bijection, then these generating functions are the same.

Theorem 1.1 (Simplified version of Theorem 2.3)**.** *If f, g are functions such that $f(\sigma) = g(w)$ whenever $\nabla_n(\sigma, \lambda) = w$ for some n, λ, then*

$$\sum_{n\geq 0}\sum_{\sigma\in S_n} f(\sigma)\frac{q^{\mathrm{inv}(\sigma)}z^n}{(q;q)_n} = \sum_{n\geq 0}\sum_{w\in\mathbb{P}^n} g(w)q^{\mathrm{sum}(w)}(z/q)^n.$$

The definition of consecutive patterns for compositions through ∇_n has at least one shortcoming. For instance, $w_i w_{i+1}$ is a consecutive occurrence of 12 in w if $w_i \leq w_{i+1}$. From the perspective of compositions, though, distinguishing between the case $w_i < w_{i+1}$ and the case $w_i = w_{i+1}$ may well be of interest. So there are consecutive pattern problems for compositions that have no analog for permutations. However, every such problem for permutations can be reformulated in terms of compositions. Therefore, methods for finding generating functions for compositions may be applied to finding generating functions for permutations.

Next, we construct a bijection between compositions and wall polyominoes, a subset of CCPs (see Figure 2.3 for a selection of common classes of CCPs). This bijection gives a correspondence between consecutive patterns on compositions and consecutive patterns on wall polyominoes. For instance, an *upper ascent* in a CCP is a pair of consecutive columns such that the top of the second is higher than the top of the first. Through the bijection, ascents in a composition correspond to upper ascents in a wall polyomino. We can often obtain a generating function for wall polyominoes based on a similar one for some other subset of CCPs. Then, we can use our bijection to find a similar result for compositions and thus for permutations.

Finally, in a similar vein, we give a bijection between CCPs and a certain set of words. We thus establish a hierarchy of consecutive pattern enumeration problems, where problems from the lower classes can be reformulated in the higher classes. The problems may then be solved by applying powerful techniques that have been developed for the higher classes. The following theorem will help illustrate the power of this viewpoint.

For a composition $w \in \mathbb{P}^n$, define the variation of w, denoted $\mathrm{var}(w)$, by

$$\mathrm{var}(w) = \sum_{k=0}^{n} |w_{k+1} - w_k|,$$

where we treat $w_0 = w_{n+1} = 0$. Define $\mathrm{asc}(w)$, $\mathrm{lev}(w)$, and $\mathrm{des}(w)$, respectively, to be the number of positions $i \in [n-1]$ such that $w_i < w_{i+1}$, $w_i = w_{i+1}$, and $w_i > w_{i+1}$. In Subsection 2.7.1, using Goulden and Jackson's cluster method from the study of words, we derive a generating function for directed column-convex polyominoes, a subset of CCPs that contains the wall polyominoes, by nine simple statistics. By restricting the generating function to wall polyominoes and then reformulating in terms of compositions, we obtain the following extension of a theorem due to Carlitz.

Theorem 1.2 (Restatement of Corollary 2.2)**.** *The generating function for compositions by ascents, levels, descents, variation, sum, and length,*

$$\sum_{n\geq 0}\sum_{w\in\mathbb{P}^n} a^{\mathrm{asc}(w)} b^{\mathrm{lev}(w)} d^{\mathrm{des}(w)} c^{\mathrm{var}(w)} q^{\mathrm{sum}(w)} z^n,$$

is given by

$$1 + \frac{c^2 \displaystyle\sum_{n\geq 0} \frac{(qz)^{n+1}}{1-c^2q^{n+1}} \prod_{k=1}^{n} \left(b + \frac{c^2dq^k}{1-c^2q^k} - \frac{a}{1-q^k}\right)}{1 - a \displaystyle\sum_{n\geq 1} \frac{(qz)^n}{1-q^n} \prod_{k=1}^{n-1} \left(b + \frac{c^2dq^k}{1-c^2q^k} - \frac{a}{1-q^k}\right)}.$$

In addition to the cluster method, we also make use of Goulden and Jackson's pattern algebra method, the machinery of finite automata, and Bousquet-Mélou's adaptation of Temperley's method for CCPs. We solve an open problem posed by Kitaev on permutations and obtain many new results on permutations, compositions, CCPs, and words.

1.2.2 Chapter 3: Generalizations of the Major Index

Garsia and Gessel [25] derived the following generating function for permutations by descents, major index, and inversions:

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\frac{x^{\operatorname{des}(\sigma)}u^{\operatorname{maj}(\sigma)}q^{\operatorname{inv}(\sigma)}z^n}{(x;u)_{n+1}(q;q)_n}=\sum_{k\geq 0}x^k\,\mathrm{e}_q(zu^k)\,\mathrm{e}_q(zu^{k-1})\cdots\mathrm{e}_q(z). \tag{1.7}$$

Several mathematicians have derived alternate versions of this result for other combinatorial objects. For example, Reiner [54] gave a version of the Garsia-Gessel formula for B_n, the hyperoctahedral group, and Mendes and Remmel [46] gave versions of the Garsia-Gessel formula for groups that are the wreath product of a cyclic group C_k and the symmetric group S_n. Fuller and Remmel [24] obtained several related results for compositions.

For example, let $w\in\mathbb{P}^n$ be a composition, and define z^w to be the monomial obtained by replacing the letters in w by $w_i=z_{w_i}$. Define $\operatorname{maj}(w)$ to be the sum of the i such that $w_i>w_{i+1}$. Then, Fuller and Remmel proved that

$$\sum_{n\geq 0}\sum_{w\in\mathbb{P}^n}\frac{x^{\operatorname{des}(w)}u^{\operatorname{maj}(w)}z^w}{(x;u)_{n+1}}=\sum_{k\geq 0}\frac{x^k}{\prod_{i\geq 1}(z_i;u)_{k+1}}. \tag{1.8}$$

Note that, under Fédou's insertion-shift bijection, $\operatorname{maj}(w)=\operatorname{maj}(\sigma)$, since $w_i>w_{i+1}$ if and only if $\sigma_i>\sigma_{i+1}$. Therefore, substituting $z_i=q^i(z/q)$ and applying Fédou's bijection, as with Theorem 1.1, shows that, although not obvious, Fuller and Remmel's result generalizes Garsia and Gessel's.

In Chapter 3, following the viewpoint presented in Chapter 2, we use a new argument to find a version of Garsia and Gessel's result for words. If s is a statement, define $\chi(s)$ to be 1 if the statement is true and 0 otherwise. Let X be an alphabet, and let $A\subset X^2$. For $w\in X^n$, we can then define the descent number and major index with respect to A:

$$\begin{aligned}\operatorname{des}_A(w) &= |\{i : w_i w_{i+1} \notin A\}| \text{ and}\\ \operatorname{maj}_A(w) &= \sum_{i=1}^{n-1} i \cdot \chi(w_i w_{i+1} \notin A).\end{aligned}$$

For example, for compositions and the ordinary definitions of $\operatorname{des}(w)$ and $\operatorname{maj}(w)$, we would use $X = \mathbb{P}$ and $A = \{w_1 w_2 : w_1 \leq w_2\}$. Let $A_n = \{w \in X^n : w_i w_{i+1} \in A \text{ for all } i \in [n-1]\} = \{w \in X^n : \operatorname{des}_A(w) = 0\}$, $a_n = \sum_{w \in A_n} w$, and $A(z) = \sum_{i \geq 0} a_i z^i$. Then, we obtain the following theorem.

Theorem 1.3 (Restatement of Theorem 3.5)**.**

$$\sum_{n \geq 0} \sum_{w \in X^n} \frac{x^{\operatorname{des}_A(w)} u^{\operatorname{maj}_A(w)}}{(x;u)_{n+1}} w = \sum_{k \geq 0} x^k A(u^k) A(u^{k-1}) \cdots A(1) \tag{1.9}$$

In the case of compositions, we see that $A(z) = \sum_{n \geq 0} \sum_{\lambda \in \Lambda_n} z^n \lambda$, where λ is sorted in increasing order. Thus, substituting $x_i = z x_i$ in Equation (1.1), we get $A(z) = \prod_{i \geq 1} (1 - z x_i)^{-1}$. Then, replacing $x_i = z_i$ and rearranging terms, we see that Theorem 1.3 generalizes Fuller and Remmel's result.

We use this theorem to find generating functions for several natural variations of $\operatorname{maj}(\sigma)$, such as in colored permutations or pairs of permutations. In other cases, we slightly modify the proof of Theorem 1.3 to obtain desired results. The following example is illustrative.

For $\sigma \in S_n$, define an alternating descent to be a descent from an even position or an ascent from an odd position, and define the alternating major index to be the sum of the positions of the alternating descents. More precisely, $\operatorname{altdes}(\sigma) = \sum_i 1 \cdot \chi(\sigma_{2i} > \sigma_{2i+1}) + 1 \cdot \chi(\sigma_{2i-1} < \sigma_{2i})$ and $\operatorname{altmaj}(\sigma) = \sum_i 2i \cdot \chi(\sigma_{2i} > \sigma_{2i+1}) + (2i-1) \cdot \chi(\sigma_{2i-1} < \sigma_{2i})$. We'd like to find

$$\sum_{n \geq 0} \sum_{\sigma \in S_n} \frac{x^{\operatorname{altdes}(\sigma)} u^{\operatorname{altmaj}(\sigma)} q^{\operatorname{inv}(\sigma)} z^n}{(x;u)_{n+1} (q;q)_n}. \tag{1.10}$$

Define $A_q(z)$ to be the matrix

$$\begin{bmatrix} \frac{1}{\cos_q(z)} & \frac{\sin_q(z)}{\cos_q(z)} \\ \frac{\sin_q(z)}{\cos_q(z)} & \frac{(\cos_q(z))^2 + (\sin_q(z))^2}{\cos_q(z)} \end{bmatrix}.$$

Then, in Theorem 3.10, we show that (1.10) is given by

$$\sum_{k\geq 0} x^k [\,1\;\;0\,] A_q(zu^k) A_q(zu^{k-1}) \cdots A_q(z) \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

1.2.3 Chapter 4: Partially Marked Patterns

As discussed previously, the bijection ∇_n from Chapter 2 sends $\mathrm{maj}(\sigma)$ and τ-$\mathrm{mch}(\sigma)$ to $\mathrm{maj}(w)$ and τ-$\mathrm{mch}(w)$. In Chapter 4, we present a method of finding bijections between sets of combinatorial objects that send statistics on one set to statistics on the other. This method was discovered in the process of solving an open problem by Claesson and Linusson [14]. We then found that the method could be applied to solve an open problem by Jones [35], which led us to formalize the method. We present the method in a very general way, so that it can be applied to a wide variety of combinatorial objects and statistics and preserve many statistics simultaneously. We state our solutions to these two open problems and present several other applications of the method. In this subsection, we will outline the method in the context of Jones's problem.

Let $w = (w_1 w_2 \cdots w_k)$ be a cycle in a permutation σ, *i.e.* a sequence of letters such that $\sigma(w_i) = w_{i+1}$ for $i \in [k-1]$ and $\sigma(w_k) = w_1$. Two such sequences are considered equivalent if one can be obtained from the other by cyclically shifting some number of letters from the end to the beginning. Thus, cycles are usually written canonically with their smallest element first. Every permutation can then be decomposed into disjoint cycles.

A cycle-match of the pattern π is a subsequence of consecutive elements of the cycle whose reduction is π, where we allow subsequences that wrap around the end of the cycle from w_k to w_1. For example, in the cycle (15374), the subsequence 7415 is a 4213-cycle-match. Let $\pi_{\mathrm{cyc}}(\sigma)$ be the total number of cycle-matches of π in the cycles of σ. Then, Jones and Remmel [36] showed a more general version of the statement that if π begins with 1, then the number of $\sigma \in S_n$ with no π-cycle-matches is the same as the number with no consecutive π-patterns. Jones conjectured that this was true for any π that cannot cover a cycle with overlapping π-cycle-matches. For example, in the cycle (14253), 3142 and 4253 are 3142-cycle-matches that cover the cycle, whereas no cycle can be covered by overlapping 2143-cycle-matches.

Jones and Remmel prove their result in the case of $\pi_1 = 1$ by providing a bijection from S_n to S_n that translates each π-cycle-match into a consecutive occurrence of π. Note that this actually implies that π_{cyc} and π-mch have the same distribution on S_n. Direct arguments to prove Jones's conjecture in general remain elusive. Instead, we consider the set of marked permutations where each π-cycle-match is either "marked" or "not marked". For example, a permutation with k π-cycle-matches would appear 2^k times in this set with different marked cycle-matches. We generate a similar set of marked permutations by either marking or not marking consecutive occurrences of π. We then present a bijection between these sets that preserves the number of marked patterns, rather than the total number of patterns. The main result of Chapter 4 and its corollary, reproduced below, then prove that π_{cyc} and π-mch have the same distribution in S_n. In addition, our proof allows us to apply a technique called the Garsia-Milne involution principle to obtain a bijection from S_n to S_n that translates each π-cycle-match into a consecutive occurrence of π.

Let X be an alphabet, and define a *pattern* P on X^* to be a set of pairs of the form $\langle a_1a_2\cdots a_k, b_1b_2\cdots b_k\rangle$, where $1 \leq a_1 < \cdots < a_k$ and $b_1b_2\cdots b_k \in X^k$ for some k. Each pair represents a set of indices and one possible sequence of letters to occupy those indices. We do not require that we use the same k for all pairs in P. An *occurrence* of the pattern P in a word $w \in X^n$ is a subsequence of indices $a_1a_2\cdots a_k$ with $a_k \leq n$ such that there exists a pair $\langle a_1a_2\cdots a_k, w_{a_1}w_{a_2}\cdots w_{a_k}\rangle \in P$. We let $P(w)$ denote the number of occurrences of the pattern P in the word w. For example, in compositions, the consecutive pattern 12 can be written as the pattern $\{a_1a_2 \in \mathbb{P}^2 : a_2 = a_1 + 1\} \times \{b_1b_2 \in \mathbb{P}^2 : b_1 \leq b_2\}$.

Define a *pattern family* to be a set of the form $\mathcal{F} = \bigcup_{n\geq 0}\langle f, g, A_n, \mathcal{F}_n, \mathcal{P}_n\rangle$, where f and g are functions mapping $\mathbb{N}$ to $\mathbb{N}$, and for each $n \geq 0$,

1. A_n is a finite alphabet,
2. $\mathcal{F}_n$ is a subset of $A_n^{f(n)}$, and

3. $\mathcal{P}_n$ is a totally ordered set of patterns $P_1, P_2, \ldots, P_{g(n)}$.

We shall be interested in the generating function

$$R_{\mathcal{F}}(t, x_1, x_2, \ldots) = \sum_{n \geq 0} t^n \sum_{w \in \mathcal{F}_n} \prod_{i=1}^{g(n)} x_i^{P_i(w)} \tag{1.11}$$

as well as its specialization

$$R_{\mathcal{F}}(t, x) = \sum_{n \geq 0} t^n \sum_{w \in \mathcal{F}_n} x^{\sum_{i=1}^{g(n)} P_i(w)}. \tag{1.12}$$

Note that we allow $g(n) = 0$, in which case we will assume that $\mathcal{P}_n = \emptyset$ and interpret $\prod_{i=1}^{g(n)} x_i^{P_i(w)}$ and $x^{\sum_{i=1}^{g(n)} P_i(w)}$ to be equal to 1.

Given a pattern family $\mathcal{F} = \bigcup_{n \geq 0} \langle f, g, A_n, \mathcal{F}_n, \mathcal{P}_n \rangle$, we can form the *partially marked pattern family* $\mathcal{PMF} = \bigcup_{n \geq 0} \langle f, g, A_n, \mathcal{PMF}_n, \mathcal{P}_n \rangle$ from $\mathcal{F}$, where if $\mathcal{P}_n = \{P_1, \ldots, P_{g(n)}\}$, then $\mathcal{PMF}_n$ is the set of all $(g(n)+1)$-tuples of the form $\langle w, H_1, \ldots, H_{g(n)} \rangle$ such that $w \in \mathcal{F}_n$ and for $i \in [g(n)]$, H_i is any subset of the occurrences of the pattern P_i in w. Thus we can think of the $(g(n)+1)$-tuple $\langle w, H_1, \ldots, H_{g(n)} \rangle$ as an element $w \in \mathcal{F}_n$ where some of the occurrences of P_i in w are "marked" for $i \in [g(n)]$. We define the weight of $\langle w, H_1, \ldots, H_{g(n)} \rangle$ to be

$$w_{\mathcal{PMF}}(w, H_1, \ldots, H_{g(n)}) = \prod_{i=1}^{g(n)} y_i^{|H_i|}, \tag{1.13}$$

where again we make the convention that if $g(n) = 0$, then we set $w_{\mathcal{PMF}}(w) = 1$. Then, we shall consider the generating function

$$MR_{\mathcal{F}}(t, y_1, y_2, \ldots) = \sum_{n \geq 0} t^n \sum_{(w, H_1, \ldots, H_{g(n)}) \in \mathcal{PMF}_n} w_{\mathcal{PMF}}(w, H_1, \ldots, H_{g(n)}) \tag{1.14}$$

as well as its specialization

$$MR_{\mathcal{F}}(t, y) = MR_{\mathcal{F}}(t, y, y, \ldots). \tag{1.15}$$

The key result of Chapter 4 is the following theorem.

Theorem 1.4 (Restatement of Theorem 4.1)**.** *Suppose that $\mathcal{F}$ is a pattern family and $\mathcal{PMF}$ is the partially marked pattern family constructed from $\mathcal{F}$. Then*

$$MR_{\mathcal{F}}(t, x_1 - 1, x_2 - 1, \ldots) = R_{\mathcal{F}}(t, x_1, x_2, \ldots), \tag{1.16}$$

so that

$$MR_{\mathcal{F}}(t, x-1) = R_{\mathcal{F}}(t, x). \tag{1.17}$$

Theorem 1.4 has the following obvious corollary.

Corollary 1.1 (Restatement of Corollary 4.1)**.** *Suppose that*

$$\mathcal{F} = \bigcup_{n \geq 0} \langle f, g, A_n, \mathcal{F}_n, \mathcal{P}_n \rangle$$

and

$$\mathcal{G} = \bigcup_{n \geq 0} \langle h, g, B_n, \mathcal{G}_n, \mathcal{Q}_n \rangle$$

are pattern families. (Here we are not insisting that $f = h$, which means that for any given n, the elements of $\mathcal{F}_n$ and $\mathcal{G}_n$ can have different lengths, but we are insisting that the number of patterns in $\mathcal{P}_n$ and $\mathcal{Q}_n$ are the same.) Let $\mathcal{PMF}$ and $\mathcal{PMG}$ be the partially marked pattern families constructed from $\mathcal{F}$ and $\mathcal{G}$, respectively. Then

$$MR_{\mathcal{F}}(t, y_1, y_2, \ldots) = MR_{\mathcal{G}}(t, y_1, y_2, \ldots) \tag{1.18}$$

implies

$$R_{\mathcal{F}}(t, x_1, x_2, \ldots) = R_{\mathcal{G}}(t, x_1, x_2, \ldots). \tag{1.19}$$

Consider again the conjecture of Jones and Remmel. In Subsection 4.3.2, we present a bijection that translates marked π-cycle-matches into marked consecutive occurrences of π. In the language of pattern families, this bijection shows that $MR_{\mathcal{F}} = MR_{\mathcal{G}}$, where $\mathcal{F}$ and $\mathcal{G}$ are the pattern families corresponding to π-cycle-matches and consecutive occurrences of π. Corollary 1.1 then shows that $R_{\mathcal{F}} = R_{\mathcal{G}}$, which are the generating functions we were originally interested in.

In Section 4.2, we give a proof of Corollary 1.1 using the Garsia-Milne involution principle. In doing so, we obtain a bijective proof of every problem we can solve using this method.

A portion of Chapter 1 has been published in the Electronic Journal of Combinatorics. Rawlings, Don; Tiefenbruck, Mark. "Consecutive Patterns: From Permutations to Polyominoes and Back", Electronic Journal of Combinatorics, Volume 17, 2010. In addition, a portion of Chapter 1 has been submitted for publication in Pure Mathematics and Applications. Remmel, Jeff; Tiefenbruck, Mark. A portion of Chapter 1 is also currently being prepared for submission for publication of the material. Remmel, Jeff; Tiefenbruck, Mark. The dissertation author is an author for all of these papers.

Chapter 2

Consecutive Patterns

2.1 Introduction

The problems of enumerating permutations, compositions, and words by patterns formed by consecutive terms (parts or letters) have been widely studied and, for the most part, their stories are separate and parallel. In contrast, the problem of enumerating column-convex polyominoes (CCPs) by consecutive patterns has received only scant and indirect consideration.

Our primary purpose is to show that these problem sets are in fact intimately related. More precisely, if $\mathcal{P}S$, $\mathcal{P}C$, $\mathcal{P}CCP$, and $\mathcal{P}W$ respectively denote the sets of consecutive pattern enumeration problems on permutations, compositions, column-convex polyominoes, and words, then

$$\mathcal{P}S \subset \mathcal{P}C \subset \mathcal{P}CCP \subset \mathcal{P}W. \tag{2.1}$$

The significance of (2.1) is that it allows powerful methods from the larger problem sets to be applied to the smaller problem sets. To illustrate, we will show how various results on words as well as Bousquet-Mélou's [4] adaptation of Temperley's [59] method for enumerating CCPs may be used to count permutations by consecutive patterns.

In particular, we exploit the perspective of (2.1) to q-count permutations by (i, d)-peaks, up-down type, uniform m-peak ranges, and (i, m)-maxima. Notably, a specialization of Corollary 2.4 provides a solution to the $(2m + 1)$-alternating pattern problem on permutations posed by Kitaev [40, Problem 1]. We will also show that the generating function for permutations by a given pattern is deducible from the generating function for a related pattern permutation set; for instance, the generating function for permutations by peaks may be obtained from the one for up-down permutations of odd length.

Our secondary purpose is to initiate the explicit study of CCPs by consecutive (or ridge) patterns. Our introduction of two-column ridge patterns provides a unifying characterization of the common subclasses of CCPs. In Subsections 2.7.1 and 2.7.2, we use results on words to enumerate directed CCPs by two-column ridge patterns and by valleys. The Temperley method as modified in [4] is employed in Subsection 2.9.2 to count CCPs by peaks.

We begin our exposé of (2.1) with a discussion of $\mathcal{P}S$ and then work our way up the sequence of inclusions. As this chapter concentrates solely on consecutive patterns, if p is a pattern or a set of patterns, then we will generally write $p(\sigma)$ instead of p-mch(σ) without concern for ambiguity. Also, we will always write partitions in increasing order.

2.2 Consecutive patterns in permutations

When a permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in S_n$ is sketched in a natural way, patterns take shape. In the sketch of $\sigma = 256143 \in S_6$ in Figure 2.1, one discerns ascents, descents, peaks, valleys, and other patterns.

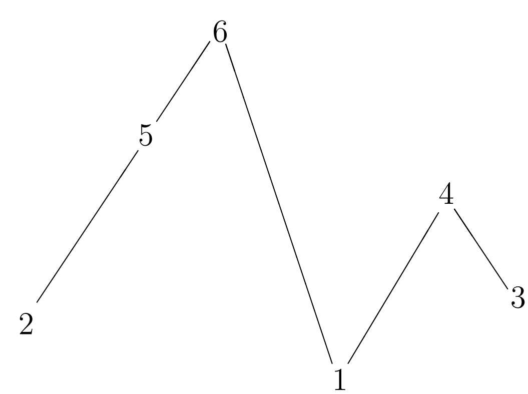


Figure 2.1: Sketch of the permutation $\sigma = 256143$

There are two standard ways of counting the number of times a given set of patterns $P \subseteq \bigcup_{m\geq 1} S_m$ occurs consecutively in a permutation $\sigma \in S_n$:

- $P(\sigma)$ = the total number of times elements of P occur consecutively in σ, and
- $P_{no}(\sigma)$ = the maximum number of non-overlapping times elements of P occur consecutively in σ.

We also define $p_{no}(\sigma)$ to be the maximum number of non-overlapping times that p occurs in σ. Note that in this case, $P_{no}(\sigma)$ is not necessarily the same as $\sum_{p\in P} p_{no}(\sigma)$, since consecutive occurrences of different patterns in P could overlap. For example, relative to Figure 2.1, $132(\sigma) = 132_{no}(\sigma) = 231(\sigma) = 231_{no}(\sigma) = 1$. However, for $P = \{132, 231\}$, note that $P(\sigma) = 2$, whereas $P_{no}(\sigma) = 1$, since the peaks $\sigma_2\sigma_3\sigma_4 = 561$ and $\sigma_4\sigma_5\sigma_6 = 143$ overlap at $\sigma_4 = 1$. We will also define $\text{peak}_{no}(\sigma)$ and $\text{val}_{no}(\sigma)$ accordingly.

For a pattern set $P \subseteq \bigcup_{m\geq 1} S_m$, two primary enumeration questions arise:

- Q1: What is the cardinality of $PS_n = P \cap S_n$? Elements of PS_n are referred to as P-pattern permutations of length n.
- Q2: How many permutations in S_n contain k consecutive P-patterns, counting overlaps?

The variation of Q2 involving the maximal number of non-overlapping patterns will be denoted by Q2_{no}. The problem of counting permutations that contain no P-patterns is known as the avoidance problem. The pattern avoidance cases ($k = 0$) of Q2 and Q2_{no} are identical as $\{\sigma \in S_n : P(\sigma) = 0\} = \{\sigma \in S_n : P_{no}(\sigma) = 0\}$.

As will be seen, there is a hierarchy between some versions of Q1, Q2, and Q2_{no}; in these cases, solving Q1 solves Q2, which in turn solves Q2_{no}. Our placement of the problem Q1 of enumerating permutations replete with P-patterns at the top of the hierarchy complements and sharply contrasts with the central role played in [38, 47] of the avoidance problem of counting permutations devoid of P in solving Q2_{no}.

2.2.1 Selected examples

In 1881, André [1] solved what has become the classic example of Q1. For $\mathrm{UD} = \bigcup_{m\geq 1}\{p \in S_m : p_1 < p_2 > p_3 < p_4 > \cdots\}$, the elements of $\mathrm{UD}S_n$ are said to be up-down permutations of length n. André showed that the exponential generating function for the number of up-down permutations of length n is given by

$$\sum_{n\geq 0} |\mathrm{UD}S_n| \frac{z^n}{n!} = \sec(z) + \tan(z). \tag{2.2}$$

As an example of Q2, we present the generating function for permutations by peaks obtained by Mendes and Remmel [47]:

$$\sum_{n\geq 0}\sum_{\sigma\in S_n} \frac{y^{\mathrm{peak}(\sigma)} z^n}{n!} = \frac{\sqrt{y-1}}{\sqrt{y-1} - \tan(z\sqrt{y-1})}. \tag{2.3}$$

Prior to [47], Kitaev [40] obtained a different form for the right side of (2.3). Incidentally, Entringer [18] enumerated "circular" permutations by peaks.

The appearance of the tangent function in both (2.2) and (2.3) is no coincidence. A general explanation is provided in Section 2.5, thereby showing that solving Q1 solves Q2.

When we compute the q-exponential generating function for permutations by inversions, we encounter many natural q-analogs of well-known results. For instance, Gessel [27] and Mendes and Remmel [47] respectively showed that

$$\sum_{n\geq 0}\sum_{\sigma\in \mathrm{UD}S_n} \frac{q^{\mathrm{inv}(\sigma)} z^n}{(q;q)_n} = \sec_q(z) + \tan_q(z) \text{ and} \tag{2.4}$$

$$\sum_{n\geq 0}\sum_{\sigma\in S_n} \frac{y^{\mathrm{peak}(\sigma)} q^{\mathrm{inv}(\sigma)} z^n}{(q;q)_n} = \frac{\sqrt{y-1}}{\sqrt{y-1} - \tan_q(z\sqrt{y-1})}, \tag{2.5}$$

where $\sec_q(z) = 1/\cos_q(z)$ and $\tan_q(z) = \sin_q z/\cos_q z$. Replacing z by $z(1-q)$ and then letting q approach 1 reduces (2.4) to (2.2); hence (2.4) is a q-analog of (2.2). Likewise, (2.5) is a q-analog of (2.3).

2.2.2 Solving Q2 solves Q2$_{no}$

In [38], Kitaev made the beautiful observation that Q2$_{no}$ for a single pattern may be reduced to the avoidance problem. Shortly thereafter, Mendes and Remmel

[47] extended Kitaev's result by tracking a set of patterns and adding the inversion number to the mix.

Theorem 2.1 (Mendes and Remmel 2007)**.** *If $P \subseteq S_m$ with $m > 1$, then*

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\frac{q^{\mathrm{inv}(\sigma)}y^{P_{no}(\sigma)}z^n}{(q;q)_n}=\frac{\mathcal{K}_q(z)}{1-y+y\left(1-z(1-q)^{-1}\right)\mathcal{K}_q(z)},$$

where $\mathcal{K}_q(z)=\sum_{n\geq 0}\left(\sum_{\sigma\in S_n}q^{\mathrm{inv}(\sigma)}0^{P(\sigma)}\right)z^n/(q;q)_n$ is the q-exponential generating function for permutations that consecutively avoid P.

Among many consequences of Theorem 2.1, Mendes and Remmel obtained a solution to $\mathrm{Q2}_{no}$ relative to peaks:

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\frac{q^{\mathrm{inv}(\sigma)}y^{\mathrm{peak}_{no}(\sigma)}z^n}{(q;q)_n}=\left(1-\frac{yz}{1-q}+\sqrt{-1}(1-y)\tan_q(z\sqrt{-1})\right)^{-1}. \quad (2.6)$$

Theorem 2.1 provides a bridge from some versions of Q2 to $\mathrm{Q2}_{no}$. For instance, setting $y=0$ in (2.5) gives the q-exponential generating function for peak-avoiding permutations, which in turn may be plugged into Theorem 2.1 to get (2.6). For this reason, our primary focus will be on Q2.

2.3 Consecutive patterns in compositions

As with permutations, a sketch of a composition $w \in \mathbb{P}^n$ reveals patterns. When $w = 377254 \in \mathbb{P}^6$ is sketched as in Figure 2.2, one observes ascents, levels, descents, peaks, valleys, and more.

In particular, we define a peak in a composition w to be a subsequence $w_iw_{i+1}w_{i+2}$ satisfying $w_i \leq w_{i+1} > w_{i+2}$. The number of peaks in w is denoted by $\mathrm{peak}(w)$. In Figure 2.2, subsequence $w_2w_3w_4 = 772$ is a peak and $\mathrm{peak}(w) = 2$.

2.3.1 Two revealing examples

Naturally, Q1 and Q2 have been considered in the context of compositions. Paralleling André [1], a composition w for which $w_1 \leq w_2 > w_3 \leq w_4 > \cdots$ is

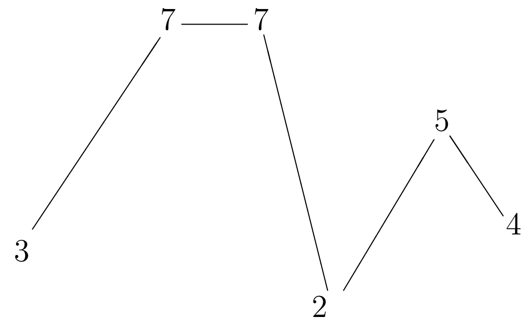


Figure 2.2: Sketch of the composition $w = 377254$

said to be up-down. If $\mathrm{UD}\mathbb{P}^n$ denotes the set of up-down compositions of length n, then

$$\sum_{n\geq 0}\sum_{w\in \mathrm{UD}\mathbb{P}^n} q^{\mathrm{sum}(w)}(z/q)^n = \sec_q(z) + \tan_q(z). \tag{2.7}$$

Carlitz [8] obtained a related result; he used $w_1 \leq w_2 \geq w_3 \leq w_4 \geq \cdots$ as the defining property of an up-down composition.

As an example of Q2 for compositions, the generating function for compositions by peaks (see Section 2.5 for a proof) is

$$\sum_{n\geq 0}\sum_{w\in \mathbb{P}^n} y^{\mathrm{peak}(w)} q^{\mathrm{sum}(w)}(z/q)^n = \frac{\sqrt{y-1}}{\sqrt{y-1} - \tan_q(z\sqrt{y-1})}. \tag{2.8}$$

Heubach and Mansour [32] obtained the distributions for compositions with parts in an arbitrary alphabet by various three-letter patterns; their result for peaks is more general than (2.8).

Comparison of (2.4) with (2.7) and of (2.5) with (2.8) strongly suggests that certain problems in $\mathcal{P}S$ and $\mathcal{P}C$ are one-in-the-same. Fédou's [20] insertion-shift bijection provides the connection.

2.3.2 Fédou's bijection: $\mathcal{P}\mathbf{S} \subset \mathcal{P}\mathbf{C}$

For $\sigma \in S_n$ and $1 \leq i \leq n$, let $\mathrm{inv}_i\,\sigma = |\{k : i < k \leq n, \sigma_i > \sigma_k\}|$, *i.e.* the number of inversions starting at σ_i. Then we define the function $\nabla_n : S_n \times \Lambda_n \to \mathbb{P}^n$ such that if $\sigma \in S_n$ and $\lambda \in \Lambda_n$, with the letters of λ sorted in increasing order, then $\nabla_n(\sigma, \lambda) = w$, where

$$w_i = \mathrm{inv}_i(\sigma) + \lambda_{\sigma_i}. \tag{2.9}$$

For example,

$$\nabla_6(256143, 224444) = 377254. \tag{2.10}$$

As communicated privately by Foata, ∇_n is the inverse of Fédou's insertion-shift bijection. In Theorem 2.2, we will prove that ∇_n is a bijection as well as a few simple properties.

Theorem 2.2. *The function* ∇_n *is a bijection. Moreover, if* $\nabla_n(\sigma, \lambda) = w$*, then*

$$\mathrm{inv}(\sigma) + \mathrm{sum}(\lambda) = \mathrm{sum}(w) \tag{2.11}$$

and

$$\sigma_i < \sigma_m \textit{ if and only if } w_i \leq w_m + |\{j : i < j < m, \sigma_i > \sigma_j\}|. \tag{2.12}$$

Proof. To prove that ∇_n is a bijection, it will suffice to show that each $w \in \mathbb{P}^n$ comes from a unique pair (σ, λ). We will proceed by induction. In the case where $n = 1$, the statement is clearly true. Now, for $n > 1$, assume that ∇_{n-1} is a bijection.

Given $\sigma \in S_n$ such that $\sigma_i = 1$, define $\sigma^- \in S_{n-1}$ by removing σ_i from σ and subtracting 1 from each remaining letter. Given $\lambda \in \Lambda_n$, define $\lambda^- = \lambda_2\lambda_3 \cdots \lambda_n$. For example, if $\sigma = 256143$ and $\lambda = 224444$, then $\sigma^- = 14532$ and $\lambda^- = 24444$. Then, $\nabla_{n-1}(\sigma^-, \lambda^-) = w^-$, where $w_j^- = w_j - 1$ if $j < i$ and $w_j^- = w_{j+1}$ if $j \geq i$. Also, we see that $\lambda_1 = w_i$ and $w_i \leq w_j$ for all j, with a strict inequality when $j < i$.

