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Inelastic final-state interaction

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Abstract

The final-state interaction in multichannel decay processes is systematically studied in the hadronic picture with application to B decay in mind. Since the final-state interaction is intrinsically interwoven with the decay interaction in this case, no simple phase theorem like "Watson's theorem" holds for experimentally observed final states. We first solve exactly the two-channel problem as a toy model in order to clarify the issues. The constraints of the two-channel approximation turns out to be too stringent for most B decay modes, but realistic multichannel problems are too complex for useful quantitative analysis at present. To alleviate the stringent constraints of the two-body problem and to cope with complexity beyond it, we introduce a method of approximation that is applicable to the case where one prominent inelastic channel dominates over all others. We illustrate this approximation method with the amplitude of the decay $B \to K\pi$ fed by the intermediate states of a charmed-meson pair. Even with our approximation we need more accurate information of strong interactions than we have now. Nonetheless we are able to obtain some insight in the issue and draw useful conclusions on general features on the strong phases.

I. INTRODUCTION

The well-known phase theorem [1] holds for the final-state interaction (FSI) of decay processes when the final state consists of a single eigenstate of scattering. While no simple nontrivial extension is known in the case of multichannel final states, some calculations were made in the past with unjustified extension of the single-channel phase theorem[2]. A two-channel problem was studied with a certain class of S-matrix and the correct observation was made that inelastic channels are the main source of strong phases in many B decay modes[3]. However, it is not easy to obtain quantitatively reliable results from the two-channel model. Taking the large limit of open channels, the statistical model [4, 5] was proposed as an alternative approach. Quantitatively, however, it is short of predictive power since it does not ask for detailed knowledge of strong interaction. When one approaches the problem in the quark-gluon picture, one faces inability or large uncertainty in computing contributions of the soft collinear constituents numerically. It is fair to say that at present we are far from successful computation of a FSI phase in multichannel decay processes.

In this paper we first study the two-channel problem in detail. The problem is solvable in a reasonably compact form without approximation or assumption if relevant information is available about strong interaction physics and decay branching fractions. The general solution to the two-channel toy-model shows how the elastic scattering phases and the channel coupling contribute to the total FSI phase. It concludes in agreement with Donoghue *et al*[3] that if a large strong phase emerges in the B decay into two light-mesons, its major source is coupling to decay channels that have large branching fractions.[1] Although it points to the source of problems in strong phases, the simple two-channel model is inapplicable to B decay. We proceed to the case of more than two decay channels. Even the three-channel problem is mathematically too complicated for solving in a compact form. On the physics side our knowledge of strong interaction at total energy 5 GeV ($\simeq m_B$) is not good enough to carry through the analysis with precision. To cope with formidable complexity of the problem, we introduce an approximation method that works in the case that, aside from the channel of our interest, one inelastic channel dominates over all others. This is different from an approximate two-channel problem. It can happen, for instance, to the two-body light-hadron decays of B meson when they couple to the charmed-meson pair states. We apply our approximation method to the decay $B \to K\pi$ and make semiquantitative analysis with knowledge of hadron physics currently available to us.

II. FRAMEWORK AND INPUT

Two basic ingredients in discussion of FSI are unitarity and time-reversal. In the Standard Model the decay interaction is sum of effective local operators $\mathcal{O}_a(a = 1, 2, 3, \cdots)$, each of which has a CP-violating (and therefore T-violating) phase factored out as

$$H_{int}e^{i\delta_\mathrm{w}} + \mathrm{h.c.}, \quad (TH_{int}T^{-1} = H_{int}). \tag{1}$$

The T-violating "weak phase" δ_w arises from the CKM elements. In computing decay amplitudes we work separately on the T-invariant part H_{int} of the decay interaction. In the

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[1] This was suggested to the author by Wolfenstein in many occasions over years.[6]

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case that T-violation is more general, we can break up the weak interactions H_w into T-even and T-odd parts as $H_{int}^{(\pm)} \equiv \frac{1}{2}(H_w \pm TH_wT^{-1})$ so that

$$TH_{int}^{(\pm)}T^{-1} = \pm H_{int}^{(\pm)}. \tag{2}$$

Then both $H_{int}^{(+)}$ and $iH_{int}^{(-)}$ are T-even since $i \to i^* = -i$ under time reversal.[2] We should compute the decay amplitudes for $H_{int}^{(+)}$ and $iH_{int}^{(-)}$ separately and take a suitable sum of them at the end. Therefore it is sufficient to consider in general only T-even weak interactions. No matter what method one may use, computation of FSI phases must always be made separately for different decay operators. Because two decay operators generate two different FSI phases for the same decay process even if the net quantum numbers of operators are identical. We consider in this paper only final-state interaction of strong interaction though it is in principle easy to include electromagnetic FSI. We shall refer to FSI phases also as strong phases in this paper.

With T-invariance, the strong S-matrix operator obeys

$$TST^{-1} = S^\dagger. \tag{3}$$

We can always choose phases of states such that the T-invariant S-matrix elements ($S_{kj} = \langle k|S|j\rangle = \langle k^{out}|j^{in}\rangle$) are not only unitary but symmetric;

$$S_{jk} = S_{kj} \tag{4}$$

since $|j^{\text{in}}\rangle \to \langle j^{\text{out}}|$ and $\langle k^{\text{out}}| \to |k^{\text{in}}\rangle$ under time reversal. It is emphasized here that the requirement of Eq. (4) fixes the phases of states except for the overall sign of ± 1.[3] Specifically for the eigenchannels of the S-matrix $|a^{in}\rangle$ and $\langle b^{out}|$, it holds by definition that

$$\langle b|S|a\rangle = \langle b^{out}|a^{in}\rangle = \delta_{ba}e^{2i\delta_a}, \tag{5}$$

where δ_a is the eigenphase shift. In the case of decay matrix elements, the initial state is a one-particle state that is stable with respect to strong interaction. Since the initial decaying state is an asymptotic state with respect to strong interaction, there is no distinction between "in" and "out" states. For B decay

$$|B\rangle \xrightarrow{T} \langle B|, \tag{6}$$

where we choose the B meson at rest. Eq. (6) removes an arbitrary unphysical phase from the state $|B\rangle$ too.

A simple relation results from time reversal of the decay matrix element $\langle a^{out}|H_{int}|B\rangle$ when the final state $\langle a^{\text{out}}|$ is an eigenstate of S. T-invariance of H_{int} leads to

$$\langle a^{out}|H_{int}|B\rangle = \langle B|H_{int}|a^{in}\rangle. \tag{7}$$

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[2] If we make this breakup for the Standard Model interaction, we would get $H_{int}^{(+)} = \cos\delta_{\text{w}}H_{int}$ and $iH_{int}^{(-)} = -\sin\delta_{\text{w}}H_{int}e^{i\delta_{\text{w}}}$ for the first term $H_w = H_{int}e^{i\delta_{\text{w}}}$ of Eq. (1).

