University of California
Santa Barbara

Topics in relative arbitrage, stochastic games and high-dimensional PDEs

A dissertation submitted in partial satisfaction
of the requirements for the degree

Doctor of Philosophy
in
Statistics and Applied Probability

by

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June 2021

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June 2021

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I dedicate this dissertation to my parents.

Acknowledgements

I would like to show my deepest gratitude to my advisor, Prof. Tomoyuki Ichiba, for introducing me to stochastic portfolio theory and mean field games, for being so generous with his advice and time, and for helping me grow through challenges in academic research patiently. Tomoyuki is an amazing mentor and a role model for me - His immense knowledge and passion for research, his authentic care towards people is what I hope to emulate in my professional and daily life.

I owe a debt of gratitude to Prof. Jean-Pierre Fouque for enlightening discussions that helped me greatly in research, pointing me to further investigate the information structures and time horizons in relative arbitrage problems. JP's enthusiastic classes and talks led me to the curious world of financial math. I was fortunate to have received invaluable research advice and career opportunities due to his mentorship. I am very grateful to Prof. Nils Detering for reading my manuscripts and kindly offering me feedback in many delightful and informative conversations. The wonderful courses from Nils about measure theory, random graphs, and systemic risk benefited me a lot.

I sincerely appreciate Prof. Xin Guo, Prof. Leonard Wong, Prof. Ruimeng Hu, Prof. Li-Hsien Sun, Dr. Mathieu Lauriére, Dr. Tao Chen, Dr. Moritz Voss, Dr. Michael Zhang, for their comments and ideas that helped improve my works in this thesis. It was a great pleasure having discussions with them, and other researchers I met in conferences and seminars. They explained interesting research topics to me and gave me insights and inspirations for research.

I wish to express my appreciation to professors Michael Ludkovski, Raya Feldman, and S. Rao Jammalamadaka, who have helped shape my understanding in probability and statistics. Raya always encourages me in my research and I learned so much from her about teaching and caring for students when I worked as a TA for several quarters with her. Many thanks to my colleagues in the PSTAT department, Aditya, Alex, Andrea, Franky, Jiajing, Jiaye, Liangchen, Lucy, Ming, Pierre-O, Victoria, Yuanbo for friendships, collaborations in projects, and sharing laughter and jokes. Special thanks to Dr. Marie Webb for helping me with academic writings.

I truly appreciate Dori, Jeong, Junliang, Raphael, and Will in Morgan Stanley who showed me mentorships and perspectives from the finance industry. Their passion and strong work ethic influences me a lot.

The years at UCSB were a beautiful experience, during which I got to know so many wonderful friends. Shout out to Yi who is always cheerful and supportive of me, and my golf buddies, my roommates, my friends in the US and China who bring tons of happiness to my life.

Finally, I'm infinitely thankful to my parents, Guizhen and Xu, and my grandparents. Their unwavering love and support encourages me at every stage of my life. I dedicate this dissertation to them, without whom this journey would not have been possible.

Curriculum Vitæ

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Education

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Recent Publications and Preprints

- Tomoyuki Ichiba, Tianjiao Yang. Relative arbitrage opportunities in N investors and mean-field regimes. arXiv preprint arXiv:2006.15158, 2021.
- Yi Ding, Radha Kumaran, Tianjiao Yang, and Tobias Höllerer. Predicting Video Affect via Induced Affection in the Wild. In Proceedings of the 2020 International Conference on Multimodal Interaction, pp. 442-451, 2020.

Invited Talks

- Western Conference on Mathematical Finance, 2021.
- Center for Financial Mathematics and Actuarial Research Seminar Series, 2020.
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Conferences and Workshops

- 10th Western Conference on Mathematical Finance, UCSB, 2021
- High Dimensional Hamilton-Jacobi PDEs, Workshop, IPAM, UCLA, 2020
- Mean Field Games, Short Course, American Mathematical Society, Denver, 2020
- SIAM Financial Math and Engineering Conference, Toronto, 2019
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Abstract

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The relative arbitrage portfolio introduced in Stochastic Portfolio Theory (SPT), outperforms a benchmark portfolio over a time-horizon with probability one. Following this concept, when an investor competes with both market and peers, does relative arbitrage opportunity exist as well? What is the best performance one can achieve? What is the impact on market dynamics and investors when a large group competes in this way?

This thesis constructs a framework of multi-agent optimization under SPT to tackle these questions. With a market model depending on stock capitalizations and targeted investors, we analyze the market behavior and optimal investment strategies to attain relative arbitrage in a large population regime under some market conditions.

We show a unique equilibrium for relative arbitrage in N-player and mean field games (MFG) with mild conditions on the equity market, by modifying extended MFG with common noise and its notion of the uniqueness in Nash equilibrium. The optimal arbitrage can be decomposed and generated using the idea of functionally generated portfolios. In this way, the constraints on relative return and investment time horizon can be specified.

The second part of the thesis studies numerical aspects of solving high dimensional PDEs with multiple solutions, and learning relative arbitrage opportunities. A grid based solution for relative arbitrage is derived in volatility stabilized market models. We then study deep learning schemes for non-unique solutions of PDEs. Experiments on solving the non-negative minimal solution of a Cauchy problem is provided.

Contents

Chapter 1

Introduction and preliminaries

This chapter introduces the background of the main topics used in this thesis, including Stochastic Portfolio Theory and Stochastic Games. The notations introduced in this chapter will be used repeatedly in the following chapters. The outline of the thesis is in the end of this chapter.

1.1 Stochastic portfolio theory

The basic assumptions and the settings of stock capitalizations in this thesis fall under Stochastic Portfolio Theory (SPT), introduced by Robert Fernholz [26], which analyzes portfolio behavior and equity market structure.

Stochastic portfolio theory assumes the existence of a local martingale deflator.The paper [65] further discusses the relation between the no-arbitrage hypothesis and Stochastic Portfolio Theory. Here, no assumption is made regarding the existence of an equivalent (local) martingale measure, i.e., arbitrage opportunities are not excluded. [28] shows that relative arbitrage can exist in equity markets that resemble actual markets, and it resulted from market diversity, a condition that prevents the concentration of all the market capital into a single stock. While [27] shows that diversity is not the proximate cause for the existence of relative arbitrage, but instead this cause appears to lie in a condition related to the variance rates of the stocks in the market. This variance-related condition can pertain even in the absence of diversity.

It is commonly used by market participants to compare the performance of an investment strategy with a benchmark index. Among different metrics and tools to capture opportunities that outperform a benchmark portfolio, relative arbitrage established in Stochastic Portfolio Theory (SPT), see Fernholz [26], is of special interest to investment and portfolio management.

1.1.1 Market and its Properties

In a market model, there are n stocks with prices-per-share driven by K independent Brownian motions $W = (W_1, \dots, W_K)$ on a filtered probability space, $K \geqslant n$. A stock price process X_i, $i = 1, \dots, n$ satisfies

$$dX_i(t) = X_i(t)(\beta(t)dt + \sum_{k=1}^{K} \sigma_{ik}(t)dW_k(t),$$

with its rates of return $\beta(\cdot)$ and the volatilities $\sigma(\cdot)$.

$\pi_i(t)$ is the proportion of wealth $V(t)$ at time t that is invested in stock i. To emphasize the dependence of wealth on the initial capitalization v and portfolio, we write $V(t) = V^{v,\pi}(t)$, and

$$\frac{dV^{v,\pi}(t)}{V^{v,\pi}(t)} = \sum_{i=1}^{n} \pi_i(t) \frac{dX_i(t)}{X_i(t)}.$$

In particular, we will use market portfolio $\mathbf{m}_i(\cdot)$ in the future, which is in proportion to the weight of each stock,

$$\mathbf{m}_i(t) := \frac{X_i(t)}{X(t)}, \quad i = 1, \ldots, n.$$

We denote $X(t)$ as the total capitalization: $X(t) = X_1(t) + \ldots + X_n(t)$.

Investing according to the portfolio process $\mathbf{m}(\cdot)$ amounts to ownership of the entire market. The resulting wealth process is

$$V^{\mathbf{m}}(t) = \frac{V^{\mathbf{m}}(0)}{X(0)} X(t).$$

Apply Ito's rule, the dynamics is

$$d\mathbf{m}_i(t) = \mathbf{m}_i(t)\left[\gamma_i^{\mathbf{m}} dt + \sum_{k=1}^{K} \tau_{ik}^{\mathbf{m}}(t) dW_k(t)\right], \quad i = 1, \ldots, n. \tag{1.1}$$

Here $\tau^{\mathbf{m}}(t)$ is the matrix with entries $\tau_{ik}^{\mathbf{m}}(t) := \sigma_{ik}(t) - \sum_{j=1}^{n} \mathbf{m}_j(t)\sigma_{jk}(t)$, $\mathbf{e}_i$ the ith unit vector in $\mathbb{R}^n$ and the vector $\gamma^{\mathbf{m}}(t)$ is with the entries $\gamma_i^{\mathbf{m}}(t) := (\mathbf{e}_i - \mathbf{m}(t))'(\beta(t) - \alpha(t)\mathbf{m}(t))$.

1.1.2 Relative Arbitrage

The relative arbitrage problem is first defined in SPT, the focus of which is to generate a strategy that outperforms a benchmark portfolio almost surely at the end of a certain time span and look for the highest relative return. It shows in [28] that relative arbitrage can exist in equity markets that resemble actual markets, and that the relative arbitrage results from market diversity, a condition that prevents the concentration of all the market capital into a single stock. Specific examples of the market including the stabilized volatility model, in which relative arbitrage exists, are introduced in [27]. To relax the assumptions about the behavior of the market imposed in SPT, [68] considers relative arbitrage in regulated markets where dividends and the merge and split of companies are taken into account.

Suppose we use portfolio ρ_1 and ρ_2 to generate wealth processes from the same initial wealth, i.e., $V^{\rho_1}(0) = V^{\rho_2}(0) = 1$. There is an arbitrage of portfolio ρ_1 relative to ρ_2 if $V^{\rho_1}(T)$ dominates $V^{\rho_2}(T)$ almost surely at the end of time span $[0, T]$

$$\mathbb{P}\big(V^{\rho_1}(T) \geqslant V^{\rho_2}(T)\big) = 1, \quad \mathbb{P}\big(V^{\rho_1}(T) > V^{\rho_2}(T)\big) > 0.$$

The concept of arbitrage in arbitrage theory can be understood as a portfolio relative to an all zero-valued strategy.

Desired properties of the market

A natural question follows is *When does relative arbitrage exist?* This section recalls some properties of the market which are used to show the existence of relative arbitrage.

Definition 1.1.1 (Non-degeneracy and bounded variance). *A market is a family $\mathcal{M} = \{X_1, \ldots, X_n\}$ of n stocks, each of which is defined as in (2.1), such that the matrix $\alpha(t)$ is nonsingular for every $t \in [0, \infty)$, a.s. The market $\mathcal{M}$ is called nondegenerate if there exists a number $\epsilon > 0$ such that for $x \in \mathbb{R}^n$*

$$\mathbb{P}(x\alpha(t)x^T \geqslant \epsilon||x||^2, \forall t \in [0, \infty)) = 1,$$

The market $\mathcal{M}$ has bounded variance from above, if there exists a number $M > 0$ such that for $x \in \mathbb{R}^n$

$$\mathbb{P}(x\alpha(t)x^T \leqslant M||x||^2, \forall t \in [0, \infty)) = 1.$$

Remark 1. *Let π be a portfolio in a nondegenerate market. Then there exists an $\epsilon > 0$*

such that for $i = 1, \ldots, n$,

$$\tau_{ii}^{\pi}(t) \geqslant \epsilon(1 - \pi_{\max}(t))^2, \forall t \in [0, \infty) \tag{1.2}$$

almost surely. Indeed, this is directly from definition 1.1.1, and $\tau_{ii}^{\pi}(t) = \alpha_{ii}(t) - 2\alpha_{i\pi}(t) + \alpha_{\pi\pi}(t)$*, where* $\alpha_{\pi\pi}(t) = \pi'(t)\alpha(t)\pi(t)$*. Details of the proof can be found in [26].*

Intuitively, no single company can ever be allowed to dominate the entire market in terms of relative capitalization.

Definition 1.1.2 (Diversity of market)**.** *The model* $\mathcal{M}$ *of* (2.1)*,* (2.2) *is diverse on the time-horizon* $[0, T]$*, with* $T > 0$ *a given real number, if there exists a number* $\eta \in (0, 1)$ *such that*

$$\max_{1 \leqslant i \leqslant n} \mathbf{m}_i := \mathbf{m}_{(1)} < 1 - \eta, \forall\ 0 \leqslant t \leqslant T \tag{1.3}$$

almost surely and $\mathcal{M}$ *is weakly diverse if there exists a number* $\eta \in (0, 1)$ *such that*

$$\frac{1}{T} \int_0^T \mathbf{m}_{(1)}(t) dt < 1 - \eta, \forall\ 0 \leqslant t \leqslant T \tag{1.4}$$

almost surely.

Optimal arbitrage

The market portfolio plays an important role as numeraire and the relative arbitrage with respect to the market is a common interest for investors. In Fernholz and Karatzas [22], the best possible investment strategy to capture relative arbitrage with respect to the market portfolio is characterized as the minimal proportion of initial market capitalization $X(0) := x_1 + \ldots + x_n$ as initial wealth to start with, so that at terminal time the wealth $V(T)$ outperforms the total market capitalization $X(T)$.

The best arbitrage opportunity is further analyzed in [23] in a market with Knightian uncertainty. The smallest proportion of the initial market capitalization is described as the min-max value of a zero-sum stochastic game between the investor and the market. Further investigation of exploiting relative arbitrage opportunities has been done in [6, 29, 64, 65]. Assuming the market is diverse and sufficiently volatile, functionally generated portfolios introduced in SPT is a tool to construct portfolios with favored return characteristics. The optimization problem from the functional generated portfolio point of view is handled in [72]. The papers [56] and [71] connect relative arbitrage with information theory and optimal transport problems.

1.2 Stochastic Games

Games are defined as mathematical models of strategic interaction among rational decision makers. In many situations, every party in the game interacts not only once. Instead, their actions are inter-temporal strategies because of ongoing interactions over time based on historical information. This type of game is modelled in repeated games.

Stochastic games first introduced by Lloyd Shapley [66] further generalize repeated games. Stochastic game models a repeated interaction between several participants in which the underlying state of the environment changes stochastically, and it depends on the decisions of the participants. The play proceeds by steps according to transition probabilities controlled jointly by the players. Each player faces a Markov decision process in which they maximize a total payoff criterion.

In most parts of this thesis, we apply models under the umbrella of stochastic differential games. Specifically, a stochastic differential game consists of the following elements:

- A set of players $\ell \in \{1, \dots, N\}$;

- State dynamics X_t on the probability space $(\Omega, \mathcal{F}, P)$ equipped with a complete and right-continuous filtration $\mathbb{F}$. The time evolution of the state is represented by a stochastic differential equation

$$dX_t = \beta(X_t, \pi_t)dt + \sigma(X_t, \pi_t)dW_t, \quad t \in (0, T],$$

$$X_0 = x_0,$$

W_t is independent $\mathbb{F}$-Brownian Motion of the same dimension as that of X_t.

- A set of action profiles for each player $\mathbb{A} = \mathbb{A}^1 \times \ldots \times \mathbb{A}^N$;

- A cost functional, which is specific to actions and players and is with the implicit assumption that players try to maximize their individual cost; Thus, each player ℓ want to choose their actions so as to minimize the expected value of the cost functional J respectively.

$$J(\pi) = g(X_T) + \int_0^T f(X_t, \pi_t)dt$$

As a special case, when $N = 1$, the model is equivalent to a stochastic control problem.

1.2.1 Notions of equilibrium

In N player games, since the controls are allowed to differ from one player to another, the expected cost functionals $J_1, \ldots, J_N$ may not be the same. Except for some very specific cases, it is hard to find controls $\pi_1, \ldots, \pi_N$ that minimize simultaneously all $J_1, \ldots, J_N$. Instead of solving the problem globally, the idea of consensus is formalized by the concept of Nash equilibrium (NE). Players choose their strategies optimally given correct beliefs about the strategies of the other players, while no player has incentive to change their strategy.

Definition 1.2.1 (Nash Equilibrium)**.** *A vector* $\pi^{\ell*} = (\pi_i^{\ell*}, \ldots, \pi_n^{\ell*})$ *of admissible strategies is a Nash Equilibrium if, for all* $\pi_i^{\ell} \in \mathcal{A}$ *and* $i = 1, \ldots, n$,

$$J^{\ell}(\pi_i^{\ell*}, \pi_i^{-\ell*}) \leqslant J^{\ell}(\pi_i^{\ell}, \pi_i^{-\ell*}),$$

where $\pi(\cdot) = (\pi^1(\cdot), \ldots, \pi^N(\cdot))$.

1.2.2 Mean field games

The introduction of "mean fields" arising from physics provides a solution to simplify the modelling of all inter-particle interactions when there are a large number of particles in a dynamic system. The pioneering work of Lasry and Lions [49] and Huang et al. [39] of Mean field game theory adapts this methodology to agents interacting through information and strategies in a game setting.

Non-zero-sum N-player games are notoriously hard to solve. With a coupled system of N differential equations, explicit solutions of equilibria are difficult to find. Furthermore, there is no existence theorem for approximate Nash equilibria in such games. The agents of mean field game theory are less sophisticated than the players of N-player game theory since they base their strategies only on the statistical state of the mass of co-agents. Mean field games are expected to be more effective and tractable than N player games because of the decoupling of PDEs rooted in differential calculus and measure theory. In return, mean field games might give us more information about the finitely many investors situation.

A special class of stochastic differential games under the following assumptions is considered, for the discussion of mean field games:

Assumption 1. *(1) All the players are indistinguishable (statistically identical) in their behavior, and a single player's influence on the outcome of the game diminishes as the number of players tends to infinity.*

(2) All the players are strongly symmetric in the sense that an individual player is only affected by the statistical distribution of the private states of the other players.

An individual's private state may have different sources of randomness - the idiosyncratic noise, which is independent from the other players; and the common noise, which is noise in the environment that affects everyone. In general, we deal with the model for a symmetric system of size N, that is given by a system of N SDEs of the form:

$$dX_t^i = b(t, X_t, \mu_t, \alpha_t)dt + \sigma(t, X_t, \mu_t, \alpha_t)dW_t^i + \sigma_0(t, X_t, \mu_t, \alpha_t)dB_t,$$

for $t \in [0, T]$, $T > 0$, $i \in \{1, \ldots, N\}$, where $(B, W^1, \ldots, W^N)$ is a sequence of independent k-dimensional Wiener processes on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. b, σ, σ_0 are measurable functions defined on $[0, T] \times \mathbb{R}^k \times \mathcal{P}(\mathbb{R}^k)$. $\mathcal{P}(\mathbb{R}^k)$ is the space of probability measures on $\mathbb{R}^k$ endowed with the topology of weak convergence. Denote $\overline{\mu}_t^N$ as the empirical distribution of the N private states, that is

$$\overline{\mu}_t^N = \frac{1}{N} \sum_{j=1}^{N} \delta_{X_t^i},$$

δ_x denotes the unit mass (Dirac measure) at $x \in E$, where E is a compact metric space. When searching for a Nash Equilibrium of a large number of players, $\overline{\mu}_t^N$ is not affected by a deviation of a single player. Furthermore, because of de Finetti's law of large numbers, we expect that these empirical measures converge when the size $N \to \infty$.

When the impact of the common noise exists, the limiting environment must be given by a stochastic flow $(\mu_t)_{t \in [0,T]}$ of probability measures describing the conditional distribution of the population in equilibrium given the realization of the common noise, owing to the theory of propagation of chaos. A general mean field game formulation in this case is

(i) Fix an adapted process $[0,T] \ni t \to \mu_t \in \mathcal{P}(\mathbb{R}^k)$.

(ii) Solve the stochastic control problem

$$\begin{aligned} &\inf_{\alpha\in\mathcal{A}} \mathbb{E}\bigg[\int_0^T f(t, X_t, \mu_t, \alpha_t)dt + g(X_T, \mu_T)\bigg] \\ &\text{subject to } \; dX_t = b(t, X_t, \mu_t, \alpha_t)dt + \sigma(t, X_t, \mu_t, \alpha_t)dW_t + \sigma_0(t, X_t, \mu_t, \alpha_t)dB_t, \\ &\qquad\qquad\quad X_0 = x_0. \end{aligned} \tag{1.5}$$

(iii) Given an optimal control, find the corresponding conditional laws $(\mu_t^\star)$, $t \in [0,T]$ of the optimally controlled state process $(X_t^\star)$, $t \in [0,T]$ given W.

(iv) Find a fixed point $(\mu_t)_{t\in[0,T]}$, such that $\mu_t = \mu_t^\star$, for all $t \in [0,T]$.

The large population system in early MFG theory is reformulated by [13] into the stochastic version to accommodate the common noise. With the notion of weak mean field games, [45] and [16] study the mean field game with common noise in the open loop equilibrium. [45] assures that the weak MFG solutions characterize the limits of approximate Nash equilibria. In an approximate Nash equilibrium, this requirement is weakened to allow the possibility that a player may have a small incentive to do something different. If a sequence of N-player approximate equilibria exists, then its limits are described by weak MFG solutions. Conversely, if a weak MFG solution exists, then it is achieved as the limit of some sequence of N-player approximate equilibria.

Intuitively speaking, even in the limit $N \to \infty$, the equilibrium distribution of the population should still feel the influence of the common noise W_0, and for that reason, it should not be deterministic. The common noise turns the forward Kolmogorov equation into a forward stochastic Kolmogorov equation.

Our paper applies the philosophy of mean field games from [15] and [35] to search for approximate Nash equilibrium when $N \to \infty$. This approach of comparing N-player game and the corresponding mean field game is also discussed in [48], where the Merton problems with constant equilibrium strategies are studied.

1.3 Outline of the thesis

Thesis organization in a nutshell

The organization of this paper is as follows. Chapter 2 introduces the market with N investors as a well-posed interacting particle system. Chapter 3 discusses the relative arbitrage problem and the benchmark we use in the rest of the thesis under a finite particle system.

In Chapter 4, the existence of relative arbitrage is proved and the optimization of relative arbitrage is derived in N-player games with different information structures. Chapter 5 proceeds the relative arbitrage equilibrium of extended mean field games and presents an example with explicit equilibrium and optimal strategies. We show that the mean field game limit is indeed a nice approximation to the N-players game.

Nevertheless, only a few specific types of mean field games under certain conditions have unique closed form equilibria solutions. We present in Chapter 6 that the functional generated portfolios (FGP) results for a large population and its connection with Nash equilibrium results in previous chapters. In Chapter 7, numerical schemes are investigated for the solvability of Nash equilibrium in N-player games and mean field games, focusing on volatility-stabilized models. Lastly, Chapter 8 discusses deep learning solutions towards the PDE systems in relative arbitrage problems.

Outline

It is commonly used by market participants to compare the performance of an investment strategy with a benchmark index. To better describe and analyze the market based on SPT, this paper investigates the following questions: How do we capture the competitive behaviors of participants in the financial market? With additional information about these investors, how do we improve the market model and make portfolio suggestions? We want to enable portfolio managers or asset management entities to customize their portfolio optimization strategies based on the preference and selection of benchmarks.

This paper forms a stochastic differential game system of equity market, where investors aim to pursue outperformance to the market index and peer investors. We introduce the mean field interaction among participants and study the relationship between the N-player game and mean field game set-up of our problem of interest.

One focus of our work is on the multi-agent optimization theory for relative arbitrages. Our model arises from the pioneering work of Fernholz and Karatzas [22], which characterizes the best possible relative arbitrage with respect to the market portfolio. We construct a general framework for multi-player portfolio optimization problems without the requirement of the existence of an equivalent martingale measure.

- **Market, investors and their mean field interactions**

  This paper first considers N investors in an equity market $\mathcal{M}$ over a time horizon $[0, T]$. We consider N is big, so that the equity trading of this group as a whole contributes to the evolution of the market; whereas individuals among the group are too "small" to bring changes to the market. These investors interact with the market through a joint distribution of their wealth and strategies, particularly for example, through the total investments of this group to the assets. There are n stocks with prices-per-share driven by n independent Brownian motions $W =$

$(W_1, \ldots, W_n)$ on a filtered probability space. The n-dimensional price process $\mathcal{X}^N = (X_1^N, \ldots, X_n^N)$ follows a nonlinear stochastic differential equation

$$d\mathcal{X}^N(t) = \mathcal{X}^N(t)\beta(t, \mathcal{X}^N(t), \nu_t^N)dt + \mathcal{X}^N(t)\sigma(t, \mathcal{X}^N(t), \nu_t^N)dW_t, \tag{1.6}$$

in which its drift β and diffusion σ coefficients also depend on the joint empirical measure ν_t^N of portfolio strategy π^ℓ and wealth V^ℓ, $\ell = 1, \ldots, N$ of N investors.

$$\nu_t^N := \frac{1}{N}\sum_{\ell=1}^{N} \boldsymbol{\delta}_{(V^\ell(t), \pi^\ell(t))} \tag{1.7}$$

We show the market model is well-posed through a finite dynamical system.

Another focus of our work is to build up the finite and infinite player game framework of relative arbitrage. This also provides a novel application to the N-player games and mean field games. After the discussion of N-player game, we establish a modified extended mean field game and a scheme to seek the mean field equilibrium: The infinite-player system involves two different fixed point conditions about the cost functional and the state processes, whereas only one of them is required to be unique. Our paper applies the philosophy of mean field games to search for approximate Nash equilibrium when $N \to \infty$.

- **Relative arbitrage as a N-player game's equilibrium**

  To specify what we mean by relative arbitrage opportunities in this problem set-up, we first define a benchmark process $\mathcal{V}^N$ by the weighted average performance of the market and the investors

$$\mathcal{V}^N(t) := \delta \cdot X^N(t) + (1-\delta) \cdot \frac{1}{N}\sum_{\ell=1}^{N} V^\ell(t), \quad 0 \leqslant t \leqslant T\,,$$

with a fixed weight $\delta \in [0,1]$. An investor achieves the relative arbitrage if his/her terminal wealth can outperform this benchmark by c_ℓ, a constant personal index for the investor ℓ, given at time 0. Furthermore, $\mathbb{A}^N$ denotes all admissible, self-financing long-only portfolios for N investors.

The first question raised in this paper is: *What is the best strategy one can take, so that the arbitrage relative to the above benchmark can be attained?* Specifically, every investor we study aims to outperform the market and their competitors, starting with as little proportion of the benchmark capital as possible. Mathematically, given the other $(N-1)$ investors' portfolios $\pi^{-\ell} \in \mathbb{A}^{N-1}$, the objective of investor ℓ, $\ell = 1, \ldots, N$, is formulated as

$$u^\ell(T) = \inf \left\{ \omega^\ell \in (0,\infty) \middle| \exists \pi^\ell(\cdot) \in \mathbb{A} \text{ such that } v^\ell = \omega^\ell \mathcal{V}^N(0),\ V^\ell(T) > e^{c_\ell} \mathcal{V}^N(T) \right\},$$

where $V^\ell(\cdot) := V^{v^\ell,\pi^\ell}(\cdot)$ is the wealth process generated by $\pi^\ell(\cdot)$ with initial wealth v^ℓ.

Since the interactions of a large group of investors are through stocks, portfolios and wealth, the next question that arises is: *Is it possible for every investor to take the optimal strategy in the market $\mathcal{M}$?* We characterize the optimal wealth one can achieve by the unique Nash equilibrium of the finite population game. Under some market conditions, $u^\ell(T-t, \mathcal{X}^N(t), \mathcal{Y}(t))$ is the smallest nonnegative solution of a Cauchy problem (3.13)-(3.15), where $\mathcal{Y}(t)$ is the empirical mean of ν_t^N, see (2.5). We distinguish the PDE characterization using open loop or closed loop controls respectively. The unique Nash equilibrium is achieved by a strategy

$$\pi_i^{\ell\star}(t) := \mathbf{m}_i(t) + X_i^N(t) D_{x_i} \tilde{v}(t) + \sum_{j=1}^{n} (\tau\sigma^{-1})_{ji} D_{y_j} \bar{v}^\ell(t),$$

where

$$\bar{v}^{\ell}(t) := \log u^{\ell}_{T-t} + \frac{1-\delta}{\delta X^N_t} \cdot \frac{1}{N} \sum_{k=1}^{N} V^k_t \log u^k_{T-t}.$$

for $i = 1, \ldots, n$, $\ell = 1, \ldots, N$. It turns out that $\pi^{\ell\star}_i$ and u^{ℓ}_{T-t} are proportional to c_ℓ for $\ell = 1, \ldots, N$. We show the existence of relative arbitrage through the Fichera drift [31].

- **Relative arbitrage as a mean field game's equilibrium**

The relative arbitrage problem provides a new application and some modifications in mean field games. Because of the special problem set-up, there are two mean field measures that evolve in different directions, while the uniqueness of Nash equilibrium depends on one of the measures. In particular, the mean field benchmark is given by

$$\mathcal{V}(T) := \delta \cdot X(T) + (1-\delta) \cdot m_T, \quad m := \mathbb{E}[V|\mathcal{F}^B].$$

On the other hand, the state processes depend on the conditional law of wealth and strategies $\nu := \text{Law}(V, \pi|\mathcal{F}^B)$ with respect to the Brownian motion B. This yields the McKean-Vlasov SDEs of stock prices

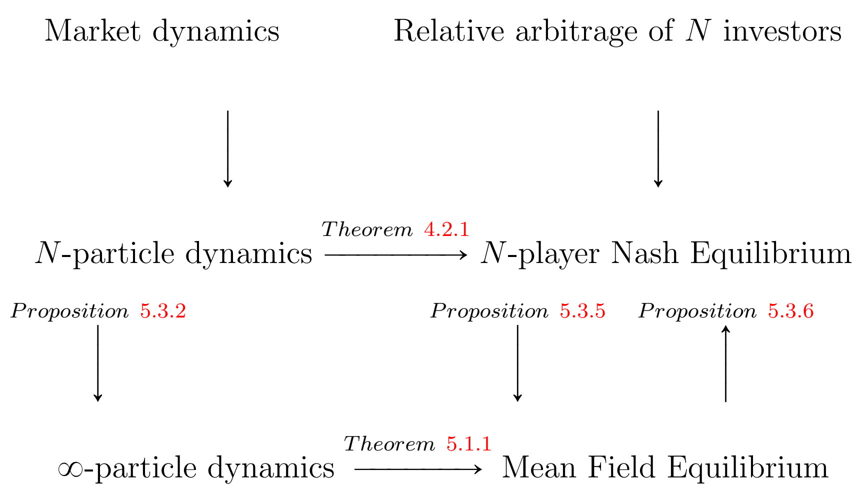
$$\begin{aligned} d\mathcal{X}_t &= \beta(\mathcal{X}_t, \nu_t, m_t)dt + s(\mathcal{X}_t, \nu_t, m_t)dB_t, \quad t \in (0, T] \\ \mathcal{X}(0) &= \mathbf{x}; \end{aligned}$$

and a representative player's wealth

$$\begin{aligned} dV_t &= \pi(t)\beta(\mathcal{X}_t, \nu_t, m_t)dt + \pi(t)\sigma(\mathcal{X}_t, \nu_t, m_t)dB_t, \quad t \in (0, T] \\ V(0) &= \tilde{u}(T)\mathcal{V}(0). \end{aligned}$$

A modified extended mean field game model with common noise is introduced. Both open and closed loop equilibrium are considered here regarding the well-posedness of the mean field system and the approximation of games.

We summarize these results in the following diagram.



Alternatively, the optimal strategies in N-player games and mean field games can be formulated as functional generated portfolios (FGP). This could be a possible remedy for the computational issues in relative arbitrage problems.

- **Numerical solutions of high-dimensional PDEs and volatility-stabilized market**

  We explore the numerical schemes for stochastic portfolio theory and high-dimensional PDEs. Both of these topics suffer intractabilities in general.

  We first develop a grid based solution for relative arbitrage in volatility stabilized models. We start from market models without influence of investors, such as the ones used in [26], [22], and [27]. For example if the stock processes follow

$$dX_i(t) = X(t)dt + \sqrt{X_i(t)X(t)}dW_i(t),$$

for $i = 1, \ldots, n$. In this case, the best investment opportunity for arbitrage relative to the market portfolio is characterized as

$$u(T-t, \mathcal{X}(t)) = \frac{X_1(t) \ldots X_n(t)}{X_1(t) + \ldots + X_n(t)} \frac{\mathbb{E}[X_1(T) + \ldots + X_n(T)]}{X_1(T) \ldots X_n(T)}, \quad t \in [0, T].$$

A numerical scheme is developed based on Bessel processes to avoid the inefficiency of a traditional finite difference scheme.

The thesis then presents deep learning schemes that work for more general market models and high dimensional PDEs with multiple solutions. Above all, the non-negative minimal continuous solution of Cauchy problems appeared in relative arbitrage problems of our interest. We first develop a deep learning scheme similiar to [67]. Then we propose a probabilistic numerical scheme where the associated reflected BSDE problem achieves an approximation of the non-negative minimal continuous solution.

Chapter 2

Finite dynamical system of equity market

This chapter serves to construct the market model and the finite dynamical system we use frequently in the rest of the thesis. The market model contains the processes of stock capitalization and trading volume on each stock. We showed in [42] that the finite-particle SDE system of stock and trading volume processes admits a unique solution under certain market conditions. We also compare several investment strategies constructed from the market model we use.

2.1 Market Model

We consider an equity market and focus on the market behavior and a group of investors in this market. The number of investors we include is large enough to affect the market as a whole. Nevertheless, there are possibly other investors apart from this very group we consider.

2.1.1 Capitalizations

For a given finite time horizon $[0,T]$, an admissible market model $\mathcal{M}$ we use in this paper consists of a given n dimensional Brownian motion $W(\cdot) := (W_1(\cdot),\ldots,W_n(\cdot))'$ on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ of the space Ω of continuous functions. Filtration $\mathbb{F}$ represents the "flow of information" in the market, where $\mathbb{F} := \{\mathcal{F}(t)\}_{0\leqslant t<\infty} = \{\sigma(\omega(s)); 0 < s < t\}$ with $\mathcal{F}(0) := \{\varnothing,\Omega\}$, mod $\mathbb{P}$. $W(\cdot)$ is adapted to the $\mathbb{P}$-augmentation of $\mathbb{F}$. All the local martingales and supermartingales are with respect to the filtration $\mathbb{F}$ if not written out specifically.

Thus, there are n risky assets (stocks) with prices-per-share $\mathcal{X}^N(\cdot) = (X_1^N(\cdot),\ldots,X_n^N(\cdot))$ driven by n independent Brownian motions as follows: for $t\in[0,T]$, $\omega\in\Omega$,

$$dX_i^N(t) = X_i^N(t)(\beta_i(t,\omega)dt + \sum_{k=1}^{n}\sigma_{ik}(t,\omega)dW_k(t)), \quad i = 1,\ldots,n, \tag{2.1}$$

or

$$X_i^N(t) = x_i^N \exp\left\{\int_0^t (\beta_i(s,\omega) - \frac{1}{2}\sum_{k=1}^{n}(\sigma_{ik}(s,\omega))^2 dt + \sum_{k=1}^{n}\int_0^t \sigma_{ik}(s,\omega)dW_k(s)\right\},$$

with the initial condition $X_i^N(0) = x_i^N$. We assume that $\dim(W(t)) = \dim(\mathcal{X}^N(t)) = n$, that is, we have exactly as many sources of randomness as there are stocks in the market $\mathcal{M}$. The market $\mathcal{M}$ is hence a complete market. The dimension n is chosen to be large enough to avoid unnecessary dependencies among the stocks we define. Here, $'$ stands for the transpose of matrices.

Here, $\beta(\cdot) = (\beta_1(\cdot),\ldots,\beta_n(\cdot))' : [0,T]\times\Omega \to \mathbb{R}^n$ as the mean rates of return for n stocks and $\sigma(\cdot) = (\sigma_{ik}(\cdot))_{n\times n} : [0,T]\times\Omega \to \mathrm{GL}(n)$ as volatilities are assumed to be invertible, $\mathbb{F}$-progressively measurable in which $\mathrm{GL}(n)$ is the space of $n\times n$ invertible real matrices. Then $W(\cdot)$ is adapted to the $\mathbb{P}$-augmentation of the filtration $\mathbb{F}$. To satisfy the integrability condition, we assume

$$\sum_{i=1}^{n}\int_0^T \Big(|\beta_i(t,\omega)| + \alpha_{ii}(t,\omega)\Big)dt < \infty, \tag{2.2}$$

where $\alpha(\cdot) := \sigma(\cdot)\sigma'(\cdot)$, and its i,j element $\alpha_{i,j}$ is the covariance process between X_i^N and X_j^N for $1 \leqslant i,j \leqslant n$.

2.1.2 Wealth and Portfolios

In the above market model, there are N *small* investors, "small" is in the sense that each individual of these N investors has very little influence on the overall system. An investor ℓ uses the proportion $\pi_i^\ell(t)$ of current wealth $V^\ell(t)$ to invest in the stock i at each time t for $\ell = 1,\dots,N$. The wealth process V^ℓ of an individual investor ℓ is

$$\frac{dV^\ell(t)}{V^\ell(t)} = \sum_{i=1}^{n}\pi_i^\ell(t)\frac{dX_i^N(t)}{X_i^N(t)}, \quad V^\ell(0) = v^\ell. \tag{2.3}$$

Since equity prices move according to the supply and demand for stock shares, we consider the average capital invested as a factor in the price processes.

Definition 2.1.1 (Investment strategy, long only portfolio and average capital invested)**.** *We define the the following items related to proportion* π^ℓ *as below:*

(1) An $\mathcal{F}$*-progressively measurable and adapted process* $\pi^\ell : [0,\infty)\times\Omega \to \mathbb{R}^n$ *is called an investment strategy if*

$$\int_0^T (|\pi^{\ell\prime}(t,\omega)\beta(t,\omega)| + \pi^{\ell\prime}(t,\omega)\alpha(t,\omega)\pi^\ell(t,\omega))dt < \infty, \quad T \in (0,\infty), \omega\in\Omega,\ a.s. \tag{2.4}$$

The strategy here is self-financing, since wealth at any point of time is obtained by trading the initial wealth according to the strategy $\pi(\cdot)$*.*

(2) *Each* $\pi^\ell(\cdot) = (\pi_1^\ell(\cdot), \dots, \pi_n^\ell(\cdot))'$ *is a long-only portfolio if it is a portfolio that takes values in the set*

$$\Delta_n := \{\pi = (\pi_1, ..., \pi_n) \in \mathbb{R}^n | \pi_1 \geqslant 0, \dots, \pi_n \geqslant 0; \pi_1 + \dots + \pi_n = 1\}.$$

An investment strategy that takes value in Δ_n *is called an admissible strategy, and we denote the admissible set as* $\mathbb{A}$*. If* $\pi^\ell \in \mathbb{A}$*, for all* $\ell = 1, \dots, N$*, then* $(\pi^1, \dots, \pi^N) \in \mathbb{A}^N$*. In the rest of the paper, we only consider strategies in the admissible set* $\mathbb{A}$*.*

(3) *Each investor* ℓ *uses the proportion* $\pi_i^\ell(t)$ *of current wealth* $V^\ell(t)$ *to invest in the* i*th stock at each time* t*. The average amount* $\mathcal{Y}_i(t)$ *invested by* N *players on stock* i *is assumed to satisfy*

$$\begin{aligned} \mathcal{Y}_i(t) &:= \frac{1}{N}\sum_{\ell=1}^N V^\ell(t)\pi_i^\ell(t) = \int_0^t \gamma_i(r,\omega)dr + \int_0^t \sum_{k=1}^n \tau_{ik}(r,\omega)dW_k(r), \quad t \in (0,\infty) \\ &\frac{1}{N}\sum_{\ell=1}^N V^\ell(0)\pi_i^\ell(0) =: y_i \end{aligned} \tag{2.5}$$

for $i = 1, \dots, n$*, where* $\gamma(\cdot)$ *and* $\tau(\cdot)$ *are assumed to satisfy*

$$\sum_{i=1}^n \int_0^T \Big(|\gamma_i(t,\omega)| + \psi_{ii}(t,\omega)\Big)dt < \infty \tag{2.6}$$

for every $T \in [0,\infty)$*,* $\psi(\cdot) := \tau(\cdot)\tau'(\cdot)$*.*

In fact, the average capitalization $\mathcal{Y}(t) := (\mathcal{Y}_1(t), \dots, \mathcal{Y}_n(t))$, $t \geqslant 0$ may depend on $\mathcal{X}^N$ and π. The process in Definition 2.1.1(3) here is defined for simplicity.

2.2 Construction of investment strategies

We investigate several investment strategies formulated from the wealth and strategies $(V^\ell(\cdot), \pi^\ell(\cdot))$ of a group of N investors.

Theorem 2.2.1. *Given investment strategies $\pi^\ell \in \Delta_n$, $\ell = 1, \ldots, N$ as in Definition 2.1.1, define the following wealth processes with the same starting capitalization $v^\ell = v$ for every $\ell = 1, \ldots, N$:*

- *The wealth $\widetilde{V}(t)$ achieved by the average of the N portfolios used by the N investors,*

$$\widetilde{V}(t) := V^{\overline{\pi}}(t), \quad \text{where} \quad \overline{\pi}(t) := \frac{1}{N}\sum_{\ell=1}^{N} \pi^\ell(t).$$

- *The arithmetic average wealth $\overline{V}(t)$ of the N investors,*

$$\overline{V}(t) := \frac{1}{N}\sum_{\ell=1}^{N} V^\ell(t).$$

- *The wealth $\check{V}(t)$ is generated by takes the proportion of capitalization of a certain stock in the market as the corresponding strategy of that stock,*

$$\check{V}(t) := V^{\check{\pi}}(t), \quad \text{where} \quad \check{\pi}_i(t) := \frac{\sum_{\ell=1}^{N} \pi_i^\ell(t) V^\ell(t)}{\sum_{i=1}^{n}\sum_{\ell=1}^{N} \pi_i^\ell(t) V^\ell(t)} = \frac{\sum_{\ell=1}^{N} \pi_i^\ell(t) V^\ell(t)}{\sum_{\ell=1}^{N} V^\ell(t)}.$$

We have

$$\widetilde{V}(t) \leqslant \overline{V}(t), \quad \widetilde{V}(t) \leqslant \check{V}(t) \quad \text{for any } t \in (0, \infty).$$

Proof.

The arithmetic average $\overline{V}(t)$ follows

$$\overline{V}(t) = \frac{v}{N}\sum_{\ell=1}^{N}\exp\Big\{\int_0^t \pi^{\ell\prime}(s,\omega)\Big(\beta(s,\omega)-\frac{\alpha(s,\omega)}{2}\Big)ds+\int_0^t \pi^{\ell\prime}(s,\omega)\sigma(s,\omega)dW_k(s)\Big\}. \quad (2.7)$$

$\widetilde{V}(t)$ turns out to be the geometric average wealth, since

$$\begin{aligned}\log\widetilde{V}(t) =&\int_0^t \frac{1}{N}\sum_{\ell=1}^{N}\pi^{\ell\prime}(s,\omega)(\beta(s,\omega)-\frac{\alpha(s,\omega)}{2})ds\\ &+\int_0^t \frac{1}{N}\sum_{\ell=1}^{N}\pi^{\ell\prime}(s,\omega)\sigma(s,\omega)dW_k(s)+\log v \quad (2.8)\\ =&\frac{1}{N}\sum_{\ell=1}^{N}\log V^\ell(t),\end{aligned}$$

so

$$\widetilde{V}(t) = \Big(\prod_{\ell=1}^{N}V^\ell(t)\Big)^{\frac{1}{N}}.$$

By the inequality of arithmetic and geometric means that for a sequence $(x_1,\ldots,x_n)$, $\sqrt[n]{x_1\cdot\ldots\cdot x_n} \leqslant \frac{1}{n}(x_1+\ldots+x_n)$, we have $\widetilde{V}(t) \leqslant \overline{V}(t)$ for $t \in [0,T]$, when the initial wealth of each investor is the same.

To prove the second inequality about the relationship between $\widetilde{V}(t)$ and $\check{V}(t)$, we first write out $\check{V}(t)$ as

$$\begin{aligned}\check{V}(t) =& v\exp\Bigg\{\int_0^t \frac{1}{\sum_{\ell=1}^{N}V^\ell(s)}\sum_{\ell=1}^{N}\pi^{\ell'}(s)V^\ell(s)(\beta(s,\omega)-\frac{\alpha(s,\omega)}{2})ds\\ &+\int_0^t \frac{1}{\sum_{\ell=1}^{N}V^\ell(s)}\sum_{\ell=1}^{N}\pi^{\ell'}(s)V^\ell(s)\sigma(s,\omega)dW(s)\Bigg\}\\ =& v\prod_{\ell=1}^{N}\exp^{\frac{V^\ell(s)}{\sum_{\ell=1}^{N}V^\ell(s)}}\Big\{\int_0^t \pi^{\ell\prime}(s)(\beta(s,\omega)-\frac{\alpha(s,\omega)}{2})ds+\int_0^t \pi^{\ell\prime}(s)\sigma(s,\omega)dW(s)\Big\}.\end{aligned}$$

The logarithm of $\breve{V}(t)$ is therefore

$$\log \breve{V}(t) = \frac{1}{\sum_{\ell=1}^{N} V^{\ell}(t)} \sum_{\ell=1}^{N} V^{\ell}(t) \log V^{\ell}(t). \tag{2.9}$$

Compare (2.8) and (2.9) by taking the ratio $\log \frac{\widetilde{V}(t)}{\breve{V}(t)}$:

$$\begin{aligned}
\log \frac{\widetilde{V}(t)}{\breve{V}(t)} &= \frac{1}{N}\Big(\sum_{\ell=1}^{N} \log V^{\ell}(t) - \frac{1}{\overline{\overline{V}}} \sum_{\ell=1}^{N} V^{\ell}(t) \log V^{\ell}(t)\Big) \\
&\leqslant \log \overline{V}(t) - \frac{1}{\overline{\overline{V}}} \cdot \frac{1}{N} \sum_{\ell=1}^{N} V^{\ell}(t) \log V^{\ell}(t) \\
&\leqslant \log \overline{V}(t) - \frac{1}{\overline{\overline{V}}} \cdot \overline{V} \log \overline{V}(t) = 0.
\end{aligned}$$

The first line is simply combining the expressions in (2.8) and (2.9); the inequality on the second line holds because of the concavity of function $\log(x)$ and Jensen's inequality. The third line is from the convexity of function $x \log(x)$, and Jensen's inequality. Therefore, $\widetilde{V}(t) \leqslant \breve{V}(t)$ for any $t \in (0, \infty)$.

A quick computation shows that the wealth process that strategy $\breve{\pi}(t)$ generates is exactly $\overline{V}(t)$. Hence building a portfolio using $\breve{\pi}(t)$ is one way to get $\overline{V}(t)$.

To sum up, for all $t > 0$, $\widetilde{V}(t) \leqslant \overline{V}(t) = \breve{V}(t)$. That is, the wealth of taking the average strategy earns less than the average wealth or the wealth of applying $\breve{\pi}_i(t)$. The equality $\widetilde{V}(t) = \overline{V}(t) = \breve{V}(t) = V^{\ell}(t)$ holds for each $\ell = 1, \ldots, N$, if each investor is of the same wealth at time t. $\square$

2.3 General finite dynamical system

The interaction among the players we consider here is of the mean field type, in that whenever an individual player (investor) has to make a decision, he or she may not be able to see the individual private information of the other players but may see the average of functions of the private states of the other players. We use the mean field interaction particle models from Statistical Physics to describe the market - We model the N investors as N particles, for fixed N.

For any metric space $(\mathbb{X}, d)$, $\mathcal{P}(\mathbb{X})$ denotes the space of probability measures on $\mathbb{X}$ endowed with the topology of the weak convergence. $\mathcal{P}_p(\mathbb{X})$ is the subspace of $\mathcal{P}(\mathbb{X})$ of the probability measures of order p, that is, $\mu \in \mathcal{P}_p(\mathbb{X})$ if $\int_{\mathbb{X}} d(x, x_0)^p \mu(dx) < \infty$, where $x_0 \in \mathbb{X}$ is an arbitrary reference point. For $p \geqslant 1$, μ, $\nu \in \mathcal{P}_p(\mathbb{X})$, the p-Wasserstein metric on $P_p(\mathbb{X})$ is defined by

$$W_p(\nu_1, \nu_2)^p := \inf_{\pi \in \Pi(\nu_1, \nu_2)} \int_{\mathbb{X} \times \mathbb{X}} d(x, y)^p \kappa(dx, dy),$$

where d is the underlying metric on the space. $\Pi(\nu_1, \nu_2)$ is the set of Borel probability measures π on $\mathbb{X} \times \mathbb{X}$ with the first marginal ν_1 and the second marginal ν_2. Precisely, $\kappa(A \times \mathbb{X}) = \nu_1(A)$ and $\kappa(\mathbb{X} \times A) = \nu_2(A)$ for every Borel set $A \subset \mathbb{X}$.