Now, given w, we may construct the composition that must be w^-. Choose i such that w_i is minimal and $w_j > w_i$ for all $j < i$. From the previous paragraph, we know this is the only possible position for which σ_i could be 1. Then, w^- is obtained by removing w_i and subtracting 1 from each w_j with $j < i$. Since ∇_{n-1} is a bijection, we may then find unique σ^- and λ^- corresponding to w^-. We know that $\sigma_i = 1$ and $\lambda_1 = w_i$, so we obtain σ by inserting a 1 in the i-th position of σ^- and adding 1 to every other letter, and $\lambda = w_i\lambda^-$. Since each of these steps was uniquely determined, so are σ and λ. Finally, we must check that λ is a partition. Since λ^- is a partition and $w_i \in \mathbb{P}$, it suffices to check that $w_i \leq \lambda_1^-$. However, λ_1^- is the minimal element of w^-, which by construction is at least w_i, so we are done.

Summing (2.9) over i from 1 to n, (2.11) is obvious. Next, if $\sigma_i < \sigma_m$, then $\lambda_{\sigma_i} \leq \lambda_{\sigma_m}$. Thus, solving (2.9) for λ_{σ_i}, we see that $w_i - \text{inv}_i(\sigma) \leq w_m - \text{inv}_m(\sigma)$, or $w_i \leq w_m + \text{inv}_i(\sigma) - \text{inv}_m(\sigma)$. However, $\text{inv}_i(\sigma) - \text{inv}_m(\sigma) = |\{j : i < j < m, \sigma_i > \sigma_j\}|$. On the other hand, if $\sigma_i > \sigma_m$, then by following similar logic, we see that $w_i \geq w_m + 1 + |\{j : i < j < m, \sigma_i > \sigma_j\}|$, and (2.12) follows.

□

Due to the second property, (2.12), ∇_n roughly transfers the overall shape and patterns of σ to the corresponding w. For example, as can be seen in Figures 2.1 and 2.2, $\sigma = 256143 \in S_6$ and $w = 377254 \in \mathbb{P}^6$, related by (2.10), are of similar shape. The peaks 561 and 143 in $\sigma = 256143$ coincide with the peaks 772 and 254 in $w = 377254$. A special case of (2.12) is that $\sigma_i < \sigma_{i+1}$ if and only if $w_i \leq w_m$. Thus, we see that ∇_n preserves peaks. Overall, (2.12) implies that consecutive patterns will be translated well from permutations to compositions.

Rather than defining consecutive patterns directly on compositions, it is then convenient to take an indirect path through ∇_n. For $p \in S_m$, the subsequence $w_k w_{k+1} \dots w_{k+m-1}$ is said to be a consecutive p-pattern in w provided the corresponding subsequence $\sigma_k \sigma_{k+1} \dots \sigma_{k+m-1}$ is a consecutive p-pattern in the unique permutation σ satisfying $w = \nabla_n(\sigma, \lambda)$. Furthermore, for $P \subseteq \cup_{m \geq 1} S_m$ and $w = \nabla_n(\sigma, \lambda)$, we define $P(w) = P(\sigma)$ and $P_{no}(w) = P_{no}(\sigma)$.

The definition of patterns for compositions through ∇_n has at least one shortcoming. For instance, $w_k w_{k+1}$ is a consecutive 12-pattern in w if $w_k \leq w_{k+1}$. From the perspective of compositions, though, distinguishing between the case $w_k < w_{k+1}$ and the case $w_k = w_{k+1}$ may well be of interest. So there are problems in $\mathcal{PC}$ that have no analog in $\mathcal{PS}$. However, $\mathcal{PS} \subset \mathcal{PC}$.

Theorem 2.3. *If $B_n \subseteq S_n$ for each n, f, g are functions such that $f(\sigma) = g(w)$ whenever $\sigma \in B_n$ and $\nabla_n(\sigma, \lambda) = w$ for some λ, then*

$$\sum_{n \geq 0} \sum_{\sigma \in B_n} f(\sigma) \frac{q^{\text{inv}(\sigma)} z^n}{(q;q)_n} = \sum_{n \geq 0} \sum_{w \in \nabla_n(B_n, \Lambda_n)} g(w) q^{\text{sum}(w)} (z/q)^n.$$

Proof. First, we claim the well-known fact that $(q;q)_n^{-1} = \sum_{\lambda \in \Lambda_n} q^{\text{sum}(\lambda) - n}$. We can show this by considering the conjugate of each $\lambda \in \Lambda_n$. The conjugate, λ', has any number of parts less than or equal to n, with at least one part of size n, while $\text{sum}(\lambda') = \text{sum}(\lambda)$. The part of size n contributes n to $\text{sum}(\lambda')$, so $\text{sum}(\lambda) - n$ gives the sum of the remaining parts. Thus, substituting $x_i = q^i$ for $i \in [n]$ and $x_i = 0$ for $i > n$ in (1.1), we obtain $(1-q)^{-1}(1-q^2)^{-1}\cdots(1-q^n)^{-1} = (q;q)_n^{-1}$, proving the claim.

By the properties of ∇_n,

$$\sum_{n\geq 0}\sum_{\sigma\in B_n} f(\sigma)\frac{q^{\text{inv}(\sigma)}z^n}{(q;q)_n} = \sum_{n\geq 0}\sum_{\sigma\in B_n}\sum_{\lambda\in\Lambda_n} f(\sigma)q^{\text{inv}(\sigma)+\text{sum}(\lambda)-n}z^n$$

$$= \sum_{n\geq 0}\sum_{w\in\nabla_n(B_n,\Lambda_n)} g(w)q^{\text{sum}(w)}(z/q)^n.$$

□

There are three immediate applications of Theorem 2.3. First, Theorem 2.3 may be used to deduce (2.8) from Mendes and Remmel's (2.5). Likewise, (2.7) follows from Gessel's (2.4). Finally, Theorem 2.3 may be used to rewrite Mendes and Remmel's Theorem 2.1 in the context of compositions.

Corollary 2.1. *If $P \subseteq S_m$ with $m > 1$, then*

$$\sum_{n\geq 0}\sum_{w\in\mathbb{P}^n} y^{P_{no}(w)}q^{\text{sum}(w)}z^n = \frac{L_q(z)}{1-y+y\left(1-zq(1-q)^{-1}\right)L_q(z)},$$

where $L_q(z) = \sum_{n\geq 0}\left(\sum_{w\in\mathbb{P}^n} q^{\text{sum}(w)}0^{P(w)}\right)z^n$ is the generating function for compositions that consecutively avoid P.

Corollary 2.1 is both more and less general than Heubach, Kitaev, and Mansour's [34] Theorem 4.1; for a pattern set of cardinality 1, their result holds for an arbitrary alphabet of positive integers. In Section 2.4, we will prove a more general version of Corollary 2.1 for words.

2.3.3 Compositions by two-term patterns and variation

In a composition w, a subsequence $w_k w_{k+1}$ is said to be an ascent, level, or descent respectively as $w_k < w_{k+1}$, $w_k = w_{k+1}$, or $w_k > w_{k+1}$. The numbers of ascents, levels, and descents in w are denoted by $\text{asc}(w)$, $\text{lev}(w)$, and $\text{des}(w)$. When sketched as in Figure 2.2, one of the more compelling features of a composition $w \in \mathbb{P}^n$ is its vertical variation defined by

$$\mathrm{var}(w) = \sum_{k=0}^{n} |w_{k+1} - w_k|,$$

where, by convention, $w_0 = w_{n+1} = 0$. As a consequence of the perspective afforded by (2.1), we obtain the following joint distribution of (asc, lev, des, var) on compositions from our Corollary 2.7 on directed column-convex polyominoes recorded in Subsection 2.7.1.

Corollary 2.2. *The generating function for compositions by ascents, levels, descents, and variation*

$$K(c,z) = \sum_{n\geq 0} \sum_{w\in\mathbb{P}^n} a^{\mathrm{asc}(w)} b^{\mathrm{lev}(w)} d^{\mathrm{des}(w)} c^{\mathrm{var}(w)} q^{\mathrm{sum}(w)} z^n$$

is given by

$$K(c,z) = 1 + \frac{c^2 \sum_{n\geq 0} \frac{(qz)^{n+1}}{1-c^2q^{n+1}} \prod_{k=1}^{n} \left(b + \frac{c^2dq^k}{1-c^2q^k} - \frac{a}{1-q^k}\right)}{1 - a\sum_{n\geq 1} \frac{(qz)^n}{1-q^n} \prod_{k=1}^{n-1} \left(b + \frac{c^2dq^k}{1-c^2q^k} - \frac{a}{1-q^k}\right)}.$$

Setting $c = 1$ in Corollary 2.2 and making use of Cauchy's q-binomial theorem gives Carlitz's [7] generating function $K(1,z)$ for compositions by ascents, levels, and descents. Heubach and Mansour [33] recently extended Carlitz's result to an arbitrary alphabet of positive integers.

The distributions of var and of closely related statistics over various combinatorial sets have been considered in [2, 45, 51, 60]. In [60], Tiefenbruck expressed the generating function for compositions with bounded parts by variation as a ratio of coefficients of basic hypergeometric series. Recently, Mansour [45] determined the generating function for the same version of var on compositions as in [2].

2.4 Factors and consecutive patterns in words

Let X be an alphabet. An element $f \in X^+$ is a factor of $w \in X^*$ if $f = w_k w_{k+1} \dots w_{k+\ell(f)-1}$ for some k. The number of times f appears as a factor in w is denoted by $f(w)$.

For a non-empty set $\mathcal{F} \subseteq X^+$, a factor f of w is a said to be a consecutive $\mathcal{F}$-pattern in w if $f \in \mathcal{F}$. The number of consecutive $\mathcal{F}$-patterns in w is denoted by $\mathcal{F}(w)$; so $\mathcal{F}(w) = \sum_{f\in\mathcal{F}} f(w)$. We refer to $\mathcal{F}$ as a factor set.

The containment $\mathcal{P}C \subset \mathcal{P}W$ in (2.1) is now evident: a composition w is just a word with letters selected from the alphabet $\mathbb{P} = \{1, 2, 3, \dots\}$. Also, each pattern p of length m defined on compositions may be naturally matched with the factor set $\mathcal{F}_p = \{f \in \mathbb{P}^m : p(f) = 1\}$. For $p = 132$ defined on compositions through Fédou's bijection as in Subsection 2.3.2, $\mathcal{F}_{132} = \{acb \in \mathbb{P}^3 : a \leq b < c\}$. In general, for a pattern set P on compositions, we define $\mathcal{F}_P = \cup_{p\in P}\mathcal{F}_p$ and note that $P(w) = \mathcal{F}_P(w)$.

As a result, any method for the set $\mathcal{P}W$ may be applied to the set $\mathcal{P}C$ and, via Theorem 2.3, to $\mathcal{P}S$. In this regard, some modifications of Goulden and Jackson's [29] result for enumerating words by factors are fundamental.

As in Stanley [58, p. 266-267], we state Goulden and Jackson's [29] result in the context of the free monoid. Following Noonan and Zeilberger [49], the stipulation that no element of the factor set $\mathcal{F}$ be a factor of another is dropped. We further drop the requirement that the alphabet be finite, and we consider restrictions on the first and last letters.

For a non-empty set $\mathcal{F} \subset X^+$, an $\mathcal{F}$-cluster is a triple (w, ν, β) in which

$$w = w_1 w_2 \dots w_{\ell(w)} \in X^+,$$
$$\nu = (f_{(1)}, f_{(2)}, \dots, f_{(k)}) \text{ for some } k \geq 1 \text{ with each } f_{(i)} \in \mathcal{F}, \text{ and}$$
$$\beta = (b_1, b_2, \dots, b_k) \text{ with each } b_i \text{ being a positive integer,}$$

where $f_{(i)} = w_{b_i} w_{b_i+1} \cdots w_{b_i+\ell(f_{(i)})-1}$, each $w_i w_{i+1}$ is a factor of some $f_{(j)}$, $b_1 \leq b_2 \leq \cdots \leq b_k$, and if $b_i = b_{i+1}$, then $\ell(f_{(i)}) < \ell(f_{(i+1)})$.

Roughly speaking, the pair (ν, β) is a recipe for covering w with $\mathcal{F}$-factors: β specifies where the factors in ν are to be "placed so as to cover" w. Accordingly, w is said to be $\mathcal{F}$-coverable and the pair (ν, β) is said to be a covering of w. We let $C_{\mathcal{F}}$ denote the set of $\mathcal{F}$-clusters.

The cluster generating function over a subset W of X^* is defined to be the formal series

$$C_{\mathcal{F}}(\mathbf{y}, W) = \sum_{\substack{(w,\nu,\beta)\in C_{\mathcal{F}} \\ w\in W}} \left(\prod_{f\in\mathcal{F}} y_f^{f(\nu)} \right) w$$

where $f(\nu)$ is the number of times f appears as a component in ν. With but trivial changes, Stanley's solution to problem 14(a) in [58, p. 266-267] establishes the following theorem.

Theorem 2.4 (Modifications of Goulden and Jackson's [29] result)**.** *If, for non-empty $L, R \subseteq X$ and a non-empty $\mathcal{F} \subseteq X^+$, we define*

$$\mathcal{L}(\mathbf{y}) = \sum_{l\in L} l + C_{\mathcal{F}}(\mathbf{y}, LX^*),$$

$$\mathcal{R}(\mathbf{y}) = \sum_{r\in R} r + C_{\mathcal{F}}(\mathbf{y}, X^*R), \text{ and}$$

$$\mathcal{X}(\mathbf{y}) = \sum_{x\in X} x + C_{\mathcal{F}}(\mathbf{y}, X^*),$$

and if the result of replacing each y_f in $\mathbf{y}$ by $y_f - 1$ is denoted by $\mathbf{y} - \mathbf{1}$, then

$$\sum_{w\in X^*} \left(\prod_{f\in\mathcal{F}} y_f^{f(w)} \right) w = (1 - \mathcal{X}(\mathbf{y}-\mathbf{1}))^{-1},$$

$$\sum_{w\in LX^*} \left(\prod_{f\in\mathcal{F}} y_f^{f(w)} \right) w = \mathcal{L}(\mathbf{y}-\mathbf{1})(1 - \mathcal{X}(\mathbf{y}-\mathbf{1}))^{-1},$$

$$\sum_{w\in X^*R} \left(\prod_{f\in\mathcal{F}} y_f^{f(w)} \right) w = (1 - \mathcal{X}(\mathbf{y}-\mathbf{1}))^{-1}\mathcal{R}(\mathbf{y}-\mathbf{1}), \text{ and}$$

$$\sum_{w\in LX^*R} \left(\prod_{f\in\mathcal{F}} y_f^{f(w)} \right) w = C_{\mathcal{F}}(\mathbf{y}-\mathbf{1}, LX^*R) + \mathcal{L}(\mathbf{y}-\mathbf{1})(1 - \mathcal{X}(\mathbf{y}-\mathbf{1}))^{-1}\mathcal{R}(\mathbf{y}-\mathbf{1}).$$

Proof. Let D_w be the multi-set of $\mathcal{F}$-factors in w. Then, it is clear that

$$\sum_{T\subseteq D_w} \prod_{f\in T} y_f = \prod_{f\in D_w} (y_f + 1).$$

Therefore, we may replace y_f with $y_f + 1$ in the statement of the theorem to obtain the equivalent statements such as

$$\sum_{w\in X^*} \sum_{T\subseteq D_w} \left(\prod_{f\in T} y_f \right) w = (1 - \mathcal{X}(\mathbf{y}))^{-1}.$$

However, for each $T \subseteq D_w$, we may decompose w uniquely into single letters and clusters consisting of all overlapping elements of T. We see that the right-hand side of each equation gives every possible decomposition of each word exactly once, so they are indeed the same. □

We will also now state a generalization of Theorem 2.1 for words, showing that Q2 solves Q2$_{no}$ in the case of a single pattern set. We drop the restriction to a set of patterns of the same length $m > 1$. When we apply Theorem 2.5 to compositions and permutations, we see that we may drop the restriction in those cases as well.

Theorem 2.5. *Let X be an alphabet, and let $\mathcal{F} \subseteq X^+$. Then,*

$$\sum_{w \in X^*} y^{\mathcal{F}_{no}(w)} w = \left(1 - y\left(K \sum_{x \in X} x - (K-1)\right)\right)^{-1} K,$$

where $K = \sum_{w \in X^*} 0^{\mathcal{F}_{no}(w)} w$ is the generating function for words that avoid $\mathcal{F}$.

Proof. Given $w \in X^*$, scanning from left to right, break w into sub-words after each letter that completes an $\mathcal{F}$-pattern within the current sub-word. Then, $\mathcal{F}_{no}(w)$ is the number of such breaks. The final sub-word avoids $\mathcal{F}$, while the other sub-words each contain one non-overlapping $\mathcal{F}$-pattern ending at the last letter. The latter type of words can be generated by starting with an $\mathcal{F}$-avoiding word v, appending any letter x, then removing those words that still avoid $\mathcal{F}$. The result follows. □

2.5 Application of Theorem 2.4 to $\mathcal{PS}$ (and $\mathcal{PC}$)

In light of Subsection 2.2.2 (solving Q2 solves Q2$_{no}$), we focus on Q2. We begin with a useful digression into the setting of compositions.

Consider the alphabet $\mathbb{P}$, let $P \subseteq \bigcup_{m \geq 1} S_m$, and let

$$D_P(\mathbf{y}; z) = \sum_{(w,\nu,\beta) \in C_{\mathcal{F}_P}} \left(\prod_{p \in P} y_p^{p(\nu)}\right) q^{\mathrm{sum}(w)} z^{\ell(w)},$$

where $p(\nu) = \sum_{f \in \mathcal{F}_p} f(\nu)$. Replacement of each letter i by $q^i z$ in the first identity of Theorem 2.4 yields

$$\sum_{n \geq 0} \sum_{w \in \mathbb{P}^n} \left(\prod_{p \in P} y_p^{p(w)} \right) q^{\mathrm{sum}(w)} z^n = \left(1 - zq(1-q)^{-1} - D_P(\mathbf{y} - \mathbf{1}; z)\right)^{-1}. \quad (2.13)$$

Besides being a practical tool for enumerating compositions by patterns, (2.13) also reveals the fact that solving Q1 solves Q2 for compositions. To illustrate both points, we deduce (2.8) from (2.7) and (2.13). Relative to $P = \{132, 231\}$, set $y_{132} = y_{231} = y$. As the P-clusters are in one-to-one correspondence with the up-down compositions of odd length greater than 1,

$$\frac{z}{1-q} + D_P(y; z/q) = \frac{1}{\sqrt{y}} \sum_{n \geq 0} \sum_{w \in \mathrm{UD}\mathbb{P}^{2n+1}} q^{\mathrm{sum}(w)} (z\sqrt{y}/q)^{2n+1}.$$

So, (2.13) with z replaced by z/q and the odd part of (2.7) imply (2.8). Thus, counting up-down compositions solves the problem of counting compositions by peaks.

Theorem 2.3 allows the considerations of the previous paragraph to be rephrased in the context of permutations. So, for permutations, solving Q1 solves Q2. Also, Theorem 2.3 applied to the lefthand side of (2.13) implies Theorem 2.6.

Theorem 2.6. *If* $P \subseteq \bigcup_{m \geq 1} S_m$, *then*

$$\sum_{n \geq 0} \sum_{\sigma \in S_n} \left(\prod_{p \in P} y_p^{p(\sigma)} \right) \frac{q^{\mathrm{inv}(\sigma)} z^n}{(q;q)_n} = \left(1 - z(1-q)^{-1} - D_P(\mathbf{y} - \mathbf{1}; z/q)\right)^{-1}.$$

Theorem 2.6 strengthens the main result in Rawlings [52] by dropping the restriction that P be permissible (that is, no $p \in P$ occurs as a consecutive pattern in another $r \in P$). The restricted result in [52] was used to extend some permutation results of Elizalde and Noy's [18] as well as to solve a few other problems in $\mathcal{P}S$. The example of Subsection 2.5.3 involves a non-permissible P.

For $P = \{p \in S_m : p_1 *_1 p_2 *_2 \cdots *_{m-1} p_m\}$ where $*_1, *_2, \ldots, *_{m-1} \in \{<, >\}$, there are two common types of problems in $\mathcal{P}S$ to be considered. The first is to track P as a whole and the second involves tracking the patterns in P individually. Relative to Q2, these respective problems are to determine

$$\sum_{\sigma \in S_n} y^{P(\sigma)} q^{\mathrm{inv}(\sigma)} \text{ and } \sum_{\sigma \in S_n} \left(\prod_{p \in P} y_p^{p(\sigma)} \right) q^{\mathrm{inv}(\sigma)}.$$

To illustrate the use of Theorem 2.6, we will apply it to deduce four new results. The examples in Subsections 2.5.1 and 2.5.2 track particular pattern sets as wholes, the example of Subsection 2.5.3 tracks two pattern sets of different lengths, and the example of Subsection 2.5.4 tracks patterns individually. In doing these examples, we must enumerate permutations by up-down type.

2.5.1 Permutations by $(\mathbf{i},\mathbf{d})$-peaks and permutations of up-down type

For $i,d \geq 2$, let $P_{i,d} = \{p \in S_{i+d-1} : p_1 < p_2 < \cdots < p_i > p_{i+1} > \cdots > p_{i+d-1}\}$. A consecutive occurrence of a $P_{i,d}$-pattern in a permutation σ is said to be an (i,d)-peak. In Figure 2.1, $\sigma_1\sigma_2\sigma_3\sigma_4 = 2561$ is a $(3,2)$-peak. Of course, a $(2,2)$-peak is just a peak as defined in Subsection 1.1.2. Theorems 2.4 and 2.6 may be used to obtain the generating function for permutations by (i,d)-peaks as rational expressions of q-Olivier functions

$$\Phi_{i,k}(z) = \sum_{n\geq 0} \frac{z^{in+k}}{(q;q)_{in+k}}.$$

To this end, for $i_1, d_1, \ldots, i_m, d_m \geq 2$ and $k \geq 1$, let $\mathrm{UD}\mathbb{P}_{i_1,d_1;\cdots;i_m,d_m;k}$ denote the set of compositions w that begin with a weakly increasing sequence $w_1 \leq w_2 \leq \cdots \leq w_{i_1}$ of length i_1, then continue with a strictly decreasing sequence $w_{i_1} > w_{i_1+1} > \cdots > w_{i_1+d_1-1}$ of length d_1, followed by a weakly increasing sequence of length i_2, then a strictly decreasing sequence of length d_2, and so on until ending with a weakly increasing sequence of length k.

We let $(j,d)^m$ denote the list $j,d;j,d;\ldots;j,d$ in which j,d appears m times. A composition in $\mathrm{UD}\mathbb{P}_{i,d;(j,d)^m;k}$, for any $m \geq 0$, is said to be of up-down type $(i,j,d;k)$. Up-down permutations of type $(i,j,d;k)$ are similarly defined.

Corollary 2.3. *If, for $i,j,d \geq 2$, we set $\mu = i+d-2$ and $\xi_m = \sqrt[m]{-1}$, then the generating function for permutations by (i,d)-peaks and inversions is*

$$\sum_{n\geq 0}\sum_{\sigma\in S_n} \frac{y^{P_{i,d}(\sigma)} q^{\mathrm{inv}(\sigma)} z^n}{(q;q)_n} = \left(1 - z(1-q)^{-1} - \frac{K_{i,i,d;1}(\sqrt[\mu]{y-1}z)}{\sqrt[\mu]{y-1}}\right)^{-1}$$

where, for $k \geq 1$*,*

$$K_{i,j,d;k}(z) = \sum_{m \geq 0} \sum_{w \in \mathbb{P}_{i,d;(j,d)^m;k}} q^{\mathrm{sum}(w)} (z/q)^{\ell(w)}.$$

Moreover, $K_{i,j,d;k}(z)$ *satisfies, for* $d \geq 3$ *and* $\nu = j + d - 2$*, the recurrence*

$$\begin{aligned} K_{i,j,d;k}(z) = {} & \frac{\xi_\nu^{-\mu} K_{i,j+1,d-1;1}(\xi_\nu z) \left(z^k (q;q)_k^{-1} + \xi_\nu^{-k} K_{j,j+1,d-1;k+1}(\xi_\nu z)\right)}{1 + K_{j,j+1,d-1;1}(\xi_\nu z)} \\ & - \xi_\nu^{-\mu-k} K_{i,j+1,d-1;k+1}(\xi_\nu z) \end{aligned}$$

with the initial condition

$$K_{i,j,2;k}(z) = \xi_j^{-i-k} \left(\frac{\Phi_{j,i}(\xi_j z) \Phi_{j,k}(\xi_j z)}{\Phi_{j,0}(\xi_j z)} - \Phi_{j,i+k}(\xi_j z) \right).$$

Before providing proof, a few examples are presented. First, the above recurrence provides a straightforward means of determining $K_{i,j,d;k}(z)$ as a rational expression of q-Olivier functions. Therefore, the generating function for permutations by (i,d)-peaks given by Corollary 2.3 is also a rational expression in q-Olivier functions. For instance,

$$\sum_{n \geq 0} \sum_{\sigma \in S_n} \frac{y^{P_{3,3}(\sigma)} q^{\mathrm{inv}(\sigma)} z^n}{(q;q)_n} = \left(1 - z(1-q)^{-1} - \frac{K_{3,3,3;1}(\sqrt[4]{y-1} z)}{\sqrt[4]{y-1}} \right)^{-1},$$

where

$$\begin{aligned} K_{3,3,3;1}(z) &= \frac{-K_{3,4,2;1}(\xi_4 z) \left(z(1-q)^{-1} + \xi_4^{-1} K_{3,4,2;2}(\xi_4 z)\right)}{1 + K_{3,4,2;1}(\xi_4 z)} + \xi_4^{-1} K_{3,4,2;2}(\xi_4 z), \\ K_{3,4,2;1}(z) &= -\frac{\Phi_{4,3}(\xi_4 z) \Phi_{4,1}(\xi_4 z)}{\Phi_{4,0}(\xi_4 z)} + \Phi_{4,4}(\xi_4 z), \text{ and} \\ K_{3,4,2;2}(z) &= \xi_4^{-1} \left(-\frac{\Phi_{4,3}(\xi_4 z) \Phi_{4,2}(\xi_4 z)}{\Phi_{4,0}(\xi_4 z)} + \Phi_{4,5}(\xi_4 z) \right). \end{aligned}$$

Second, Corollary 2.3 and the comments of Subsection 2.2.2 may be used to solve the $\mathrm{Q2}_{no}$ version of counting permutations by (i,d)-peaks. We illustrate by obtaining Mendes and Remmel's [47] result for the case $(i,2)$. First, note that the initial condition at the end of Corollary 2.3 implies

$$K_{i,i,2;k}(z) = \frac{\xi_i^{-k} \Phi_{i,k}(\xi_i z)}{\Phi_{i,0}(\xi_i z)} - \frac{z^k}{(q;q)_k}. \tag{2.14}$$

Substituting $K_{i,i,2;1}(z)$ into Corollary 2.3, setting $y = 0$, and plugging the result into Theorem 2.1 gives Mendes and Remmel's result, namely

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\frac{y^{P_{i,2no}(\sigma)}q^{\mathrm{inv}(\sigma)}z^n}{(q;q)_n}=\left(1-yz(1-q)^{-1}-\frac{(1-y)\Phi_{i,1}(z)}{\Phi_{i,0}(z)}\right)^{-1}.$$

Third, by definition, $K_{i,j,d;k}(z)$ is the generating function for up-down compositions of type $(i,j,d;k)$. The classic result $z/(1-q)+K_{2,2,2;1}(z)=\tan_q(z)$ for up-down compositions of odd length is evident in (2.14). Similarly, $z/(1-q)+K_{3,3,3;1}(z)$ is the generating function for the so-called up-up-down-down compositions. Prodinger and Tshifhumulo [50] gave another recurrence, without obtaining a closed form, for the generating function for up-up-down-down compositions. With "$\geq$" in place of "$>$", Carlitz [8] determined the generating function for up-down compositions of type $(i,i,2;k)$.

Finally, we again underscore the value of Theorem 2.3 in transcribing pattern results between the settings of compositions and permutations. For instance, replacing z in the first part of Corollary 2.3 with qz and invoking Theorem 2.3 gives the generating function for compositions by (i,d)-peaks.

Likewise, $K_{i,j,d;k}(z)$ transcribes as the generating function for permutations by up-down type $(i,j,d;k)$ and by inversion number. So, $z/(1-q)+K_{3,3,3;1}(z)$ is a q-analog of Carlitz and Scoville's [10] result for up-up-down-down permutations. Using another method, Mendes, Remmel, and Riehl [48] obtained generating functions for up-down permutations of type $(i,j,2;k)$ with $k\leq j$. For up-down type $(0,j,2;k)$, see Carlitz [6].

Proof of Corollary 2.3. The relevant cluster generating function is

$$D_{P_{i,d}}(y;z/q)=\sum_{(w,\nu,\beta)\in C_{\mathcal{F}_{P_{i,d}}}}q^{\mathrm{sum}(w)}y^{P_{i,d}(\nu)}(z/q)^{\ell(w)}.$$

Clearly, a composition is $P_{i,d}$-coverable if and only if it belongs to $\mathrm{UD}\mathbb{P}_{(i,d)^m;1}$ for some $m\geq 1$. Moreover, each $w\in\mathrm{UD}\mathbb{P}_{(i,d)^m;1}$ has but one $P_{i,d}$-covering. It follows that

$$D_{P_{i,d}}(y;z/q)=\frac{1}{\sqrt[u]{y}}\sum_{m\geq 1}\sum_{w\in\mathrm{UD}\mathbb{P}_{(i,d)^m;1}}q^{\mathrm{sum}(w)}(\sqrt[u]{y}z/q)^{\ell(w)}=\frac{K_{i,i,d;1}(\sqrt[u]{y}z)}{\sqrt[u]{y}}.$$

The above equality and Theorem 2.6 imply the first part of Corollary 2.3.

There are several theoretical frameworks (including the pattern algebra of Goulden and Jackson [30] described in Section 2.8) for determining $K_{i,j,d;k}(z)$. We will use Theorem 2.4; in this approach, up-down compositions having strictly descending runs of length d are exchanged for "straighter" up-down compositions having strictly descending runs of length $d-1$.

Let $\mathbb{P}_{i,d} = \{w \in \mathbb{P}^{i+d-1} : w_1 \le w_2 \le \cdots \le w_i > w_{i+1} > \cdots > w_{i+d-1}\}$. For any word w in $\mathbb{P}^*$ or in $\mathbb{P}_{i,d}^*$, the symbol $\ell(w)$ is always to be interpreted as the length of w relative to the alphabet $\mathbb{P}$.

Relative to the alphabet $X_{d-1} = \mathbb{P}_{k,1} \bigcup \left(\bigcup_{l\ge 2} \mathbb{P}_{l,d-1}\right)$, let $\mathcal{F}_{d-1}$ denote the set of words of the form uv where $u, v \in X_{d-1}$ whose last letter in the factor u is less than or equal to the first letter in v. For a word $w = u_{(1)}u_{(2)}\ldots u_{(n)}$ with each $u_{(m)} \in X_{d-1}$, let $\mathrm{asc}(w) = \sum_{f\in\mathcal{F}_{d-1}} f(w)$.

As $\mathrm{UD}\mathbb{P}_{i,d;(j,d)^m;k} = \{w \in \mathbb{P}_{i,d-1}\mathbb{P}_{j,d-1}^*\mathbb{P}_{k,1} : \mathrm{asc}(w) = 0\}$, Theorem 2.4 leads to

$$K_{i,j,d;k}(z) = \sum_{m\ge 0} \sum_{w\in\mathbb{P}_{i,d-1}\mathbb{P}_{j,d-1}^m\mathbb{P}_{k,1}} 0^{\mathrm{asc}(w)} q^{\mathrm{sum}(w)} (z/q)^{\ell(w)} = A_{1,d} + \frac{A_{2,d}A_{3,d}}{1+A_{4,d}},$$

where

$$A_{1,d} = \sum_{m\ge 0} (-1)^{m+1} \sum_{\substack{w\in\mathbb{P}_{i,d-1}\mathbb{P}_{j,d-1}^m\mathbb{P}_{k,1} \\ \mathrm{asc}(w)=m+1}} q^{\mathrm{sum}(w)} (z/q)^{\ell(w)},$$

$$A_{2,d} = \sum_{m\ge 0} (-1)^{m} \sum_{\substack{w\in\mathbb{P}_{i,d-1}\mathbb{P}_{j,d-1}^m \\ \mathrm{asc}(w)=m}} q^{\mathrm{sum}(w)} (z/q)^{\ell(w)},$$

$$A_{3,d} = \sum_{m\ge 0} (-1)^{m} \sum_{\substack{w\in\mathbb{P}_{j,d-1}^m\mathbb{P}_{k,1} \\ \mathrm{asc}(w)=m}} q^{\mathrm{sum}(w)} (z/q)^{\ell(w)}, \text{ and}$$

$$A_{4,d} = \sum_{m\ge 1} (-1)^{m} \sum_{\substack{w\in\mathbb{P}_{j,d-1}^m \\ \mathrm{asc}(w)=m-1}} q^{\mathrm{sum}(w)} (z/q)^{\ell(w)}.$$

Completion of the proof is now just a matter of determining the sums $A_{l,d}$. Being of similar nature, only a few are evaluated here.

As an example of the case $d = 2$, note that

$$1 + A_{4,2} = 1 + \sum_{m\geq 1}(-1)^m z^{jm} \sum_{0\leq w_1\leq\cdots\leq w_{jm}} q^{w_1+\cdots+w_{jn}} = \sum_{m\geq 0}\frac{(-1)^m z^{jm}}{(q;q)_{jm}} = \Phi_{j,0}(\xi_j z).$$

For $A_{1,d}$ with $d \geq 3$, note that $\ell(w) = \mu + \nu m + k$ and that $\{w \in \mathbb{P}_{i,d-1}\mathbb{P}^m_{j,d-1}\mathbb{P}_{k,1} : \mathrm{asc}(w) = m+1\} = \mathrm{UD}\mathbb{P}_{i,d-1;(j+1,d-1)^m;k+1}$. Thus, $A_{1,d} = -\xi_\nu^{-\mu-k} K_{i,j+1,d-1;k+1}(\xi_\nu z)$.

□

2.5.2 Uniform range distributions

For $i, d \geq 2$ and $m \geq 1$, let $P_{(i,d)^m}$ denote the set of patterns $p \in S_{(i+d-2)m+1}$ that begin with an increasing sequence $p_1 < \cdots < p_i$ of length i, continue with a decreasing sequence $p_i > p_{i+1} > \cdots > p_{i+d-1}$ of length d, followed by an increasing sequence $p_{i+d-1} < p_{i+d} < \cdots < p_{2i+d-2}$ of length i, and so on so as to form m consecutive (i,d)-peaks. The consecutive occurrence of a $p \in P_{(i,d)^m}$ in a permutation σ is said to be a uniform m-peak range of type (i,d). The following result extends Corollary 2.3 to uniform ranges.