[3] If one multiplies the states with some phases as $|j\rangle \to e^{i\alpha}|j\rangle$ and $|k\rangle \to e^{i\beta}|k\rangle$, S_{jk} and S_{kj} would acquire phases of opposite signs $e^{\pm i(\beta-\alpha)}$ so that the equality $S_{jk} = S_{kj}$ would break down.

Inserting the completeness relation $\sum_b |b^{out}\rangle\langle b^{out}| = I$ next to H_{int} in the right-hand side, we obtain from Eq. (7) with Eq. (5)

$$\langle a^{out}|H_{int}|B\rangle = \sum_b \langle B|H_{int}|b^{out}\rangle\langle b^{out}|a^{in}\rangle = e^{2i\delta_a}\langle a^{out}|H_{int}|B\rangle^*. \tag{8}$$

With the decay amplitude $\langle a^{out}|H_{int}|B\rangle$ into the eigenchannel a denoted by A_a, this relation reads

$$A_a = e^{2i\delta_a}A_a^*, \tag{9}$$

namely, $A_a = \pm e^{i\delta_a}|A_a|$. This is the well-known phase theorem usually referred to as Watson's theorem[1]. It is a powerful theorem when the final state is an eigenstate of S. It has been an important tool of analysis in hyperon decay and $K \to \pi\pi$ decay.

But this relation is of little use when rescattering has inelasticity. We mean by *inelasticity* that observable final states are linear combinations of eigenstates whose weights are *not* simply determined by the Clebsch-Gordan coefficients of isospin symmetry or SU(3) coefficients. In such cases the phase of the elastic scattering amplitude has little to nothing to do with the FSI phase of the corresponding decay amplitude, as we shall see it in a moment.

Usefulness of the phase theorem is thus limited to the decay of low-mass particles where rescattering is purely elastic up to isospin structure. If the K meson mass were sufficiently above 1 GeV, for instance, $\pi\pi$ of definite isospin would no longer be an eigenstate of S-matrix even approximately. The state $\rho\rho$ and $\omega\omega$ would enter an S-matrix eigenchannel with $\pi\pi$ and composition of such an eigenstate depends on low-energy dynamics of the transition among $\pi\pi$, $\rho\rho$, and $\omega\omega$. In the case of B decay an experimentally observed final state is a linear combination of many different S-matrix eigenstates so that the net FSI phase results from the eigenphases weighted with the decay amplitudes of $B \to$ eigenstate. Take. for instance, the $\pi\pi$ final state in $I = 0$ of $\overline{B}^0$ decay. The state $|\pi\pi\rangle_{I=0}$ (in s-wave) is far from being an S-matrix eigenstate at energy m_B. If we want to use the phase theorem Eq. (9), we must expand $|\pi\pi\rangle$ in the strong S-matrix eigenstates at m_B. However, we have little knowledge of these eigenstates since their composition depends sensitively on strong interaction at long and intermediate distances. Experimentally the two-body channels do not account for all final states in B decay. Three and four particle final states of the same J^{PC} may be significant. Unless there is a good reason to believe that channel coupling is negligible among these final states, the strong S-matrix eigenstates at m_B are made of many different particle states ($\pi\pi$, $\rho\rho$, $K\overline{K}$, $\pi\pi K\overline{K}$, $\cdots$). Therefore, if we expand the $\pi\pi$ state at total energy m_B, it is a linear combination of many different eigenstates of strong S-matrix. We would have to know these expansion coefficients in order to determine the FSI phase of $\pi\pi$ final state with the phase theorem.

Let us formulate what we have described above. When an observable final state $|i^{out}\rangle$ is not S-matrix eigenstate, we expand it in the eigenstates of S-matrix $|a^{out}\rangle$ as[4]

$$|i^{out}\rangle = \sum_a O_{ia}|a^{out}\rangle. \tag{10}$$

We choose that the S-matrix is symmetric ($S_{ij} = S_{ji}$) in the basis of the observable states. (cf Eq./ (4)) Then the expansion coefficients O_{ia} are real, that is, the transformation matrix

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[4] We represent the S-matrix eignestates by $|a\rangle$, $|b\rangle$, $\cdots$ and the observed particle states by $|i\rangle$, $|j\rangle$, $|k\rangle$, $\cdots$.

$\mathcal{O}$ in Eq. (10) is an orthogonal matrix. (See Appendix if proof is needed.) With this expansion and Eq. (9),

$$A_i = \sum_a O_{ia} A_a = \sum_a O_{ia} e^{2i\delta_a} A_a^*. \tag{11}$$

We are able to write the right-hand side of Eq. (11) in terms of observable decay amplitudes A_j's as

$$
\begin{aligned}
A_i &= \sum_{aj} O_{ia} e^{2i\delta_a} O_{ja} A_j^*, \\
&= \sum_j S_{ij} A_j^*,
\end{aligned}
\tag{12}
$$

where $S_{ij} = O_{ia} e^{2i\delta_a} O_{ja}$ has been used. This is the fundamental relation in discussion of FSI. The physical picture is simple and clear. (Fig.1) The input is unitarity and time-reversal of the S-matrix aside from choice of unphysical phases of states. Dynamical information of strong interaction is fed through the eigenphases δ_a and the orthogonal mixing matrix $\mathcal{O}$. In addition, we must provide relative magnitude of the amplitudes A_i as independent pieces of input from weak interaction. Consequently the phase of A_i depends not only on strong interaction but also on weak interaction. It is clear here that the FSI phase of a decay amplitude has virtually nothing to do with the phase of the elastic scattering amplitude $a_J(s)$ of $J = 0$ at 5 GeV. Although it looks almost futile to go any further, the purpose of this paper is to extract something useful for B decay out of Eq. (12).
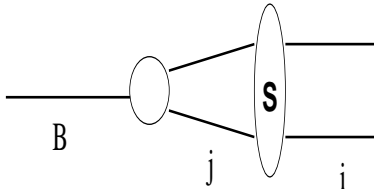

FIG. 1: The final-state interaction relation in diagrams. Eq. (12).

III. DYNAMICAL INPUT

We focus on the FSI phases of the two-body decay modes of the B meson. The phases of three-body decay amplitudes depend on the sub-energies of three particles. It is only the phases integrated over the sub-energies with the total energy fixed to m_B that enter Eq. (12). We need the S-matrix elements S_{ij} from experiment. To be concrete, let us consider the elastic $\pi^+\pi^-$ scattering amplitude as an example. The argument below is identical for other two-body channels of light mesons. In B decay the relevant partial-wave channel is $\pi^+\pi^-$ in s-wave ($J^P = 0^+$) with isospin $I = 0$ and $I = 2$.