Also, denote by $C([0, T]; \mathbb{R}^{d_0})$ the space of continuous functions from $[0, T]$ to $\mathbb{R}^{d_0}$. In this paper, we often take $\mathbb{X} = \mathbb{R}^{d_0}$ when considering a real-valued random variable or take $\mathbb{X}$ as the path space $\mathbb{X} = C([0, T]; \mathbb{R}^{d_0})$ for a process, where a fixed number d_0 will be specified later. $\mathcal{P}_p(\mathbb{R}^{d_0})$ equipped with the Wasserstein distance $\mathcal{W}_p$ is a complete separable metric space, since $\mathbb{R}^{d_0}$ is complete and separable.

Definition 2.3.1 (Empirical measure in the finite N-particle system)**.** *Consider* $(V^\ell, \pi^\ell) \in C([0, T]; \mathbb{R}_+) \times C([0, T]; \mathbb{A})$ *that are* $\mathcal{F}$*-measurable random variables, for every investor* $\ell = 1, \ldots, N$. *We define empirical measures* $\nu^N \in \mathcal{P}^2(C([0, T], \mathbb{R}_+) \times C([0, T], \mathbb{A})) \cong$

$\mathcal{P}^2(C([0,T],\mathbb{R}_+\times\mathbb{A}))$ *of the random vectors* $(V^\ell(t),\pi^\ell(t))$ *as*

$$\nu_t^N := \frac{1}{N}\sum_{\ell=1}^{N}\boldsymbol{\delta}_{(V^\ell(t),\pi^\ell(t))}, \quad t \geqslant 0,$$

where $\boldsymbol{\delta}_x$ *is the Dirac delta mass at* $x\in\mathbb{R}_+\times\mathbb{A}$. *Thus for any Borel set* $A\subset\mathbb{R}_+\times\mathbb{A}$,

$$\nu_t^N(A) = \frac{1}{N}\sum_{\ell=1}^{N}\boldsymbol{\delta}^A_{(V^\ell(t),\pi^\ell(t))} = \frac{1}{N}\cdot\#\{\ell\leqslant N : (V^\ell(t),\pi^\ell(t))\in A\},$$

where $\#\{\cdot\}$ *represents the cardinality of the set.*

Denote $\mathcal{X}_t^N = (X_1^N(t),\ldots,X_n^N(t))$, $\mathbf{V}_t = (V^1(t),\ldots,V^N(t))$ for $t\geqslant 0$. For a fixed N, with ν_t^N in definition 2.3.1 that generalizes $\mathcal{Y}(t)$, we can generalize the $(n+N)$-dimensional system as

$$d\mathcal{X}_t^N = \mathcal{X}_t^N\beta(t,\mathcal{X}_t^N,\nu_t^N)dt + \mathcal{X}_t^N\sigma(t,\mathcal{X}_t^N,\nu_t^N)dW_t; \quad \mathcal{X}_0^N = \mathbf{x}_0^N \tag{2.10}$$

and for $\ell = 1,\ldots,N$,

$$dV_t^\ell = V_t^\ell\pi_t^\ell\beta(t,X_t^N,\nu_t^N)dt + V_t^\ell\pi_t^\ell\sigma(t,X_t^N,\nu_t^N)dW_t; \quad V_0^\ell = v^\ell. \tag{2.11}$$

A strong solution of the conditional Mckean-Vlasov system (2.10)-(2.11) is a triplet

$$(\mathcal{X}^N,\mathbf{V},\nu^N)\in(C([0,T],\mathbb{R}_+^n),C([0,T],\mathbb{R}_+^N),\mathcal{P}^2(C([0,T],\mathbb{R}_+\times\mathbb{A}))).$$

We made the following assumptions on the triplet to ensure that the system (2.10)-(2.11) is well-posed. In the following sections we shall assume the well-posedness of the system despite the assumptions on system coefficients or strategy processes for simplicity.

Assumption 2. *The initial wealth and strategies of the N players are i.i.d samples from*

ν_0^N the distribution of (v_0, π_0). The stock price vector $\mathbf{x}_0^N$ at time 0 has a finite second moment, $\mathbb{E}|\mathbf{x}_0^N|^2 < \infty$, and is independent of Brownian motion $\{W_t\}$.

$|\cdot|$ denotes Euclidean norm of $\mathbb{R}^d$ valued process and Frobenius norm of $\mathbb{R}^{d\times n}$ values processes, $d = n$ or N in particular. Let

$$b_i(t, x, \nu) := x_i \beta_i(t, x, \nu), \quad s_{ik}(t, x, \nu) = x_i \sigma_{ik}(t, x, \nu).$$

Assumption 3. *Assume the Lipschitz continuity and linear growth condition are satisfied with Borel measurable mappings $b(t, x, \nu)$, $s(t, x, \nu)$ from $[0, T] \times C([0, T], \mathbb{R}_+^n) \times \mathcal{P}^2(C([0, T], \mathbb{R}_+^N \times \mathbb{A}^N))$ to $\mathbb{R}^n$. That is, there exists a constant $L \in (0, \infty)$, such that*

$$|b(t, x, \nu) - b(t, \widetilde{x}, \widetilde{\nu})| + |s(t, x, \nu) - s(t, \widetilde{x}, \widetilde{\nu})| \leqslant L[|x - \widetilde{x}| + \mathcal{W}_2(\nu, \widetilde{\nu})]$$

for a constant $C^G \in (0, \infty)$,

$$|x\beta(t, x, \nu)| + |x\sigma(t, x, \nu)| \leqslant C^G(1 + |x| + M_2(\nu)),$$

where

$$M_2(\nu) = \left(\int_{C([0,T],\mathbb{R}_+\times\mathbb{A})} |x|^2 d\nu(x) \right)^{1/2}; \quad \nu \in \mathcal{P}_2(C([0, T], \mathbb{R}_+ \times \mathbb{A})).$$

Assume the following Lipschitz continuity and boundedness, $L, B \in (0, \infty)$

$$|v^\ell \beta(t, x, \nu) - \widetilde{v}^\ell \beta(t, \widetilde{x}, \widetilde{\nu})| + |v^\ell \sigma(t, x, \nu) - \widetilde{v}^\ell \sigma(t, \widetilde{x}, \widetilde{\nu})| \leqslant L[|x - \widetilde{x}| + n|v^\ell - \widetilde{v}^\ell| + \mathcal{W}_2(\nu, \widetilde{\nu})],$$

$$|v^\ell \beta(t, x, \nu)| + |v^\ell \sigma(t, x, \nu)| \leqslant B,$$

for every $v^\ell \in \mathbb{R}_+$, $\ell = 1, \ldots, N$; $t \in [0, T]$; $x, \widetilde{x} \in \mathbb{R}_+^n$; $\nu, \widetilde{\nu} \in \mathcal{P}_2(C([0, T], \mathbb{R}_+^N \times \mathbb{A}^N)$.

The admissible strategies $\pi(t)$ might have different structures given the accessible information at time t.

Definition 2.3.2. *A control $\pi(t) \in \mathbb{A}$ is an* ***open loop control*** *if it is a function of time t and initial states v_0. It is called a* ***closed loop feedback control*** *if $\pi(t) \in \mathbb{A}$ is a function of time t and states of every controller $\mathbf{V}(t)$. We denote $\mathbf{V}(t)$ as $\mathbf{V}_t$ for simplicity. It is specified by feedback functions $\phi^\ell : [0,T] \times \Omega \times \mathbb{R}_+^N \to \mathbb{A}$, for $\ell = 1, \ldots, N$, to be evaluated along the path of the state process.*

In this thesis we focus on open loop controls and closed loop feedback controls.

Assumption 4. *Let $\ell = 1, \ldots, N$. For a closed loop feedback control, we assume π^ℓ is Lipschitz in v, i.e., there exists a mapping $\phi^\ell : \mathbb{R}_+^N \to \mathbb{A}$ such that $\pi_t^\ell = \phi^\ell(\mathbf{V}_t)$.*

$$|\phi^\ell(v) - \phi^\ell(\widetilde{v})| \leqslant nL|v - \widetilde{v}|$$

for every $v, \widetilde{v} \in \mathbb{R}_+^N$.

Proposition 2.3.1. *Under Assumption 3 and 4, the $(n+N)$-dimensional SDE system (2.10)-(2.11) admits a unique strong solution, for each n, N.*

Proof. We restrict the discussion on the time homogeneous case, whereas the inhomogeneous case can be proved in the same fashion. Rewrite the system as a $(n+N)$-dimension

SDE system:

$$d\begin{pmatrix}\mathcal{X}_t^N\\ \mathbf{V}_t\end{pmatrix} = \begin{pmatrix} X_1^N(t)\beta_1(\mathcal{X}_t^N,\nu_t^N)dt + X_1(t)\sum_{k=1}^n \sigma_{1k}(\mathcal{X}_t^N,\nu_t^N)dW_k(t)\\ \dots \\ X_n^N(t)\beta_n(\mathcal{X}_t^N,\nu_t^N)dt + X_n(t)\sum_{k=1}^n \sigma_{nk}(\mathcal{X}_t^N,\nu_t^N)dW_k(t)\\ V_t^1\pi_t^{1\prime}\beta(\mathcal{X}_t^N,\nu_t^N)dt + V_t^1\pi_t^{1\prime}\sigma(\mathcal{X}_t^N,\nu_t^N)dW_t\\ \dots \\ V_t^N\pi_t^{N\prime}\beta(\mathcal{X}_t^N,\nu_t^N)dt + V_t^N\pi_t^{N\prime}\sigma(\mathcal{X}_t^N,\nu_t^N)dW_t \end{pmatrix} \tag{2.12}$$
$$:= f(\mathcal{X}_t^N,\mathbf{V}_t,\nu_t^N)dt + g(\mathcal{X}_t^N,\mathbf{V}_t,\nu_t^N)dW_t,$$

where $f(\mathcal{X}_t^N,\mathbf{V}_t,\nu_t) = (f_1(\cdot),\dots,f_{n+N}(\cdot))$, $f_i(\cdot) = X_i^N(t)\beta_i(\cdot)$ for $i=1,\dots,n$, $f_j(\cdot) = \pi_t^{j-n}\beta(\cdot)$ for $j=n+1,\dots,n+N$. Similiarly, $g(\mathcal{X}_t^N,\mathbf{V}_t,\nu_t) = (g_1(\cdot),\dots,g_{n+N}(\cdot))$, $g_i(\cdot) = X_i^N(t)\sigma_i(\cdot)$ for $i=1,\dots,n$, $g_j = V_t^{j-n}\pi_t^{j-n}\sigma(\mathcal{X}_t^N,\nu_t)$ for $j=n+1,\dots,n+N$.

Let us consider a closed loop strategy $\pi_t^\ell = \phi^\ell(\mathbf{V}_t)$. Open loop strategies case can be shown in the same way. Define a mapping $L_N : \mathbb{R}_+^N \to \mathcal{P}^2(C([0,T],\mathbb{R}_+\times\mathbb{A}))$,

$$L_N(\mathbf{V}_t) = \frac{1}{N}\sum_{\ell=1}^N \boldsymbol{\delta}_{(V_t^\ell,\phi^\ell(\mathbf{V}_t))} = \nu_t^N.$$

Define $F:\mathbb{R}_+^{N+n}\to\mathbb{R}^{N+n}$, $G:\mathbb{R}_+^{N+n}\to\mathbb{R}^{N+n}\times\mathbb{R}^n$, with

$$F(\mathcal{X}_t^N,\mathbf{V}_t) = f(\mathcal{X}_t,\mathbf{V}_t,L_N(\mathbf{V}_t));\quad G(\mathcal{X}_t^N,\mathbf{V}_t) = g(\mathcal{X}_t,\mathbf{V}_t,L_N(\mathbf{V}_t)).$$

Let $(x,v) = (x_1,\dots,x_n,v^1,\dots,v^N)$ and $(y,u) = (y_1,\dots,y_n,u^1,\dots,u^N)$ be two pairs of values of $(\mathcal{X}^N(\cdot),\mathbf{V}(\cdot))$ and define a constant $L_m := \max\{1,L,B\}$. By the inequality

$(a+b)^2 \leqslant 2(a^2+b^2)$, uniformly boundedness and Lipschitz condition of β_i and ϕ^ℓ,

$$\begin{aligned}
&|F(x,v)-F(y,u)|^2\\
\leqslant&\sum_{i=1}^{n}|b_i(x,L_N(v))-b_i(y,L_N(u))|^2\\
&+\sum_{\ell=1}^{N}|v^\ell\phi^\ell(v)\beta(x,L_N(v))-u^\ell\phi^\ell(u)\beta(y,L_N(u))|^2\\
\leqslant&2L_m^2[|x-y|^2+\mathcal{W}_2^2(L_N(v),L_N(u)]+2nL_m^2|v-u|^2\\
&+8NL_m^2[|x-y|^2+n|v^\ell-u^\ell|^2+\mathcal{W}_2^2(L_N(v),L_N(u)],
\end{aligned}$$

If the strategies are of the form $\phi^\ell(v^\ell)$ and is Lipchitz continuous $|\phi^\ell(v^\ell)-\phi^\ell(\tilde{v}^\ell)| < nL|v^\ell-\tilde{v}^\ell|$, the last inequality above should be instead

$$2L_m^2[|x-y|^2+\mathcal{W}_2^2(L_N(v),L_N(u))]+8NL_m^2|x-y|^2+10nL_m^2|v-u|^2+8NL_m^2\mathcal{W}_2^2(L_N(v),L_N(u))].$$

Denote the empirical measure induced by the joint distribution of random variable u and v by

$$\tilde{\pi}=\frac{1}{N}\sum_{\ell=1}^{N}\boldsymbol{\delta}_{(u^\ell,v^\ell)}.$$

It is a coupling of the function $L_N(v)$ and $L_N(u)$. By the definition of Wasserstein distance,

$$\mathcal{W}_2^2(L_N(v),L_N(u))\leqslant\int_{\mathbb{R}^N\times\mathbb{R}^N}|(v,\phi(v))-(u,\phi(u))|^2\tilde{\pi}(dv,du)\leqslant\frac{n}{N}(L^2+1)|v-u|^2$$

Therefore,

$$|F(x,v)-F(y,u)|^2\leqslant(2+8N)L_m^2[|x-y|^2+n|v-u|^2],$$

when $L_m<\sqrt{\frac{4(N-1)^2-5}{1+4N}}$. In the same vein, we conclude the Lipschitz continuity of $G(\cdot)$. Thus according to the existence and uniqueness conditions of multi-dimensional SDEs, the system (2.10)-(2.11) is well-defined. □

We will use the system in Proposition 2.3.1 for N-investor relative arbitrage problem in Chapter 3-4. We study this $(n + N)$-dimensional system (2.10)-(2.11) when $N \to \infty$ in Chapter 5 for mean field games.

Chapter 3

Relative arbitrage in a finite particle system

In this chapter we construct an investment framework that arises from the pioneering work [22] about the optimal arbitrage opportunities relative to the market portfolio. To analyze the market and information from investors, we propose a model in which the market dynamics depend on a certain group of investment entities. The portfolio of these entities of interest is determined by a relative arbitrage benchmark.

3.1 Benchmark of the market and investors

We first recall the definition of relative arbitrage in Stochastic Portfolio Theory.

Definition 3.1.1 (Relative Arbitrage). *Given two investment strategies $\pi(\cdot)$ and $\rho(\cdot)$, with the same initial capital $V^{\pi}(0) = V^{\rho}(0) = 1$, we shall say that $\pi(\cdot)$ represents an arbitrage opportunity relative to $\rho(\cdot)$ over the time horizon $[0, T]$, with a given $T > 0$, if*

$$\mathbb{P}\big(V^{\pi}(T) \geqslant V^{\rho}(T)\big) = 1 \quad \textit{and} \quad \mathbb{P}\big(V^{\pi}(T) > V^{\rho}(T)\big) > 0.$$

The *market portfolio* $\mathbf{m}$ is used to describe the behavior of the market: By investing in proportion to the market weight of each stock,

$$\pi_i^{\mathbf{m}}(t) := \frac{X_i^N(t)}{X^N(t)}, \quad i = 1, \ldots, n, \quad t \geqslant 0. \tag{3.1}$$

Consider the wealth process $V^{\mathbf{m}}(\cdot)$ generated by the market portfolio. Let $V^{\mathbf{m}}(0) = x_0$, and since

$$\frac{dV^{\mathbf{m}}(t)}{V^{\mathbf{m}}(t)} = \sum_{i=1}^{n} \pi_i^{\mathbf{m}}(t) \cdot \frac{dX_i^N(t)}{X_i^N(t)} = \frac{dX^N(t)}{X^N(t)}, \quad t \geqslant 0, \tag{3.2}$$

the market portfolio amounts to the ownership of the entire market - the total capitalization

$$X^N(t) = X_1^N(t) + \ldots + X_n^N(t), t \in (0, T]; \quad X^N(0) := x_0.$$

The performance of a portfolio is measured with respect to the market portfolio and other factors. For example, asset managers improve not only absolute performance compared to the market index, but also relative performance with respect to all collegial managers - they try to exploit strategies that achieve an arbitrage relative to market and peer investors. We next define the benchmark of the overall performance.

Definition 3.1.2 (Benchmark)**.** *Relative arbitrage benchmark* $\mathcal{V}^N(T)$*,* $T \in (0, \infty)$*, which is the weighted average of performances of the market portfolio and the average portfolio of* N *investors, is defined as*

$$\mathcal{V}^N(T) = \delta \cdot X^N(T) + (1 - \delta) \cdot \frac{1}{N} \sum_{\ell=1}^{N} V^{\ell}(T), \tag{3.3}$$

with a given constant weight $0 \leqslant \delta \leqslant 1$*.*

The second term $(1 - \delta) \cdot \frac{1}{N} \sum_{\ell=1}^{N} V^{\ell}(T)$, is the average amount of wealth at T.

We assume each investor measures the logarithmic ratio of their own wealth at time

T to the benchmark in (3.3), and searches for a strategy with which the logarithmic ratio is above a personal level of preference almost surely. For $\ell = 1, \ldots, N$, we denote the investment preference of investor ℓ by c_ℓ, a real number given at $t = 0$. Note that c_ℓ is an investor-specific constant, and so it might be different among individuals $\ell = 1, \ldots, N$. An arbitrary investor ℓ tries to achieve

$$\log \frac{V^\ell(T)}{\mathcal{V}^N(T)} > c_\ell, \quad \text{a.s.} \quad \text{or equivalently,} \quad V^\ell(T) \geqslant e^{c_\ell} \mathcal{V}^N(T), \quad \text{a.s.} \tag{3.4}$$

Thus $\mathcal{V}^N(T)$ is the benchmark and an investor ℓ aims to match $e^{c_\ell}\mathcal{V}^N(T)$ based on their preferences.

Assumption 5. *Assume that the preferences of investors c_ℓ are statistically identical and independent samples from a common distribution Law(c).*

Proposition 3.1.1. *We have the following properties of c_ℓ and δ.*

1. *If every investor achieves relative arbitrage opportunity in the sense of (3.4), then we must have*
$$\frac{(1-\delta)}{N} \sum_{\ell=1}^{N} e^{c_\ell} < 1; \tag{3.5}$$

2. *Relative arbitrage in the sense of (3.4) is guaranteed, if $(c_1, \ldots, c_N)$ satisfies that*

$$e^{c_\ell} \leqslant \frac{V^\ell(T)}{\min\{X^N(T), V^1(T), \ldots, V^N(T)\}} \quad \textit{for every } \ell = 1, \ldots, N \quad \textit{a.s.} \tag{3.6}$$

Its proof is given in Appendix A.1. A special case that $c_\ell = c$ (constant) for every $\ell = 1, \ldots, N$. We already know from [22] and [23] that any $c_\ell \leqslant 0$ is a valid level of satisfaction. (3.5) in Proposition 3.1.1 tells us that c_ℓ can be a small positive number. Investors pursuing relative arbitrage should follow the condition (3.5) for c_ℓ.

The following theorem shows that benchmark $\mathcal{V}^N$ is a valid wealth process

Theorem 3.1.1. *Benchmark* $\mathcal{V}^N(t) = \delta X^N(t) + (1-\delta)\frac{1}{N}\sum_{\ell=1}^N V^\ell(t)$ *can be generated from a strategy* $\Pi(\cdot) := (\Pi_1(\cdot), \ldots, \Pi_n(\cdot)) \in \mathbb{A}$,

$$\Pi_i(t) = \frac{\delta X_i^N(t) + (1-\delta)\mathcal{Y}_i(t)}{\mathcal{V}^N(t)},$$

where $\mathcal{Y}_i(t)$ *is defined in* (8.16).

Proof. To show $\mathcal{V}(t)$ is a wealth process generated by a strategy, we use (2.3) and get

$$\begin{aligned}\frac{d\mathcal{V}^N(t)}{\mathcal{V}^N(t)} &= \frac{1}{\mathcal{V}^N(t)}\left(\delta dX(t) + \frac{1}{N}(1-\delta)\sum_{\ell=1}^N\sum_{i=1}^n V^\ell \pi_i^\ell(t)\frac{dX_i(t)}{X_i(t)}\right)\\ &= \sum_{i=1}^n \Pi_i(t)\frac{dX_i(t)}{X_i(t)}, \quad \text{for } t \in (0,T],\end{aligned}$$

and

$$\mathcal{V}^N(0) = \delta x_0 + \frac{1-\delta}{N}\sum_{\ell=1}^N v^\ell,$$

where

$$\begin{aligned}\Pi_i(t) &= \frac{\delta X^N(t)}{\mathcal{V}^N(t)}\mathbf{m}_i(t) + \frac{(1-\delta)}{N\mathcal{V}^N(t)}\sum_{\ell=1}^N V^\ell \pi_i^\ell(t)\\ &= \frac{\delta X_i^N(t) + (1-\delta)\mathcal{Y}_i(t)}{\mathcal{V}^N(t)}.\end{aligned}$$

Further computations show that $\Pi_i(t)$ satisfies self-financing condition (2.4). $\Pi \in \mathbb{A}$ since $\sum_{i=1}^N \Pi_i(t) = 1$ and $0 < \Pi_i(t) < 1$, $i = 1, \ldots, n$. □

3.2 Optimization in relative arbitrage

Assumption 6. *We assume the existence of a market price of risk process* $\theta : [0,\infty) \times \Omega \times \mathcal{P}^2(C([0,T], \mathbb{R}_+ \times \mathbb{A})) \to \mathbb{R}^n$, *an* $\mathbb{F}$*-progressively measurable process such that for any*

$(t, \omega, \nu) \in [0, \infty) \times \Omega \times \mathcal{P}^2(C([0,T], \mathbb{R}_+ \times \mathbb{A}))$,

$$\sigma(t,\omega,\nu)\theta(t,\omega,\nu) = \beta(t,\omega,\nu), \quad \tau(t,\omega,\nu)\theta(t,\omega,\nu) = \gamma(t,\omega,\nu); \tag{3.7}$$

$$\mathbb{P}\bigg(\int_0^T ||\theta(t,\omega,\nu)||^2 dt < \infty, \forall T \in (0,\infty) \bigg) = 1.$$

In the scope of complete market, Assumption 6 shows that the price of risk process $\theta(t)$ governs both the risk premium per unit volatility of stocks and trading volumes, since the market is simultaneously defined by the stocks and the investors. The group of investors we consider in this paper influences the stock capitalization through the trading volumes driven by the same $W(\cdot)$. Thus it does not bring an extra risk factor to the market. In future sections, We shall see the relationship in (3.7) is a key to more tractable and practical results in game formations. We take $\mathbb{F} = \mathbb{F}^{\mathcal{X}^N, \mathcal{Y}} = \mathbb{F}^W$ from now on.

Next we define the deflator based on the market price of the risk process.

Definition 3.2.1. *We define a local martingale $L(t)$,*

$$dL(t) = \theta(t)L(t)dW_t, \quad t \geqslant 0.$$

Equivalently,

$$L(t) := \exp\Big\{ -\int_0^t \theta'(s)dW(s) - \frac{1}{2}\int_0^t ||\theta(s)||^2 ds \Big\}, \quad 0 \leqslant t < \infty.$$

Thus under Assumption 6, the market is endowed with the existence of a local martingale L with $\mathbb{E}[L(T)] \leqslant 1$. We denote the discounted processes $\widehat{V}^{\ell}(\cdot) := V^{\ell}(\cdot)L(\cdot)$, and

$\widehat{X}(\cdot) := X(\cdot)L(\cdot)$. $\widehat{V}^\ell(\cdot)$ admits

$$d\widehat{V}^\ell(t) = dV^\ell(t)L(t) = \widehat{V}^\ell(t)\big(\pi^{\ell\prime}(t)\sigma(t) - \theta'(t)\big)dW(t); \quad \widehat{V}^\ell(0) = \widehat{v}_\ell. \tag{3.8}$$

Remark 2. *With Assumption 6, assume the market $\mathcal{M}$ has bounded variance. Denote $\sum_{\ell=1}^N v^\ell := \bar{v}$. On $[0,T]$, given the existence of relative arbitrage in the sense of (3.4) and Definition 3.1.2, if*

$$c_\ell < \log v_\ell - \log(\delta x_0 + (1-\delta)\bar{v}), \tag{3.9}$$

then the process $L(\cdot)$ is a strict local martingale, i.e., $\mathbb{E}[L(T)] < 1$.

This can be proved by contradiction, assuming $L(T)$ is a martingale. Then by Girsanov theorem, $\mathbb{Q}_T(A) := \mathbb{E}[L(T)1_A], A \in \mathcal{F}$ defines a probability measure that is equivalent to $\mathbb{P}$. We can show $\Delta^\ell(t) := V^\ell(t) - e^{c_\ell}(\delta X(t) + (1-\delta)\frac{1}{N}\sum_{\ell=1}^N V^\ell(t))$ is a martingale under $\mathbb{Q}_T$. Thus the existence of relative arbitrage opportunities implies

$$\mathbb{E}^{Q_T}[\Delta^\ell(T)] = \mathbb{E}^{Q_T}[\Delta^\ell(0)] = v_\ell - e^{c_\ell}\delta x_0 - e^{c_\ell}(1-\delta)\bar{v} \geqslant 0,$$

contradicting to (3.9). This is a generalization of Proposition 6.1 in [24] where the single investor case is studied.

Conversely, for a real number $T > 0$, if $L(T)$ is a martingale and $c_\ell \geqslant \log v_\ell - \log(\delta x_0 + (1-\delta)\bar{v})$ for $\ell = 1, \dots, N$, then no arbitrage relative to the market and investors is possible on the time horizon $[0,T]$.

Now, we shall answer the questions posed in Chapter 1: Given the portfolios

$$\pi^{-\ell}(\cdot) := (\pi^1(\cdot), \dots, \pi^{\ell-1}(\cdot), \pi^{\ell+1}(\cdot), \dots, \pi^N(\cdot)),$$

of all but investor ℓ, what is the best strategy to achieve relative arbitrage for investor

$\ell = 1, \ldots, N$, and if there exists such an optimal strategy, is it possible for all N investors to follow it? We first utilize an idea in the same vein of optimal relative arbitrage in [22], i.e., using the optimal strategy $\pi^{\ell\star}$, the investor ℓ will start with the least amount of the initial capital (or initial cost) relative to $\mathcal{V}^N(0)$, in order to match or exceed the benchmark $e^{c_\ell}\mathcal{V}^N(T)$ at the terminal time T, that is, given $\pi^{-\ell}(\cdot)$, each investor ℓ optimizes

$$u^\ell(T) = \inf\left\{\omega^\ell \in (0,\infty) \,\middle|\, \exists \pi^\ell(\cdot) \in \mathbb{A} \text{ such that } v^\ell = \omega^\ell \mathcal{V}^N(0),\ V^{v^\ell,\pi^\ell}(T) \geqslant e^{c_\ell} \cdot \mathcal{V}^N(T)\right\}. \tag{3.10}$$

Specifically, by (3.2), if everyone uses market portfolio with same initial wealth $v^\ell = v$ for a constant $v > 0$ and every ℓ, their wealth is then $V^\ell(t) = \frac{v}{x_0} X^N(t)$. When the investor adopts the same initial amount of benchmark, i.e., $v = x_0$, and $c_\ell = 0$, then

$$\mathcal{V}^N(t) = \delta X(t) + (1-\delta)\frac{v}{x_0} X(t) = X(t).$$

Therefore in this case a single investor or multiple non-distinguishable investor with market portfolio will match the market.

The following proposition characterizes one's best relative arbitrage opportunities by the customized benchmark $e^{c_\ell} \cdot \mathcal{V}^N(T)$, for any $\ell = 1, \ldots, n$, T and N are fixed real numbers.

Proposition 3.2.1. *$u^\ell(T)$ in (3.10) can be derived as $e^{c_\ell}\mathcal{V}^N(T)$'s discounted expected values over $\mathbb{P}$*

$$u^\ell(T) = \mathbb{E}\big[e^{c_\ell}\mathcal{V}^N(T)L(T)\big] \,/\, \mathcal{V}^N(0). \tag{3.11}$$

This result is essential to the PDE characterization of the objective $u^\ell(T)$ in Section 3.3. It is derived from the supermartingale property of $\widehat{V}^\ell(\cdot)$ and martingale representation theorem, see Appendix A.2 for the details of the proof. To use the martingale representation results in a complete market, we shall assume that $\mathcal{F} = \mathcal{F}^{\mathcal{X}^N,\mathcal{Y}} = \mathcal{F}^W$, where $\mathcal{F}^{\mathcal{X}^N,\mathcal{Y}}$ is the filtration generated by the σ-fields $\{\sigma(\mathcal{X}^N(s), \mathcal{Y}(s); 0 < s < t), t \geqslant 0\}$.

3.3 PDE characterization of the best relative arbitrage

Assumption 7. *We assume $\beta(\cdot)$, $\sigma(\cdot)$, $\gamma(\cdot)$ and $\tau(\cdot)$ take values in $\mathbb{R}^n_+ \times \mathbb{R}^n_+$, are time-homogeneous and the process $(\mathcal{X}^N(t), \mathcal{Y}(t)), t \geqslant 0$ in Definition 2.1.1 is Markovian, i.e.,*

$$X_i^N(t)\beta_i(t) = b_i(\mathcal{X}^N(t), \mathcal{Y}(t)),$$

$$X_i^N(t)\sigma_{ik}(t) = s_{ik}(\mathcal{X}^N(t), \mathcal{Y}(t)), \quad \sum_{k=1}^{n} s_{ik}(t)s_{jk}(t) = a_{ij}(\mathcal{X}^N(t), \mathcal{Y}(t)),$$

$$\gamma_i(t) = \gamma_i(\mathcal{X}^N(t), \mathcal{Y}(t)), \quad \tau_{ik}(t) = \tau_{ik}(\mathcal{X}^N(t), \mathcal{Y}(t)),$$

where $b_i, s_{ik}, a_{ij}, \gamma_i, \tau_i : (0,\infty)^n \times (0,\infty)^n \to \mathbb{R}$ are Hölder continuous.

We define $\tilde{u}^\ell : (0,\infty) \times (0,\infty)^n \times (0,\infty)^n \to (0,\infty)$ from the processes $(\mathcal{X}^N(\cdot), \mathcal{Y}(\cdot))$ starting at $(\mathbf{x}, \mathbf{y}) \in (0,\infty)^n \times (0,\infty)^n$, and write the terminal values

$$\tilde{u}^\ell(T) := \tilde{u}^\ell(T, \mathbf{x}, \mathbf{y}); \quad \ell = 1, \dots N. \tag{3.12}$$

We use the notation D_i and D_{ij} for the partial and second partial derivative with respect to the ith or the ith and jth variables in $\mathcal{X}^N(t)$, respectively; D_p and D_{pq} for the first and second partial derivative in $\mathcal{Y}(t)$.

Assumption 8. *There exist a function $H : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$ of class C^2, such that*

$$b(\mathbf{x}, \mathbf{y}) = 2a(\mathbf{x}, \mathbf{y})D_x H(\mathbf{x}, \mathbf{y}), \quad \gamma(\mathbf{x}, \mathbf{y}) = 2\psi(\mathbf{x}, \mathbf{y})D_y H(\mathbf{x}, \mathbf{y}),$$

i.e., $b_i(\cdot) = \sum_{j=1}^n a_{ij}(\cdot)D_jH(\cdot)$, $\gamma_p(\cdot) = \sum_{q=1}^n \psi_{pq}(\cdot)D_qH(\cdot)$ *in component wise for* $i, p = 1, \ldots, n$.

After the direct calculation based on (3.20) and (9) in the next section, $\tilde{u}^\ell(\cdot)$ follows a Cauchy problem

$$\frac{\partial \tilde{u}^\ell(\tau, \mathbf{x}, \mathbf{y})}{\partial \tau} = \mathcal{A}\tilde{u}^\ell(\tau, \mathbf{x}, \mathbf{y}), \quad \tau \in (0, \infty), \ (\mathbf{x}, \mathbf{y}) \in (0, \infty)^n \times (0, \infty)^n, \tag{3.13}$$

$$\tilde{u}^\ell(0, \mathbf{x}, \mathbf{y}) = e^{c_\ell}, \qquad (\mathbf{x}, \mathbf{y}) \in (0, \infty)^n \times (0, \infty)^n, \tag{3.14}$$

where

$$\begin{aligned}\mathcal{A}\tilde{u}^\ell(\tau, \mathbf{x}, \mathbf{y}) =& \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n a_{ij}(\mathbf{x}, \mathbf{y})\Big(D^2_{ij}\tilde{u}^\ell(\tau, \mathbf{x}, \mathbf{y}) + \frac{2\delta D_i\tilde{u}^\ell(\tau, \mathbf{x}, \mathbf{y})}{\delta\mathbf{x}\cdot\mathbf{1} + (1-\delta)\mathbf{y}\cdot\mathbf{1}}\Big) \\ &+ \frac{1}{2}\sum_{p=1}^n\sum_{q=1}^n \psi_{pq}(\mathbf{x}, \mathbf{y})\Big(D^2_{pq}\tilde{u}^\ell(\tau, \mathbf{x}, \mathbf{y}) + \frac{2(1-\delta)D_p\tilde{u}^\ell(\tau, \mathbf{x}, \mathbf{y})}{\delta\mathbf{x}\cdot\mathbf{1} + (1-\delta)\mathbf{y}\cdot\mathbf{1}}\Big) \\ &+ \sum_{i=1}^n\sum_{p=1}^n (s\tau^T)_{ip}(\mathbf{x}, \mathbf{y})D^2_{ip}\tilde{u}^\ell(\tau, \mathbf{x}, \mathbf{y}) \\ &+ \sum_{i=1}^n\sum_{p=1}^n (s\tau^T)_{ip}(\mathbf{x}, \mathbf{y})\frac{\delta D_p\tilde{u}^\ell(\tau, \mathbf{x}, \mathbf{y}) + (1-\delta)D_i\tilde{u}^\ell(\tau, \mathbf{x}, \mathbf{y})}{\delta\mathbf{x}\cdot\mathbf{1} + (1-\delta)\mathbf{y}\cdot\mathbf{1}}.\end{aligned} \tag{3.15}$$

We emphasize that (3.13) is determined entirely from the volatility structure of $\mathcal{X}^N(\cdot)$ and $\mathcal{Y}(\cdot)$. Moreover, c_ℓ enters into the initial condition (3.14).

3.3.1 Proof and computational details of PDE characterization

We first show some main steps of computing (3.13)-(3.15).

Hence the infinitesimal operator for the process $(\mathcal{X}^N(\cdot), \mathcal{Y}(\cdot))$ can be written as

$$\begin{aligned}\mathcal{L}f =& \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\mathbf{x},\mathbf{y})\big[\frac{1}{2}D_{ij}f + 2D_i f D_j H(\mathbf{x},\mathbf{y})\big] \\ &+ \sum_{p=1}^{n}\sum_{q=1}^{n} \psi_{pq}(\mathbf{x},\mathbf{y})\big[\frac{1}{2}D_{pq}f + 2D_p f D_q H(\mathbf{x},\mathbf{y})\big] \\ &+ \frac{1}{2}\sum_{i=1}^{n}\sum_{p=1}^{n}(s\tau^T)_{ip}(\mathbf{x},\mathbf{y})D_{ip}f + \frac{1}{2}\sum_{i=1}^{n}\sum_{p=1}^{n}(\tau s^T)_{pi}(\mathbf{x},\mathbf{y})D_{pi}f,\end{aligned}$$

where $(\tau s^T)_{pi}(\mathbf{x},\mathbf{y}) = (s\tau^T)_{ip}(\mathbf{x},\mathbf{y}) = \sum_{k=1}^{K} s_{ik}(\mathbf{x},\mathbf{y})\tau_{pk}(\mathbf{x},\mathbf{y})$ and by the definition of $\theta(\cdot)$ in (3.7) and Assumption 8,

$$\theta(\mathbf{x},\mathbf{y}) = 2s(\mathbf{x},\mathbf{y})D_x H(\mathbf{x},\mathbf{y}) = 2\tau(\mathbf{x},\mathbf{y})D_y H(\mathbf{x},\mathbf{y}),$$

or

$$\theta(\mathbf{x},\mathbf{y}) = s^T(\mathbf{x},\mathbf{y})D_x H(\mathbf{x},\mathbf{y}) + \tau^T(\mathbf{x},\mathbf{y})D_y H(\mathbf{x},\mathbf{y}). \tag{3.16}$$

Then it follows from (3.16) and Itô's lemma applying on $H(\cdot)$ that

$$\begin{aligned}&\int_0^{\cdot} \theta'(\mathcal{X}^N(t),\mathcal{Y}(t))dW(t) \\ =&\int_0^{\cdot} \big(s'(\mathcal{X}^N(t),\mathcal{Y}(t))D_x H(\mathcal{X}^N(t),\mathcal{Y}(t)) + \tau'(\mathcal{X}^N(t),\mathcal{Y}(t))D_y H(\mathcal{X}^N(t),\mathcal{Y}(t))\big)dW(t) \\ =&H(\mathcal{X}^N(\cdot),\mathcal{Y}(\cdot)) - H(\mathbf{x},\mathbf{y}) - \int_0^{\cdot} \mathcal{L}H(\mathcal{X}^N(t),\mathcal{Y}(t))dt,\end{aligned} \tag{3.17}$$

$$
\begin{aligned}
&\frac{1}{2}\int_0^{\cdot} ||\theta(t)||^2 dt \\
=&\frac{1}{2}\int_0^{\cdot} ||s'(\mathcal{X}^N(t),\mathcal{Y}(t))D_xH(\mathcal{X}^N(t),\mathcal{Y}(t)) + \tau'(\mathcal{X}^N(t),\mathcal{Y}(t))D_yH(\mathcal{X}^N(t),\mathcal{Y}(t))||^2 dt \\
=&\frac{1}{2}\int_0^{\cdot} \sum_{i=1}^n\sum_{j=1}^n a_{ij}(\mathbf{x},\mathbf{y})D_iH(\mathbf{x},\mathbf{y})D_jH(\mathbf{x},\mathbf{y})dt \\
&+ \frac{1}{2}\int_0^{\cdot} \sum_{p=1}^n\sum_{q=1}^n \psi_{pq}(\mathbf{x},\mathbf{y})D_pH(\mathbf{x},\mathbf{y})D_qH(\mathbf{x},\mathbf{y})dt \\
&+ \int_0^{\cdot} \sum_{i=1}^n\sum_{p=1}^n (s\tau^T)_{ip}(\mathbf{x},\mathbf{y})D_iH(\mathbf{x},\mathbf{y})D_qH(\mathbf{x},\mathbf{y})dt.
\end{aligned}
$$

Thus

$$
\begin{aligned}
L(t) &= \exp\bigg\{ -\int_0^t \theta^T(s)dW(s) - \frac{1}{2}\int_0^t ||\theta(s)||^2 ds\bigg\} \\
&= \exp\bigg\{ -H(\mathcal{X}^N(t),\mathcal{Y}(t)) + H(\mathbf{x},\mathbf{y}) - \int_0^t (k(\mathcal{X}^N(s),\mathcal{Y}(s)) + \tilde{k}(\mathcal{X}^N(s),\mathcal{Y}(s)))ds\bigg\},
\end{aligned}
$$

where

$$
\begin{aligned}
k(\mathbf{x},\mathbf{y}) &:= -\sum_{i=1}^n\sum_{j=1}^n \frac{a_{ij}(\mathbf{x},\mathbf{y})}{2}[D_{ij}^2H(\mathbf{x},\mathbf{y}) + 3D_iH(\mathbf{x},\mathbf{y})D_jH(\mathbf{x},\mathbf{y})], \\
\tilde{k}(\mathbf{x},\mathbf{y}) &: = -\sum_{i=1}^n\sum_{j=1}^n \frac{\psi_{pq}(\mathbf{x},\mathbf{y})}{2}[D_{pq}^2H(\mathbf{x},\mathbf{y}) + 3D_pH(\mathbf{x},\mathbf{y})D_qH(\mathbf{x},\mathbf{y})] \\
&\quad + \sum_{i=1}^n\sum_{p=1}^n (s\tau^T)_{ip}D_iH(\mathbf{x},\mathbf{y})D_pH(\mathbf{x},\mathbf{y})
\end{aligned}
$$

for $(\mathbf{x},\mathbf{y}) \in (0,\infty)^n \times (0,\infty)^n$. Since

$$
\sum_{\ell=1}^N V^\ell(t) = \sum_{i=1}^n\sum_{\ell=1}^N V^\ell(t)\pi_i^\ell(t), \tag{3.18}
$$

we have the expression of benchmark

$$
\mathcal{V}^N(0) = \delta\sum_{i=1}^n x_i + (1-\delta)\sum_{i=1}^n y_i = \delta\mathbf{x}\cdot\mathbf{1} + (1-\delta)\mathbf{y}\cdot\mathbf{1}, \tag{3.19}
$$

where $\mathbf{1}$ is $1 \times n$ column vector having all n elements equal to one. Let us denote

$$g(\mathbf{x}, \mathbf{y}) := \mathcal{V}^N(0)e^{-H(\mathbf{x},\mathbf{y})} = \big(\delta \mathbf{x} \cdot \mathbf{1} + (1-\delta)\mathbf{y} \cdot \mathbf{1}\big)e^{-H(\mathbf{x},\mathbf{y})},$$

$$G(T, \mathbf{x}, \mathbf{y}) := \mathbb{E}^{\mathbb{P}}\big[g(\mathcal{X}^N(T), \mathcal{Y}(T))e^{-\int_0^T k(\mathcal{X}^N(t)) + \tilde{k}(\mathcal{Y}(t))dt}\big].$$

Denote $\tau := T - t$. (3.11) can be rewritten as

$$\tilde{u}^\ell(\tau, \mathbf{x}, \mathbf{y}) = e^{c_\ell}\frac{G(\tau, \mathbf{x}, \mathbf{y})}{g(\mathbf{x}, \mathbf{y})}. \tag{3.20}$$

Assumption 9. *Assume that $g(\cdot)$ is Hölder continuous, uniformly on compact subsets of $\mathbb{R}^n_+ \times \mathbb{R}^n_+$, $\ell = 1, \ldots, N$; $G(\cdot)$ is continuous on $(0, \infty) \times (0, \infty)^n \times (0, \infty)^n$, of class $C^2((0, \infty) \times (0, \infty)^n \times (0, \infty)^n)$.*

The function $G(\cdot)$ yields the following dynamics by Feynman-Kac formula,

$$\begin{aligned}\frac{\partial G}{\partial \tau}(\tau, \mathbf{x}, \mathbf{y}) = \mathcal{L}G(\tau, \mathbf{x}, \mathbf{y}) - \big(k(\mathbf{x}, \mathbf{y}) + \tilde{k}(\mathbf{x}, \mathbf{y})\big)G(\tau, \mathbf{x}, \mathbf{y}), \quad & (\tau, \mathbf{x}, \mathbf{y}) \in \mathbb{R}_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+, \\ G(0, \mathbf{x}, \mathbf{y}) = g(\mathbf{x}, \mathbf{y}), \ (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+. &\end{aligned} \tag{3.21}$$

Under Assumption 9, $\tilde{u}^\ell(\tau, \mathbf{x}, \mathbf{y}) \in C^2((0, \infty) \times (0, \infty)^n \times (0, \infty)^n)$ is bounded on $K \times (0, \infty)^n \times (0, \infty)^n$ for each compact $K \subset (0, \infty)$.

Plugging (3.20) in the above equations set and using the Markovian property of $g(\cdot)$ gives

$$\frac{\partial \tilde{u}^\ell(t, \mathbf{x}, \mathbf{y})}{\partial t}g(\mathbf{x}, \mathbf{y}) = \mathcal{L}(\tilde{u}^\ell(t, \mathbf{x}, \mathbf{y})g(\mathbf{x}, \mathbf{y})) - \big(k(\mathbf{x}, \mathbf{y}) + \tilde{k}(\mathbf{x}, \mathbf{y})\big)\tilde{u}^\ell(t, \mathbf{x}, \mathbf{y})g(\mathbf{x}, \mathbf{y}).$$

For simplicity, we write $\tilde{u}^\ell(t)$ in place of $\tilde{u}^\ell(t,\mathbf{x},\mathbf{y})$. It follows

$$\begin{aligned}
\frac{\partial \tilde{u}^\ell(t)}{\partial t} &= \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(\mathbf{x},\mathbf{y})\bigg(D^2_{ij}\tilde{u}^\ell(t) + 2D_i\tilde{u}^\ell(t)\frac{D_j g(\mathbf{x},\mathbf{y})}{g(\mathbf{x},\mathbf{y})} + \tilde{u}^\ell(t)\frac{D_{ij}g(\mathbf{x},\mathbf{y})}{g(\mathbf{x},\mathbf{y})}\bigg)\\
&\quad + 2\sum_{i,j=1}^{n} a_{ij}(\mathbf{x},\mathbf{y})\bigg(D_i\tilde{u}^\ell(t) + \tilde{u}^\ell(t)\frac{D_i g(\mathbf{x},\mathbf{y})}{g(\mathbf{x},\mathbf{y})}\bigg)D_jH(\mathbf{x},\mathbf{y})\\
&\quad + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}[D^2_{ij}H(\mathbf{x},\mathbf{y}) + 3D_iH(\mathbf{x},\mathbf{y})D_jH(\mathbf{x},\mathbf{y})]\tilde{u}^\ell(t)\\
&\quad + \frac{1}{2}\sum_{p,q=1}^{n} \psi_{pq}(\mathbf{x},\mathbf{y})\bigg(D^2_{pq}\tilde{u}^\ell(t) + 2D_p\tilde{u}^\ell(t)\frac{D_q g(\mathbf{x},\mathbf{y})}{g(\mathbf{x},\mathbf{y})} + \tilde{u}^\ell(t)\frac{D_{pq}g(\mathbf{x},\mathbf{y})}{g(\mathbf{x},\mathbf{y})}\bigg)\\
&\quad + 2\sum_{p,q=1}^{n} \psi_{pq}(\mathbf{x},\mathbf{y})\bigg(D_p\tilde{u}^\ell(t) + \tilde{u}^\ell(t)\frac{D_p g(\mathbf{x},\mathbf{y})}{g(\mathbf{x},\mathbf{y})}\bigg)D_qH(\mathbf{x},\mathbf{y})\\
&\quad + \frac{1}{2}\sum_{p,q=1}^{n} \psi_{pq}[D^2_{pq}I(\mathbf{y}) + 3D_pH(\mathbf{x},\mathbf{y})D_qH(\mathbf{x},\mathbf{y})]\tilde{u}^\ell(t)\\
&\quad + \sum_{i,p=1}^{n} (s\tau^T)_{ip}(\mathbf{x},\mathbf{y})\bigg(D^2_{ip}\tilde{u}^\ell(t) + D_i\tilde{u}^\ell(t)\frac{D_p g(\mathbf{x},\mathbf{y})}{g(\mathbf{x},\mathbf{y})} + D_p\tilde{u}^\ell(t)\frac{D_i g(\mathbf{x},\mathbf{y})}{g(\mathbf{x},\mathbf{y})}\\
&\qquad\qquad + \tilde{u}^\ell(t)\frac{D_{ip}g(\mathbf{x},\mathbf{y})}{g(\mathbf{x},\mathbf{y})}\bigg)\\
&\quad - \sum_{i,p=1}^{n} (s\tau^T)_{ip}(\mathbf{x},\mathbf{y})D_iH(\mathbf{x},\mathbf{y})D_pH(\mathbf{x},\mathbf{y})\tilde{u}^\ell(t).
\end{aligned}$$

We can simplify this equation with the following computations.