Corollary 2.4. *If $i, d \geq 2$, $m \geq 1$, and $\nu = i + d - 2$, then the generating function for permutations by uniform m-peak ranges and inversions is*

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\frac{y^{P_{(i,d)^m}(\sigma)} q^{\mathrm{inv}(\sigma)} z^n}{(q;q)_n} = \left(1 - \frac{z}{1-q} - \sum_{n\geq m} A_{n,m}(y-1)B_n(q)z^{n\nu+1}\right)^{-1},$$

where

$$A_{n,m}(y) = \left.\frac{yz^m(1-z)}{1-z-yz(1-z^m)}\right|_{z^n} \quad \textit{and}$$

$$B_n(q) = K_{i,i,d;1}(z)|_{z^{n\nu+1}},$$

with $K_{i,i,d;1}(z)$ as determined in Corollary 2.3.

The case $i = d = 2$ with $y = 0$ of Corollary 2.4 provides a solution to the problem posed by Kitaev [40, Problem 1] of counting permutations that consecutively avoid $(2m+1)$-reverse-alternating patterns (which, as noted in [40], is the same as the number of permutations that consecutively avoid $(2m+1)$-alternating patterns). The case for even-length alternating patterns may be dealt with similarly.

Proof of Corollary 2.4. First, note that

$$D_{P_{(i,d)^m}}(y; z/q) = \sum_{(w,\nu,\beta)\in C_{\mathcal{F}_{P_{(i,d)^m}}}} q^{\mathrm{sum}(w)} y^{P_{(i,d)^m}(\nu)} (z/q)^{\ell(w)}.$$

Next, observe that a composition is $P_{(i,d)^m}$-coverable if and only if it belongs to the set of up-down compositions $\bigcup_{n\geq m} \mathrm{UD}\mathbb{P}_{(i,d)^n;1}$. Moreover, there may be multiple $P_{(i,d)^m}$-coverings (ν, β) for such a composition. For instance, $w = 231423221 \in \mathrm{UD}\mathbb{P}_{(2,2)^4;1}$ is $P_{(2,2)^2}$-covered by $((23142, 23221); (1, 5))$, but it is also covered by $((23142, 14232, 23221); (1, 3, 5))$.

For $n \geq m \geq 1$ and $k \geq 1$, let $a_{n,m,k}$ denote the number of times that a given $w \in \mathrm{UD}\mathbb{P}_{(i,d)^n;1}$ appears in a $P_{(i,d)^m}$-cluster (w, ν, β) with $P_{(i,d)^m}(\nu) = k$. Of course, $a_{n,m,k}$ is independent of the choice of $w \in \mathrm{UD}\mathbb{P}_{(i,d)^n;1}$.

For $n \geq 1$, let

$$A_{n,m}(y) = \sum_{k\geq 1} a_{n,m,k} y^k \text{ and}$$
$$B_n(q) = \sum_{w\in \mathrm{UD}\mathbb{P}_{(i,d)^n;1}} q^{\mathrm{sum}(w)}/q^{n\nu+1}.$$

Evidently, $B_n(q) = K_{i,i,d;1}(z)|_{z^{n\nu+1}}$. It follows that

$$D_{P_{(i,d)^m}}(y; z/q) = \sum_{n\geq m} A_{n,m}(y) B_n(q) z^{n\nu+1}.$$

In view of Theorem 2.6, we need only establish the formula for $A_{n,m}(y)$. Note that a typical $P_{(i,d)^m}$-cluster that contributes to the count $a_{n,m,k}$ is of the form $(w, \nu, (b_1, b_2, \ldots, b_k))$ with b_2 equaling $r(i + d - 2) + 1$ for some $r \in [m]$. So, for $n \geq m \geq 1$ and $k \geq 2$, $a_{n,m,k} = \sum_{j=1}^{m} a_{n-j,m,k-1}$. Routine computations then lead to the fact that

$$\sum_{n\geq m} \sum_{k\geq 1} a_{n,m,k} y^k z^n = \frac{yz^m(1-z)}{1-z-yz(1-z^m)}.$$

Thus, $A_{n,m}(y) = yz^m(1-z)\left(1-z-yz(1-z^m)\right)^{-1}|_{z^n}$. □

2.5.3 Permutations by peaks and twin peaks

Let $\mathrm{tpeak} = \{p \in S_5 : p_1 < p_2 > p_3 < p_4 > p_5\}$. A consecutive occurrence of $p \in \mathrm{tpeak}$ in a permutation is referred to as a twin peak. The set $P = \mathrm{peak} \cup \mathrm{tpeak}$

is not permissible and therefore not within the scope of the theorem in [52]. However, Theorem 2.6 makes the joint enumeration of permutations by peaks and twin peaks straightforward; we just need to determine

$$D_P(x,y;z/q) = \sum_{(w,\nu,\beta)\in C_{\mathcal{F}_P}} x^{\mathrm{peak}(\nu)} y^{\mathrm{tpeak}(\nu)} q^{\mathrm{sum}(w)} (z/q)^{\ell(w)}. \tag{2.15}$$

To this end, first note that the set of P-coverable compositions corresponds to the set of up-down compositions $\bigcup_{n\geq 1} \mathrm{UD}\mathbb{P}^{2n+1}$.

For $w \in \mathrm{UD}\mathbb{P}^{2n+1}$, let $a_{n,l,k}$ denote the number of P-coverings (ν,β) of w by l peaks and k twin peaks. Set $A_n(x,y) = \sum_{l,k\geq 0} a_{n,l,k} x^l y^k$. From the easily deduced recurrence relationship

$$a_{n,l,k} = a_{n-1,l-1,k} + a_{n-1,l-1,k-1} + a_{n-2,l-1,k-1} + a_{n-1,l,k-1} + a_{n-2,l,k-1},$$

we find that $A_n(x,y) = (xz + yz^2 + xyz^2)(1 - xz - xyz - xyz^2 - yz - yz^2)^{-1}|_{z^n}$.

Let $B_n(q) = \sum_{w\in \mathrm{UD}\mathbb{P}^{2n+1}} q^{\mathrm{sum}(w)}/q^{2n+1} = \tan_q(z)|_{z^{2n+1}}$. In view of (2.15), we have

$$D_P(x,y;z/q) = \sum_{n\geq 1} A_n(x,y) B_n(q) z^{2n+1}.$$

Finally, the last equality and Theorem 2.6 imply that the generating function for permutations by peaks and twin peaks,

$$\sum_{n\geq 0} \sum_{\sigma\in S_n} \frac{q^{\mathrm{inv}(\sigma)} x^{\mathrm{peak}(\sigma)} y^{\mathrm{tpeak}(\sigma)} z^n}{(q;q)_n},$$

is given by

$$\left(1 - \frac{z}{1-q} - \sum_{n\geq 1} A_n(x-1,y-1) B_n(q) z^{2n+1}\right)^{-1}.$$

Another solution to the joint peak and twin peak problem is given in Subsection 2.8.5.

2.5.4 Permutations and up-down permutations by $(\mathbf{i},\mathbf{m})$-maxima

For $i \geq 2$ and $1 \leq m \leq i$, let $p_{(m)}$ denote the unique permutation in S_{i+1} with $p_{(m)1} < p_{(m)2} < \cdots < p_{(m)i}$ and $p_{(m)i+1} = i+1-m$. Also, let $P_i =$

$\{p_{(1)}, p_{(2)}, \ldots, p_{(i)}\}$. A consecutive occurrence of $p_{(m)} \in P_i$ in a permutation σ is said to be an (i,m)-maximum. Carlitz and Scoville [9] refer to $(2,1)$-maxima and $(2,2)$-maxima respectively as rising and falling maxima; they expressed the joint distribution of $\{12, 132, 231, 21\}$ over $0S_n0 = \{0\sigma 0 : \sigma \in S_n\}$ in terms of a second order differential equation.

The statement of our next result requires the q-binomial coefficient, which for integers n and k is defined as

$$\left[\begin{array}{c} n \\ k \end{array}\right] = \begin{cases} (q^{n-k+1};q)_k/(q;q)_k & \text{if } n \geq k \geq 0 \\ 0 & \text{otherwise .} \end{cases}$$

Corollary 2.5. *If $i \geq 2$ and $1 \leq m \leq i$, then the generating function for permutations by (i,m)-maxima and inversions is*

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\left(\prod_{m=1}^{i} y_m^{p_{(m)}(\sigma)}\right)\frac{q^{\mathrm{inv}(\sigma)}z^n}{(q;q)_n} = \left(1 - \frac{\Phi_{i,1}(\mathbf{y}-\mathbf{1};\xi_i z)}{\xi_i\Phi_{i,0}(\mathbf{y}-\mathbf{1};\xi_i z)}\right)^{-1},$$

where $\xi_i = \sqrt[i]{-1}$ and

$$\Phi_{i,k}(y_1,\ldots,y_i;z) = \sum_{n\geq 0}\frac{z^{in+k}}{(q;q)_{in+k}}\prod_{j=0}^{n-1}\left(y_i + \sum_{m=1}^{i-1}(y_i - y_m)q^m\left[\begin{array}{c} ij+k+m-1 \\ m \end{array}\right]\right).$$

For $i = 2$, $y_1 = y$, and $y_2 = 1$, Corollary 2.5 gives the q-analog obtained in [47, 52] of Elizalde and Noy's [18] result for permutations by $p = 132$:

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\frac{y^{132(\sigma)}q^{\mathrm{inv}(\sigma)}z^n}{(q;q)_n} = \left(1 - \sum_{n\geq 0}\frac{(y-1)^n q^n z^{2n+1}}{(q^2;q^2)_n(1-q^{2n+1})(1-q)^n}\right)^{-1}.$$

Proof of Corollary 2.5. By Theorem 2.6, we need to compute

$$D_{P_i}(\mathbf{y};z/q) = \sum_{(w,\nu,\beta)\in C_{\mathcal{F}}}\left(\prod_{m=1}^{i} y_m^{p_{(m)}(\nu)}\right)q^{\mathrm{sum}(w)}(z/q)^{\ell(w)},$$

where $\mathcal{F} = \bigcup_{p\in P_i}\mathcal{F}_p$.

Define $\mathcal{L}_i$ to be the set of compositions w of length $in+1$ for any $n \geq 0$ such that $w_j > w_{j+1}$ if and only if j is a positive multiple of i. Note that a composition w is P_i-coverable if and only if $w \in \mathcal{L}_i$ and $\ell(w) > 1$. Moreover, such a composition has but one P_i-covering. Thus,

$$\frac{z}{1-q} + D_{P_i}(\mathbf{y};z/q) = \sum_{w\in\mathcal{L}_i}\left(\prod_{p\in P_i} y_p^{p(w)}\right)q^{\mathrm{sum}(w)}(z/q)^{\ell(w)}.$$

Let $\mathcal{L}_i(y_1, y_2, \ldots, y_i; z)$ denote the right side of the above equality. For $w \in \mathcal{L}_i$ of length $in+1$, observe that $p_{(1)}(w) + p_{(2)}(w) + \cdots + p_{(i)}(w) = n$. So

$$\mathcal{L}_i(y_1, \ldots, y_i; z) = y_i^{(-1/i)} \mathcal{L}_i(y_1/y_i, \ldots, y_{i-1}/y_i, 1; z\sqrt[i]{y_i}).$$

We therefore only need to determine $\mathcal{L}_i(y_1, \ldots, y_{i-1}, 1; z)$.

To determine $\mathcal{L}_i(y_1, \ldots, y_{i-1}, 1; z)$, we appeal to Theorem 2.4. We work with the factor set $\mathcal{G} = \{uv : u \in \Lambda_i, v \in \mathbb{P}\}$. For each $uv \in \mathcal{G}$, define

$$y_{uv} = \begin{cases} t & \text{if } u_i \leq v, \\ y_m & \text{if } u_{i+1-m} > v \geq u_{i-m} \text{ for } 1 \leq m \leq i-1, \text{ and} \\ 1 & \text{if } u_1 > v. \end{cases}$$

For $w = u_{(1)}u_{(2)} \ldots u_{(n)}$ with $u_{(1)}, \ldots, u_{(n-1)} \in \Lambda_i$ and $u_{(n)} \in \mathbb{P}$, let $\operatorname{asc}(w)$ be the number of indices j such that the last letter of $u_{(j)}$ is less than or equal to the first letter of $u_{(j+1)}$. Note that $\prod_{g \in \mathcal{G}} y_g^{g(w)} = t^{\operatorname{asc}(w)} \prod_{m=1}^{i-1} y_m^{p_{(m)}(w)}$.

Since

$$\mathcal{L}_i(y_1, \ldots, y_{i-1}, 1; z) = \sum_{w \in \Lambda_i^* \mathbb{P}} 0^{\operatorname{asc}(w)} q^{\operatorname{sum}(w)} (z/q)^{\ell(w)} \prod_{m=1}^{i-1} y_m^{p_{(m)}(w)},$$

Theorem 2.4 implies that $\mathcal{L}_i(y_1, \ldots, y_{i-1}, 1; z)$ is given by

$$\frac{\displaystyle\sum_{n \geq 0} (-1)^n z^{in+1} \sum_{0 \leq \operatorname{sum}(\alpha) \leq n} \prod_{m=1}^{i-1} (1-y_m)^{\alpha_m} \sum_{w \in \mathcal{C}_{i,1,n;\alpha}} q^{\operatorname{sum}(w)-in-1}}{\displaystyle 1 + \sum_{n \geq 0} (-1)^n z^{in+i} \sum_{0 \leq \operatorname{sum}(\alpha) \leq n} \prod_{m=1}^{i-1} (1-y_m)^{\alpha_m} \sum_{w \in \mathcal{C}_{i,i,n;\alpha}} q^{\operatorname{sum}(w)-in-i}}, \tag{2.16}$$

where the sums on the right are over $\alpha \in \mathbb{N}^{i-1}$ and w in $\mathcal{C}_{i,k,n;\alpha} = \{w \in \Lambda_i^n \Lambda_k : p_{(i)}(w) = 0 \text{ and } p_{(m)}(w) = \alpha_m \text{ for } 1 \leq m \leq i-1\}$.

The proof is completed by showing that the numerator and denominator in (2.16) are respectively $\Phi_{i,1}(y_1, \ldots, y_{i-1}, 1; \xi_i z)/\xi_i$ and $\Phi_{i,0}(y_1, \ldots, y_{i-1}, 1; \xi_i z)$. □

Another generating function for permutations by (i, m)-maxima is derived in Subsection 2.9.4. Notably, Theorem 2.3 applied to the lefthand side of (2.16) yields the generating function for the set $\mathrm{UD}S_{i,i,2;1}$ of up-down permutations of type $(i, i, 2; 1)$ by (i, m)-maxima. Setting $y_1 = y_2 = \ldots = y_i = 1$, replacing z by $(1-q)z$, and letting $q \to 1$ in Corollary 2.6 gives a result due to Carlitz [6].

Corollary 2.6. *For $i \geq 2$, the generating function for permutations of up-down type $(i,i,2;1)$ by (i,m)-maxima is given by*

$$\frac{z}{1-q} + \sum_{\sigma \in UDS_{i,i,2;1}} \left(\prod_{m=1}^{i} y_m^{p_{(m)}} \right) \frac{q^{\mathrm{inv}(\sigma)} z^{\ell\sigma}}{(q;q)_{\ell\sigma}} = \frac{\Phi_{i,1}(y_1,\ldots,y_{i-1},1;\xi_i z)}{\xi_i \Phi_{i,0}(y_1,\ldots,y_{i-1},1;\xi_i z)}.$$

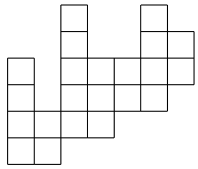
2.6 Ridge patterns in CCPs

The enumeration of CCPs and of subclasses of CCPs by various statistics has been widely studied. Polyomino enumeration is surveyed in Delest [15], Guttmann [31], Rensburg [56], and Viennot [61]. Our purpose here is to essentially initiate the study of CCPs by consecutive (or ridge) patterns.

The simplest ridge patterns are formed between two adjacent columns. For a column-convex polyomino Q, we say that an upper ascent (respectively upper level, upper descent) occurs at index k if the top cell in Q_k is lower than (respectively level with, higher than) the top cell in Q_{k+1}. Lower ascents, lower levels, and lower descents are similarly defined along the lower ridge. In Figure 1.2, Q has lower descents at indices 1,2,5, and 7. The numbers of upper ascents, upper levels, upper descents, lower ascents, lower levels, and lower descents in Q are respectively denoted by $\mathrm{uasc}(Q)$, $\mathrm{ulev}(Q)$, $\mathrm{udes}(Q)$, $\mathrm{lasc}(Q)$, $\mathrm{llev}(Q)$, and $\mathrm{ldes}(Q)$. In Figure 1.2, $\mathrm{uasc}(Q) = 2$ and $\mathrm{llev}(Q) = 1$.

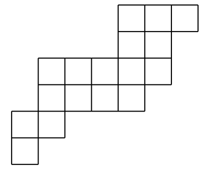
As displayed in Figure 2.3, the two-column ridge patterns may be used to characterize many of the common subclasses of CCPs. More complex consecutive patterns along either the lower or upper ridges are formed by subsequences of 3 or more columns.

The relative height of a CCP Q, denoted by $\mathrm{relh}(Q)$, is defined to be the y-ordinate of the top edge in the rightmost column of Q. In Figure 1.2, $\mathrm{relh}(Q) = -1$. The relative height of a parallelogram polyomino is known as its row number.



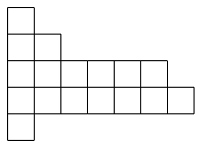
Directed Column-Convex Polyomino

No lower descents



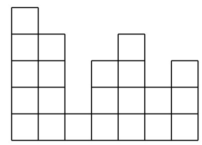
Parallelogram Polyomino

No lower or upper descents



Stack Polyomino

No upper ascents or lower descents



Wall Polyomino

No lower ascents or descents

Figure 2.3: Common classes of CCPs

2.6.1 Verification that $\mathcal{PC} \subset \mathcal{PCCP} \subset \mathcal{PW}$

Let WP_n be the set of wall polyominoes with n columns. The map $\gamma_n : \mathbb{P}^n \to \mathrm{WP}_n$ defined by $\gamma_n(w) = Q$ where Q_k has w_k cells is a bijection such that

$$\mathrm{area}(Q) = \mathrm{sum}(w) \text{ and } \mathrm{per}(Q) - 2\ell(Q) = \mathrm{var}(w). \tag{2.17}$$

For example, γ_7 maps the composition $w = 5413423 \in \mathbb{P}^7$ to the wall polyomino displayed in Figure 2.3. Interestingly, the second part of (2.17) relates the variation of a composition to the perimeter of a wall polyomino, and (2.11) together with the first part of (2.17) provides a connection between the inversion number of a permutation and the area of a wall polyomino.

Through γ_n, a consecutive p-pattern in a composition w induces an upper ridge p-pattern in the associated wall polyomino Q. For instance, $Q_kQ_{k+1}Q_{k+2}$ is deemed a 132-pattern in Q if $w_kw_{k+1}w_{k+2}$ is a 132-pattern in the associated w; that is, $Q_kQ_{k+1}Q_{k+2}$ is a 132-pattern if $Q_{k+1}Q_{k+2}$ is an upper descent and if the top cell in Q_{k+2} is level with or above the top cell in Q_k. The number of times an upper ridge pattern p occurs in Q is denoted by $p(Q)$.

The bijection γ_n immediately implies $\mathcal{PC} \subset \mathcal{PCCP}$: If $P \subseteq \cup_{m\geq 1} S_m$ and if $B_n \subseteq \mathbb{P}^n$, then

$$\sum_{n\geq 0}\sum_{w\in B_n} c^{\mathrm{var}(w)} q^{\mathrm{sum}(w)} \left(\prod_{p\in P} y_p^{p(w)}\right) z^n = \sum_{n\geq 0}\sum_{Q\in\gamma_n(B_n)} c^{\mathrm{per}(Q)} q^{\mathrm{area}(Q)} \left(\prod_{p\in P} y_p^{p(Q)}\right) \frac{z^n}{c^{2n}}. \tag{2.18}$$

Of course, as with Theorem 2.3, (2.18) also holds for any functions that are preserved by γ_n.

To see the inclusion $\mathcal{PCCP} \subset \mathcal{PW}$, consider the alphabet of biletters $X = \left\{\binom{j}{m} : j, m \in \mathbb{P}\right\}$ and let

$$\mathcal{Y} = \bigcup_{n\geq 0}\left\{\binom{j_1 j_2 \dots j_n}{m_1 m_2 \dots m_n} \in X^n : m_n = 1 \text{ and } j_k + j_{k+1} > m_k \text{ for } 1 \leq k < n\right\}.$$

For a column-convex polyomino Q with n columns, define

$$\delta(Q) = \binom{j_1 j_2 \dots j_n}{m_1 m_2 \dots m_n} \tag{2.19}$$

where j_k is the number of cells in Q_k, $m_n = 1$, and, for $1 \leq k < n$, m_k is the change in the y-ordinate from the bottom edge of Q_{k+1} to the top edge of Q_k. For Q in Figure 1.2, $\delta(Q) = \binom{23644532}{35526541}$.

The map δ is a bijection from CCP to $\mathcal{Y}$. As such, δ allows CCPs to be viewed as words. Such a viewpoint is implicit in Temperley [59] and explicit in Bousquet-Mélou and Viennot [3]. Thus, a problem in $\mathcal{PCCP}$ may readily be converted into a problem in $\mathcal{PW}$; so $\mathcal{PCCP} \subset \mathcal{PW}$.

2.7 Application of Theorem 2.4 to the set $\mathcal{PCCP}$

The inclusion $\mathcal{PCCP} \subset \mathcal{PW}$ means that Theorem 2.4 may be applied to solving problems in $\mathcal{PCCP}$. We present two examples on directed column-convex polyominoes.

2.7.1 DCCPs by two-column ridge patterns

Our first example enumerates DCCPs by the five two-column ridge patterns, perimeter, relative height, area, and length.

Corollary 2.7. *The generating function*

$$G = \sum_{Q \in DCCP} a_u^{\mathrm{uasc}(Q)} a_l^{\mathrm{lasc}(Q)} b_u^{\mathrm{ulev}(Q)} b_l^{\mathrm{llev}(Q)} c^{\mathrm{per}(Q)} d^{\mathrm{udes}(Q)} h^{\mathrm{relh}\, Q} q^{\mathrm{area}(Q)} z^{\ell(Q)}$$

is given by

$$G = \frac{c^2 h \sum_{n \geq 0} \frac{(c^2 qz)^{n+1}}{1 - c^2 h q^{n+1}} \prod_{k=1}^{n} \left(b_l + \frac{a_l c^2 h q^k}{1 - c^2 h q^k} \right) \left(b_u + \frac{c^2 d q^k}{1 - c^2 q^k} - \frac{a_u}{1 - q^k} \right)}{1 - a_u \sum_{n \geq 1} \frac{(c^2 qz)^n}{1 - q^n} \prod_{k=1}^{n} \left(b_l + \frac{a_l c^2 h q^k}{1 - c^2 h q^k} \right) \prod_{k=1}^{n-1} \left(b_u + \frac{c^2 d q^k}{1 - c^2 q^k} - \frac{a_u}{1 - q^k} \right)}.$$

Proof. Define

$$H(b_u, b_l, d, z) = \sum_{Q \in \mathrm{DCCP}} b_u^{\mathrm{ulev}(Q)} b_l^{\mathrm{llev}(Q)} c^{\mathrm{per}(Q)} d^{\mathrm{udes}(Q)} h^{\mathrm{relh}(Q)} q^{\mathrm{area}(Q)} z^{\ell(Q)}. \tag{2.20}$$

As $\mathrm{uasc}(Q) = \ell(Q) - \mathrm{ulev}(Q) - \mathrm{udes}(Q) - 1$ and $\mathrm{lasc}(Q) = \ell(Q) - \mathrm{llev}(Q) - 1$, it follows that

$$G = \frac{1}{a_u a_l} H(b_u/a_u, b_l/a_l, d/a_u, a_u a_l z). \tag{2.21}$$

It then suffices to determine H.

Consider the alphabet $X = \left\{ \binom{j}{m} : j, m \in \mathbb{P}, j \geq m \geq 1 \right\}$. Then, let $R = \left\{ \binom{j}{m} \in X : m = 1 \right\}$ and, for a statement S, let $\chi(S)$ be 1 if S is true and 0 otherwise. An element $\binom{j_1 j_2 \ldots j_n}{m_1 m_2 \ldots m_n} \in X^n$ will be abbreviated by $\binom{j}{m}$; so the kth letter in $\binom{j}{m}$ is $\binom{j}{m}_k = \binom{j_k}{m_k}$.

Let $\mathcal{F} = \left\{ \binom{j}{m} \in X^2 : m_1 \geq j_2 \right\}$. For $f = \binom{j}{m} \in \mathcal{F}$, we will substitute $y_f = c^{2(m_1 - j_2)} d (b_u d^{-1})^{\chi(m_1 = j_2)}$. When restricted to DCCP, the map δ in (2.19) is a bijection onto $X^* R$. Moreover, if $Q = Q_1 Q_2 \ldots Q_n \in \mathrm{DCCP}$ and $\delta(Q) = \binom{j}{m} \in X^{n-1} R$, then

$$\begin{gathered} \mathrm{area}(Q) = \mathrm{sum}(j), \mathrm{per}(Q) = 2\left(n + \mathrm{relh}(Q) + S\right), \\ \mathrm{relh}(Q) = \mathrm{sum}(j) - \mathrm{sum}(m) + 1, \text{ and } b_u^{\mathrm{ulev}(Q)} d^{\mathrm{udes}(Q)} c^{2S} = \prod_{f \in \mathcal{F}} y_f^{f\binom{j}{m}} \end{gathered} \tag{2.22}$$

where $S = \sum_{k=1}^{n} (m_k - j_{k+1}) \chi(m_k > j_{k+1})$. The facts in (2.22) regarding area and relative height were observed by Bousquet-Mélou and Viennot [3].

It follows from (2.20) and (2.22) that

$$H = c^2h \sum_{\binom{j}{m}\in X^*R} q^{\text{sum}(j)}(c^2h)^{\text{sum}(j)-\text{sum}(m)}(c^2z)^{\ell\binom{j}{m}} \left(\prod_{f\in\mathcal{F}} y_f^{f\binom{j}{m}}\right) \prod_{k=1}^{\ell\binom{j}{m}-1} b_l^{\chi(j_k=m_k)}. \tag{2.23}$$

An $\mathcal{F}$-cluster $\left(\binom{j}{m},\nu,\beta\right)$ has $\binom{j}{m}\in X^n$, $\nu = \left(\binom{j_1 j_2}{m_1 m_2},\binom{j_2 j_3}{m_2 m_3},\ldots,\binom{j_{n-1} j_n}{m_{n-1} m_n}\right)$, and $\beta = (1,2,\ldots,n-1)$ for some $n\geq 2$, so an application of Theorem 2.4 to (2.23) yields

$$H = \frac{c^2h\sum_{n\geq 0}(c^2z)^{n+1}\sum_T(n)}{1-\sum_{n\geq 1}(c^2z)^n\sum_B(n)} \tag{2.24}$$

where if

$$\prod(n) = \prod_{k=1}^{n} b_l^{\chi(j_k=m_k)}\left(c^{2(m_k-j_{k+1})}d(b_ud^{-1})^{\chi(m_k=j_{k+1})}-1\right),$$

then

$$\sum_T(n) = \sum_{\binom{j}{m}}(c^2h)^{\text{sum}(j)-\text{sum}(m)}q^{\text{sum}(j)}\prod(n)$$

summed over $\binom{j}{m}$ satisfying $j_1\geq m_1\geq j_2\geq\cdots\geq j_{n+1}\geq m_{n+1}=1$, and

$$\sum_B(n) = \sum_{\binom{j}{m}}(c^2h)^{\text{sum}(j)-\text{sum}(m)}q^{\text{sum}(j)}b_l^{\chi(j_n=m_n)}\prod(n-1)$$

summed over $\binom{j}{m}$ satisfying $j_1\geq m_1\geq j_2\geq\cdots\geq j_n\geq m_n\geq 1$. Both $\sum_T(n)$ and $\sum_B(n)$ are nested geometric sums. As such, they are easily determined. For instance,

$$\begin{aligned}\sum_T(1) &= q^2\sum_{j_2\geq 1}(c^2hq^2)^{j_2-1}\sum_{m_1\geq j_2}q^{m_1-j_2}\left(c^{2(m_1-j_2)}d(b_ud^{-1})^{\chi(m_1=j_2)}-1\right)\\ &\quad\cdot\sum_{j_1\geq m_1}b_l^{\chi(j_1=m_1)}(c^2hq)^{j_1-m_1}\\ &= \frac{q^2}{1-c^2hq^2}\left(b_u+\frac{c^2dq}{1-c^2q}-\frac{1}{1-q}\right)\left(b_l+\frac{c^2hq}{1-c^2hq}\right).\end{aligned}$$

In general,

$$\sum_T(n) = \frac{q^{n+1}}{1-c^2hq^{n+1}}\prod_{k=1}^{n}\left(b_l+\frac{c^2hq^k}{1-c^2hq^k}\right)\left(b_u+\frac{c^2dq^k}{1-c^2q^k}-\frac{1}{1-q^k}\right)\quad\text{and}$$

$$\sum_B(n) = \frac{q^n}{1-q^n}\prod_{k=1}^{n}\left(b_l+\frac{c^2hq^k}{1-c^2hq^k}\right)\prod_{k=1}^{n-1}\left(b_u+\frac{c^2dq^k}{1-c^2q^k}-\frac{1}{1-q^k}\right).$$

The last two equalities for $\sum_T(n)$ and $\sum_B(n)$ together with (2.24) and (2.21) complete the proof. □

Corollary 2.7 (with $a_u = a$, $b_u = b$, $a_l = 0$, $b_l = h = 1$, and z replaced by z/c^2) with (2.18) implies Corollary 2.2 of Subsection 2.3.3. Corollary 2.7 also implies many known results, a few of which are displayed in Table 2.1. The noted

Table 2.1: Results implied by Corollary 2.7

Polyominoes	Distribution	Reference
DCCP	(area, per, relh, udes, col) $a_u, a_l, b_u, b_l = 1$	Rawlings [51]
DCCP	(area, per, relh, col) $a_u, a_l, b_u, b_l, d = 1$	Bousquet-Melou [4]
PP	(area, uasc, lasc, col) $a_u, a_l, c, h = 1; d = 0$	Delest, Dubernard, and Dutour[17]
PP	(area, col) $a_u, a_l, b_u, b_l, c, h = 1; d = 0$	Delest and Fédou[16]

distribution of Delest, Dubernard, and Dutour [17] also tracked the height of the leftmost column; their notion of corners coincides exactly with upper and lower ascents. Bousquet-Mélou's entry included both the left and right column heights.

2.7.2 DCCPs by valleys along the upper ridge

A column-subsequence $Q_kQ_{k+1}Q_{k+2}$ in a column-convex polyomino Q is said to be a valley provided that Q_kQ_{k+1} is an upper descent and $Q_{k+1}Q_{k+2}$ is an upper ascent or an upper level. The number of valleys in Q is denoted by val(Q). Furthermore, Q is said to be down-up provided that Q_kQ_{k+1} is an upper descent when k is odd and is an upper ascent or an upper level when k is even. Let DU_n denote the set of down-up directed column-convex polyominoes of length n.

Corollary 2.8. *The generating function for DCCPs by valleys, area, and length,*

$$\sum_{Q\in DCCP} y^{\mathrm{val}(Q)} q^{\mathrm{area}(Q)} z^{\ell(Q)},$$

is given by

$$\frac{\displaystyle\sum_{n\geq 0}\frac{(1-y)^n q^{(n+1)(2n+1)} z^{2n+1}}{(q;q)_{2n+1}(q;q)_{2n}}}{\displaystyle\sum_{n\geq 0}\frac{(1-y)^n q^{n(2n+1)} z^{2n}}{(q;q)_{2n}^2} - \sum_{n\geq 0}\frac{(1-y)^n q^{(n+1)(2n+1)} z^{2n+1}}{(q;q)_{2n+1}^2}}.$$

The proof of Corollary 2.8 consists of first using Theorem 2.4 to express the generating function for DCCPs by valleys in terms of down-up DCCPs of odd lengths. Theorem 2.4 is then applied again in a manner analogous to the second half of the proof of Corollary 2.3 to show that

$$\sum_{n\geq 0}\sum_{Q\in\ \mathrm{DU}_{2n+1}} q^{\mathrm{area}(Q)} z^{2n+1} = \frac{\displaystyle\sum_{n\geq 0}\frac{(-1)^n q^{(n+1)(2n+1)} z^{2n+1}}{(q;q)_{2n+1}(q;q)_{2n}}}{\displaystyle\sum_{n\geq 0}\frac{(-1)^n q^{n(2n+1)} z^{2n}}{(q;q)_{2n}^2}}.$$

2.8 The pattern algebra method

Goulden and Jackson's pattern algebra method [30, section 4.3] is a powerful method for solving Q1 and Q2 on words for pattern sets with a particular structure. That is, if X is an alphabet and $\pi_1 \subset X^2$, then we will consider unions of factor sets $\mathcal{F} = \{w \in X^m\}$ with any number of restrictions of the form $w_i w_{i+1} \in \pi_1$ or $w_i w_{i+1} \notin \pi_1$. For example, let $X = \mathbb{P}$ and $\pi_1 = \{w_1 w_2 \in \mathbb{P}^2 : w_1 \leq w_2\}$. Then, the factor set for peaks is $\{w \in \mathbb{P}^3 : w_1 w_2 \in \pi_1, w_2 w_3 \notin \pi_1\}$, and the factor set for up-down compositions is $\bigcup_{m\geq 1}\{w \in \mathbb{P}^m : w_{2i-1}w_{2i} \in \pi_1 \text{ and } w_{2i}w_{2i+1} \notin \pi_1 \text{ for all } i\}$.

The pattern algebra method is capable of tracking many of these pattern sets individually, with any number being tracked by non-overlapping occurrences. In this section, we use the pattern algebra method to obtain a q-analog of Kitaev's [40] Theorem 30 and to deduce a better generating function for permutations by peaks and twin peaks, among other results.

The essentials of the pattern algebra method follow. Let X be an alphabet, $\pi_1 \subset X^2$, and $\pi_2 = X^2 \setminus \pi_1$. Suppose $\alpha = \sum_{w\in X^*} c_w w$ is a formal series, and for given $x, y \in X$, let $X_{x,y} = \{w \in X^+ : w_1 = x, w_{\ell(w)} = y\}$. Then, the incidence matrix $I(\alpha)$ is a matrix with rows and columns indexed by X such that $I(\alpha)_{x,y} =$

$\sum_{w\in X_{x,y}} c_w(w/y)$, *i.e.* the restriction of α to words in $X_{x,y}$, except the final y has been removed from each word. For $U \subseteq X^*$, we also define $I(U) = I(\sum_{w\in U} w)$ and note that $I(X) = I$, the identity matrix.

For the remainder of this section, we let $A = I(\pi_1)$, $B = I(\pi_2)$, and $W = I(X^2)$. In particular, $W = A + B$. It is crucial to note that, for formal series α and β, $I(\alpha)I(\beta) = I(\gamma)$, where γ is formed by concatenating words u and v from α and β, respectively, where the last letter of u is the first letter of v, and removing one copy of the repeated letter.