Although the high-energy $\pi\pi$ scattering cannot be directly measured in experiment, we can make a reasonable estimate about elastic $\pi\pi$ scattering since the center-of-momentum energy 5 GeV is in the high-energy asymptotic region well above the $\pi\pi$ resonances.[5] The

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[5] The excited charmonia exist at mass not far below 5 GeV but their coupling to light hadrons is suppressed by QCD. Coupling of $\pi\pi$ to the open charm channels[7] is one of interesting subjects of our study later in this paper.
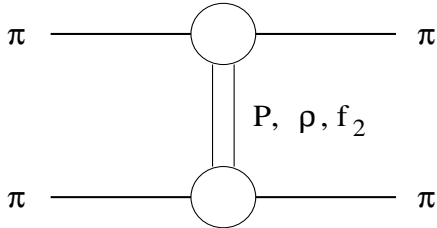
FIG. 2: The Regge exchange in elastic $\pi\pi$ scattering.

two-body light-hadron scattering in the high energy asymptotic region was studied theoretically and experimentally in the 1960's. The Regge theory describes this physics well. The properties of the Regge trajectories can be deduced from meson-nucleon and nucleon-nucleon scattering even though we have no meson-meson scattering experiment. We give a very brief review[8] of our knowledge in this area of the past time since it is an important input in our study of FSI.

First of all, the Pomeron exchange dominates in high-energy elastic scattering. (Fig 2) The Pomeron may include a cut and can be more a complicated singularity than a simple pole in J-plane unlike the non-Pomeron trajectories such as ρ and f_2. At the level of numerical accuracy of our discussion, however, we treat the Pomeron as a simple pole at $J = \alpha(t)$ with the intercept $\alpha(0) = 1$ and a vanishingly small slope. This entails factorization of the J-plane residue into product of two vertices. Since isospin is zero for the Pomeron, it contributes equally to the $I = 0$ and $I = 2$ states of the crossed channels $\pi\pi$. With these properties of the Pomeron we obtain necessary pieces of information on $\pi\pi$ scattering from πp and pp scattering at high energies.

The invariant amplitude for the asymptotic elastic scattering is parametrized with the Regge parameters of the Pomeron in the form

$$A^{\pi\pi}(s,t) = -\beta_P^{\pi\pi}(t)\frac{1 + e^{-i\pi\alpha_P(t)}}{\sin \pi\alpha_P(t)}\left(\frac{s}{s_0}\right)^{\alpha_P(t)} \tag{13}$$

where $\beta_P^{\pi\pi}(t) = (\beta_P^{\pi}(t))^2$ and $\alpha_P(t) \simeq 1 + \alpha_P'(0)t + \cdots$ are the factorized residue and the trajectory of the Pomeron, respectively, in terms of the invariant momentum transfer t.[6] We can fix $\beta_P^{\pi\pi}(0)$ by the optical theorem $\sigma_{tot} = \mathrm{Im}A(s,0)/s$ and the factorization relation $\sigma_{tot}^{\pi\pi} \simeq (\sigma_{tot}^{\pi p})^2/\sigma_{tot}^{pp} \simeq 22mb$ at $\sqrt{s} = m_B$ [8, 9]. The value $22mb$ is in line with the empirical quark counting rule, $\sigma_{tot}^{\pi\pi} \simeq \frac{2}{3}\sigma_{tot}^{\pi p} \simeq (\frac{2}{3})^2\sigma_{tot}^{pp}$[10]. The width of the forward peak is more or less universal to all elastic hadron scatterings[11, 12]. We approximate the Pomeron slope to zero ($\alpha_P'(0) \simeq 0$) and fit the forward elastic peak with the standard exponential form $\exp(t/t_0)$. The diffraction peak parameter is fixed by experiment to $t_0 \simeq (0.22 \sim 0.29)\mathrm{GeV}^2$ by the elasticity σ_{el}/σ_{tot} at $\sqrt{s} = m_B$[9, 13, 14]. This value of t_0 reproduces the t dependence of $d\sigma_{el}/dt$ that falls by roughly three orders of magnitude from $t = 0$ to $t \simeq -1$ GeV^2 for pp and πp scattering[11, 12]. It is justified by factorization to choose the $\pi\pi$ forward peak parameter equal to that of pp and πp. The $\pi\pi$ invariant amplitude at high energies is thus

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[6] The possible small $\log^2 s$ rise of $\sigma_{tot}(s)$ does not set in at $\sqrt{s} \simeq$5 GeV. We may safely ignore it in our numerical work. We have included in $\beta_P^{\pi\pi}(t)$ the t-dependence of some other kinematical factors such as $2\alpha_P(t) + 1$. The value of s_0 is at our choice, normally chosen to be 1 GeV^2.

set to

$$A^{\pi\pi}(s,t) \simeq 22 \text{ mb} \times i \, s \, e^{\frac{t}{t_0}}, \tag{14}$$

where we choose $t_0 = (0.253 \pm 0.033) \text{GeV}^2$. The uncertainty in t_0 is primarily due to whether one estimates it with σ_{el}/σ_{tot} of $\pi^{\pm}p$ or pp and $p\overline{p}$.

The partial-wave amplitudes $a_l(s)$ can be projected out of $A^{\pi\pi}(s,t)$. The result for the s-wave is:

$$a_0^{\pi\pi}(s) = (0.282 \pm 0.037)i, \quad (\text{at } s = m_B^2), \tag{15}$$

which leads to the s-wave S-matrix with $2|\mathbf{p}_{\text{cm}}|/\sqrt{s} \simeq 1$,

$$S_0^{\pi\pi}(s) = 1 + 2ia_0^{\pi\pi}(s). \tag{16}$$

Hereafter we shall often parametrize strength of elastic scattering by ϵ as

$$S_0(s) = 1 - \epsilon, \tag{17}$$

With Eq. (15), the Pomeron contribution to the S-matrix of $\pi\pi$ scattering at m_B is

$$S_0^{\pi\pi} \simeq 1 - (0.56 \pm 0.07). \tag{18}$$

The partial-wave amplitudes $a_l(s)$ extracted from the flat Pomeron amplitude are purely imaginary for all l. The amplitude $a_0(s)$ approaches asymptotically the imaginary axis below the center of the Argand diagram (shown schematically for $I = 0$ in Fig. 3). If one described the high inelasticity of high-energy $\pi\pi$ scattering by an absorptive black sphere potential, one would have $S_0^{\pi\pi}(s) \to 0$ (i.e., $a_0(s) \to 0.5i$). In this limit $\sigma_{\text{el}} \to \frac{1}{2}\sigma_{\text{tot}}$ for all l's by shadow scattering effect, which is in disagreement with experiment. Although the numerical value in the right-hand side of Eq. (18) has been extracted for the $\pi\pi$ channel, it is much the same for other two-meson channels. With a help of the Kp cross section σ_{tot}^{Kp}[9] we obtain

$$\begin{aligned} S_0^{\pi K} &\simeq 1 - 0.51, \quad (\pi K) \\ S_0^{K\overline{K}} &\simeq 1 - 0.45, \quad (K\overline{K}), \end{aligned} \tag{19}$$

where uncertainties are comparable to ± 0.07 quoted for $\pi\pi$ in Eq. (18) or a little larger.