By (3.19), and the definition of $g(\cdot)$,

$$\frac{D_i g(\mathbf{x},\mathbf{y})}{g(\mathbf{x},\mathbf{y})} = -D_iH(\mathbf{x},\mathbf{y}) + \frac{\delta}{\delta\mathbf{x}\cdot\mathbf{1} + (1-\delta)\mathbf{y}\cdot\mathbf{1}},$$

$$\frac{D_p g(\mathbf{x},\mathbf{y})}{g(\mathbf{x},\mathbf{y})} = -D_pH(\mathbf{x},\mathbf{y}) + \frac{1-\delta}{\delta\mathbf{x}\cdot\mathbf{1} + (1-\delta)\mathbf{y}\cdot\mathbf{1}}.$$

Th second order derivative with respect to $\mathbf{x}$ is

$$\frac{D_{ij}g(\mathbf{x},\mathbf{y})}{g(\mathbf{x},\mathbf{y})} = -\frac{\delta(D_iH(\mathbf{x},\mathbf{y}) + D_jH(\mathbf{x},\mathbf{y}))}{\delta\mathbf{x}\cdot\mathbf{1} + (1-\delta)\mathbf{y}\cdot\mathbf{1}} - D^2_{ij}H(\mathbf{x},\mathbf{y}) + D_iH(\mathbf{x},\mathbf{y})D_jH(\mathbf{x},\mathbf{y}),$$

and the counterpart of second order derivative with respect to $\mathbf{y}$ is of the same structure

$$\frac{D_{pq}g(\mathbf{x},\mathbf{y})}{g(\mathbf{x},\mathbf{y})} = -\frac{(1-\delta)(D_pH(\mathbf{x},\mathbf{y}) + D_qH(\mathbf{x},\mathbf{y}))}{\delta\mathbf{x}\cdot\mathbf{1} + (1-\delta)\mathbf{y}\cdot\mathbf{1}} - D^2_{pq}H(\mathbf{x},\mathbf{y}) + D_pH(\mathbf{x},\mathbf{y})D_qH(\mathbf{x},\mathbf{y}),$$

$$\frac{D_{ip}g(\mathbf{x},\mathbf{y})}{g(\mathbf{x},\mathbf{y})} = -\frac{\delta D_iH(\mathbf{x},\mathbf{y}) + (1-\delta)D_pH(\mathbf{x},\mathbf{y}))}{\delta\mathbf{x}\cdot\mathbf{1} + (1-\delta)\mathbf{y}\cdot\mathbf{1}} - D^2_{ip}H(\mathbf{x},\mathbf{y}) + D_iH(\mathbf{x},\mathbf{y})D_pH(\mathbf{x},\mathbf{y}).$$

As a result when the drift $\gamma(\cdot)$ and volatility term $\tau(\cdot)$ in (8.16) is given, (3.13) - (3.15) are satisfied. We distinguish the optimal arbitrage objective $\tilde{u}^\ell(\cdot)$ under different information structures in the next section when (8.16) is specified with the function of wealth processes and strategies.

Theorem 3.3.1. *Under Assumption 7,8, and 9, the function* $\tilde{u}^\ell : [0,\infty) \times (0,\infty)^n \times (0,\infty)^n \to (0,1]$ *is the smallest non-negative continuous function, of class* C^2 *on* $(0,\infty) \times (0,\infty)^n$*, that satisfies* $\tilde{u}^\ell(0,\cdot) \equiv e^{c_\ell}$ *and*

$$\frac{\partial \tilde{u}^\ell(\tau,\mathbf{x},\mathbf{y})}{\partial t} \geqslant \mathcal{A}\tilde{u}^\ell(\tau,\mathbf{x},\mathbf{y}), \tag{3.22}$$

where $\mathcal{A}(\cdot)$ *follows* (3.15)*.*

Proof of this theorem can be found in Appendix A.2.

3.3.2 Cauchy problem in different information structure

To see the difference of PDE characterization from using open and closed loop controls, we first study the dynamics of the trading volume $\mathcal{Y}_i(t)$, $i = 1, \dots, n$ which is assumed to follow (2.5) previously.

In general, the investment strategy is a function ϕ of time, wealth or noise depending on the information structure,

$$d\mathcal{Y}_i(t) = \frac{1}{N} \sum_{\ell=1}^{N} dV^\ell(t) \phi_i^\ell(t, \mathbf{V}).$$

By Itô's formula on $\phi^\ell(t, \mathbf{V}) : [0, T] \times \mathbb{R}_+^N \to \mathbb{A}$,

$$\begin{aligned} & dV^\ell(t)\pi_i^\ell(t) \\ =& V^\ell(t)\big(\phi_i^\ell(t, \mathbf{V})\phi^\ell(t, \mathbf{V})\beta(t) + D_t\phi_i^\ell(t, \mathbf{V}) + \sum_{\ell=1}^{N} V^\ell(t)\phi^\ell(t, \mathbf{V})\beta(t) D_\ell \phi_i^\ell(t, \mathbf{V})\big)dt \\ & + \frac{1}{2} V^\ell(t) \sum_{\ell, m=1}^{N} V^\ell(t) V^m(t) \phi^\ell(t, \mathbf{V}) \alpha(t) \phi^{m\prime}(t, \mathbf{V}) \partial^2_{\ell m} \phi_i^\ell(t, \mathbf{V}) dt \\ & + \sum_{m=1}^{N} \phi^\ell(t, \mathbf{V}) \alpha(t) \phi^{m\prime}(t, \mathbf{V}) D_m \phi_i^\ell(t, \mathbf{V}) dt \\ & + V^\ell(t)\big(\phi_i^\ell(t, \mathbf{V})\phi^\ell(t, \mathbf{V})\sigma(t) + \sum_{\ell=1}^{N} V^\ell(t)\phi^\ell(t)\sigma(t) D_\ell \phi_i^\ell(t, \mathbf{V})\big) dW_t. \end{aligned} \tag{3.23}$$

As a result, when searching for Nash equilibrium, we get a more specific form of the optimal strategy $\pi^{\ell\star}$.

Characterization with open loop controls

Recall definition 2.3.2, specifically we consider here controls $\pi^\ell(t)$ given by the deterministic functions $\phi^\ell : [0,T] \times \Omega \to \mathbb{A}$, $\ell = 1, \ldots, N$,

$$\pi^\ell(t) = \phi^\ell(t, \mathbf{v}, W_{[0,t]}), \tag{3.24}$$

for every $t \geqslant 0$, $\mathbf{v} := (v^1, \ldots, v^N)$, $v^\ell = \tilde{u}^\ell(T)\mathcal{V}^N(0)$, $W_{[0,t]}$ is the path of n-dimensional Wiener process between time 0 and time t. From (3.23),

$$\begin{aligned} d\mathcal{Y}_i(t) =& \frac{1}{N}\sum_{\ell=1}^N dV^\ell(t)\pi_i^\ell(t) \\ =& \frac{1}{N}\sum_{\ell=1}^N \big(\phi_i^\ell(t)V^\ell(t)\phi^\ell(t)\beta(t) + V^\ell\partial_t\phi_i^\ell(t)\big)dt + \frac{1}{N}\sum_{\ell=1}^N \phi_i^\ell(t)V^\ell(t)\phi^\ell(t)\sigma(t)dW(t). \end{aligned} \tag{3.25}$$

Thus we can write out explicitly the coefficients $\tau(\cdot)$, $\psi(\cdot)$ in (8.16) and the Cauchy problem of objective $\tilde{u}^\ell$ for each ℓ

$$\frac{\partial \tilde{u}^\ell(\tau, \mathbf{x}, \mathbf{y})}{\partial \tau} = \mathcal{A}\tilde{u}^\ell(\tau, \mathbf{x}, \mathbf{y}), \quad \tau \in (0, \infty),\ (\mathbf{x}, \mathbf{y}) \in (0,\infty)^n \times (0,\infty)^n,$$

$$\tilde{u}^\ell(0, \mathbf{x}, \mathbf{y}) = e^{c_\ell}, \qquad (\mathbf{x}, \mathbf{y}) \in (0,\infty)^n \times (0,\infty)^n,$$

where

$$
\begin{aligned}
\mathcal{A}\tilde{u}^{\ell}(\tau,\mathbf{x},\mathbf{y}) =& \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\mathbf{x},\mathbf{y})\Big(D_{ij}^{2}\tilde{u}^{\ell}(\tau,\mathbf{x},\mathbf{y}) + \frac{2\delta D_i\tilde{u}^{\ell}(\tau,\mathbf{x},\mathbf{y})}{\delta\mathbf{x}\cdot\mathbf{1}+(1-\delta)\mathbf{y}\cdot\mathbf{1}}\Big) \\
&+ \frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n} \psi_{pq}(\mathbf{x},\mathbf{y})\Big(D_{pq}^{2}\tilde{u}^{\ell}(\tau,\mathbf{x},\mathbf{y}) + \frac{2(1-\delta)D_p\tilde{u}^{\ell}(\tau,\mathbf{x},\mathbf{y})}{\delta\mathbf{x}\cdot\mathbf{1}+(1-\delta)\mathbf{y}\cdot\mathbf{1}}\Big) \\
&+ \sum_{i=1}^{n}\sum_{p=1}^{n}(s\tau^{T})_{ip}(\mathbf{x},\mathbf{y})D_{ip}^{2}\tilde{u}^{\ell}(\tau,\mathbf{x},\mathbf{y}) \\
&+ \sum_{i=1}^{n}\sum_{p=1}^{n}(s\tau^{T})_{ip}(\mathbf{x},\mathbf{y})\frac{\delta D_p\tilde{u}^{\ell}(\tau,\mathbf{x},\mathbf{y})+(1-\delta)D_i\tilde{u}^{\ell}(\tau,\mathbf{x},\mathbf{y})}{\delta\mathbf{x}\cdot\mathbf{1}+(1-\delta)\mathbf{y}\cdot\mathbf{1}}.
\end{aligned}
\tag{3.26}
$$

The process $\tau(\cdot)$ and $\psi(\cdot)$ follow

$$\tau_{ik} = \frac{1}{N}\sum_{\ell=1}^{N}\phi_i^{\ell}(t)V^{\ell}(t)\sum_{i=1}^{n}\phi_i^{\ell}(t)\sigma_{ik}(t),$$

$$\psi_{pq}(\mathbf{x},\mathbf{y}) = \sum_{\ell,m=1}^{N} V^{\ell}(t)V^{m}(t)\phi^{\ell}(t)\phi^{m}(t)\alpha(t)\phi_i^{\ell}(t)\phi_j^{m}(t). \tag{3.27}$$

Characterization with closed loop controls

However, in closed loop control, a player at time t has complete information of the states of all the other players at time t. A general closed loop control is given by the form

$$\hat{\pi}^{\ell}(t) = \hat{\phi}^{\ell}(t,\hat{\mathbf{V}}_{[0,t]}).$$

While a closed loop Markovian control is given by

$$\pi^{\ell}(s) = \phi^{\ell}(s,\mathbf{V}_s^{t,x}), \tag{3.28}$$

for each (t,x), where $\phi^\ell : [0,T] \times \Omega \times \mathbb{R}^n_+ \to \mathbb{A}$, $\mathbf{V}^{t,x}_s := (V^1(s), \ldots, V^N(s))^{t,x}$, and $(V^\ell(s))_{t \leqslant s \leqslant T}$ is the unique solution of

$$\frac{dV^\ell(s)}{V^\ell(s)} = \sum_{i=1}^n \pi_i^\ell(s) \frac{dX_i^N(s)}{X_i^N(s)}, \quad V^\ell(t) = v_t^\ell, \quad t \leqslant s \leqslant T.$$

In particular, for $\ell = 1, \ldots, N$, if $\pi_\ell(t)$ is of the form $\phi^\ell(t, V_t^\ell)$, then by Itô's formula

$$\begin{aligned} & dV^\ell(t)\pi_i^\ell(t) \\ =& V^\ell(t)\big(\phi_i^\ell(t,V^\ell)\phi^\ell(t,V^\ell)\beta(t) + D_t\phi_i^\ell(t,V^\ell) + V^\ell(t)\phi^\ell(t,V^\ell)\beta(t)D_\ell\phi_i^\ell(t,V^\ell)\big)dt \\ &+ \frac{1}{2}\mathrm{Tr}(\phi(t,V^\ell)\alpha(t)\phi(t,V^\ell))(V^\ell(t))^2\big(V^\ell(t)D^2_{\ell\ell}\phi_i^\ell(t,V^\ell) + 2D_\ell\phi_i^\ell(t,V^\ell)\big)dt \\ &+ V^\ell(t)\big(\phi_i^\ell(t,V^\ell)\phi^\ell(t,V^\ell)\sigma(t) + V^\ell(t)\phi^\ell(t,V^\ell)\sigma(t)D_\ell\phi_i^\ell(t,V^\ell)\big)dW_t. \end{aligned} \tag{3.29}$$

Hence the Cauchy problem in the closed loop feedback case is

$$\frac{\partial \tilde{u}^\ell(\tau,\mathbf{x},\mathbf{y})}{\partial \tau} = \mathcal{A}\tilde{u}^\ell(\tau,\mathbf{x},\mathbf{y}), \quad \tau \in (0,\infty),\ (\mathbf{x},\mathbf{y}) \in (0,\infty)^n \times (0,\infty)^n,$$

$$\tilde{u}^\ell(0,\mathbf{x},\mathbf{y}) = e^{c_\ell}, \qquad (\mathbf{x},\mathbf{y}) \in (0,\infty)^n \times (0,\infty)^n,$$

where

$$\begin{aligned} \mathcal{A}\tilde{u}^\ell(\tau,\mathbf{x},\mathbf{y}) =& \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n a_{ij}(\mathbf{x},\mathbf{y})\Big(D^2_{ij}\tilde{u}^\ell(\tau,\mathbf{x},\mathbf{y}) + \frac{2\delta D_i\tilde{u}^\ell(\tau,\mathbf{x},\mathbf{y})}{\delta\mathbf{x}\cdot\mathbf{1} + (1-\delta)\mathbf{y}\cdot\mathbf{1}}\Big) \\ &+ \frac{1}{2}\sum_{p=1}^n\sum_{q=1}^n \psi_{pq}(\mathbf{x},\mathbf{y})\Big(D^2_{pq}\tilde{u}^\ell(\tau,\mathbf{x},\mathbf{y}) + \frac{2(1-\delta)D_p\tilde{u}^\ell(\tau,\mathbf{x},\mathbf{y})}{\delta\mathbf{x}\cdot\mathbf{1} + (1-\delta)\mathbf{y}\cdot\mathbf{1}}\Big) \\ &+ \sum_{i=1}^n\sum_{p=1}^n (s\tau^T)_{ip}(\mathbf{x},\mathbf{y})D^2_{ip}\tilde{u}^\ell(\tau,\mathbf{x},\mathbf{y}) \\ &+ \sum_{i=1}^n\sum_{p=1}^n (s\tau^T)_{ip}(\mathbf{x},\mathbf{y})\frac{\delta D_p\tilde{u}^\ell(\tau,\mathbf{x},\mathbf{y}) + (1-\delta)D_i\tilde{u}^\ell(\tau,\mathbf{x},\mathbf{y})}{\delta\mathbf{x}\cdot\mathbf{1} + (1-\delta)\mathbf{y}\cdot\mathbf{1}}. \end{aligned} \tag{3.30}$$

Denote $\tilde{\phi}_i^\ell(t, V^\ell) := \phi_i^\ell(t, V^\ell) + V^\ell(t) D_\ell \phi_i^\ell(t, V^\ell)$,

$$\psi_{pq}(\mathbf{x}, \mathbf{y}) = \sum_{\ell,m=1}^{N} V^\ell(t) V^m(t) \phi^\ell(t, V^\ell) \phi^m(t, V^\ell) \alpha(t) \tilde{\phi}_i^\ell(t, V^\ell) \tilde{\phi}_j^m(t, V^\ell). \tag{3.31}$$

3.4 Existence of Relative Arbitrage

The Cauchy problem (3.13)-(3.14) admits a trivial solution $\tilde{u}^\ell(\tau, \mathbf{x}, \mathbf{y}) \equiv e^{c_\ell}$. Thus we need $\tilde{u}^\ell(\tau, \mathbf{x}, \mathbf{y})$ to take values less than e^{c_ℓ}, indicating that the uniqueness of Cauchy problem fails.

Through the Föllmer exit measure [30] we can relate the solution of Cauchy problem $u^\ell(\cdot)$ to the maximal probability of a supermartingale process staying in the interior of the positive orthant through $[0, T]$. Following the route suggested by [22] and [64], there exists a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, such that $\mathbb{P}$ is locally absolutely continuous with respect to $\mathbb{Q}$: $\mathbb{P} << \mathbb{Q}$, $\Lambda(T)$ is a $\mathbb{Q}$-martingale, and $d\mathbb{P} = \Lambda(T) d\mathbb{Q}$ holds on each $\mathcal{F}_T$, $T \in (0, \infty)$. We can characterize $\tilde{u}^\ell(t)$ by an auxiliary diffusion which takes values in the nonnegative orthant $[0, \infty)^{2n} / \{\mathbf{0}\}$.

Definition 3.4.1 (Auxiliary process and the Fichera drift). *We define the following*

1. *The auxiliary process $\zeta = (\zeta_1, \ldots, \zeta_{2n})$ is defined as*

$$d\zeta_i(\cdot) = \hat{b}_i(\zeta(\cdot))dt + \hat{\sigma}_{ik}(\zeta(\cdot))dW_k, \quad \zeta_i(0) = \zeta_i, \quad i = 1, \ldots, 2n,$$

where

$$\hat{b}_i(\mathbf{x}, \mathbf{y}) = \begin{cases} \frac{\delta \sum_{j=1}^n a_{ij}(\mathbf{x}, \mathbf{y})}{\delta \mathbf{x} \cdot \mathbf{1} + (1-\delta) \mathbf{y} \cdot \mathbf{1}} & \text{if } i = 1, \ldots, n, \\ 0 & \text{if } i = n+1, \ldots, 2n, \end{cases}$$

$$\hat{a}_{ij}(\mathbf{x},\mathbf{y}) = \begin{cases} a_{ij}(\mathbf{x},\mathbf{y}) & \text{if } i,j = 1, \ldots, n, \\ \psi_{ij}(\mathbf{x},\mathbf{y}) & \text{if } i,j = n+1, \ldots, 2n, \\ 0 & \text{otherwise.} \end{cases}$$

2. *The Fichera drift is defined as*

$$f_i(\cdot) := \hat{b}_i(\mathbf{x},\mathbf{y}) - \frac{1}{2}\sum_{j=1}^{n} D_j \hat{a}_{ij}(\mathbf{x},\mathbf{y}),$$

for $i = 1, \ldots, 2n$, $(\mathbf{x},\mathbf{y}) \in (0,\infty)^n \times (0,\infty)^n$.

Assumption 10. *The functions* $b_i(\cdot), \sigma_{ik}(\cdot)$ *are of class* $C^1((0,\infty)^n \times (0,\infty)^n)$ *and satisfy the linear growth condition*

$$||b(\mathbf{x},\mathbf{y})|| + ||s(\mathbf{x},\mathbf{y})|| \leqslant C(1 + ||\mathbf{x}|| + ||\mathbf{y}||), \quad (\mathbf{x},\mathbf{y}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+. \tag{3.32}$$

$a_{ij}(\cdot)$ *satisfy the nondegeneracy condition, i.e., if there exists a number* $\epsilon > 0$ *such that*

$$a_{ij}(\mathbf{x},\mathbf{y}) \geqslant \epsilon(||\mathbf{x}||^2 + ||\mathbf{y}||^2), (\mathbf{x},\mathbf{y}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+.$$

Definition 3.4.2. *If Assumption 10 holds and*

$$\sum_{i=1}^{n}\left(\hat{b}_i(\mathbf{x},\mathbf{y})\mathbf{n}_i + \frac{1}{2}\sum_{j=1}^{n} D_j \hat{a}_{ij}(\mathbf{x},\mathbf{y})\right) \leqslant 0, \tag{3.33}$$

where $\mathbf{n} = (\mathbf{n}_1, \ldots, \mathbf{n}_{2n})$ *is the outward normal vector to* ∂G*, then* ∂G *is an obstacle from inside.* ∂G *is an obstacle from outside if the (3.33) is replaced by*

$$\sum_{i=1}^{n}\left(\hat{b}_i(\mathbf{x},\mathbf{y})\mathbf{n}_i + \frac{1}{2}\sum_{j=1}^{n} D_j \hat{a}_{ij}(\mathbf{x},\mathbf{y})\right) \geqslant 0. \tag{3.34}$$

Assumption 11. *The system of* $\zeta^\ell(\cdot)$ *admits a unique-in-distribution weak solution with values in* $[0,\infty)^n \times [0,\infty)^n/\{\mathbf{0}\}$.

We set $\mathcal{T}^\ell := \{t \geqslant 0 | \zeta^\ell(t) \in \mathcal{O}^{2n}\}$ as the first hitting time of auxiliary process $\zeta^\ell(\cdot)$ to $\mathcal{O}^{2n}$, the boundary of $[0,\infty)^{2n}$.

Proposition 3.4.1. *With the nondegeneracy condition of* a_{ij}, *suppose that the functions* $\hat{\sigma}_{ik}(\cdot)$ *are continuously differentiable on* $(0,\infty)^{2n}$; *that the matrix* $\hat{a}(\cdot)$ *degenerates on* $\mathcal{O}^{2n}$; *and that the Fichera drifts for the process* $\zeta^\ell(\cdot)$ *can be extended by continuity on* $[0,\infty)^{2n}$. *For an investor* ℓ, *if* $f_i(\cdot) \geqslant 0$ *holds on each face of the orthant, then* $\tilde{u}^\ell(\cdot,\cdot) \equiv e^{c_\ell}$, *and no arbitrage with respect to the market portfolio exists on any time-horizon. If* $f_i(\cdot) < 0$ *on each face* $\{x_i^N = 0\}$, $i = 1,\dots,n$ *and* $\{y_i = 0\}$, $i = n+1,\dots,2n$ *of the orthant, then* $\tilde{u}^\ell(\cdot,\cdot) < e^{c_\ell}$ *and arbitrage with respect to the market portfolio exists, on every time-horizon* $[0,T]$ *with* $T \in (0,\infty)$.

Proof. With the nondegeneracy condition of covariance $(a_{ij})_{1\leqslant i,j\leqslant n}$, Theorem 2 in [22] suggests that

$$\tilde{u}^\ell(T,\mathbf{x},\mathbf{y}) = e^{c_\ell}\mathbb{Q}[\mathcal{T}^\ell > T], \quad (T,\mathbf{x},\mathbf{y}) \in [0,\infty)\times[0,\infty)^n\times[0,\infty)^n.$$

For the first claim, we only need to show the probability $\mathbb{Q}[\mathcal{T}^\ell > T] \equiv 1$, for $(T,\mathbf{x},\mathbf{y}) \in [0,\infty)\times[0,\infty)^n\times[0,\infty)^n$. Denote a bounded and connected C^3 boundary $G_R := \{z \in \mathbb{R}^{2n}, z_i < 0, ||z|| < R\}$, and R can be arbitrarily large. Then from Theorem 9.4.1 (or Corollary 9.4.2) of [31], since

$$\sum_{i=1}^n \left(\hat{b}_i(\mathbf{x},\mathbf{y}) - \frac{1}{2}\sum_{j=1}^n D_j \hat{a}_{ij}(\mathbf{x},\mathbf{y})\right)\mathbf{n}_i \leqslant 0,$$

in which $\mathbf{n} = (\mathbf{n}_1,\dots,\mathbf{n}_{2n})$ is the outward normal vector at $(\mathbf{x},\mathbf{y})$ to $\mathcal{O}^{2n}$, the boundary $\mathcal{O}^{2n}$ is an obstacle from outside of G_R, i.e., $\mathcal{G} := B_R(0)/G_R$. The Fichera vector field points toward the domain interior at the boundary. Let $R \to \infty$, the boundary is not attainable almost surely for $(\mathbf{x}, \mathbf{y}) \in [0, \infty)^{2n}$.

If $f_i(\cdot) < 0$ on each face $\{z_i = 0\}$, $i = 1, \ldots, 2n$, then

$$\sum_{i=1}^{n} \Big(\hat{b}_i(\mathbf{x}, \mathbf{y}) - \frac{1}{2} \sum_{j=1}^{n} D_j \hat{a}_{ij}(\mathbf{x}, \mathbf{y}) \Big) \mathbf{n}_i \geqslant 0,$$

and the Fichera drift at $\mathcal{O}^{2n}$ points toward the exterior of $[0, \infty)^{2n}$. It is equivalent to show that $\mathbb{Q}[\mathcal{T}^{\ell} > T] < 1$, for $(T, \mathbf{x}, \mathbf{y}) \in [0, \infty) \times [0, \infty)^n \times [0, \infty)^n$, we only need to show $\mathbb{Q}[\mathcal{T}^{\ell} < T] > 0$, i.e., the boundary $\{z_i = 0\}$, $i = 1, \ldots, 2n$, is attainable by $\zeta^{\ell}(\cdot)$.

From Chapter 11 and 13 in [31], every point in $\partial \mathcal{G}$ is a regular point, and thus

$$\lim_{z \to z_0, z \in \mathcal{G}} \mathbb{Q}_z(\tau^g < \infty, ||\zeta^{\ell}(\tau^g) - z_0|| < \delta) = 1,$$

where τ^g is the exit time from $\bar{\mathcal{G}}$. Therefore, if $z_0 \in \Sigma := \cup_{i=1}^{2n} \{z \in \mathbb{R}^{2n} : z_i = 0\} \cap \overline{\mathcal{G}}$, for a fixed δ such that $B^{+}_{\delta}(z_0) := \cap_{i=1}^{2n} \{z \in \mathbb{R}^{2n} : z_i > 0\} \cap B_{\delta}(z_0)$ is a proper subset of $\mathcal{G}$, we have

- If $||\zeta^{\ell}_i - z_0|| \leqslant \eta$,
$$\mathbb{Q}(\tau^g < \infty, \ \zeta^{\ell}(\tau^g) \in \Sigma) > 0.$$

- If $||\zeta^{\ell}_i - z_0|| > \eta$,
$$\inf_{z \in A} \mathbb{Q}_z(\zeta^{\ell}(\tau^g) \in B_{\delta}(z_0), \tau^g < \infty) > \frac{1}{2},$$
where
$$A := \bigcap_{i=1}^{2n} \{z \in \mathbb{R}^{2n} : z_i > 0, ||z - z_0|| = \eta\}.$$

Now take $r \in A$ and a continuous sample path $\omega_{\star}$ such that $\omega_{\star}(0) = z_0$, $\omega_{\star}(\tau_{\star}) = r$, and $\omega_{\star}(s) \notin A$ for $0 \leqslant s < \tau_{\star}$, where $\tau_{\star} := \inf\{t > 0 : \zeta^{\ell}(t) \in A\}$. Consider an ϵ-neighborhood

$N_{\epsilon,\omega_\star}$ of $\omega_\star \in C(\mathcal{G})$,

$$N_{\epsilon,\omega_\star} = \{\omega \in C(\mathcal{G}) :\ \omega(0) = \zeta_i^\ell,\ ||\omega - \omega_\star|| < \epsilon,\ \omega(\tau_\star) = r\} \subset \{\omega \in \Omega : \zeta^\ell(\tau_\star, \omega) \in A\},$$

then the support theorem in [69] shows that

$$\mathbb{Q}_{\zeta_i^\ell}(N_{\epsilon,\omega_\star}) > 0,$$

where $\phi : [0, \infty) \to \mathbb{R}^{2n}$ is continuously differentiable, and $|| \cdot ||_T^s$ is the supremum norm $||\omega_1 - \omega_2|| = \sup_{0 \leqslant s \leqslant \tau_\star} |\omega_1 - \omega_2|$, ω_1, $\omega_2 \in C(\mathcal{G})$. Hence

$$\mathbb{Q}_{z_0}(N_{\epsilon,\omega_\star}) \leqslant \mathbb{Q}_{z_0}(\tau_\star < \infty,\ \zeta(\tau_\star) \in A).$$

Therefore

$$\begin{aligned}
\mathbb{Q}_{\zeta_i^\ell 0}(\zeta^\ell(\tau^g) \in \Sigma, \tau^g < \infty) &\geqslant \mathbb{Q}_{\zeta_i^\ell}(\zeta^\ell(\tau^g) \in \Sigma, \tau^g < \infty) \\
&\geqslant \mathbb{E}_{\zeta_i^\ell}[\mathbb{Q}_{z_0}(\zeta^\ell(\tau^g) \in \Sigma, \tau^g < \infty) \cdot \mathbf{1}(\zeta^\ell(\tau_\star),\ \tau_\star < \infty)|\mathcal{F}_{\tau_\star}] \\
&= \mathbb{E}_{\zeta_i^\ell}[\mathbb{Q}_{\zeta^\ell(\tau_\star)}(\zeta^\ell(\tau^g) \in \Sigma, \tau^g < \infty) \cdot \mathbf{1}(\zeta^\ell(\tau_\star) \in A,\ \tau_\star < \infty)] \\
&\geqslant \mathbb{E}_{\zeta_i^\ell}[\inf_{z \in A} \mathbb{Q}_z(\zeta^\ell(\tau^g) \in \Sigma, \tau^g < \infty) \cdot \mathbf{1}(\zeta^\ell(\tau_\star) \in A,\ \tau_\star < \infty)] \\
&\geqslant \frac{1}{2}\mathbb{Q}_{\zeta_i^\ell}(\zeta^\ell(\tau_\star) \in A,\ \tau_\star < \infty).
\end{aligned}$$

The equality in the above expressions is from the strong Markov property of $\zeta^\ell(\cdot)$.

In conclusion, the process $\zeta^\ell(\cdot)$ attain the set $\cup_{i=1}^{2n}\{z_i = 0\}$ with positive probability, so $\tilde{u}^\ell(\cdot, \cdot) < 1$ when Fichera drift $f_i(\cdot) < 0$. □

Therefore investor ℓ can find relative arbitrage opportunities with a unique $\tilde{u}^\ell$, the minimal solution of (3.22) given $f_i(\cdot) < 0$ on each face of $\mathcal{O}^{2n}$.

Chapter 4

Relative arbitrage in N-player games

Investors aiming to achieve relative arbitrage in the market model introduced earlier are characterized in N-player games in the cases with open and closed loop information structures. We pay special attention to the uniqueness of Nash equilibrium and the relationship between mean field terms $\mu^{N\star}$, $\nu^{N\star}$. We then provide approaches to search for Nash equilibrium of N investors seeking best arbitrage opportunities. The approach employing Markovian condition of market coefficients will also be useful in the formulation of mean field games.

4.1 N-player games set-up

As we have seen in the previous sections, the stock prices and investors' wealth are coupled. Variation of one investor's strategies contributes to the change of the trading volume of each stock, and thus the change of stock prices. Consequently, the wealth of others is affected by this investor. In addition, all the investors considered here are competitive. They attempt to not only behave better than the market index but also beat the performance of peers exploiting similar opportunities - everyone simultaneously wishes to optimize their initial wealth to achieve a relative arbitrage.

Investors interact with each other, adopt a plan of actions after analyzing other people's options, and finally, make decisions. This motivates us to model the investors as participants in a N-player game.

4.1.1 Construction of Nash equilibrium

The solution concept of this N-player game is Nash equilibrium. In this spirit, assuming that the others have already chosen their own strategies, a typical player computes the best response to all the other players, which amounts to the solution of an optimal control problem to minimize the expected cost $\tilde{u}^\ell$. Specifically, when investor ℓ assumes the wealth of other players are fixed, they wish to take the solution of (3.13) and (3.14) as their wealth to begin with so that

$$V^\ell(T) \geqslant e^{c_\ell}\mathcal{V}^N(T) = \delta \cdot e^{c_\ell} X^N(T) + (1-\delta)\cdot e^{c_\ell}\frac{1}{N}\sum_{\ell=1}^{N} V^\ell(T).$$

We articulate the definition of Nash equilibrium in this problem.

Definition 4.1.1 (Nash Equilibrium). *A vector $\pi^{\ell *} = (\pi_i^{\ell *}, \dots, \pi_n^{\ell *})$ of admissible strategies in Definition 2.1.1 is a Nash Equilibrium, if for all $\pi_i^\ell \in \mathbb{A}$ and $i = 1, \dots, n$,*

$$J^\ell(\pi_i^{\ell *}, \pi_i^{-\ell *}) \leqslant J^\ell(\pi_i^{\ell}, \pi_i^{-\ell *}), \tag{4.1}$$

where the cost to investor ℓ yields

$$J^\ell(\pi) := \inf\left\{\omega^\ell > 0 \;\middle|\; V^{\omega^\ell \mathcal{V}^N(0), \pi^\ell}(T) \geqslant e^{c_\ell}\mathcal{V}^N(T)\right\},$$

where $\pi(\cdot) = (\pi^1(\cdot), \dots, \pi^N(\cdot))$*. Hence,*

$$\inf_{\pi^\ell \in \mathbb{A}} J^\ell(\pi) = u^\ell(T). \tag{4.2}$$

Since $v^\ell = \omega^\ell e^{c_\ell} \mathcal{V}^N(0)$, the infimum is attained, and

$$J^\ell(\pi; 0, x_0) = e^{c_\ell} \frac{\mathcal{V}^N(T)}{\mathcal{V}^N(0)} \exp^{-1} \left\{ \int_0^T \pi_t^{\ell\prime}(\beta_t - \frac{1}{2}\alpha_t \pi_t^\ell)dt + \int_0^T \pi_t^{\ell\prime} \sigma_i(t) dW_t \right\} \leqslant \omega^\ell. \tag{4.3}$$

Each individual aims to minimize the relative amount of initial capital, beginning with which one can match or exceed the benchmark.

In addition, we recall the information structure and the types of actions that players take in a game. It is an *open loop Nash equilibrium* if the admissible strategies satisfy the conditions of Definition 4.1.1, with the control $\pi^\ell(t)$ given by the form

$$\pi^\ell(t) = \phi^\ell(t, \mathbf{v}, W_{[0,t]}), \tag{4.4}$$

for every $t \geqslant 0$, $\mathbf{v} := (v^1, \dots, v^N)$, $v^\ell = \tilde{u}^\ell(T)\mathcal{V}^N(0)$, $W_{[0,t]}$ is the path of the Wiener process between time 0 and time t deterministic functions $\phi^\ell : [0,T] \times \Omega \to \mathbb{A}$, $\ell = 1, \dots, N$. Here, $\pi^{-\ell}$ is the process with the same trajectories as the $(\pi^{1\star}, \dots, \pi^{\ell\star}, \dots, \pi^{N*})$, even after player ℓ changes strategy from $\pi^{\ell *}$ to π^ℓ. Thus the strategies π^k for $k \neq \ell$ of the other players are not affected by the deviation of player ℓ.

However, in closed loop equilibria, the trajectory of the state of the system enters the strategies, then when ℓ change $\pi^{\ell *}(t)$ to $\pi^\ell(t)$, other players are likely to be affected. Players at time t have complete information of the states of all the other players at time t, or in other words we allow feedback strategies. As a special case in closed loop equilibria, a Markovian equilibrium is the admissible strategies profile $\pi^\star = (\pi^{1\star}, \dots, \pi^{\ell\star}, \dots, \pi^{N\star})$

of the form

$$\pi^\ell(s) = \phi^\ell(s, \mathbf{V}_s^{t,x}), \tag{4.5}$$

for each (t, x), where $\phi^\ell : [0, T] \times \Omega \times \mathbb{R}_+^n \to \mathbb{A}$, $\mathbf{V}_s^{t,x} := (V^1(s), \ldots, V^N(s))^{t,x}$, and $(V^\ell(s))_{t \leqslant s \leqslant T}$ is the unique solution of

$$\frac{dV^\ell(s)}{V^\ell(s)} = \sum_{i=1}^n \pi_i^\ell(s) \frac{dX_i^N(s)}{X_i^N(s)}, \quad V^\ell(t) = v_t^\ell, \quad t \leqslant s \leqslant T.$$

4.1.2 The uniqueness of Nash equilibrium

Subsequently, we clarify the notion of unique Nash equilibrium we will apply in this paper. Investors pay more attention to the change of the wealth processes than the change of the strategies, since two different strategy processes may result in the same wealth at time T. Therefore we investigate the uniqueness in distribution of wealth, and we use the strong uniqueness here because it satisfies the nature of the investment goal in this paper.

Definition 4.1.2. *With the same conditions in Definition 2.3.1, we define empirical measures of the random vectors $\left(V^\ell(t)\right)_{\ell=1,\ldots,N} \in \mathbb{R}_+^N$, given the initial measure $\mu_0^N \in \mathcal{P}^2(\mathbb{R}_+)$,*

$$\mu_t^N := \frac{1}{N} \sum_{\ell=1}^N \boldsymbol{\delta}_{V^\ell(t)}.$$

We denote the measure flow $\mu^N := (\mu_t^N)_{t \in [0,T]}$.

We give the following notion of the uniqueness of N-player game Nash equilibrium. We do not require the optimal control to be unique.

Definition 4.1.3. *We say that the uniqueness holds for Nash equilibrium if any two* *solutions* μ_a^N, μ_b^N, *defined on* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, *with the same initial law* $\mu_0^N \in \mathcal{P}^2(\mathbb{R}_+)$,

$$\mathbb{P}[\mu_a^N = \mu_b^N] = 1,$$

where μ^N *is the empirical distribution of wealth processes as in definition 4.1.2.*

Denote the mean field interactions μ^N and ν^N under the Nash equilibrium of N-player games as $\mu^{N\star}$ and $\nu^{N\star}$. Generally, the uniqueness of $\mu^{N\star}$ is a less restricted condition than the uniqueness of $\nu^{N\star}$. Starting from a uniquely fixed optimal $\nu^{N\star}$ or $\mu^{N\star}$, we analyze whether the counterpart $\mu^{N\star}$ or $\nu^{N\star}$ is unique. We elaborate on this point as follows.

- If there is a unique optimal $\nu^{N\star}$ or $\frac{1}{N}\sum_{\ell=1}^{N} \boldsymbol{\delta}_{(V^{\ell\star}(t), \pi_i^{\ell\star}(t))}$ in the sense of Definition 4.1.1 then it implies that its marginal distribution $\mu^{N\star}$ or $\frac{1}{N}\sum_{\ell=1}^{N} \boldsymbol{\delta}_{V^{\ell\star}(t)}$ is the unique Nash equilibrium defined by Definition 4.1.3.

- However, the converse is not true - A unique $\mu^{N\star}$ does not necessarily give unique optimal $\nu^{N\star}$. When searching for NE, suppose the optimal path $V^{\ell\star} \in C([0,T]; \mathbb{R}_+^n)$ is unique for each ℓ. By (2.12), the process follows

$$V^{\ell\star}(t) = v^\ell \exp\left\{ \int_0^t \pi^{\ell\prime}(s)\big(\beta(s) - \frac{\alpha(s)}{2}\pi(s)\big)ds + \int_0^t \pi^{\ell\prime}(s)\sigma(s)dW_s \right\}. \tag{4.6}$$

  Solving the corresponding $\{\pi_i^\ell(t)\}_{i=1,\dots,n}$ of (4.6) may rely on Malliavin calculus and the solution can be written by different stochastic processes. Thus there could be multiple possible quantities of optimal measure $\nu^{N\star}$.

To put this conclusion another way, there could be multiple solutions of $\pi^{\ell\star}(\cdot)$ that generated the unique $V^{\ell\star}$ or $\mu^{N\star}$ for $\ell = 1, \dots, N$. In the next section (Proposition 4.2.1 and Proposition 4.2.2), we show methods to attain one solution of the optimal strategies $\{\pi_i^{\ell\star}(t)\}_{i=1,\dots,n}$ that generate the unique optimal wealth $V^{\ell\star}$. It assumes each $dW_i(t)$ term of $d\hat{V}^{\star}(t)$ is identical to that of $d\mathcal{V}^N(t)u^{\ell}(T-t, \mathcal{X}^N(t), \mathcal{Y}(t))$.

4.2 Optimal arbitrage opportunities in N-player game

From the relationship of $\mu^{N\star}$ and $\nu^{N\star}$ we can see that the key to search for Nash equilibrium is the fixed point condition on the control space. Since requiring $V^{\ell\star}$ to be unique does not determine $\pi^{\star}$ or $\nu^{N\star}$, both $\mu^{N\star}$ and $\nu^{N\star}$ should be fixed through $\pi^{\ell\star}$.

Thus we search for the equilibrium in the control space by fixed point argument - Assume all controls $\pi^k(\cdot)$, $k \neq \ell$ are chosen, player ℓ will choose the optimal strategy $\pi^{\star}$ that achieves optimal value function. Then one solves the equation of wealth processes (2.3) and trading volume (2.5) with the equation of optimal cost function (3.22). If the corresponding optimal strategy $\pi^{\star}$ agrees with π, then the associated μ^N is the Nash equilibrium. We will see in the next chapter that this also provides a route for searching mean field equilibrium.

We specify the methodology below.

Searching Nash equilibrium in N-player game

1. Suppose we start with a given set of control processes $\pi := (\pi^1, \dots, \pi^N)$. With the empirical distribution μ^N and ν^N, solve the N-particle system (2.10) and (2.11).

2. We get $J^{\ell}(\cdot)$ from μ^N and ν^N. Solve $\tilde{u}^{\ell}(T) := \inf_{\pi \in \mathbb{A}} J^{\ell}(\pi)$ and the corresponding optimal control $\pi^{\star}$. We find a function Φ so that $\pi^{\star} = \Phi(\pi)$.

3. If there exists $\hat{\pi}$, such that $\hat{\pi} = \Phi(\hat{\pi})$, then $\mu^{N\star} := \frac{1}{N}\sum_{\ell=1}^{N} \boldsymbol{\delta}_{(V^{v^{\ell}, \hat{\pi}^{\ell}})}$ is the Nash equilibrium.

We have the following result of Nash equilibrium strategies. As in Chapter 3, we consider investment decisions based upon the current market environment only, in order to preserve the solution of (3.13)-(3.15), $u^\ell(T-t, \mathcal{X}^N(t), \mathcal{Y}(t))$ in Markovian market model.

Proposition 4.2.1. *Under Assumption 7,8, and 9, Nash equilibrium is attained when the strategies yield*

$$\pi_i^{\ell\star} = \mathbf{m}_i(t) + X_i^N(t) D_{x_i} \bar{v}^\ell(t) + \sum_{j=1}^{n} (\tau\sigma^{-1})_{ji} D_{y_j} \bar{v}^\ell(t) \tag{4.7}$$

for $\ell = 1, \ldots, N$, where

$$\bar{v}^\ell(t) = \log \tilde{u}^\ell(T-t, \mathcal{X}^N(t), \mathcal{Y}(t)) + \frac{1-\delta}{\delta N X_t^N} \sum_{\ell=1}^{N} V^\ell(t) \log \tilde{u}^\ell(T-t, \mathcal{X}^N(t), \mathcal{Y}(t)), \tag{4.8}$$

and

$$D_{x_i}\bar{v}^\ell(t) = D_{x_i} \log \tilde{u}^\ell(T-t, \mathcal{X}^N(t), \mathcal{Y}(t)) + \frac{1-\delta}{\delta N X_t^N} \sum_{\ell=1}^{N} V^\ell(t) D_{x_i} \log \tilde{u}^\ell(T-t, \mathcal{X}^N(t), \mathcal{Y}(t)),$$

$$D_{y_i}\bar{v}^\ell(t) = D_{y_i} \log \tilde{u}^\ell(T-t, \mathcal{X}^N(t), \mathcal{Y}(t)) + \frac{1-\delta}{\delta N X_t^N} \sum_{\ell=1}^{N} V^\ell(t) D_{y_i} \log \tilde{u}^\ell(T-t, \mathcal{X}^N(t), \mathcal{Y}(t)).$$

$\tilde{u}^\ell(t)$ is the smallest nonnegative solution in the Cauchy problem.

Proof. For a given choice of $\pi \in \mathbb{A}$, $\tilde{u}^\ell := \inf_{\pi\in\mathbb{A}} J^\ell(\pi)$ is uniquely determined by the smallest nonnegative solution of (3.22). A choice of $\pi \in \mathbb{A}$ satisfies (2.11), i.e.,

$$dV_t^\ell = V_t^\ell \pi_t^\ell \beta(t, X_t^N, \nu_t^N)dt + V_t^\ell \pi_t^\ell \sigma(t, X_t^N, \nu_t^N) dW_t, \quad V_0^\ell = v^\ell.$$

For simplicity we denote $\tilde{u}^\ell(T-t, \mathcal{X}_t^N, \mathcal{Y}_t)$ as $\tilde{u}^\ell(T-t)$. Assuming that all controls $\pi^k(\cdot)$, $k \neq \ell$ are chosen, player ℓ will choose the optimal strategy $\pi^\star$ that achieves

$$V^{\ell\star}(\cdot) = e^{c_\ell} \mathcal{V}^N(t) \tilde{u}^\ell(T-\cdot). \tag{4.9}$$

Suppose every player $\ell = 1, \dots, N$ follows the relationship $V^{\ell\star}(\cdot) = e^{c_\ell}\mathcal{V}^N(\cdot)\tilde{u}^\ell(T-\cdot)$, then sum up the left hand side of (4.9) with respect to ℓ, i.e., $\sum_{\ell=1}^N V^{\ell\star}(\cdot)$ yields

$$\sum_{\ell=1}^N V^{\ell\star}(\cdot) = \sum_{\ell=1}^N \big[e^{c_\ell}(\delta X^N(\cdot) + (1-\delta)\sum_{\ell=1}^N V^{\ell\star}(\cdot))\tilde{u}^\ell(T-\cdot)\big].$$

Solve $\sum_{\ell=1}^N V^{\ell\star}(\cdot)$ from above and plug the solution back to (4.9). Then it follows

$$\begin{aligned} V^{\ell\star}(t) =& e^{c_\ell}\tilde{u}^\ell(T-t)\delta X^N(t)\left(1 + \frac{(1-\delta)\sum_{\ell=1}^N e^{c_\ell}\tilde{u}^\ell(T-t)}{N-(1-\delta)\sum_{\ell=1}^N e^{c_\ell}\tilde{u}^\ell(T-t)}\right) \\ =& \frac{e^{c_\ell}\tilde{u}^\ell(T-t)\delta X^N(t)}{1-(1-\delta)\frac{1}{N}\sum_{\ell=1}^N e^{c_\ell}\tilde{u}^\ell(T-t)}. \end{aligned} \tag{4.10}$$

Without loss of generality, assume $X(0) = V^{\mathbf{m}}(0)$, then

$$\log X(t) = \log \frac{x_0}{V^{\mathbf{m}}(0)} V^{\mathbf{m}}(t) = \log x_0 + \int_0^t \mathbf{m}_s'(\beta_s - \frac{1}{2}\alpha_s\mathbf{m}_s)ds + \int_0^t \mathbf{m}_t'\sigma(s)dW_s.$$

Thus equivalently we can write,

$$\begin{aligned} \log V^{\ell\star}(t) =& \log(e^{c_\ell}\delta) + \log X^N(t) + \log\tilde{u}^\ell(T-t) - \log\big(1-(1-\delta)\frac{1}{N}\sum_{\ell=1}^N e^{c_\ell}\tilde{u}^\ell(T-t)\big) \\ =& \log v_0^{\ell\star} + \int_0^t \mathbf{m}_s'(\beta_s - \frac{1}{2}\alpha_s\mathbf{m}_s)ds + \int_0^t \mathbf{m}_t'\sigma(s)dW_s \\ & + \log\tilde{u}^\ell(T-t) - \log\tilde{u}^\ell(T) \\ & - \log\big(1-(1-\delta)\frac{1}{N}\sum_{\ell=1}^N e^{c_\ell}\tilde{u}^\ell(T-t)\big) + \log\big(1-(1-\delta)\frac{1}{N}\sum_{\ell=1}^N e^{c_\ell}\tilde{u}^\ell(T)\big), \end{aligned} \tag{4.11}$$

and

$$v_0^{\ell\star} = \frac{\delta x_0 e^{c_\ell}\tilde{u}^\ell(T)}{1-(1-\delta)\frac{1}{N}\sum_{\ell=1}^N e^{c_\ell}\tilde{u}^\ell(T)}.$$

With a fixed set of control processes π^ℓ, we solve $\tilde{u}^\ell_{T-t}$, and expect that the optimal strategy $\pi^{\ell\star}$ will coincide with the fixed $\pi^{\ell}(\cdot)$. Thus we can find the Nash equilibrium strategy by comparing $V^{\ell\star}$ in (4.11) and V^{ℓ} defined in (2.11). By Ito's formula on $\tilde{u}^{\ell}(\cdot)$ as a function of $\mathcal{X}_t^N$ and $\mathcal{Y}_t$, we obtain

$$d\tilde{u}^{\ell}(T-t) = (\mathcal{L}\tilde{u}^{\ell} - \frac{\partial \tilde{u}^{\ell}}{\partial \rho})(T-t)dt + \sum_{k=1}^{n} R_k^{\ell}(T-t, \mathcal{X}_t^N, \mathcal{Y}_t)dW_k(t),$$

where $\rho = T-t$, for $t \in [0,T]$, $\mathcal{L}$ is the infinitesimal generator of $(\mathbf{x}, \mathbf{y}) \in (0,\infty)^n \times (0,\infty)^n$, i.e.,

$$\begin{aligned}\mathcal{L}\tilde{u}^{\ell}(\rho) =& b(\mathbf{x},\mathbf{y}) \cdot \partial_x \tilde{u}^{\ell}(\rho) + \gamma(\mathbf{x},\mathbf{y}) \cdot \partial_y \tilde{u}^{\ell}(\rho) \\ &+ \frac{1}{2}\mathrm{tr}\big[a(\mathbf{x},\mathbf{y}) \cdot \partial_{xx}^2 \tilde{u}^{\ell}(\rho) + \psi(\mathbf{x},\mathbf{y}) \cdot \partial_{yy}^2 \tilde{u}^{\ell}(\rho) + (s\tau' + \tau s')(\mathbf{x},\mathbf{y}) \cdot \partial_{xy}^2 \tilde{u}^{\ell}(\rho)\big]\end{aligned}$$

and

$$R_k^{\ell}(T-t, \mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} \sigma_{ik}(\mathbf{x},\mathbf{y}) x_i D_i \tilde{u}^{\ell}(T-t) + \sum_{p=1}^{n} \tau_{pk}(\mathbf{x},\mathbf{y}) D_p \tilde{u}^{\ell}(T-t).$$

Thus the local martingale term in (4.11) is

$$\begin{aligned}&\int_0^t \mathbf{m}_i'(s)\sigma(s)dW(s) + \int_0^t \frac{1}{\tilde{u}^{\ell}(T-s)} \sum_{k=1}^{n} R_k^{\ell}(T-s)dW_k(s) \\ &+ \frac{(1-\delta)\mathcal{V}^N(t)}{N\delta X^N(t)} \int_0^t \sum_{\ell=1}^{N}\sum_{k=1}^{n} e^{c_\ell} R_k^{\ell}(T-s)dW_k(s).\end{aligned}$$

By comparing the drift and volatility of (2.11) and (4.11), we arrive at (4.7). Notice the consistency condition is in the space of control as indicated in Section 4.1.2. □

Remark 3. *The intuition of the proof above is to form a fixed point problem on the control space by comparing the strategy $\hat{\pi}$ that generated dynamics (2.11) and the optimal strategy generated (4.9). The optimal strategy is a mapping of the $\hat{\pi}$, we denote it as $\Phi(\hat{\pi})$. (4.7) is a solution of $\hat{\pi} = \Phi(\hat{\pi})$.*

As can be seen from the proof, the above result also holds true without the Markovian *assumption 7. We consider functions depending on the path* $(\mathcal{X}^N_{[0,t]}, \mathcal{Y}_{[0,t]})$ *hence the optimal strategy of the Nash equilibrium* (4.7) *can be obtained with the solution of optimal arbitrage in the form* $u^\ell(T-t, \mathcal{X}^N_{[0,t]}, \mathcal{Y}_{[0,t]})$, *which is defined in* (3.11).