Finally, we define the operator Ψ, which converts an incidence matrix back to a formal series, by $\Psi(I(\sum_{w\in X^*} c_w w)) = \sum_{w\in X^+} c_w w$. The empty word has been removed in the process, as it is not accounted for in the incidence matrix. Note that Ψ is linear and, for incidence matrices F and G, $\Psi(FWG) = \Psi(F)\Psi(G)$.

2.8.1 A general strategy

We will consider the problem of enumerating words by factor sets whose incidence matrices can be written as a rational function of A, W, and B. Such a problem can be solved by the following process, which is different, yet equal in scope, to that given by Goulden and Jackson.

1. Define a variable to be the incidence matrix for the desired formal series, and then devise a system of linear equations to describe it. The design of this system should mimic that of a regular grammar in that each variable will be multiplied by at most one of A, W, and B and always on the same side. We will name our variables F_i and use right multiplication in this section.

2. Substitute either $A = W - B$ or $B = W - A$ and solve the system, treating F_iW terms as constants. Using $B = W - A$, we obtain a system of the form $F_i = f_i(A) + \sum_j F_jWf_{ij}(A)$, where the fs are rational functions.

3. Finally, apply Ψ to the entire system, noting that $\Psi(FWG) = \Psi(F)\Psi(G)$, and solve for $\Psi(F_i)$.

We will typically compute $\Psi(f(A))$ by expanding $f(A)$ as a power series in A and using the linearity of Ψ to obtain a sum of the $\Psi(A^n)$. We then may substitute various quantities for the letters in X to obtain various generating functions. We compute the image of $\Psi(A^n)$ under some common substitutions in the next subsection.

For DCCPs, we must also compute $\Psi(F_i Z)$, where Z is an incidence matrix that restricts the last letter of each word. To do so, we add an extra step to the end of the previous strategy. In this step, we take the result of step 2, multiply on the right by Z, and apply Ψ to both sides. This gives us $\Psi(F_i Z)$ in terms of $\Psi(F_i)$, which we computed in step 3. We will then also need to compute the image of $\Psi(A^n Z)$ under the common substitutions for DCCPs.

2.8.2 Key formulas

As $\Psi(A^n)$ and $\Psi(B^n)$ show up frequently, it is prudent to give their values under a few common homomorphisms.

For compositions, we will usually want to consider $\pi_1 = \{w_1 w_2 \in \mathbb{P}^2 : w_1 \leq w_2\}$. Then, $\Psi(A^{n-1}) = \sum_{\lambda \in \Lambda_n} \lambda$, where λ is written in increasing order. As shown in the proof of Theorem 2.3, if we substitute $q^i(z/q)$ for i, then we get

$$\Psi(A^{n-1}) = z^n/(q;q)_n. \tag{2.25}$$

Similarly, $\Psi(B^{n-1})$ is the generating function for the set of $w \in \mathbb{P}^n$ that strictly decrease between letters. Therefore, if we subtract $n-i$ from w_i, then we get the set of words that weakly decrease, *i.e.* Λ_n, and we have subtracted $\binom{n}{2}$ from sum(w). Therefore, if we substitute $q^i(z/q)$ for i, then we get

$$\Psi(B^{n-1}) = z^n q^{\binom{n}{2}}/(q;q)_n. \tag{2.26}$$

Of course, using Theorem 2.3, we may use (2.25) and (2.26) to apply our results to permutations.

For DCCPs, we consider the alphabets $X = \left\{\binom{j}{m} : j, m \in \mathbb{P}, j \geq m \geq 1\right\}$ and $R = \{\binom{j}{m} \in X : m = 1\}$. Then, through the bijection δ from (2.19), $\pi_1 = \{\binom{j_1 j_2}{m_1 m_2} : j_2 \geq m_1\}$ corresponds to upper ascents or levels. From the discussion at the end of Subsection 2.8.1, we will use $Z = I(R)$.

With this setup, $\Psi(B^{n-1})$ is the generating function for the set of $\binom{j}{m} \in X^n$ such that $0 < m_n \le j_n < m_{n-1} \le j_{n-1} < \cdots < m_1 \le j_1$. Thus, we can consider the word $j_1 m_1 j_2 m_2 \cdots j_n m_n$ as a partition. Taking the conjugate, we have a partition with parts of size at most $2n$ with at least one part of size $2i$ for all $i \in [n]$. Also, we know that $\mathrm{sum}(j)$ corresponds to the area of the DCCP, $\mathrm{sum}(j) - \mathrm{sum}(m)$ contributes to the relative height, and each letter where $j_i > m_i$ corresponds to a lower ascent. As before, let q track the area, h track the relative height, and a_l track lower ascents. Then, in the conjugate, we want to replace each part of size $2i-1$ with $q^i h$ and each part of size $2i$ with q^i. We also need to modify Equation (1.1) to account for the required even parts and to include a factor of a_l whenever there is at least one part of an odd size. That is, for even $i \in [2n]$, we need $x_i + x_i x_i + x_i x_i x_i + \cdots = x_i(1-x_i)^{-1}$, and for odd $i \in [2n]$, we need $1 + a_l(x_i + x_i x_i + \cdots) = 1 + a_l x_i (1-x_i)^{-1} = (1-(1-a_l)x_i)(1-x_i)^{-1}$. Therefore, with our substitutions, we get

$$\Psi(B^{n-1}) = \frac{((1-a_l)hq;q)_n q^{\binom{n+1}{2}} z^n}{(hq;q)_n (q;q)_n}. \tag{2.27}$$

Now, $\Psi(B^{n-1}Z)$ is the generating function for the set of $\binom{j}{m} \in X^n$ such that $1 = m_n \le j_m < \cdots < m_1 \le j_1$. We now employ a similar argument, except there must be exactly one part of size $2n$, we do not count a lower ascent if $j_n > 1$, and we add one to the relative height. Therefore, with our substitutions, we get

$$\Psi(B^{n-1}Z) = \frac{((1-a_l)hq;q)_{n-1} hq^{\binom{n+1}{2}} z^n}{(hq;q)_n (q;q)_{n-1}}. \tag{2.28}$$

The formal series $\Psi(A^{n-1})$ and $\Psi(A^{n-1}Z)$ are somewhat more difficult to compute. Theorem 2.7 addresses this issue.

Theorem 2.7. *Given $A(x) = 1 + \Psi(A^0)x + \Psi(A^1)x^2 + \cdots$, $B(x) = 1 + \Psi(B^0)x + \Psi(B^1)x^2 + \cdots$, $AZ(x) = \Psi(Z) + \Psi(AZ)x + \Psi(A^2Z)x^2 + \cdots$, and $BZ(x) = \Psi(Z) + \Psi(BZ)x + \Psi(B^2Z)x^2 + \cdots$, then*

$$A(x) = (B(-x))^{-1} \tag{2.29}$$

and

$$AZ(x) = A(x)BZ(-x). \tag{2.30}$$

Proof. Let $F = x(I - xA)^{-1}$, so that $A(x) = 1 + \Psi(F)$. Multiplying on the right by $(I - xA)$, we get $F - xFA = xI$. Set $A = W - B$ and solve for F, treating FW as a constant. It follows that

$$F = x(I + xB)^{-1} + xFW(I + xB)^{-1}.$$

Applying Ψ and using the fact that $\Psi(FWG) = \Psi(F)\Psi(G)$ yields

$$\begin{aligned}\Psi(F) &= x\Psi((I + xB)^{-1}) + x\Psi(F)\Psi((I + xB)^{-1})\\ &= (1 + \Psi(F))x\Psi((I + xB)^{-1}).\end{aligned}$$

Add 1 to both sides, and then solve for $1 + \Psi(F)$ to obtain

$$\begin{aligned}1 + \Psi(F) &= (1 - x\Psi((I + xB)^{-1}))^{-1}\\ &= (B(-x))^{-1}.\end{aligned}$$

Now, let $G = (I - xA)^{-1}Z$, so that $AZ(x) = \Psi(G)$. Following the same strategy, we get $G - xAG = Z$. Set $A = W - B$ and solve for G, treating WG as a constant. It follows that

$$G = (I + xB)^{-1}Z + x(I + xB)^{-1}WG.$$

Applying Ψ and using the fact that $\Psi(FWG) = \Psi(F)\Psi(G)$ yields

$$\Psi(G) = \Psi((I + xB)^{-1}Z) + x\Psi((I + xB)^{-1})\Psi(G).$$

Finally, solve for $\Psi(G)$ to get

$$\begin{aligned}\Psi(G) &= (1 - x\Psi((I + xB)^{-1}))^{-1}\Psi((I + xB)^{-1}Z)\\ &= (B(-x))^{-1}BZ(-x)\\ &= A(x)BZ(-x).\end{aligned}$$

□

Applying Theorem 2.7 to (2.27) and (2.28), we find that

$$\Psi(A^{n-1}) = \left(\sum_{k\geq 0} \frac{((1-a_l)hq;q)_k q^{\binom{k+1}{2}}(-x)^k}{(hq;q)_k(q;q)_k}\right)^{-1}\Bigg|_{x^n}$$

and that

$$\Psi(A^{n-1}Z) = \left. \frac{\displaystyle\sum_{k\geq 0} \frac{((1-a_l)hq;q)_k hq^{\binom{k+2}{2}}(-x)^k}{(hq;q)_{k+1}(q;q)_k}}{\displaystyle\sum_{k\geq 0} \frac{((1-a_l)hq;q)_k q^{\binom{k+1}{2}}(-x)^k}{(hq;q)_k(q;q)_k}} \right|_{x^{n-1}}. \tag{2.31}$$

Note that (2.31) counts the number of parallelogram polyominoes with n columns by area, relative height, and lower ascents. If we set $h = a_l = 1$, we obtain the result of Delest and Fédou [16].

2.8.3 Up-down and down-up words

Our first example will be to find a version of (2.4) for words. For given X and $\pi_1 \subset X^2$, we will define an up-down word to be a word w such that $w_{2i-1}w_{2i} \in \pi_1$ and $w_{2i}w_{2i+1} \notin \pi_1$ for all i. Likewise, we define a down-up word to be a word w where $w_{2i-1}w_{2i} \notin \pi_1$ and $w_{2i}w_{2i+1} \in \pi_1$ for all i. Let UDX^* be the set of up-down words, and let DUX^* be the set of down-up words. Then, we have the following theorem.

Theorem 2.8. *Given an alphabet X and $\pi \subset X^2$, then*

$$\sum_{w\in UDX^*} w = (1-\Psi((I+A^2)^{-1}A))^{-1}(1+\Psi((I+A^2)^{-1})), \tag{2.32}$$

and

$$\sum_{w\in DUX^*} w = 1-\Psi((I+A^2)^{-1}A) + \Psi((I+A^2)^{-1})(1-\Psi((I+A^2)^{-1}A))^{-1}(1+\Psi((I+A^2)^{-1})). \tag{2.33}$$

Proof. Let F_0 be the incidence matrix for the set of up-down words of odd length, and let F_1 be the incidence matrix for the set of up-down words of even length (not including ϵ). Then, F_0 and F_1 satisfy the following system of equations:

$$F_0 = I + F_1 B,$$
$$F_1 = F_0 A.$$

Replace $B = W - A$, and solve the system of equations, treating F_1W as a constant, to obtain

$$F_0 = (I + A^2)^{-1} + F_1W(I + A^2)^{-1},$$
$$F_1 = (I + A^2)^{-1}A + F_1W(I + A^2)^{-1}A.$$

Apply Ψ to both sides of the equation, recalling that $\Psi(FWG) = \Psi(F)\Psi(G)$, then solve for $\Psi(F_0)$ and $\Psi(F_1)$ to obtain

$$\Psi(F_0) = \Psi((I + A^2)^{-1}) + \Psi((I + A^2)^{-1}A)(1 - \Psi((I + A^2)^{-1}A))^{-1}\Psi((I + A^2)^{-1}),$$
$$\Psi(F_1) = \Psi((I + A^2)^{-1}A)(1 - \Psi((I + A^2)^{-1}A))^{-1}.$$

Equation (2.32) is then given by $1 + \Psi(F_0) + \Psi(F_1)$.

Next, let G_0 be the incidence matrix for the set of down-up words of odd length, and let G_1 be the incidence matrix for the set of down-up words of even length (not including ϵ). Then, G_0 and G_1 satisfy the following equations:

$$G_0 = I + G_1A,$$
$$G_1 = G_0B.$$

Following the same steps, solve for $\Psi(G_0)$ and $\Psi(G_1)$, then note that Equation (2.33) is given by $1 + \Psi(G_0) + \Psi(G_1)$. □

As a consequence of Theorem 2.8, we now obtain the generating function for down-up permutations, $\mathrm{DU}S_n$, defined in the obvious way.

Corollary 2.9. *The generating function for down-up permutations by inversions is given by*

$$\sum_{\sigma \in DUS_n} \frac{q^{\mathrm{inv}(\sigma)} z^n}{(q;q)_n} = \frac{\sin_q^2(z) + \cos_q^2(z)}{\cos_q(z)} + \frac{\sin_q(z)}{\cos_q(z)}. \tag{2.34}$$

Proof. Apply Equation (2.33) to compositions using $X = \mathbb{P}$ and $\pi_1 = \{w_1w_2 \in \mathbb{P}^2 : w_1 \leq w_2\}$. Expand the right-hand side of the equation as a series in A, then for each letter $i \in \mathbb{P}$, substitute $q^i(z/q)$ as in Subsection 2.8.2. Finally, apply Theorem 2.3 to obtain the result. □

It is interesting to note that when we substitute $z(1-q)$ for z and let q approach 1, we obtain $\sec(z)+\tan(z)$, the same as André's original result. This is because subtracting each letter from $n+1$ exchanges down-up and up-down permutations. However, this operation changes the number of inversions, which is why we get a result that is partially different from (2.4).

2.8.4 A q-analog of a distribution due to Kitaev

In this subsection, we use the pattern algebra method to obtain a q-analog of Kitaev's Theorem 30 in [40] that enumerates permutations that avoid consecutive $P=\{4312,4213,4123,3214,3124,2134\}$-patterns. We begin by deducing the relevant generating function on words.

Theorem 2.9. *If $\mathcal{F}\subset X^4$ such that $I(\mathcal{F})=AWB$, then*

$$\sum_{w\in X^*} y^{\mathcal{F}(w)}w=(1-\alpha\beta^{-1})^{-1},$$

where

$$\begin{aligned}
\alpha &= \Psi(U)-\Psi(BU)\gamma,\\
\beta &= 1+(y-1)^2\Psi(B^3U)-\Psi(U)\gamma,\\
\gamma &= (y-1)(1+(y-1)\Psi(B^2U+(y-1)B^3U))^{-1}\Psi(BU+(y-1)B^2U), \text{ and}\\
U &= (I-(y-1)^2B^4)^{-1}.
\end{aligned}$$

Proof. For all $f\in\mathcal{F}$, let $y_f=y$, and let $F_0=I(X)+I(C_{\mathcal{F}}(\mathbf{y}-\mathbf{1},X^*))$, where $C_{\mathcal{F}}$ is the cluster generating function as defined in Section 2.4. By Theorem 2.4, we then seek to compute $(1-\Psi(F_0))^{-1}$. Following our general strategy in Subsection 2.8.1, we construct a system of equations.

F_0 is the incidence matrix for the set of one-letter words and $\mathcal{F}$-clusters. These words are generated by arbitrary products of the incidence matrices AAB, ABB, and $AABB$. Thus, noting that each $\mathcal{F}$-factor needs to be counted by $y-1$, we could say that $F_0=I+F_0((y-1)AAB+(y-1)ABB+(y-1)^2AABB)$. However, in order to use our general strategy, we must write a system of equations where each unknown incidence matrix appears with only one A, B, or W to its right. Thus, we introduce a few extra unknown incidence matrices to obtain the system

$$
\begin{aligned}
F_0 &= I + (y-1)F_2B + (y-1)F_3B,\\
F_1 &= F_0A,\\
F_2 &= F_1A,\\
F_3 &= F_1B + (y-1)F_2B.
\end{aligned}
$$

Next, we substitute $A = W - B$ and, since there were no F_2A or F_3A terms in the system, immediately eliminate F_2 and F_3, giving

$$
\begin{aligned}
F_0 &= I + (y-1)F_1W(B + (y-1)B^2) - (y-1)^2F_1B^3,\\
F_1 &= F_0W - F_0B.
\end{aligned}
$$

Treating F_0W and F_1W as constants, solve for F_0 and F_1 to get

$$
\begin{aligned}
F_0 &= (I + (y-1)F_1W(B + (y-1)B^2) - (y-1)^2F_0WB^3)U,\\
F_1 &= (F_0W - B - (y-1)F_1W(B^2 + (y-1)B^3))U.
\end{aligned}
$$

Using the property $\Psi(FWG) = \Psi(F)\Psi(G)$, take Ψ of both sides to obtain

$$
\begin{aligned}
\Psi(F_0) &= \Psi(U) + (y-1)\Psi(F_1)\Psi(BU + (y-1)B^2U) - (y-1)^2\Psi(F_0)\Psi(B^3U),\\
\Psi(F_1) &= \Psi(F_0)\Psi(U) - \Psi(BU) - (y-1)\Psi(F_1)\Psi(B^2U + (y-1)B^3U).
\end{aligned}
$$

Finally, solve the system for $\Psi(F_0)$ and compute $(1 - \Psi(F_0))^{-1}$. □

The generating function $\mathcal{K}$ in the following result gives a q-analog of Theorem 30 in [40].

Corollary 2.10. *If* $P = \{1243, 1342, 1432, 2341, 2431, 3421\}$*, then*

$$
\sum_{n\geq 0}\sum_{\sigma\in S_n} \frac{y^{P_{no}(\sigma)}q^{\mathrm{inv}(\sigma)}z^n}{(q;q)_n} = \frac{\mathcal{K}}{1 - y + y(1 - \frac{z}{1-q})\mathcal{K}},
$$

where

$$
\mathcal{K} = \frac{e_q(z)e_q(-z) + \cos_q^2(z) + \sin_q^2(z) + 2e_q(-z)\cos_q(z) + (e_q(z) + e_q(-z))\sin_q(z)}{4e_q(-z)\cos_q(z)}.
$$

Proof. Let $T = \{4312, 4213, 4123, 3214, 3124, 2134\}$. Then, the "complement-reversal" bijection $c_n : S_n \to S_n$ given by $c_n(\sigma_1\sigma_2\cdots\sigma_n) = (n+1-\sigma_n)(n+1-\sigma_{n-1})\ldots(n+1-\sigma_1)$ shows that $\sum_{\sigma\in S_n} y^{P(\sigma)}q^{\mathrm{inv}(\sigma)} = \sum_{\sigma\in S_n} y^{T(\sigma)}q^{\mathrm{inv}(\sigma)}$. By Theorem 2.3,

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\frac{y^{T(\sigma)}q^{\mathrm{inv}(\sigma)}z^n}{(q;q)_n} = \sum_{n\geq 0}\sum_{w\in\mathbb{P}^n} y^{T(w)}q^{\mathrm{sum}(w)}(z/q)^n. \tag{2.35}$$

Using $\pi_1 = \{w_1w_2 \in \mathbb{P}^2 : w_1 > w_2\}$, note that B^n is equivalent to what A^n is when $\pi_1 = \{w_1w_2 \in \mathbb{P}^2 : w_1 \leq w_2\}$. The right-hand side of (2.35) may then be evaluated by applying (2.25) to the result in Theorem 2.9. Finally, set $y = 0$ and apply Corollary 2.1. □

Note that we could have just used (2.26) with Theorem 2.9 to obtain the result in Corollary 2.10. However, the resulting generating function would be much more difficult to manipulate. Alternatively, we could have substituted $B = W - A$ in proving Theorem 2.9 to obtain an equivalent result. However, the resulting system of equations would have required more work to solve. The complement-reversal bijection will frequently save us effort in this manner. This bijection also gives an explanation for why $\frac{\sin_q(z)}{\cos_q(z)}$ appears in both (2.4) and (2.34).

2.8.5 The twin peak problem revisited

Define a peak to be a single word from $\Psi(AB)$ and a twin peak to be one from $\Psi(ABAB)$. For $w \in X^*$, let peak(w) be the number of peaks in w and tpeak(w) be the number of twin peaks. Theorem 2.10 solves Q2 on words for peaks and twin peaks; as corollaries, we obtain corresponding results on permutations and DCCPs.

Theorem 2.10. *For any non-empty alphabet X,*

$$\sum_{w\in X^*} x^{\mathrm{peak}(w)}y^{\mathrm{tpeak}(w)}w = (1-\alpha\beta^{-1})^{-1},$$

where

$$\alpha = \Psi(U) + x\Psi(B^2U)(y - 1 - \gamma),$$
$$\beta = 1 + x(y - 1)\Psi(B^3U) - \Psi(BU)(x - 1 + x\gamma),$$
$$\gamma = x(y - 1)(1 - x(y - 1)\Psi(BU - B^3U))^{-1}\Psi(BU), \text{ and}$$
$$U = (I + (xy - 1)B^2 - x(y - 1)B^4)^{-1}.$$

Proof. Let $\mathcal{F}$ be the set of peaks and twin peaks, $y_f = x$ if $f \in \mathcal{F}$ is a peak and $y_f = y$ if f is a twin peak. Then, let $F_0 = I(X) + I(C_{\mathcal{F}}(\mathbf{y} - \mathbf{1}, X^*))$, where $C_{\mathcal{F}}$ is the cluster generating function of Section 2.4. By Theorem 2.4, we then seek to compute $(1 - \Psi(F_0))^{-1}$. Following our general strategy in Subsection 2.8.1, we deduce a system of equations.

As in Theorem 2.9, F_0 is the incidence matrix for the set of one-letter words and $\mathcal{F}$-clusters. In this case, those words are generated by arbitrary products of the incidence matrix AB, which are up-down words of odd length. As in Subsection 2.5.3, however, we must be careful to distinguish all of the possible ways to cover each word with peaks and twin peaks.

We will consider all possible ways to form prefixes of valid clusters, counting peaks or twin peaks when they are completed. Starting from F_0, we may only multiply by A, so let $F_1 = F_0A$. Starting from F_1, we may only multiply by B, but there are three ways we could count this transition: as part of a peak but not a twin peak, as part of a peak and a twin peak, or only as part of a twin peak. In the first case, we have completed an $\mathcal{F}$-cluster, so this contributes $(x - 1)F_1B$ to F_0. Let F_2 be the incidence matrix for cluster prefixes that are half-way through a twin peak. Then, the second and third cases contribute $(x-1)F_1B + F_1B = xF_1B$ to F_2. Starting from F_2, we may only multiply by A, so let $F_3 = F_2A$. Starting from F_3, we may only multiply by B, but there are four ways we could count this transition: any combination of being part or not being part of a peak, as well as being in the middle of a new twin peak or not. Combining these appropriately, we contribute $x(y-1)F_3B$ to F_0 and F_2, obtaining the following system:

$$\begin{aligned}
F_0 &= I + (x-1)F_1B + x(y-1)F_3B,\\
F_1 &= F_0A,\\
F_2 &= xF_1B + x(y-1)F_3B,\\
F_3 &= F_2A.
\end{aligned}$$

Substitute $A = W - B$ and eliminate F_1 and F_3 to obtain

$$\begin{aligned}
F_0 &= I + (x-1)F_0WB - (x-1)F_0B^2 + x(y-1)F_2WB - x(y-1)F_2B^2,\\
F_2 &= xF_0WB - xF_0B^2 + x(y-1)F_2WB - x(y-1)F_2B^2.
\end{aligned}$$

Treating F_0W and F_2W as constants, solve for F_0 and F_2:

$$\begin{aligned}
F_0 &= (I + x(y-1)B^2 + F_0W((x-1)B - x(y-1)B^3) + x(y-1)F_2WB)U,\\
F_2 &= x(-B^2 + F_0WB + (y-1)F_2W(B - B^3))U. \qquad (2.36)
\end{aligned}$$

Apply Ψ, recall that $\Psi(FWG) = \Psi(F)\Psi(G)$, and solve for $\Psi(F_0)$. □

The method of deriving the system of equations in Theorem 2.10 is quite general. It is equivalent to the process of constructing a non-deterministic finite automaton to read clusters and output $x-1$ when it matches a peak and $y-1$ when it matches a twin peak (technically, this is a non-deterministic Mealy machine). From this perspective, it is clear that our use of the cluster generating function is unnecessary. That is, we could just as well have designed a machine to read arbitrary sequences of As and Bs and output x or y when it reads a peak or twin peak. However, using the cluster generating function, when possible, will generally save quite a few terms in the resulting system of equations.

When deriving a result for permutations, we generally prefer to set $B = W - A$ instead of $A = W - B$, since $\Psi(A^n)$ has a nicer formula than $\Psi(B^n)$ with our usual substitutions. However, as the following paragraph shows, when we make a substitution that does not depend on the order of the letters, A and B may be switched in Theorem 2.10 without affecting the result.

Let $\mathcal{F}'$ be the set of words from $\Psi(BA+BABA)$, $y_f = x$ if f is from $\Psi(BA)$ and $y_f = y$ if f is from $\Psi(BABA)$. Let (w, ν, β) be an $\mathcal{F}$-cluster with $w \in X^n$.

Then, if $w_n w_1 \in \pi_1$, let $w' = w_2 w_3 \dots w_n w_1$; otherwise, let $w' = w_n w_1 w_2 \dots w_{n-1}$. Then, (w', ν', β) is an $\mathcal{F}'$-cluster, where ν' contains factors of w' with lengths and positions matching those in ν. Moreover, since no other word of the form $w_i \dots w_n w_1 \dots w_{i-1}$ is $\mathcal{F}'$-coverable, this is a bijection. Then, after substituting the letters in X, we find that $C_{\mathcal{F}}(\mathbf{y}, \mathbf{X}^*) = \mathbf{C}_{\mathcal{F}'}(\mathbf{y}, \mathbf{X}^*)$, since w' is a rearrangement of the letters of w. This fact gives another explanation for why $\frac{\sin_q(z)}{\cos_q(z)}$ appears in both (2.4) and (2.34).

Using the above fact, we obtain the following alternative solution to the problem considered in Subsection 2.5.3.

Corollary 2.11. *The generating function for permutations by peaks, twin peaks, and inversions is*

$$\sum_{n\geq 0}\sum_{\sigma\in S_n} \frac{q^{\mathrm{inv}(\sigma)} x^{\mathrm{peak}(\sigma)} y^{\mathrm{tpeak}(\sigma)} z^n}{(q;q)_n} = \frac{1}{1 - \frac{s_+ \sin_q(z\sqrt{r_+})}{2\sqrt{r_+}\cos_q(z\sqrt{r_+})} - \frac{s_- \sin_q(z\sqrt{r_-})}{2\sqrt{r_-}\cos_q(z\sqrt{r_-})}},$$

where $r_\pm = (xy - 1 \pm \sqrt{D})/2$, $s_\pm = 1 \pm (2x - xy - 1)/\sqrt{D}$, *and* $D = (xy+1)^2 - 4x$.

Proof. Let $X = \mathbb{P}$ and $\pi_1 = \{w_1 w_2 \in \mathbb{P}^2 : w_1 \leq w_2\}$. In Theorem 2.10, replace each B with an A, expand U as a series in A, substitute $q^i(z/q)$ for each letter $i \in \mathbb{P}$, then apply (2.25) and simplify. □

A peak in a CCP Q is a sequence of columns $Q_i Q_{i+1} Q_{i+2}$ such that $Q_i Q_{i+1}$ is an upper ascent or level and $Q_{i+1} Q_{i+2}$ is an upper descent. A twin peak is a sequence $Q_i Q_{i+1} Q_{i+2} Q_{i+3} Q_{i+4}$ such that $Q_i Q_{i+1} Q_{i+2}$ and $Q_{i+2} Q_{i+3} Q_{i+4}$ are peaks. Let peak(Q) and tpeak(Q) respectively denote the number of peaks and twin peaks in Q. Corollary 2.12 applies Theorem 2.10 to DCCPs.

Corollary 2.12. *The generating function for DCCPs by peaks, twin peaks, area, and length is*

$$\sum_{Q\in DCCP} x^{\mathrm{peak}(Q)} y^{\mathrm{tpeak}(Q)} q^{\mathrm{area}(Q)} z^{\ell(Q)} = \frac{\Psi(F_0 Z)}{1 - \Psi(F_0)},$$

where X, $\Psi(B^n)$, *and* $\Psi(B^n Z)$ *are given for DCCPs in Subsection 2.8.2,* $\Psi(F_0)$

and U are as in Theorem 2.10, and

$$\begin{aligned}\Psi(F_0Z) &= \Psi(UZ + x(y-1)B^2UZ) + \Psi(F_0)\Psi((x-1)BUZ - x(y-1)B^3UZ) \\ &\quad + x(y-1)\Psi(F_2)\Psi(BUZ), \\ \Psi(F_2) &= x(-\Psi(B^2U) + \Psi(F_0)\Psi(BU))(1 - x(y-1)\Psi(BU - B^3U))^{-1}.\end{aligned}$$

Proof. The version of Theorem 2.4 for words ending with a particular letter implies that we seek $(1-\Psi(F_0))^{-1}\Psi(F_0Z)$ in Theorem 2.10. To evaluate $\Psi(F_0Z)$, right-multiply both sides of the first equality in (2.36) by Z, apply Ψ, and solve. □

2.9 The Temperley method

In short, the Temperley [59] method involves introducing a variable to track the size of the last letter in a word, deriving a recurrence for the generating function using that variable, and then solving the recurrence. By streamlining Temperley's method, Bousquet-Mélou [4] developed a powerful tool for enumerating CCPs by practically any desired set of statistics. The Temperley method does not work at the word level, so it cannot derive results as general as the pattern algebra method. However, when restricted to CCPs, the Temperley method can be used to solve all of the same problems and more. In this final section, we illustrate the use of the Temperley method as modified in [4] in solving a few select consecutive pattern problems on CCPs, compositions, and permutations.

2.9.1 Two-column ridge patterns in CCPs

We begin our exposition of the Temperley method by stating a trivial modification of a result by Bousquet-Mélou [4].

Lemma 2.1. *If*

$$F(b) = zr(b) + zs(b)F(1) + zt(b)F(1/h) + zy(b)F(qb), \tag{2.37}$$

then $F(b)$ is given by

$$R(b) + \frac{S(b)[R(1) + T(1)R(\frac{1}{h}) - R(1)T(\frac{1}{h})] + T(b)[R(\frac{1}{h}) + R(1)S(\frac{1}{h}) - S(1)R(\frac{1}{h})]}{1 - S(1) - T(\frac{1}{h}) - T(1)S(\frac{1}{h}) + S(1)T(\frac{1}{h})},$$

where

$$R(b) = \sum_{n \geq 0} z^{n+1} y(b)y(qb)\cdots y(q^{n-1}b)r(q^n b),$$
$$S(b) = \sum_{n \geq 0} z^{n+1} y(b)y(qb)\cdots y(q^{n-1}b)s(q^n b), \text{ and}$$
$$T(b) = \sum_{n \geq 0} z^{n+1} y(b)y(qb)\cdots y(q^{n-1}b)t(q^n b).$$

Proof. The proof is essentially identical to Bousquet-Mélou's [4]. First, obtain an equation for $F(qb)$ by substituting qb for b in Equation (2.37), then substitute this expression for the $F(qb)$ in equation (2.37). Repeat this process for the $F(q^2b)$ that was introduced in this manner, then $F(q^3b)$, and so on, to obtain

$$F(b) = R(b) + S(b)F(1) + T(b)F(1/h). \tag{2.38}$$

Next, substitute $b = 1$ and $b = \frac{1}{h}$ in Equation (2.38) to obtain two more equations relating $F(b)$, $F(1)$, and $F(\frac{1}{h})$. Solve the system of equations for $F(b)$. □

The following theorem gives a recurrence for the generating function for CCPs by all six two-column patterns, perimeter, relative height, area, length, and height of the last column. It should be clear that the solution to this recurrence is given by Lemma 2.1.

Theorem 2.11. *If we define $F(b)$ by*

$$\sum_{Q \in CCP} a_u^{\mathrm{uasc}(Q)} b_u^{\mathrm{ulev}(Q)} d_u^{\mathrm{udes}(Q)} a_l^{\mathrm{lasc}(Q)} b_l^{\mathrm{llev}(Q)} d_l^{\mathrm{ldes}(Q)} c^{\mathrm{per}(Q)} q^{\mathrm{area}(Q)} h^{\mathrm{relh}(Q)} b^{|Q_{\ell(Q)}|} z^{\ell(Q)},$$

then,

$$
\begin{aligned}
F(b) = {} & \frac{zqc^4b}{1-qc^2bh} + zc^2\left(\frac{q^2b^2c^2a_ua_lh}{(1-qb)(1-qbc^2h)} + \frac{qba_lb_u}{1-qb} - \frac{qbd_ua_l}{(1-h)(1-qb)}\right)F(1) \\
& + zc^2\left(\frac{qbh}{1-qbh}d_ub_l + \frac{q^2b^2c^2h}{(1-qbh)(1-qbc^2)}d_ud_l + \frac{qbh^2}{(1-h)(1-qbh)}d_ua_l\right)F(\frac{1}{h}) \\
& + zc^2\left(\frac{qbc^2h}{1-qbc^2h}a_ub_l + \frac{q^2b^2c^4h}{(1-qbc^2)(1-qbc^2h)}a_ud_l - \frac{qbc^2h}{(1-qb)(1-qbc^2h)}a_ua_l\right. \\
& + b_ub_l + \frac{qbc^2}{1-qbc^2}b_ud_l - \frac{1}{1-qb}b_ua_l - \frac{1}{1-qb}d_ub_l - \frac{qbc^2}{(1-qbh)(1-qbc^2)}d_ud_l \\
& \left. + \frac{1}{(1-qb)(1-qbh)}d_ua_l\right)F(qb).
\end{aligned}
$$

Proof. Note that b keeps track of the height of the last column in P.

There are ten cases to consider. The first case is when P consists of one column. The other nine cases are those where the last pair of columns form each combination of upper descent, upper ascent, upper level, lower descent, lower ascent, and lower level.