The values in Eqs. (18) and (19) are the Pomeron contribution alone. The nonleading Regge exchanges generate a small imaginary part for S_0. The relevant trajectories are the ρ and f_2 in the case of $\pi\pi$. Their contributions can be estimated with a few additional theoretical inputs[7] from the cross section difference $\sigma_{\pi^-p} - \sigma_{\pi^+p} \simeq 1.6mb$ at $\sqrt{s} = m_B$[14]. Within the uncertainty due to t-dependence of the residue $\beta(t)$, their contributions to $\text{Im}S_0$ are at the level of $0.05i$ for $\pi\pi$. Some may wonder about validity of extracting the s-wave amplitude from the forward peak region alone. The s-wave amplitude has a flat angular dependence so that the contribution of $a_0(s)$ extends equally to all directions ($P_0(\cos\theta) = 1$). On the other hand experiment shows that the forward peak falls off by more than three orders of magnitude and there is no sign of the s-wave contribution at large angles. But this is no surprise. The s-wave amplitude at large angles is canceled by the partial-wave amplitudes of up to $l = O(\sqrt{s})$ which are rapidly oscillatory in angular dependence as $P_l(\cos\theta) \sim \sin[(l + \frac{1}{2})\theta + \frac{\pi}{4})]/\sqrt{l\sin\theta}$ (for $l\theta \gg 1$).

Now our task is to extract useful pieces of information from Eq. (12) with the high-energy elastic S-matrix of Eq. (18) or Eq. (19).

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[7] The inputs are the isospin-current coupling of ρ and the exchange degeneracy of ρ and f_2.
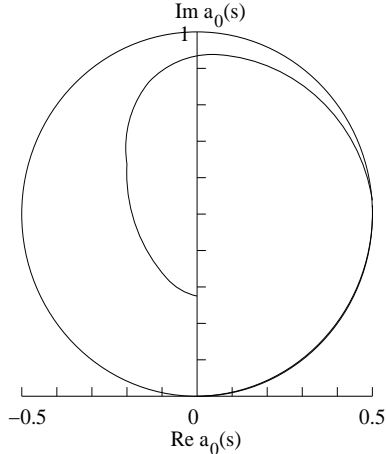
FIG. 3: Energy dependence of $a_0(s)$ in the complex plane (Argand diagram).

IV. FINAL STATE INTERACTION OF TWO COUPLED CHANNELS

Let us first count how many dynamical quantities are involved in the most general $(n \times n)$ FSI relation, Eq. (12). The unitary and symmetric S matrix contains $\frac{1}{2}n(n+1)$ independent parameters; n eigenphase shifts and $\frac{1}{2}n(n-1)$ rotation angles of $\mathcal{O}$. To solve for A_i's of observable channels in Eq. (12), therefore, we must feed the $\frac{1}{2}n(n+1)$ dynamical parameters of strong interaction. This is not sufficient to determine A_i uniquely. Although the FSI relation Eq. (12) may look as if it introduced $2n$ constraints through the real and imaginary parts, only a half of them, namely n of them are actually independent.[8] We must provide the relative magnitudes of A_a or A_i as an additional input in order to determine the FSI phases uniquely. The magnitude of a decay amplitude is determined primarily by weak interaction, *i.e.*, the property of decay operators. Knowledge of strong interaction alone can never determine multichannel FSI phases. We need to know interplay of strong and weak interactions.

Nobody is capable of tackling this problem for a general value n. We will therefore be content with studying the FSI relation first in the simple manageable case of $n = 2$ and then searching a sensible approximation in more complicated and realistic cases.

Although the two-channel problem is the next to the simplest, there has been no serious attempt to study this case in the past, probably with a good reason as we see below. Although it may not look much relevant to the B decay of the real world, we have a chance to see through general characteristics of coupled channel effects. For instance, how is the FSI phase of $\pi\pi$ channel affected by the $\rho\rho$ channel ? If one of the charm-anticharm channels such as $D^{(*)}\overline{D}^{(*)}$ strongly couples to the $\pi\pi$ channel, how does this channel affect the FSI of the $\pi\pi$ channel ?[9] While simple-minded perturbative calculations have been undertaken in the past, we would like to study these questions systematically with the two-channel toy

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[8] In the case of a single channel, the relation $A_1 = e^{2i\delta_1}A_1^*$ gives a constraint only on the phase of A_1, not its magnitude. In the case of n channels, something similar happens: The phases of A_a for *eigenchannels* are determined when S_{ij} are completely specified, but their magnitudes $|A_a|$ are not.

[9] For some dynamical reason the branching fraction to $\rho\rho$ is an order of magnitude larger than that to $\pi\pi$. The $D^*\overline{D}^*$ channel has a huge branching fraction because of the robust $b \to c$ transition.

model that incorporates unitarity.