Remark 4. *We have the following constraints on the solution of optimal strategies and optimal wealth.*

First, since the wealth processes should be nonnegative, the wealth $V^\ell \geqslant 0$ *in* (4.10), *we have for* $\ell = 1, \dots, N$.

$$V^{\ell\star}(t) = \frac{e^{c_\ell}\tilde{u}^\ell(T-t)\delta X^N(t)}{1-(1-\delta)\frac{1}{N}\sum_{\ell=1}^N e^{c_\ell}\tilde{u}^\ell(T-t)} \geqslant 0,$$

hence we get the constraint

$$\frac{1}{N}\sum_{\ell=1}^N e^{c_\ell}\tilde{u}^\ell(T-t) \leqslant \frac{1}{1-\delta}.$$

Second, optimal strategies (4.7) *should satisfy* $\sum_{i=1}^n \pi_i^{\ell\star}(t) = 1$. *We look further into this constraint in Chapter 6.*

The end of Section 3.3 suggests that optimal strategies are linearly dependent on e^{c_ℓ}, $\ell = 1, \dots, N$. To illustrate, the investors pursuing relative arbitrage end up with the terminal wealth $V^\ell(T)$ proportional to e^{c_ℓ} if starting from the same initial wealth. However, at every time t, the information of every $V^\ell(t)$, $\ell = 1, \dots, N$ is required to pinpoint the optimal strategy. Therefore, a mean field regime is discussed in the next chapter to resolve the complexity in N-player game.

We conclude from the above arguments that although we start from an open loop control as defined in Definition 4.4, we end up with closed loop feedback strategies in Nash equilibrium. One possible reason for this result is that the average trading volume of investors is in the stock price dynamics, so a change of strategy of one investor would give rise to the change of stock capitalization and thus influence the other players. In this way, the players at time t have information of the states of all the other players at time t in a latent way.

As mentioned in Remark 3, the approach in Proposition 4.2.1 is suitable for either Markovian or non-Markovian controls. Next, we provide another approach to solve specifically for controls of closed loop Markovian or open loop form. This approach will be useful when we derive the mean field equilibrium in the next section.

Proposition 4.2.2. *Under Assumption 7,8, and 9, when controls of a closed loop Markovian form (4.5), or an open loop $\phi(t, \mathbf{v}, W_t)$ are adopted, there is a Nash equilibrium $\pi^\star = (\pi^{1\star}, \dots, \pi^{N\star})$, where for $\ell = 1, \dots, N$, $\pi^{\ell\star}$ follows (4.7).*

Proof.

The Markovian condition in Assumption 7 gives

$$\frac{\mathbb{E}^{\mathbb{P}}\big[\mathcal{V}^N(T)L(T)|\mathcal{F}(t)\big]}{\mathcal{V}^N(t)L(t)} = \tilde{u}^\ell(T-t, \mathcal{X}^N(t), \mathcal{Y}(t)),$$

where $\tilde{u}^\ell(\cdot)$ is the minimal nonnegative solution of (3.22). Again we use the property for $0 \leqslant t \leqslant T$ that $V^\ell(t) = \mathcal{V}^N(t)\tilde{u}^\ell(T-t, \mathcal{X}^N(t), \mathcal{Y}(t))$, the deflated wealth process

$$\hat{V}^\ell(t) := V^\ell(t)L(t) = \mathbb{E}^{\mathbb{P}}\big[\mathcal{V}^N(T)L(T)|\mathcal{F}_t\big]$$

is a martingale. As a result, the dt terms in $d\hat{V}^\ell(t)$ will vanish, namely,

$$\hat{V}^\ell(t) = \hat{V}^\ell(0) + \sum_{k=1}^{n} \int_0^t \hat{V}^\ell(s) B_k(T-s, \mathcal{X}(s), \mathcal{Y}(s)) dW_k(s), \;\; 0 \leqslant t \leqslant T, \tag{4.12}$$

where

$$\begin{aligned}B_k(t,x,\pi) &= \sum_{i=1}^n \sigma_{ik}(\mathbf{x},\mathbf{y})x_i D_i \log \tilde{u}^\ell(T-t,\mathbf{x},\mathbf{y}) + \sum_{m=1}^n \tau_{mk}(\mathbf{x},\mathbf{y}) D_m \log \tilde{u}^\ell(T-t,\mathbf{x},\mathbf{y}) \\ &+ \sum_{i=1}^n \frac{\delta X^N(t)}{\mathcal{V}^N(t)}\Big(\frac{x_i}{\sum_{i=1}^n x_i}\sigma_{ik}(t) - \Theta_k(\mathbf{x},\mathbf{y})\Big) \\ &+ \frac{(1-\delta)/N}{\mathcal{V}^N(t)}\sum_{i=1}^n\sum_{\ell=1}^N \Big(V^\ell(t)\pi_i^\ell\sigma_{ik}(t) - V^\ell(t)\Theta_k(\mathbf{x},\mathbf{y})\Big).\end{aligned}$$

Thus we have the fixed point problem

$$\begin{aligned}\pi_i^{\ell\star}(t) &= X_i^N(t) D_i \log \tilde{u}^\ell(T-t,\mathbf{x},\mathbf{y}) + \tau_i(\mathbf{x},\mathbf{y})\sigma^{-1}(\mathbf{x},\mathbf{y}) D_k \log \tilde{u}^\ell(T-t,\mathbf{x},\mathbf{y}) \\ &+ \frac{\delta X^N(t)}{\mathcal{V}^N(t)}\mathbf{m}_i(t) + \frac{(1-\delta)}{N\mathcal{V}^N(t)}\sum_{\ell=1}^N V^{\ell\star}(t)\pi_i^{\ell\star}(t),\end{aligned} \tag{4.13}$$

where $V^{\ell\star}(t)$ is generated from $\pi^{\ell\star}(t)$.

Next, we check the consistency condition of $\pi^\star$ in (4.13) and π we start with. Define a map $\Phi : \mathbb{A} \to \mathbb{A}$, we want to find a fixed point so that $\Phi(\pi) = \pi$. By Brouwer's fixed-point theorem, since $\mathbb{A}$ is a compact convex set, there exists a fixed point for the mapping Φ. In Nash equilibrium, we assume that all players follow the strategy $\pi^\star$ - if we multiply both sides by V^ℓ and then summing over $\ell = 1, \dots, n$ in (4.13), it gives

$$\begin{aligned}\sum_{\ell=1}^N V^\ell(t)\pi_i^\ell(t) = \frac{\mathcal{V}^N(t)}{\delta X^N(t)}\Big[& X_i^N(t)\sum_{\ell=1}^N V^\ell(t) D_{x_i}\log\tilde{u}^\ell(T-t,\mathbf{x},\mathbf{y}) \\ &+ \sum_{j=1}^n (\tau\sigma^{-1})_{ji}(\mathbf{x},\mathbf{y})\sum_{\ell=1}^N V^\ell(t) D_{y_j}\log\tilde{u}^\ell(T-t,\mathbf{x},\mathbf{y}) \\ &+ \frac{\delta X^N(t)}{\mathcal{V}^N(t)}\mathbf{m}_i(t)\frac{1}{N}\sum_{\ell=1}^N V^\ell(t)\Big].\end{aligned}$$

After some computations we conclude

$$\pi_i^{\ell\star} = \mathbf{m}_i(t) + X_i(t) D_{x_i}\bar{v}^N(t) + \sum_{j=1}^{n} (\tau\sigma^{-1})_{ji}(t) D_{y_j}\bar{v}^N(t), \tag{4.14}$$

where $\tilde{v}^N(t)$ satisfies (4.8). □

Remark 5. *In (4.13) the last two terms*

$$\frac{\delta X^N(t)}{\mathcal{V}^N(t)}\mathbf{m}_i(t) + \frac{(1-\delta)}{N\mathcal{V}^N(t)} \sum_{\ell=1}^{N} V^{\ell\star}(t)\pi_i^{\ell\star}(t)$$

is of the same expression as the strategy used to generate benchmark $\mathcal{V}^N(t)$ in Theorem 3.1.1.

Proposition 4.2.1 and 4.2.2 provide the general method to search for Nash equilibrium and a set of optimal strategies achieving the Nash equilibrium. We prove next that the Nash equilibrium is unique.

Proposition 4.2.3. *A sufficient condition of unique Nash equilibrium $\mu^{N\star}$ in the sense of Definition 4.1.3 is the first exit time from the set K_t is greater than T, i.e., $\tau^K > T$ where*

$$K_t = \left(0, \frac{\left(N - (1-\delta)\sum_{\ell=1}^{N} e^{c_\ell}\tilde{u}^\ell(T-t)\right)^2}{N\delta\left|\sum_{\ell=1}^{N} e^{c_\ell} D_m \tilde{u}^\ell(T-t)\right|}\right), \quad \tau^K = \inf\{t \geqslant 0; X^N(t) \notin K_t\}. \tag{4.15}$$

$\tilde{u}^\ell(T-t)$ is the solution of the Cauchy problem (3.13) - (3.15).

Proof. To investigate the uniqueness of Nash equilibrium, we look at a mapping on the empirical mean of wealth m_t^N. As discussed in Remark 3, with the optimal strategies $\hat{\pi}$ as the solution of a fixed point problem, such that $\hat{\pi} = \Phi^a(\hat{\pi})$, $\Phi^a : \mathbb{A} \to \mathbb{A}$, we get $m_t^{N\star} := \frac{1}{N}\sum_{\ell=1}^{N} \boldsymbol{\delta}_{(\hat{V}_t^\ell, \hat{\pi}_t^\ell)}$ for every $t \in [0,T]$ is the Nash equilibrium.

Now we search for the Nash equilibrium on the space of the empirical mean of wealth m_t^N. Define $\mathcal{Y}_i(t) := \mathbb{E}[\hat{V}^\ell(t)\hat{\pi}_i^\ell(t)|\mathcal{F}_t^W]$ for every ℓ, $t \in [0,T]$ and filtration $\mathcal{F}_t^W$ generated by common noises W. Solve $\tilde{u}^\ell := \inf_{\pi \in \mathbb{A}} J^\ell(\pi)$. Thus we get (4.10) as the solution $\hat{V}^\ell(t)$ as a function of the given $\Phi(m_t^N)$, where $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$. So the Nash equilibrium is achieved if there exists a fixed point mapping $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\Phi(m_t^N) = m_t^N$. We will derive the function $\Phi(\xi)$ in (4.17).

Let $\mathbf{y}^{-n} := (y_1, y_2, \dots, y_{n-1})$. To clarify the mapping and contraction argument of the fixed point problem, we do a transformation on $\tilde{u}^\ell(T-t, \mathbf{x}, \mathbf{y})$ such that we look at a Cauchy problem of $\tilde{u}^\ell(T-t, \mathbf{x}, \tilde{\mathbf{y}})$ instead, where $\tilde{\mathbf{y}} := (\mathbf{y}^{-n}, \sum_{i=1}^n y_i)$. We have $\sum_{i=1}^n \mathcal{Y}_i(t) = m_t^N$ from (3.18).

In Cauchy problem (3.13) - (3.15), for $p = 1, \dots, n$, if $p < n$,

$$D_p \tilde{u}^\ell(T-t, \mathbf{x}, \tilde{\mathbf{y}}) = \frac{\partial \tilde{u}^\ell(T-t, \mathbf{x}, \tilde{\mathbf{y}})}{\partial \tilde{y}_p} + \frac{\partial \tilde{u}^\ell(T-t, \mathbf{x}, \tilde{\mathbf{y}})}{\partial \tilde{y}_n};$$

If $p = n$,

$$D_p \tilde{u}^\ell(T-t, \mathbf{x}, \tilde{\mathbf{y}}) = \frac{\partial \tilde{u}^\ell(T-t, \mathbf{x}, \tilde{\mathbf{y}})}{\partial \tilde{y}_n}.$$

So the Cauchy problem of $\tilde{u}^\ell(T-t, \mathbf{x}, \tilde{\mathbf{y}})$ yields

$$\begin{aligned}
\mathcal{A}\tilde{u}^\ell(\tau, \mathbf{x}, \tilde{\mathbf{y}}) =& \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n a_{ij}(\mathbf{x}, \tilde{\mathbf{y}}) \Big(D_{ij}^2 \tilde{u}^\ell(\tau, \mathbf{x}, \tilde{\mathbf{y}}) + \frac{2\delta D_i \tilde{u}^\ell(\tau, \mathbf{x}, \tilde{\mathbf{y}})}{\delta \mathbf{x}\cdot \mathbf{1} + (1-\delta)\tilde{\mathbf{y}}\cdot\mathbf{1}} \Big) \\
&+ \frac{1}{2}\sum_{p=1}^n \sum_{q=1}^n \psi_{pq}(\mathbf{x}, \tilde{\mathbf{y}}) \Big(D_{pq}^2 \tilde{u}^\ell(\tau, \mathbf{x}, \tilde{\mathbf{y}}) + \frac{2(1-\delta) D_p \tilde{u}^\ell(\tau, \mathbf{x}, \tilde{\mathbf{y}})}{\delta \mathbf{x}\cdot \mathbf{1} + (1-\delta)\tilde{\mathbf{y}}\cdot\mathbf{1}} \Big) \\
&+ \sum_{i=1}^n \sum_{p=1}^n (s\tau^T)_{ip}(\mathbf{x}, \tilde{\mathbf{y}}) D_{ip}^2 \tilde{u}^\ell(\tau, \mathbf{x}, \tilde{\mathbf{y}}) \\
&+ \sum_{i=1}^n \sum_{p=1}^n (s\tau^T)_{ip}(\mathbf{x}, \tilde{\mathbf{y}}) \frac{\delta D_p \tilde{u}^\ell(\tau, \mathbf{x}, \tilde{\mathbf{y}}) + (1-\delta) D_i \tilde{u}^\ell(\tau, \mathbf{x}, \tilde{\mathbf{y}})}{\delta \mathbf{x}\cdot \mathbf{1} + (1-\delta)\tilde{\mathbf{y}}\cdot\mathbf{1}}.
\end{aligned} \tag{4.16}$$

From (4.10), the empirical mean of optimal wealth $m_t^N = \frac{1}{N}\sum_{\ell=1}^N \hat{V}^\ell(t)$ should satisfy the consistency condition $m_t^N = \Phi(m_t^N)$ where $\Phi(\cdot)$ is

$$\Phi(\xi) := \frac{\delta X^N(t) \sum_{\ell=1}^N e^{c_\ell} \tilde{u}^\ell(T-t, \mathbf{x}, \mathbf{y}^{-n}, \xi)}{N - (1-\delta) \sum_{\ell=1}^N e^{c_\ell} \tilde{u}^\ell(T-t, \mathbf{x}, \mathbf{y}^{-n}, \xi)}. \tag{4.17}$$

Denote D_m as the partial derivative with respect to m_t^N, $D_m \tilde{u}^\ell(T-t) = \frac{\partial \tilde{u}^\ell(T-t, \mathbf{x}, \tilde{\mathbf{y}})}{\partial \tilde{y}_n}$. Thus the derivative of $\Phi(m_t^N)$ given $X^N(t)$ is

$$\Phi'(m_t^N) = \frac{N\delta X^N(t) \sum_{\ell=1}^N e^{c_\ell} D_m \tilde{u}^\ell(T-t)}{\big(N - (1-\delta) \sum_{\ell=1}^N e^{c_\ell} \tilde{u}^\ell(T-t)\big)^2}.$$

We denote $A_t = N - (1-\delta) \sum_{\ell=1}^N e^{c_\ell} \tilde{u}^\ell(T-t)$. In addition, $0 < \tilde{u}^\ell(T-t) \leqslant 1$, c_ℓ is bounded by $\max\{c_1, \dots, c_N\}$ for a fixed N. Hence $|\Phi'(m_t^N)| < 1$ is satisfied when

$$0 < X^N(t) < \frac{A_t^2}{D_t}, \tag{4.18}$$

for every t. For simplicity, we set $D_t = N\delta |\sum_{\ell=1}^N e^{c_\ell} D_m \tilde{u}^\ell(T-t)|$, and $K_t := (0, \frac{A_t^2}{D_t})$. By mean value theorem, and since Φ is continuous, we get Φ is a contraction of m_t^N.

The first exit time for the interval K_t is $\tau^K = \inf\{t \geqslant 0; X^N(t) \notin K_t\}$ as in (4.15). If $\tau^K > T$ then Nash equilibrium generated by (4.7) is unique using Banach fixed point theorem.

□

Given x^N, if $\beta(t)$ and $\alpha(t)$ are deterministic processes, $X^N(t)$ is a log-normal distribution where

$$\log X^N(t) \sim N\bigg(\log x^N + \int_0^t \big(\beta(t) - \frac{1}{2}\alpha(t)\big) dt,\ \int_0^t \alpha(t) dt \bigg).$$

As a result, with the solution $\tilde{u}$ of (3.22), the probability of attaining the unique Nash equilibrium is

$$P(X^N(t) \in K_t) = \mathcal{N}\left(\frac{\log \frac{A_t^2}{D_t x^N} - \int_0^t \big(\beta(t) - \frac{1}{2}\alpha(t)\big)dt}{\int_0^t \alpha(t)dt}\right),$$

where $\mathcal{N}$ is the cumulative distribution function of a standard Gaussian distribution. In general this is not the case. For example we discuss market dynamics in volatility-stablized models in Chapter 7, where market dynamics can be constructed from time changed Bessel processes.

Chapter 5

Mean field relative arbitrage problem

It is expected that in large population games, a mean field formulation is more tractable than the N-player games and might help disclose more about the finitely many investors situation. Section 5.1 establishes the optimization of relative arbitrage using extended mean field games. Section 5.3 constructs a Mckean-Vlasov SDE of the form that the coefficients of the diffusion depend on the joint distribution of the state processes and the control, and show the propagation of chaos holds to provide proofs of the market model used in Section 5.1. Through approximate Nash equilibrium, we justify that the mean field formulation is an appropriate generalization of N-player relative arbitrage problem. In the last section, we extend the results in previous sections using smooth functions of probability measure flows.

5.1 Extended Mean Field Games

We have observed that it is unlikely to get a tractable equilibrium from N-player game, especially when N is large. In this section, we study relative arbitrage for the infinite limit population of players. With propagation of chaos results provided, a player in a large game limit should feel the presence of other players through the statistical distribution of states and actions. Then they make decisions through a modified objective that involves mean field as $N \to \infty$. For this reason, we expect the MFG framework to be more tractable than N-player games.

5.1.1 Formulation of Extended Mean Field Games

We formulate the model on $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ which support Brownian motion B, a n-dimensional common noise, equally distributed as W. The systemic effect of random noises towards the market might be different when we consider a finite or infinite group of investors interacting with the market. B is adapted to the $\mathbb{P}$-augmentation of $\mathbb{F}$ and can explain the limiting random noises in the market $\mathcal{M}$ when $N \to \infty$. We denote the natural filtration induced by $(B_t)_{t\geqslant 0}$ as $\mathcal{F}^B_t = \sigma(B_s : 0 \leqslant s \leqslant t)$. The admissible strategies $\pi(\cdot) \in \mathbb{A}$ follow similar conditions as (2.4) and is $\mathcal{F}^B$-progressively measurable.

In general, the capitalization and state processes depend on the joint distribution of (V^ℓ, π^ℓ_i), $\ell = 1, \ldots, N$, while the cost function is related to the empirical distribution of the private states. With a given initial condition $\mu_0 \in \mathcal{P}^2(C([0,T];\mathbb{R}_+))$ as a degenerate distribution of value 1, we define the conditional law of $V(t)$ given $\mathcal{F}^B$ as

$$\mu_t := Law\big(V(t)\big|\mathcal{F}^B_t\big), \tag{5.1}$$

and the conditional law of $(V(t), \pi(t))$ given $\mathcal{F}^B$, with a given initial condition $\nu_0 \in$

$\mathcal{P}^2(C([0,T];\mathbb{R}_+ \times \mathbb{A}))$, is

$$\nu_t := Law(V(t), \pi(t)|\mathcal{F}_t^B).$$

The mean field game model is constructed upon McKean-Vlasov SDEs of stocks and wealth

$$d\mathcal{X}(t) = \mathcal{X}(t)\beta(\mathcal{X}(t), \nu_t)dt + \mathcal{X}(t)\sigma(\mathcal{X}(t), \nu_t)dB_t, \quad X_0 = \mathbf{x}; \tag{5.2}$$

$$\frac{dV(t)}{V(t)} = \sum_{i=1}^{n} \pi_i(t)\frac{dX_i(t)}{X_i(t)}. \tag{5.3}$$

A player competes with the market and the entire group with respect to the benchmark

$$\mathcal{V}(T) = \delta \cdot X(T) + (1-\delta) \cdot \int_\Omega f(v)\ d\mu_T(v), \tag{5.4}$$

is the weighted sum of total capitalization and the first moment of measure μ. $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a Borel function with $f(V(t)) \in L^1$ for $t \in [0,T]$.

We give the following Assumption 12 for the mean field set-up.

Assumption 12. *We assume the following items for the capitalization processes, wealth and preference of investors.*

(1) Assume $\mathbf{x} \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}_+^n)$, *and* $\mathbb{E}[\sup_{0\leqslant t\leqslant T} ||(V(t), X(t))||^2] \leqslant \infty$.

(2) In addition to Assumption 5, the preference c *assumed to be independently and identically distributed as* c^ℓ *from a distribution* L_c *and are independent with common noise* B.

With infinitely many investors of interest, the relative arbitrage problem can be modelled as mean field games and the players become indistinguishable. This is explain briefly in the following Remark 6. The detail of theoretical support for the McKean-Vlasov system and mean field game models is presented in Section 5.3.

Remark 6. *From Proposition 5.3.1-5.3.3, we show that the above McKean-Vlasov problem admits a unique solution, where* $\nu_t := Law(V(t), \pi(t)|\mathcal{F}_t^B)$. *Furthermore, the weak limit of measure flow* $\nu^N \in \mathcal{P}^2(C([0,T];\mathbb{R}_+ \times \mathbb{A}))$ *in Definition 2.3.1 is exactly* ν_t. *Thus* $V^\ell(t)$ *is asymptotically identical independent copies given the common noise* B *when* $\ell = 1, \dots, N$, $N \to \infty$. *Hence we consider a representative player which is randomly selected from the infinite number of investors in mean field set-up. Small deviations of a single player would not influence the entire system given the common noise* B.

However, notice that the results in Section 5.3 are based on the Lipschitz conditions of market coefficients and function of the strategies $\phi(\cdot)$. *But in this section, we consider the McKean-Vlasov system (5.2)-(5.3) when* $\{\pi_i(t)\}_{t\in[0,T]}$, *for* $i = 1, \dots, n$ *is fixed. Thus we avoid the assumption of a Lipschitz condition for* $\phi(\cdot)$.

5.1.2 Mean Field Equilibrium

Every player tries to minimize the relative amount of initial capital with respect to that of the benchmark $\mathcal{V}(T)$. That is, a representative player seeks to minimize the objective

$$J^{\mu,\nu}(\pi; 0, x_0) := \inf\left\{\omega > 0 \;\Big|\, V^{\omega e^c \mathcal{V}(0),\pi}(T) \geqslant e^c \mathcal{V}(T)\right\}. \tag{5.5}$$

We define the mean field equilibrium below, which appears as a fixed point of best response function.

Definition 5.1.1. *(Mean Field Equilibrium) Let* $\pi^\star(\cdot) \in \mathbb{A}$ *be an admissible strategy, then it gives mean field equilibrium (MFE) if* $J^{\mu,\nu}$ *in (5.5) satisfies*

$$J^{\mu,\nu}(\pi^\star) = \inf_{\pi \in \mathbb{A}} J^{\mu,\nu}(\pi).$$

Analogous to Definition 4.1.3, we do not require the optimal control to be unique in

Definition 5.1.2.

Definition 5.1.2. *We say that uniqueness holds for the MFG equilibrium if any two solutions* μ^a, μ^b, *defined on filtered probabilistic set-ups* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, *with the same initial law* $\mu_0 \in \mathcal{P}^2(\mathbb{R}_+)$,

$$\mathbb{P}[\mu^a = \mu^b] = 1,$$

where μ *is the distribution of wealth processes as in* (5.1).

Specifically, we consider the mean field interaction is through the expected investments of an investor on assets - the conditional expectation of the product of wealth and controls, i.e.,

$$d\mathcal{Z}(t) = d\mathbb{E}(V(t)\pi(t)|\mathcal{F}_t^B) = \gamma(\mathcal{X}(t), \mathcal{Z}(t))dt + \tau(\mathcal{X}(t), \mathcal{Z}(t))dB_t, \quad Z_0 = z_0, \tag{5.6}$$

where

$$dX_i(t) = X_i(t)\beta(\mathcal{X}(t), \mathcal{Z}(t)))dt + X_i(t)\sigma(\mathcal{X}(t), \mathcal{Z}(t))dB_t, \quad X_i(0) = x_i.$$

and a representative player's wealth $V(t)$ is generated from a strategy $\pi(t) \in \mathbb{A}$ through (2.3). We take $\mathbb{F} = \mathbb{F}^{X,Z} = \mathbb{F}^B$ in order to characterize the optimal arbitrage for the rest of the sections.

In particular, $\hat{\mathbb{A}} = \arg\inf_{\pi \in \mathbb{A}} J^{\mu,\nu}(\pi)$ denotes the set of optimal controls. In the control problem, the flow of measure $(m_T, \mathcal{Z}(T))$ is frozen conditional on the common noise. $(m_T, \mathcal{Z}(T))$ is an equilibrium if there exists $\pi^\star \in \hat{\mathbb{A}}$ such that the fixed point of the mean field measure exists, i.e., $m_T = \mathbb{E}[V_T^\star|\mathcal{F}_T^B]$; $\mathcal{Z}(T) = \mathbb{E}[\mathcal{Z}^\star(T)|\mathcal{F}_T^B]$. In this section, we consider

$$\inf_{\pi \in \mathbb{A}} J^{\mu,\nu}(\pi) = \inf_{\pi \in \mathbb{A}} J^{m,Z}(\pi),$$

and after computations we have that the objective function follows

$$J^{m,Z}(\pi) = e^c \frac{\delta X(T) + (1-\delta)m_T}{\delta \sum_{i=1}^n x_i + (1-\delta)m_0} \exp^{-1}\bigg\{ \int_0^T \pi'(t)(\beta(t, \mathcal{X}_t, \mathcal{Z}_t) - \frac{1}{2}\alpha_t \pi(t))dt + \int_0^T \pi'(t)\sigma(t, \mathcal{X}_t, \mathcal{Z}_t)dB_t \bigg\}. \tag{5.7}$$

The representative agent's optimal initial proportion to achieve relative arbitrage can be characterized as

$$u(T) := \inf_{\pi \in \mathbb{A}} J^{\mu,\nu}(\pi) = \mathbb{E}[e^c \mathcal{V}(T)L(T)]/\mathcal{V}(0), \tag{5.8}$$

Assumption 13. *There exist a function* $H : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}_+^n$ *of class* C^2*, such that*

$$b(\mathbf{x}, \mathbf{z}) = 2a(\mathbf{x}, \mathbf{z})D_x H(\mathbf{x}, \mathbf{z}), \quad \gamma(\mathbf{x}, \mathbf{z}) = 2\psi(\mathbf{x}, \mathbf{z})D_y H(\mathbf{x}, \mathbf{z}),$$

Using an analogous proof in Section 3.1.1 about $\tilde{u}^\ell(\cdot)$,

$$L(t) = \exp\bigg\{ - H(\mathcal{X}(t), \mathcal{Z}(t)) + H(\mathbf{x}, \mathbf{z}) - \int_0^t (k(\mathcal{X}(s)) + \tilde{k}(\mathcal{Z}(s)))ds \bigg\}, \tag{5.9}$$

where

$$k(\mathbf{x}, \mathbf{z}) := -\sum_{i=1}^n \sum_{j=1}^n \frac{a_{ij}(\mathbf{x}, \mathbf{z})}{2}[D_{ij}^2 H(\mathbf{x}, \mathbf{z}) + 3D_i H(\mathbf{x}, \mathbf{z}) D_j H(\mathbf{x}, \mathbf{z})],$$

$$\tilde{k}(\mathbf{x}, \mathbf{z}) : = -\sum_{i=1}^n \sum_{j=1}^n \frac{\psi_{pq}(\mathbf{x}, \mathbf{z})}{2}[D_{pq}^2 H(\mathbf{x}, \mathbf{z}) + 3D_p H(\mathbf{x}, \mathbf{z}) D_q H(\mathbf{x}, \mathbf{z})] + \sum_{i=1}^n \sum_{p=1}^n (s\tau^T)_{ip} D_i H(\mathbf{x}, \mathbf{z}) D_p H(\mathbf{x}, \mathbf{z})$$

for $(\mathbf{x}, \mathbf{z}) \in (0, \infty)^n \times (0, \infty)^n$.

Under Assumption 13, and assuming the Markovian market coefficients $\beta(\cdot)$, $\sigma(\cdot)$,$\gamma(\cdot)$,

$\tau(\cdot)$ for $(\mathbf{x}, \mathbf{z}) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$ as in Assumption 7,

$$\tilde{u}(\tau, \mathbf{x}, \mathbf{z}) = e^c \frac{G(\tau, \mathbf{x}, \mathbf{z})}{g(\mathbf{x}, \mathbf{z})}, \tag{5.10}$$

where

$$g(\mathbf{x}, \mathbf{z}) := \Big(\delta \sum_{i=1}^n x_i + (1-\delta)m\Big) e^{-H(\mathbf{x},\mathbf{z})},$$

$$G(T, \mathbf{x}, \mathbf{z}) := \mathbb{E}^{\mathbb{P}}\big[g(\mathcal{X}(T), \mathcal{Z}(T)) e^{-\int_0^T k(\mathcal{X}(t),\mathcal{Z}(t)) + \tilde{k}(\mathcal{X}(t),\mathcal{Z}(t))dt}\big],$$

where $m = \sum_{i=1}^n z_i$.

Assumption 14. *Assume that $g(\cdot)$ is Hölder continuous, uniformly on compact subsets of $\mathbb{R}_+^n \times \mathbb{R}_+^n$, $\ell = 1, \ldots, N$; $G(\cdot)$ is continuous on $(0,\infty) \times (0,\infty)^n \times (0,\infty)^n$, of class $C^2((0,\infty) \times (0,\infty)^n \times (0,\infty)^n)$.*

The function $G(\cdot)$ yields the following dynamics by Feynman-Kac formula,

$$\begin{aligned} \frac{\partial G}{\partial \tau}(\tau, \mathbf{x}, \mathbf{z}) = \mathcal{L}G(\tau, \mathbf{x}, \mathbf{z}) - \big(k(\mathbf{x}, \mathbf{z}) + \tilde{k}(\mathbf{x}, \mathbf{z})\big)G(\tau, \mathbf{x}, \mathbf{z}), \qquad (\tau, \mathbf{x}, \mathbf{z}) \in \mathbb{R}_+ \times \mathbb{R}_+^n \times \mathbb{R}_+^n, \\ G(0, \mathbf{x}, \mathbf{z}) = g(\mathbf{x}, \mathbf{z}), \ \ (\mathbf{x}, \mathbf{z}) \in \mathbb{R}_+^n \times \mathbb{R}_+^n. \end{aligned} \tag{5.11}$$

Under Assumption 14, $\tilde{u}(\tau, \mathbf{x}, \mathbf{z}) \in C^2((0,\infty) \times (0,\infty)^n \times (0,\infty)^n)$ is bounded on $K \times (0,\infty)^n \times (0,\infty)^n$ for each compact $K \subset (0,\infty)$. We derive that (5.8) solves a single Cauchy problem as opposed to the N-dimensional PDEs system in N-player game,

$$\frac{\partial \tilde{u}(\tau, \mathbf{x}, \mathbf{z})}{\partial \tau} \geqslant \mathcal{A}\tilde{u}(\tau, \mathbf{x}, \mathbf{z}), \quad \tilde{u}(0, \mathbf{x}, \mathbf{z}) = e^c, \tag{5.12}$$

$$\text{where}\quad \mathcal{A}\tilde{u}(\tau,\mathbf{x},\mathbf{z}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\mathbf{x},\mathbf{z})\Big(D_{ij}^2\tilde{u}^{\ell}(\tau,\mathbf{x},\mathbf{z}) + \frac{2\delta D_i\tilde{u}^{\ell}(\tau,\mathbf{x},\mathbf{z})}{\delta\mathbf{x}\cdot\mathbf{1}+(1-\delta)\mathbf{z}\cdot\mathbf{1}}\Big)$$
$$+ \frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n} \psi_{pq}(\mathbf{x},\mathbf{z})\Big(D_{pq}^2\tilde{u}(\tau,\mathbf{x},\mathbf{z}) + \frac{2(1-\delta)D_p\tilde{u}(\tau,\mathbf{x},\mathbf{z})}{\delta\mathbf{x}\cdot\mathbf{1}+(1-\delta)\mathbf{z}\cdot\mathbf{1}}\Big)$$
$$+ \sum_{i=1}^{n}\sum_{p=1}^{n}(s\tau^T)_{ip}(\mathbf{x},\mathbf{z})D_{ip}^2\tilde{u}(\tau,\mathbf{x},\mathbf{z})$$
$$+ \sum_{i=1}^{n}\sum_{p=1}^{n}(s\tau^T)_{ip}(\mathbf{x},\mathbf{z})\frac{\delta D_p\tilde{u}(\tau,\mathbf{x},\mathbf{z}) + (1-\delta)D_i\tilde{u}^{\ell}(\tau,\mathbf{x},\mathbf{z})}{\delta\mathbf{x}\cdot\mathbf{1}+(1-\delta)\mathbf{z}\cdot\mathbf{1}},$$

for $\tau \in (0,\infty)$, $(\mathbf{x},\mathbf{z}) \in (0,\infty)^n \times (0,\infty)^n$.

The steps of searching equilibrium for extended mean field game with joint measure of state and control is formulated in [15]. The paper [17] manifests an example of extended mean field games with application in price anarchy. They use two different measures as law of the state processes and the law of control.

The steps to seek equilibrium we introduce is different in that a modified version of extended mean field game is discussed, where the state processes and cost functional depend on different measures, and uniqueness of Nash equilibrium is specified here. In the following, we show the steps to attain a unique equilibrium in an open loop or closed loop Markovian form.

In the same vein of the arguments in Section 4.1.2, we can show the uniqueness of $\nu^\star$ leads to the uniqueness of $\mu^\star$ while the reverse way is not true. More explicitly, $\mathcal{Z}^\star = \mathbb{E}[V^\star\pi^\star|\mathcal{F}^B]$ is not expected to be unique. Since the diffusion process of $\mathcal{Z}(T)$ is given by Definition 2.1.1(3) and (5.6), we consider the fixed point over the control space when it comes to $\mathcal{Z}(T) = \mathbb{E}[\mathcal{Z}_T^\star|\mathcal{F}_T^B]$.

Steps of Solving Mean Field Game

(i) Start with a fixed ϕ such that $\pi = (\pi(t))_{0\leqslant t\leqslant T} = \phi(\mathbf{v}, B_{[0,T]})$ or $\phi(V(t))$, the open loop and feedback function respectively, and solve

$$dV(t) = \pi(t)\beta(\mathcal{X}(t), \mathcal{Z}(t))dt + \pi(t)\sigma(\mathcal{X}(t), \mathcal{Z}(t))dB_t, \quad V(0) = \tilde{u}(T)\mathcal{V}(0) := v_0,$$

$$dX_i(t) = X_i(t)\beta_i(\mathcal{X}(t), \mathcal{Z}(t))dt + X_i(t)\sum_{k=1}^{n}\sigma_{ik}(\mathcal{X}(t), \mathcal{Z}(t))dB_k(t), \quad i = 1, \ldots, n,$$

where $Z_i(t) = \mathbb{E}[V(t)\pi_i(t)|\mathcal{F}_t^B]$ for $0 \leqslant t \leqslant T$, $\mathcal{Z}(t) = (Z_1(t), \ldots, Z_n(t))$.

(ii) For all $0 \leqslant t \leqslant T$, $m_t = \mathbb{E}[V_t|\mathcal{F}_t^B]$ on $\mathbb{R}_+$, where V is the path generated from the fixed ϕ. Thus given $m = (m_t)_{0\leqslant t\leqslant T}$, solve

$$\inf_{\pi\in\mathbb{A}} J^{m,Z}(\pi) = u(T) = \mathbb{E}\big[e^c\big(\delta X(T) + (1-\delta)m_t\big)L(T)\big]\,/\,\mathcal{V}(0),$$

using $X(T)$ from step (i). The corresponding optimum

$$\phi^\star := \arg\inf_{\pi\in\mathbb{A}} J^{\mu,\nu}(\pi) = \arg\inf_{\pi\in\mathbb{A}} J^{m,Z}(\pi).$$

Define the mapping $\phi^\star = \Phi(\phi)$.

(iii) Find $\hat{\phi}$ such that $\hat{\phi}^\star = \Phi(\hat{\phi})$.

By Itô's formula we have

$$\hat{V}(t) = \hat{V}(0) + \sum_{k=1}^{n}\int_0^t \hat{V}(s)I_k(T-s, \mathcal{X}(s), \mathcal{Z}(s))dB_k(s),\ 0 \leqslant t \leqslant T, \tag{5.13}$$

where

$$I_k(\rho, x, z) = \sum_{i=1}^{n} \sigma_{ik}(\mathbf{x}, \mathbf{z}) x_i D_{x_i} \log \tilde{u}(\rho, \mathbf{x}, \mathbf{z}) + \sum_{j=1}^{n} \tau_{jk}(\mathbf{x}, \mathbf{z}) D_{z_j} \log \tilde{u}(\rho, \mathbf{x}, \mathbf{z})$$
$$+ \sum_{i=1}^{n} \frac{\delta X(t)}{\mathcal{V}(t)} \left(\frac{x_i}{\sum_{i=1}^{n} x_i} \sigma_{ik}(t) - \theta_k(\mathbf{x}, \mathbf{z}) \right) + \frac{(1-\delta) m_t}{\mathcal{V}(t)} \left(\pi_i \sigma_{ik}(t) - \theta_k(\mathbf{x}, \mathbf{z}) \right). \tag{5.14}$$

Remark 7. *We use vol to represent the volatility of a process to simplify the notations from now on. Let $\rho = T - t$, $t \in [0, T]$. In general, if searching for Nash equilibrium when m_t is fixed, then*

$$I_k(\rho, x, z) = \sum_{i=1}^{n} \sigma_{ik}(\mathbf{x}, \mathbf{z}) x_i D_{x_i} \log \tilde{u}(\rho, \mathbf{x}, \mathbf{z}) + \sum_{j=1}^{n} \tau_{jk}(\mathbf{x}, \mathbf{z}) D_{z_j} \log \tilde{u}(\rho, \mathbf{x}, \mathbf{z})$$
$$+ \sum_{i=1}^{n} \frac{\delta X(t)}{\mathcal{V}(t)} \left(\frac{x_i}{\sum_{i=1}^{n} x_i} \sigma_{ik}(t) - \theta_k(\mathbf{x}, \mathbf{z}) \right) + \frac{(1-\delta)}{\mathcal{V}(t)} vol(dL_t m_t) L_t^{-1}.$$

If the coefficients $\gamma(\cdot)$ and $\tau(\cdot)$ in $\mathcal{Z}(t)$ dynamics can not be observed, it can be distinguished in the same sense as (3.27) and (3.31). Then (5.14) and Proposition 5.1.1 can be expressed more explicitly from there.

In the following theorem we derive the mean field equilibrium by fixed point conditions on the control space.

Proposition 5.1.1. *Under Assumption 3, 7, 12, and 14, there exists a Mean Field Equilibrium $\mu^\star$. It is attained at*

$$\pi_i^\star(t) = X_i^\star(t) D_{x_i} \log \tilde{u}(T-t) + \sum_{j=1}^{n} (\tau \sigma^{-1})_{ji}(\mathbf{x}, \mathbf{z}) D_{z_j} \log \tilde{u}(T-t)$$
$$+ \frac{\delta X^\star(t)}{\mathcal{V}^\star(t)} \mathbf{m}_i^\star(t) + \frac{(1-\delta) m_t}{\mathcal{V}^\star(t)} \pi_i^\star(t), \tag{5.15}$$

or equivalently

$$\pi_i^\star(t) = \mathbf{m}_i^\star(t) + \frac{1}{\delta}\mathbf{m}_i^\star(t)\mathcal{V}^\star(t)D_{x_i}\log\tilde{u}(T-t) + \frac{\mathcal{V}^\star(t)}{\delta X(t)}\sum_{j=1}^{n}(\tau\sigma^{-1})_{ji}(\mathbf{x},\mathbf{z})D_{z_j}\log\tilde{u}(T-t).$$

If the corresponding Nash equilibrium $\mu^\star$ is unique in the sense of Definition 5.1.2, then the first exit time from the interval $\tilde{K}_t$ is greater than T, i.e., $\tilde{\tau}^K > T$ where

$$\tilde{K}_t = \left(0, \frac{\big(1-(1-\delta)\mathbb{E}[e^c\tilde{u}(T-t)|\mathcal{F}_t^B]\big)^2}{\delta\big|\mathbb{E}[e^c D_m\tilde{u}(T-t)|\mathcal{F}_t^B]\big|}\right), \quad \tilde{\tau}^K = \inf\{t \geqslant 0; X(t) \notin \tilde{K}_t\}. \tag{5.16}$$

Proof. We adopt a similiar path to obtain a solution of optimal strategy $\pi^\star(\cdot)$ and the uniqueness of equilibrium in Proposition 4.2.1 and 4.2.2. When searching for mean field equilibrium, we start from a given choice of $\pi(t) \in \mathbb{A}$, for any $t \in [0,T]$. The McKean-Vlasov system (5.2)-(5.3) can be solved with the given $\pi(t) \in \mathbb{A}$ and the optimal value $\tilde{u}(T)$ of the minimization problem (5.8). Every player in the mean field game acts optimally by following

$$V^\star(t) = e^c\mathcal{V}^\star(t)\tilde{u}(T-t), \tag{5.17}$$

with the rest of the pack assumed to be fixed.

Thus by

$$\mathcal{V}^\star(t) = \delta X_t^\star + (1-\delta)\mathbb{E}[e^c\mathcal{V}^\star(t)\tilde{u}(T-t)|\mathcal{F}_t^B],$$

we solve (5.17) which yields

$$V^\star(t) = \frac{e^c\delta\tilde{u}(T-t)}{1-(1-\delta)e^c\tilde{u}(T-t)}\Big(X^\star(t) - \delta(1-\delta)\big(\mathbb{E}[X^\star(t)|\mathcal{F}_t^B] - X^\star(t)\big)\Big). \tag{5.18}$$

After comparing $\log V^\star(t)$ in (5.3) and (5.18), this yields

$$\pi_i^\star(t) = \mathbf{m}_i(t) + \frac{1}{\delta}\mathbf{m}_i^\star(t)\mathcal{V}^\star(t)D_{x_i}\log\tilde{u}(T-t) + \frac{\mathcal{V}^\star(t)}{\delta X(t)}\sum_{j=1}^{n}(\tau\sigma^{-1})_{ji}(\mathbf{x},\mathbf{z})D_{z_j}\log\tilde{u}(T-t).$$

$\tilde{u}(T-t,\mathbf{x},\mathbf{z})$ is the smallest nonnegative solution in (5.12).

The derivation of $\pi^\star$ ensures that it generates a wealth process $V^\star$. Thus with conditions for $\pi^\star \in \Delta_n$, it follows that (5.15) gives the admissible optimal strategy $\pi^\star \in \mathbb{A}$.

Next, we show the equilibrium is unique. We fix the process m solve the optimal control problem for $V^\star$. We first perform a transformation of $\tilde{u}(\cdot)$ on variables $\mathbf{z} = (z_1, \ldots, z_n)$. The transformed vector

$$\tilde{\mathbf{z}} = (\tilde{z}_1, \ldots, \tilde{z}_n) = (z_1, \ldots, z_{n-1}, m),$$

where denote $m = \sum_{i=1}^{n} z_i$, since $m_t = \sum_{i=1}^{n} Z_i(t)$.

To simplify, let $D_m\tilde{u}(T-t) = D_m\tilde{u}(T-t,\mathbf{x},\mathbf{z}^{-n},m)$. Thus taking derivative with respect to m follows

$$D_m\tilde{u}(T-t) = \frac{\partial\tilde{u}(T-t,\mathbf{x},\tilde{\mathbf{z}})}{\partial\tilde{z}_n}. \tag{5.19}$$

Denote $\Phi(m_t) := \mathbb{E}[V(t)|B]$, it is equivalent to show that there is the unique fixed point mapping $\Phi(m_t) = m_t$, where $\Phi(\cdot)$ yields

$$\Phi(\xi) = \frac{\delta X^\star(t)\mathbb{E}[e^c\tilde{u}(T-t,\mathbf{x},\mathbf{z}^{-n},\xi)|\mathcal{F}_t^B]}{1-(1-\delta)\mathbb{E}[e^c\tilde{u}(T-t,\mathbf{x},\mathbf{z}^{-n},\xi)|\mathcal{F}_t^B]},$$

and the derivative

$$\Phi'(\xi) = \frac{\delta X^\star(t)\mathbb{E}[e^cD_\xi\tilde{u}(T-t,\mathbf{x},\mathbf{z}^{-n},\xi)|\mathcal{F}_t^B]}{\big(1-(1-\delta)\mathbb{E}[e^c\tilde{u}(T-t,\mathbf{x},\mathbf{z}^{-n},\xi)|\mathcal{F}_t^B]\big)^2}.$$

By the smoothness of $\tilde{u}(T-t)$ based on Assumption 15, $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function of m. Furthermore, we set

$$\tilde{A}_t = 1 - (1-\delta)\mathbb{E}[e^c \tilde{u}(T-t)|\mathcal{F}_t^B], \quad \tilde{D}_t = \delta\big|\mathbb{E}[e^c D_m \tilde{u}(T-t)|\mathcal{F}_t^B]\big|,$$

$D_m\tilde{u}(T-t)$ is derived in (5.19). By mean value theorem, Φ is a contraction of m_t if $\tilde{\tau}^K > T$, where $\tilde{\tau}^K = \inf\{t \geqslant 0; X(t) \notin \tilde{K}_t\}$, $\tilde{K}_t := [0, \frac{\tilde{A}_t^2}{\tilde{D}_t})$. As a result, the mean field equilibrium generated by (5.15) is unique when the first exit time from $\tilde{K}$ is less than T. □

Remark 8. *Following from Remark 7, if when searching for the equilibrium on the space of m_t, then*

$$\begin{aligned}\pi_i^\star(t) &= X_i^\star(t) D_{x_i} \log \tilde{u}(T-t) + \sum_{j=1}^n (\tau\sigma^{-1})_{ji}(\mathbf{x},\mathbf{z}) D_{z_j} \log \tilde{u}(T-t) \\ &+ \frac{\delta X^\star(t)}{\mathcal{V}^\star(t)} \mathbf{m}_i(t) + \frac{(1-\delta)}{\mathcal{V}^\star(t)} \big(vol(dL_t m_t)\sigma^{-1}\big)_i(\mathbf{x},\mathbf{z}),\end{aligned}$$

$vol(dL_t m_t)$ in the above is from (5.14). In particular, (5.15) with open loop control can be expressed as

$$\begin{aligned}\pi_i^\star(t) &= X_i^\star(t) D_{x_i} \log \tilde{u}(T-t) + \sum_{j=1}^n \pi_i(t)\pi_j(t) D_{z_j} \log \tilde{u}(T-t) \\ &+ \frac{\delta X^\star(t)}{\mathcal{V}^\star(t)} \mathbf{m}_i(t) + \frac{(1-\delta)}{\mathcal{V}^\star(t)} L_t(m_t(\theta_t'\sigma^{-1})_i - V_t\pi_i(t)),\end{aligned} \tag{5.20}$$

$$\begin{aligned}F(\tilde{u}(\cdot),\mathbf{x},\mathbf{z}) &= \sum_{j=1}^n \pi_j D_j \log \tilde{u}(T-t) \\ &= \frac{\sum_{j=1}^n X_j^\star(t) D_{x_j} \log \tilde{u}(T-t) D_{z_j} \log \tilde{u}(T-t) + \sum_{j=1}^n D_{z_j} \log \tilde{u}(T-t)}{1 - \sum_{j=1}^n D_j \log \tilde{u}(T-t)} \\ &+ \frac{(1-\delta) L_t m_t \sum_{j=1}^n (\theta_t'\sigma^{-1})_j D_{z_j} \log \tilde{u}(T-t)}{\mathcal{V}^\star(t)(1 - \sum_{j=1}^n D_j \log \tilde{u}(T-t))}.\end{aligned}$$

Hence

$$\pi_i^\star(t) = \frac{X_i^\star(t) D_{x_i} \log \tilde{u}(T-t) + \frac{\delta X_t}{\mathcal{V}_t}\mathbf{m}_i(t) + \frac{1-\delta}{\mathcal{V}} L_t m_t (\theta_t' \sigma^{-1})_i}{\frac{\delta X_t}{\mathcal{V}_t} - F(\tilde{u}(\cdot), \mathbf{x}, \mathbf{z})}.$$

It is clear from (5.15) that the mean field strategy actually depends on $(\mathcal{X}(t), \mathcal{Z}(t))$, which means the optimal strategies are driven by capitalization and trading volumes, regardless of the information structure of the strategies we set up in the beginning. Similiar to the observations in Chapter 4 of N-player game, π is independent of preference c, meaning that the representative player's preference level c is not a crucial factor when exploiting strategies.