1. single column: In this case, the top and bottom of the column contribute 2 to the perimeter. Then, each cell in the column contributes 2 more to the perimeter, 1 to the area, 1 to the relative height, and 1 to the height of the last column. There is at least one cell in the column, so these CCPs are counted by
$$\frac{zqc^4b}{1-qc^2bh}. \tag{2.39}$$
2. upper level, lower level: In this case, we start with any CCP Q, then add a new column with the same number of cells as the last column of Q. For each duplicated cell, we must add 1 to the area. Also, we must add 1 to the number of columns, 1 to the number of lower levels, 1 to the number of upper levels, and 2 to the perimeter. Therefore, these CCPs are counted by
$$zb_ub_lc^2F(qb). \tag{2.40}$$
3. upper ascent, lower level: In this case, we start with any CCP Q, then add a new column with the same number of cells as the last column of Q, then

add any positive number of cells above it. For each of the duplicated cells, we must add 1 to the area. Also, we must add 1 to the number of columns, 1 to the number of lower levels, 1 to the number of upper ascents, and 2 to the perimeter. For each additional cell, we must add 1 to the area, 1 to the relative height, 1 to the height of the last column, and 2 to the perimeter. Therefore, the CCPs in this case are counted by

$$za_u b_l c^2 \frac{qbc^2 h}{1 - qbc^2 h} F(qb). \tag{2.41}$$

4. upper level, lower descent: This case is very similar to the previous case. We get

$$zb_u d_l c^2 \frac{qbc^2}{1 - qbc^2} F(qb). \tag{2.42}$$

5. upper level, lower ascent: For this case, we start with any CCP Q and add a new column of any positive height, aligned to form an upper level. Then, we subtract the ones that are at least the same height as the last column of Q. Each cell in the new column adds 1 to the area and contributes 1 to the height of the last column. Also, we must add 1 to the number of columns, 1 to the number of upper levels, 1 to the number of lower ascents, and 2 to the perimeter. Combining the two cases, we get

$$zb_u a_l c^2 \left(\frac{qb}{1 - qb} F(1) - \frac{1}{1 - qb} F(qb) \right). \tag{2.43}$$

6. upper descent, lower level: This case is very similar to the previous case. However, instead of setting $b = 1$ to ignore the height of the last column of Q, we set $b = 1/h$ to subtract it from the relative height, then add 1 back for each cell in the new last column. We get

$$zd_u b_l c^2 \frac{qbhF(\frac{1}{h}) - F(qb)}{1 - qbh}. \tag{2.44}$$

7. upper ascent, lower descent: This case is straightforward. Given $Q \in$ CCP, we copy the last column, then add at least one cell below it and at least one cell above it. We get

$$za_u d_l c^2 \frac{q^2 b^2 c^4 h}{(1 - qbc^2)(1 - qbc^2 h)} F(qb). \tag{2.45}$$

8. upper ascent, lower ascent: For this case, we repeat the argument for upper level, lower ascent, then add any positive number of cells to the top of the last column to form an upper ascent. We get

$$za_ua_lc^2\left(\frac{q^2b^2c^2h}{(1-qb)(1-qbc^2h)}F(1)-\frac{qbc^2h}{(1-qb)(1-qbc^2h)}F(qb)\right). \quad (2.46)$$

9. upper descent, lower descent: This case is very similar to the previous case. We get

$$zd_ud_lc^2\left(\frac{q^2b^2c^2h}{(1-qbh)(1-qbc^2)}F(1/h)-\frac{qbc^2}{(1-qbh)(1-qbc^2)}F(qb)\right). \quad (2.47)$$

10. upper descent, lower ascent: For this case, we start with any CCP Q, add a column of any positive height, aligning it to form an upper descent. Then, we subtract the ones that form a lower level or descent. The first part is tricky, so we'll pause to address it. We start by choosing a cell to be the top of the new column. This gives us the following possible values for the change in relative height: $-1, -2, -3, \ldots, -n+1$, where n is the height of the last column of Q. We can accomplish this by removing n by setting $b = 1/h$, adding any positive number, then substracting the cases where the new relative height is at least as high as $\mathrm{relh}(Q)$. Therefore, this piece of our result is given by

$$\frac{qb(hF(\frac{1}{h})-F(1))}{1-h}.$$

Next, we append any number of cells below this one cell, giving

$$\frac{qb(hF(\frac{1}{h})-F(1))}{(1-h)(1-qb)}.$$

Now, we are ready to subtract the results with a lower level or descent. We have derived these results already, so we repeat that analysis and obtain

$$zd_ua_lc^2\left(\frac{qb(hF(\frac{1}{h})-F(1))}{(1-h)(1-qb)}-\frac{qbhF(\frac{1}{h})-F(qb)}{(1-qbh)(1-qb)}\right). \quad (2.48)$$

Adding Equations (2.39) through (2.48) gives the result. □

2.9.2 CCPs by peaks

The generating functions for column-convex polyominoes by various upper ridge patterns, area, width, and relative height are always rational functions. Theorem 2.12 provides an example. Let peak(Q) be as in Corollary 2.12.

Theorem 2.12. *The generating function for CCPs by peaks, area, and length is*

$$\sum_{Q\in\mathrm{CCP}} y^{\mathrm{peak}(Q)} q^{\mathrm{area}(Q)} z^{\ell(Q)} = \frac{(\frac{zq}{1-q} + \frac{2z^2q^3}{(1-q)^3})(1 + \frac{2zq}{(1-q)^2})}{(1 - \frac{zq^2}{(1-q)^2})(1 + \frac{zq}{(1-q)^2}) - \frac{2yz^2q^3}{(1-q)^4}}.$$

Proof. Define $F(b) = \sum_{Q\in\mathrm{CCP}} y^{\mathrm{peak}(Q)} q^{\mathrm{area}(Q)} z^{\ell(Q)} h^{\mathrm{relh}(Q)} b^{|Q_{\ell(Q)}|}$. Let $F_0(b)$ be the restriction of $F(b)$ to Q such that $\ell(Q) \geq 2$ and the last pair of columns form an upper level or ascent, and let $F_1(b)$ be the restriction to the rest. Then, we seek $F(1) = F_0(1) + F_1(1)$, with $h = 1$. As with the pattern algebra method, we derive a system of equations that F_0 and F_1 must satisfy. However, in this case, we will not be able to use the cluster generating function effectively, since CCP cannot essentially be written as X^* for some alphabet X.

The analysis in Theorem 2.11 makes most of the work straightforward. The CCPs contributing to F_0 are obtained by adding a column with an upper ascent or level to any CCP. The CCPs contributing to F_1 fall into three cases: they have one column, they end with a peak, or they end with two upper descents. The CCPs ending in a peak come from CCPs contributing to F_0 followed by an upper descent. The CCPs ending in two upper descents come from CCPs contributing to F_1 followed by an upper descent. Thus, we obtain the following system of equations:

$$F_0(b) = \frac{zqb}{(1-qb)(1-qbh)}(F_0(1) + F_1(1)),$$
$$F_1(b) = \frac{zqbh}{1-qbh} + \frac{zqb}{(1-qb)(1-h)}(yhF_0(1/h) - yF_0(1) + hF_1(1/h) - F_1(1)).$$

Substituting $b = 1$ and $b = 1/h$ into the above system gives four linear equations in the four unknown functions $F_0(1)$, $F_1(1)$, $F_0(1/h)$, and $F_1(1/h)$. Solve this system, compute $F_0(1) + F_1(1)$, and set $h = 1$. $\square$

2.9.3 Compositions by mesas

A mesa in a composition w is a subsequence $w_i w_{i+1} w_{i+2} w_{i+3}$ satisfying $w_i < w_{i+1} = w_{i+2} > w_{i+3}$. Let mesa$(w)$ be the number of mesas in w.

Corollary 2.13. *The generating function for compositions by mesas is*

$$\sum_{n\geq 0}\sum_{w\in\mathbb{P}^n} y^{\mathrm{mesa}(w)} q^{\mathrm{sum}(w)}(z/q)^n = \left(1-\frac{\displaystyle\sum_{n\geq 0}\frac{(-1)^n(y-1)^n z^{3n+1}q^{3n(n+1)/2}}{(q;q^3)_{n+1}(q^3;q^3)_n}}{\displaystyle\sum_{n\geq 0}\frac{(-1)^n(y-1)^n z^{3n}q^{n(3n+1)/2}}{(q^2;q^3)_n(q^3;q^3)_n}}\right)^{-1}.$$

Proof. Again, we deduce a system of equations. We will define $F_0(b)$ to be a generating function for length-one words and words that can be covered by overlapping mesas. If $w \in \mathbb{P}^n$ is such a word, then w will contribute a factor of $(y-1)$ for each mesa as well as $q^{\mathrm{sum}(w)}(z/q)^n b^{w_n - 1}$ to $F_0(b)$. Note that b essentially tracks the last letter of w, the hallmark of the Temperley method. Then, by Theorem 2.4, we seek $(1-F_0(1))^{-1}$.

For our convenience, we will say that $w \in F_0(b)$ if w contributes a non-zero term to $F_0(b)$. We will consider all ways to form prefixes of the words $w \in F_0(b)$. The only way to extend such a w by one letter is to add an ascent. Thus, we let $F_1(b)$ be the generating function for $w \in F_0(b)$ with an ascent added. Similarly, the only way to extend a $w \in F_1(b)$ is by appending a level. Thus, we let $F_2(b)$ be the generating function for $w \in F_1(b)$ with a level added. Finally, the only way to extend a $w \in F_2(b)$ is by appending a descent, completing a new mesa. We may then deduce relationships between $F_0(b)$, $F_1(b)$, and $F_2(b)$ as we did with CCPs to obtain the system

$$\begin{aligned} F_0(b) &= \frac{z}{1-qb} + \frac{(y-1)z}{1-qb}(F_2(1) - F_2(qb)), \\ F_1(b) &= \frac{zqb}{1-qb}F_0(qb), \\ F_2(b) &= zF_1(qb). \end{aligned}$$

Elimination of $F_1(b)$ and $F_2(b)$ gives

$$F_0(b) = \frac{z}{1-qb} + \frac{(y-1)z^3q^2}{(1-q^2)(1-qb)}F_0(q^2) - \frac{(y-1)z^3q^3b}{(1-qb)(1-q^3b)}F_0(q^3b).$$

At this point, we could use Lemma 2.1 to write a formula for $F_0(b)$. However, it is easy enough to solve this equation directly. Eliminating $F_0(q^3b)$ by repeated substitution, we obtain

$$F_0(b) = \left(z + \frac{(y-1)z^3q^2}{1-q^2}F_0(q^2)\right)\sum_{n\geq 0}\frac{(-1)^n(y-1)^n z^{3n}q^{3n(n+1)/2}b^n}{(qb;q^3)_{n+1}(q^3b;q^3)_n}. \qquad (2.49)$$

Substituting $b = q^2$ and some algebra yields

$$z + \frac{(y-1)z^3q^2}{1-q^2}F_0(q^2) = \frac{z}{\displaystyle\sum_{n\geq 0}\frac{(-1)^n(y-1)^n z^{3n}q^{n(3n+1)/2}}{(q^2;q^3)_n(q^3;q^3)_n}}.$$

Substituting the above into (2.49) and setting $b = 1$ completes the proof. □

2.9.4 Permutations by (i, m)-maxima revisited

The modified Temperley method in [4] may be applied to obtain an alternative version of the generating function in Corollary 2.5. In fact, for fixed i, d, we may apply the Temperley method to track all of the different (i, d)-peaks individually. However, in the following corollary, we will only address the (i, m)-maxima, namely the $(i, 2)$-peaks. This will be sufficient to demonstrate the method.

Theorem 2.13. *If $i \geq 2$ and $1 \leq m \leq i$, then the generating function for permutations by (i, m)-maxima and inversions,*

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\left(\prod_{m=1}^{i} y_m^{p_{(m)}(\sigma)}\right)\frac{q^{\mathrm{inv}(\sigma)}z^n}{(q;q)_n},$$

is given by

$$\left(1 - \frac{\displaystyle\sum_{n\geq 0}\frac{z^{in+1}}{1-q^{in+1}}\prod_{k=0}^{n-1}T(q^{ik})}{\displaystyle 1 - \frac{y_i - 1}{(q;q)_{i-1}}\sum_{n\geq 1}\frac{z^{in}}{1-q^{in}}\prod_{k=1}^{n-1}T(q^{ik-1})}\right)^{-1},$$

where

$$T(b) = \sum_{m=1}^{i-1}\frac{(y_m-1)q^m}{(q;q)_m(q^{m+1}b;q)_{i-m}} - \frac{y_i - 1}{(q;q)_{i-1}(1-qb)}.$$

Proof. Let $\mathcal{F} = \cup_{p\in P_i}\mathcal{F}_p$. As in Theorem 2.13, we define $F(b)$ to be the generating function for $\mathcal{F}$-clusters and length-one words. If $w \in \mathbb{P}^n$ is such a word, then it contributes $q^{\mathrm{sum}(w)}(z/q)^n b^{w_n-1}$ times a factor of $y_m - 1$ for each (i,m)-maxima in w to $F(b)$. By Theorem 2.6, we seek $(1-F(1))^{-1}$. As usual, we derive a recurrence for $F(b)$. Note that (i,m)-maxima can never overlap except at the first or last letter, so the only way to extend an $\mathcal{F}$-cluster is to append an (i,m)-maximum. For $1 \le m < i$, the generating function for words from $F(b)$ with an (i,m)-maximum appended is

$$\frac{(y_m-1)q^m z^i}{(q;q)_m (q^{m+1}b;q)_{i-m}} F(q^i b). \tag{2.50}$$

The generating function for words from $F(b)$ with an extra (i,i)-maximum appended is

$$\frac{(y_i-1)z^i}{(q;q)_{i-1}(1-qb)}(F(q^{i-1}) - F(q^i b)). \tag{2.51}$$

Summing (2.50) over m and adding in (2.51) and the one letter words, we obtain a recurrence for $F(b)$:

$$F(b) = \frac{z}{1-qb} + \frac{(y_i-1)z^i}{(q;q)_{i-1}(1-qb)} F(q^{i-1}) + z^i T(b) F(q^i b).$$

We solve this equation in the usual way. Elimination of $F(q^i b)$ leads to

$$F(b) = \left(z + \frac{(y_i-1)z^i}{(q;q)_{i-1}} F(q^{i-1})\right) \sum_{n\ge 0} \frac{z^{ni}}{1-q^{1+ni}b} \prod_{k=0}^{n-1} T(q^{ki}b). \tag{2.52}$$

Substituting $b = q^{i-1}$ and some algebra gives

$$z + \frac{(y_i-1)z^i}{(q;q)_{i-1}} F(q^{i-1}) = z\left(1 - \frac{y_i-1}{(q;q)_{i-1}} \sum_{n\ge 1} \frac{z^{ni}}{1-q^{ni}} \prod_{k=1}^{n-1} T(q^{ki-1})\right)^{-1}.$$

Substitute into (2.52) and set $b = 1$ to obtain the final result. □

2.9.5 Permutations by left-to-right minima and patterns beginning with 1

Given a permutation $\sigma \in S_n$, we say that σ_i is a left-to-right minimum if $\sigma_j > \sigma_i$ for all $j < i$. Let $\mathrm{ltrmin}(\sigma)$ denote the number of left-to-right minima in σ. If we define left-to-right minima in compositions through Fédou's bijection, it turns out that w_i is a left-to-right minimum in w if $w_j > w_i$ for all $j < i$. We'll define $\mathrm{ltrmin}(w)$ accordingly. Theorem 2.14 gives a method of computing generating functions by ltrmin and consecutive patterns that start with 1.

Theorem 2.14. *Let $P \subset \bigcup_{m\geq 1} S_m$ such that $p_1 = 1$ for each $p \in P$. Define $f(z)$ to be the generating function for compositions starting with 1 by the consecutive patterns in P, sum, and length, i.e.*

$$f(z) = \sum_{n\geq 1} \sum_{w\in\{1\}\mathbb{P}^{n-1}} \left(\prod_{p\in P} y_p^{p(w)}\right) q^{\mathrm{sum}(w)} z^n. \tag{2.53}$$

Then,

$$\sum_{n\geq 0} \sum_{w\in\mathbb{P}^n} \left(\prod_{p\in P} y_p^{p(w)}\right) x^{\mathrm{ltrmin}(w)} q^{\mathrm{sum}(w)} z^n = \prod_{i\geq 0} (1 + xf(q^i z)). \tag{2.54}$$

Proof. Given a composition w, break w into sub-words before each left-to-right minimum. Then, each sub-word begins with its smallest letter, and the first letters strictly decrease between sub-words. Therefore, for each $k \in \mathbb{P}$, w has either 1 or 0 sub-words beginning with k, and any set of such sub-words may be uniquely combined to give a corresponding w. A sub-word beginning with the letter $k \in \mathbb{P}$ may be obtained from one beginning with 1 by adding $k-1$ to each letter, and this leaves the relative order of the letters unchanged, thus preserving the occurrences of each $p \in P$. Therefore, the generating function for sub-words beginning with k is $f(q^{k-1}z)$. Finally, since each $p \in P$ begins with 1 and the first letter of each sub-word is smaller than all letters before it, no consecutive occurrence of p can involve letters of w in more than one sub-word. Thus, $p(w)$ is the sum of the number of consecutive occurrences of p over the sub-words of w. The result follows. □

Theorem 2.15 gives an example of the use of Theorem 2.14.

Theorem 2.15. *Let $f(z)$ be given by*

$$f(z) = \frac{z\sum_{n\geq 0} \frac{q^n z^n}{(1-q)^n (q;q^2)_n}}{1 - z\sum_{n\geq 0} \frac{q^n z^n}{(1-q)^n (q;q^2)_{n+1}}}. \tag{2.55}$$

Then,

$$\sum_{n\geq 0} \sum_{\sigma\in S_n} x^{\mathrm{ltrmin}(\sigma)} y^{132(\sigma)} \frac{q^{\mathrm{inv}(\sigma)} z^n}{(q;q)_n} = \prod_{i\geq 0} (1 + xf(q^i z)).$$

Proof. Clearly, in order to obtain this result, we must show that $f(z)$ is the generating function for compositions beginning with 1 by consecutive 132-patterns. To that end, define $F(b)$ to be the generating function for compositions w beginning with 1 by $q^{\text{sum}(w)}(z/q)^{\ell(w)}b^{w_{\ell(w)}-1}$, and we will derive an equation similar to those earlier in this section.

Given a word w, we may extend it in one of two ways: by adding a single letter, or by adding two letters to construct a consecutive 132-pattern at the end of the word. However, this process will create some words in more than one way. For example, the word 132 may be created starting with 1 by adding a 3, then adding a 2, or it may be created by adding 32 at once. Thus, when we add two letters at once, we will count those 132-patterns with $y-1$, so the combined contribution from the two methods gives a factor of y. The only word not obtained in one of these two ways is 1. Thus, we obtain the equation

$$F(b) = z + \frac{z}{1-qb}F(1) + \frac{qz}{(1-q)(1-qb)}F(q^2b). \tag{2.56}$$

We solve Equation (2.56) in the usual way. Eliminating $F(q^2b)$, we obtain

$$F(b) = z\sum_{n\geq 0}\frac{q^n z^n}{(1-q)^n(qb;q^2)_n} + zF(1)\sum_{n\geq 0}\frac{q^n z^n}{(1-q)^n(qb;q^2)_{n+1}}.$$

Substituting $b=1$ and solving for $f(1)$, we obtain the result in Equation (2.55).

□

The method of deriving Equation (2.56) used in Theorem 2.15 is equivalent in spirit to using the cluster generating function. In fact, we could have used this method in all of the examples of Sections 2.8 and 2.9, including Theorem 2.12, where we could not use the cluster generating function. This method is particularly valuable for CCPs and other words that are not essentially X^* for some alphabet X.

Jones and Remmel [36] showed that if $f(z)$ is the exponential generating function for permutations by patterns beginning with 1, then the exponential generating function for the same patterns and left-to-right minima is given by $(f(z))^x = e^{x\log(f(z))}$. Our final theorem of this chapter generalizes this result to include inversions by showing that we can obtain the generating function for compositions starting with 1 from the generating function for all compositions when we're tracking patterns that begin with 1.

Theorem 2.16. *For* $P \subset \bigcup_{m\geq 1} S_m$ *such that each* $p \in P$ *begins with* 1*, define* $f(z)$ *by*

$$f(z) = \sum_{n\geq 0} \sum_{w\in \mathbb{P}^n} \left(\prod_{p\in P} y_p^{p(w)} \right) q^{\mathrm{sum}(w)} (z/q)^n.$$

Then,

$$\sum_{n\geq 1} \sum_{w\in \{1\}\mathbb{P}^{n-1}} \left(\prod_{p\in P} y_p^{p(w)} \right) q^{\mathrm{sum}(w)} (z/q)^n = \frac{f(z)}{f(qz)} - 1.$$

Proof. By the argument in Section 2.5, if $F(z) = \frac{z}{1-q} + D_P(\mathbf{y}-\mathbf{1}, \mathbf{z}/\mathbf{q})$, then $f(z) = (1-F(z))^{-1}$. Therefore, we can solve for $F(z)$ in terms of $f(z)$, namely $F(z) = 1 - (f(z))^{-1}$. Now, we wish to apply Theorem 2.4 again to obtain the generating function for words beginning with a restricted subset, *i.e.* $\{1\}$. In order to apply the theorem, we need the generating function for clusters and one-letter words beginning with 1. To get this, we start with $F(z)$ and subtract the words that begin with a letter greater than 1. Thus, the clusters and one-letter words beginning with 1 are counted by $F(z) - F(qz)$. Finally, by Theorem 2.4, we seek $(F(z) - F(qz))f(z)$. The result follows. □

Chapter 2, in part, has been published in the Electronic Journal of Combinatorics. Rawlings, Don; Tiefenbruck, Mark. "Consecutive Patterns: From Permutations to Polyominoes and Back", Electronic Journal of Combinatorics, Volume 17, 2010. The dissertation author was an author of this paper.

Chapter 3

Generalizations of the Major Index

3.1 Introduction

Gessel gave a generating function for descents, major index, and inversions both in his thesis [27] and in a paper coauthored with Garsia [25]. Their result is stated below.

Theorem 3.1 (Garsia and Gessel)**.**

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\frac{x^{\mathrm{des}(\sigma)}u^{\mathrm{maj}(\sigma)}q^{\mathrm{inv}(\sigma)}z^n}{(x;u)_{n+1}(q;q)_n}=\sum_{k\geq 0}x^k\,\mathrm{e}_q(zu^k)\,\mathrm{e}_q(zu^{k-1})\cdots\mathrm{e}_q(z). \tag{3.1}$$

Let $(a,b;p,q)_n = (a-b)(ap-bq)\cdots(ap^{n-1}-bq^{n-1})$ be the p,q-shifted factorial of n. Then it is easy to see that

$$\frac{p^{n+\mathrm{coinv}(\sigma)}q^{\mathrm{inv}(\sigma)}}{(p,q;p,q)_n}=\frac{(q/p)^{\mathrm{inv}(\sigma)}}{(q/p;q/p)_n}$$

for all $\sigma\in S_n$. Therefore, substituting q/p for q and z/p for z in (3.1) (or the results in Chapter 2) gives the p,q-exponential generating function that includes co-inversions. We can make similar substitutions to include the co-major index and ascent number. For clarity and brevity, we thus omit these statistics from the results in this chapter while noting that it is straightforward to include them.

As in Chapter 2, in order to use words to study questions about permutations, we will use Fédou's insertion-shift bijection. As a consequence of Equation (2.12), we see that ∇_n preserves descents. That is, $w_i > w_{i+1}$ if and only if $\sigma_i > \sigma_{i+1}$. Therefore, ∇_n also preserves the major index. Thus, we may apply Theorem 2.3 to enumeration problems involving the major index.

Several authors have extended the Garsia-Gessel formula (3.1) to other groups. For example, Reiner [54] gave a B_n version of the Garsia-Gessel formula where B_n is the hyperoctahedral group, and Mendes and Remmel [46] gave versions of the Garsia-Gessel formula for groups that are the wreath product of a cyclic group C_k and the symmetric group.

Fuller and Remmel [24] studied analogues of Theorem 3.1 in compositions. Given a composition $w \in \mathbb{P}^n$, let z^w be the monomial $z_{w_1} \cdots z_{w_n}$. Since compositions can have repeated entries, it is natural to have analogues of des and maj where we replace $>$ by $\geq$ or $=$ in the definition of des and maj. That is, we let $\mathrm{Des}(w) = \{i : w_i > w_{i+1}\}$, $\mathrm{WDes}(w) = \{i : w_i \geq w_{i+1}\}$, and $\mathrm{Lev}(w) = \{i : w_i = w_{i+1}\}$. Then we define

$$\begin{aligned}
\mathrm{des}(w) = |\,\mathrm{Des}(w)| \text{ and } \mathrm{maj}(w) &= \sum_{i \in \mathrm{Des}(w)} i, \\
\mathrm{wdes}(w) = |\,\mathrm{WDes}(w)| \text{ and } \mathrm{wmaj}(w) &= \sum_{i \in \mathrm{WDes}(w)} i, \text{ and} \\
\mathrm{lev}(w) = |\,\mathrm{Lev}(w)| \text{ and } \mathrm{levmaj}(w) &= \sum_{i \in \mathrm{Lev}(w)} i.
\end{aligned}$$

Fuller and Remmel [24] proved the following results.

Theorem 3.2 (Fuller and Remmel)**.**

$$\sum_{n \geq 0} \sum_{w \in N^n} \frac{x^{\mathrm{des}(w)} u^{\mathrm{maj}(w)} z^w}{(x;u)_{n+1}} = \sum_{k \geq 0} \frac{x^k}{\prod_{i \geq 1} (z_i; u)_{k+1}}.$$

Theorem 3.3 (Fuller and Remmel)**.**

$$\sum_{n \geq 0} \sum_{w \in N^n} \frac{x^{\mathrm{wdes}(w)} u^{\mathrm{wmaj}(w)} z^w}{(x;u)_{n+1}} = \sum_{k \geq 0} x^k \prod_{i \geq 1} (-z_i; u)_{k+1}.$$

Theorem 3.4 (Fuller and Remmel)**.**

$$\sum_{n\geq 0}\sum_{w\in N^n}\frac{x^{\mathrm{lev}(w)}u^{\mathrm{levmaj}(w)}z^w}{(x;u)_{n+1}}=\sum_{k\geq 0}\frac{x^k}{\prod_{j=0}^k(\sum_{n\geq 0}(-u^j)^n p_n)},$$

where $p_n = p_n(z_1, z_2, \ldots) = \sum_{i\geq 1} z_i^n$ *is the power symmetric function.*

The goal of this chapter is to prove a common generalization of the results of Fuller and Remmel [21], and then we will show how we can use this result to not only recover the results of Garsia and Gessel [25], Reiner [54], Mendes and Remmel [46], and others but also to prove several new analogues of (3.1).

The outline of the chapter is as follows. In Section 3.2, we shall state and prove our main theorem that generalizes (3.1) as well as Fuller and Remmel's results. In Section 3.3, we will present many extensions of this theorem and apply it to a variety of combinatorial objects and variations on the major index statistic.

3.2 The main theorem

In this section, we will derive a general version of Garsia and Gessel's result. As with the Pattern Algebra in Section 2.8, we let X be an alphabet, with $A \subset X^2$. We will typically think of the case $X = \mathbb{P}$ and $A = \{w_1w_2 : w_1 \leq w_2\}$, but our results will hold in general. For $w \in X^n$, we can then define the descent number and major index with respect to A:

$$\mathrm{des}_A(w) = |\{i : w_iw_{i+1} \notin A\}| \text{ and}$$
$$\mathrm{maj}_A(w) = \sum_{i=1}^{n-1} i\cdot\chi(w_iw_{i+1}\notin A)$$

When the meaning is clear, we will drop the subscript A. Further, let $A_n = \{w \in X^n : w_iw_{i+1} \in A \text{ for all } i \in [n-1]\} = \{w \in X^n : \mathrm{des}_A(w) = 0\}$, $a_n = \sum_{w\in A_n} w$, and $A(z) = a_0 + a_1z + a_2z^2 + \cdots$. We will let $B = X^2 \setminus A$ and define B_n, b_n, and $B(z)$ accordingly. We can now state a general result.

Theorem 3.5.

$$\sum_{n\geq 0}\sum_{w\in X^n}\frac{x^{\mathrm{des}_A(w)}u^{\mathrm{maj}_A(w)}}{(x;u)_{n+1}}w=\sum_{k\geq 0}x^kA(u^k)A(u^{k-1})\cdots A(1). \tag{3.2}$$

Proof. Fix $w \in X^n$, and define the partition $\mu(w)$ such that if $w_i w_{i+1} \in B$, then $\mu(w)$ has a part of size i. Clearly $\mathrm{sum}(\mu(w)) = \mathrm{maj}(w)$ and $\ell(\mu(w)) = \mathrm{des}(w)$. Define $\bar{\Lambda}^{\leq n}$ to be the set of weak partitions with parts less than or equal to n. By the same method as for (1.1), we find that

$$\sum_{\lambda \in \bar{\Lambda}^{\leq n}} x^{\ell(\lambda)} u^{\mathrm{sum}(\lambda)} = (1-x)^{-1}(1-xu)^{-1} \cdots (1-xu^n)^{-1} = \frac{1}{(x;u)_{n+1}}.$$

Then, for $\lambda \in \bar{\Lambda}^{\leq n}$, form the weak partition $\lambda + \mu(w)$ by merging λ and $\mu(w)$; that is, for all i, $\lambda + \mu(w)$ will have one part of size i for each such part in either λ or $\mu(w)$. It is then clear that the left side of (3.2) is

$$\sum_{n \geq 0} \sum_{w \in X^n} \sum_{\lambda \in \bar{\Lambda}^{\leq n}} x^{\ell(\lambda+\mu(w))} u^{\mathrm{sum}(\lambda+\mu(w))} w. \tag{3.3}$$

Now, fix $\nu = \lambda + \mu(w)$ for some $\lambda \in \bar{\Lambda}^{\leq n}$ and sort the letters of ν in increasing order. We can use ν to factor w into sub-words as $w = w^{(0)} w^{(1)} \cdots w^{(k)}$, where $w^{(i)}$ starts at position $\nu_i + 1$, using $\nu_0 = 0$, and extends as far as possible without overlapping. For example, Figure 3.2 displays ν and the corresponding sub-words of $w = 021044203$ when λ is the empty partition. Note that the descents of w

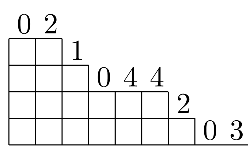


Figure 3.1: ν and $w^{(i)}$ for $w = 021044203$ and $\lambda = \epsilon$

have been broken into distinct sub-words, so that each sub-word has no descents. Figure 3.2 shows the sub-words when $\lambda = 001169$. Clearly $\mathrm{des}(w^{(j)}) = 0$ for each j, regardless of λ, since $\mu(w)$ (and thus ν) contains a part i if $w_i w_{i+1} \in B$.

Now, fix k and choose $w^{(i)}$ such that $\mathrm{des}(w^{(i)}) = 0$ for $0 \leq i \leq k$. Then, one can compute the corresponding ν by letting $\nu_i = \ell(w^{(0)} \cdots w^{(i-1)})$ for $1 \leq i \leq k$. Thus, the set of ν with k parts is in bijection with the set of k-tuples of words with no descents. Further, we see that each letter of $w^{(i)}$ has $k-i$ squares beneath it in the corresponding diagram, so $\mathrm{sum}(\nu) = \sum_{i=0}^{k}(k-i)\ell(w^{(i)})$. It is thus clear that (3.3) is also equal to the right side of (3.2).

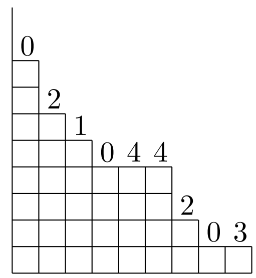


Figure 3.2: ν and $w^{(i)}$ for $w = 021024203$ and $\lambda = 001169$

□

In the case of $X = \mathbb{P}$ and $A = \{w_1w_2 : w_1 \leq w_2\}$, we see that A_n is the set of partitions of length n, sorted in increasing order. Thus, if we substitute q^i for the letter $i \in \mathbb{P}$, then $A(z)$ becomes $\mathrm{e}_q(z)$, so that Theorems 3.5 and 2.3 imply (3.1). If we vary the choice of A and substitute z_iq^i for the letter i, Theorem 3.5 also implies the results of Fuller and Remmel.

3.3 Examples and extensions

In this section, we will provide many examples utilizing Theorem 3.5. Some of our results will translate the notion of major index to other common combinatorial objects such as colored permutations or directed column-convex polyominoes. Other results will modify the notion of major index to sum the positions of features such as alternating descents or descents that occur at specified positions in the permutation.

3.3.1 Colored permutations

A natural extension of the major index is to the set of colored permutations. If you have c colors, a colored permutation is a permutation such that each letter has been assigned a color, indexed by the set $[c] = \{1, 2, \ldots, c\}$. Colored permutations of length n can be thought of as elements of the set $[c]^n \times S_n$. When $c = 2$, the colored permutations are closely related to the signed permutations B_n, for which (3.1) has been treated separately (and distinctly) by Reiner [54] and by Mendes and Remmel [46]. We will give yet another interpretation.

Let $\pi = (v, \sigma) \in [c]^n \times S_n$. We will be interested in the generating function

$$\sum_{n\geq 0} \sum_{\pi \in [c]^n \times S_n} \frac{x^{\mathrm{des}(\pi)} u^{\mathrm{maj}(\pi)} q^{\mathrm{inv}(\pi)} z^v}{(x;u)_{n+1}(q;q)_n}. \tag{3.4}$$

However, we need to define the various colored permutation statistics.

We could define des(π), maj(π), and inv(π) to be des(σ), maj(σ), and inv(σ), respectively. However, in that case, (3.4) can be obtained from (3.1) merely by replacing z with $z_1 + z_2 + \cdots + z_c$, since the colors do not affect any of the statistics. Instead, we will use a lexicographic order on the letters of π, so that $\binom{v_i}{\sigma_i} > \binom{v_j}{\sigma_j}$ if $v_i > v_j$ or if $v_i = v_j$ and $\sigma_i > \sigma_j$.

Now, if we define des(π), maj(π), and inv(π) analogously to permutations, then (3.4) is still obtained from (3.1) by substituting $z_1 + \cdots + z_c$ for z. This is because we can replace the underlying permutation σ with $\tau \in S_n$ such that $\tau_i > \tau_j$ if and only if $\pi_i > \pi_j$. This replacement does not affect the number of times each $\sigma \in S_n$ appears with a particular value of z^v or the relative order of the letters in π. However, now des(π) = des(τ), maj(π) = maj(τ), and inv(π) = inv(τ), as in the previous paragraph.

We will instead seek the generating function

$$\sum_{n\geq 0} \sum_{\pi \in [c]^n \times S_n} \frac{x^{\mathrm{des}(\pi)} u^{\mathrm{maj}(\pi)} q^{\mathrm{inv}(\sigma)} z^v}{(x;u)_{n+1}(q;q)_n}, \tag{3.5}$$

where we track the inversions of the underlying permutation σ. First, we will prove a necessary extension to Theorem 3.5.