We can write the general T-invariant S-matrix of 2×2 with three parameter ($\frac{1}{2}n(n+1)=3$ for $n=2$) in the form of

$$S = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e^{2i\delta_1} & 0 \\ 0 & e^{2i\delta_2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$
$$= \begin{pmatrix} e^{2i\delta_1}c_\theta^2 + e^{2i\delta_2}s_\theta^2 & (e^{2i\delta_1} - e^{2i\delta_2})c_\theta s_\theta \\ (e^{2i\delta_1} - e^{2i\delta_2})c_\theta s_\theta & e^{2i\delta_1}s_\theta^2 + e^{2i\delta_2}c_\theta^2, \end{pmatrix} \tag{20}$$

where $\cos\theta$ and $\sin\theta$ are abbreviated as c_θ and s_θ in the second line. Substituting this S-matrix in the FSI relation Eq. (12), we obtain the constraints on the real and imaginary parts or the magnitudes and phases of the decay amplitudes defined by[10]

$$A_j = a_j + ib_j = |A_j|e^{\Delta_j}, \quad (j = 1, 2) \tag{21}$$

where the phases $\Delta_{1,2}$ are the FSI phases (the strong phases) of channel 1 and 2. We have in mind $j = 1$ for $\pi\pi$ and $j = 2$ for either $\rho\rho$ or $D^{(*)}\overline{D}^{(*)}$ of $I = 0$. The constraining equation of Eq. (12) can be written out for the real and imaginary parts as

$$\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \mathrm{Re}S & \mathrm{Im}S \\ \mathrm{Im}S & -\mathrm{Re}S \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix}. \tag{22}$$

where $\mathrm{Re}S$ and $\mathrm{Im}S$ are the 2×2 matrices. To be explicit,

$$\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} c_{2\delta_1}c_\theta^2 + c_{2\delta_2}s_\theta^2 & (c_{2\delta_1} - c_{2\delta_2})c_\theta s_\theta & s_{2\delta_1}c_\theta^2 + s_{2\delta_2}s_\theta^2 & (s_{2\delta_1} - s_{2\delta_2})c_\theta s_\theta, \\ (c_{2\delta_1} - c_{2\delta_2})c_\theta s_\theta & c_{2\delta_1}s_\theta^2 + c_{2\delta_2}c_\theta^2 & (s_{2\delta_1} - s_{2\delta_2})c_\theta s_\theta & s_{2\delta_1}s_\theta^2 + s_{2\delta_2}c_\theta^2 \\ s_{2\delta_1}c_\theta^2 + s_{2\delta_2}s_\theta^2 & (s_{2\delta_1} - s_{2\delta_2})c_\theta s_\theta & -(c_{2\delta_1}c_\theta^2 + c_{2\delta_2}s_\theta^2) & -(c_{2\delta_1} - c_{2\delta_2})c_\theta s_\theta, \\ (s_{2\delta_1} - s_{2\delta_2})c_\theta s_\theta & s_{2\delta_1}s_\theta^2 + s_{2\delta_2}c_\theta^2 & -(c_{2\delta_1} - c_{2\delta_2})c_\theta s_\theta & -(c_{2\delta_1}s_\theta^2 + c_{2\delta_2}c_\theta^2) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix}. \tag{23}$$

As we have pointed out above, Eq. (23) contains only two independent constraints, not four. Indeed, one can show that two eigenvalues of the 4×4 matrix in the right-hand side are unity and generate no constraint. The two remaining eigenvalues are -1 and generate constraints.

We now feed our dynamical input Eq. (17) in the slightly different notation;

$$S_{11} = 1 - \epsilon, \quad (|S_{11}| < 1) \tag{24}$$

where ϵ is real but not necessarily very small in magnitude. For our discussion later we choose ϵ to be $\simeq 0.5$, as suggested by the Pomeron dominance in elastic scattering. It is easy to include the nonleading Regge contributions and relax the condition $\epsilon^* = \epsilon$. Fixing ϵ amounts to setting two parameters in S-matrix of Eq. (20) so that we are left with one parameter out of three. When we fix the S_{11} component as in Eq. (24), it is more convenient to parametrize the S-matrix in the form

$$S = \begin{pmatrix} 1 - \epsilon & i\sqrt{2\epsilon - \epsilon^2}e^{i\chi} \\ i\sqrt{2\epsilon - \epsilon^2}e^{i\chi} & (1 - \epsilon)e^{2i\chi} \end{pmatrix}, \tag{25}$$

---

[10] Recall that the arbitrary unphysical phases of states have been fixed up to an overall $\pm$ sign by the symmetry condition $S_{ij} = S_{ji}$ on the S-matrix.

where the angle χ is the remaining single parameter of the S-matrix. It is related to two eigenphase shifts $\delta_{1,2}$ defined in Eq. (20) by

$$\sin 2\chi = \frac{\sin^2 2\delta_1 - \sin^2 2\delta_2}{\sin 2(\delta_1 - \delta_2)}. \tag{26}$$

Note that once S_{11} is given, magnitude of the channel coupling $|S_{12}| = \sqrt{2\epsilon - \epsilon^2}$ in Eq. (25) is fixed by unitarity and no longer a free parameter.

A.   Case of $|S_{22} - 1| > |S_{11} - 1|$

Let us first study the case where the partial-wave amplitude $a_0^{\pi\pi}(s)$ of rescattering is stronger in the second channel than in the first channel, namely, i.e., $|S_{22} - 1| > |S_{11} - 1|$. This may serve as a toy-model of $\pi\pi$ and $D^*\overline{D}^*$;[11] The $D^*\overline{D}^*$ scattering of $I = 0$ is presumably strong because of the near-threshold enhancement and/or broad excited charmonium resonances. For illustration we consider the extreme case that the rescattering in the second channel is maximally strong relative to that in the first channel. This is realized by choosing $e^{2i\chi} = -1(\chi = \frac{\pi}{2})$ in Eq. (25). The symmetric unitary S-matrix takes the form of

$$S = \begin{pmatrix} 1 - \epsilon & -\sqrt{2\epsilon - \epsilon^2} \\ -\sqrt{2\epsilon - \epsilon^2} & -1 + \epsilon \end{pmatrix}, \quad (\epsilon^* = \epsilon). \tag{27}$$

In terms of partial-wave amplitudes, this gives $a_0^{\pi\pi}(s) = \frac{1}{2}i\epsilon$ and $i(1 - \frac{1}{2}\epsilon) \approx \frac{3}{2}i\epsilon$ for the first and second channels, respectively, with $\frac{1}{2}\epsilon \approx 0.25$. By substituting this S-matrix in the FSI relations, we obtain (see Eq. (21))

$$A_1 = a_1 + ib_1, \tag{28}$$
$$A_2 = -\sqrt{\frac{\epsilon}{2 - \epsilon}}\left(a_1 - i\frac{2 - \epsilon}{\epsilon}b_1\right).$$

The real and the imaginary parts of A_1 are still independent of each other. The phases $\Delta_{1,2}$ cannot be determined uniquely even after the S-matrix is fully specified. But the phase Δ_1 is related to Δ_2 by

$$\begin{aligned} \tan\Delta_1 &= -\frac{\epsilon}{2 - \epsilon}\tan\Delta_2 \\ &\simeq -\frac{1}{3} \times \tan\Delta_2. \end{aligned} \tag{29}$$

Even when χ of $S_{22} = |S_{22}|e^{2i\chi}$ is equal to $\frac{1}{2}\pi$ (normally called as "resonant"), Δ_2 is not necessarily equal to $\frac{1}{2}\pi$. Although this may look puzzling at the first sight, it is not. To determine $\Delta_{1,2}$ uniquely, we need to feed one more piece of information. For instance, if the value of the ratio $|A_2/A_1|$ is supplied, we can determine Δ_1 and Δ_2 individually. By eliminating Δ_2 from Eq. (29) we obtain the relation that determines Δ_1 in terms of $|A_2/A_1|$ and ϵ:

$$\sin^2\Delta_1 = \frac{\epsilon}{4(1 - \epsilon)}\left[(2 - \epsilon)\frac{|A_2|^2}{|A_1|^2} - \epsilon\right]. \tag{30}$$

---

[11] The $D^*\overline{D}^*$ branching is larger than that of $D\overline{D}$. The $D^*\overline{D}$ and $D\overline{D}^*$ channels cannot make $J^P = 0^+$.