5.2 Generalized results with flows of measure

As in the beginning of this chapter, we denote μ_t in (5.1) as the conditional law of $V(t)$ given $\mathcal{F}^B$. That is, $\mu_t := Law\big(V(t)\big|\mathcal{F}_t^B\big)$, with a given initial condition $\mu_0 \in \mathcal{P}^2(C([0,T];\mathbb{R}_+))$. The conditional law of $(V(t), \pi(t))$ given $\mathcal{F}^B$, i.e.,

$$\nu_t := Law(V(t), \pi(t)|\mathcal{F}_t^B)$$

with a given initial condition $\nu_0 \in \mathcal{P}^2(C([0,T];\mathbb{R}_+ \times \mathbb{A}))$. In general, players interact through the measure flow μ and ν instead of the conditional expectations $\mathbb{E}[V(t)|B]$ and $\mathbb{E}[V(t)\pi(t)|B]$, respectively. In this case the benchmark is as defined in (5.4). The player feels the presence of the group through the joint distribution of wealth and strategies.

In this section, we show how the results about the mean field Cauchy problem (5.12) changed according to the replacement of continuous functions by probability measures. We first recall some basic notions of the differentiability of a function f with respect to probability measures.

Definition 5.2.1 (Linear derivative)**.** *For any $f : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R}$, the linear derivative*

$\frac{\delta f(x,\mu,v)}{\delta \mu} : \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^n$ *is a bounded continuous function defined as follows. For every* $\mu,\ \mu' \in \mathcal{P}_2(\mathbb{R}^n)$, *denote* $\mu^\lambda = \lambda\mu + (1-\lambda)\mu'$,

$$f(x,\mu) - f(x,\mu') = \int_0^1 \int_{\mathbb{R}^n} \frac{\delta f(x,\mu^\lambda,v)}{\delta \mu} d(\mu-\mu')(v) d\lambda.$$

In [12], L-derivative provides a vector space structure by lifting functions f of probability measures in $\mathcal{P}_2(\mathbb{R}^n)$ to flat vector space $L^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^n)$. Let the function $\tilde{f}$ defined on the Hilbert space, and we have $\tilde{f}(\mathbf{x}) = f(\text{Law}(\mathbf{x}))$, $\mathbf{x} \in L^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^n)$.

Definition 5.2.2 (L-derivative)**.** *L-derivative of f at μ_0, denoted as $(\partial_\mu f)(\mu_0,)$ satisfies*

$$f(\mu) = f(\mu_0) + \mathbb{E}[\partial_\mu f(Law(X_0))(X_0)\cdot(X-X_0)] + o(||X-X_0||_2).$$

f on $\mathcal{P}_2(\mathbb{R}^d)$ is said to be L-differentiable if its lift function $\tilde{f}$ is Fréchet differentiable at X_0.

The relationship between L-derivative and linear derivative is

$$\partial_\mu f(x,\mu,v) = \partial_v \frac{\delta f(x,\mu,v)}{\delta \mu}.$$

The above definitions come into use because we need a chain rule for derivatives of functions of capitalization processes $\mathcal{X}(t)$, wealth process $V(t)$, trading volume $V(t)\pi_i(t)$. We simplify the notations as

$$dX_t^i = \bar{b}_i(t)dt + \bar{s}_i(t)dB_t$$

$$dV_t = \bar{B}(t)dt + \bar{S}(t)dB_t$$

$$dV(t)\pi_i(t) = \bar{b}_i^y(t)dt + \bar{s}_i^y(t)dB_t,$$

where $\pi = (\pi(t))_{0\leqslant t\leqslant T} = \phi(\mathbf{v}, B_{[0,T]})$ or $\phi(V(t))$, the open loop and feedback function respectively.

By Itô's formula with flow of measure, $F \in C^{1,1}([0,T]\times\mathbb{R}^n_+\times\mathcal{P}^2(\mathbb{R}_+\times\mathbb{A})\times\mathcal{P}_2(\mathbb{R}_+))$, we have

$$\begin{aligned} dF(t,X_t,\mu_t,\nu_t) = \partial_t F(t,X_t,\mu_t,\nu_t)dt \\ &+ \big(\mathcal{L}F(X_t,\mu_t,\nu_t) + \mathcal{L}_\mu F(X_t,\mu_t,\nu_t) + \mathcal{L}_\nu F(t,X_t,\mu_t,\nu_t)\big)dt \\ &+ \sum_{i=1}^n \partial_{x_i} F(r,X_t,\mu_t)dB_t + \int_{\mathbb{R}^n} \partial_\mu F(t,X_t,\mu_t,\nu_t,v)\bar{s}(v,\mu_t)\mu_t(dv)dB_t \end{aligned} \tag{5.21}$$

where

$$\mathcal{L}F(t,X_t,\mu_t,\nu_t) = \sum_{i=1}^n \partial_{x_i} F(t,X_t,\mu_t,\nu_t)\bar{b}(X_t,\mu_t) + \frac{1}{2}\sum_{i,j=1}^n \partial_{x_i}\partial_{x_j} F(t,X_t,\mu_t,\nu_t)\bar{a}(X_t,\mu_t),$$

$$\begin{aligned} \mathcal{L}_\mu F(t,X_t,\mu_t,\nu_t) = &\int_{\mathbb{R}^n} \partial_\mu F(t,X_t,\mu_t,\nu_t,v)\bar{B}(v,\mu_t)\mu_t(dv)dt \\ &+ \frac{1}{2}\int_{\mathbb{R}^n} \mathrm{Tr}[\partial_\mu\partial_v F(t,X_t,\mu_t,\nu_t,v)\bar{A}(v,\mu_t)]\mu_t(dv) \\ &+ \frac{1}{2}\int\int_{\mathbb{R}^n\times\mathbb{R}^n} \mathrm{Tr}[\partial_\mu^2\partial_v F(t,X_t,\mu_t,\nu_t,v,\tilde{v})\bar{A}(v,\mu_t)]\mu_t(dv)\mu_t(d\tilde{v}) \\ &+ \int_{\mathbb{R}^n} \mathrm{Tr}[\partial_\mu\partial_\nu F(t,X_t,\mu_t,\nu_t,v)\bar{s}(v,\mu_t)\bar{s}^y(v,\mu_t)]\mu_t(dv)\nu_t(dv). \end{aligned}$$

The counterpart $\mathcal{L}_\nu F(t,X_t,\mu_t,\nu_t)$ is similar to the above.

Assume the Markovian market coefficients $\beta(\cdot)$, $\sigma(\cdot)$, $\gamma(\cdot)$, $\tau(\cdot)$ for $(\mathbf{x},\nu)\in\mathbb{R}^n_+\times\mathcal{P}^2(\mathbb{R}^n_+)$. We modify function $H(\cdot)$ defined first in Assumption 13. Assume there exist a function $H : \mathbb{R}^n_+\times\mathcal{P}^2(\mathbb{R}_+\times\mathbb{A})\to\mathbb{R}^n_+$ of class C^2, such that

$$b(\mathbf{x},\nu) = 2a(\mathbf{x},\nu)D_x H(\mathbf{x},\nu), \quad \gamma(\mathbf{x},\nu) = 2\psi(\mathbf{x},\nu)D_\nu H(\mathbf{x},\nu),$$

where $D_\nu H(\mathbf{x},\nu)$ is the L-derivative of $H(\cdot)$ with respect to ν as defined in Definition 5.2.2.

$$\tilde{u}(\tau, \mathbf{x}, \mu, \nu) = e^c \frac{G(\tau, \mathbf{x}, \mu, \nu)}{g(\mathbf{x}, \mu, \nu)}, \tag{5.22}$$

where

$$g(\mathbf{x}, \mu, \nu) := \Big(\delta \sum_{i=1}^{n} x_i + (1-\delta) \int_{\Omega} f(v)\ d\mu(v)\Big) e^{-H(\mathbf{x},\nu)},$$

$f : \mathbb{R}_+ \to \mathbb{R}_+$ be a Borel function with $f(V(t)) \in L^1$ for $t \in [0, T]$.

$$G(T, \mathbf{x}, \mu, \nu) := \mathbb{E}^{\mathbb{P}}\big[g(\mathcal{X}(T), \mu_T, \nu_T) e^{-\int_0^T k(\mathcal{X}(t)) + \tilde{k}(\nu_t) dt}\big],$$

where $m = \sum_{i=1}^{n} z_i$.

Assumption 15. *Assume that $g(\cdot)$ is Hölder continuous, uniformly on compact subsets of $\mathbb{R}_+^n \times \mathcal{P}^2(\mathbb{R}_+ \times \mathbb{A})$, $\ell = 1, \ldots, N$; $G(\cdot)$ is continuous on $\mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathcal{P}^2(\mathbb{R}_+ \times \mathbb{A})$, of class $C^2(\mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathcal{P}^2(\mathbb{R}_+ \times \mathbb{A}))$.*

The function $G(\cdot)$ yields the following dynamics by Feynman-Kac formula,

$$\begin{aligned} \frac{\partial G}{\partial \tau}(\tau, \mathbf{x}, \mathbf{z}) = \mathcal{L}G(\tau, \mathbf{x}, \mathbf{z}) - \big(k(\mathbf{x}, \mathbf{z}) + \tilde{k}(\mathbf{x}, \mathbf{z})\big) G(\tau, \mathbf{x}, \mathbf{z}), \qquad (\tau, \mathbf{x}, \mathbf{z}) \in \mathbb{R}_+ \times \mathbb{R}_+^n \times \mathbb{R}_+^n, \\ G(0, \mathbf{x}, \mathbf{z}) = g(\mathbf{x}, \mathbf{z}),\ \ (\mathbf{x}, \mathbf{z}) \in \mathbb{R}_+^n \times \mathbb{R}_+^n. \end{aligned} \tag{5.23}$$

Under Assumption 15, $\tilde{u}(\tau, \mathbf{x}, \mu, \nu) \in C^2(\mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathcal{P}^2(\mathbb{R}_+ \times \mathbb{A}) \times \mathcal{P}_2(\mathbb{R}_+))$ is bounded on $K \times \mathbb{R}_+^n \times \mathcal{P}^2(\mathbb{R}_+ \times \mathbb{A}) \times \mathcal{P}_2(\mathbb{R}_+)$ for each compact $K \subset \mathbb{R}_+^n$. Especially, we note that the difference from the previous chapters is that here $u(t, \mathbf{x}, \mu, \nu)$ continuously L-differentiable at μ and ν; $\partial_v \partial_\mu u(t, \mathbf{x}, \mu, \nu)(v) \in \mathbb{R}^{n \times n}$, is locally bounded and is jointly continuous for any $(t, \mathbf{x}, \mu, \nu, v)$ in $\mathbb{R}_+ \times \mathbb{R}_+^n \times \mathcal{P}^2(\mathbb{R}_+ \times \mathbb{A}) \times \mathcal{P}_2(\mathbb{R}_+) \times \mathbb{R}_+^n$.

By Feynman-Kac formula with flows of measure and (5.22), we get

$$
\begin{aligned}
&\frac{\partial \tilde{u}^{\ell}(t,\mathbf{x},\mu,\nu)}{\partial t} g(\mathbf{x},\nu) \\
=&\mathcal{L}(\tilde{u}^{\ell}(t,\mathbf{x},\mu,\nu)g(\mathbf{x},\nu)) + \mathcal{L}_{\nu}(\tilde{u}^{\ell}(t,\mathbf{x},\mu,\nu)g(\mathbf{x},\nu)) + \mathcal{L}_{\mu}(\tilde{u}^{\ell}(t,\mathbf{x},\mu,\nu)g(\mathbf{x},\nu)) \\
&- (k(\mathbf{x},\nu) + \tilde{k}(\mathbf{x},\nu))\tilde{u}^{\ell}(t,\mathbf{x},\mu,\nu)g(\mathbf{x},\nu),
\end{aligned}
$$

where

$$
\begin{aligned}
k(\mathbf{x},\nu) &= -\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{a_{ij}(\mathbf{x},\nu)}{2}[D^2_{ij}H(\mathbf{x},\nu) + 3D_iH(\mathbf{x},\nu)D_jH(\mathbf{x},\nu)], \\
\tilde{k}(\mathbf{x},\nu) &:= -\sum_{p=1}^{n}\sum_{q=1}^{n} \frac{\psi_{pq}(\mathbf{x},\nu)}{2}[D^2_{pq}H(\mathbf{x},\nu) + 3D_pH(\mathbf{x},\nu)D_qH(\mathbf{x},\nu)] \\
&\quad + \sum_{i=1}^{n}\sum_{p=1}^{n}(s\tau^T)_{ip}D_iH(\mathbf{x},\nu)D_pH(\mathbf{x},\nu).
\end{aligned}
$$

We can also adapt the mean-field Cauchy problem (5.12) and mean field equilibrium with the above modifications about probability measure flows μ and ν. Following from Definition 5.3.1, when searching for mean field equilibrium, we start from each arbitrary strategy π and solve the McKean-Vlasov system with measured-valued stochastic process $\nu = (\nu_t)_{0 \leqslant t \leqslant T}$ on $\mathbb{R}_+ \times \mathbb{A}$, adapted to the filtration generated by the random measure B. We search for the mean field equilibrium through the fixed point problem on the control space.

5.3 Connecting N-player game and mean field game of relative arbitrage optimization

5.3.1 The limit of dynamical systems

In this section we attempt to show that in the limit $N \to \infty$, stock capitalization vector $\mathcal{X}(t) := (X_1(t), \ldots, X_n(t))$ and the wealth of a representative player will satisfy Mckean-Vlasov SDEs. The paper [52] provides weak and strong uniqueness results of McKean-Vlasov equation under relaxed regularity conditions. Differentiating from the usual Mckean-Vlasov SDEs of the form that the coefficients of the diffusion depend on the distribution of the solution itself, we here consider the joint distribution of the state process $V(\cdot)$ and the control $\pi \in \mathbb{A}$, and show the propagation of chaos holds.

Definition 5.3.1. *Let $(X_t)_{0 \leqslant t \leqslant T}$ be a solution of (5.24) on the tuple $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. The random variable $\mathcal{L}(\hat{V}(t), \hat{\pi}(t))$ provides a conditional law of $(\hat{V}(t), \hat{\pi}(t))$ given $\mathcal{F}^B$. $\nu^i = \{\nu^i_t\} = \{Law(\hat{V}(t), \hat{\pi}(t))|\mathcal{F}^B)\}_{0 \leqslant t \leqslant T} : \Omega \in \omega \to \mathcal{L}(\hat{V}(\omega, \cdot), \hat{\pi}(\omega, \cdot))$ be the flow of marginal conditional distributions of X given the common source of noise.*

We consider the market under the Markovian model as in Assumption 7. The goal is to study the McKean-Vlasov system

$$d\mathcal{X}(t) = \mathcal{X}(t)\beta(\mathcal{X}(t), \nu_t)dt + \mathcal{X}(t)\sigma(\mathcal{X}(t), \nu_t)dB_t, \quad \mathcal{X}_0 = \mathbf{x}; \tag{5.24}$$

$$dV(t) = \pi(t)\beta(\mathcal{X}(t), \nu_t)dt + \pi(t)\sigma(\mathcal{X}(t), \nu_t)dB_t, \quad V(0) = v_0, \tag{5.25}$$

where $B(\cdot) = (B_1(\cdot), \ldots, B_n(\cdot))$ is n-dimensional Brownian motion. $\nu := Law(V, \pi|\mathcal{F}^B_t)$.

Remark 9. *Same as the finite dynamical system in Chapter 2, we analyze a McKean-Vlasov system with initial states given, v_0 is with the same law as v^ℓ. v_0 is supported on*

$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

However, when solving relative arbitrage problems in mean field games, the initial value is obtained using the objective $\tilde{u}(T)$, *that is,* $v_0 = \tilde{u}(T)\mathcal{V}(0)$.

The following proposition shows that ν^N has a weak limit $\nu \in \mathcal{P}^2(C([0,T];\mathbb{R}_+ \times \mathbb{A}))$ with $\mathcal{W}_2$ distance. We denote $\mathcal{C}^A = C([0,T];\mathbb{R}_+ \times \mathbb{A})$ as the path space equipped with the supremum norm $||x|| = \sup_{t\in[0,T]} |x_t|$. We assume the boundedness of coefficients as in the finite system.

Assumption 16. *Consider the control process* $\pi(\cdot)$ *in open loop or closed loop Markovian form as* (4.4) *or* (4.5)*, respectively.*

In particular, we can generalize the strategy in closed loop Markovian form as $\pi_t = \phi(V_t, \nu_t)$. *We assume* π *is Lipschitz in* v*, i.e., there exists a mapping* $\phi : C([0,T], \mathbb{R}_+^N) \times \mathcal{P}^2(C([0,T], \mathbb{R}_+^N \times \mathbb{A})) \to \mathbb{A}$ *such that .*

$$|\phi(v,\nu) - \phi(\tilde{v},\tilde{\nu})| \leqslant L[n|v-\tilde{v}| + \mathcal{W}_2(\nu,\tilde{\nu})]$$

for every $v, \tilde{v} \in \mathbb{R}_+^N$, $\nu, \tilde{\nu} \in \mathcal{P}^2(C([0,T], \mathbb{R}_+^N \times \mathbb{A}))$.

Proposition 5.3.1. *Under Assumption 3, 12, and 16, there exists a unique strong solution of the Mckean-Vlasov system* (5.24)-(5.25).

Proof. Define the truncated supremum norm $||x||_t$ and the truncated Wasserstein distance on $\mathcal{P}^2(\mathcal{C}^A)$ as in [46]. $||x||_t^2 := \sup_{0\leqslant s\leqslant t} |x_s|$,

$$d_t^2(\mu,\nu) = \inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathcal{C}^n\times\mathcal{C}^n} ||x-y||_t^2 \pi(dx,dy).$$

Define $\Phi : \mathcal{P}^2(\mathcal{C}^A) \to \mathcal{P}^2(\mathcal{C}^A)$ so that

$$\Phi(\nu) = \text{Law}(V^\nu, \pi^\nu | \mathcal{F}^B). \tag{5.26}$$

We need to show (5.24) and (5.25) uniquely solve the fixed point problem (5.26).

We take two arbitrary measures $\nu^a, \nu^b \in \mathcal{P}^2(\mathcal{C}^A)$, and denote the wealth involving measure ν as V^ν, and stock capitalization vector involving ν as $\mathcal{X}^\nu$. By Cauchy-Schwarz, Jensen's inequality and Lipschitz conditions in Assumption 3 and 16, it follows

$$
\begin{aligned}
&\mathbb{E}\Big[||(V^{\nu^a}, \mathcal{X}^{\nu^a}) - (V^{\nu^b}, \mathcal{X}^{\nu^b})||_t^2 | \mathcal{F}_t^B\Big] \\
\leqslant & 4t\mathbb{E}\bigg[\int_0^t |V^{\nu^a}(r)\pi^{\nu^a}(r)\beta(\mathbf{x}, \nu^a) - V^{\nu^b}(r)\pi^{\nu^b}(r)\beta(\mathbf{x}, \nu^b)|^2 + |b(\mathbf{x}, \nu^a) - b(\mathbf{x}, \nu^a)|^2 dr \bigg| \mathcal{F}_t^B\bigg] \\
&+ 4\mathbb{E}\bigg[\sup_{0\leqslant s\leqslant t}\bigg|\int_0^s V^{\nu^a}(r)\pi^{\nu^a}(r)\sigma(\mathbf{x}, \nu^a) - V^{\nu^b}(r)\pi^{\nu^b}(r)\sigma(\mathbf{x}, \nu^b)dB_r\bigg|^2 \\
&\qquad + \sup_{0\leqslant s\leqslant t}\bigg|\int_0^s \mathcal{X}_r^{\nu^a}\sigma(\mathbf{x}, \nu^a) - \mathcal{X}_r^{\nu^b}\sigma(\mathbf{x}, \nu^b)dB_r\bigg|^2 \bigg| \mathcal{F}_t^B\bigg] \\
\leqslant & 4(t+4)L^2\mathbb{E}\bigg[\int_0^t (|V_r^{\nu^a} - V_r^{\nu^b}|^2 + |\mathcal{X}_r^{\nu^a} - \mathcal{X}_r^{\nu^b}|^2 + \mathcal{W}_2^2(\nu_r^a, \nu_r^b))dr \bigg| \mathcal{F}_t^B\bigg]
\end{aligned}
$$

By Gronwall's inequality,

$$
\mathbb{E}[||V^{\nu^a} - V^{\nu^b}||_t^2 | \mathcal{F}_t^B] \leqslant \mathbb{E}[||(V^{\nu^a}, \mathcal{X}^{\nu^a}) - (V^{\nu^b}, \mathcal{X}^{\nu^b})||_t^2 | \mathcal{F}_t^B] \leqslant C_T \mathbb{E}[\int_0^t \mathcal{W}_2^2(\nu_r^a, \nu_r^b)dr], \tag{5.27}
$$

where $C_T = 4(T+4)L^2 \exp(4(T+4)L^2)$.

Define a mapping $L : \mathbb{R}_+ \to \mathcal{P}^2(C([0,T], \mathbb{R}_+ \times \mathbb{A}))$,

$$
L(V_t) = Law(V_t, \phi(V_t)) = \nu_t,
$$

then it follows that for $(v, u) \in \mathbb{R}_+ \times \mathbb{R}_+$,

$$
\mathcal{W}_2^2(L(v), L(u)) \leqslant n(2L^2+1)|v-u|^2 + 2L^2\mathcal{W}_2^2(L(v), L(u)),
$$

and hence

$$
\mathcal{W}_2^2(\nu_t^a, \nu_t^b) \leqslant C_n \mathbb{E}[||V^{\nu^a} - V^{\nu^b}||_t^2].
$$

where $C_n = \frac{n(1+2L^2)}{1-2L^2}$.

If $\pi(\cdot)$ is an open loop control, i.e., $\pi(t) = \phi(v_0, \nu_t, B_{[0,T]})$,

$$\mathbb{E}[||\pi^{\nu^a} - \pi^{\nu^b}||_t^2 | \mathcal{F}_t^B] \leqslant L^2 \mathbb{E}[\mathcal{W}_2^2(\nu^a, \nu^b) | \mathcal{F}_t^B] \leqslant L^2 \mathbb{E}[\mathcal{W}_2^2(\nu_t^a, \nu_t^b)]. \tag{5.28}$$

If $\pi(\cdot)$ is in the closed loop Markovian form, i.e., $\pi(t) = \phi(t, V(t), \nu_t)$,

$$\begin{aligned}\mathbb{E}[||\pi^{\nu^a} - \pi^{\nu^b}||_t^2 | \mathcal{F}_t^B] &\leqslant 2L^2 \mathbb{E}[||V^{\nu^a} - V^{\nu^b}||_t^2 + \mathcal{W}_2^2(\nu_t^a, \nu_t^b)] \\ &\leqslant 8(T+4)L^4 \exp(4(T+4)L^2) \mathbb{E}[\int_0^t \mathcal{W}_2^2(\nu_r^a, \nu_r^b) dr] + 2L^2 \mathbb{E}[\mathcal{W}_2^2(\nu_t^a, \nu_t^b)].\end{aligned} \tag{5.29}$$

Then the coupling of $\Phi(\nu_1), \Phi(\nu_2)$ gives

$$\begin{aligned}\mathbb{E}[d_T^2(\Phi(\nu_T^a), \Phi(\nu_T^b))] &\leqslant \mathbb{E}[||(V^{\nu^a}, \pi^{\nu^a}) - (V^{\nu^b}, \pi^{\nu^b})||_T^2 | \mathcal{F}_T^B] \\ &\leqslant C_F \mathbb{E}[\int_0^T d_r^2(\nu_r^a, \nu_r^b) dr],\end{aligned} \tag{5.30}$$

where $C_F = (3 + 2L^2 C_n)C_T$ for closed loop Markovian controls, and $C_F = (1 + L^2 C_n)C_T$ for open loop controls. After induction, we get

$$\mathbb{E}[d_T^2(\Phi^k(\nu_T^a), \Phi^k(\nu_T^b))] \leqslant \frac{(C_F T)^k}{k!} \mathbb{E}[d_T^2(\nu_T^a, \nu_T^b)].$$

Following Banach's fixed point theorem we conclude that Φ has a unique fixed point. □

Subsequently, we show in the following proposition that MFE strategies coincide with the limit of optimal empirical measure in the weak sense.

Proposition 5.3.2. *Under Assumption 3, 12, and 16, the limits* $\nu_t = \lim_{N \to \infty} \nu_t^N$, $\mu_t = \lim_{N \to \infty} \mu_t^N$ *exist in the weak sense for* $t \in [0, T]$ *with respect to the 2-Wasserstein distance, where* $\nu^N \in \mathcal{P}^2(\mathcal{C}^A)$, $\mu^N \in \mathcal{P}^2(C([0,T]; \mathbb{R}_+))$.

Proof. Let wealth process (U^{ℓ}) be the solution of (5.25) with $\pi(\cdot)$ as closed loop Markovian dynamics $\phi^{\ell} : [0, T] \times \mathbb{R}_+ \to \mathbb{A}$,

$$dU^{\ell}(t) = U^{\ell}(t)\phi^{\ell}(t, U^{\ell}(t))\beta(\mathcal{X}_t, \nu_t)dt + U^{\ell}(t)\phi^{\ell}(t, U^{\ell}(t))\sigma(\mathcal{X}_t, \nu_t)dB_t, \quad U^{\ell}(0) = v^{\ell}, \tag{5.31}$$

or of open loop dynamics

$$dU^{\ell}(t) = U^{\ell}(t)\phi^{\ell}(v^{\ell}, B_{[0,T]})\beta(X_t, \nu_t)dt + U^{\ell}(t)\phi^{\ell}(v^{\ell}, B_{[0,T]})\sigma(X_t, \nu_t)dB_t, \quad U^{\ell}(0) = v^{\ell}. \tag{5.32}$$

for $\ell = 1, \ldots, N$. The initial states v^{ℓ} are i.i.d copies of v. The initial value of $U^{\ell}(0)$ is of the same law with $V^{\ell}(0)$ by Assumption 12.

$$\mathbb{E}[||(V^{\ell}, \phi^{\ell}(V^{\ell})) - (U^{\ell}, \phi^{\ell}(U^{\ell}))||_t^2] \leqslant C_F \mathbb{E}\Big[\int_0^t \mathcal{W}_2^2(\nu_r^N, \nu_r)dr\Big] \leqslant C_F \mathbb{E}\Big[\int_0^t d_r^2(\nu^n, \nu)dr\Big] \tag{5.33}$$

for $t \in [0, T]$, C_F is defined in Proposition 5.3.1. For simplicity, let us discuss in the case of closed loop dynamics, the result of which can be generalized to open loop dynamics.

We follow the coupling arguments in [15], the empirical measure of (V^{ℓ}, U^{ℓ}) is a coupling of the N-player empirical measure ν^N defined in Definition 2.3.1 and $\tilde{\nu}^N$, where $\tilde{\nu}^N$ are the empirical measure of N i.i.d samples U^{ℓ} in (5.31) or (5.32). Thus

$$d_t^2(\nu^N, \tilde{\nu}^N) \leqslant \frac{1}{N}\sum_{\ell=1}^{N} ||(V^{\ell}, \phi^{\ell}(V^{\ell})) - (U^{\ell}, \phi^{\ell}(U^{\ell}))||_t^2, \quad \text{a.s..} \tag{5.34}$$

By the triangle inequality and (5.33), (5.34),

$$\mathbb{E}[d_t^2(\nu^N, \nu)dr] \leqslant 2\mathbb{E}[d_t^2(\tilde{\nu}^N, \nu)] + 2C_F \mathbb{E}[\int_0^t d_r^2(\nu^N, \nu)dr],$$

and then by Gronwall's inequality and set $t = T$, it follows

$$\mathbb{E}[\mathcal{W}_2^2(\nu^N, \nu)] \leqslant 2e^{2C_F T}\mathbb{E}[\mathcal{W}_2^2(\tilde{\nu}^N, \nu)]. \tag{5.35}$$

Since (U^ℓ, π^ℓ), $\ell = 1, \ldots, N$ is independent given the noise B, use conditional law of large numbers (See Theorem 3.5 in [51]),

$$\mathbb{P}\big(\lim_{n\to\infty}\sum_{\ell=1}^{N} f(U^\ell, \pi^\ell) - \mathbb{E}[f(U^\ell, \pi^\ell)|\mathcal{F}^B]\big) = 1, \quad \text{for every} f \in C_b(\mathbb{R}^n)$$

We then apply Theorem 6.6 in [57], which gives that on a separable metric space, $\nu^N \to \nu$ weakly,

$$\lim_{N\to\infty}\int_{\mathbb{R}^N} d(x, x_0)^2\nu^N(dx) = \int_{\mathbb{R}^N} d(x, x_0)^2\nu(dx) \quad \text{a.s.},$$

which lead us to

$$\mathbb{E}[\mathcal{W}_2^2(\tilde{\nu}^N, \nu)] \to 0.$$

Therefore by using triangle inequality, it leads to $\mathbb{E}[\mathcal{W}_2^2(\nu^N, \nu)] \to 0$. We can use similar methods to derive $\mathbb{E}[\mathcal{W}_2^2(\mu^N, \mu)] \to 0$. □

Next we show the convergence of n-dimensional continuous stochastic process $\mathcal{X}^N(t)$.

Assumption 17. *On the probability space $(\Omega, \mathbb{F}, \mathbb{P})$, there exists positive constants δ such that the following conditions on β and σ:*

$$\int_s^t |x\beta_i(r, x, \nu)|dr \leqslant \eta(x, \nu)|t - s|^{\frac{1+\delta}{2}},$$

$$\int_s^t |x\sigma_{ij}^2(r, x, \nu)|dr \leqslant \xi(x, \nu)|t - s|^{\frac{1+\delta}{2}},$$

where $t, s \in [0, T]$, and η, ξ being $\mathcal{F}$-measurable functions with values in $(0, \infty) \times \mathcal{P}^2(\mathcal{C}^A)$

such that $\mathbb{E}[\eta(x,\nu)^2] < \infty$, $\mathbb{E}[\xi(x,\nu)^2] < \infty$.

Proposition 5.3.3. *If Assumption 12 and 17 holds, then there exists n dimensional continuous process $\mathcal{X}$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $\{\mathbb{P}^{\mathcal{X}^N}\}$ converges weakly to $\{\mathbb{P}^{\mathcal{X}}\}$ as $N \to \infty$, $\mathcal{X}(t) = \lim_{N\to\infty} \mathcal{X}^N(t)$ exists a.s. for all $t \in [0,T]$. $\mathcal{X}^N(t)$ is defined in (2.10).*

Proof. First we show that $\{\mathbb{P}^{\mathcal{X}^N}\}$ is tight. A sequence of measures μ^N on $\mathcal{P}^2(C([0,T];\mathbb{R}_+))$ is tight if and only if

(i) there exist positive constants M_x and γ such that $\mathbb{E}\{|\mathbf{x}^N|^\gamma\} \leqslant M_x$ for every $N = 1, 2, \ldots$,

(ii) there exist positive constants M_k and δ_1, δ_2 such that $\mathbb{E}\{|\mathcal{X}^N(t) - \mathcal{X}^N(s)|^{\delta_1}\} \leqslant M_k|t-s|^{1+\delta_2}$ for every N, $t, s \in [0,k]$, $k = 1, 2, \ldots$.

The proof of this can be found in [43].

With $\mathbf{x} \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^n_+)$ in Assumption 12, condition (i) in the above statement holds.

To check condition (ii), by Cauchy–Schwarz inequality,

$$\begin{aligned}
|\mathcal{X}^N(t) - \mathcal{X}^N(s)|^2 =& |X_1^N(t) - X_1^N(s)|^2 + \ldots + |X_n^N(t) - X_n^N(s)|^2 \\
=& \sum_{i=1}^n \Big| \int_s^t X_i^N(r)\beta_i(r)dr + \sum_{k=1}^n \int_s^t X_i^N(r)\sigma_{ik}(r)dW_k(r)\Big|^2 \\
\leqslant& \sum_{i=1}^n \bigg(\eta(\omega,\nu)^2|t-s|^{1+\delta} + \sum_{k=1}^n \Big|\int_s^t \sigma_{ik}(r)dW_k(r)\Big|^2\bigg).
\end{aligned}$$

Then for any constant δ in Assumption 17, by Itô's isometry,

$$\begin{aligned}
\mathbb{E}[|\mathcal{X}^N(t) - \mathcal{X}^N(s)|^2] \leqslant& (n+1)\sum_{i=1}^n \bigg(\mathbb{E}[\eta(\omega,\nu)^2]|t-s|^{1+\delta} + \mathbb{E}\Big[\int_s^t |\sigma_{ik}(r)|^2 dW_k(r)\Big]\bigg) \\
\leqslant& n(n+1)\big(\mathbb{E}[\eta(\omega,\nu)^2] + \mathbb{E}[\xi(\omega,\nu)^2]\big)|t-s|^{1+\delta},
\end{aligned}$$

where $\mathbb{E}[\eta(\omega,\nu)^2]+\mathbb{E}[\xi(\omega,\nu)^2]<\infty$. Thus condition (ii) follows.

By Prokhorov theorem [8], tightness implies relative compactness, which means here that each subsequence of $\{\mathcal{X}^N\}$ contains a further subsequence converging weakly on the space $C([0,T];\mathbb{R}^n_+)$. As a result, a subsequence exists such that $\mathcal{X}(t)=\lim_{N\to\infty}\mathcal{X}^N(t)$ a.s.. Then if every finite dimensional distribution of $\{\mathbb{P}^{\mathcal{X}^N}\}$ converges, then the limit of $\{\mathbb{P}^{\mathcal{X}^N}\}$ is unique and hence $\{\mathbb{P}^{\mathcal{X}^N}\}$ converges weakly to $\{\mathbb{P}^{\mathcal{X}}\}$ as $N\to\infty$. □

Proposition 5.3.4. *Under Assumption 3, 12, 16 and 17, The solution of Mckean-Vlasov system* (5.24)-(5.25) *is the weak limit of the solution of N-particle system* (2.10)-(2.11).

Proof. For (5.24) and (2.10), it is equivalent to show that the drift and volatility of $\mathcal{X}_t$ matches the weak limit of that of $\mathcal{X}^N(t)$, i.e.,

$$\int_0^t \beta(s,\mathcal{X}(s),\nu_s)ds=\lim_{N\to\infty}\int_0^t \beta(s,\mathcal{X}^N(s),\nu^N(s))ds, \tag{5.36}$$

$$\int_0^t \sigma(s,\mathcal{X}(s),\nu_s)ds=\lim_{N\to\infty}\int_0^t \sigma(s,\mathcal{X}^N(s),\nu^N(s))ds. \tag{5.37}$$

in the weak sense.

By Assumption 3, with the Lipschitz constant L,

$$\begin{aligned}&\Big|\Big|\int_0^t \big(\beta(s,\mathcal{X}^N_s,\nu^N_s)-\beta(s,\mathcal{X}_s,\nu_s)\big)ds\Big|\Big|^2_{L^2}\\ \leqslant&\int_0^t ||\beta(s,\mathcal{X}^N_s,\nu^N_s)-\beta(s,\mathcal{X}_s,\nu_s)||^2_{L^2}ds\\ \leqslant&L^2\Big(\int_0^t \mathbb{E}[|\mathcal{X}^N_s-\mathcal{X}_s|^2]ds+\int_0^t \mathbb{E}[\mathcal{W}_2(\nu^N_s,\nu_s)^2]ds\Big).\end{aligned}$$

By Itô's isometry, we have

$$\begin{aligned}
&\Big|\Big|\int_0^t \sigma(s,\mathcal{X}_s^N,\nu_s^N)dW_s - \int_0^t \sigma(s,\mathcal{X}_s,\nu_s)dB_s\Big|\Big|_{L^2}^2\\
\leqslant&\Big|\Big|\int_0^t \sigma(\mathcal{X}_s^N,\nu_s^N)dW_s - \int_0^t \sigma(\mathcal{X}_s,\nu_s)dW_s\Big|\Big|_{L^2}^2 + \Big|\Big|\int_0^t \sigma(\mathcal{X}_s,\nu_s)dW_s - \int_0^t \sigma(\mathcal{X}_s,\nu_s)dB_s\Big|\Big|_{L^2}^2\\
=&\mathbb{E}\Big[\int_0^t |\sigma(s,\mathcal{X}_s^N,\nu_s^N) - \sigma(s,\mathcal{X}_s,\nu_s)|^2 ds\Big]\\
\leqslant& L^2\mathbb{E}\Big[\int_0^t \big(|\mathcal{X}_s^N - \mathcal{X}_s|^2 + \mathcal{W}_2^2(\nu_s^N,\nu_s)\big)ds\Big].
\end{aligned}$$

From the results in Proposition 5.3.2 and 5.3.3, by Lebesgue dominated convergence theorem,

$$\lim_{N\to\infty}\Big|\Big|\int_0^t \big(\beta(s,\mathcal{X}_s^N,\nu_s^N) - \beta(s,\mathcal{X}_s,\nu_s)\big)ds\Big|\Big|_{L^2}^2 = 0,$$

$$\lim_{N\to\infty}\Big|\Big|\int_0^t \big(\sigma(s,\mathcal{X}_s^N,\nu_s^N) - \sigma(s,\mathcal{X}_s,\nu_s)\big)ds\Big|\Big|_{L^2}^2 = 0.$$

We conclude (5.36)-(5.37).

Hence we conclude that the limit of the finite particle system exists in the weak sense, and matches the solution of the Mckean-Vlasov SDEs system.

When $N \to \infty$, the limiting system is driven by X_t and $\nu_t := \text{Law}(V(t),\pi(t)|\mathcal{F}_t^B)$. The stock market follows

$$d\mathcal{X}_t = \mathcal{X}_t\beta(\mathcal{X}_t,\nu_t))dt + \mathcal{X}_t\sigma(\mathcal{X}_t,\nu_t)dB_t, \quad X_0 = \mathbf{x},$$

and a generic player's wealth is

$$dV(t) = \pi(t)\beta(\mathcal{X}_t,\nu_t))dt + \pi(t)\sigma(\mathcal{X}_t,\nu_t)dB_t, \quad V(0) = v_0. \tag{5.38}$$

π is fixed in (5.25) and (2.11). □

Remark 10. *It also follows that $\mathcal{Z}(t)$ is the weak limit of $\mathcal{Y}(t)$ in Proposition 5.3.4.*

With the notations in Definition 2.1.1 (3), if we consider the mean $\mathcal{Z}(t)$ of the measure $Law(V(t), \pi(t)|\mathcal{F}_t^B)$, we can get $\mathcal{Z}(t) = \lim_{N\to\infty} \mathcal{Y}(t)$ exists in the weak sense, and the limit $\mathcal{Z}(t)$ match the solution of the Mckean-Vlasov SDE

$$d\mathcal{Z}(t) = \gamma(\mathcal{X}_t, \mathcal{Z}(t))dt + \tau(\mathcal{X}_t, \mathcal{Z}(t))dB_t. \tag{5.39}$$

5.3.2 Approximate N-player Nash equilibrium and mean field equilibrium

In this section we justify if mean field game is an appropriate generalization of N-player relative arbitrage problem.

From the optimal strategy (5.15) derived in MFE and Definition 2.3.2. In mean field games with mean field interactions as the distributions μ_t and ν_t, for $i = 1, \ldots, n$, the players adopt

$$\pi_i^\star(t) = \phi^\star(\mathcal{X}_t, \mu_t, \nu_t). \tag{5.40}$$

Assumption 18. *Let the function of strategy $\pi^\ell(\cdot)$ be $\phi^\ell(\cdot) : C([0,T], \mathbb{R}_+^N) \times \mathcal{P}^2(C([0,T], \mathbb{R}_+^N \times \mathbb{A})) \to \mathbb{A}$. We assume ϕ^ℓ is Lipschitz , i.e.,*

$$|\phi^\ell(\mathbf{x}, \mu, \nu) - \phi^\ell(\widetilde{\mathbf{x}}, \widetilde{\mu}, \widetilde{\nu})| \leqslant L[|\mathbf{x} - \widetilde{\mathbf{x}}| + n\mathcal{W}_2(\mu, \widetilde{\mu}) + \mathcal{W}_2(\nu, \widetilde{\nu})]$$

for every $v, \widetilde{v} \in \mathbb{R}_+^N$, $\nu, \widetilde{\nu} \in \mathcal{P}^2(C([0,T], \mathbb{R}_+^N \times \mathbb{A}))$

We conclude in the following proposition that the MFE we obtain agrees with the limit of the finite equilibrium, and that the optimal arbitrage in the sense of (3.10) strongly converges to optimal arbitrage in the mean field game setting (5.8).

Proposition 5.3.5. *Under Assumption 3, 7, 12, 17, 18, and suppose the first exit time from the random intervals K_t and $\tilde{K}_t$ satisfies $\min\{\tau^K, \tilde{\tau}^K\} > T$ in (4.15) and (5.16).*

Then $u(T) = \lim_{N\to\infty} u^{\ell}(T)$ *a.s, for every* ℓ, $\ell = 1,\ldots,N$, *and* $T \in (0,\infty)$.

Proof.

We want to show that as $N \to \infty$, the Nash equilibrium of the N-player game in the system (2.10)-(2.11) converges to the mean field equilibrium in the system (5.24)-(5.25). The optimal strategy $\pi^{\ell\star}(\cdot)$ as functions $\phi^{\star}(\mathcal{X}_t, \mu_t, \nu_t)$ and $\phi^{\star}(\mathcal{X}^N_t, \mu^N_t, \nu^N_t)$ are used in Nash equilibrium of N-player game and mean field game, respectively. Thus, we look at the limit of the optimal cost in N-player game and the mean field optimal cost.

We get $\mathbb{P} \circ (\mathcal{X}^N, \mathbf{V}, \nu^N)$ is tight on the space $C([0,T];\mathbb{R}^n_+) \times C([0,T];\mathbb{R}^N_+) \times \mathcal{P}^2(\mathcal{C}^A)$ and the weak limit of $\mathbb{P} \circ (\mathcal{X}^N, \mathbf{V}, \nu^N)$ exists, following from Proposition 5.3.1 - 5.3.3. Note that the results in Proposition 5.3.1 - 5.3.3 under Assumption 16 can be generalized with Assumption 18 instead.

By using the Markovian property of $\pi(\cdot)$, and Assumption 7 on $\beta(\cdot)$ and $\sigma(\cdot)$, we have

$$u^{\ell}(T-t) = \frac{\mathbb{E}^{\mathbb{P}^{\mathcal{X}^N}}\big[e^{c_\ell}\mathcal{V}^N(T)L(T)|\mathcal{F}^W_t\big]}{\mathcal{V}^N(t)L(t)},$$

$$u(T-t) = \frac{\mathbb{E}^{\mathbb{P}}\big[e^{c}\mathcal{V}(T)L(T)|\mathcal{F}^B_t\big]}{\mathcal{V}(t)L(t)},$$

where $\mathcal{F}^{\mathcal{X}^N,\mathcal{Y}} = \mathcal{F}^W$, $\mathcal{F}^{\mathcal{X},\mathcal{Z}} = \mathcal{F}^B$, and

$$\mathcal{V}^N(t) = \delta\sum_{i=1}^{n} X^N_i(t) + (1-\delta)\sum_{i=1}^{n}\mathcal{Y}_i(t),$$

$$\mathcal{V}(t) = \delta\sum_{i=1}^{n} X_i(t) + (1-\delta)\sum_{i=1}^{n}\mathcal{Z}_i(t).$$

Given the current states of $\mathcal{X}^N(t)$, $\mathcal{Y}(t)$, $\mathcal{X}(t)$ and $\mathcal{Z}(t)$, then by the uniform integrability of $X(\cdot)$ and $V(\cdot)$, and Lebesgue dominated convergence theorem, we get the deflator $L(\mathcal{X}(t),\mathcal{Z}(t)) = \lim_{N\to\infty} L(\mathcal{X}^N(t),\mathcal{Y}(t))$ a.s., and $\mathcal{V}(T-t) = \lim_{N\to\infty}\mathcal{V}^N(T-t)$

in the weak sense for $t \in [0, T]$. c_ℓ is i.i.d samples from $Law(c)$ by Assumption 5.

Therefore as $N \to \infty$,

$$u^\ell(T) := \inf_{\pi \in \mathbb{A}} J^\ell(\pi^{\ell\star}) \to \inf_{\pi \in \mathbb{A}} J^{\mu,\nu}(\pi^\star) = u(T)$$

in the weak sense, and $u(T-t)$ is the weak limit of $u^\ell(T-t)$ when the trajectories of the current values are fixed. □

Next, we show here that MFE can be used to construct an approximate Nash equilibrium for the N-player game. Since we derive strong equilibrium in both N-player and mean field game, μ^N and μ are measurable with respect to the information generated by W and B, respectively.

Definition 5.3.2. *For $\epsilon_N \geqslant 0$, an open-loop ϵ_N-equilibrium is a tuple of admissible controls*

$$\phi^N := (\phi^{N,1}(t), \ldots, \phi^{N,N}(t))_{0 \leqslant t \leqslant T}, \quad \phi^{N,\ell}(t) \in \mathbb{A} \subset \Delta_n,$$

for every ℓ, such that

$$J^\ell(\phi^N) \leqslant \inf_{p \in \mathbb{A}} J^\ell(p, \phi^{N,-\ell}) + \epsilon_N,$$

where $p \in \mathbb{A}$ is an open loop control, and ϕ is of the form in (5.40).

An closed-loop ϵ_N-equilibrium is a tuple ϕ^N such that

$$J^\ell(\phi^N) \leqslant \inf_{p \in \mathbb{A}} J^\ell(p^N) + \epsilon_N,$$

where each component in ϕ^N is defined in (5.40); $p^N := (p(U_{[0,t]}), \phi^{N,-\ell}(U_{[0,t]}))$, in which U_t is the N-vector of wealth processes generated by this strategy, $p : [0,T] \times C([0,T]; \mathbb{R}_+^N) \to \mathbb{A}$ is of the form $(p(t, U_{[0,t]}))_{0 \leqslant t \leqslant T}$, $\phi^{N,-\ell}$ is defined in (5.40). For any $\ell = 1, \ldots, N$, $\phi^{N,\ell}$ and p are $\mathbb{F}$-progressively measurable functions.

To check if the strategies in mean field equilibrium can form an approximate open loop N-player Nash equilibrium, we consider N strategies that use the same $\phi^\star$ in the mean field case. A simplified situation is when μ_t, ν_t fixed from mean field equilibrium, then the corresponding strategies ϕ^ℓ for $\ell = 1, \dots, N$ in N-player game is

$$\phi^\ell = \phi^\star(\mathcal{X}_t, \mu_t, \nu_t). \tag{5.41}$$

A more realistic construction is to use μ_t^N or ν_t^N instead, that is,

$$\phi^\ell = \phi^\star(\mathcal{X}_t, \mu_t^N, \nu_t^N). \tag{5.42}$$

Although closed loop controls (5.40) are distributed, when considering players in the N-player game adopt the optimal strategy from the associated mean field game, the strategies in the form (5.41) or (5.42) are not distributed closed feedback form, $i = 1, \dots, n$. In fact (5.41) yields an open loop Nash equilibrium. Both (5.41) and (5.42) may depend on the past trajectory of common noises.