Theorem 3.6. *Let X be an alphabet with $A \subset X^2$. Let $Y = [c] \times X$ and define $C = \{\binom{v_1 v_2}{w_1 w_2} \in Y^2 : v_1 < v_2$ or $v_1 = v_2$ and $w_1 w_2 \in A\}$. If $C^{(i)} = \{\binom{i\ i}{w_1 w_2} : w_1 w_2 \in A\}$, then*

$$C(z) = C^{(1)}(z) C^{(2)}(z) \cdots C^{(c)}(z),$$

and thus

$$\sum_{n\geq 0} \sum_{w \in Y^n} \frac{x^{\mathrm{des}_C(w)} u^{\mathrm{maj}_C(w)}}{(x;u)_{n+1}} w = \sum_{k\geq 0} x^k C(u^k) C(u^{k-1}) \cdots C(1).$$

Proof. By definition, any word $\binom{v}{w}$ with no C-descents must have the letters of v weakly increasing. Further, for letters with the same element of $[c]$, the corresponding sub-word of w must have no A-descents. Thus, the first part of the theorem is clear. The second is a direct application of Theorem 3.5. $\square$

Corollary 3.1.

$$\sum_{n\geq 0}\sum_{\pi=(v,\sigma)\in[c]^n\times S_n}\frac{x^{\mathrm{des}(\pi)}u^{\mathrm{maj}(\pi)}q^{\mathrm{inv}(\sigma)}z^v}{(x;u)_{n+1}(q;q)_n}=\sum_{k\geq 0}x^k\prod_{i=1}^{c}\mathrm{e}_q(z_iu^k)\,\mathrm{e}_q(z_iu^{k-1})\cdots\mathrm{e}_q(z_i).$$

Proof. In order to use Theorem 3.6 to find (3.5), we must modify Fédou's bijection to accommodate colored permutations. We do it in the obvious way, assigning the color of σ_i to w_i. That is, for $\binom{v}{\sigma}\in[c]^n\times S_n$ and $\lambda\in\Lambda_n$, we will associate the colored composition $\binom{v}{\nabla_n(\sigma,\lambda)}\in([c]\times\mathbb{P})^n$. Then, it is clear how Theorem 2.3 extends to the colored setting. Finally, substituting q^az_b for the letter $\binom{b}{a}\in[c]\times\mathbb{P}$, $C(z)$ becomes $\prod_{i=1}^{c}\mathrm{e}_q(z_iz)$. Applying Theorem 3.6 then gives the desired result. $\square$

3.3.2 Pairs of permutations by common descents

Another natural extension of the major index is to common descents. That is, for $\sigma,\tau\in S_n$, let $\mathrm{comdes}(\sigma,\tau)$ be the number of positions i such that $\sigma_i>\sigma_{i+1}$ and $\tau_i>\tau_{i+1}$, and let $\mathrm{commaj}(\sigma,\tau)$ be the sum of those i. This type of statistic has been studied by Carlitz and Scoville [9], together and along with Vaughan [11], in the 1970s and revisited by Fédou and Rawlings [21, 22] in the 1990s. Mendes and Remmel [46] derived the generating function analogous to (3.1), which we will re-derive here.

Theorem 3.7. *Let X be an alphabet with corresponding A,B as defined in Section 3.2, and let Y be another alphabet with corresponding pair C,D. Let $Z=X\times Y$, $F=\{\binom{x_1x_2}{y_1y_2}:x_1x_2\in B$ and $y_1y_2\in D\}$, and $E=Z^2\setminus F$. Then*

$$\sum_{n\geq 0}\sum_{w\in Z^n}\frac{x^{\mathrm{des}_E(w)}u^{\mathrm{maj}_E(w)}}{(x;u)_{n+1}}w=\sum_{k\geq 0}x^kE(u^k)E(u^{k-1})\cdots E(1),$$

where $E(z)=(F(-z))^{-1}$ and $F_n=B_n\times D_n$.

Proof. In Theorem 2.7, we proved the relationship $A(z) = (B(-z))^{-1}$ for all pairs A, B, so $E(z), F(z)$ satisfy the same relationship. It should also be noted that the same result implies $B(z) = (A(-z))^{-1}$ and $D(z) = (C(-z))^{-1}$, from which B_n and D_n can be obtained. The claim $F_n = B_n \times D_n$ is obvious. Apply Theorem 3.5 to complete the proof. □

It should be noted that, if X and Y are partially ordered sets (posets) such that $B = \{w_1w_2 \in X : w_1 \leq w_2\}$ and $D = \{w_1w_2 \in Y : w_1 \leq w_2\}$, then F is the product order on $X \times Y$. Since Theorem 3.5 may be applied equally well to posets, Theorem 3.7 thus gives us an easy way to compute results for product posets.

Corollary 3.2.

$$\sum_{n\geq 0}\sum_{\sigma,\tau\in S_n} \frac{x^{\mathrm{comdes}(\sigma,\tau)}u^{\mathrm{commaj}(\sigma,\tau)}q_1^{\mathrm{inv}(\sigma)}q_2^{\mathrm{inv}(\tau)}z^n}{(x;u)_{n+1}(q_1;q_1)_n(q_2;q_2)_n} = \sum_{k\geq 0} x^k E(zu^k)E(zu^{k-1})\cdots E(z),$$

where

$$E(z) = \left(\sum_{n\geq 0} \frac{q_1^{\binom{n}{2}}q_2^{\binom{n}{2}}(-z)^n}{(q_1;q_1)_n(q_2;q_2)_n}\right)^{-1}.$$

Proof. To apply Theorem 3.7, we must compute B_n in the standard case $X = \mathbb{P}$, $A = \{w_1w_2 : w_1 \leq w_2\}$. Substituting $q^i(z/q)$ for each $i \in \mathbb{P}$, we computed in (2.26) that B_n (and thus D_n) is given by $q^{\binom{n}{2}}/(q;q)_n$. Applying Theorem 2.3 to both σ and τ completes the proof. □

It is straightforward to see how Theorem 3.7 and Corollary 3.2 extend to more than two sets of words. We also see that Theorem 3.7 can be applied to the colored permutations and words in the previous subsection to obtain a different result.

3.3.3 Descents at positions congruent to i mod j

Another interesting extension of the major index is to count only those descents that occur at specific positions. In this subsection, we will consider the particular case where we count descents that occur at positions congruent to i mod j. That is, for $1 \leq i \leq j$, let $\mathrm{des}_{i \upharpoonright j}(\sigma)$ be the number of k such that $\sigma_{i+jk} > \sigma_{i+jk+1}$, and let $\mathrm{maj}_{i\Gamma j}(\sigma)$ be the sum of those $i + jk$. Then, we will be interested in computing

$$\sum_{n\geq 0}\sum_{\sigma\in S_n} \frac{x^{\mathrm{des}_{i\Gamma j}(\sigma)}u^{\mathrm{maj}_{i\Gamma j}(\sigma)}q^{\mathrm{inv}(\sigma)}z^n}{(x;u)_{n+1}(q;q)_n}. \tag{3.6}$$

However, as we will see, it will be easier to compute a slightly different variant of the generating function.

Theorem 3.8. *Let X be an alphabet with corresponding A, B as defined in Section 3.2. Then,*

$$\sum_{n\geq 0}\sum_{w\in X^n} \frac{x^{\mathrm{des}_{i\Gamma j}(w)}u^{\mathrm{maj}_{i\Gamma j}(w)}}{(xu^i;u^j)_{\lfloor\frac{n-i}{j}\rfloor+1}}w = A_{(0)} + \sum_{k\geq 1} x^k A_1(u^k)A_j(u^{k-1})\cdots A_j(u)A_{(k)}, \tag{3.7}$$

where

$$\begin{aligned}
A_{(0)} &= (1-a_1^{i+1})(1-a_1)^{-1} + a_1^{i-1}a_2(1-a_1^{j-2}a_2)^{-1}(1-a_1^j)(1-a_1)^{-1},\\
A_1(z) &= a_1^i z^i + a_1^{i-1}a_2 z^{i+1}(1-a_1^{j-2}a_2z^j)^{-1}a_1^{j-1}z^{j-1},\\
A_j(z) &= 1 + a_1^j z^j + a_1^{j-1}a_2 z^{j+1}(1-a_1^{j-2}a_2z^j)^{-1}a_1^{j-1}z^{j-1}, \text{ and}\\
A_{(k)} &= 1 + a_1(1-a_1^{j-2}a_2)^{-1}(1-a_1^j)(1-a_1)^{-1}.
\end{aligned}$$

Proof. Consider the proof of Theorem 3.5. For the statistic $\mathrm{des}_{i\Gamma j}(w)$, $\mu(w)$ can only have parts congruent to i mod j. Therefore, instead of merging $\mu(w)$ with all $\lambda \in \bar{\Lambda}^{\leq n}$, it is sufficient to merge only with those λ with parts congruent to i mod j. Following the usual argument, these are counted by $(xu^i;u^j)^{-1}_{\lfloor\frac{n-i}{j}\rfloor+1}$. Following the proof of Theorem 3.5, we will come to the conclusions in the following paragraph.

For $k = 0$, we see that w is a word with no descents at positions congruent to i mod j, which are counted by $A_{1n}(z)$. If $k > 0$, then:

- $w^{(0)}$ has length congruent to i mod j and no descents at positions congruent to i mod j,
- for $1 \leq i < k$, $w^{(i)}$ has length congruent to 0 mod j and no descents at positions congruent to 0 mod j, and
- $w^{(k)}$ has arbitrary length and no descents at positions congruent to 0 mod j.

These three cases are counted respectively by $A_1(z)$, $A_j(z)$, and $A_n(z)$. ∎

It is clear how to apply Theorem 3.8 to permutations. Apply Theorem 3.8 to compositions, letting the letter $i \in \mathbb{P}$ be replaced by $q^i(z/q)$, so that $a_1 = \frac{z}{1-q}$ and $a_2 = \frac{z^2}{(q;q)_2}$. Fédou's bijection completes the result.

Now, let $S_n^{i\Rsh j}$ be the set of permutations whose only descents occur at positions congruent to $i \bmod j$. Mendes, Remmel, and Riehl [48] derived the generating function for permutations in $S_n^{i\Rsh j}$ by descents and inversions. In these permutations, $\mathrm{des}_{i\Rsh j}(\sigma) = \mathrm{des}(\sigma)$ and $\mathrm{maj}_{i\Rsh j}(\sigma) = \mathrm{maj}(\sigma)$, so restricting (3.7) to $S_n^{i\Rsh j}$, we obtain the natural extension of their result to include the major index.

Theorem 3.9. *Let X be an alphabet with corresponding A, B as defined in Section 3.2, and let $X^n_{i\Rsh j} = \{w \in X^n : w_k w_{k+1} \in B$ implies $k = i \bmod j\}$. Then,*

$$\sum_{n\geq 0}\sum_{w\in X^n_{i\Rsh j}} \frac{x^{\mathrm{des}_{i\Rsh j}(w)} u^{\mathrm{maj}_{i\Rsh j}(w)}}{(xu^i;u^j)_{\lfloor\frac{n-i}{j}\rfloor+1}} w = A(1) + \sum_{k\geq 1} x^k A_1(u^k) A_j(u^{k-1}) \cdots A_j(u) A(1),$$

where

$$A_1(z) = a_i z^i + a_{i+j} z^{i+j} + a_{i+2j} z^{i+2j} + \cdots \quad \text{and}$$
$$A_j(z) = 1 + a_j z^j + a_{2j} z^{2j} + \cdots .$$

Proof. The proof is identical to that of Theorem 3.8, except that now the subwords $w^{(i)}$ may not have any descents. It is clear then that $A_{(0)}(z)$ and $A_{(k)}(z)$ reduce to $A(1)$ and that $A_1(z)$ and $A_j(z)$ are as given. □

It is again clear how to apply Theorem 3.9 to permutations. We see that a_n becomes $\frac{z^n}{(q;q)_n}$, so that $A_1(z)$ and $A_j(z)$ can actually be written as linear combinations of complex q-exponential functions.

It should be noted that both Theorem 3.8 and Theorem 3.9 can be thought of as special cases of a more general result. That is, in the context of compositions, we can rewrite w in the proofs as a word w' on the alphabets $\mathbb{P}^j$ or Λ_j, respectively, except with modifications to the first and last letters of w'.

3.3.4 Permutations by alternating descents

Chebikin [12] defined the alternating descent set of a permutation σ by $\mathrm{AltDes}(\sigma) = \{2i-1 : \sigma_{2i-1} < \sigma_{2i}\} \cup \{2i : \sigma_{2i} > \sigma_{2i+1}\}$. In words, it is the set of odd positions that are ascents and the even positions that are descents. One could also think of the set of common ascents and descents with an up-down permutation. We then define $\mathrm{altdes}(\sigma) = |\,\mathrm{AltDes}(\sigma)|$ and $\mathrm{altmaj}(\sigma) = \sum_{i \in \mathrm{AltDes}(\sigma)} i$. We wish to compute

$$\sum_{n \geq 0} \sum_{\sigma \in S_n} \frac{x^{\mathrm{altdes}(\sigma)} u^{\mathrm{altmaj}(\sigma)} q^{\mathrm{inv}(\sigma)} z^n}{(x;u)_{n+1}(q;q)_n}. \tag{3.8}$$

Remmel [55] computed the version of (3.8) with inversions omitted.

Theorem 3.10. *Let X be an alphabet with corresponding A, as defined in Section 3.2. Define*

$$A^{\cos}(z) = \sum_{n \geq 0} (-1)^n a_{2n} z^{2n} \text{ and}$$
$$A^{\sin}(z) = \sum_{n \geq 0} (-1)^n a_{2n+1} z^{2n+1},$$

and let $A^M(z)$ be the matrix

$$A^M(z) = \begin{bmatrix} (A^{\cos}(z))^{-1} & (A^{\cos}(z))^{-1} A^{\sin}(z) \\ A^{\sin}(z)(A^{\cos}(z))^{-1} & A^{\cos}(z) + A^{\sin}(z)(A^{\cos}(z))^{-1} A^{\sin}(z) \end{bmatrix}.$$

Then,

$$\sum_{n \geq 0} \sum_{w \in X^n} \frac{x^{\mathrm{altdes}(w)} u^{\mathrm{altmaj}(w)}}{(x;u)_{n+1}} w = \sum_{k \geq 0} x^k [1 \;\; 0] A^M(u^k) A^M(u^{k-1}) \cdots A^M(1) \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{3.9}$$

Proof. Consider the proof of Theorem 3.5. In this case, the set of words that may be used for $w^{(i)}$ depends on whether $w^{(i)}$ starts at an even or odd position in w. Therefore, we need to compute the generating functions for words with no alternating descents for each combination of even and odd length and starting at an even or odd position. We have already obtained these generating functions in Theorem 2.8. The set of words starting at odd positions corresponds to the set of up-down words, whose generating function is given by (2.32). The set of words starting at even positions is the set of down-up words, whose generating function is given by (2.33).

We have separated the even and odd terms and placed them in the matrix $A^M(z)$. The entries in the first row contain the sub-words starting at odd positions, while the second row contains sub-words starting at even positions. The first column contains the sub-words ending at even positions, and the second column contains the sub-words ending at odd positions. This placement ensures that the matrix multiplication in (3.9) results in sub-words being lined up properly. The leading $[\,1\ \ 0\,]$ restricts the sum to words that start at an odd position, *i.e.* 1, whereas the trailing vector combines words of all lengths. The remainder of the proof remains unchanged. $\square$

Corollary 3.3. *With* $\sin_q(z)$ *and* $\cos_q(z)$ *defined as before,*

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\frac{x^{\mathrm{altdes}(\sigma)}u^{\mathrm{altmaj}(\sigma)}q^{\mathrm{inv}(\sigma)}z^n}{(x;u)_{n+1}(q;q)_n}=\sum_{k\geq 0}x^k[\,1\ \ 0\,]A^M(zu^k)A^M(zu^{k-1})\cdots A^M(z)\begin{bmatrix}1\\1\end{bmatrix},$$

where

$$A^M(z)=\begin{bmatrix}\frac{1}{\cos_q(z)} & \frac{\sin_q(z)}{\cos_q(z)}\\ \frac{\sin_q(z)}{\cos_q(z)} & \frac{(\cos_q(z))^2+(\sin_q(z))^2}{\cos_q(z)}\end{bmatrix}.$$

Proof. Apply Theorem 3.10 to the case $X=\mathbb{P}$, $A=\{w_1w_2 : w_1\leq w_2\}$, then use Theorem 2.3. $\square$

We note that a similar method could have been used to compute (3.6).

3.3.5 Compositions with number of even-to-odd and odd-to-even transitions

Theorem 3.5 has enough generality to keep track of extra statistics that sum over sub-words, such as the number of occurrences of a particular letter. However, we may use the approach of the previous subsection to track some more interesting statistics, such as the number of transitions between subsets of the alphabet X. While we could state a very general result, we will content ourselves with a simple example.

Theorem 3.11. *For* $w\in\mathbb{P}^n$, *let* $\mathrm{eo}(w)$ *be the number of* i *such that* w_i *is odd and* w_{i+1} *is even, and let* $\mathrm{oe}(w)$ *be the number of* i *such that* w_i *is even and* w_{i+1} *is* *odd. Also, recall the q-binomial coefficient* $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}$*, and define*

$$\begin{aligned}
A^{oo}(z) &= \sum_{i\geq 0} r^i s^i \sum_{n\geq 1} \frac{q^{\binom{2i+1}{2}} \begin{bmatrix} n-1 \\ 2i \end{bmatrix}_q z^n}{(q^2;q^2)_n}, \\
A^{oe}(z) &= \sum_{i\geq 0} r^{i+1} s^i \sum_{n\geq 1} \frac{q^{\binom{2i+2}{2}} \begin{bmatrix} n-1 \\ 2i+1 \end{bmatrix}_q z^n}{(q^2;q^2)_n}, \\
A^{eo}(z) &= \frac{s}{r} A^{oe}(zq), \text{ and} \\
A^{ee}(z) &= A^{oo}(zq),
\end{aligned}$$

and define the matrices

$$\begin{aligned}
A^M(z) &= \begin{bmatrix} 1 + A^{ee}(z) + sA^{oe}(z) & A^{eo}(z) + sA^{oo}(z) \\ rA^{ee}(z) + A^{oe}(z) & 1 + rA^{eo}(z) + A^{oo}(z) \end{bmatrix} \text{ and} \\
A_1^M(z) &= \begin{bmatrix} A^{ee}(z) & A^{eo}(z) \\ A^{oe}(z) & A^{oo}(z) \end{bmatrix}.
\end{aligned}$$

For $w \in \mathbb{P}^n$*, define the* weight *of* w*,* $\mathrm{wt}(w)$*, to be* $r^{\mathrm{oe}(w)} s^{\mathrm{eo}(w)} q^{\mathrm{sum}(w)} (z/q)^n$*. Then,*

$$\sum_{w\in N^*} x^{\mathrm{des}(w)} u^{\mathrm{maj}(w)} \,\mathrm{wt}(w) = 1 + \sum_{k\geq 0} \frac{x^k}{1-x} [\,1\ \ 1\,] A_1^M(zu^k) A^M(zu^{k-1}) \cdots A^M(z) \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Proof. Similar to the proof of Theorem 3.10, we will need generating functions for the sub-words that begin or end with all combinations of even or odd numbers. Moreover, these generating functions will need to track $eo(w^{(i)})$ and $oe(w^{(i)})$ themselves.

Consider $\mu \in \Lambda_n$ that starts with at least one 1, followed by at least one 2, etc., up to at least one i. Let λ be a weak partition of length n with all even parts. By varying μ and λ and letting $\nu_j = \mu_j + \lambda_j$, we can form every partition $\nu \in \Lambda_n$ starting with an odd part and switching between odd and even $i-1$ times. Removing one 1, one 2, etc. from μ, we obtain an arbitrary partition with $n-i$ parts of size up to i. It is well-known that these are counted by $z^{n-i} \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q$, and the parts we removed are counted by $z^i q^{\binom{i}{2}}$. To count sub-words starting with an even number, add 1 to each letter. The generating function for weak partitions with n even parts by sum is $(q^2;q^2)_n^{-1}$. Putting this analysis together, we therefore obtain the functions $A^{ee}(z), A^{eo}(z), A^{oe}(z), A^{oo}(z)$ given above.

Finally, the matrix $A_1^M(z)$ ensures that the sub-word $w^{(0)}$ is non-empty, so that we can count transitions properly. We adjust for this by adding the factor of $\frac{1}{1-x}$. The matrix $A^M(z)$ merely ensures that the sub-words are joined in a way that we can track the transitions. In the top row, we assume the previous letter was even, and in the left column, the new last letter will be even. □

3.3.6 Directed column-convex polyominoes

Directed column-convex polyominoes (DCCPs) were introduced in Section 2.6. Namely, they are the subset of CCPs with no lower descents. As such, the set of DCCPs corresponds, under the bijection δ from Subsection 2.6.1, to words on the alphabet $Y = \left\{\binom{j}{m} : j, m \in \mathbb{P}, j \geq m\right\}$ whose last letters have $m = 1$. Using δ, we can derive a version of (3.1) for DCCPs.

Theorem 3.12. *For $Q \in DCCP$, let* $\operatorname{udes}(Q)$ *be the number of upper descents in Q and* $\operatorname{umaj}(Q)$ *be the sum of their positions. Then the generating function for DCCPs by upper descents, upper major index, area, and length,*

$$\sum_{Q \in DCCP} x^{\operatorname{udes}(Q)} u^{\operatorname{umaj}(Q)} q^{\operatorname{area}(Q)} z^{\ell(Q)},$$

is given by

$$1 + \sum_{k \geq 0} \sum_{j=0}^{k} x^k A(zu^k) A(zu^{k-1}) \cdots A(zu^{j+1}) A_n(zu^j),$$

where

$$A(z) = \left(\sum_{k \geq 0} \frac{(-z)^k q^{\binom{k+1}{2}}}{(q;q)_k^2} \right)^{-1} \quad \textit{and}$$

$$A_n(z) = z \left(\sum_{k \geq 0} \frac{(-z)^k q^{\binom{k+2}{2}}}{(q;q)_{k+1}(q;q)_k} \right) \left(\sum_{k \geq 0} \frac{(-z)^k q^{\binom{k+1}{2}}}{(q;q)_k^2} \right)^{-1}.$$

Proof. Following the proof of Theorem 3.5, we need generating functions for words on Y with no upper descents by area and length, with and without the restriction that $m_n = 1$. We computed them in Subsection 2.8.2, obtaining $A(z)$ and $A_n(z)$

above. The inclusion of the sum over j handles the condition that the final non-empty sub-word must have $m_n = 1$. □

We will note two other methods by which we could have dealt with the restriction on the final letter. First, we could have written the expression using a product of matrices. Second, we could have modified the proof of Theorem 3.5 as we did in Theorem 3.8. That is, when $\mu(w)$ is used to factor w into sub-words, w_n always appears in the final sub-word. Therefore, we could have restricted λ to $\bar{\Lambda}^{\leq n-1}$. In that case, $w^{(k)}$ would always be non-empty, allowing us to modify only the final term in the product.

Chapter 3, in full, has been submitted for publication in Pure Mathematics and Applications. Remmel, Jeff; Tiefenbruck, Mark. The dissertation author is an author of this paper.

Chapter 4

Partially Marked Patterns

4.1 Introduction

In this chapter, we shall show how one can use the involution principle of Garsia and Milne [26] to modify a class of bijections that preserve marked occurrences of patterns in certain sequences to obtain bijections that preserve the number of occurrences of patterns in such sequences.

To serve as an example of what we have in mind, let $\sigma \in S_n$ be a permutation. Then we say that σ_i is a *consecutive ascent top* in σ if $\sigma_i = \sigma_{i-1} + 1$. Here we use $\sigma_0 = 0$ so that 1 is a consecutive ascent top if and only if $\sigma_1 = 1$. We let $\mathrm{CAT}(\sigma)$ denote the set of σ_i such that σ_i is a consecutive ascent top in σ and $\mathrm{cat}(\sigma) = |\mathrm{CAT}(\sigma)|$. Recall that σ_j is a *left-to-right minimum* if $\sigma_i > \sigma_j$ for all $i < j$. We say that σ_i is *part of an ascent* if either $\sigma_{i-1} < \sigma_i$ or $\sigma_i < \sigma_{i+1}$, where defined. We let $\mathrm{NALRmin}(\sigma)$ denote the set of σ_i such that σ_i is a left-to-right minimum in σ that is not part of an ascent and $\mathrm{nalrmin}(\sigma) = |\mathrm{NALRmin}(\sigma)|$. For example, Table 4.1 gives the values of the statistics $\mathrm{cat}(\sigma)$ and $\mathrm{nalrmin}(\sigma)$ for each $\sigma \in S_3$.

We claim that for all $n \geq 1$,

$$\sum_{\sigma \in S_n} x^{\mathrm{cat}(\sigma)} = \sum_{\sigma \in S_n} x^{\mathrm{nalrmin}(\sigma)}. \tag{4.1}$$

Thus for each $n \geq 1$, there must be a bijection $\theta_n : S_n \to S_n$ such that for all $\sigma \in S_n$, $\mathrm{cat}(\sigma) = \mathrm{nalrmin}(\theta_n(\sigma))$. We have not been able to find a simple description of

Table 4.1: Table of values of $\mathrm{cat}(\sigma)$ and $\mathrm{nalrmin}(\sigma)$ for S_3

σ	$\mathrm{cat}(\sigma)$	$\mathrm{nalrmin}(\sigma)$
123	3	0
132	1	0
213	0	1
231	1	1
312	1	1
321	0	3

such a bijection θ_n. However, it is far easier to construct a bijection between marked consecutive ascent tops and marked left-to-right minima that are not part of ascents. That is, let MCAT_n denote the set of all $\sigma \in S_n$ where certain elements of $\mathrm{CAT}(\sigma)$ are marked. For example, if $\sigma = 13245867$, then $\mathrm{CAT}(\sigma) = \{1, 5, 7\}$. Thus if we mark the 5 and 7 in $\mathrm{CAT}(\sigma)$, then we shall indicate this by underlining them, which would produce the marked permutation $\sigma = 1324\underline{5}86\underline{7}$. Similarly, let $\mathrm{MNALRmin}_n$ denote the set of $\sigma \in S_n$ where certain elements of $\mathrm{NALRmin}(\sigma)$ are marked. For example, if $\tau = 54762138$, then $\mathrm{NALRmin}(\tau) = \{2, 5\}$. If we mark the $5 \in \mathrm{NALRmin}(\tau)$, then we would indicate this by over-lining it, which would produce the marked permutation $\tau = \overline{5}4762138$. Thus for example,

$$\begin{aligned}
\mathrm{MCAT}_1 &= \{1, \underline{1}\}, \\
\mathrm{MNALRmin}_1 &= \{1, \overline{1}\}, \\
\mathrm{MCAT}_2 &= \{12, \underline{1}2, 1\underline{2}, \underline{12}, 21\}, \text{ and} \\
\mathrm{MNALRmin}_2 &= \{12, 21, \overline{2}1, 2\overline{1}, \overline{21}\}.
\end{aligned}$$

Given an element $\sigma \in \mathrm{MCAT}_n$, let $\mathrm{mcat}(\sigma)$ denote the number of $i \in \mathrm{CAT}(\sigma)$ that are marked. Similarly, given an element $\sigma \in \mathrm{MNALRmin}_n$, let $\mathrm{mnalrmin}(\sigma)$ denote the number of $i \in \mathrm{NALRmin}(\sigma)$ that are marked. We claim that it is easy to recursively construct a bijection $\Gamma_n : \mathrm{MCAT}_n \to \mathrm{MNALRmin}_n$ such that for all $\sigma \in \mathrm{MCAT}_n$, $\mathrm{mcat}(\sigma) = \mathrm{mnalrmin}(\Gamma_n(\sigma))$.

Clearly, we want $\Gamma_1(1) = 1$ and $\Gamma_1(\underline{1}) = \overline{1}$. Next suppose that we have defined Γ_{n-1}. For any $\tau = \tau_1 \cdots \tau_{n-1} \in S_{n-1}$, there are n positions where we can insert n to create a permutation in S_n, namely, either directly in front τ_i for $i = 1, \ldots, n-1$ or at the end. We let $\tau^{(i)}$ be the permutation that arises from τ by inserting n such that n occupies the i-th position. For example, if $\tau = 2341$, then $\tau^{(3)} = 23541$ and $\tau^{(5)} = 23415$. Now suppose we are given $\alpha \in \text{MCAT}_{n-1}$ such that $\text{mcat}(\alpha) = k$. We can create $n-k$ new elements $\gamma \in \text{MCAT}_n$ with $\text{mcat}(\gamma) = k$ from α by inserting n in front of any element of α that is not marked or by adding n at the end. Let $E_n(\alpha)$ be the set of elements that are created from α in this way. We can also create one $\tau \in \text{MCAT}_n$ from α with $\text{mcat}(\tau) = k+1$ by inserting $\underline{n}$ immediately after $n-1$. We let $A_n(\alpha)$ denote $\{\tau\}$ in this case. For example, if $\alpha = \underline{1}34\underline{5}26 \in \text{MCAT}_6$, then $E_7(\alpha)$ would consist of the following 5 elements of MCAT_7 with 2 marked consecutive ascent tops:

$$\begin{aligned}
\gamma^{(2)} &= \underline{1}734\underline{5}26, \\
\gamma^{(3)} &= \underline{1}374\underline{5}26, \\
\gamma^{(5)} &= \underline{1}34\underline{5}726, \\
\gamma^{(6)} &= \underline{1}34\underline{5}276, \text{ and} \\
\gamma^{(7)} &= \underline{1}34\underline{5}267.
\end{aligned}$$

$A_7(\alpha)$ consists of the permutation obtained by inserting $\underline{7}$ immediately after 6, namely $\underline{1}34\underline{5}26\underline{7}$.

Next suppose we are given $\beta \in \text{MNALRmin}_{n-1}$ such that $\text{mnalrmin}(\sigma) = k$. We can create $n-k$ new elements $\delta \in \text{MNALRmin}_n$ with $\text{mnalrmin}(\delta) = k$ from β by inserting n immediately after any β_i that is not marked or at the start of the permutation. Let $F_n(\beta)$ be the set of elements that are created from β in this way. We can also create one $\sigma \in \text{MNALRmin}_n$ from β with $\text{mnalrmin}(\sigma) = k+1$ by inserting $\overline{n}$ at the start. We then let $B_n(\beta) = \{\sigma\}$. For example, if $\beta = \overline{5}2643\overline{1} \in \text{MNALRmin}_6$, then $F_7(\beta)$ consists of the following 5 elements of MNALRmin_7 with 2 marked left-to-right minima that are not part of ascents:

$$\begin{aligned}
\delta^{(1)} &= 7\overline{5}2643\overline{1}, \\
\delta^{(3)} &= \overline{5}27643\overline{1}, \\
\delta^{(4)} &= \overline{5}26743\overline{1}, \\
\delta^{(5)} &= \overline{5}26473\overline{1}, \text{ and} \\
\delta^{(6)} &= \overline{5}26437\overline{1}.
\end{aligned}$$

Similarly, $B_n(\beta)$ consists of the permutation obtained by inserting $\overline{7}$ at the start of β, namely $\sigma = \overline{7}\overline{5}26431\overline{1}$.

Note that the elements of $E_n(\alpha)$ are naturally ordered by saying that γ comes before γ' in $E_n(\alpha)$ if and only if the position of n in γ precedes the position of n in γ'. Similarly, the elements of $F_n(\beta)$ are naturally ordered by saying that δ comes before δ' in $F_n(\beta)$ if and only if the position of n in δ precedes the position of n in δ'. Then, Γ_n is defined from Γ_{n-1} as follows. Suppose that we are given an $\alpha \in \text{MCAT}_{n-1}$ with $\text{mcat}(\alpha) = k$ so that $\text{mnalrmin}(\Gamma_{n-1}(\alpha)) = k$. Thus $|E_n(\alpha)| = |F_n(\Gamma_{n-1}(\alpha))| = n-k$, so we define Γ_n to map the i-th element of $E_n(\alpha)$ to the i-th element of $F_n(\Gamma_{n-1}(\alpha))$ for $i = 1, \ldots, n-k$. We also ensure that Γ_n maps the element in $A_n(\alpha)$ to the element in $B_n(\Gamma_{n-1}(\alpha))$.

In fact, one can easily use this recursive definition to construct the image of any $\alpha \in \text{MCAT}_n$ under Γ_n by recursively removing $n, n-1, \ldots, 2$ and recording which choice one made at each step. For example, if $\alpha = \underline{1}34\underline{5}267$ and we record the position of the marked permutation in E_n or A_n we used at each stage, we would end up with following:

$$\begin{aligned}
\underline{1}34\underline{5}267 &\to E_7 : 5,\\
\underline{1}34\underline{5}26 &\to E_6 : 4,\\
\underline{1}34\underline{5}2 &\to A_5,\\
\underline{1}342 &\to E_4 : 2,\\
\underline{1}32 &\to E_3 : 1,\\
\underline{1}2 &\to E_2 : 1, \text{ and}\\
\underline{1} &\to A_1.
\end{aligned}$$

Thus to define $\Gamma_7(\alpha)$, we use a similar procedure to build up a permutation by choosing the corresponding elements of F_n and B_n. That is, since for α, 2 was inserted to obtain the first element of $E_2(\underline{1})$, we choose the first element of $F_2(\overline{1})$, namely $2\overline{1}$. Since for α, 3 was inserted to obtain the first element of $E_3(\underline{1}2)$, we choose the first element of $F_3(2\overline{1})$, namely $32\overline{1}$. Continuing on in this way, we would construct the following sequence:

$$\begin{aligned} \overline{1} &\leftarrow B_1, \\ 2\overline{1} &\leftarrow F_2 : 1, \\ 32\overline{1} &\leftarrow F_3 : 1, \\ 342\overline{1} &\leftarrow F_4 : 2, \\ \overline{5}342\overline{1} &\leftarrow B_5, \\ \overline{5}3426\overline{1} &\leftarrow F_6 : 4, \text{ and} \\ \overline{5}34267\overline{1} &\leftarrow F_7 : 5. \end{aligned}$$

Thus $\Gamma_7(\underline{1}34\underline{5}267) = \overline{5}34267\overline{1}$.

The main purpose of this chapter is to show that there is a general mechanism to construct our desired bijections θ_n from the bijections Γ_n. We consider the following setup. Let X be an alphabet, and define a *pattern* P on X to be a set of pairs of the form $\langle a_1a_2\cdots a_k, b_1b_2\cdots b_k\rangle$, where $1 \le a_1 < \cdots < a_k$ and $b_1b_2\cdots b_k \in X^k$. Each pair represents a set of indices and one possible sequence of letters to occupy those indices. Here we do not require that the set of indices $a_1a_2\cdots a_k$ that appear as the first elements of pairs in P are all of the same length. An *occurrence* of the pattern P in a word $w \in X^n$ is a subsequence of indices $a_1a_2\cdots a_k$ with $a_k \le n$ such that there exists a pair $\langle a_1a_2\cdots a_k, w_{a_1}w_{a_2}\cdots w_{a_k}\rangle \in P$. We let $P(w)$ denote the number of occurrences of the pattern P in the word w. For example, in compositions, the consecutive pattern 12 can be written as the pattern $\{a_1a_2 \in \mathbb{P}^2 : a_2 = a_1 + 1\} \times \{b_1b_2 \in \mathbb{P}^2 : b_1 \le b_2\}$.

Define a *pattern family* to be a set of the form $\mathcal{F} = \bigcup_{n\ge0}\langle f, g, A_n, \mathcal{F}_n, \mathcal{P}_n\rangle$, where f and g are functions mapping $\mathbb{N}$ to $\mathbb{N}$, and for each $n \ge 0$,

1. A_n is a finite alphabet,
2. $\mathcal{F}_n$ is a subset of $A_n^{f(n)}$, and
3. $\mathcal{P}_n$ is a totally ordered set of patterns $P_1, P_2, \ldots, P_{g(n)}$.