The ratio $|A_2/A_1|$ contains information of weak interactions. In multichannel decay, weak interaction plays a very important role in determining the FSI phases.

There is one shortcoming of this two-channel toy model: As one sees in Eq. (30), the ratio $|A_2/A_1|$ must lie in the range of

$$\sqrt{\frac{\epsilon}{2-\epsilon}} \leq \frac{|A_2|}{|A_1|} \leq \sqrt{\frac{2-\epsilon}{\epsilon}} \tag{31}$$

for this model to be applicable. At the lower boundary of $|A_2/A_1|$, it happens that $\Delta_1 = \Delta_2 = 0$, while $\Delta_1 = -\Delta_2 = \pm 90°$ at the upper boundary of $|A_2/A_1|$. When $|A_2/A_1|$ is in between, both Δ_1 and Δ_2 take nonzero values even though all elements of the S-matrix are real. This is an important point to be emphasized. The phase of A_1 can arise from the process $B \to 2 \to 1$ through the intermediate state 2. Some may wonder why the ratio $|A_2/A_1|$ is constrained in the two-channel toy model. With channel coupling present, one channel feeds the other by FSI to the direction to equalize magnitudes of $|A_1|$ and $|A_2|$. The FSI not only generates phases for A_1 and A_2 but also alters their magnitudes. Highly asymmetric $|A_1|$ and $|A_2|$ are incompatible with the FSI connecting the two channels unless $\epsilon \to 0$, i.e., $|S_{12}| \to 0$.

B. Case of $S_{22} \simeq S_{11}$

When channel 2 is $\rho\rho$, we expect that the elastic $\rho\rho$ scattering is asymptotic at $\sqrt{s} \simeq 5\text{GeV}$ and very similar to the elastic $\pi\pi$ scattering:

$$S_{22} = S_{11} = 1 - \epsilon, \quad (\epsilon^* = \epsilon) \tag{32}$$

In this case unitarity and symmetry require that the off-diagonal element S_{12} should be purely imaginary ($\chi = 0$ in Eq. (25)):

$$S = \begin{pmatrix} 1 - \epsilon & \sqrt{2\epsilon - \epsilon^2}i \\ \sqrt{2\epsilon - \epsilon^2}i & 1 - \epsilon \end{pmatrix}. \tag{33}$$

This is the case that was studied by Donoghue *et al*[3]. The FSI relation Eq. (12) leads us to

$$\begin{aligned} A_1 &= a_1 + ib_1 \\ A_2 &= \sqrt{\frac{2-\epsilon}{\epsilon}}\left(b_1 + i\frac{\epsilon}{2-\epsilon}a_1\right), \\ \tan\Delta_1 \tan\Delta_2 &= \frac{\epsilon}{2-\epsilon} \simeq \frac{1}{3}. \end{aligned} \tag{34}$$

Here again the FSI phases are uniquely determined only after the ratio $|A_2|/|A_1|$ is given. The phase Δ_1 is expressed in terms of $|A_2/A_1|$ and ϵ by the same relation as Eq. (30). The ratio $|A_2/A_1|$ lies in the same range as Eq. (31). In contrast to the case of the maximum $|S_{22} - 1|$, the amplitude A_2 and A_1 are 90° out of phase, one real and the other purely imaginary, at the lower and upper boundaries of the range for $|A_2/A_1|$.

C. General two-channels

Once we have explored the two extreme cases above, it is not difficult to find the general solution for an arbitrary value of χ in Eq. (25). By rewriting the FSI relation in $A_2' \equiv e^{-i\chi}A_2$, we can reduce it to the case of $S_{22} = S_{11}$ above. We obtain the solution for general χ as

$$
\begin{aligned}
A_1 &= a_1 + ib_1 \\
A_2 &= \sqrt{\frac{2-\epsilon}{\epsilon}}e^{i\chi}\left(b_1 + i\frac{\epsilon}{2-\epsilon}a_1\right), \\
\tan\Delta_1 \tan(\Delta_2 - \chi) &= \frac{\epsilon}{2-\epsilon}.
\end{aligned} \tag{35}
$$

The first and second lines of Eq. (35) reduce to those of the previous cases; $|1 - S_{22}| = \max$ and $S_{11} = S_{22}$ as $\chi \to \frac{1}{2}\pi$ and $\chi \to 0$, respectively. The third line relates the FSI phases Δ_1 and Δ_2 to each other with parameter ϵ and χ. The ratio $|A_2/A_1|$ is related to Δ_1 and ϵ exactly in the same way as in Eq. (30). Consequently $|A_2/A_1|$ is also restricted to the same range, Eq. (31).

We can actually find the exact solution even when we include the very small imaginary part of $1 - \epsilon$ in S_{11} due to the low-ranking Regge trajectories in elastic scattering. In this most general case it is convenient to write the S-matrix in the form

$$
S = \begin{pmatrix} (1-\epsilon)e^{2i\chi_1} & i\sqrt{2\epsilon - \epsilon^2}e^{i(\chi_1+\chi_2)} \\ i\sqrt{2\epsilon - \epsilon^2}e^{i(\chi_1+\chi_2)} & (1-\epsilon)e^{2i\chi_2} \end{pmatrix}, \quad (\epsilon^* = \epsilon). \tag{36}
$$

Unitarity and symmetry fixes the phase of S_{12} as shown above once those of S_{11} and S_{22} are given. For this reason one earlier literature[15] assigned the angles χ_1 and χ_2 to the "initial-state" and "final-state" interactions of scattering, hinting that the phases of decay amplitudes A_1 and A_2 acquire χ_1 and χ_2, respectively, by FSI. Unfortunately this interpretation was wrong. As we see below, the phases of $A_{1,2}$ are $\chi_{1,2}$ *plus* additional contributions that depend on mixing of the channels and weak interaction. It should also be pointed out that the simple phase relation like $\arg S_{ij} = \arg\sqrt{(-i)^2 S_{ii}S_{jj}}$ holds only for the S-matrix of 2×2, not of more than two channels.