Therefore besides approximate open loop Nash equilibrium, to check if an approximate closed loop Nash equilibrium can be constructed from (5.40), we prove approximate Nash equilibrium result in "semi-closed loop form" introduced in [15].

We show the details of approximate Nash equilibrium in Proposition 5.3.6.

Proposition 5.3.6. *Under Assumption 3, and 18, there exists a sequence of positive real numbers $(\epsilon_N)_{N\geqslant 1}$ converging to 0, such that any admissible strategy $\pi^\ell = (\pi_t^\ell)_{t\in[0,T]}$ for the first player*

$$J^{N,\ell}(p^1, \pi^{2\star}, \dots, \pi^{N,\star}) \geqslant J - \epsilon_N, \quad \ell = 1, \dots, N.$$

Proof. This proposition is proved for both open and closed loop problems in the N-player game, adapted from the methods in [15] and [47]. Strategies in the form of (5.41) are easier to deal with than (5.42), and it only gives rise to the approximate open loop equilibrium. Therefore in this proof we focus on strategies (5.42).

- **Use MFE to approximate open loop Nash equilibrium.**

  Without loss of generality, by the symmetry of the game, we only need to focus on player 1. For a fixed number of players N, each player utilizes the optimal strategy $\pi^\star$ from (5.40) in the associated mean field game. Thus the actual strategy π^ℓ for player ℓ is (5.41) or (5.42), $\ell = 1, \ldots, N$.

  Let $\pi_t^{(N)} := (\pi^1(t), \ldots, \pi^N(t)) \in \mathbb{A}^N$. We want to show the cost functional of MFE is the limit of N-player game cost with $\pi^{(N)}$ and player 1 cannot be too better off in the approximate equilibrium sense when he/she deviates from $\pi^{(N)}$, in a fixed number N-player game, namely, $J^N(\pi^{(N)})$ is indeed an ϵ-Nash equilibrium with $\epsilon_N \to 0$ as $N \to \infty$, i.e.,

$$\lim_{N\to\infty} J^N(\pi^{(N)}) = J^\infty(\pi^\star) \quad \text{and} \quad J^N(\pi^{(N)}) \geqslant J^{N\star} - \epsilon_N \text{ with } \lim_{N\to\infty} \epsilon_N = 0,$$

  where $J^{N\star}$ is the optimal cost in N-player game, for a fixed N.

  Under $\pi^{(N)}$, the corresponding wealth follows

$$dV^{\ell\star}(t) = V^{\ell\star}(t)\pi_t^\star\big(\beta(X_t, \nu_t^{N\star})dt + \sigma(X_t, \nu_t^{N\star})dW_t\big), \quad \ell = 1, \ldots, N,$$

  and the joint empirical measure of the wealth processes and MFE control functions is

$$\nu^{N\star} = \frac{1}{N}\sum_{\ell=1}^{N} \delta_{(V^{\ell\star}(t), \pi^\star)}.$$

If $\pi^{(N)}$ deviates to (p, π^{-1}), the state processes are

$$dV^1(t) = V^1(t)p_t(\beta(X_t, \nu_t^N)dt + \sigma(X_t, \nu_t^N)dW_t),$$
$$dV^\ell(t) = V^\ell(t)\pi_t^\star(\beta(X_t, \nu_t^N)dt + \sigma(X_t, \nu_t^N)dW_t), \quad \ell \neq 1,$$

and the empirical measures are

$$\mu_t^N = \frac{1}{N}\left(\delta_{V^1(t)} + \sum_{\ell=2}^N \delta_{V^{\star\ell}(t)}\right), \quad \nu_t^N = \frac{1}{N}\left(\delta_{(V^1(t),p)} + \sum_{\ell=2}^N \delta_{(V^{\ell\star}(t),\pi^\star)}\right).$$

Let

$$\mu_t^{N-1} = \frac{1}{N-1}\sum_{\ell=2}^N \delta_{V^{\star\ell}(t)}, \quad \nu_t^{N-1} = \frac{1}{N-1}\sum_{\ell=2}^N \delta_{(V^{\ell\star}(t),\pi^\star)}.$$

We can see that for $\ell > 1$, the difference of $V^{\ell\star}(t)$ and $V^\ell(t)$ is solely on the measure ν_t^N and $\nu_t^{N\star}$.

By the definition of Nash equilibrium, with an arbitrary $p \in \mathbb{A}$, we trivially have

$$\epsilon_N := J_1^N(\pi^{(N)}) - \inf J_1^N(p, \pi^{-1}) \geqslant 0,$$

for some fixed ϵ_N. $J^\infty(\pi^\star) \geqslant J^\infty(p)$ by the definition of Mean Field Equilibrium. It suffices to show that as $N \to \infty$

$$J_1^N(p, \pi^{-1}) \to J^\infty(p) \tag{5.43}$$

uniformly in p, where $J^\infty(p)$ is subject to

$$dV_t^p = V_t^p p_t(\beta(X_t, \nu_t)dt + \sigma(X_t, \nu_t)dB_t).$$

From the expression of cost functional J in (4.3), we only need to show $(\mu_t^N, \nu_t^N) \to$

(μ_t, ν_t). Since $\nu_t^{N\star} \to \nu_t^\star$ as $N \to \infty$ in the weak sense with respect to $\mathcal{W}_2^2$ as proved in Proposition 5.3.2. By (5.35),

$$\mathbb{E}[\sup_{s\in[0,T]} \mathcal{W}_2^2(\nu_s^{N\star}, \nu_s^{N-1\star})] \leqslant \mathbb{E}[\mathcal{W}_2^2(\nu^{N\star}, \nu^{N-1\star})] \leqslant \frac{C}{N}, \tag{5.44}$$

where μ_t is the time-t marginal of $\mu \in \mathcal{P}^2(\mathcal{C}^A)$. C is a constant depend on C_F, T and ν.

Use (5.27),

$$\begin{aligned}
&\mathbb{E}[\sup_{s\in[0,T]} \mathcal{W}_2^2(\nu_s^{N,-1\star}, \nu_s^N)] \\
\leqslant&\mathbb{E}[\sup_{s\in[0,T]} \frac{1}{n-1}\sum_{\ell=2}^{N}\big(|V_s^{\star\ell} - V_s^\ell|^2 + |\pi_s^{\ell\star}(\mathcal{X}_s, \mu_s^{N,\star}, \nu_s^{N,\star}) - \pi_s^{\ell\star}(\mathcal{X}_s, \mu_s^N, \nu_s^N)|^2\big)] \\
\leqslant&\frac{C}{N} + \frac{C}{N-1}\sum_{\ell=2}^{N}\mathbb{E}\Big[\int_0^t \sup_{r\in[0,s]} \mathcal{W}_2^2(\nu_r^{N,-1\star}, \nu_r^N) ds\Big],
\end{aligned}$$

The constant C is different from the above value of C, but it does not depend on N. By Gronwall's inequality and (5.44), it follows

$$\mathbb{E}[\sup_{s\in[0,T]} \mathcal{W}_2^2(\nu_s^{N\star}, \nu_s^N)] \leqslant \frac{C}{N}.$$

We showed in Proposition 5.3.2, $\nu_t^{N\star}$ weakly converges to ν_t. Therefore we get $\nu_t^N \to \nu_t$ weakly, as $N \to \infty$. Thus $(\mu_t^N, \nu_t^N) \to (\mu_t, \nu_t)$ follows.

The objective we use here is a stochastic function of stochastic processes. We have $\mu_0^N \overset{d}{=} \mu_0$, $\mathbf{x}_0^N \overset{d}{=} \mathbf{x}$. We can find a subsequence $\{N_{(k)}\}$ such that

$$\lim_{N\to\infty} \int_0^T \sigma_i(X_t^N, \nu_t^N) dW_t = \int_0^T \sigma_i(X_t, \nu_t) dB_t$$

weakly through Itô's isometry and L_2 convergence. As a result, by (4.3) and (5.7),

$$\begin{aligned}&\lim_{N\to\infty} J_N(p,\pi^{-1})\\ =&\frac{\mathcal{V}(T)}{\mathcal{V}(0)}\exp\Big\{-\int_0^T p(t)'(\beta(X_t,\nu_t)-\frac{1}{2}\alpha(X_t,\nu_t)p(t))dt-\int_0^T p(t)'\sigma_i(X_t,\nu_t)dB_t\Big\}\\ =&J^{\mu,\nu}(p).\end{aligned}$$

- **Use MFE to approximate semi-closed loop Nash equilibrium:**

From the expression of cost functional J in (4.3), we need to show $(U_t^1,\mu_t^N,\nu_t^N)\to(V_t^p,\mu_t,\nu_t)$.

The distinctive characteristic for a closed loop control is that it depends on the wealth and the deviation of the strategy in turn influence all the investors' wealth. Hence when $\pi_t^{(N)}$ deviate to $(p_t,\pi_t^{-1}):=(p(U_t^1),\pi^\star(U_t^2),\ldots,\pi^\star(U_t^N))$, and the state processes become

$$\begin{aligned}dU_t^1&=U_t^1p(U_t^1)(b(X_t,\nu_t^N)dt+s(X_t,\nu_t^N)dW_t),\\ dU_t^\ell&=U_t^\ell\pi^\star(U_t^\ell)(b(X_t,\nu_t^N)dt+s(X_t,\nu_t^N)dW_t),\quad \ell\neq 1.\end{aligned}$$

and the empirical measure is

$$\nu_t^N=\frac{1}{N}\Big(\delta_{(U_t^1,p_t)}+\sum_{\ell=2}^N\delta_{(U_t^\ell,\pi_t^\star)}\Big),$$

while the state process with p_t in mean field game is

$$dV_t^p=V_t^pp_t(\beta(X_t,\nu_t)dt+\sigma(X_t,\nu_t)dB_t).$$

By (5.29), (5.30),

$$\mathbb{E}[\sup_{s\in[0,T]} \mathcal{W}_2^2(\nu_s^{N,-1\star}, \nu_s^N)] \leqslant \mathbb{E}\bigg[\sup_{s\in[0,T]} \frac{1}{N-1}\sum_{\ell=2}^{N}\Big(|V_s^{\star\ell} - U_s^\ell|^2 + |\pi_s^{\star\ell}(V_s^{\star\ell}, \mu_s^{N\star}, \nu_s^{N\star}) - \pi_s^{\ell\star}(U_s^\ell, \mu_s^N, \nu_s^N)|^2]\Big)\bigg]$$

$$\leqslant \frac{C}{N-1}\sum_{\ell=2}^{N}\mathbb{E}\bigg[\int_0^t \sup_{r\in[0,s]} \mathcal{W}_2^2(\nu_r^{N\star}, \nu_r^N)ds\bigg]$$

$$\leqslant \frac{C}{N} + C\mathbb{E}\bigg[\int_0^t \sup_{r\in[0,s]} \mathcal{W}_2^2(\nu_r^{N,-1\star}, \nu_r^N)ds\bigg],$$

it follows

$$\mathbb{E}[\sup_{s\in[0,T]} \mathcal{W}_2^2(\nu_s^{N,-1\star}, \nu_s^N)] \leqslant \frac{C}{N}.$$

The constant C does not depend on N.

Next we want to show $U_t^1 \to V_t^p$, since $p \in \mathbb{A}$ is bounded,

$$\begin{aligned}
&|X_s^U - X_s^V|^2 + |U_t^1 - V_t^p|^2 \\
\leqslant&(\int_0^t |X_s^U b(s,\nu_s^N) - X_s^V b(s,\nu_s)|ds)^2 + (\int_0^t |U_s^1 p_s b(s,\nu_s^N) - V_s^p p_s b(s,\nu_s)|ds)^2 \\
&+ (\int_0^t |U_s^1 p_s \sigma(s,\nu_s^N) - V_s^p p_s \sigma(s,\nu_s)|dW_t)^2 + (\int_0^t |X_s^U \sigma(s,\nu_s^N) - X_s^V \sigma(s,\nu_s)|dW_t)^2 \\
\leqslant& \int_0^t ||X_s^U b(s,\nu_s^N) - X_s^V b(s,\nu_s)||^2 + ||X_s^U \sigma(s,\nu_s^N) - X_s^V \sigma(s,\nu_s)||^2 \\
&+ ||U_s^1 p_s b(s, X_t^U, \nu_s^N) - V_s^p p_s b(s, X_t^V, \nu_s)||^2 + ||U_s^1 p_s \sigma(s,\nu_t^N) - V_s^p p_s \sigma(s,\nu_t)||^2 ds \\
\leqslant& C_0 L^2 \int_0^t \Big(|X_s^U - X_s^V|^2 + |U_t^1 - V_t^p|^2 + W_2^2(\nu_s^N, \nu_s)\Big)ds.
\end{aligned}$$

C_0 is a constant depending on the Lipschitz constant of coefficients, C_F and C above.

By Gronwall's inequality,

$$|U_t^1 - V_t^p|^2 \leqslant |X_s^U - X_s^V|^2 + |U_t^1 - V_t^p|^2 \leqslant W_2^2(\nu_s^N, \nu_s)$$

Therefore we conclude $(U_t^1, \mu_t^N, \nu_t^N) \to (V_t^p, \mu_t, \nu_t)$. Then $p(U_t^1, \mu_t^N, \nu_t^N) \to p(V_t^p, \mu_t, \nu_t)$.

$$\lim_{N \to \infty} J_N(p, \pi^{-1}) = J^{\mu,\nu}(p)$$

□

Thus we have the propagation of chaos for N-player games of relative arbitrage problems, and the corresponding mean field games can be used to approximate finite-player games. The last section justifies that the influence of each single player on the whole system is diminishing as N gets larger. Asymptotically we can consider a representative player and solve a single optimization problem instead.

Chapter 6

Functionally Generated Portfolios (FGP)

A versatile tool introduced in Stochastic portfolio theory is portfolio generating functions. This class of functions is usually smooth functions of market weights, which allows us to create portfolios with well-defined return characteristics and obtain probability-one constraints on the return relative to the market portfolio. In this chapter, we extend the current characterization of FGP to FGP in market models influenced by investors.

We start from a review of the original formulation of FGP. Portfolio generating functions can create well-performing portfolios that have little requirement on the estimation of the drifts or volatilities of the stocks.

$$\log\left(\frac{V^{\pi}(T)}{V^{\mathbf{m}}(T)}\right) = \log\left(\frac{\mathbf{G}(\mathbf{m}(T))}{\mathbf{G}(\mathbf{m}(0))}\right) + \int_0^T \mathbf{g}(t)dt, \quad T \in [0, \infty),$$

with

$$\mathbf{g}(t) := -\frac{1}{2\mathbf{G}(\mathbf{m}(t))} \sum_{i=1}^{m} \sum_{j=1}^{m} D^2_{ij}\mathbf{G}(\mathbf{m}(t)) \cdot \mathbf{m}_i(t)\mathbf{m}_j(t)\psi^{\mathbf{m}}_{ij}(t),$$

where $\psi^{\mathbf{m}}(\cdot)$ is defined by $\tau^{\mathbf{m}}(\cdot)$ in the dynamics (1.1),

$$\psi_{ij}^{\mathbf{m}}(t) = \sum_{k=1}^{K} \tau_{ik}^{\mathbf{m}}(t)\tau_{jk}^{\mathbf{m}}(t).$$

This formulation if based on the fact that with a strategy π, $\log V^{\pi}(t)$ depends solely on $\mathbf{m}(0)$, $\mathbf{m}(t)$ and a finite variation process related to time-aggregated market volatility. Using this almost sure pathwise decomposition formula, it was able to formulate conditions under which the portfolio outperforms the market portfolio with probability 1 for all sufficiently long horizons. With appropriate generating functions, this allows us to obtain probability-one constraints on the relative return.

[25] proved the characterization of the portfolios that are functionally generated. [56], [71] shows essentially that no other portfolio functions, other than those that are functionally generated, can beat the market in the long run without additional assumptions in discrete time.

6.1 FGP in N-player market

In this section, we construct portfolio generating functions to create portfolios with relative arbitrage. The idea is that deterministic functions defined on Δ^n can be used to generate portfolios, and in this way, we will be able to get information on the behavior of portfolios.

Since we use the benchmark $\mathcal{V}(t)$, we first define the relative return of a stock versus the weighted average of market portfolio and sum of investors' portfolio.

Definition 6.1.1. *(Relative return) For portfolios π, the relative return process of π*

versus a benchmark $\eta(t)$ is defined by

$$\log \frac{V^{\pi}(t)}{\eta(t)}$$

A benchmark that is commonly used is the market portfolio. Based on definition 3.1.1, with the weighted average of market portfolio and average portfolio as a benchmark, we develop the following definitions which use certain real-valued functions of the market weights and functions of the strategies of investors to generate portfolios. The goal is to find conditions so that the wealth process of investors ℓ dominates a big proportion of the weighted average of the market and the investors.

Therefore, by the similar notation in definition 6.1.1, we look for a decomposition of the relative return of π^{ℓ} versus $e^{c_{\ell}}\mathcal{V}(t)$ as the sum of generating functions of μ and π, and a drift process.

Recall that the market portfolio follows the dynamic (1.1), we rewrite it as follows and define $\tau^{m}(t)$ and $\gamma_i^{m}(t)$ for future use.

$$d\mathbf{m}_i(t) = \mathbf{m}_i(t)\left[\gamma_i^{m} dt + \sum_{k=1}^{n} \tau_{ik}^{m}(t) dW_k(t)\right], \quad i = 1, \ldots, n. \tag{6.1}$$

Here $\tau^{m}(t)$ is the matrix with entries $\tau_{ik}^{m}(t) := \sigma_{ik}(t) - \sum_{j=1}^{n} \mathbf{m}_j(t)\sigma_{jk}(t)$, $\mathbf{e}_i$ is the ith unit vector in $\mathbb{R}^n$ and the vector $\gamma^{m}(t)$ is with the entries $\gamma_i^{m}(t) := (\mathbf{e}_i - \mathbf{m}(t))'(\beta(t) - \alpha(t)\mathbf{m}(t))$.

Let I be the identity matrix of size n, and $\mathbf{1}$ be the n-dimension column of ones.

Theorem 6.1.1. *Let $\mathbf{G}_1^{\ell}$, $\mathbf{G}_2^{\ell} : U \to (0,\infty)$ be positive C^2 functions defined on a neighborhood U of $\mathbb{A}$ such that for all i, $x_i D_i \log \mathbf{G}_1^{\ell}(x)$, $x_i D_i \log \mathbf{G}_2^{\ell}(x)$ are bounded on $\mathbb{A}$. For $t \in [0,T]$, $\mathbf{G}_1^{\ell}$, $\mathbf{G}_2^{\ell}$ generate the portfolio*

$$\pi^{\ell}(t) = \widetilde{G}_1^{\ell}(t) + \widetilde{G}_2^{\ell}(t) + \mathcal{R}(t) \tag{6.2}$$

where

$$\widetilde{G}_1^\ell(t) = (D_i \log \mathbf{G}_1^\ell(\mathbf{m}(t))\mathbf{m}_i(t))_n(I - \mathbf{1}\mathbf{m}(t));\ \widetilde{G}_2^\ell(t) = D \log \mathbf{G}_2^\ell(\mathcal{Y}(t))\tau(t)\sigma^{-1}(t);$$

$$\mathcal{R}(t) = \frac{\delta\mathcal{X}^N(t) + (1-\delta)\mathcal{Y}(t)}{\mathcal{V}^N(t)}.$$

The process

$$d\log\frac{V^\ell(t)}{e^{c_\ell}\mathcal{V}^N(t)} = d\log\mathbf{G}_2(\mathcal{Y}(t)) + d\log\mathbf{G}_1(\mathbf{m}(t)) + d\Xi_t, t \in [0,T], a.s. \tag{6.3}$$

is with a drift process $\Xi(\cdot)$ *such that a.s., for* $t \in [0,T]$,

$$\begin{aligned}\frac{d\Xi^\ell(t)}{dt} =& \widetilde{G}_1^\ell(t)\alpha(t)\mathbf{m}(t) + \widetilde{G}_2^\ell(t)\alpha(t)\pi^\ell(t) - \frac{1}{2}\Big(||\widetilde{G}_1^\ell(t)\sigma||^2 + ||\widetilde{G}_2^\ell(t)\sigma||^2 - ||\pi^{\ell\prime}\sigma||^2\Big)\\ &+ \frac{1}{2\mathbf{G}_1^\ell(\mathbf{m}(t))}\sum_{i,j=1}^n D_{ij}\mathbf{G}_1^\ell(\mathbf{m}(t))\mathbf{m}_i(t)\mathbf{m}_j(t)(\sum_{k=1}^n \tau_{ik}^m(t)\tau_{jk}^m(t))\\ &+ \frac{1}{2\mathbf{G}_2^\ell(\mathcal{Y}(t))}\sum_{i,j=1}^n D_{ij}\mathbf{G}_2^\ell(\mathcal{Y}(t))\psi_{ij}(t).\end{aligned} \tag{6.4}$$

We denote $D\log\mathbf{G}_2^\ell(\mathcal{Y}(t))$ *as the row vector* $(D_i\log\mathbf{G}_2^\ell(\mathcal{Y}(t)))_n$.

Proof. By Ito's lemma,

$$\begin{aligned}d\log\frac{V^\ell(t)}{e^{c_\ell}\mathcal{V}^N(t)} =& \left(\pi^{\ell\prime}(t)\left(\beta(t) - \frac{\alpha(t)}{2}\pi^\ell(t)\right) - \mathcal{R}'(t)\beta(t) + \frac{1}{2}||\mathcal{R}'(t)\sigma(t)||^2\right)dt\\ &+ \big(\pi^{\ell\prime}(t) - \mathcal{R}'(t)\big)\sigma(t)dW(t).\end{aligned} \tag{6.5}$$

Since $\mathbf{G}_1$ and $\mathbf{G}_2$ are twice continuously differentiable function, it follows

$$\begin{aligned}D_{ij}\log\mathbf{G}_1^\ell(\mathbf{m}(t)) &= \frac{D_{ij}\mathbf{G}_1^\ell(\mathbf{m}(t))}{\mathbf{G}_1^\ell(\mathbf{m}(t))} - D_i\log\mathbf{G}_1^\ell(\mathbf{m}(t))D_j\log\mathbf{G}_1^\ell(\mathbf{m}(t)),\\ D_{ij}\log\mathbf{G}_2^\ell(\mathcal{Y}(t)) &= \frac{D_{ij}\mathbf{G}_2^\ell(\mathcal{Y}(t))}{\mathbf{G}_2^\ell(\mathcal{Y}(t))} - D_i\log\mathbf{G}_2^\ell(\mathcal{Y}(t))D_j\log\mathbf{G}_2^\ell(\mathcal{Y}(t))\end{aligned} \tag{6.6}$$

Then using (6.6) and Ito's lemma,

$$
\begin{aligned}
&d\log \mathbf{G}_1^\ell(\mathbf{m}(t)) + d\log \mathbf{G}_2^\ell(\mathcal{Y}(t))\\
&= \sum_{i=1}^n D_i \log \mathbf{G}_1^\ell(\mathbf{m}(t)) d\mathbf{m}_i(t)\\
&+ \frac{1}{2\mathbf{G}_1^\ell(\mathbf{m}(t))} \sum_{i,j=1}^n D_{ij}\mathbf{G}_1^\ell(\mathbf{m}(t))\mathbf{m}_i(t)\mathbf{m}_j(t)(\sum_{k=1}^n \tau_{ik}^m(t)\tau_{jk}^m(t))dt\\
&- \frac{1}{2}\sum_{i,j=1}^n D_i \log \mathbf{G}_1^\ell(\mathbf{m}(t)) D_j \log \mathbf{G}_1^\ell(\mathbf{m}(t))\mathbf{m}_i(t)\mathbf{m}_j(t)(\sum_{k=1}^n \tau_{ik}^m(t)\tau_{jk}^m(t))dt\\
&+ \sum_{i=1}^n D_i \log \mathbf{G}_2^\ell(\mathcal{Y}(t)) d\mathcal{Y}_i(t)\\
&+ \frac{1}{2\mathbf{G}_2^\ell(\mathcal{Y}(t))} \sum_{i,j=1}^n D_{ij}\mathbf{G}_2^\ell(\mathcal{Y}(t))\psi_{ij}(t)dt\\
&- \frac{1}{2}\sum_{i,j=1}^n D_i \log \mathbf{G}_2^\ell(\mathcal{Y}(t)) D_j \log \mathbf{G}_2^\ell(\mathcal{Y}(t))\psi_{ij}(t)dt,
\end{aligned}
\tag{6.7}
$$

The local martingale part of (6.5) and (6.7) are the same, and this leads to

$$
\pi^{\ell\prime}(t) = \big[(D_i \log \mathbf{G}_1^\ell(\mathbf{m}(t))\mathbf{m}_i(t))_n'(I - \mathbf{1}m'(t)) + D\log \mathbf{G}_2^\ell(\mathcal{Y}(t))\tau(t)\sigma^{-1}(t)\big] + \mathcal{R}(t),
$$

for $t \in [0, T]$, and for each ℓ. Substitute this result into (6.5),

$$
\begin{aligned}
d\log\frac{V^\ell}{e^{c_\ell}\mathcal{V}(t)} =& d\log \mathbf{G}_2^\ell(\mathcal{Y}(t)) + d\log \mathbf{G}_1^\ell(\mathbf{m}(t))\\
&- \Big\{\widetilde{G}_1^\ell(t)(-\alpha(t)\mathbf{m}(t)) + \widetilde{G}_2^\ell(t)(-\alpha(t)\pi^\ell(t))\\
&+ \frac{1}{2}\big(||\widetilde{G}_1^\ell(t)\sigma||^2 + ||\widetilde{G}_2^\ell(t)\sigma||^2 - ||\pi^{\ell'}\sigma||^2\big)\\
&- \frac{1}{2\mathbf{G}_1^\ell(\mathbf{m}(t))}\sum_{i,j=1}^n D_{ij}\mathbf{G}_1^\ell(\mathbf{m}(t))\mathbf{m}_i(t)\mathbf{m}_j(t)(\sum_{k=1}^n \tau_{ik}^m(t)\tau_{jk}^m(t))\\
&- \frac{1}{2\mathbf{G}_2^\ell(\mathcal{Y}(t))}\sum_{i,j=1}^n D_{ij}\mathbf{G}_2^\ell(\mathcal{Y}(t))\psi_{ij}(t)\Big\}dt.
\end{aligned}
$$

Thus we conclude (6.2) and (6.3). □

6.2 Optimal arbitrage strategies and equilibrium using FGP

We characterize the strategies (4.14) achieved in Nash equilibrium using functionally generated portfolios. FGP methodology could be easier to use than the dynamics of the portfolios since there is randomness in the model, which is difficult for analysis.

In (4.14), the strategy is generated by a function of market portfolio and average trading volume,

$$\pi^{\ell\prime}(t) = \big[(\frac{\partial}{\partial \mathbf{m}_i} \log \mathbf{u}^\ell(\mathbf{m},\mathbf{y})\mathbf{m}_i)'_n(I - \mathbf{1}m'(t)) + \frac{\partial}{\partial \mathbf{y}} \log \mathbf{u}^\ell(\mathbf{m},\mathbf{y})\tau(t)\sigma^{-1}(t)\big] + \mathcal{R}(t). \quad (6.8)$$

More importantly, the notion of optimal strategies (4.14) can be treated through Theorem 6.1.1. Let $\mathbf{G}_1^\ell,\ \mathbf{G}_2^\ell : U \to (0,\infty)$ be positive C^2 functions defined on a neighborhood U of $\mathbb{A}$ such that for all i, $x_i D_i \log \mathbf{G}_1^\ell(x)$, $x_i D_i \log \mathbf{G}_2^\ell(x)$ are bounded on $\mathbb{A}$. We write $\tilde{u}^\ell(t,\mathbf{x},\mathbf{y}) = \mathbf{u}^\ell(t,\mathbf{m},\mathcal{Y})$, then by taking the derivatives of $\mathcal{X}^N(t)$, $\mathcal{Y}(t)$, it follows

$$X_i^N(t) D_i \log \tilde{u}^\ell(T-t, \mathcal{X}^N(t), \mathcal{Y}(t)) = \left[D_i \log \mathbf{G}_1^\ell(\mathbf{m}(t)) - \sum_{i=1}^n D_i \log \mathbf{G}_1^\ell(\mathbf{m}(t))\mathbf{m}_i(t) \right] \mathbf{m}_i(t);$$

$$\sum_{j=1}^n (\tau\sigma^{-1})_{ji}(\mathbf{x}^N, \mathbf{y}) D_{y_j} \log \tilde{u}^\ell(T-t, \mathcal{X}^N(t), \mathcal{Y}(t)) = (D_j \log \mathbf{G}_2^\ell(\mathcal{Y}(t)))_n \tau(t)\sigma^{-1}(t)\mathbf{e}'_i.$$

When we express $\tau(\cdot)$ explicitly using the methodology in Section 3.3.2, we can simplify the above to a functional characterization without the coefficients $\tau(\cdot)$ and $\sigma(\cdot)$. For example, if open loop strategies (4.4) is used, then by the dynamics of $\mathcal{Y}$ in (3.25), we have

$$\sum_{j=1}^{n}(\tau\sigma^{-1})_{ji}(\mathbf{x}^N,\mathbf{y})D_{y_j}\log\tilde{u}^{\ell}(T-t,\mathcal{X}^N(t),\mathcal{Y}(t)) = \frac{1}{N}(D_j\log\mathbf{G}_2^{\ell}(\mathcal{Y}(t)))_n\sum_{\ell=1}^{N}\pi_i^{\ell}(t)V^{\ell}(t)\pi^{\ell\prime}(t).$$

Furthermore, we can use portfolio generating functions to find conditions on investment strategies by $\sum_{i=1}^{n}\pi_i(t) = 1$, $t \in [0,T]$. We get

$$\frac{1-\delta}{N\delta X^N(t)}\sum_{\ell=1}^{N}V_t^{\ell}w_t^{\ell} = w_t^{\ell},$$

where $w_t^{\ell} := X_i(t)D_i\log\tilde{u}^{\ell}(t) + \tau_i(t)\sigma^{-1}(t)D_{p_i}\log\tilde{u}^{\ell}(t)$. Hence $\sum_{\ell=1}^{N}V^{\ell}(t)w^{\ell}(t) = 0$ or $\delta X^N(t) = (1-\delta)\frac{1}{N}\sum_{\ell=1}^{N}V^{\ell}(t)$. The latter indicates that the market consists of the N investors we considered. If $w^{\ell}(t) = 0$, then every investor is the same, and their strategy follows the market portfolio. If $w^{\ell}(t) \neq 0$, then $\mathbf{1}'\widetilde{G}_2(t) = 0$, and

$$\sum_{j=1}^{n}\sum_{i=1}^{n}D_i\log\mathbf{G}_2(t)(\tau(t)\sigma^{-1}(t))_{ji} = 0. \tag{6.9}$$

Remark 11. (6.9) *comes from the fact that* $\mathbf{1}'\widetilde{F}(t) = 0$ *and* $\mathbf{1}'\widetilde{R}(t) = 1$.

In general, the function $G(\cdot)$ *is chosen so that*

$$\mathbf{1}'\widetilde{G}(t) = -n \cdot dc_{\ell}(t).$$

We will show later that c_{ℓ} *has to be a constant or a stochastic process.* (6.2) *and* (6.4) *only depends on the volatility structure of* $X_i(t)$ *and* $\Pi_i(t)$.

Remark 12. *[26] discuss the case when the benchmark is the market portfolio. It constructs generating functions that allow us to obtain portfolios with well defined relative returns with respect to the market. Following the same conditions as above theorem,* F

generates the portfolio π *with weights*

$$\pi(t) = (D_i \log \mathbf{F}(\mathbf{m}(t))\mathbf{m}_i(t))'_n - [(D \log \mathbf{F}(\mathbf{m}(t)))'\mathbf{m}(t) + \mathbf{1}']\mathbf{m}(t) \tag{6.10}$$

From this functionally generated aspect, we can use the portfolio representation (6.8) to analyze relative arbitrage in a more specific problem with relaxed constraint: An investor expects to reach the goal of relative arbitrage by a specific terminal time, as opposed to every fixed $T \in (0, \infty)$ in previous chapters.

6.3 Applications

Portfolio generating functions can create portfolios with desirable well-defined return characteristics.

We showed in Section 6.2 the arbitrage opportunities in terms of portfolio generated functions over time horizon $[0, T]$ for any $T > 0$. In this section we use an example to demonstrate another type of relative arbitrage - long term relative arbitrage. This type of goal provides more flexibility when setting up a model and relaxes some of the assumptions on stock dynamics. For example, certain forms of drifts and volatilities are required for the Fichera drift argument in Section 3.4.

Now we want to show $\mathcal{M}$ contains strong arbitrage opportunities relative to the performance benchmark, at least for sufficiently large real numbers $T > 0$. We illustrate this path by example 6.3.1. We employ the idea of functionally generated portfolios [26] to seek optimal strategies. By doing so, we may reduce the intractability of the N-player game problem.

If the model $\mathcal{M}$ of (2.1), (2.2) is weakly diverse over the time-horizon $[0, T]$, and if strong non-degeneracy condition holds, then $\mathcal{M}$ contains strong arbitrage opportunities relative to the market portfolio, at least for sufficiently large real numbers $T > 0$. Denote $\mathbf{e}_i$ the ith unit vector in $\mathbb{R}^n$.

Example 6.3.1. *Suppose that $\mathcal{M}$ is nondegenerate, weakly diverse in $[0, T]$, and has bounded variance, see Definition 1.1.1-1.1.2. We assume for $t \in [0, T]$, there exists constants $c_0, N_c, M_\pi > 0$ such that*

$$\widehat{V}^\ell(t)/\widehat{V}^\ell(0) \geqslant c_0 X^N(t) L(t);$$

$$\mathcal{Y}_i(t)/\mathbf{m}_i(t) \leqslant N_c, \quad i = 1, \ldots, n;$$

$$|\sum_{i=1}^{n} \gamma_i(t)| \leqslant M_\pi.$$

Consider the function $\mathbf{G}_1$ and $\mathbf{G}_2$ are defined by

$$\mathbf{G}_1(x) = \prod_{i=1}^{n} x_i, \quad \mathbf{G}_2(x) = 1 - \frac{1}{2}\sum_{i=1}^{n} x_i^2.$$

$\mathbf{G}_1$ and $\mathbf{G}_2$ generate the portfolio

$$\pi_i^\ell(t) = 1 - \left(n + \frac{\delta X^N(t)}{\mathcal{V}^N(t)}\right)\mathbf{m}_i(t) + \frac{(1-\delta)\mathcal{Y}_i(t)}{\mathcal{V}^N(t)} + \left(\frac{-\mathcal{Y}(t)}{\mathbf{G}_2(\mathcal{Y}(t))}\right)' \tau(t)\sigma^{-1}(t)\mathbf{e}_i, \quad i = 1, \ldots, n. \tag{6.11}$$

Then π^ℓ strictly dominates $\mathcal{V}^N(t)$ in (3.3) if

$$T \geqslant \frac{nN^2 - 2n^2 - 2}{-2\epsilon n + 2M_\pi n^2 - \frac{2n^2 M_0}{1-\frac{N^2}{2n}} + Mn^2\left(n(n-1) + \frac{N^2}{1-\frac{N^2}{2}}\frac{\lambda^2_{\max}(\tau)}{\lambda^2_{\min}(\sigma^{-1})}\right)}.$$

The notations of constants and details of the proof can be found in Appendix 6.1.

Lemma 6.3.1. *A matrix A is semi-definite if and only if $(xAy')^2 \leqslant (xAx')(yAy')$ for all x, y in $\mathbb{R}^n$. The equality holds if and only if xA and yA are linearly dependent.*

Lemma 6.3.2. *If $A = (a_{ij})$ is a positive semi-definite matrix, then there is an index k such that $a_{kk} \geqslant a_{ij}$, for any i and j. In other words, the largest entry of the matrix A appears on the diagonal.*

We show here the derivation in Example 6.3.1.

Proof of Example 6.3.1. Let $\mathcal{M}$ be a market without dividends. Suppose that $\mathcal{M}$ is nondegerate and has bounded variance. Suppose $\mathcal{M}$ is weakly diverse in [0,T]. Consider the function $\mathbf{G}_1$ and $\mathbf{G}_2$ are defined as in example 6.3.1.

$\sqrt{\prod_{i=1}^n \mathbf{m}_i} \leqslant \frac{\sum_{i=1}^n \mathbf{m}_i}{n} \leqslant \sqrt{\frac{\sum_{i=1}^n \mathbf{m}_i^2}{n}}$ implies that

$$0 \leqslant \mathbf{G}_1(\mathbf{m}) \leqslant \frac{1}{n^2}, \quad 1 - \frac{N^2}{2} \leqslant \mathbf{G}_2(\mathcal{Y}(t)) \leqslant 1 - \frac{N^2}{2n}$$

then

$$1 - \frac{N^2}{2} \leqslant \log \mathbf{G}_1(\mathbf{m}) + \log \mathbf{G}_2(\mathcal{Y}(t)) \leqslant \frac{1}{n^2} + 1 - \frac{N^2}{2n}.$$

The portfolio (6.11) generated by $\mathbf{G}_1$ and $\mathbf{G}_2$ implies

$$\pi_i^\ell > \max\{0, 1 - (n + \delta X^N(t)/\mathcal{V}^N(t))(1-\eta) + \left(\frac{-\mathcal{Y}(t)}{\mathbf{G}_2(\mathcal{Y}(t))}\right)' \tau(t)\sigma^{-1}(t)\mathbf{e}_i\}; \tag{6.12}$$

$$\pi_i^\ell < \min\{1 + \frac{1-\delta}{N}(V^\ell(t))'_N(\pi_i^\ell(t))_N/\mathcal{V}^N(t) + \left(\frac{-\mathcal{Y}(t)}{\mathbf{G}_2(\mathcal{Y}(t))}\right)' \tau(t)\sigma^{-1}(t)\mathbf{e}_i, 1\}. \tag{6.13}$$

Denote $\max_{i=1,\dots,n} \mathbf{m}_i = \mathbf{m}_{(1)}$, $\min_{i=1,\dots,n} \mathbf{m}_i = \mathbf{m}_{(n)}$, $\max_{i=1,\dots,n} \pi_i = \pi_{(1)}$, $\min_{i=1,\dots,n} \pi_i = \pi_{(n)}$, and the eigenvalues of $\alpha(t)$: $\max_{i=1,\dots,n} \lambda_i = \lambda_{(1)}$, $\min_{i=1,\dots,n} \lambda_i = \lambda_{(n)}$. $\mathbf{m}_{\max} := (\mathbf{m}_{(1)}, \mathbf{m}_{(1)}, \dots, \mathbf{m}_{(1)})$.

We'll use the following results to simplify $\Xi(T)$:

(i) $\mathcal{M}$ is nondegenerate, weakly diverse and has bounded variation;

(ii) $\frac{1}{n} \leqslant \sum_{i=1}^n \mathbf{m}_i^2 \leqslant 1$ implies that $0 \leqslant ||(\mathbf{1} - n\mathbf{m})|| \leqslant \sqrt{n(n-1)}$;

$\sum_{i=1}^{n}(\pi_i^\ell)^2 \leqslant 1$ implies $||\sum_{\ell=1}^{N}\pi^{\ell\prime}(t)\tau\sigma^{-1}||_2 \leqslant ||\sum_{\ell=1}^{N}\pi^{\ell\prime}(t)||_2 \cdot ||\tau||_2 \cdot ||\sigma^{-1}||_2 \leqslant N\sqrt{\frac{\lambda_{max}(\psi)}{\lambda_{min}(\alpha)}}$, where the norm for τ and σ^{-1} is matrix induced norm. For a matrix $A \in \mathbb{R}^{m\times n}$, $\sqrt{\text{Trace}(AA')} = ||A||_F \leqslant \sqrt{n}||A||_2$, where $||\cdot||_2$ is the matrix induced norm. $\text{Trace}(\tau\tau^{\pi\prime}) = \sum_{i=1}^{n}\sum_{\ell=1}^{N}\tau_{ii}^\ell \geqslant n\epsilon\sum_{\ell=1}^{N}(1-\pi_{(1)}^\ell)^2$, then $||\tau||_2 \geqslant \epsilon\sum_{\ell=1}^{N}(1-\pi_{(1)}^\ell)^2$;

(iii) $|\beta_i|$ and $|\alpha_{ij}|$ for any i and j is bounded by lemma 6.3.2, thus we could easily get $\mathcal{Y}(t)\tau(t)\sigma^{-1}(t)\beta(t) > M_0$; By lemma 6.3.1, $\mathbf{e}_i'\alpha(t)\mathbf{m}(t) \leqslant (\mathbf{e}_i'\alpha(t)\mathbf{e}_i)(\mathbf{m}'(t)\alpha(t)\mathbf{m}(t)) \leqslant MM'||\mathbf{m}(t)||^2 \leqslant MM'$, where $\mathbf{e}_i'\alpha(t)\mathbf{e}_i \leqslant M||\mathbf{e}_i||^2$, $\mathbf{m}'(t)\alpha(t)\mathbf{m}(t) \leqslant M'||\mathbf{m}(t)||^2$.

$$
\begin{aligned}
\Xi(T) &= \int_0^T \left\{(\mathbf{e}_i - \mathbf{m}(t))'\alpha(t)\mathbf{m}(t) + \frac{-\mathcal{Y}(t)}{\mathbf{G}_2(\mathcal{Y}(t))}\big(\gamma^\pi(t) - \tau\sigma^{-1}(t)\beta(t)\big) + d^\ell(t)\frac{\alpha(t)}{2}d^{\ell\prime}(t)\right\}dt \\
&\overset{(i)}{\leqslant} \int_0^T \left\{\mathbf{e}_i'\alpha(t)\mathbf{m}_{(1)}(t)\mathbf{1} - \epsilon||\mathbf{m}||^2 + \frac{1}{\mathbf{G}_2(\mathcal{Y}(t))}\sum_{\ell=1}^{N}\pi^{m\prime}(t)\big(\gamma^\pi(t) - \tau\sigma^{-1}(t)\beta(t)\big)\right. \\
&\qquad \left. + \frac{M}{2}\Big[||\mathbf{1}-n\mathbf{m}||^2 + \frac{1}{\mathbf{G}_2(\mathcal{Y}(t))}||\sum_{\ell=1}^{N}\pi^{\ell\prime}(t)\tau^\ell\sigma^{-1}||^2\Big] - \frac{\epsilon}{2}||\pi^\ell||^2\right\}dt \\
&\overset{(ii,iii)}{\leqslant} T\Big[MM' - \frac{\epsilon}{n} + \frac{M_\pi}{1-\frac{N^2}{2}} - \frac{M_0}{1-\frac{N^2}{2n}} + \frac{M}{2}\Big(n(n-1) + \frac{N^2}{(1-\frac{N^2}{2})^2}\frac{\lambda_{(1)}^2(\tau)}{\lambda_{(n)}^2(\sigma^{-1})}\Big)\Big] \\
&\quad - \frac{\epsilon}{2}\int_0^T \max_i |\pi_i^\ell|^2 dt
\end{aligned}
$$

where

$$
d^\ell(t) := \mathbf{1} - n\mathbf{m} - \pi^\ell(t) + \frac{-\mathcal{Y}(t)}{\mathbf{G}_2(\mathcal{Y}(t))}\tau\sigma^{-1},
$$

$$
\max_i |\pi_i^\ell|^2 > \Big[\max\{0, 1 - (n + c_\ell\delta\frac{X^N(t)}{\mathcal{V}(t)})(1-\eta) - \frac{\mathcal{Y}_i(t)}{\mathbf{G}_2(\mathcal{Y}(t))}\tau(t)\sigma^{-1}(t)\mathbf{e}_i\}\Big]^2.
$$

Hence, for $t \in [0, T]$,

$$\begin{aligned}\log \frac{V^\ell}{e^{c_\ell}\mathcal{V}^N(t)} &= \log \mathbf{G}_2(\mathcal{Y}(t)) + \log \mathbf{G}_1(\mathbf{m}(t)) + \Xi_t \\ &\leqslant 1 + \frac{1}{n^2} - \frac{N^2}{2n} + T\Big[MM' - \frac{\epsilon}{n} + \frac{M_\pi}{1-\frac{N^2}{2}} - \frac{M_0}{1-\frac{N^2}{2n}} \\ &\qquad + \frac{M}{2}\Big(n(n-1) + \frac{N^2}{1-\frac{N^2}{2}}\frac{\lambda^2_{\max}(\tau)}{\lambda^2_{\min}(\sigma^{-1})}\Big)\Big].\end{aligned}$$

Then π strictly dominates the weighted average $\mathcal{V}^N(t)$ if

$$T \geqslant \frac{nN^2 - 2n^2 - 2}{-2\epsilon n + 2M_\pi n^2 - \frac{2n^2 M_0}{1-\frac{N^2}{2n}} + Mn^2\Big(n(n-1) + \frac{N^2}{1-\frac{N^2}{2}}\frac{\lambda^2_{\max}(\tau)}{\lambda^2_{\min}(\sigma^{-1})}\Big)}.$$

□

Thus to solve an optimization problem for relative arbitrage opportunities, alternatively, we can study the optimization of generating functions. The researches in [41], [73], [20] bridge connections between functionally generated portfolios and Cover's universal portfolios [19]. Universal portfolio is the average of all constant-weighted portfolios weighted by their performances.

Note that the wealth $\check{V}(t)$ defined in Theorem 2.2.1 is the average of every investor's portfolio weighted by their performances when $N \to \infty$. The relationship $\widetilde{V}(t) \leqslant \check{V}(t)$ satisfies Cover's celebrated result when π is constant portfolios.

Chapter 7

Volatility-stabilized model (VSM) and its numerical methods

This chapter starts to look at the tractability of a single-player relative arbitrage model - the problem studied in [22]. In Section 7.1, we summarize some important properties in stock capitalization and market portfolio dynamics, which are closely related to Bessel processes. In Section 7.2, we give a numerical solution for optimal arbitrage in the volatility-stabilized market by simulating stocks from Bessel bridges and using the tool of finite differences and interpolations. With the support of the first two sections, we derive in Section 7.3 a numerical scheme for multi-player optimal arbitrage problems.

7.1 Volatility-stabilized market model

The volatility stabilized model is introduced in [27], it possesses similar characteristics in real markets such as the *leverage effect*, where rates of return and volatility have a negative correlation with the stock capitalization relative to the market $\{\mathbf{m}_i(t)\}_{i=1,\dots,n}$.

In a market described in Section 1.1.1 where investors have no influence on the stock price processes nor other investors. A general form of the capitalizations in volatility stabilized model is

$$dX_i(t) = \kappa X(t)dt + \sqrt{X_i(t)X(t)}dW_i(t), \quad i = 1, \ldots, n, \tag{7.1}$$

where $n \geqslant 2$, $\kappa \in [\frac{1}{2}, 1]$.

7.1.1 Bessel process

We study the Bessel process and some of its properties as the Bessel process is closely related to the stock capitalizations.

Definition 7.1.1. *For every $m \geqslant 0$ and $x \geqslant 0$, the unique strong solution of the equation*

$$dQ_t = mdt + 2\sqrt{|Q_s|}dW_t,\ Q_0 = x$$

is called the square of m-dimensional Bessel process started at x. W_t is a linear BM, $\langle W, W\rangle_t = t$.

Based on process Q_t, the m-dimensional Bessel process follows

$$R_t := sgn(Q_t)\sqrt{|Q_t|},\ R_0 = sgn(x)\sqrt{|x|}.$$

Let $T > 0$, denote the process $X := \{X_s, s \in [0, T]\}$ as the m-dimensional Bessel bridge with $X_0 = x$ and $X_T = c \in \mathbb{R}$. Loosely speaking, X is the process R conditioned to take the value c at time T. When $m \geqslant 2$, the Bessel bridge between 0 and 0 over $[0, 1]$ is the unique solution to

$$dX_t = \left(\frac{m-1}{2X_t} - \frac{X_t}{T-t}\right)dt + dW_t, \quad X_0 = x > 0.$$

The squared Bessel bridge of dimension m is the unique solution to

$$X_t = 2\int_0^t \sqrt{X_s}dW_s + \int_0^t \left(m - \frac{2X_t}{1-s}\right)ds.$$

See [63] for more details. Generally we may use a change of measure approach to get diffusion of Bessel bridge.

7.1.2 Bessel and Jacobi processes in volatility-stabilized models

The stock capitalization $\mathcal{X}(t)$ as the unique-in-distribution solution of (7.1), can be written as a time changed squared Bessel process. In [34], Bessel process and volatility-stabilized processes with time change are studied in detail. The total market capitalization is

$$X(t) = xe^{\frac{(1+\zeta)n-1}{2}t+B(t)},$$

where $\zeta = 2\kappa - 1$, $B(\cdot)$ is a Brownian motion

$$B(t) = \sum_{i=1}^{n}\int_0^t \sqrt{\mathbf{m}_i(s)}dW_i(s),$$

its quadratic variation $\langle B\rangle_t = t$, and $W_1(\cdot), \ldots, W_n(\cdot)$ are independent standard Brownian motions.