Let $w_{\mathcal{F}_n}$ be a function on the words in $\mathcal{F}_n$. We shall be interested in the generating function

$$R_{\mathcal{F}}(t, x_1, x_2, \ldots) = \sum_{n \geq 0} t^n \sum_{w \in \mathcal{F}_n} w_{\mathcal{F}_n}(w) \prod_{i=1}^{g(n)} x_i^{P_i(w)} \tag{4.2}$$

as well as its specialization

$$R_{\mathcal{F}}(t, x) = \sum_{n \geq 0} t^n \sum_{w \in \mathcal{F}_n} w_{\mathcal{F}_n}(w) x^{\sum_{i=1}^{g(n)} P_i(w)}. \tag{4.3}$$

We will typically take $w_{\mathcal{F}_n}(w) = 1$ for all w to obtain the ordinary generating function or $w_{\mathcal{F}_n}(w) = \frac{1}{n!}$ to obtain the exponential generating function. However, $w_{\mathcal{F}_n}$ could track other statistics on w as well. Note that we allow $g(n) = 0$, in which case we will assume that $\mathcal{P}_n = \emptyset$ and interpret $\prod_{i=1}^{g(n)} x_i^{P_i(w)}$ and $x^{\sum_{i=1}^{g(n)} P_i(w)}$ to be equal to 1.

For example, suppose that we wanted to formulate the problem of finding the distribution of consecutive ascent tops as a pattern family. Then we would consider the pattern family $\mathcal{F} = \bigcup_{n \geq 0} \langle f, g, A_n, \mathcal{F}_n, \mathcal{P}_n \rangle$ where $f(n) = n$ for all $n \geq 0$, $g(0) = 0$, $g(n) = 1$ for all $n \geq 1$, and

1. $A_0 = \{0\}$,
2. $A_n = \{1, \ldots, n\}$ for all $n \geq 1$,
3. $\mathcal{F}_0 = \{\epsilon\}$,
4. $\mathcal{F}_n = S_n$ for all $n \geq 1$, and
5. $\mathcal{P}_n = \{P\}$ where P is the pattern consisting of the pair $\langle 1, \{1\} \rangle$ plus the pairs $\langle i(i+1), j(j+1) \rangle$ for $1 \leq i, j \leq n-1$.

Similarly, if we wanted to formulate the problem of finding the distribution of left-to-right minima that are not part of ascents as a pattern family, we would consider the pattern family $\mathcal{G} = \bigcup_{n \geq 0} \langle f, g, A_n, \mathcal{F}_n, \mathcal{Q}_n \rangle$ where $f, g, A_n, \mathcal{F}_n$ are as before, and $\mathcal{Q}_n = \{Q\}$ where Q is the pattern consisting of pairs $\langle 12 \cdots i(i+1), \sigma_1 \cdots \sigma_{i+1} \rangle$ where $\sigma_i > \sigma_{i+1}$ and $\sigma_j > \sigma_i$ for all $j < i$ as well as the pairs $\langle 12 \cdots n, \sigma_1 \cdots \sigma_n \rangle$ where $\sigma_n = 1$.

Given a pattern family $\mathcal{F} = \bigcup_{n \geq 0} \langle f, g, A_n, \mathcal{F}_n, \mathcal{P}_n \rangle$, we can form the *partially marked pattern family* $\mathcal{PMF} = \bigcup_{n \geq 0} \langle f, g, A_n, \mathcal{PMF}_n, \mathcal{P}_n \rangle$ from $\mathcal{F}$, where if $\mathcal{P}_n = \{P_1, \ldots, P_{g(n)}\}$, then $\mathcal{PMF}_n$ is the set of all $(g(n)+1)$-tuples of the form $\langle w, H_1, \ldots, H_{g(n)}\rangle$ such that $w \in \mathcal{F}_n$ and for $i \in [g(n)]$, H_i is a possibly empty set of occurrences of the pattern P_i in w. Thus we can think of the $(g(n)+1)$-tuple $\langle w, H_1, \ldots, H_{g(n)}\rangle$ as an element $w \in \mathcal{F}_n$ where some of the occurrences of P_i in w are "marked" for $i \in [g(n)]$. We define the weight of $\langle w, H_1, \ldots, H_{g(n)}\rangle$ to be

$$w_{\mathcal{PMF}}(w, H_1, \ldots, H_{g(n)}) = \prod_{i=1}^{g(n)} y_i^{|H_i|}, \tag{4.4}$$

where again we make the convention that if $g(n) = 0$, then we set $w_{\mathcal{PMF}}(w) = 1$. Then, we shall consider the generating function

$$MR_{\mathcal{F}}(t, y_1, y_2, \ldots) = \sum_{n\geq 0} t^n \sum_{(w,H_1,\ldots,H_{g(n)})\in\mathcal{PMF}_n} w_{\mathcal{F}_n}(w) w_{\mathcal{PMF}}(w, H_1, \ldots, H_{g(n)}) \tag{4.5}$$

as well as its specialization

$$MR_{\mathcal{F}}(t, y) = MR_{\mathcal{F}}(t, y, y, \ldots). \tag{4.6}$$

The key result of this chapter is the following theorem.

Theorem 4.1. *Suppose that* $\mathcal{F} = \bigcup_{n\geq 0}\langle f, g, A_n, \mathcal{F}_n, \mathcal{P}_n\rangle$ *is a pattern family and* $\mathcal{PMF} = \bigcup_{n\geq 0}\langle f, g, A_n, \mathcal{PMF}_n, \mathcal{P}_n\rangle$ *is the partially marked pattern family constructed from* $\mathcal{F}$*. Then*

$$MR_{\mathcal{F}}(t, x_1 - 1, x_2 - 1, \ldots) = R_{\mathcal{F}}(t, x_1, x_2, \ldots), \tag{4.7}$$

so that

$$MR_{\mathcal{F}}(t, x - 1) = R_{\mathcal{F}}(t, x). \tag{4.8}$$

We prove Theorem 4.1 via a simple involution. Now Theorem 4.1 has the following obvious corollary.

Corollary 4.1. *Suppose that*

$$\mathcal{F} = \bigcup_{n\geq 0}\langle f, g, A_n, \mathcal{F}_n, \mathcal{P}_n\rangle$$

and

$$\mathcal{G} = \bigcup_{n \geq 0} \langle h, g, B_n, \mathcal{G}_n, \mathcal{Q}_n \rangle$$

are pattern families. (Here we are not insisting that $f = h$, which means that for any given n, the elements of $\mathcal{F}_n$ and $\mathcal{G}_n$ can have different lengths, but we are insisting that the number of patterns in $\mathcal{P}_n$ and $\mathcal{Q}_n$ are the same.) Let $\mathcal{PMF}$ and $\mathcal{PMG}$ be the partially marked pattern families constructed from $\mathcal{F}$ and $\mathcal{G}$, respectively. Then

$$MR_{\mathcal{F}}(t, y_1, y_2, \ldots) = MR_{\mathcal{G}}(t, y_1, y_2, \ldots) \tag{4.9}$$

implies

$$R_{\mathcal{F}}(t, x_1, x_2, \ldots) = R_{\mathcal{G}}(t, x_1, x_2, \ldots). \tag{4.10}$$

We can give a completely bijective proof of Corollary 4.1 by combining our proof of Theorem 4.1 with the involution principle of Garsia and Milne [26]. That is, for all $n \geq 0$, if there is a bijection $\Gamma_n : \mathcal{PMF}_n \to \mathcal{PMG}_n$ such that for all $(s, H_1, \ldots, H_{g(n)}) \in \mathcal{PMF}_n$,

$$w_{\mathcal{PMF}}(s, H_1, \ldots, H_{g(n)}) = w_{\mathcal{PMG}}(\Gamma_n((s, H_1, \ldots, H_{g(n)})),$$

then we can construct a bijection $\theta_n : \mathcal{F}_n \to \mathcal{G}_n$ from Γ_n such that for all $s \in \mathcal{F}_n$,

$$\prod_{i=1}^{g(n)} x_i^{P_i(s)} = \prod_{i=1}^{g(n)} x_i^{Q_i(\theta_n(s))},$$

where $\mathcal{P}_n = \{P_1, \ldots, P_{g(n)}\}$ and $\mathcal{Q}_n = \{Q_1, \ldots, Q_{g(n)}\}$.

The outline of this chapter is as follows. In Section 4.2, we shall give our proofs of Theorem 4.1 and Corollary 4.1. Then in Section 4.3, we shall give several examples where we can construct bijections between partially marked pattern families that preserve the number of marked patterns, which can be automatically exploited to give bijections between their corresponding pattern families that preserve the number of occurrences of each pattern.

4.2 Main results

In this section, we shall give a proof of Theorem 4.1 and a bijective proof of Corollary 4.1.

Proof of Theorem 4.1. Clearly to prove Theorem 4.1, we need only show that, for fixed $s \in \mathcal{F}_n$,

$$\sum_{\langle s,H_1,\ldots,H_{g(n)}\rangle \in \mathcal{PMF}_n} \prod_{i=1}^{g(n)} (x_i-1)^{|H_i|} = \prod_{i=1}^{n} x_i^{P_i(s)}, \tag{4.11}$$

where $\mathcal{P}_n = \{P_1,\ldots,P_{g(n)}\}$. Let $C_i(s)$ be the set of all P_i-patterns that occur in s, such that $H_i \subseteq C_i(s)$. We then consider the space $\mathcal{UF}_n$ containing all $(2g(n)+1)$-tuples $\langle s,H_1,\ldots,H_{g(n)},X_1,\ldots,X_{g(n)}\rangle$ such that $\langle s,H_1,\ldots,H_{g(n)}\rangle \in \mathcal{PMF}_n$ and $X_i \subseteq H_i$ for $i=1,\ldots,n$. For $u = \langle s,H_1,\ldots,H_{g(n)},X_1,\ldots,X_{g(n)}\rangle$, we then define the weight of u, $W_{\mathcal{UF}}(u)$, to be

$$W_{\mathcal{UF}}(u) = \prod_{i=1}^{g(n)} x_i^{|X_i|} \tag{4.12}$$

and the sign of u, $sgn(u)$, to be

$$sgn(s,H_1,\ldots,H_{g(n)},X_1,\ldots,X_{g(n)}) = \prod_{i=1}^{g(n)} (-1)^{|H_i|-|X_i|}. \tag{4.13}$$

For fixed s, it is then easy to see that

$$\sum_{u=\langle s,H_1,\ldots,H_{g(n)},X_1,\ldots,X_{g(n)}\rangle \in \mathcal{UF}_n} sgn(u)W_{\mathcal{UF}}(u) = \sum_{\langle s,H_1,\ldots,H_{g(n)}\rangle \in \mathcal{PMF}_n} \prod_{i=1}^{g(n)} (x_i-1)^{|H_i|}. \tag{4.14}$$

Next we define an involution I on $\mathcal{UF}_n$ as follows. Given $u \in \mathcal{UF}_n$, let e be the least d such that $C_d(s) \setminus X_d$ is not empty. Then look for the least pair $\delta = \langle a_1 \ldots, a_k, b_1, \ldots, b_k\rangle$ in $C_e(s) \setminus X_e$, where we order the pairs in P_e using a graded lexicographic order on the first element, followed by a lexicographic order on the second element when the first elements are equal. Then (i) if $\delta \in H_e$, we let

$$I(u) = \langle s,H_1,\ldots,H_{e-1},H_e-\{\delta\},H_{e+1},\ldots H_{g(n)},X_1,\ldots,X_{g(n)}\rangle$$

and (ii) if $\delta \notin H_e$, we let

$$I(u) = \langle s, H_1, \ldots, H_{e-1}, H_e \cup \{\delta\}, H_{e+1}, \ldots H_{g(n)}, X_1, \ldots, X_{g(n)} \rangle.$$

If there is no such e, then it must be the case that $C_i(s) = H_i = X_i$ for all i, in which case we define $I(u) = u$. It is easy to see that I is an involution and that if $I(u) \neq u$, then $sgn(u)W_{\mathcal{UF}}(u) = -sgn(I(u))W_{\mathcal{UF}}(I(u))$. Hence I shows that

$$\sum_{u \in \mathcal{UF}_n} sgn(u)W_{\mathcal{UF}}(u) = \sum_{\substack{u \in \mathcal{UF}_n \\ I(u)=u}} sgn(u)W_{\mathcal{UF}}(u).$$

Thus we must examine the fixed points of I. Note that if $I(u) = u$ where $u = \langle s, H_1, \ldots, H_{g(n)}, X_1, \ldots, X_{g(n)} \rangle$, then we must have $C_i(s) = H_i = X_i$ for all i and hence $sgn(u) = 1$ and

$$W_{\mathcal{UF}}(u) = \prod_{i=1}^{g(n)} x_i^{|X_i|} = \prod_{i=1}^{g(n)} x_i^{|C_i(s)|} = \prod_{i=1}^{g(n)} x_i^{P_i(s)}.$$

Thus it follows that

$$\sum_{\substack{u \in \mathcal{UF}_n \\ I(u)=u}} sgn(u)W_{\mathcal{UF}}(u) = \prod_{s \in \mathcal{F}_n} x_i^{P_i(s)}, \tag{4.15}$$

which is what we wanted to prove. □

Garsia and Milne [26] proved the following theorem.

Theorem 4.2 (Involution Principle)**.** *Suppose S (resp. S') is a finite set of signed objects, and I (resp. I') is a sign-reversing involution on S (resp. S') such that the set T (resp. T') of fixed points of I (resp. I') consists of positive objects. Suppose also that $f : S \to S'$ is a sign-preserving bijection. Then there is a bijection $g : T \to T'$ constructed canonically from I, I', and f. To compute $g(x)$ for $x \in T$, we repeatedly apply f, then I', then f^{-1}, then I, until an application of f yields an element of T'. See Figure 4.1.*

Next we present our bijective proof of Corollary 4.1.

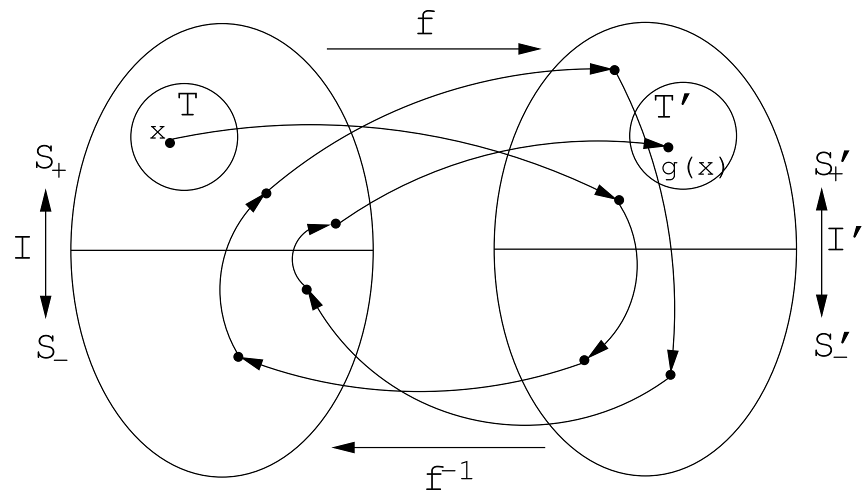


Figure 4.1: Schematic setup for the involution principle

Proof of Corollary 4.1. Suppose that we are given pattern families $\mathcal{F}$ and $\mathcal{G}$ such that

$$MR_{\mathcal{F}}(t, y_1, y_2, \ldots) = MR_{\mathcal{G}}(t, y_1, y_2, \ldots). \tag{4.16}$$

Then there exists a bijection $\Gamma_n : \mathcal{PMF}_n \to \mathcal{PMG}_n$ such that for each $(g(n)+1)$-tuple $\langle s, H_1, \ldots, H_{g(n)} \rangle \in \mathcal{PMF}_n$,

$$w_{\mathcal{PMF}}(s, H_1, \ldots, H_{g(n)}) = w_{\mathcal{PMG}}(\Gamma_n(s, H_1, \ldots, H_{g(n)})).$$

Let $\mathcal{UF}_n$ be the set of all $(2g(n)+1)$-tuples $\langle s, H_1, \ldots, H_{g(n)}, X_1, \ldots, X_{g(n)} \rangle$ such that $\langle s, H_1, \ldots, H_{g(n)} \rangle \in \mathcal{PMF}_n$ and $X_i \subseteq H_i$ for $i = 1, \ldots, n$. Similarly, let $\mathcal{UG}_n$ be the set of all $(2g(n)+1)$-tuples $\langle s, H_1, \ldots, H_{g(n)}, X_1, \ldots, X_{g(n)} \rangle$ such that $\langle s, H_1, \ldots, H_{g(n)} \rangle \in \mathcal{PMG}_n$ and $X_i \subseteq H_i$ for $i = 1, \ldots, n$.

We can use Γ_n to obtain a bijection $f_n : \mathcal{UF}_n \to \mathcal{UG}_n$ as follows. First suppose that $\Gamma_n(s, H_1, \ldots, H_n) = (t, K_1, \ldots, K_n)$. Since we are assuming that $w_{\mathcal{PMF}}(s, H_1, \ldots, H_{g(n)}) = w_{\mathcal{PMG}}(t, K_1, \ldots, K_{g(n)})$, we know that $|H_i| = |K_i|$ for $i = 1, \ldots, g(n)$. In the proof of Theorem 4.1, we defined a total order on the pairs $\langle a_1 \ldots a_k, b_1 \ldots b_k \rangle$ in P_i for all i, where $\mathcal{P}_n = \{P_1, \ldots, P_{g(n)}\}$. We can define a similar total order on the pairs in Q_i for all i, where $\mathcal{Q}_n = \{Q_1, \ldots, Q_{g(n)}\}$. We use these total orders to give a total order to the pairs in H_i and to the pairs in

K_i. This given, we define

$$f_n(\langle s, H_1, \ldots, H_{g(n)}, X_1, \ldots, X_{g(n)} \rangle) = \langle t, K_1, \ldots, K_{g(n)}, Y_1, \ldots, Y_{g(n)} \rangle,$$

where Y_i is the set of pairs in K_i with the same positions in K_i's total order as the pairs in X_i have in H_i's. It is then easy to see that f_n is a sign- and weight-preserving bijection. That is, for all such tuples related by f_n,

$$\begin{aligned} \prod_{i=1}^{g(n)} x_i^{|X_i|} &= \prod_{i=1}^{g(n)} x_i^{|Y_i|} \text{ and} \\ \prod_{i=1}^{g(n)} (-1)^{|H_i|-|X_i|} &= \prod_{i=1}^{g(n)} (-1)^{|K_i|-|Y_i|}. \end{aligned}$$

Let I be the sign-reversing, weight-preserving involution on $\mathcal{UF}_n$, as defined in the proof of Theorem 4.1, whose fixed points are the set of all $(2g(n)+1)$-tuples $\langle s, H_1, \ldots, H_n, X_1, \ldots, X_n \rangle$ such that $C_i(s) = H_i = X_i$ for all i. Note the sign of such fixed points is 1. We can similarly define a sign-reversing, weight-preserving involution J on $\mathcal{UG}_n$ whose fixed points are the set $\langle t, K_1, \ldots, K_n, Y_1, \ldots, Y_n \rangle$ such that $D_i(t) = K_i = Y_i$ for all i, where $D_i(t)$ is the set of all pairs $\langle a_1 \ldots a_k, b_1 \ldots b_k \rangle$ that occur in t. This gives all the ingredients to invoke the involution principle of Garsia and Milne.

Thus from f_n, I, and J, we construct a bijection g_n that maps the fixed points of I onto the fixed points of J, such that if $g_n(\langle s, H_1, \ldots, H_n, X_1, \ldots, X_n \rangle) = \langle t, K_1, \ldots, K_n, Y_1, \ldots, Y_n \rangle$, then $|C_i(s)| = |H_i| = |K_i| = |D_i(t)|$ for all i. We then define a map $\theta_n : \mathcal{F}_n \to \mathcal{G}_n$ by $\theta_n(s) = t$ if and only if g_n maps the unique fixed point of I starting with s to the unique fixed point of J starting with t. Clearly θ_n is a bijection with our desired weight-preserving property, namely, for all $s \in \mathcal{F}_n$, if $\theta_n(s) = t$, then

$$\prod_{i=1}^{g(n)} x_i^{P_i(s)} = \prod_{i=1}^{g(n)} x_i^{Q_i(t)}.$$

□

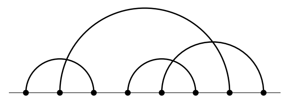
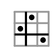


Figure 4.2: The matching $M = \{(1,3),(2,7),(4,6),(5,8)\}$

4.3 Examples

4.3.1 A conjecture due to Claesson and Linusson

Following Claesson and Linusson [14], we make the following definitions. A *matching* on $[2n]$ is a partition of that set into blocks of size 2. An example of a matching is $M = \{(1,3),(2,7),(4,6),(5,8)\}$. Figure 4.2 shows a graphical representation of M with an *arc* connecting i with j precisely when $(i,j) \in M$. A *nesting* of M is a pair of arcs (a_1, a_2) and (b_1, b_2) with $a_1 < b_1 < b_2 < a_2$. We call such a nesting a *left-nesting* if $b_1 = a_1 + 1$. Similarly, we call it a *right-nesting* if $a_2 = b_2 + 1$. A given nesting may be a left-nesting, a right-nesting, both, or neither. M has one nesting, formed by the arcs $(2,7)$ and $(4,6)$. It is a right-nesting.

In a permutation $\pi = \pi_1 \cdots \pi_n$, an occurrence of the pattern  is a subsequence $\pi_i \pi_{i+1} \pi_j$ such that $\pi_j + 1 = \pi_i < \pi_{i+1}$. As an example, the permutation $\pi = 351426$ contains one such occurrence, 352. It can be seen graphically in Figure 4.3.

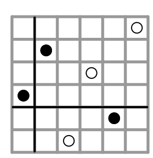
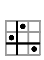
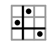


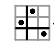
Figure 4.3: Occurrence of the pattern in $\pi = 351426$

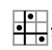
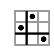
Bousquet-Mélou et al. [5] gave bijections between matchings on $[2n]$ with no left- or right-nestings and three other classes of combinatorial objects, thus proving that they are equinumerous. The other classes were unlabeled $(\mathbf{2}+\mathbf{2})$-free posets (or interval orders) on n nodes; permutations in S_n avoiding the pattern  ; and ascent sequences of length n. Claesson and Linusson conjectured that the distribution of right-nestings in matchings on $[2n]$ with no left-nestings is equal to the distribution of occurrences of  in S_n.

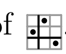
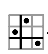
In order to use Corollary 4.1, we first must express our sets as pattern families. While this is straightforward for occurrences of  in S_n, matchings on $[2n]$ do not easily fit into this framework. We will first need to define an encoding of matchings with no left-nestings as words, then define a pattern that corresponds to right-nestings. There are a few obvious encodings, but Claesson and Linusson [14] give a bijection between these matchings and inversion tables that will better suit our needs.

Let NLN_n be the set of matchings on $[2n]$ with no left-nestings, and let $M \in \mathrm{NLN}_n$. If $i < j$ and $\alpha = (i, j)$ is an arc in M, then we call i the *opener* of α and j the *closer* of α. Order the arcs of M by closer, so for example, the arc with closer $2n$ is the n-th arc. An *inversion table* is a weak composition where the i-th letter must be strictly less than i. Let $\mathcal{I}_n$ be the set of inversion tables of length n. Then the function $f : \mathrm{NLN}_n \to \mathcal{I}_n$, given by $f(M) = w$, where w_i is the number of closers less than the opener of the i-th arc of M, is a bijection.

Now we must determine what pattern on inversion tables corresponds to a right-nesting in the associated matching. Suppose that $a_1 < b_1 < b_2 < a_2$ and that (a_1, a_2) and (b_1, b_2) are arcs in M that form a right-nesting. By definition, $a_2 = b_2 + 1$, so these arcs are consecutive when ordered by closer, with (b_1, b_2) first. Moreover, since M has no left-nestings, there must be a closer between a_1 and b_1, so there is at least one more closer to the left of b_1 than a_1. Thus, if (b_1, b_2) is the i-th arc, then $w_i > w_{i+1}$. Further, there must be no openers between b_2 and a_2, so if $j > i + 1$, then $w_j \neq i$. It is also clear that whenever two arcs satisfy these properties, they form a right-nesting. We will call the position i a *proscriptive descent*. We see that proscriptive descents in $\mathcal{I}_n$ correspond to right-nestings in NLN_n.

We will now use this setup to construct the bijection Γ_n. Our construction will be recursive, so define Γ_0 to map the empty inversion table to the empty permutation, and suppose that Γ_{n-1} has already been defined. Let $w \in \mathcal{I}_{n-1}$, and let H be a set of marked proscriptive descents in w. Let $(\sigma, K) = \Gamma_{n-1}(I, H)$ be the corresponding permutation in S_{n-1} and set of marked occurrences of .

We consider the ways that w can be extended to an element of $\mathcal{I}_n$ with the same set H: w_n may be chosen with $0 \leq w_n \leq n-1$, as long as $w_n \notin H$ for a total of $n-|H|$ ways. Also, σ can be extended to an element of S_n with essentially the same set K by inserting a new smallest element, as in the proof of Theorem 2.2. We say "essentially" because every element of σ is increased by 1 and some move to the right. For example, inserting into the second position of 24315 yields 351426. However, if 241 was a marked occurrence of , then 352 is the new marked occurrence. If $j > i+1$ and $\sigma_j + 1 = \sigma_i < \sigma_{i+1}$ is an occurrence of  in K, then we may not insert a new smallest element at position $i+1$. Thus, there are $n-|K|$ places we may insert a new smallest element. For both w and σ, we can order the choices we have to extend each word. For w, we will sort them by w_n. For σ, we will sort them by the distance from the end of σ that 1 is inserted. Then, Γ_n will associate the corresponding items in each sorted list.

It is also possible to extend w while adding a marked proscriptive descent to H: w_n may be chosen with $0 \leq w_n < w_{n-1}$, as long as $w_n \notin H$. Thus, if w_{n-1} was chosen as the k-th valid choice, then there will be $k-1$ valid choices for w_n. We can also extend σ while adding a marked occurrence of . We must insert the new smallest element more than one position to the right of the previous smallest element, and we still must not place it between the elements forming an ascent in a marked occurrence of . Thus, if the 1 in σ had been placed k-th valid position from the right, then there will be $k-1$ valid positions for the new smallest element. We will sort the possibilities as in the previous case. Then, Γ_n will associate the corresponding items in each sorted list.

This completes the description of the bijection Γ_n. We may now apply Corollary 4.1 to obtain a bijection θ_n that proves Claesson and Linusson's conjecture.

4.3.2 A conjecture due to Jones

Let $w = (w_1 w_2 \ldots w_n)$ be an n-cycle in a permutation, and let $\pi \in S_m$, where $m \leq n$. A *cycle-match* of the pattern π in w is a sequence of consecutive elements of w, where w_1 follows w_n, that reduce to π. For example, in the cycle (34769), 693 is a 231-cycle-match. We will let $\pi_{\text{cyc}}(\sigma)$ denote the total number of π-cycle-matches in the cycles of σ. That is, if $\text{Cyc}(\sigma)$ denotes the set of cycles in σ and $\pi_{\text{cyc}}(w)$ denotes the number of π-cycle-matches in the cycle w, then $\pi_{\text{cyc}}(\sigma) = \sum_{w \in \text{Cyc}(\sigma)} \pi_{\text{cyc}}(w)$. If P is a set of permutations, we will define $P_{\text{cyc}}(\sigma)$ and $P_{\text{cyc}}(w)$ accordingly. Also, we will say that two consecutive letters of a cycle form a cycle-descent if the first is greater than the second, and we say that cycles with one element have one cycle-descent. Thus, we define $\text{des}_{\text{cyc}}(\sigma)$ to be the number of cycle-descents in the cycles of σ and note that $\text{des}_{\text{cyc}}(\sigma) = \sum_{w \in \text{Cyc}(\sigma)}(1+\text{des}(w_1 \cdots w_{\ell(w)}))$. Finally, we will let $\text{cyc}(\sigma)$ denote the number of cycles in σ.

Let $\text{NM}_n(\pi)$ be the set of $\sigma \in S_n$ such that $\pi\text{-mch}(\sigma) = 0$, and let $\text{NCM}_n(\pi)$ be the set of $\sigma \in S_n$ such that $\pi_{\text{cyc}}(\sigma) = 0$. Then, Jones and Remmel [36] showed that if $\pi_1 = 1$, then

$$\sum_{n \geq 0} \sum_{\sigma \in \text{NM}_n(\pi)} x^{\text{ltrmin}(\sigma)} y^{\text{des}(\sigma)+1} \frac{t^n}{n!} = \sum_{n \geq 0} \sum_{\sigma \in \text{NCM}_n(\pi)} x^{\text{cyc}(\sigma)} y^{\text{des}_{\text{cyc}}(\sigma)} \frac{t^n}{n!}. \tag{4.17}$$

Further, if C_n is the set of $\sigma \in S_n$ such that $\text{cyc}(\sigma) = 1$, then they showed that

$$\sum_{n \geq 0} \sum_{\sigma \in \text{NCM}_n(\pi)} x^{\text{cyc}(\sigma)} y^{\text{des}_{\text{cyc}}(\sigma)} \frac{t^n}{n!} = e^{x \sum_{\sigma \in C_n \cap \text{NCM}_n(\pi)} y^{\text{des}_{\text{cyc}}(\sigma)} \frac{t^n}{n!}}.$$

However, their proofs were actually sufficient to conclude that

$$\sum_{n \geq 0} \sum_{\sigma \in S_n} p^{\pi\text{-mch}(\sigma)} x^{\text{ltrmin}(\sigma)} y^{\text{des}(\sigma)+1} \frac{t^n}{n!} = \sum_{n \geq 0} \sum_{\sigma \in S_n} p^{\pi_{\text{cyc}}(\sigma)} x^{\text{cyc}(\sigma)} y^{\text{des}_{\text{cyc}}(\sigma)} \frac{t^n}{n!} \tag{4.18}$$

and

$$\sum_{n \geq 0} \sum_{\sigma \in S_n} p^{\pi_{\text{cyc}}(\sigma)} x^{\text{cyc}(\sigma)} y^{\text{des}_{\text{cyc}}(\sigma)} \frac{t^n}{n!} = e^{x \sum_{\sigma \in C_n} p^{\pi_{\text{cyc}}(\sigma)} y^{\text{des}_{\text{cyc}}(\sigma)} \frac{t^n}{n!}}.$$

In fact, π can be replaced by a pattern set P where each $p \in P$ begins with 1, and we can track each pattern in P individually.

We will say that a cycle $w = (w_1 w_2 \cdots w_n)$ is covered by a set of P-cycle-matches if each pair $w_i w_{i+1}$ and $w_n w_1$ belongs to some P-cycle-match from the set. For example, the cycle (14253) is covered by the 3142-cycle-matches 3142 and 4253. Jones [35] conjectured that (4.17) is true with $x = y = 1$ for any π such that no cycle can be covered by π-cycle-matches. We see that all π such that $\pi_1 = 1$

satisfy this criterion, since $w_n w_1$ can never occur in the middle of a π-cycle-match if w_1 is the smallest element of the cycle. Jones also conjectured that (4.17) is true with $x = 1$ for certain π. We will show that (4.18) is true with $x = y = 1$ for any set P that cannot cover any cycle with P-cycle-matches. Further, (4.18) will be true whenever $P \cup \{12\}$ has that property.

Fix P so that no cycle can be covered by P-cycle-matches. It is clear that consecutive P-patterns and P-cycle-matches in permutations in S_n can be converted to the framework of pattern families. For $\sigma \in S_n$ and H a set of marked P-cycle-matches in σ, we need to define $\Gamma_n(\sigma, H) = (\tau, K)$ such that K is a set of marked consecutive P-patterns in τ.

First, order the cycles of σ in decreasing order by smallest element, and write each cycle so that its smallest element is written first. When $P = \{\pi\}$ with $\pi_1 = 1$, Jones and Remmel showed that concatenating the cycles in this order and removing the parentheses gives a bijection that sends π-cycle-matches to consecutive occurrences of π, thus proving (4.17). In order to reverse the process, we split the permutation before each left-to-right minimum and declare each subword to be a cycle of a new permutation. This bijection is known as Foata's First Fundamental Transformation [23]. However, when $\pi_1 > 1$, this method no longer works. For example, the cycle (1432) has one 2143-cycle-match, but its image 1432 has no consecutive occurrences of 2143. Likewise, 2143 has one consecutive occurrence, but its preimage (2)(143) has none. We will thus continue our description of Γ_n.

If any P-cycle-match in H wraps around the end of its cycle, then rewrite the cycle so that it begins with that P-cycle-match. Repeat until no element of H wraps around the end of its cycle. This is always possible because of the condition on P. Now remove the parentheses from the cycles of σ to obtain τ and set $J = H$. For example, if $P = \{2143\}$, $\sigma = (1542)(36)$, and $H = \{2154\}$, then we first write $\sigma = (36)(1542)$, then $\sigma = (36)(2154)$, then $\tau = 362154$. On the other hand, if we had not marked the 2143-cycle-match, *i.e.* if $H = \emptyset$, then $\tau = 361542$ as in Foata's transformation.

In order to reverse Γ_n, start with τ and K, and break τ before its left-to-right minima. However, if we have broken τ in the middle of one of its marked P-patterns, then we move that break to the left until it is no longer contained in a marked P-pattern. We note that this could move multiple break points to the same position, so that the bijection no longer identifies left-to-right minima of τ with cycles of σ. Finally, take the sub-words delimited by the breaks in τ, and declare them to be cycles of σ, setting $H = K$.

In this case, our bijection tells us even more: since Γ_n sets $J = H$, $\theta_n(\sigma)$ has the exact same P-patterns as σ's P-cycle-matches, rather than just the same number. Also, note that every element of σ either begins a cycle-descent or a 12-cycle-match, so that $\mathrm{des}_{\mathrm{cyc}}(\sigma) = \ell(\sigma) - 12_{\mathrm{cyc}}(\sigma)$. Also, since $\mathrm{des}(\sigma) = \ell(\sigma) - 1 - 12\text{-mch}(\sigma)$, we see that $\mathrm{des}(\theta_n(\sigma)) + 1 = \mathrm{des}_{\mathrm{cyc}}(\sigma)$ whenever $12\text{-mch}(\theta_n(\sigma)) = 12_{\mathrm{cyc}}(\sigma)$. Therefore, if $P \cup \{12\}$ cannot cover any cycle with cycle-matches, then (4.18) will be true with $x = 1$. We conclude with two interesting classes of sets P that cannot cover a cycle.

If $\sigma \in S_n$, define $\sigma.^* = \{\tau : \tau \in \bigcup_{m\geq n} S_m, \tau_i = \sigma_i \text{ for } 1 \leq i \leq n\}$ and $\sigma.^+ = \{\tau : \tau \in \bigcup_{m>n} S_m, \tau_i = \sigma_i \text{ for } 1 \leq i \leq n\}$. Then, for any k,

$$P_k = \bigcup_{\sigma\in S_k} \sigma.^+ \text{ and} \tag{4.19}$$

$$Q_k = k(k-1)\cdots 1.^+ \bigcup 12.^* \bigcup 213.^* \bigcup \cdots \bigcup (k-1)(k-2)\cdots 1k.^* \tag{4.20}$$

cannot cover a cycle. We shall now prove these assertions.

Let τ be a permutation that is covered by n overlapping P_k-patterns. We will prove by induction that any k consecutive letters of τ after the first must contain at least one letter greater than the first k letters of τ, and the last letter of τ must be one of these. Then, it is impossible to add a pattern to complete a cycle. Our claim is clearly true for $n = 1$. Now, suppose it is true for all τ covered by $n-1$ overlapping patterns. Sort the patterns covering τ by starting position, then length. If the n-th pattern begins at position 1, then the claim is trivially true. Otherwise, by the inductive hypothesis, one of the first k letters of the n-th pattern must be greater than the first k letters of τ. However, the $k+1$-th through the last letter of the n-th pattern must be greater than that. Therefore, the last letter of τ is greater than the first k letters of τ, and every k consecutive letters after the first still contain one greater than the first k letters of τ.