To solve the FSI relation with Eq. (36) we factor out the "elastic phases" χ_1 and χ_2 of the diagonal S-matrix elements S_{ii} (*not* of the partial-wave amplitudes $a_l(s)$) from the decay amplitudes by introducing A_i' by $A_i' = e^{-i\chi_i}A_i$ $(i = 1, 2)$. Then the FSI relation $A' = SA'^*$ reduces to the form identical to Eq. (33). Therefore the solution for $A_{1,2}$ can be immediately written as Eq. (34):

$$
\begin{aligned}
A_1 &= e^{i\chi_1}(a_1 + ib_1) \\
A_2 &= \sqrt{\frac{2-\epsilon}{\epsilon}}e^{i\chi_2}\left(b_1 + i\frac{\epsilon}{2-\epsilon}a_1\right), \\
\tan(\Delta_1 - \chi_1)\,\tan(\Delta_2 - \chi_2) &= \frac{\epsilon}{2-\epsilon},
\end{aligned} \tag{37}
$$

where a_1 and b_1 are real The first and the second line of Eq. (37) requires $|A_2/A_1|$ to remain in the same range as in the previous two special cases. (cf Eq. (31)). The relation of Eq. (30) is trivially modified as

$$
\sin^2(\Delta_1 - \chi_1) = \frac{\epsilon}{4(1-\epsilon)}\left[(2-\epsilon)\frac{|A_2|^2}{|A_1|^2} - \epsilon\right]. \tag{38}
$$

This relation is the bottom line of the general two-channel toy model: The total FSI phase Δ_1 of A_1 is sum of the rescattering phase χ_1 of $\sqrt{S_{11}}$, not of the elastic partial-wave amplitude $(a_l = \frac{1}{2i}(S_l - 1))$, plus the the phase due to rescattering through the second channel. It cannot be over-emphasized that in the presence of inelasticity the phase of $\sqrt{S_{11}}$ is very different from the phase of the partial-wave amplitude. For Pomeron-dominated scattering, for instance, $\arg a_0(s) = \frac{1}{2}\pi$ since $a_l(s)$ is purely imaginary for all l, but $\arg \sqrt{S_{11}} = 0$ or $\pm\pi$ since S_{11} is real and positive for $0 < \mathrm{Im}\, a_0 < 0.5$. The phases of $\sqrt{S_l}$ and $a_l(s)$ would be equal only in the elastic limit where $S_{11} = e^{2i\delta_1}$ and $a_l(s) = (1/2i)(S_{11} - 1) = e^{i\delta_1}\sin\delta_1$. It makes no sense whatsoever even as an approximation to equate the FSI phase with the phase of the elastic partial-wave amplitude a_l in B decay.

It is worth mentioning here that the solutions of the two-channel problem, Eqs. (37) and (38), apply to the Ω^- decay into ΛK^- and $(\Xi\pi)_{I=1/2}$. The ΛK^- and $\Xi\pi$ yields add up to over 99% of the observed nonleptonic final states. In this case the lower ranking Regge trajectories contribute more to the diagonal S-matrix elements than in the light hadrons channels of B decay.

While the two-channel toy model casts light on many important issues, it has one undesirable feature that the ratio $|A_2/A_1|$ is restricted within the rather narrow range set by Eq. (31). Numerically,

$$0.58 \leq |A_2/A_1| \leq 1.73. \tag{39}$$

This constraint limits applicability of the two-channel toy-model to B decay modes. We must extend to more than two channels to study B decay. However, as the coupled channels increases, the number of dynamical unknowns quickly increases in the FSI relation. In order to keep our problem manageable, we must introduce some approximation that keeps mathematical complexity under control.

V. TRUNCATED MULTICHANNEL PROBLEM

Going back to our fundamental equation Eq. (12), we consider the situation where one inelastic channel makes a dominant feed back to the elastic channel (channel 1) and all other inelastic channels are either unimportant individually or largely cancel among them. We are specifically interested in the case,

$$|S_{21}| \ll |S_{11}|, \quad \text{but} \quad |A_2| \gg |A_1| \tag{40}$$

such that

$$|S_{12}A_2^*| = O(|S_{11}A_1^*|), \quad |\sum_{j\geq 3} S_{1j}A_j^*| \ll |S_{12}A_2^*|. \tag{41}$$

In this case we can truncate the sum over the inelastic channels at $j = 2$;

$$A_1 = \sum_{j=1,2} S_{1j}A_j^* + \sum_{j\geq 3} S_{1j}A_j^* \simeq S_{11}A_1^* + S_{12}A_2^*. \tag{42}$$

The FSI relation for the channel 2 reads

$$A_2 = S_{21}A_1^* + S_{22}A_2^* + \sum_{j\geq 3} S_{2j}A_j^*. \tag{43}$$

Since unlike the off-diagonal S_{21} the diagonal S-matrix element $S_{22} = 1 + 2ia_0^{22}(s)$ contains the term unity, we expect that S_{22} is $O(1)$ or a substantial fraction of it unless the term 1 is

cancelled accidentally by $2ia_0^{22}(s)$ with high accuracy. In comparison, S_{12} represents a small leakage into a dominant inelastic channel in the present case. Therefore

$$|S_{21}A_1^*/S_{22}A_2^*| = |S_{21}/S_{22}| \times |A_1^*/A_2^*|, \tag{44}$$

where the both factors in the right-hand side are small. In other words, the effect of the single elastic channel back on a robust inelastic channel j is negligible when the coupling S_{j1} between them is feeble. Therefore, magnitude of the first term $S_{21}A_1^*$ in the right-hand side of Eq. (43) is much smaller than that of $|S_{22}A_2^*|$ by the assumptions made in Eqs. (40) and (41). Therefore, we may drop the first term in Eq. (43). Then no information of the channel 1 is needed to solve Eq. (43) for A_2. Therefore we solve only Eq. (42) and obtain a relation between A_1 and A_2. Solving Eq. (43) for A_2 may be hard. In the numerical exercise later we do not attempt to compute for A_2 theoretically in terms of other inelastic channels, but resort to experiment for information of A_2.