We define a continuous, strictly increasing time change

$$\Lambda(t) := \frac{1}{4}\int_0^t X(s)ds.$$

For each process $R_i(t) = \sqrt{X_i(\Lambda^{-1}(t))}$, $i = 1, \ldots, n$, $\Lambda^{-1}(t)$ serves as the clock.

$\Lambda^{-1}(t) = 4\int_0^u \frac{ds}{R^2(s)}$. We write out the independent Bessel processes $R_1(\cdot), \ldots, R_n(\cdot)$

$$dR_i(u) = \frac{m-1}{2R_i(u)}du + d\hat{W}_i(u),$$

where $m = 4\kappa = 2(1+\zeta)$, $\{\hat{W}_i(\cdot)\}_{i=1,\ldots,n}$ are independent Brownian motions,

$$\hat{W}_i(u) := \int_0^{\Lambda^{-1}(u)} \sqrt{\Lambda'(\xi)} dW_i(\xi), 0 \leqslant u < \infty,$$

and $\langle \hat{W}_i, \hat{W}_j \rangle(u) = u\delta_{ij}$. It follows that $Q_i(\cdot) = R_i^2(\cdot)$ for $i = 1, \ldots, n$ are independent squared Bessel processes with order m. $Q(\cdot) = Q_1(\cdot) + \ldots + Q_n(\cdot)$ is a Bessel process with order mn.

The market weights $\{\mathbf{m}_i\}_{i=1,\ldots,n}$ in the volatility-stabilized market models is the Jacobi diffusion process, or Wright-Fisher diffusion in general.

$$d\mathbf{m}_i(t) = (1-\zeta)(1 - n\mathbf{m}_i(t))dt + \sqrt{\mathbf{m}_i(t)(1-\mathbf{m}_i(t))}d\tilde{W}_i(t).$$

with $\sum_{i=1}^n \mathbf{m}_i = 1$, $\tilde{W}_i(t)$ is negatively correlated with

$$cov(\tilde{W}_i(t), \tilde{W}_j(t)) = \begin{cases} -\frac{1}{(1-\mathbf{m}_i)(1-\mathbf{m}_j)}, & \text{if } i \neq j \\ 1, & i = j \end{cases}$$

The joint density of market weights, at fixed times and suitable stopping times is derived by [55].

7.2 Grid-based numerical solution of optimal arbitrage in VSM

We start from a one-player relative arbitrage problem over $[0, T]$, which is studied in [22]: Consider the process

$$dX_i(t) = X(t)dt + \sqrt{X_i(t)X(t)}dW_i(t), \tag{7.2}$$

or equivalently

$$d(\log X_i(t)) = \frac{1}{2\mu_i(t)}dt + \sqrt{\frac{1}{\mu_i(t)}}dW_i(t).$$

Let $\mathcal{X}(t) := (X_1(t), \ldots, X_n(t))$. Recall that under Assumption 6, we defined the local martingale $L(t)$ in Definiton 3.2.1.

The best investment opportunity for arbitrage relative to the market portfolio is characterized as

$$u(T) := \inf\left\{w > 0 \middle| \exists \pi(\cdot) \in \mathcal{A} \text{ s.t. } V^{wX(0),\pi}(T) \geqslant X(T), \text{ a.s.}\right\}, \tag{7.3}$$

where $X(T) = X_1(T) + \ldots + X_n(T)$.

It has shown in [22] that with Markovian market coefficients, (7.3) can be expressed as

$$u(T-t, \mathcal{X}(t)) = \frac{\mathbb{E}[L(T)X(T)|\mathcal{F}_t]}{L(t)X(t)}. \tag{7.4}$$

If the market follows (7.2), the resulting solution $u(\cdot)$ is

$$u(T-t, \mathcal{X}(t)) = \frac{\mathbb{E}[L(T)X(T)|\mathcal{F}_t]}{L(t)X(t)} = \frac{X_1(t)\ldots X_n(t)}{X_1(t)+\ldots+X_n(t)}\mathbb{E}\left[\frac{X_1(T)+\ldots+X_n(T)}{X_1(T)\ldots X_n(T)} \,\middle|\, \mathcal{F}_t\right]. \tag{7.5}$$

7.2.1 Challenges of the estimation through finite differences

The optimal quantity u in (7.3) is the minimal non-negative continuous solution $u \in C^{1,2}([0,T] \times \mathbb{R}^n)$ of the semi-linear parabolic Cauchy problem,

$$\frac{\partial U}{\partial \tau}(\tau, \mathbf{x}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\mathbf{x}) D_{ij}^2 U(\tau, \mathbf{x}) + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{a_{ij}(\mathbf{x}) D_i U(\tau, \mathbf{x})}{x_1 + \ldots + x_n}. \tag{7.6}$$

$$u(0) = 1. \tag{7.7}$$

In the case of (7.2), $a_{ij} = \sqrt{X_i(t)X(t)}\delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise. To tackle (7.6) - (7.7) with finite difference methods yield several challenges. First, $U(\tau, \mathbf{x})$ is on an unbounded domain of $(\tau, \mathbf{x}) \in \mathbb{R}_+ \times \mathbb{R}_+^n$. Some artificial boundary conditions of $U(\tau, \mathbf{x})$ are required for the implementation. Second, The solutions of (7.6) - (7.7) are not unique. It is a delicate issue to select the correct minimal nonnegative solution, especially with a constant initial condition (7.7) and extra boundary conditions. Last but not least, the grid based numerical schemes are notoriously expensive in terms of its computational costs, known as "curse of dimensionality".

One way to reduce these obstacles under grid-based schemes is to consider the probabilistic representation (7.5), where only first order derivative are required to discretize the processes $X_i(\cdot)$, $i = 1, \ldots, n$.

The solution of (7.2) takes values in $(0, \infty)^n$ as noted in [27], thus $\mathcal{X}(t)$ and should not explode or goes to zero. However simulating $\mathcal{X}(t)$ by discretizing (7.2) with, for example, Euler scheme, produces $\mathcal{X}(t)$ values that are very close to zero inevitably. This cause numerical overflow when approximating $u(T - t, \mathcal{X}(t))$ by (7.5), especially the product terms $X_1(\cdot) \ldots X_n(\cdot)$. In numerical experiments we found the approximated solution explode or goes to zero quite frequently even when the dimension of stocks is as small as $n = 2$.

Fortunately thanks to special properties of VSM, we can simulate $\{\mathcal{X}(t)\}_{t\in[0,T]}$, by time-changed Bessel processes.

7.2.2 Simulation Algorithm

The m dimensional Bessel process is generated from the Euclidean norm $|\cdot|$ of m-dimensional Brownian motion $\tilde{W}$.

$$R_t(\omega) = |\tilde{W}(t,\omega)| = (\tilde{W}_1(t,\omega)^2 + \ldots + \tilde{W}_n(t,\omega)^2)^{\frac{1}{2}}.$$

In between of the interpolation points, we model it by the Bessel bridge X of dimension 2 starting at X_{θ_k} such that it finishes at $X_{\theta_{k+1}}$ at time $T = t_{k+1}$. We simulate Bessel bridges R^b_t based on Brownian bridges,

$$R^b_t = |Y_t| = \big[\sum_{i=1}^{m}(y^i_t)^2\big]^{\frac{1}{2}},$$

where $Y_t := (Y_t)$ is the Brownian bridge from $a \in \mathbb{R}^n$ to $b \in \mathbb{R}^n$ over $[0,T)$,

$$dY_t = \frac{b - Y_t}{T-t}dt + dB_t, \quad Y_0 = a.$$

Hence $R^b_0 = |a|$, and by Itô's formula

$$\begin{aligned} dR^b_t &= \left(\frac{m-1}{2R^b_t} - \frac{R^b_t}{T-t} + \frac{z\sum_{i=1}^m Y^{(i)}_t}{R^b_t(T-t)}\right)dt + \sum_{i=1}^{m}\frac{Y^{(i)}_t}{R^b_t}dW^{(i)}_t \\ &= \left(\frac{m-1}{2R^b_t} - \frac{R^b_t}{T-t} + \frac{z\sum_{i=1}^m Y^{(i)}_t}{R^b_t(T-t)}\right)dt + dZ_t, \end{aligned} \tag{7.8}$$

where $z = |b|$. For $i = 1,\ldots,n$, $\{W^{(i)}_t\}$ is a sequence of independent Brownian motions. Z_t is a standard Brownian motion since the Levy's theorem follows that the volatility term $Z_t = \sum_{i=1}^{n} \int \frac{Y_t^{(i)}}{R_t^b} dW_t^{(i)}$ is a standard Brownian motion. As a simple example, assume $R_0 = 0$, $m = 1$, the 1 dimensional Bessel bridge can be generated from the Brownian bridge $Y_0 = a$ and

$$dY_t = \frac{x - Y_t}{T - t} dt + dB_t, \quad x = b \text{ or } -b, \quad \text{for } t \in (0, T).$$

Apart from these methods, in [50], the author looks for the exact simulation by sampling from the probability distribution of squared Bessel bridges and Bessel processes by randomized gamma distribution. The PDF of the squared Bessel bridge $(X_t)_{0 \leqslant t \leqslant T}$ conditional on $X_0 = a$ and $X_T = c$,

$$p(0, T, t; x, z, y) = \frac{T}{2t(T - t)} \exp\left(- \frac{\bar{x} + \bar{y}}{2t} - \frac{\bar{z}t}{2} \right) \frac{I_\nu(\sqrt{xy}/t) I_\nu(\sqrt{yz}/(T - t))}{I_\nu(\sqrt{\bar{x}\bar{z}}/T)},$$

where $\bar{x} = \frac{x(T-t)}{T}$, $\bar{y} = \frac{yT}{T-t}$, $\bar{z} = \frac{z}{T(T-t)}$, $0 < t < T$.

Details of the algorithm

The goal is to estimate u through

$$u(T - t, \mathcal{X}(t)) = \frac{\mathbb{E}[L(T)X(T)|\mathcal{F}(t)]}{X(t)L(t)}$$

We develop an algorithm to use $\mathcal{X}(t)$ as input and simulate $\{\mathcal{X}(s)\}_{t \leqslant s \leqslant T}$ so that $L(T)$ and thus $u(T - t)$ can be obtained.

Given the starting time $s \in [0, T]$, the time changed stock capitalization processes follow squared Bessel processes in m dimension,

$$Y_i(t_k) = R_i^2(t_k) = X_i(\Lambda^{-1}(t_k)) = X_i(\Lambda^{-1}(t_0)) + \big(\sum_{\ell=1}^{k} \Delta W(t_\ell) \big)^2$$

with uniform mesh $t_k := s + k\Delta t$, $i = 1, \ldots, n$. The time-changed total capitalization of the stocks is $Y(t_k) := \sum_{i=1}^{n} Y_i(t_k)$. Here, denote the clock as θ. The mapping of θ_k to t_k is

$$\theta_k := \Lambda^{-1}(t_k) = \sum_{\ell=1}^{k} \frac{4}{Y(t_\ell)} \Delta t.$$

With this, we can find the required range of uniform mesh so that $\theta_{N-1} \leqslant T \leqslant \theta_N$ for an appropriate N.

Next we refine the last segment $[\theta_{N-1}, \theta_N]$ by the Bessel bridge process (7.8) between $(\theta_{N-1}, X(\theta_{N-1}))$ and $(\theta_N, X(\theta_N))$.

$$X_T^s = R^b_{X(\theta_{N-1}),X(\theta_N)}(T).$$

Thus the desired $X(T)$ is solved by interpolating $R^b_{X(\theta_{N-1}),X(\theta_N)}(\theta_{N-1})$ and $R^b_{X(\theta_{N-1}),X(\theta_N)}(\theta_N)$. For example, a linear interpolating result is

$$X(\theta) = \frac{\theta_N - \theta}{\theta_N - \theta_{N-1}} R^b_{X(\theta_{N-1}),X(\theta_N)}(\theta_{N-1}) + \frac{\theta - \theta_{N-1}}{\theta_N - \theta_{N-1}} R^b_{X(\theta_{N-1}),X(\theta_N)}(\theta_N),$$

and let $\theta = T$, we get the estimation of $X_i(T)$ and $X(T)$.

Now we can calculate the optimal arbitrage which is a conditional expectation in (7.5). As an example, (7.2) corresponds to squared Bessel processes of dimension $m = 4$. Then the optimal arbitrage objective at terminal time is

$$u(T^k, x) = \frac{x_1 \ldots x_n}{x_1 + \ldots + x_n} \mathbb{E}\left[\frac{X(T^j)}{X_1(T^j) \ldots X_n(T^j)}\right].$$

To solve u_{T-s,X_s} in general, we have

$$u_{T-s} = \frac{X_1(s) \ldots X_n(s)}{X(s)} \mathbb{E}\left[\frac{X(T)}{X_1(T) \ldots X_n(T)}\right].$$

We demonstrate the steps of the algorithm in Algorithm 1.

Algorithm 1 Solve u by simulating bessel processes in VSM

Input: $n = \#$ of estimating stock processes, $N_T = \#$ of uniform meshes on $[0, T]$, $m =$ dimension of the time changed Bessel processes to model $X_i(t)$ for $i = 1, \dots, n$.

for $s \leftarrow 0$ to N_T **do**
  Initialize the states $\mathcal{X}(s) := (x_1, \dots, x_n)$, $\theta_0 = s$.
  $k \leftarrow 0$
  while $\theta_k \leqslant T$ **do**
    $k \leftarrow k + 1$, $t_k := s + k\Delta t$.
    Generate n_p samples of m-dimensional independent Brownian Motion $W(t_k)$
    Simulate n_p independent, m-dimensional squared Bessel processes $Y_i(t_k) = x_i + \left(\left(\sum_{\ell=1}^{k} W(t_\ell)\right)^2\right)$ and mn-dimensional squared Bessel processes $Y(t_k) = \sum_{i=1}^{n} Y_i(t_k)$.
    $\theta_{k+1} = \theta_k + \frac{4}{Y(t_k)}\Delta t$, where $\Delta t := T/N_T$.
  end while
  Collect $\{\theta_0, \dots, \theta_{k+1}\}$; Simulate the squared Bessel bridge $R^b_{X(\theta_k), X(\theta_{k+1})}(T)$. Evaluate $X_i(T)$ by interpolation techniques between non-uniform mesh points (θ_k, θ_{k+1}).
  Compute deflator $L(\cdot)$, and apply it to obtain $u(T - s, \mathcal{X}_s)$.
end for
return The optimal arbitrage path $u(T - t, \mathcal{X}(t))$ for $t := s\Delta t$, $s = 0, 1, \dots, N_T$.

The implementation of Algorithm 1 and numerical examples are carried out in Python. For the sake of consistency with previous sections and simplicity, the numerical examples are based on (7.2), but we can implement Algorithm 1 under other VSM as well.

Figure 7.1 shows the evolution of $u(T - t, \mathbf{x})$ with respect to stock values respectively in one simulation. We generate Bessel processes of $m = 4$ dimension and illustrate the

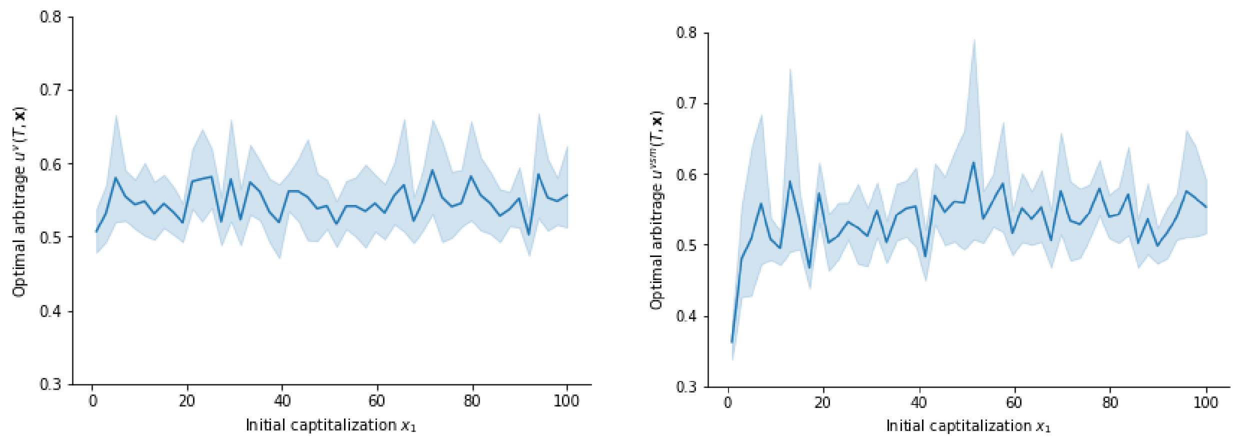


Figure 7.1: Approximated $u(T, \mathbf{x})$ along mesh grids of $\mathbf{x}$. The comparison of computing without (left) or with (right) Bessel bridges. $T = 1$ with $n = 2$ stocks.

performance improvements of simulating Bessel bridges for interpolation in both figures.

We approximate $u(T - t, \mathcal{X}(t))$ conditional on fixed $\mathcal{X}(t)$ by one Monte Carlo simulation $u^v(T - t, \mathcal{X}(t))$. The given $\mathcal{X}(t)$ is sampled from a Gamma distribution. The left subfigure shows the simulated results from a time changed squared Bessel process. In the right subfigure, we refine the simulation in the last time grid $[\theta_N - 1, \theta_N]$ by using the Bessel bridge as Algorithm 1 does. By comparing the two figures, we can see the need for further refinement of the last time grid: We can achieve a more accurate simulated result especially when $\mathbf{x}$ takes values close to zero without increasing the computing time too much.

In Figure 7.1, the grid size $\Delta t = 0.01$, and the refinement by Bessel bridges is using time increment $\Delta_t^b = 0.001$. The number of sample paths $m = 200$ of Brownian motions is used to generate Bessel processes.

We then repeat the simulations multiple times and take the average of the results to improve accuracy. We present the result in the plots below.

We compute the statstics of $u^{vsm}(T, \mathbf{x})$ across $x_i \in (2, 100)$, for $i = 1, 2$. In 60 times of simulations, the mean of $u^{vsm}(T, \mathbf{x})$ is 0.539 and the standard deviation is 0.203. In 100

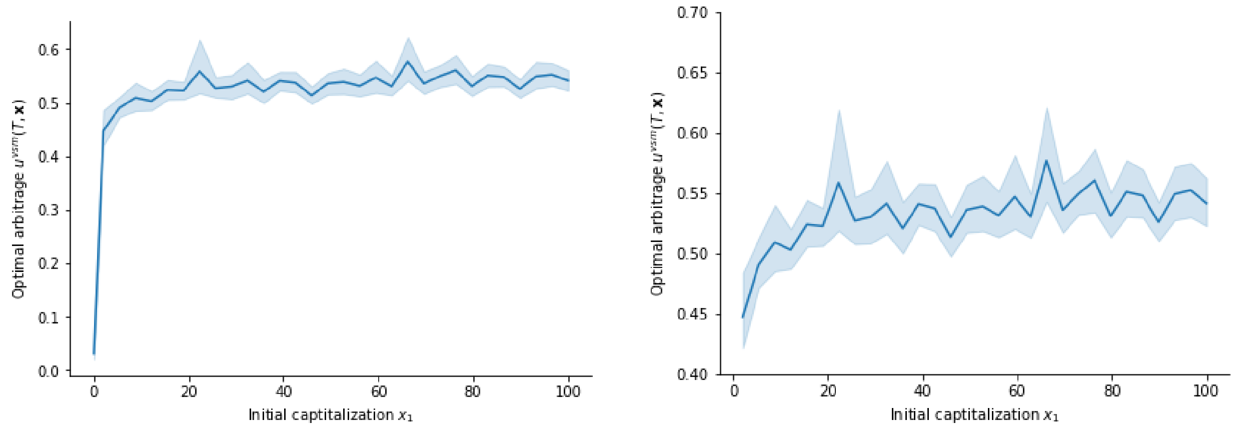


Figure 7.2: Approximated $u(T, x)$ with (x_1, x_2) generated from a Gamma distribution. The comparison of computing with (left) or without (right) the initial condition $u(0, \mathbf{x}) = 1$ when $t = T$. $\Delta t = 0.01$, interpolation with Bessel bridges is using time increment $\Delta_t^b = 0.001$; number of sample paths $m = 200$ of Brownian motions to generate Bessel processes. $T = 1$ with $n = 2$ stocks.

times of simulations, the mean of $u^{vsm}(T, \mathbf{x})$ is 0.532 and the standard deviation is 0.126. In 250 times of simulations, the mean of $u^{vsm}(T, \mathbf{x})$ is 0.531 and the standard deviation is 0.122.

Next, we summarize the result from the statistics of the output in 100 times of Monte Carlo simulations. Figure 7.2 and Figure 7.3 show the evaluated quantity of $u^{vsm}(T-t, \mathbf{x})$ along $\mathbf{x}$ and time t axis respectively.

We recognize in Figure 7.2 that when stock capitalization tend to a very small positive values, the optimal arbitrage quantity u also decreases significantly.

When study the evolution along the time axis as demonstrated from Figure 7.3, the optimal arbitrage quantity $u(T - t)$ stays near zero before terminal time T and sees a immediate surge to 1 since when time to maturity equals zero, or $t = T$, the given condition is $u(0, \mathbf{x}) = 1$. Therefore, boundary conditions might can be added to (7.6)-(7.7) in numerical computations,

$$u(\tau, \mathbf{x}) = 0, \quad (\tau, \mathbf{x}) \in (0, T] \times \mathcal{O}^n / \{\mathbf{0}\}, \tag{7.9}$$

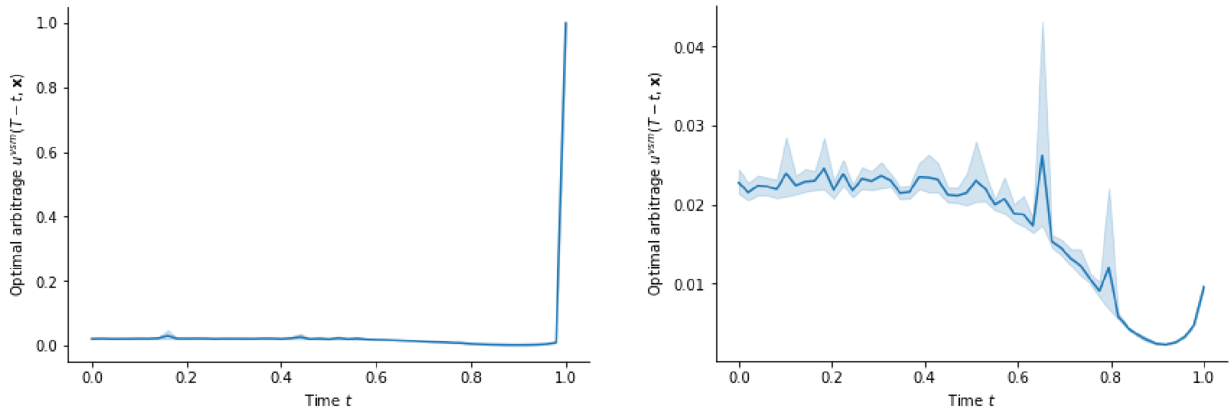


Figure 7.3: Approximated $u(T-t, x)$ with (x_1, x_2) randomly generated from a Gamma distribution. The comparison of computing with (left) or without (right) the close-to-zero initial values. $\Delta t = 0.01$, interpolation with Bessel bridges is using time increment $\Delta_t^b = 0.001$; number of sample paths $m = 200$ of Brownian motions to generate Bessel processes. $T = 1$ with $n = 2$ stocks.

where $\mathcal{O}^n$ is the boundary of the domain $[0, \infty)^n$.

7.3 A mean field relative arbitrage result

Next, we encompass a class of market models for mean field regimes, where the models exhibit selected characteristics of real equity markets and provide a tractable mean field equilibrium.

The smaller stocks tend to have greater volatility than the larger stocks. We construct the stock capitalization coefficients using this similar idea in VSM. Meanwhile, the trading volume and the volatility of a stock tends to be negatively correlated. The parameters β, σ, γ, τ in $\mathcal{M}$ are set to the following specific forms which agree with these market behaviors. For $1 \leqslant i, j \leqslant n$, with infinite number of investors,

$$\beta_i(t) = (1+\zeta)\frac{1}{2\mathbf{m}_i(t)Z_i(t)}, \quad a_{ij} = \frac{X_i(t)}{Z_i(t)}\delta_{ij};$$

$$\gamma_i(t) = Z_i(t); \quad \psi_{ij}(t) = Z_i(t)\delta_{ij}.$$

From (5.12),

$$\frac{\partial \tilde{u}(\tau, \mathbf{x}, \mathbf{z})}{\partial \tau} \geqslant \mathcal{A}\tilde{u}(\tau, \mathbf{x}, \mathbf{z}), \quad \tilde{u}(0, \mathbf{x}, \mathbf{z}) = e^c,$$

$$\begin{aligned}
\text{where} \quad \mathcal{A}\tilde{u}(\tau, \mathbf{x}, \mathbf{z}) = {} & \frac{1}{2}\sum_{i=1}^{n} \frac{x_i}{z_i}\left(D^2_{ii}\tilde{u}(\tau, \mathbf{x}, \mathbf{z}) + \frac{2\delta D_i\tilde{u}(\tau, \mathbf{x}, \mathbf{z})}{\delta\mathbf{x}\cdot\mathbf{1} + (1-\delta)\mathbf{z}\cdot\mathbf{1}}\right) \\
& + \frac{1}{2}\sum_{p=1}^{n} z_p\left(D^2_{pp}\tilde{u}(\tau, \mathbf{x}, \mathbf{z}) + \frac{2(1-\delta)D_i\tilde{u}(\tau, \mathbf{x}, \mathbf{z})}{\delta\mathbf{x}\cdot\mathbf{1} + (1-\delta)\mathbf{z}\cdot\mathbf{1}}\right) \\
& + \sum_{i=p=1}^{n} \sqrt{x_i}\frac{\delta D_p\tilde{u}(\tau, \mathbf{x}, \mathbf{z}) + (1-\delta)D_i\tilde{u}^{\ell}(\tau, \mathbf{x}, \mathbf{z})}{\delta\mathbf{x}\cdot\mathbf{1} + (1-\delta)\mathbf{z}\cdot\mathbf{1}} \\
& + \sum_{i=p=1}^{n} \sqrt{x_i}D^2_{ip}\tilde{u}(\tau, \mathbf{x}, \mathbf{z}),
\end{aligned}$$

for $\tau \in (0, \infty)$, $(\mathbf{x}, \mathbf{z}) \in (0, \infty)^n \times (0, \infty)^n$.

We can check that the Fichera drift $f_i(\cdot) < 0$. Similiarly to Proposition 3.4.1, we can get $\tilde{u}(\cdot) < 1$. When $\zeta = 1$,

$$D_x H_i(\mathbf{x}, \mathbf{y}) := \frac{b_i(\mathbf{x}, \mathbf{y})}{a_{ii}(\mathbf{x}, \mathbf{y})} = \frac{X_i(t)\beta_i(\mathbf{x}, \mathbf{y})}{a_{ii}(\mathbf{x}, \mathbf{y})} = \frac{1}{x_i},$$

$$D_y H_i(\mathbf{x}, \mathbf{y}) := \frac{\gamma_i(\mathbf{x}, \mathbf{y})}{\psi_{ii}(\mathbf{x}, \mathbf{y})} = 1$$

The benchmark in this case is $\mathcal{V}(t) = \delta X(t) + (1-\delta)m_t$.

$$L(t) = \frac{x_1 \ldots x_n}{X_1(t) \ldots X_n(t)};$$

$$m^\star_t = \mathbb{E}[V^\star(t)|B] = \frac{e^c\delta\widetilde{u}_{T-t}}{1-(1-\delta)e^c\mathbb{E}[\widetilde{u}_{T-t}|B]}.$$

Thus by (5.22),

$$
\begin{aligned}
u(T-t) =& \frac{X_1(t)\dots X_n(t)}{\mathcal{V}(t)}\mathbb{E}[\frac{\mathcal{V}(T)}{X_1(T)\dots X_n(T)}|\mathcal{F}_t^B] \\
=& \frac{1-(1-\delta)e^c\mathbb{E}[u_{T-t}|B]}{1-(1-\delta)e^c}\frac{X_1(t)\dots X_n(t)}{X(t)}\mathbb{E}[\frac{X(T)}{X_1(T)\dots X_n(T)}|\mathcal{F}_t^B].
\end{aligned}
$$

After taking the conditional expectation of $u(T-t)$ given $\mathcal{F}_t^B$ in the above equation, we obtain the result

$$
u(T-t) = \frac{\widetilde{u}_{T-t}^X}{1-(1-\delta)e^c(1-\widetilde{u}_{T-t}^X)}, \tag{7.10}
$$

where

$$
\widetilde{u}_{T-t}^X = \frac{X_1(t)\dots X_n(t)}{X(t)}\mathbb{E}[\frac{X(T)}{X_1(T)\dots X_n(T)}|\mathcal{F}_t^B].
$$

By Theorem 5.1.1, the optimal strategy $\pi_i^{\ell\star}$ of investor ℓ in a mean field game is

$$
\pi_i^\star(t) = \mathbf{m}_i^\star(t) + \frac{1}{\delta}\mathbf{m}_i^\star(t)\mathcal{V}^\star(t)D_{x_i}\log\tilde{u}(T-t) + \frac{\mathcal{V}^\star(t)}{\delta X(t)\sqrt{X_i(t)}}D_{z_i}\log\tilde{u}(T-t).
$$

We denote p_t as the conditional density of $V(t)$ given B_t, which follows

$$
\begin{aligned}
dp_t =& \big[-\partial_v(V(t)\pi(t)\beta(t)p_t) + \frac{1}{2}(V(t)\pi(t)\sigma(t))^2\partial_{xx}p_t\big]dt - V(t)\pi(t)\sigma(t)(\partial_x p_t)dW_t \\
=& \Big\{-\partial_v\big[V(t)\sum_{i=1}^n \pi_i(t)\frac{1}{\mathbf{m}_i(t)Z_i(t)}p_t\big] + \frac{1}{2}V_t^2\sum_{i=1}^n \pi_i^2(t)\frac{1}{X_i(t)Z_i(t)}\partial_{vv}p_t\Big\}dt \\
& - V(t)\sum_{i=1}^n \pi_i(t)\sqrt{\frac{1}{X_i(t)Z_i(t)}}\partial_v p_t dB_t.
\end{aligned}
$$

Next, plug $\pi_i^\star(t)$ into the equation of p_t, and let $m_t = \int vp_t(v)dv$, i.e., the consistency condition, we can obtain a closed form solution of $\pi^\star(t)$ in terms of $\mathcal{X}(t)$, $\mathcal{Z}(t)$, $\tilde{u}_{T-t}$.

Remark 13. *We show here the approach to compute the explicit dynamics of m_t when the function of $\gamma(\cdot)$ and $\tau(\cdot)$ are given. This same approach can be used when $\gamma(\cdot)$ and $\tau(\cdot)$ need to be solved using (3.23).*

Grid-based numerical schemes in the mean-field case are similar to Algorithm 1, but without the same coefficient structure in VSM, we should generate processes $(\mathcal{X}(\cdot), \mathcal{Z}(\cdot))$ and the objective (7.10) differently. It is worth mentioning that the dimension of stocks, the number of time discretization and the number of sample paths all add the complexity in the algorithm and cause the increase of computation time significantly. When it comes to a multi-investor problem we discussed in Chapter 4-5, more dimensions of complexities is included besides the finite difference issue we addressed in Section 7.2: N-player problem requires all estimations of optimal function u^{ℓ}, $\ell = 1, \dots, N$; while mean field problem needs estimations of mean field measures. The boundary conditions for $\mathbf{z}$ is harder to be understood intuitively or to be characterized.

Chapter 8

Numerical approaches to High-dimensional PDEs

This chapter starts another important topic of the thesis. We study deep learning schemes to deal with multiple solutions of high-dimensional PDEs. We carry out experiments on solving the non-negative minimal solution of relative arbitrage Cauchy problems as an example.

8.1 Introduction on learning high dimensional PDE and stochastic games

Traditional ways to solve PDEs usually rely on evolution of operators along spatio-temporal grids. This poses expensive computational costs especially for high-dimensional PDEs or the so-called "curse of dimensionality": the memory requirements and complexity grow exponentially with the dimension.

The following works give mesh-free methods on the probabilistic approximation methods for PDEs based on suitable deep learning approximations for BSDEs. The deep

BSDE approach introduced in [37] tackles a class of high-dimensional semilinear PDEs by reformulating the PDEs using backward stochastic differential equations. Independent realizations of a standard Brownian motion will act as training data, and the gradient of the unknown solution is approximated by neural networks. There is a related approach for FBSDEs in [62], where the parameters of the neural network are learned by minimizing the loss function over the full time horizon.

To adapt the deep BSDE method for nonlinear PDEs, [40] proposes deep backward schemes to solve high dimensional nonlinear PDEs. At each time step, the solution and its gradient are estimated simultaneously by the minimization of sequential loss functions through backward induction. [7] explains the connection between fully nonlinear second-order PDEs and 2BSDEs, and introduces the deep 2BSDE scheme.

The deep learning algorithm, or "Deep Galerkin Method" (DGM) first studied in [67], uses a deep neural network to combine least squares of differential operators and conditions. By randomly sampling spatial points and time points, it is free of the need of a global mesh. See [61] and [32] for the papers that use this spirit . Their algorithm estimates simultaneously by backward time induction the solution and its gradient by multi-layer neural networks, while the Hessian is approximated by automatic differentiation of the gradient at the previous step.

The second topic of primary interest is numerical methods to solve large population stochastic differential games. A lot of the learning algorithms in high-dimensional PDEs we just introduced can be applied to numerical computation of stochastic differential games.

To solve mean field games problems numerically, one possible way is to solve the discretized system of forward-backward PDEs. The finite-difference scheme is first introduced by [1], focusing on stationary and evolutive versions of MFG models. Existence and uniqueness properties and the bounds for the solutions of the discrete schemes are also proved. The paper [2] extends the aforementioned finite difference scheme to extended mean field games where the players interact through both the states and controls. Similar to Section 8.1, to avoid computational difficulties grid-based characterization of the Nash equilibrium to two coupled equations: a Hamilton-Jacobi-Bellman equation and a Fokker-Planck equation. It discusses in [18] about deep neural networks for solving MFGs, with a particular Deep Galerkin Method architecture, to approximate the density and the value function by NNs separately.

A different way is to simulate the learning and decision-making process in mean field games. Following the similarity of the process of reinforcement learning and stochastic games, there is literature using Markov decision process (MDP) and reinforcement learning algorithms to solve MFG problems in a model-free way - no knowledge of an exact mathematical model of the MDP is required. In [36] proposes a simulator based Q-learning algorithm with Boltzmann policy (GMF-Q). [70] proposes a policy-gradient based algorithm for MFC and a two-timescale approach to solve MFG with finite state and action spaces. A unified two-timescale Mean Field Q-learning for MFG and MFC is studied in [4]. The paper [11] connects the theories of MFGs and GANs, where two neural networks for HJB equation and FP equation are trained in an adversarial direction of time.

A different aspect to solve stochastic games is through the nature of the evolution of optimal strategy through a learning process with iterative interaction among players, called fictitious play. It is first introduced by [9] and is adapted to learning mean field games in [14]. Literature utilizing this idea with deep neural networks can be found in [38] for N-player stochastic differential games and [60] for MFG in continuous time.

8.2 Solving optimal arbitrage by deep learning based methods

From previous chapters we deal with Cauchy problems with non-unique solutions. Motivated by that, we discuss deep learning schemes to solve high dimension PDEs with non-unique solutions.

We recall briefly about some concepts in neural networks and notations. Mathematically, a neural network can be defined as a directed graph with vertices representing neurons and edges representing links. The dimension of each layer depends on the number of neurons of that layer. One basic form of neural networks is the feedforward neural network. It is composed of each layer's affine transformation A_ℓ and nonlinear transformations ϕ, where

$$A_\ell(x) = \mathcal{W}_\ell x + \beta_\ell, \tag{8.1}$$

where $\mathcal{W}_\ell$ and β_ℓ are the weight and bias term of the layer ℓ, $\ell = 1, \dots, L$. These parameters in neural networks are usually trained using gradient based optimizers iteratively.

8.2.1 Learning Algorithms

We first use a mesh-free method similar to the idea in [67], in order to deal with higher dimension problems. We randomly sample time and spatial pairs and use deep learning to solve the PDE problems.

We use the system (8.2)-(8.3), which admits multiple solutions, as an example.

$$\frac{\partial u(\tau, \mathbf{x})}{\partial \tau} - \mathcal{A}u(\tau, \mathbf{x}) = 0; \quad u(0, \mathbf{x}) = 1 \tag{8.2}$$

where

$$\mathcal{A}u(\tau, \mathbf{x}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\mathbf{x})D_{ij}^2 u(\tau, \mathbf{x}) + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{a_{ij}(\mathbf{x})D_i u(\tau, \mathbf{x})}{x_1 + \ldots + x_n} \tag{8.3}$$

Inputs and Network Structure

The training data for the designed neural network consists of random samples from the internal region on which the PDE is defined; and the random sample from the terminal condition.

Since the input $\mathcal{X}(\cdot)$ follows the stock capitalization processes, we can include this information in the sampling distribution. For VSM models, we sample $\{X_i(t)\}_{i=1,\ldots,n}$ following a Bessel process. Various approaches to get samples from Bessel processes are discussed in Section 7.2. This guarantees the sampling region is general enough to represent the true values of stock capitalization. This also reduces the problem of over-fitting since the samples used for training can be generated many times from Bessel processes to satisfy the requirement of generalization.

The hidden layers use the similiar idea of LSTM layers. Each layer produces weights based on the last layer, determining how much of the information gets passed to the next layer.

Loss function

With samples (t_i, x_i), the goal is to approximate u with an approximating function $f(t, x; \Theta)$ given by a deep neural network with parameter set Θ in every layer from (8.1)

$$\Theta = \{\mathcal{W}^\ell, \beta^\ell\}_{\ell=1}^{L}.$$

The parameter set Θ is optimized with respect to the loss function with regularization terms.

We can write a general differential equation as

$$F(t, u(t,x), \partial_t u, \partial_x u, \partial_{xx} u) = 0. \tag{8.4}$$

The first loss term $L_1(\Theta_n)$ is for the distance of the actual (zero) value of the operator F in (8.4) and the value of $F(t, \hat{u}, \partial_t \hat{u}, \partial_x \hat{u}, \partial_{xx} \hat{u})$ with estimated $\hat{u}$. Let

$$||f||^2_{[0,T]\times\mathbb{R}^n_+} = \sum_{(t_n, x_n)} |f(t_n, x_n)|^2.$$

Therefore for (8.2),

$$L_1(\Theta_n) = ||\frac{\partial \hat{u}(T - t_n, \mathbf{x}_n; \Theta_n))}{\partial t} - \mathcal{A}\hat{u}(T - t_n, \mathbf{x}_n; \Theta_n)||, \tag{8.5}$$

where the operator $\mathcal{A}$ is defined in (8.3). $\frac{\partial \hat{u}(T - t_n, \mathbf{x}_n; \Theta_n))}{\partial t}$ and the counterparts of derivatives with respect to x is computed by automatic differentiation.

The second term is for the initial condition so that the output at $\tau = 0$ satisfies $u(\tau, \mathbf{x}) = 1$.

$$L_2(\Theta_n) = ||\hat{u}(0, \mathbf{x}_N; \Theta_n)) - 1||_{\mathbb{R}^n_+}, \tag{8.6}$$

$\mathbf{x}_N \in \mathbb{R}^n_+$ is the space sample at time 0. Note that we sometimes need to imposes more penalty to the deviations from the initial conditions, and this can be done by scaling the loss terms.

Multiple solution

Another tricky part in algorithm design here is that the algorithm needs to distinguish multiple solutions and search for the non-negative minimal solution. Most of the works on solving high-dimensional differential equations focus on equations that admit a unique solution. However it is of a great practical need to tackle an ODE or PDE problem that might have multiple solutions, or to model problems as differential inequalities. One of the applications in ODE is the Bratu equation also mentioned in [33], which has been used to model combustion phenomena as well as temperatures in the sun's core.

One general goal is to find all possible solutions of an equation that permits non-unique solutions. The paper [33] deals with this goal for one dimensional ODE. Since the N different solutions of an equation can be trained by a neural network with the same architecture, the same number of layers and units, but with different weights, an additional loss function term that characterizes the pairwise distances of solutions can be added.

We demonstrate outcomes that correspond to multiple solutions of the Cauchy problems and corresponding relative error for different number of iterations in Figure 8.1. In particular the relative error is constantly zero when the trivial solution $\hat{u}(0, \mathbf{x}) \equiv 1$ is achieved. We did not distinguish the correspondence of solution curves and relative error curves here since that does not give us more information about solutions.

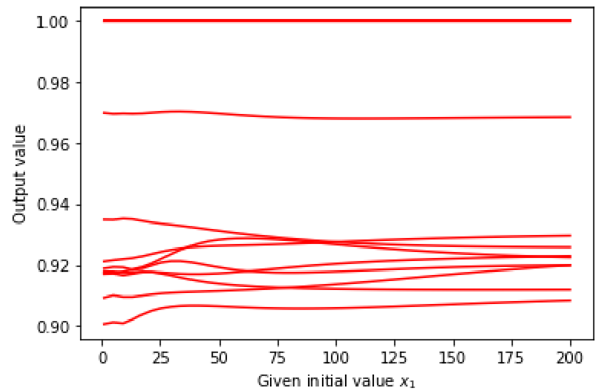
We can see that to learn a solution from a PDE system with nonunique solutions, more information and knowledge about the PDE itself is needed.

If we take the observed boundary condition of volatility stabilized model into account, we learn the PDEs with the artificial boundary condition (7.9) and loss $L_3(\Theta_n)$,

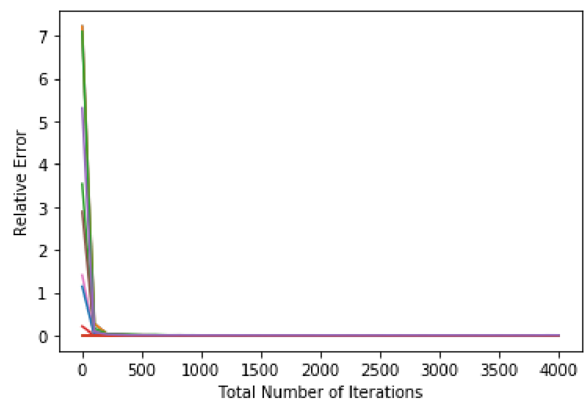
$$L_3(\Theta_n) = ||\hat{u}(T-t, \mathbf{x}_0; \Theta_n))||_{(0,T]\times\mathcal{O}^n/\{\mathbf{0}\}} \tag{8.7}$$

i.e., the new loss function is

$$L(\Theta_n) = L_1(\Theta_n) + L_2(\Theta_n) + L_3(\Theta_n).$$



(a) Training output $\hat{u}(T, \mathbf{x})$ curve with respect to different values of (x_1, x_2).



(b) Relative error regarding number of iterations for all epochs.

Figure 8.1: Training results of (7.6)-(7.7) on time horizon $[0, T]$ where $T = 1$. It uses uniform meshes of $x_i \in (0, 200]$, $i = 1, 2$. The size of sample pairs $(t, \mathbf{x})$ in the internal area and the initial time area is 500 and 100, respectively.

We summarize this method in Algorithm 2. The approach is similar to Deep Galerkin Method, but we improve some details in the method in order to solve the nonuniqueness of PDEs.

Algorithm 2 Learning Cauchy problem based on deep Galerkin (CDG)

1: Generate random samples pairs $(t_n, \mathbf{x}_n) \in [0,T] \times \mathbb{R}_+^n$. For initial conditions, draw random samples $\mathbf{x} \in \mathbb{R}_+^n$ when $t = 0$.

2: Compute the loss function $L(\cdot)$ by

$$L(\Theta_n) = L_1(\Theta_n) + L_2(\Theta_n) + L_3(\Theta_n), \tag{8.8}$$

where L_1, L_2 is defined in (8.5)-(8.6), $L_3(\Theta_n)$ in (8.7).

3: Take stochastic gradient descent at a point (s_n, x_n),

$$\Theta_{n+1} = \Theta_n - \alpha_n \nabla_\Theta L(\Theta, s_n).$$

Repeat stage 1-3 until convergence criterion is satisfied.

8.2.2 Numerical Experiment

We carry out the experiment using Pytorch in Python. We explain the details in the algorithm implementation below.

The choice of distributions and functions that we draw the samples (t, x) from is important to the performance of learning. The performance of the deep learning model in Algorithm 2 can be boosted by incorporating some special characteristics in the associated differential equations. For example in VSM, we know x behave like Bessel process, so for a pair of interior sample (t_n, x_n), we first sample t_n from a uniform distribution

$$t_n = t_0 + (1 - t_0)\mathcal{U}, \tag{8.9}$$

$t_0 = 1e - 10$, $\mathcal{U}$ is a random number sampled from standard uniform distribution. Then use t_n we get the corresponding

$$x_n = x_0 + (500 - x_0) \cdot t_n \cdot \mathcal{N}^2, \tag{8.10}$$

where $x_0 = 1e - 10$, $\mathcal{N}$ is a random number sampled from standard normal distribution. We name Algorithm 2 with input in the form (8.9) - (8.10) as **CDG-BE**. We will justify this choice of input by comparing it with x_n sampled from other distributions that do not depend on time or different dependence with time variables. One example we use is a normal distribution independent of t_n, named as **CDG-RN**.

Specifically, the loss function that we use is $L(\Theta_n) = L_1(\Theta_n) + 10 * L_2(\Theta_n) + L_3(\Theta_n)$. To avoid the cliff region of gradients in parameter updates, especially because of the initial condition $u(0) = 1$ in our algorithm, we clip the gradient of loss functions with the threshold value 100.

The layer parameters (weight and bias) are defined and initialized using Xavier initialization. The network was trained for a number of iterations (epochs), with random re-sampling of points for the interior and terminal conditions every 1000 iterations. Parameter optimization is updated by Adam optimizer, with a learning rate $\alpha_n = 1e - 03$ and MultiplicativeLR learning rate scheduler. In terms of hyperparameters, we use the number of layers $L = 3$, the number of nodes in each layer $M = 50$. We use the sigmoid function as the activation function in every layer.

We run the algorithm with epoch = 50 (sampling stages), and the number of iterations in each epoch is 100. Each time the sample size of the internal domain $n_1 = 500$, and the sample size of the initial domain $n_2 = 100$.

We first present a preliminary training result with a uniform mesh in Figure 8.2.

We show an example of how training with prior information improves the efficiency and accuracy of learning the solutions. It contains the output curves from multiple times

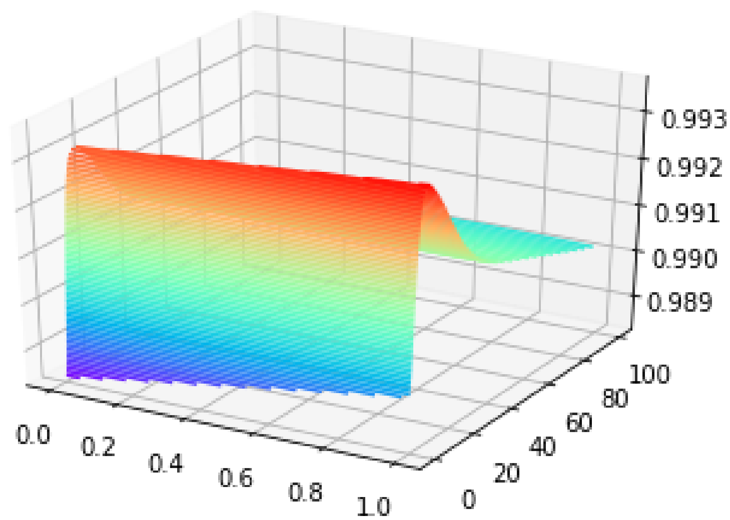
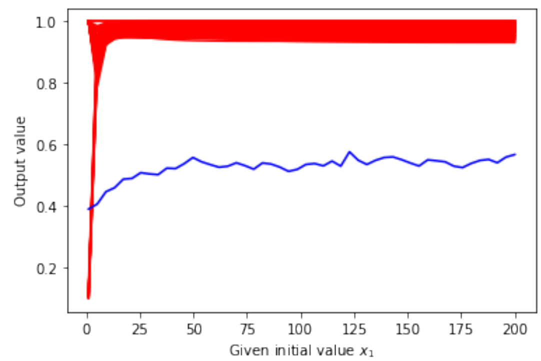


Figure 8.2: A trained curve of $u(\tau, \mathbf{x})$ on $\tau \in [0,1]$ with $n = 10$ stocks. The space variables $x_i \in [1, 200]$, for $i = 1, \ldots, n$

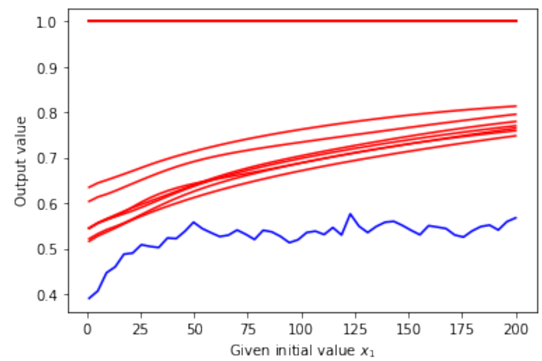
of training in Figure 8.3. The following Figure 8.3a is a curve trained with sample of x generated from normal distribution.