For the second assertion, let τ be a permutation that is covered by n overlapping Q_k-patterns. If the first j letters of τ are part of descents, then we will prove by induction that the letter after the 1 in any of these patterns must be greater than or equal to τ_{j+1}. Then, it is impossible to complete a cycle from τ. Our claim is clearly true for $n = 1$. Now, suppose it is true for all τ covered by $n-1$ overlapping patterns. Sort the patterns covering τ by starting position, then length. If the n-th pattern begins in the initial string of descents of another pattern, then they may only overlap with their 1s coinciding, in which case the inductive hypothesis applies. Otherwise, the first letter of the n-th pattern must be greater than or equal to τ_{j+1}, since it belongs to some pattern after that pattern's 1. The letter after the 1 of the n-th pattern must be greater than that, so our claim is true.

4.3.3 Permutation patterns

If w is a composition, then define $.^*w$ to be the set of compositions that end with w. For $s, t \in [m]$, let $P_m^{st} = .^*st \cap S_m$. We'll say a position i in a permutation σ is an S_m^{st}-pattern if σ_i plays the role of s in at least one P_m^{st}-pattern; more precisely, if we have $i_1 < \cdots < i_m$ such that $\mathrm{red}(\sigma_{i_1} \cdots \sigma_{i_m}) \in P_m^{st}$, then position i_{m-1} is an S_m^{st}-pattern. We let $S_m^{st}(\sigma)$ denote the number of such positions in σ.

Theorem 4.3. *If $t \in \{1, s-2, s-1, s+1, s+2, m\}$, then the distribution of $S_m^{st}(\sigma)$ in S_n is independent of s, t.*

Proof. Rather than considering permutations where some subset of the S_m^{st} occurrences have been marked, we're going to mark a subset of the positions that do not match the pattern. We'll recursively construct these permutations, showing that the number of choices we have at each step is independent of the particular choices of s and t, provided they satisfy the given constraints. This construction will clearly give rise to a bijection.

We'll build permutations from left to right. When a permutation has $n-1$ letters, we may choose to add a new last letter, taking a value σ_n from 1 to n. All of the previous letters greater than or equal to σ_n are then increased by one. For example, adding 3 to 2413 results in 25143. When we add a new letter, we may also choose to mark that letter, meaning that it will never play the role of s in a P_m^{st}-pattern. Thus, marking a letter may restrict the options available for letters added later.

Given a marked permutation $\sigma \in S_{n-1}$, suppose there are k options available for σ_n. Consider the consequences of a particular choice of σ_n. We will show that the following claims are true:

- If $n < m - 1$ or we do not mark the letter, then there will be $k + 1$ options available for σ_{n+1}.
- Otherwise, the choices of σ_n can be ordered such that the first results in $m - 1$ options for σ_{n+1}, and each successive choice adds one more option, up to a maximum of $k + 1$, which is then repeated.

For the first claim, if $n < m - 1$, then σ does not have enough letters for σ_n to play the role of s. Therefore, marking it cannot create any restrictions, so the number of options will increase by one as normal. For the rest of the claim, if we do not mark σ_n, then we certainly do not create any new restrictions. However, from the perspective of the marked letters in σ, adding σ_n splits its position into two: σ_n and $\sigma_n + 1$. Either of these choices for σ_{n+1} would play the same role in any permutation pattern not using σ_n. All other options remain unchanged from the perspective of the marked letters. Therefore, the number of options for σ_{n+1} is $k + 1$.

Now we'll consider the second claim. Without loss of generality, assume $t > s$. The other cases can be treated by taking the complement. Regardless of s, t, the lowest $t - 1$ positions and highest $m - t$ positions will always be available, as these cannot play the role of t in any P_m^{st}-pattern. Choosing $\sigma_n = s$ will make all other positions unavailable for σ_{n+1}. This gives our lower bound of $m - 1$ options. For the rest of the claim, we will deal with each valid choice of t separately.

Suppose $t = m$. In this case, for a σ_n to be unavailable means that it would be the largest of a particular choice of letters in σ. Thus, any larger value would also be unavailable, so the k choices for σ_n must be $1, 2, \ldots, k$. Starting from $\sigma_n = s$, incrementing σ_n allows σ_{n+1} to be placed one higher, so the number of options ascends to k. At that point, all remaining choices for σ_n cannot play the role of s in any P_m^{st}-pattern, so they result in $k+1$ options, as in the case where σ_n is unmarked.

Suppose $t = s + 1$. In this case and the next, the options for σ_n are not necessarily consecutive. Choosing σ_n results in all higher numbers, apart from the top $m - t$, becoming unavailable for σ_{n+1}. Therefore, starting with $\sigma_n = s$, incrementally choosing the next available option greater than σ_n results in one more option for σ_{n+1}, up to k. All remaining choices of σ_n cannot play the role of s in a P_m^{st}-pattern, so they result in $k+1$ options.

Suppose $t = s + 2$. Choosing σ_n results in all numbers higher than $\sigma_n + 1$, apart from the top $m - t$, becoming unavailable for σ_{n+1}. However, after adding σ_n, the positions corresponding to all the higher options are incremented by one, so these are all now greater than $\sigma_n + 1$ and thus become unavailable. Therefore, incrementally choosing the next available option greater than σ_n results in one more option for σ_{n+1}, up to k. All remaining choices of σ_n cannot play the role of s in a P_m^{st}-pattern, so they result in $k+1$ options. □

Theorem 4.3 may be extended somewhat. Fix k, m, and s, and let T be a subset of k consecutive numbers from $[m] \setminus \{s\}$ such that T contains at least one of $\{1, s-2, s-1, s+1, s+2, m\}$. Then let $P_m^{sT} = \bigcup_{t \in T} P_m^{st}$. We say a position i in a permutation σ is a S_m^{sT}-pattern if σ_i plays the role of s in at least one P_m^{st}-pattern, and we let $S_m^{sT}(\sigma)$ denote the number of such positions in σ. Then, for fixed k, the distribution of S_m^{sT} in S_n is independent of s and T. The proof is essentially the same as the proof of Theorem 4.3. The main difference is that as soon as one position becomes unavailable, k positions simultaneously become unavailable.

This extension has an application to so-called quadrant marked mesh patterns [42]. Given $\sigma \in S_n$, we graph σ by placing a point at (i, σ_i) for each $i \in [n]$. Then, we say that a position i is an occurrence of the quadrant marked mesh pattern $Q^{(a,b,c,d)}$ in σ if, when we draw coordinate axes through the point (i, σ_i), there are at least a, b, c, and d points in the graph of σ in the first, second,

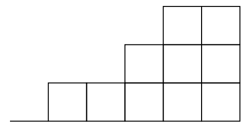


Figure 4.4: The Ferrers board F(0,1,1,2,3,3)

third, and fourth quadrants, respectively. Then, we see that an $S_m^{s[s-1]}$-pattern is equivalent to a $Q^{(0,m-s,s-2,1)}$-pattern. Based on the study of quadrant marked mesh patterns, Kitaev and Remmel [42] conjectured that the number of permutations in S_n avoiding $\{4132, 4123, 3124, 3142\}$ is the same as the number avoiding $\{4231, 4213, 3241, 3214\}$. After reversing each permutation, we see this is the same as avoiding $P_4^{1\{3,4\}}$ or $P_4^{2\{3,4\}}$, respectively, which is equivalent to avoiding $S_4^{1\{3,4\}}$ or $S_4^{2\{3,4\}}$. The extension of Theorem 4.3 proves that these have the same distribution; as a special case, then, there are the same number of permutations avoiding each pattern, proving the conjecture.

4.3.4 Rook placements

The theory of rook polynomials was introduced by Kaplansky and Riordan [37] and developed further by Riordan [57]. We refer the reader to Stanley [58, Chap. 2] for a nice exposition of some of the basics of rook polynomials and permutations with forbidden positions.

We shall consider rectangular boards $\mathbb{B}_{n,m} = [n] \times [m]$ consisting of n columns of height m. We label the rows of $\mathbb{B}_{n,m}$ from bottom to top with $1, \ldots, m$ and the columns from left to right with $1, 2, 3, \ldots, n$ and let (i, j) denote the square in the i-th column and j-th row. Given $b_1, \ldots, b_n \leq k$, we let $F(b_1, \ldots, b_n)$ denote the board consisting of all the cells $\{(i, j) : 1 \leq i \leq n, 1 \leq j \leq b_i\}$. If a board B is of the form $B = F(b_1, \ldots, b_n)$, where $b_1 \leq b_2 \leq \cdots \leq b_n$, then we say that B is a *Ferrers board*. For example, Figure 4.4 pictures the Ferrers board $F(0, 1, 1, 2, 3, 3)$.

Given a board $B \subseteq \mathbb{B}_{n,m}$, we let $\mathcal{N}_k(B)$ denote the set of all placements of k rooks in B such that no two rooks lie in the same row or column. We let $r_k(B) = |\mathcal{N}_k(B)|$. Let $\mathcal{F}_{n,m}$ denote the set of all one-to-one functions $f : [n] \to [m]$. We identify each f with the rook placement $\{(i, f(i)) : i = 1, \ldots, n\}$ on

$\mathbb{B}_{n,m}$. Figure 4.5 gives an example rook placement corresponding to the function $f = \{(1,1),(2,6),(3,5),(4,2),(5,4)\} \in \mathcal{F}_{5,6}$. We let

$$h_k^{(n,m)}(B) = |\{f \in \mathcal{F}_{n,m} : |f \cap B| = k\}|.$$

We shall shall refer to $h_k^{(n,m)}(B)$ as the k-th generalized hit number for B with respect to $\mathbb{B}_{n,m}$. In the special case where $n = m$, then $\mathcal{F}_{n,m}$ can be thought of as the set of permutations in S_n, and then the $h_k^{(n,m)}(B)$ are called hit numbers.

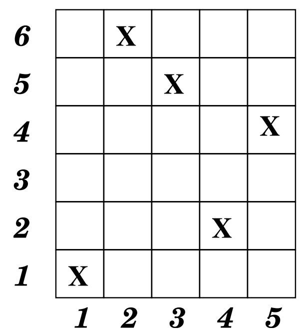


Figure 4.5: The rook placement associated with a function $f \in \mathcal{F}_{5,6}$

Following Kaplansky and Riordan [37], we have the following fundamental relationship between the rook numbers and generalized hit numbers for a board $B \subseteq \mathbb{B}_{n,m}$. Let $(n)\downarrow_0 = 1$ and $(n)\downarrow_k = n(n-1)\cdots(n-k+1)$ for $k \geq 1$.

Theorem 4.4. *For any board $B \subseteq \mathbb{B}_{n,m}$,*

$$\sum_{k=0}^{n} h_k^{(n,m)}(B)x^k = \sum_{k=0}^{n} r_k(B)(m-k)\downarrow_{n-k} (x-1)^k. \tag{4.21}$$

Proof. Replacing x by $x+1$ in equation (4.21), we see that it is enough to prove

$$\sum_{k=0}^{n} h_k^{(n,m)}(B)(x+1)^k = \sum_{k=0}^{n} r_k(B)(m-k)\downarrow_{n-k} x^k. \tag{4.22}$$

To interpret the left-hand side of (4.22), consider the set of objects O obtained by first picking a rook placement on $\mathbb{B}_{n,m}$ associated with some function $f \in \mathcal{F}_{n,m}$, and then for each rook r in $f \cap B$, we either circle the rook or not. For each such object O, we define the weight of O as $x^{\mathrm{circ}(O)}$ where $\mathrm{circ}(O)$ is the number of circled rooks in O. If $\mathcal{O}_B$ is the set of objects constructed in this way, it is easy to see that

$$\sum_{O\in\mathcal{O}_B} x^{\mathrm{circ}(O)} = \sum_{k=0}^{n} h_k^{(n,m)}(B)(x+1)^k. \tag{4.23}$$

However, we can also obtain the left-hand side of (4.23) by first picking the number k of circled rooks, then picking the k circled rooks as a placement P of k non-attacking rooks on B, which can be done in $r_k(B)$ ways, and finally extending P to a placement corresponding to a function $f \in \mathcal{F}_{n,m}$ by adding $n-k$ non-circled rooks, which can done in $(m-k)\downarrow_{n-k}$ ways. Thus

$$\sum_{O\in\mathcal{O}_B} x^{\mathrm{circ}(O)} = \sum_{k=0}^{n} r_k(B)(m-k)\downarrow_{n-k} x^k, \tag{4.24}$$

which proves (4.22). □

Theorem 4.4 tells us that if $B^{(1)}$ and $B^{(2)}$ are boards contained in $\mathbb{B}_{n,m}$ and $r_k(B^{(1)}) = r_k(B^{(2)})$ for $k = 0,\ldots,n$, then it must be the case that $h_k^{(n,m)}(B^{(1)}) = h_k^{(n,m)}(B^{(2)})$ for $k = 0,\ldots,n$. In the special case where $n = m$, Loehr and Remmel [43] used the involution principle to give a bijective proof of this fact. That is, if we are given bijections $f_k : \mathcal{N}_k(B^{(1)}) \rightarrow \mathcal{N}_k(B^{(2)})$ for $k = 0,\ldots,n$, then Loehr and Remmel showed how one can use these bijections to construct bijections $\Theta_k : \{\sigma \in S_n : |\sigma \cap B^{(1)}| = k\} \rightarrow \{\sigma \in S_n : |\sigma \cap B^{(2)}| = k\}$ for $k = 0,\ldots,n$. Thus if one has bijections showing that $r_k(B^{(1)}) = r_k(B^{(2)})$ for all k, then one can use these bijections to construct bijections showing that the hit number $h_k^{(n,n)}(B^{(1)})$ equals the hit number $h_k^{(n,n)}(B^{(2)})$ for all k.

There are many examples of different rook boards that have the same rook numbers. A large number of such examples for Ferrers boards can be obtained from the following theorem of Goldman, Joichi, and White [28].

Theorem 4.5. *(Goldman, Joichi, and White [28])*
Let $B = F(b_1,\ldots,b_n)$ be a Ferrers board. Then

$$\prod_{i=1}^{n}(x + b_i - (i-1)) = \sum_{k=0}^{n} r_{n-k}(B)(x)\downarrow_k . \tag{4.25}$$

Thus if $B^{(1)} = F(b_1, \ldots, b_n)$ and $B^{(2)} = F(c_1, \ldots, c_n)$ are such that the multisets $\{b_i - i + 1 : i = 1, \ldots, n\}$ and $\{c_i - i + 1 : i = 1, \ldots, n\}$ are equal, then Theorem 4.5 tells us $r_k(B^{(1)}) = r_k(B^{(2)})$ for all k. For example, $B^{(1)} = F(0,0,0,3,3,3)$ and $B^{(2)} = F(0,1,1,2,2,3)$ both give rise to the multiset $\{0,0,-1,-1,-2,-2\}$, so $r_k(B^{(1)}) = r_k(B^{(2)})$ for all k. In fact, Loehr and Remmel gave a bijective proof of Theorem 4.5 using the involution principle, so there are many examples of pairs of boards $B^{(1)}$ and $B^{(2)}$ where we can construct bijections $f_k : \mathcal{N}_k(B^{(1)}) \to \mathcal{N}_k(B^{(2)})$ for $k = 0, \ldots, n$. Here we will present a general bijection f_k for Ferrers boards that does not require the involution principle.

Let $B^{(1)}$ and $B^{(2)}$ be Ferrers boards satisfying the property that the multisets defined above are equal. Then we will define the bijection f_k recursively on the number of cells in the boards $B^{(1)}$ and $B^{(2)}$ (summing over the multiset, it is clear that both boards have the same size). If both boards are empty, the bijection is clear. Thus, suppose that both boards are non-empty and consider a placement of k rooks on $B^{(1)}$. Let i_1 be a column of $B^{(1)}$ with positive height such that $b_{i_1} - i_1 + 1$ is maximal, and define i_2 likewise for $B^{(2)}$. For consistency, we choose i_1 and i_2 to be minimal, but the choice is arbitrary. By hypothesis, $b_{i_1} - i_1 + 1 = c_{i_2} - i_2 + 1$.

If the top cell in column i_1 does not contain a rook, then remove the cell from $B^{(1)}$ to obtain the board $B'^{(1)}$ and also remove the top cell from column i_2 to obtain $B'^{(2)}$. Since $b_{i_1} - i_1 + 1$ is maximal, the column to the left of i_1 must contain fewer cells, so the result is still a Ferrers board. We have then subtracted one from two equal entries in the multisets, so the resulting multisets are still equal, and we can apply f_k to the placement on $B'^{(1)}$ to obtain a placement on $B'^{(2)}$. Adding the top cell of column i_2 back, we obtain a placement of k rooks on $B^{(2)}$.

If the top cell in column i_1 does contain a rook, then remove the cell and every other cell in its row and column. Move each column to the right of i_1 left by one, and move each row above the one removed down by one. Call the resulting board $B''^{(1)}$. Each column to the right has had its column number and height reduced by one, so that $b_i - i + 1$ remains the same. Repeat this operation on board $B^{(2)}$ with the top cell in column i_2 to obtain the board $B''^{(2)}$. As we have removed two equal entries in the multisets, the resulting multisets are still equal, and we can apply f_{k-1} to the placement on $B''^{(1)}$ to obtain a placement on $B''^{(2)}$. Insert the appropriate column and row and place a rook in the top cell of column i_2 to obtain a placement of k rooks on $B^{(2)}$.

The previous paragraphs give bijections f_k between Ferrers boards that preserve the rook numbers r_k. We claim that Corollary 4.1 gives us a very simple way to prove the analogous result for generalized hit numbers. That is, suppose that $B^{(1)}, B^{(2)} \subseteq \mathbb{B}_{n,m}$ and we are given bijections $f_k : \mathcal{N}_k(B^{(1)}) \to \mathcal{N}_k(B^{(2)})$ for $k = 0, \ldots, n$. To form a pattern family for $B^{(1)}$, we let $A_n = [m]$ and $\mathcal{F}_n$ be the set of all $(f(1), \ldots, f(n))$ such that $f \in \mathcal{F}_{n,m}$. If $k \neq n$, we let $A_k = \mathcal{F}_k = \emptyset$. The pattern family $\mathcal{P}_n$ is just the set of all $P_{i,j}$ such that (i,j) is in $B^{(1)}$, where $P_{i,j}$ occurs in $(f(1), \ldots, f(n))$ if and only if $f(i) = j$, so $P_{i,j} = \{\langle i,j\rangle\}$. For the ordering of the patterns, we just use the lexicographic order on the cells $(i,j) \in B^{(1)}$. Thus a partially marked pattern $\langle f, H_1, \ldots, H_{|B^{(1)}|}\rangle$ can be viewed as the rook placement corresponding to $f \in \mathcal{F}_{n,m}$ where some of the cells in $f \cap B^{(1)}$ are marked.

For $B^{(2)}$, we let $B_n = [m]$ and $\mathcal{G}_n$ denote the set of all $(g(1), \ldots, g(n))$ such that $g \in \mathcal{F}_{n,m}$. If $k \neq n$, we let $B_k = \mathcal{G}_k = \emptyset$. The pattern family $\mathcal{Q}_n$ is just the set of all $P_{i,j}$ such that $(i,j) \in B^{(2)}$. For the ordering of the patterns, we just use the lexicographic order on the cells $(i,j) \in B^{(2)}$. Thus a partially marked pattern $\langle g, K_1, \ldots, K_{|B^{(2)}|}\rangle$ can be viewed as the rook placement corresponding to $g \in \mathcal{F}_{n,m}$ where some of the cells in $g \cap B^{(2)}$ are marked. Note the fact that there is a bijection $f_1 : \mathcal{N}_1(B^{(1)}) \to \mathcal{N}_1(B^{(2)})$ implies that $|B^{(1)}| = |B^{(2)}|$, so $\mathcal{P}_n$ and $\mathcal{Q}_n$ have the same cardinality.

We claim that the bijections $f_k : \mathcal{N}_k(B^{(1)}) \to \mathcal{N}_k(B^{(2)})$ for $k = 0, \ldots, n$ allow us to bijectively prove that $\mathcal{MR}_{\mathcal{F}}(t,y) = \mathcal{MR}_{\mathcal{G}}(t,y)$. Let $b = |B^{(1)}| = |B^{(2)}|$ and consider the coefficient $\mathcal{MR}_{\mathcal{F}}(t,y)|_{y^s}$. This coefficient is the number of partial marked sequences $\langle f, H_1, \ldots, H_b\rangle$ such that $f \in \mathcal{F}_{n,m}$ and exactly s cells are marked. Note that no two of these cells lie in the same row or column. Thus the s cells correspond to a rook placement P of size s in $B^{(1)}$. We code the marked rook placement $\langle f, H_1, \ldots, H_b\rangle$ as a pair (P, U) as follows.

1. P is the rook placement corresponding to the s marked cells in $H_1, \ldots, H_n$ and

2. U is a sequence $(u_1, \ldots, u_{n-s})$ that codes the remaining values of f as follows. Let $1 \leq p_1 < \cdots < p_{n-s} \leq n$ be the columns in $\mathbb{B}_{n,m}$ that do not contain rooks in P. Suppose that $f(p_i) = t_i$ for $i = 1, \ldots, n-s$. If row t_1 is the k-th row reading from bottom to top that does not contain a rook in P, then we let $u_1 = k$. Having defined $u_1, \ldots, u_m$ where $1 \leq m < n-s$, then we let $u_{m+1} = k$ if t_{m+1} is in the k-th row reading from bottom to top that does not contain a rook in $P \cup \{(p_i, t_i) : i = 1, \ldots, m\}$.

For example, in Figure 4.6, we show the rook board $B = F(2,2,2,2,4,5,5)$ in shaded cells and a rook placement $f \in \mathcal{F}_{7,10}$ where the cells corresponding to the marked patterns are indicated by circling the rooks in those cells. Thus $f = (7,2,9,1,6,5,3)$ and the circled cells $(2,2)$ and $(6,5)$ correspond to the pattern sets H_4 and H_{17} since $(2,2)$ is the 4th cell of B in lexicographic order and $(6,5)$ is the 17th cell of B in lexicographic order. Thus $H_4 = P_{2,2}$, $H_{17} = P_{6,5}$, and $H_i = \emptyset$ otherwise. Thus $\text{code}(f, H_1, \ldots, H_{22}) = (P, U)$, where $P = \{(2,2),(6,5)\}$ and $U = (u_1, \ldots, u_5) = (5,6,1,3,1)$. Here u_1 is equal to 5, since the rook in column 1, $(1,7)$, is in the 5th available row, reading from bottom to top, that does not contain a rook in P. Similarly $u_2 = 6$, since the rook in column 3, $(3,9)$, is in the 6th available row, reading from bottom to top, that does not contain a rook in $P \cup \{(1,7)\}$, and so on.

Similarly, the coefficient of $\mathcal{MR}_{\mathcal{G}}(t,y)|_{y^s}$ is the number of partial marked sequences $\langle g, K_1, \ldots, K_b \rangle$ such that exactly s cells are marked. Again no two of these cells lie in the same row or column. Thus the s cells correspond to a rook placement Q of size s in $B^{(2)}$.

Given a $(b+1)$-tuple $W = \langle f, H_1, \ldots, H_b \rangle$ with s marked cells, we map W to a $(b+1)$-tuple $\theta_s(W) = \langle g, K_1, \ldots, K_b \rangle$ with s marked cells if and only if $\text{code}(f, H_1, \ldots, H_b) = (P, U)$ and $\text{code}(g, K_1, \ldots, K_b) = (f_s(P), U)$. We can think of this map as follows. First, $\theta_s(W)$ takes the rook placement P of the s marked patterns in W and uses f_s to map it to the placement Q of s non-attacking rooks in $B^{(2)}$. Then it uses U to extend Q to a rook placement corresponding to a function in $\mathcal{F}_{n,m}$.

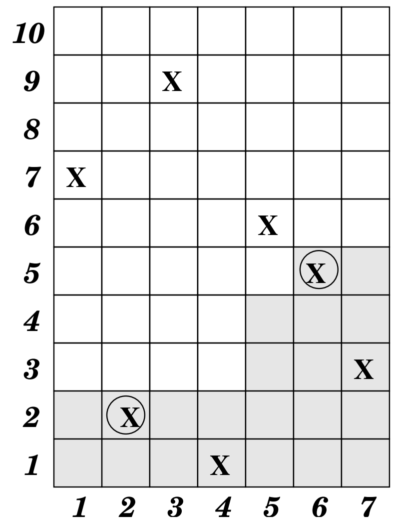


Figure 4.6: The code of a marked rook placement in $\mathcal{F}_{7,10}$

In this situation, we have shown that

$$\mathcal{MR}_{\mathcal{F}}(t,y)|_{y^s} = r_s(B^{(1)})(m-s)\downarrow_{n-s} \text{ and}$$
$$\mathcal{MR}_{\mathcal{G}}(t,y)|_{y^s} = r_s(B^{(2)})(m-s)\downarrow_{n-s},$$

and the map θ_s shows that $r_s(B^{(1)})(m-s)\downarrow_{n-s} = r_s(B^{(2)})(m-s)\downarrow_{n-s}$. Thus $\theta = \bigcup_{s=0}^{n} \theta_s$ shows that $\mathcal{MR}_{\mathcal{F}}(t,y) = \mathcal{MR}_{\mathcal{G}}(t,y)$. It then follows that our bijective proof of Corollary 4.1 allows us to construct bijections showing that $h_k^{(n,m)}(B^{(1)}) = h_k^{(n,m)}(B^{(2)})$ for $k = 0, \ldots, n$.

We note that we could further generalize the notion of hit numbers to count the number of placements of j rooks in $\mathbb{B}_{n,m}$ with k rooks inside the board B. We must merely extend our code to allow for empty columns, which is straightforward. We can also extend our results to cycle-counting rook and hit numbers. For the purposes of exposition, we will restrict ourselves to $n \times n$ boards $\mathbb{B}_{n,n}$, but this is not necessary.

With each rook placement $P \in \mathcal{N}_k(\mathbb{B}_{n,n})$, we can associate a directed graph G_P, whose vertices are labeled by $[n]$ and whose edges are the set of (i,j) such that P has a rook in cell (i,j). Figure 4.7 shows the graph associated with a rook placement in the 6×6 board. For any rook placement P, we let $\text{cyc}(P)$ denote the number of cycles in the graph of P. We note that if the placement P corresponds to a permutation σ, then the cycles of P are equivalent to the cycles of σ.

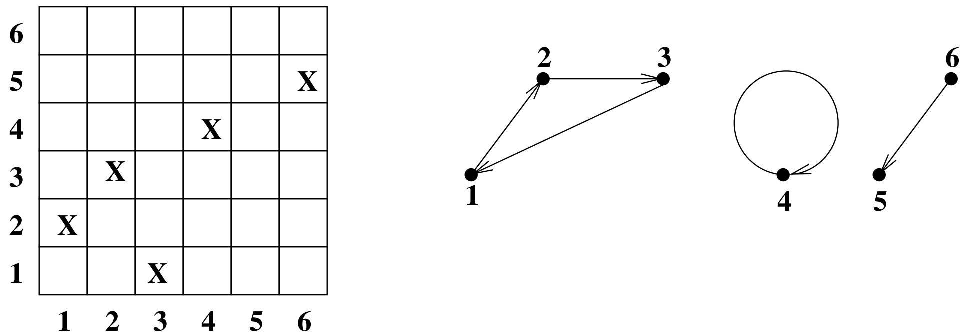


Figure 4.7: Directed graph associated with a rook placement

For any board $B \subseteq \mathbb{B}_{n,n}$, we let

$$\begin{aligned} r_k(B,y) &= \sum_{P \in \mathcal{N}_k(B)} y^{\mathrm{cyc}(P)} \text{ and} \\ h_k^{(n,n)}(B,y) &= \sum_{\substack{\sigma \in S_n \\ |\sigma \cap B| = k}} y^{\mathrm{cyc}(P)}. \end{aligned}$$

We let $(y)\uparrow_0 = 1$ and, for $k \geq 1$, we let $(y)\uparrow_k = y(y+1)\cdots(y+k-1)$. We then have the following analog of Theorem 4.4.

Theorem 4.6. *For any board* $B \subseteq \mathbb{B}_{n,n}$,

$$\sum_{k=0}^{n} h_k^{(n,n)}(B,y)x^k = \sum_{k=0}^{n} r_k(B,y)(y)\uparrow_{n-k}(x-1)^k. \tag{4.26}$$

Proof. Replace x by $x+1$ in Equation (4.26). Thus we must prove

$$\sum_{k=0}^{n} h_k^{(n,n)}(B,y)(x+1)^k = \sum_{k=0}^{n} r_k(B,y)(y)\uparrow_{n-k} x^k. \tag{4.27}$$

For (4.27), we consider configurations C that consist of a rook placement corresponding to a permutation $\sigma \in S_n$, and we circle some of the rooks that fall in $B \cap \sigma$. We then let $\mathrm{cyc}(C)$ denote the number of cycles in the graph of the underlying rook placement of C and $\mathrm{circ}(C)$ denote the number of circled rooks in C. It is then easy to see that the left-hand side of (4.27) can be interpreted as summing $y^{\mathrm{cyc}(C)}x^{\mathrm{circ}(C)}$ over all such configurations.

The right-hand side of (4.27) can be interpreted as follows. First pick the k circled rooks, which correspond to a placement $Q \in \mathcal{N}_k(B)$. Then we need to compute

$$A(Q, y) = \sum_{C} y^{\mathrm{cyc}(C)}, \tag{4.28}$$

where the sum runs over all configurations whose set of circled rooks equals Q. But this sum is easy to compute. Let i be the first column that does not contain a rook in Q. Then there are $n-k$ rows in which to place a rook in column i that do not contain rooks in Q. We claim that there is exactly one row r where placing a rook in cell (i, r) completes a cycle in the graph of Q. That is, if there is no rook in Q in row i, then i is an isolated vertex in the graph of Q, so adding a rook in cell (i, i) will give a loop on vertex i and hence increase the number of cycles by 1. Clearly in such a situation, placing a rook in cell (i, j) for $j \neq i$ cannot complete a cycle. If there is a rook in Q in row i, then there must be a maximal length path p in the graph of Q that ends in vertex i, since there are no edges out of i in the graph of Q. If this path starts in vertex j, then there is no rook in row j in Q. Hence if we add a rook to cell (i, j), then we will complete a cycle. Clearly, adding a rook to any other row in column i will not complete a cycle in this case. Thus the placement of a rook in column i will contribute a factor of $(y + n - k - 1)$ to $A(Q, y)$. But then we can repeat the argument for every placement Q' that arises from Q by adding a rook in the next empty column, say column j. That is, for each such Q', the addition of a rook in column j will contribute a factor of $(y + n - k - 2)$ to $A(Q, y)$. Continuing on in this way, we see that

$$A(Q, y) = y^{\mathrm{cyc}(Q)}(y + n - k - 1)(y + n - k - 2) \cdots (y) = y^{\mathrm{cyc}(Q)}(y) \uparrow_{n-k}.$$

Thus another way to sum $y^{\mathrm{cyc}(C)} x^{\mathrm{circ}(C)}$ over all configurations is

$$\begin{aligned}
\sum_{k=0}^{n} x^k \sum_{Q \in \mathcal{N}_k(B)} A(Q, y) &= \sum_{k=0}^{n} x^k \sum_{Q \in \mathcal{N}_k(B)} y^{\mathrm{cyc}(Q)}(y) \uparrow_{n-k} \\
&= \sum_{k=0}^{n} x^k (y) \uparrow_{n-k} \sum_{Q \in \mathcal{N}_k(B)} y^{\mathrm{cyc}(Q)} \\
&= \sum_{k=0}^{n} r_k(B, y)(y) \uparrow_{n-k} x^k.
\end{aligned}$$

□

Chung and Graham [13] proved that for Ferrers boards $F(b_1, \ldots, b_n) \subseteq \mathbb{B}_{n,n}$, we have the following factorization theorem.

Theorem 4.7. *Let $B = F(b_1, \ldots, b_n) \subseteq \mathbb{B}_{n,n}$ be a Ferrers board. Then*

$$\prod_{i:b_i<i}(x+b_i-i+1)\prod_{i:b_i\geq i}(x+b_i-i+y)=\sum_{k=0}^{n} r_{n-k}(B,y)(x)\downarrow_k . \tag{4.29}$$

As a consequence of Theorem 4.7, we see that two Ferrers boards $B^{(1)}$ and $B^{(2)}$ will have the same $r_k(B, y)$ if and only if they give rise to the same multiset $\{b_i - i + 1 : i \in [n]\}$. Thus, there exist bijections $f_k : \mathcal{N}_k(B^{(1)}) \to \mathcal{N}_k(B^{(2)})$ that preserve the number of cycles. Given these bijections, we can modify the bijection θ_s to preserve the number of cycles by simply changing the sequence U in the code of a placement.

Let P be the placement corresponding to the s marked cells of the marked placement $\langle f, H_1, \ldots, H_{|B^{(1)}|}\rangle$. U encodes the remaining rooks as follows. Let $1 \leq p_1 < \cdots < p_{n-s} \leq n$ be the columns in $\mathbb{B}_{n,n}$ that do not contain rooks in P. Suppose that $f(p_i) = t_i$ for each $i \in [n-s]$. By the argument in the proof of Theorem 4.6, considering only the rooks in P, column p_1 has exactly one cell that would complete a cycle. Label that cell with a y, then label the remaining cells where a rook could be placed from bottom to top with 1 to $n-s-1$. Let u_1 be the label on cell (p_1, t_1). In general, column p_m has exactly one cell that would complete a cycle among P and the cells (p_i, t_i) for $1 \leq i < m$. Label that cell with a y, then label the remaining cells where a rook could be placed in that column from bottom to top with 1 to $n-s-m$, and let u_m be the label on cell (p_m, t_m).

For example, in Figure 4.6, cell $(1, 1)$ would complete a cycle in column 1, so $(1, 7)$ is the 4th remaining cell. Thus, we would set $u_1 = 4$. Similarly, $(3, 3)$ and $(4, 4)$ would complete cycles, so $u_2 = 5$ and $u_3 = 1$. In column 5, however, cell $(5, 6)$ completes a cycle with the cell $(6, 5) \in P$, so $u_4 = y$. In column 7, cell $(7, 4)$ would complete a cycle with $(4, 1)$ and $(1, 7)$, so $u_5 = 1$. Thus, the code of the marked rook placement in Figure 4.6 is (P, U), where $P = \{(2, 2), (6, 5)\}$ and $U = (4, 5, 1, y, 1)$.

To describe the bijection $\theta_s(W)$, we take the rook placement P of the s marked patterns in W and use f_s to map it to a placement Q on $B^{(2)}$ with the same number of cycles. Then, we use U to extend the placement Q to one on $\mathbb{B}_{n,n}$ with the same number of cycles.

Chapter 4, in full, is currently being prepared for submission for publication of the material. Remmel, Jeff; Tiefenbruck, Mark. The dissertation author is an author of this material.

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