In general when a PDE has no analytical result and grid-based method is also hard to compute in high dimensions. We have seen in Chapter 7 that there is a class of models with probabilistic representation that is implementable by finite difference method, that is, volatility-stabilized model, We compare our machine learning result with the result in Chapter 7 $u^{\text{vsm}}(t, \mathbf{x})$ as a metric through uniform grids of time and space. In Table 8.1 we record the error for all sample pairs $(t_n, \mathbf{x}_n)$ between $u_{\min}(t_n, \mathbf{x}_n)$ and $u^{\text{vsm}}(t_n, \mathbf{x}_n)$ to evaluate the performance. $u_{\min}(t_n, \mathbf{x}_n)$ is evaluated differently with prior information as (8.9)-(8.10) or without (sampling $\mathbf{x}$ from a normal distribution). This comparison is done by repeatedly running the algorithm for 20, 80, and 500 times.

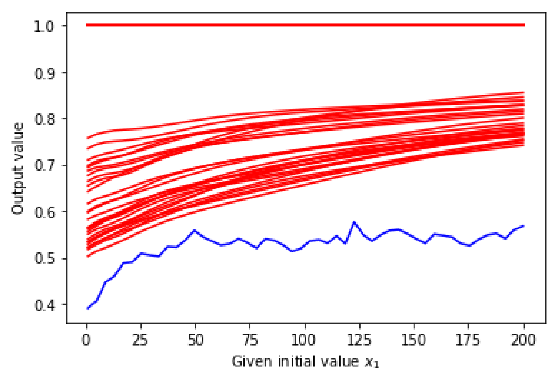
We omit the output curve of all zero values in the plot in order to have a closer observation on the plausible minimal solutions, since the output curve of all zero values is not a solution of (7.6)-(7.7). However, we can solve this issue easily by using the



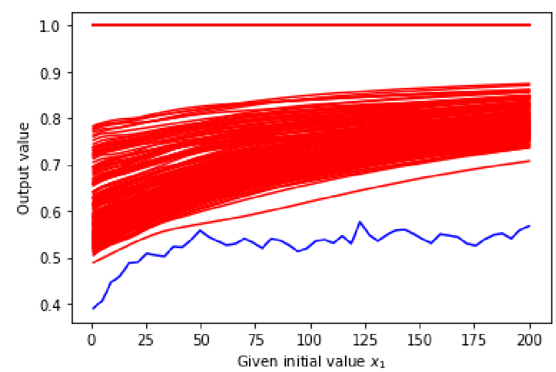
(a) Training output $\hat{u}(T, \mathbf{x})$ curve repeated 80 times of training with CDG-RN.



(b) Training output $\hat{u}(T, \mathbf{x})$ curve repeated 20 times of training with CDG-BE.



(c) Training output $\hat{u}(T, \mathbf{x})$ curve repeated 80 times of training CDG-BE.



(d) Training output $\hat{u}(T, \mathbf{x})$ curve repeated 500 times of training CDG-BE.

Figure 8.3: The comparison of trained curves $\hat{u}(T, \mathbf{x})$ with the boundary condition on $x_i \in [1, 200]$, $i = 1, 2$. The training is repeated 20, 80, 500 times. Time horizon $[0, T]$ where $T = 1$.

relative error.

The non-zero curves have higher MAE/MSE and achieve all-one values more often than using prior information about capitalization processes.

We see that when the training times increased from 20 times to 500 times as in Figure 8.3b-8.3d, the minimal solution curve comes closer to the baseline result u^{vsm}. Meanwhile, we do not see a significant improvement from running 500 times to running 1000 times.

Table 8.1: Accuracy metric of minimal curve $\hat{u}_{\min}$ from training with multiple number of times, using (8.10) input (CDG-BE) or normal input (CDG-RN), comparing with the baseline solution u^{vsm}

	Training times (M) for CDG-BE			**Training times (N) for CDG-RN**		
	M=20	**M=80**	**M=500**	**N=20**	**N=80**	**N=500**
MAE	0.13304	0.12170	0.09092	0.39900	0.40116	0.38187
MSE	0.01965	0.01713	0.01000	0.16000	0.16187	0.14668

Remark 14. *To better distinguish different solution outputs, we can define a metric*

$$-\sum_{n} \ell(u_i(t_n, x_n), u_j(t_n, x_n)).$$

to compare the value of two outputs $u_i(\cdot)$, $u_j(\cdot)$ from learning algorithms, where $\ell(\cdot,\cdot)$ is a given distance metric.

Note that another way to distinguish training outcomes is to compare the corresponding parameters except the single bias term that represents the constant in a solution. A major drawback of this technique is that training is inefficient, since different set of model parameters return the same set of solutions.

8.3 Minimal solution of high-dimensional PDEs

The challenges in the aforementioned method to solve PDEs with multiple solutions is obvious: There are several sources of the uncertainties and inaccuracies while learning the objective $u(T-t, \mathbf{x})$.

Each time we carry out the entire training with given number of epochs, we obtain an optimal outcome $u(\cdot)$,

$$\hat{u}(T-t, \mathbf{x}; \Theta) = f^{(L)}(f^{(L-1)}(\dots f^{(2)}(f^{(1)}(\mathbf{x})))), \tag{8.11}$$

which can be perceived as either a global optimum or a local optimum in the minimization problem (8.8). $f^{(1)}(\cdot), \ldots, f^{(L)}$ are the functions of L layers.

We run the entire training repeatedly for 15 times and illustrate the 15 outputs in Figure 8.1. The learned (8.11) and the consisting parameters might be approximations of different solutions of (8.2)-(8.3). However, it is hard to distinguish outputs as the same or different solution of the PDE by the parameters, because even for the same output u, it could be induced by a different set of parameters. Under the unsupervised nature, it is hard to find a standard to indicate whether it learned a correct solution of (8.2)-(8.3).

In certain model set-ups, for instance, in the class of volatility-stabilized models, we can compare the deep learning outputs with the Monte-Carlo type of solution in Chapter 7. However generally, there is a lack of analytical results or good approximations to compare the performance of our algorithm to.

The system (8.2)-(8.3) admits multiple solutions and its continuous minimal non-negative solution is the unique outcome we intend to solve. So in this section we focus on the minimal solution of a PDE system, and there is no longer a need to distinguish the multiple solutions as in Section 8.2. We propose a method based on reflected BSDEs to solve the minimal solution of PDEs. This method can be applied to a class of differential equations of which the solution is forced to stay above a given stochastic process, called the obstacle. Hence we can try to seek a lower bound of (8.2)-(8.3).

8.3.1 BSDEs characterization

We first review some related concepts of BSDEs. For simplicity of notations, we adopt the specific dimensions of processes as previous relative arbitrage problems.

Recall that a solution pair $\{(Y_t, Z_t); 0 \leqslant t \leqslant T\}$ of adapted processes with values in

$\mathbb{R} \times \mathbb{R}^n$ such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^2 + \int_0^T |Z_t|^2 dt\right] < \infty, \tag{8.12}$$

solves almost surely the BSDE

$$Y_t = \xi + \int_t^T f(Y_s, Z_s)ds - \int_t^T Z_s dW_s, t \leqslant s \leqslant T. \tag{8.13}$$

$f(y, z, \omega) : \mathbb{R} \times \mathbb{R}^n \times \Omega \to \mathbb{R}$, $\xi \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$.

The assumption below ensures the existence and uniqueness of the solution of BSDE (8.13). But note that the major goal of this chapter is to deal with PDEs and BSDEs that admit the nonuniqueness of the solutions.

Assumption 19. *The driver f is Lipschitz in (y, z) uniformly in $(t, \omega) \in [0, T] \times \Omega$, i.e.,*

$$\forall y_1, y_2 \in \mathbb{R}^m,\ \forall z_1, z_2 \in \mathbb{R}^{m\times n},\ |f(t, \omega, y_1, z_1) - f(t, \omega, y_2, z_2)| \leqslant C_f(|y_1 - y_2| + |z_1 - z_2|)$$

for every $(t, \omega) \in [0, T] \times \Omega$. $\{f(t, 0, 0)\}_{0\leqslant t\leqslant T}$ is progressively measurable on $[0, T] \times \Omega$ and

$$\mathbb{E}\int_0^T |f(t, 0, 0)|^2 ds < \infty.$$

We also recall the comparison principle in [58].

Proposition 8.3.1. *Let Assumption 19 hold. We suppose also $g_0 - g_1$ is bounded and nonnegative, $f_0 - f_1$ is bounded and nonnegative. Let $(Y_1(\cdot), Z_1(\cdot)) \in (\mathbb{R}^m, \mathbb{R}^{m\times n})$ solves*

$$Y_1(t) = g_1 + \int_t^T [f(s, Y_1(s), Z_1(s))]ds - \int_t^T Z_1(s)dW(s),$$

and $(Y_0(\cdot), Z_0(\cdot))$ *solves*

$$Y_0(t) = g_0 + \int_t^T [f(Y_0(s), Z_0(s), s) + f_0(s)]ds - \int_t^T Z_0(s)dW(s).$$

Then

$$Y_1(t) \leqslant Y_0(t), \quad 0 \leqslant t \leqslant T \quad a.s.$$

Definition 8.3.1. *The BSDE (8.13) has a minimal solution* (X_t, Y_t, Z_t) *if for any other solution* (X_t', Y_t', Z_t') *of (8.13), we have* $Y_t \leqslant Y_t'$ *a.s., for all* $t \leqslant T$.

Minimal solutions of Cauchy problem

Next we connect the minimal solution of parabolic differential equations with the minimal solution of its BSDE formulation. We use the Cauchy problem in the relative arbitrage model here to explain, but it can extend to other parabolic differential equations.

We start from a simplified relative arbitrage problem over $[0, T]$ as we did in Chapter 7. The optimal arbitrage u is the minimal non-negative continuous solution $u \in C^{1,2}([0,T] \times \mathbb{R}^n)$ of the semi-linear parabolic Cauchy problem (7.6)-(7.7),

$$\frac{\partial u}{\partial \tau}(\tau, \mathbf{x}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\mathbf{x}) D_{ij}^2 u(\tau, \mathbf{x}) + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{a_{ij}(\mathbf{x}) D_i u(\tau, \mathbf{x})}{x_1 + \ldots + x_n},$$

$$u(0, \mathbf{x}) = 1.$$

We derive here the connection of the nonnegative minimal solution $u(\tau, \mathbf{x})$ of Cauchy problem under Markovian assumption where τ is the time to maturity, to the nonnegative minimal solution of an uncoupled FBSDE. We follow a similiar route of the nonlinear Feynman-Kac theorem proved in [54], where it connects the unique solution of BSDE and quasilinear parabolic PDEs.

Proposition 8.3.2. *Suppose* $u \in C^{1,2}([0,T] \times \mathbb{R}^n)$ *is a solution of* (7.6). *Define* $\mathcal{X}(t) := (X_1(t), \ldots, X_n(t))$, *for each* $X_i(t)$, $i = 1, \ldots, n$,

$$dX_i(t) = b_i(t)dt + \sum_{k=1}^{n} s_{ik}(t)dW_k(t).$$

Then $\{\mathcal{F}_t\}_{t\in[0,T]}$*-adapted processes* $\{\mathcal{X}_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\} := \{\mathcal{X}_r^{t,x}, u(T-r, \mathcal{X}_r^{t,x}), (s\nabla u)(T-r, \mathcal{X}_r^{t,x})\}$ *solves*

$$u(T-t, \mathcal{X}_t^{t,x}) = u(0) - \int_t^T f(\mathcal{X}_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_t^T (Z_r^{t,x})^T dW(r), \tag{8.14}$$

where

$$\begin{aligned} f(t, \mathcal{X}_t, Y_t, Z_t) =& b(\mathbf{x})(s^T(t, \mathcal{X}_t))^{-1} s^T(t, \mathcal{X}_t)(D_x u(\tau, \mathbf{x}))^T \\ &- \frac{1}{x_1 + \ldots + x_n} s \cdot s^T(t, \mathcal{X}_t)(D_x u(\tau, \mathbf{x}))^T \\ =& b(\mathbf{x})(s^T(t, \mathcal{X}_t))^{-1} Z_t - \frac{1}{x_1 + \ldots + x_n} \mathbf{1}' s(t, \mathcal{X}_t) Z_t. \end{aligned} \tag{8.15}$$

Equivalently, we can rewrite (8.14) as

$$u(T-t) = u(T) + \int_0^t f(\mathcal{X}(s), u, Du)ds + \int_0^t Z_s dW(s).$$

Proof. Use Ito's formula on $u(\tau, \mathbf{x})$

$$du(T-t, \mathcal{X}(t)) = (\mathcal{L}u - \frac{\partial u}{\partial \tau})(T-t, \mathcal{X}(t))dt + \sum_{k=1}^{n} R_k(T-t, \mathcal{X}(t))dW_k(t),$$

where $R(\tau, \mathbf{x})$ is n-dimensional vector with elements $R_k(\tau, \mathbf{x}) = \sum_{i=1}^{n} x_i s_{ik}(\mathbf{x}) D_i u(\tau, \mathbf{x})$.

Plug the Cauchy problem (7.6) in, we get the minimal non-negative continuous solution of the above equation satisfies $U(0) = 1$,

$$du(T-t, \mathcal{X}(t)) = f(\mathcal{X}(t), u, Du)dt + \sum_{k=1}^{n} R_k(T-t, \mathcal{X}(t))dW_k(t),$$

where $f(\mathbf{x}, U, DU) = \sum_{i=1}^{n} b_i(\mathbf{x})D_i u(\tau, \mathbf{x}) - \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{a_{ij}(\mathbf{x})D_i u(\tau,\mathbf{x})}{x_1+\ldots+x_n}$. $Z_r := R(T-r, \mathbf{x})$. For $\forall t \leqslant T$, integrate $du(T-r, \mathcal{X}(r))$ with respect to time r, $r \in [t, T]$,

$$u(T-t) = u(0) - \int_t^T f(\mathcal{X}(r), u, Du)dr - \int_t^T Z_r^T dW(r).$$

Therefore $\{\mathcal{X}_r^{t,x}, u(r, \mathcal{X}_r^{t,x}), (s\nabla u)(r, \mathcal{X}_r^{t,x})\}$ where $u \in C^{1,2}([0,T] \times \mathbb{R}^n)$ solves (8.13).

Consider any continuous function $u : [0, \infty) \times (0, \infty)^n \to [0, \infty)$ solves (7.6)-(7.7). $u^\star(\tau, \mathbf{x}) \leqslant u(\tau, \mathbf{x})$, for every $(\tau, \mathbf{x})$. Thus by Definition 8.3.1, for any triple

$$\{\mathcal{X}_r^{t,x}, u(r, \mathcal{X}_r^{t,x}), (\nabla us)(r, \mathcal{X}_r^{t,x})\},$$

we have $u^\star(\tau, \mathbf{x}) \leqslant u(\tau, \mathbf{x})$. Therefore $\{\mathcal{X}_r^{t,x}, u^\star(r, \mathcal{X}_r^{t,x}), (\nabla u^\star s)(r, \mathcal{X}_r^{t,x})\}$ is the minimal solution among non-negative continuous solutions. □

Proposition 8.3.3. *As the previous set-up, under probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, *a solution triple* $\{(X_t, Y_t, Z_t); 0 \leqslant t \leqslant T\}$ *of* $\{\mathcal{F}_t\}_{t\in[0,T]}$*-adapted processes with values in* $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ *of*

$$Y_s^{t,x} = g(\mathcal{X}_T^{t,x}) - \int_s^T f(\mathcal{X}_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x} dW_r, t \leqslant s \leqslant T. \tag{8.16}$$

$\{\mathcal{X}_s^{t,x}\}_{t\leqslant s\leqslant T}$ *the unique solution of the forward SDE*

$$\mathcal{X}_t = \mathbf{x} + \int_0^t b(r)dr + \int_0^t s(r)dW(r) \tag{8.17}$$

such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t|^2+\int_0^T|Z_t|^2dt\right]<\infty, \tag{8.18}$$

where $f(x,y,z,\omega):\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\times\Omega\to\mathbb{R}$ *satisfies* (8.15), $g(\omega):\Omega\to\mathbb{R}^n$.

Then $u(T-t,\mathcal{X}_t)=Y_t$ *solves* (7.6) *and* $u\in C^{1,2}([0,T]\times\mathbb{R}^n)$.

Proof. $u(T-t,\mathbf{x})=Y_t^{t,\mathbf{x}}$, $u(0,\mathbf{x})=g(\mathbf{x})$ where $(\mathcal{X},Y,Z)$ is a solution of BSDE (8.13). From [54], for some deterministic $u\in C^{0,2}([0,T]\times\mathbb{R}^n)$.

$$u(T-t,\mathcal{X}_t)-u(T-s,\mathcal{X}_s)=-\int_s^t f(\mathcal{X}_r,Y_r,Z_r)dr-\int_s^t Z_r dW_r,$$

thus

$$u(T-(t+h),\mathcal{X}_{t+h}^{t,x})-u(T-t,\mathbf{x})=-\int_t^{t+h}f(\mathcal{X}_r,Y_r,Z_r)dr-\int_t^{t+h}Z_rdW_r$$

As a result

$$\begin{aligned}g(\mathbf{x})-u(T-t,\mathbf{x})&=\sum_{j=0}^{m-1}\big(u(T-t_{j+1},\mathbf{x})-u(T-t_{j+1},\mathcal{X}_{t_{j+1}}^{t_j,\mathbf{x}})\\&\qquad+u(T-t_{j+1},\mathcal{X}_{t_{j+1}}^{t_j,\mathbf{x}})-u(T-t,x)\big)\\&=\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}[\mathcal{L}u(T-t_{j+1},\mathcal{X}_r^{t_j})+f(\mathcal{X}_r^{t_j},Y_r^{t_j},Z_r^{t_j})]dr\\&\quad+\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}[Z_r^{t_j}-(\nabla u\sigma)(t_{j+1},\mathcal{X}_r^{t_j})],\end{aligned}$$

where $t=t_0<t_1<\ldots<t_m=T$. Use a sequence of time such that $\lim_{m\to\infty}\sup_{j\leqslant m-1}(t_{j+1}-t_j)=0$, and we get

$$u(T-t,\mathbf{x})=g(\mathbf{x})-\int_t^T[\mathcal{L}u(T-r,\mathbf{x})+f(\mathbf{x},u(T-r,\mathbf{x}),(\nabla us)(r,\mathbf{x}))]dr,$$

where $g(\mathbf{x}) = u(0)$. Thus we see $u \in C^{1,2}([0,T] \times \mathbb{R}^n)$ solves (7.6).

Thus if we have minimal solution of BSDE (8.16), $Y_t^\star \leqslant Y_t$ then correspondingly it holds $u_{T-t}^\star \leqslant u_{T-t}$. □

In the following part, we explain a possible way to approach the minimal solution of BSDEs.

Reflected BSDE Implementation

Consider the reflected BSDE, i.e., constraint of lower bound $u(t) = Y_t \geqslant S_t$, $0 \leqslant t \leqslant T$. S_t is a continuous obstacle process.

$$Y_t = g(\mathcal{X}_T) + \int_t^T f(\mathcal{X}_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s + K_T - K_t, t \leqslant s \leqslant T. \tag{8.19}$$

$\{K_t\}$ is continuous nondecreasing predictable process, such that $K_0 = 0$,

$$\int_0^T (Y_t - S_t)dK_t = 0. \tag{8.20}$$

(8.20) acts as a minimal push since the push happens only when the constraint is attained $Y_t = S_t$.

The minimal solution of (8.19) (Y, Z, K) is in the sense that for any other solution $(\tilde{Y}, \tilde{Z}, \tilde{K})$, $Y \leqslant \tilde{Y}$.

Regarding the Cauchy problem of our interest, its corresponding reflected BSDE solution $\{(Y_t, Z_t, K_t), 0 \leqslant t \leqslant T\}$ of $\mathcal{F}_t$-progressive measurable processes take values in $(\mathbb{R}, \mathbb{R}^n, \mathbb{R}_+)$.

Recall that we can use the penalization method in [21] to approximate the minimal solution with (Y^N, Z^N, K^N). Let $(x(t))^- = max(-x(t), 0)$, for continuous and increasing process K_t, it follows

$$K_t = \sup_{0 \leqslant s \leqslant t} \left(g(\mathcal{X}_t) + \int_s^t f(\mathcal{X}_r, Y_r, Z_r)dr - \int_s^t Z_r dW_r \right)^-,$$

$$K_T - K_t = \sup_{t \leqslant s \leqslant T} \left(g(\mathcal{X}_T^{t,x}) + \int_s^T f(\mathcal{X}_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x} dW_r \right)^-.$$

An open question here is to rigorously investigate the relationship between minimal non-negative continuous solution $u^\star$ of Cauchy problem (7.6) and reflected BSDE (8.19) with constraint $\widetilde{Y}_t \geqslant 0$, for all $t \in [0, T]$. By Proposition 8.3.2 - 8.3.3, the former solution of (7.6) can be written as $\{Y_t, 0 \leqslant t \leqslant T\}$, a solution of the forward SDE from (8.16),

$$Y_t = Y_0 + \int_0^t f(\mathcal{X}_r, Y_r, Z_r)dr + \int_0^t Z_r dW_r,\ Y_t \geqslant 0,\ 0 \leqslant t \leqslant T. \tag{8.21}$$

We compare the above with the solution of (8.19), which can be written as

$$\widetilde{Y}_t = \widetilde{Y}_0 + \int_0^t f(\mathcal{X}_r, \widetilde{Y}_r, \widetilde{Z}_r)dr + \int_0^t \widetilde{Z}_r dW_r - K_t,\ \text{s.t. } \widetilde{Y}_t \geqslant 0,\ 0 \leqslant t \leqslant T. \tag{8.22}$$

However the solutions of (8.21) and (8.22) are not unique, the comparison principles cannot be applied.

Remark 15. *To learn the solution of* (8.14)*, we should avoid modeling the evolution of $\mathcal{X}$ using the finite difference approximations $\mathcal{X}(t_k)$, where $0 = t_0 < \ldots < t_K = T$, $k = 0, \ldots, K$. For example, recall the dynamics* (7.2)*. As mentioned in Chapter 7, although we could ensure positive values of discretized $\mathcal{X}(t_k)$, $k = 0, \ldots, K$ through logarithmic characterization of the dynamics, it can happen quite frequently that the discretized dynamics evolve to be a small quantity that goes near zero which causes overflow and loss of precision problem. In volatility stabilized models, we can model $\mathcal{X}(\cdot)$ using Bessel processes to solve this issue.*

If we consider (8.19)-(8.20) *with* $Y_t \geqslant 0$, $0 \leqslant t \leqslant T$, *we can implement the BSDE and use penalization method in [21] to approximate the minimal solution with* (Y^N, Z^N, K^N). *For example, with* n*-dimensional Brownian motion* W *on a mesh of* $[0,T]$, *i.e.,* $0 = t_0 < \dots, t_N = T$, *a Euler scheme yields*

$$Y^N_{t_n+1} - Y^N_{t_n} = f(Y^N_n)(t_{n+1} - t_n) - \langle Z^N(t_n), W_{t_{n+1}} - W_{t_n}\rangle - (K^N_{t_n+1} - K^N_{t_n}),$$

The initial value $u(t=0, x=x_0)$ *is given. Thus*

$$\begin{aligned} K^N_{t_{n+1}} - K^N_{t_n} &= N\int_{t_n}^{t_{n+1}} (Y^N_s - S_s)^- ds \\ &= N\big[(Y^N_{t_{n+1}} - S_{t_{n+1}})^- - (Y^N_{t_n} - S_{t_n})^-\big]\Delta t. \end{aligned}$$

8.3.2 Learning minimal solutions of Cauchy problem

To learn the solutions and especially the minimal solutions of PDEs, we analyze the PDE and its related obstacle problem instead of dealing with BSDE and reflected BSDE directly.

Obstacle problems

An obstacle $\{S_t, 0 \leqslant t \leqslant T\}$ is a continuous progressively measurable real-valued process satisfying that S_T is bounded almost surely, and

$$\mathbb{E}[\sup_{0\leqslant t\leqslant T} S_t^2] < \infty.$$

Given the obstacle process S_t, the viscosity solution of obstacle problem (8.23) is shown in [21] to be equivalent to the associate reflected BSDE (8.19).

As is explained in Chapter 3, the solution Y_t of (8.13) is constrained to stay above

$Y_t \geqslant S_t$. Thus, we can instead study the probabilistic representation of solutions of some obstacle problems for PDEs. $S_t = h(\tau, \mathcal{X}_t)$, $\tau = T - t$.

$$\min[u(\tau, \mathbf{x}) - h(\tau, \mathbf{x}), \partial_\tau u - \mathcal{A}u(\tau, \mathbf{x})] = 0, \quad (\tau, \mathbf{x}) \in (0, T) \times \mathbb{R}^n.$$

$$u(0, \mathbf{x}) = g(x), \quad x \in \mathbb{R}^n$$

Or equivalently

$$\begin{aligned} 0 = \partial_\tau u - \mathcal{A}u(\tau, \mathbf{x}), & \qquad \{(\tau, x) : u(\tau, \mathbf{x}) > h(t, \mathbf{x})\}, \\ u(\tau, \mathbf{x}) \geqslant h(\tau, \mathbf{x}), & \qquad (\tau, x) : (0, T) \times \mathbb{R}^n, \\ u(\tau, \mathbf{x}) \in C^{1,2}([0, T] \times \mathbb{R}^n), & \qquad \{(\tau, x) : u(\tau, \mathbf{x}) = h(\tau, \mathbf{x})\}, \\ u(0, \mathbf{x}) = g(x), & \qquad x \in \mathbb{R}^n. \end{aligned} \tag{8.23}$$

While the non-negative solution of a parabolic PDE of our interest is

$$\begin{aligned} 0 = \partial_\tau u - \mathcal{A}u(\tau, \mathbf{x}), & \qquad (\tau, x) : (0, T) \times \mathbb{R}^n, \\ u(\tau, \mathbf{x}) \geqslant h(\tau, \mathbf{x}), & \qquad (\tau, x) : (0, T) \times \mathbb{R}^n, \\ u(\tau, \mathbf{x}) \in C^{1,2}([0, T] \times \mathbb{R}^n), & \qquad (\tau, x) : (0, T) \times \mathbb{R}^n, \\ u(0, \mathbf{x}) = g(x), & \qquad x \in \mathbb{R}^n. \end{aligned} \tag{8.24}$$

Since we are looking for the minimal solutions satisfying the above (8.23) and (8.24) correspondingly, we have optimization problems

$$\begin{aligned} & \min u(\tau, \mathbf{x}) \\ & \text{subject to (8.23)}. \end{aligned} \tag{8.25}$$

and

$$\begin{aligned} &\min u(\tau, \mathbf{x}) \\ &\text{subject to } (8.24). \end{aligned} \tag{8.26}$$

As we can see, the domain $\mathcal{D}_1$ of (8.25) is a subset of the domain $\mathcal{D}_2$ of (8.26), $\mathcal{D}_1 \subset \mathcal{D}_2$. In other words, (8.23) relaxes the constraints (8.24). Denote $u^{r\star}(\tau, \mathbf{x}) := \min u^r(\tau, \mathbf{x})$ as the optimal value of (8.25) and $u^\star(\tau, \mathbf{x}) : \min u(\tau, \mathbf{x})$ as the optimal value of (8.26). If both problems are feasible, then $u^{r\star}(\tau, \mathbf{x}) \leqslant u^\star(\tau, \mathbf{x})$.

We earlier proved that the optimal arbitrage opportunity exists because of the Fichera drift. Therefore in order to search $u^\star$ we put a constraint that $u^\star > 0$. Hence $h(\tau, \mathbf{x}) \equiv 0$ here. Further investigation can be carried out for $u^\star \leqslant 1$.

Algorithm for searching minimal solution of parabolic PDEs

We investigate the deep learning based solutions of the obstacle problem (8.23) and the Cauchy problem with the non-negativity constraint of solutions (8.24). This is equivalent to learning BSDE (8.14) and reflected BSDE (8.19)-(8.20) and more implementable. Here (8.20) is

$$Y_t \geqslant 0, \quad 0 \leqslant t \leqslant T. \tag{8.27}$$

We use the same network structure and similiar implementation details about input, hyperparameters and loss functions in Section 8.2. Then we add different loss functions in order to implement (8.23) and (8.24).

Next, we use a smooth penalty function to restrict the trained terminal value $\hat{u}(T, \mathbf{x}_n; \Theta_n) \in (0, 1)$. In Figure 8.4, we show a sketch of the penalty function

$$p(x) = (a - x)\text{sigmoid}(h_1(a - x)) \cdot h_2 + (x - b)\text{sigmoid}(h_3(x - b)) \cdot h_4, \tag{8.28}$$

Here let $a = 0$, $b = 1$, $h_1 = h_3 = 10$, $h_2 = h_4 = 20$ in (8.28) and the loss term follows

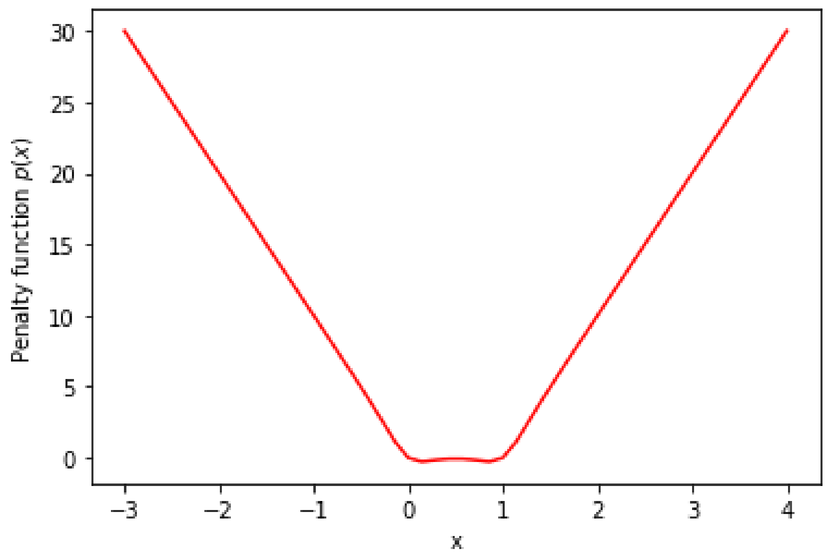


Figure 8.4: The penalty function $p(x; a, b)$ between $[a, b] = [0, 1]$, $h_1 = h_2 = h_3 = h_4 = 10$.

and

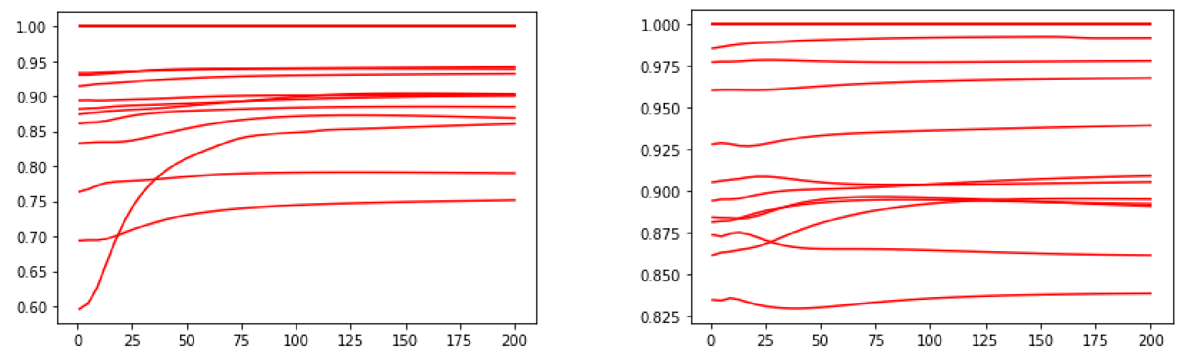
$$L_4(\Theta_n) = p(\hat{u}(t_n, \mathbf{x}_n; \Theta_n)), \tag{8.29}$$

to guarantee that it does not create extra local minimums. Local minimums increase the time of searching for the goal significantly, especially in this problem where we have multiple solutions of the PDE system.

The loss function used for optimizing parameters here is then consisted of $L_1(\cdot)$ in (8.5), $L_2(\cdot)$ in (8.6), $L_3(\cdot)$ in (8.7), $L_4(\cdot)$ in (8.29),

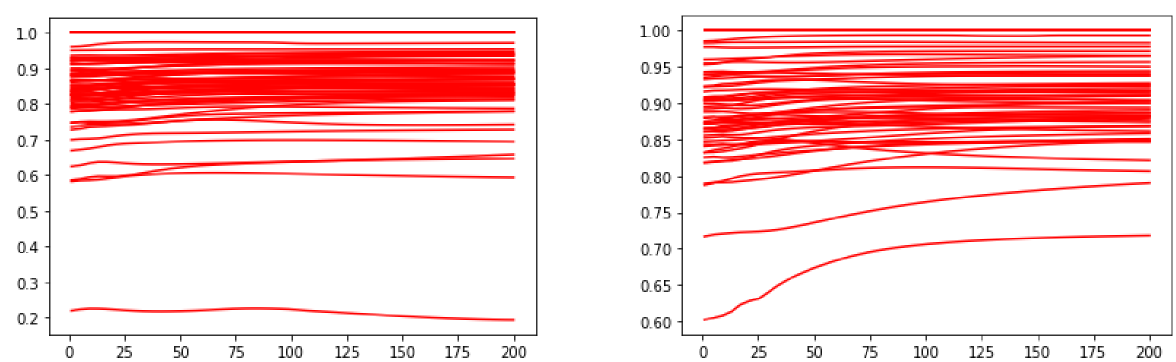
$$L(\Theta_n) = L_1(\Theta_n) + L_2(\Theta_n) + L_3(\Theta_n) + L_4(\Theta_n). \tag{8.30}$$

We carry out the deep learning scheme in Algorithm 2 for 80 times of training and compare with or without the loss term $L_4(\Theta_n)$. In the following graphs, Figure 8.5a and Figure 8.6a correspond to multiple solutions $u^{obs}(\Theta_n)$ of the obstacle problem ((8.19) with obstacle inequality (8.27)). While the obstacle inequality is not presented in Figure 8.5b and Figure 8.6b, i.e., these two plots correspond to multiple solutions of (7.6)-(7.7). We can see that learning the associated obstacle problem tends to give smaller solutions than the original PDEs. In addition, the smallest solution curve of (8.19) and (8.27) is lower than the smallest curve of (7.6)-(7.7). More importantly, as we will see more clearly in the end of this section, modelling the associated obstacle problem/ reflected BSDE pushes the trained solution significantly closer to the benchmark minimal solution from Monte-Carlo methods.



(a) Training output $\hat{u}^{obs}(T, \mathbf{x})$ curve of (8.23). (b) Training output $\hat{u}(T, \mathbf{x})$ curve of (8.24).

Figure 8.5: The comparison of trained curves $\hat{u}(T, \mathbf{x})$ with and without an inequality constraint on $x_i \in [1, 200]$, $i = 1, 2$. The training is repeated 15 times.



(a) Training output $\hat{u}(T, \mathbf{x})$ curve of (8.23). (b) Training output $\hat{u}(T, \mathbf{x})$ curve of (8.24).

Figure 8.6: The comparison of trained curves $\hat{u}(T, \mathbf{x})$ with and without an inequality constraint on $x_i \in [1, 200]$, $i = 1, 2$. The training is repeated 80 times.

We summarize the process of learning the minimal solution of Cauchy problems by obstacle problem in Algorithm 3. As in Algorithm 2, t_n is sampled from uniform distributions, and x_n is sampled from another distribution that is a function of the corresponding time sample point t_n. The sufficient amount of repetitions in Algorithm 2 is in the sense that no more significantly smaller output curves can be learned from more repetitions of training.

Algorithm 3 Obstacle Cauchy deep Galerkin (O-CDG)

1: Generate random samples pairs $(t_n, \mathbf{x}_n) \in [0,T] \times \mathbb{R}^n_+$, and sample pairs $(\tau_n, \mathbf{x}^0_n) \in [0,T] \times \mathcal{O}^n/\{\mathbf{0}\}$. For initial conditions, draw random samples $\mathbf{x} \in \mathbb{R}^n_+$ when $t = 0$.

2: Compute the loss function $L(\cdot)$ by

$$L(\Theta_n) = \sum_{i=1}^{4} L_i(\Theta_n), \tag{8.31}$$

where L_1, L_2 is defined in (8.5)-(8.6), $L_3(\cdot)$ is in (8.7) and $L_4(\cdot)$ in (8.29).

3: Take stochastic gradient descent at a point s_n,

$$\Theta_{n+1} = \Theta_n - \alpha_n \nabla_\Theta L(\Theta, s_n).$$

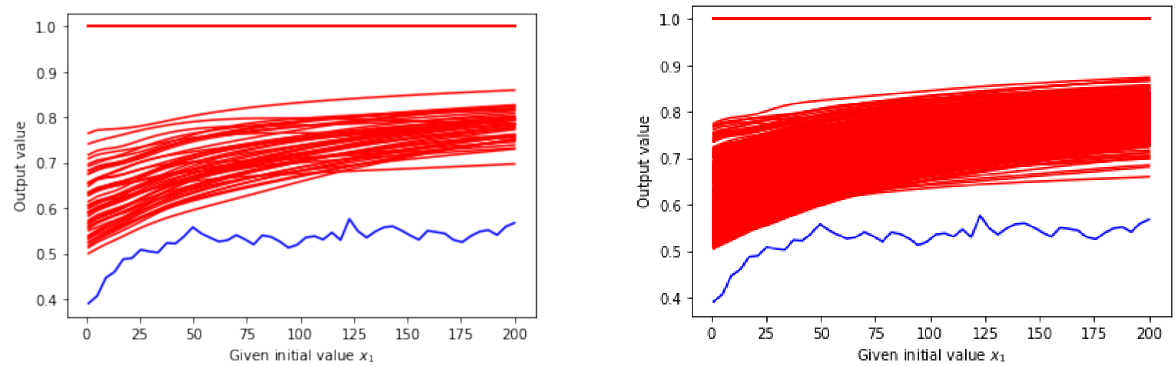
4: Repeat stage 1-3 until convergence criterion is satisfied.

5: Repeat stage 1-4 for sufficient amount of times and take the smallest training output.

The training is on the time horizon $[0,T]$ where $T = 1$. The size of sample pairs (t,x) in internal area and initial time area is 500 and 100, respectively.

We lay out the accuracy metrics of the current obstacle method O-CDG with previous methods (CDG-BE and CDG-RN) in Table 8.1 in order to present the performance improvements.

We can extend this chapter's result to the mean field problem in Chapter 5. More specifically, (5.8) and the non-negative minimal continuous solution of (5.12).



(a) Training output $\hat{u}(T, \mathbf{x})$ curve repeated 80 times of training.

(b) Training output $\hat{u}(T, \mathbf{x})$ curve repeated 500 times of training.

Figure 8.7: The comparison of trained curves $\hat{u}(T, \mathbf{x})$ of (8.23) with the boundary condition on $x_i \in [1, 200]$, $i = 1, 2$. The training is repeated 80 and 500 times.

Table 8.2: Accuracy comparison of minimal curve u with the current state of the art u^{vsm}. We show the performance metrics for O-CDG, CDG-BE and CDG-RN in 80 and 500 times of training.

	Training times					
	O-CDG		**CDG - BE**		**CDG - RN**	
	M = 80	**M = 500**	**M = 80**	**M = 500**	**N = 80**	**N = 500**
MAE	0.11115	0.09199	0.12170	0.09092	0.40116	0.38187
MSE	0.01371	0.00898	0.01713	0.01000	0.16187	0.14668

Conclusion

High-dimensional PDEs are very widely used in science and engineering models. In the beginning of this chapter, we introduced the current literature on solving high-dimensional PDEs which focuses on the PDEs with unique solution.

Generally PDEs we deal with may admit to multiple solutions. The uniqueness of the solution usually requires specific conditions on the operators of PDEs and their initial and boundary conditions. However PDEs with multiple solutions is a challenging topic both theoretically in that it is hard to have a comparison principle alike result, and numerically, in that it adds more complexity onto high-dimensional problems. We can train multiple times to get approximations to different solutions of the PDEs. But how to efficiently determine and distinguish different solutions from a large number of outputs remains a challenge.

We propose several remedies for learning PDEs with non-unique solutions.

Firstly, since the nonuniqueness of PDEs is often caused by insufficient boundary and initial conditions, adding artificial conditions that can be justified for specific problems can help. For example, in the relative arbitrage problem of volatility stabilized market, we add artificial boundary conditions to the PDE. A suitable artificial boundary condition is also helpful in grid-based methods when solving PDEs numerically.

Secondly, when using deep learning methods, applying prior information of specific problems to the learning would improve the training performance in a large scale. In deep Galerkin method we used, the inputs rely on sampling from time and space variables. Specify a suitable distribution of samples follow is a substantial factor to the training performance in this case.

Thirdly, we propose to use the associated reflected BSDE or obstacle problem of the PDE to clarify the range of multiple solutions. O-CDG method is helpful for finding the minimal solution as well as narrow the range of true solutions in PDEs with non-unique solutions.

The approach we propose in this chapter unfold more general forms of PDE problems one can approximate the solutions by deep learning. In particular, we provide an application of this deep learning approach, that is, the minimal non-negative solution of a Cauchy PDE in relative arbitrage problems.

We list some interesting topics that can be studied in the future.

Open questions

- There can be various ways to model the interaction of markets and investors. First, let us consider the same market model we provide in this thesis, with information of a group of investors coming into the market dynamics. Then a different information, or investors compete with others in a more general way - a general benchmark. Second, we can consider mean field control problems with cooperative players or a different game model similar to the one in [23], where the market opposed to an investor is constructed as a zero-sum 2-player game.

- From the numerical aspect, we have seen that the volatility-stabilized market model provides us with a computationally easy way to simulate the solution of Cauchy problems of our interest. However in general a deep learning scheme for PDEs might not have a plausible baseline solution and thus it is difficult to check the performance of deep learning results. How do we solve this issue? What metrics are we able to use in a more general case?

- We discussed relative arbitrage problems when given some terminal time $T \geqslant 0$, i.e., the goal of the investors we consider is to realize relative arbitrage over a fixed horizon $[0, T]$. We can relax this by taking terminal time T^ℓ, an investor-wise input.

  Another concept *short term relative arbitrage* has been discussed in several references to show relative arbitrage opportunities over arbitrarily short time horizons. Short term relative arbitrages in certain volatility-stabilized market models are discussed in [5] using functionally generated portfolios. We can think about the short term relative arbitrage opportunities in our model.

  Moreover, based on the current results from Nash equilibrium and functionally generated portfolio, can we relieve the requirements on coefficient estimation for practical use?

Appendix A

Appendix

A.1 Market dynamics and conditions

Proof of Propositions 3.1.1. Since everyone follows $V^{\ell}(T) \geqslant e^{c_\ell}\mathcal{V}^N(T)$, we sum up this expression for $\ell = 1, \ldots, N$ to get an inequality of $\sum_{\ell=1}^{N} V^{\ell}(t)/N$, and (3.5) follows immediately in Proposition 3.1.1. Next, (3.6) in Proposition 3.1.1 can be easily derived from Definition 3.1.1 that if

$$c_\ell \leqslant \log\left(\frac{V^{\ell}(T)}{\mathcal{V}^N(T)}\right) = \log\left(\frac{V^{\ell}(T)}{\delta X^N(t) + (1-\delta)\frac{1}{N}\sum_{\ell=1}^{N} V^{\ell}(T)}\right), \quad \ell = 1, \ldots, N,$$

then the relative arbitrage exists in the sense of (3.4). □

A.2 Relative arbitrage and Cauchy problem

Proof of Proposition 3.2.1. From Ito's formula, discounted process $\widehat{V}^{\ell}(\cdot)$ admits

$$d\widehat{V}^{\ell}(t) = \widehat{V}^{\ell}(t)\big(\pi^{\ell\prime}(t)\sigma(t) - \theta'(t)\big)dW(t); \quad \widehat{V}^{\ell}(0) = \widehat{v}_\ell,$$

and $\widehat{V}^\ell(\cdot)$ is a supermartingale. For this reason, we get from (3.10) that for an arbitrary ω^ℓ,

$$\omega^\ell \mathcal{V}^N(0) \geqslant \mathbb{E}\big[\widehat{V}^\ell\big] \geqslant \mathbb{E}\Big[\widehat{X}(T)\delta e^{c_\ell} + L(T)(1-\delta)e^{c_\ell}\frac{1}{N}\sum_{\ell=1}^{N} V^\ell(T)\Big] := p^\ell.$$

Hence, $u^\ell(T) \geqslant p^\ell$.

To prove the opposite direction $u^\ell(T) \leqslant p^\ell$, we use martingale representation theorem (Theorem 4.3.4, [53]) to find

$$U^\ell(t) := \mathbb{E}\big[e^{c_\ell}\mathcal{V}^N(T)L(T)|\mathcal{F}_t\big] = \int_0^t \tilde{p}'(s)dW_s + p^\ell, \quad 0 \leqslant t \leqslant T, \tag{A.1}$$

where $\tilde{p} : [0,T] \times \Omega \to \mathbb{R}^k$ is $\mathbb{F}$-progressively measurable and almost surely square integrable. Next, construct a wealth process $V_*(\cdot) = U^\ell(\cdot)/L(\cdot)$, it satisfies $V_*(0) = p^\ell$, $V_*(T) = e^{c_\ell}\mathcal{V}^N(T)$. If we plug a trading strategy

$$h_*(\cdot) = \frac{1}{L(\cdot)V^\ell(\cdot)}\alpha^{-1}(\cdot)\sigma(\cdot)[\tilde{p}(\cdot) + U^\ell(\cdot)\Theta(\cdot)],$$

into (3.8), further calculations imply $V_*(\cdot) \equiv V^{p,h_*}(\cdot) \geqslant 0$ a.s. $V^{p,h_*}(\cdot)$ is the wealth process from $h_*(\cdot)$. Therefore, $h_*(\cdot) \in \mathbb{A}$ with exact replication property $V^{p,h_*}(T) = e^{c_\ell}\mathcal{V}^N(T)$ a.s. Consequently, $p^\ell \geqslant u^\ell(T)$ for

$$\frac{p^\ell}{\mathcal{V}^N(0)} \in \big\{\omega > 0 | \exists \pi^\ell \in \mathbb{A}, \text{given } \pi^{-\ell}(\cdot) \in \mathbb{A}^{N-1}, \text{ s.t.} V^{\omega\mathcal{V}^N(0),\pi^\ell} \geqslant e^{c_\ell}\mathcal{V}^N(T)\big\}.$$

Thus, we proved $u^\ell(T) = \mathbb{E}\big[e^{c_\ell}\mathcal{V}^N(T)L(T)\big] / \mathcal{V}^N(0)$. □

Proof of Theorem 3.3.1. Suppose a solution of (3.22) and (3.14) is $\tilde{w}^\ell : C^2((0,\infty) \times (0,\infty)^n \times (0,\infty)^n) \to (0,\infty)$. Define $\tilde{N}(t) := \tilde{w}^\ell(T-t, \mathcal{X}^N_{[0,t]}, \mathcal{Y}_{[0,t]})e^{c_\ell}\mathcal{V}^N(t)L(t)$, $0 \leqslant t \leqslant T$.

By calculating $d\tilde{N}(t)/\tilde{N}(t)$ and using the inequality (3.22), we get that the dt terms in $d\tilde{N}(t)/\tilde{N}(t)$ is always no greater than 0. $\tilde{N}(t)$ is a local supermartingale. And since

$\tilde{N}(t) = \tilde{w}^{\ell}(T-t, \mathcal{X}_{[0,t]}, \mathcal{Y}_{[0,t]}) e^{c_\ell} \mathcal{V}^N(t) L(t) \geqslant 0$, $\tilde{N}(t)$ is a supermartingale.

Hence $\tilde{N}(0) = \tilde{w}^{\ell}(T, \mathbf{x}, \mathbf{y}) \mathcal{V}^N(0) \geqslant \mathbb{E}^{\mathbb{P}}[\tilde{N}(t)] = \mathbb{E}^{\mathbb{P}}[e^{c_\ell} \mathcal{V}^N(T) L(T)]$ holds for every $(T, \mathbf{x}, \mathbf{y}) \in (0, \infty) \times (0, \infty)^n \times (0, \infty)^n$. Then $\tilde{w}^{\ell}(T, \mathbf{x}, \mathbf{y}) \geqslant \mathbb{E}^{\mathbb{P}}\big[e^{c_\ell} \mathcal{V}^N(T) L(T)\big] / \mathcal{V}^N(0) = \tilde{u}^{\ell}(T, \mathbf{x}, \mathbf{y})$. $\square$

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