The S-matrix now need not satisfy unitarity in the subsector of channel 1 and 2. It can be written in general as

$$S = \begin{pmatrix} (1-\epsilon)e^{2i\chi_1} & i\kappa e^{i(\chi_1+\chi_2+\chi_\kappa)} \\ i\kappa e^{i(\chi_1+\chi_2+\chi_\kappa)} & \lambda e^{2i\chi_2} \end{pmatrix}, \quad (0 < \kappa < \sqrt{2\epsilon-\epsilon^2},\ 0 < \lambda < \sqrt{1-\kappa^2}), \tag{45}$$

where two real parameters κ and λ have been introduced to describe inelasticity of scattering. When the channel 2 is also a two-body light-hadron channel, the value of λ is $\simeq 1-\epsilon$ and $\chi_2 \simeq \chi_1$. In B decay the branching fractions to two-body light-mesons are much smaller than those to charmed meson pairs by the property of weak interaction. The decay $B \to K\pi$ through $D^*D_s^*$ is a typical example since $|\kappa|$ is much smaller than $1-\epsilon$ but $|A_2|$ is much larger than $|A_1|$. It has been speculated that the presence of the $D^*D_s^*$ channel may generate a large FSI phase for $K\pi$[7]. We will examine this possibility later.

The FSI relation can be solved for $A_{1,2}$ even in the presence of the additional parameter κ by the rephasing technique that we have used earlier. When we express A_1 and A_2 in terms of a_2 and b_2 instead of a_1 and b_1, the solution of Eq. (12) with Eq. (45) is:

$$A_1 = \kappa e^{i\chi_1}\left(\frac{1}{\epsilon}[-a_2\sin(\chi_2+\chi_\kappa) + b_2\cos(\chi_2+\chi_\kappa)] + \frac{i}{2-\epsilon}[a_2\cos(\chi_2+\chi_\kappa) + b_2\sin(\chi_2+\chi_\kappa)]\right)$$
$$A_2 = a_2 + ib_2, \tag{46}$$

As we have pointed out, the parameter λ is not needed to express the relation between A_1 and A_2 in the truncated approximation. Although χ_κ enters A_1, we can express the phase Δ_1 without χ_κ by using experimental knowledge of $|A_2/A_1|$. Rewriting Eq. (46) with the magnitudes and phases, we obtain a simple generalization of the previous relation,

$$\sin^2(\Delta_1 - \chi_1) = \frac{1}{4(1-\epsilon)}\left(\kappa^2\frac{|A_2|^2}{|A_1|^2} - \epsilon^2\right). \tag{47}$$

The right-hand side of Eq. (47) gives the contribution of the channel coupling that is to be added to the elastic contribution χ_1. The ratio $|A_2/A_1|$ is now bounded as

$$\frac{\epsilon}{\kappa} \le \frac{|A_2|}{|A_1|} \le \frac{2-\epsilon}{\kappa} \tag{48}$$

If κ is small, that is, if the leakage into the channel 2 is small, large values can be accommodated for $|A_2/A_1|$. Therefore the truncated model is applicable to more general situations

than the two-channel toy model that we have discussed. The ratio $|A_2/A_1|$ contains information of weak interaction. It is amusing to see in Eq. (47) that the strong phase Δ_1 coincides with the small "elastic phase" χ_1 of $\sqrt{S_{11}}$ when A_2 takes the smallest value, $|A_2/A_1| = \epsilon/\kappa$, in the allowed range of Eq. (48). The channel coupling effect $|\Delta_1 - \chi_1|$ is the strongest when $|A_2/A_1|$ takes the upper limit $(2 - \epsilon)/\kappa$ in Eq. (48).

If we proceed further and include two prominent inelastic channels, the FSI relation for A_1 is:

$$A_1 \simeq S_{11}A_1^* + S_{12}A_2^* + S_{13}A_3^*. \tag{49}$$

If we continue along this line and incorporate more inelastic channels, the FSI equation for channel 1 turns into

$$a_1 + ib_1 = S_{11}(a_1 + ib_1)^* + \sum_{j \geq 2} S_{1j}A_j^*. \tag{50}$$

The first term in the right-hand side can be viewed as counteraction of elastic rescattering. It affects not only on the phase of A_1 but also causes long-distance enhancement or suppression on magnitude of A_1 depending on the force in the elastic channel. With $S_{11} = 1 - \epsilon$, Eq. (50) reveals an interesting general feature of the multichannel FSI. Separating the real and imaginary parts of Eq. (50), we have in the case of real S_{11}

$$
\begin{aligned}
a_1 &= \frac{1}{\epsilon} \mathrm{Re} \sum_{j \geq 2} S_{1j}A_j^*, \\
b_1 &= \frac{1}{2 - \epsilon} \mathrm{Im} \sum_{j \geq 2} S_{1j}A_j^*.
\end{aligned} \tag{51}
$$

Eq. (51) shows that elastic rescattering S_{11} enhances the inelastic rescattering effect by $1/\epsilon$ ($\simeq 2$) for the real part a_1 of A_1 and and suppresses it for the imaginary part b_1 by $1/(2 - \epsilon)$ ($\simeq \frac{2}{3}$). This characteristic depends only on S_{11}, which we belive we know fairly accurately.

Despite its simplicity Eq. (51) contains useful information. For instance, *if* the transition to and from channel 1 can be described by the Born terms of t- and u-channel exchanges, the off-diagonal partial-wave amplitudes are real so that the off-diagonal S-matrix S_{1j} ($j \geq 2$) $= 2ia_0^{(1j)}$ are purely imaginary. Therefore the real decay amplitudes A_j ($j \geq 2$) of the inelastic channels contribute to the imaginary part b_1 of channel 1 in this case. Even if there is no resonance in the process, the phase Δ_1 can be very large in this way. If furthermore the "inelastic" decay amplitudes A_j ($j \geq 2$) happen to be all real, the phase Δ_1 would be $\pm 90°$. This is not surprising: In the language of dispersion theory the on-shell inelastic intermediate states generate an absorptive part for A_1 that turns out to be purely imaginary in this situation. Our Eq. (51) involves one dynamical input: The elastic scattering amplitudes of light mesons are almost purely imaginary (Pomeron-dominated).

VI. NUMERICAL EXERCISE

Some numerical exercise is called for to show relevance of our endeavor to the B decay of the real world. Contrary to the initial optimism that had prevailed before B physics experiment started, analysis of experiment seems to suggest that the FSI phases of some two-body decay amplitudes appear to be much larger than what we expected in the original short-distance picture[20]. The word $K\pi$ puzzle has been coined for the unexpectedly large tree-contribution and/or FSI phases in $K\pi$ modes. As the perturbative technique has become

more sophisticated, people have come to agree that emission and absorption of soft and collinear quarks and gluons plays an important role in many decay modes[21, 22]. Such soft constituents contribute to the FSI phases involving long-distance physics. While one can parametrizes such contributions in the soft-collinear theory, one cannot evaluate them numerically in perturbative argument. Our S-matrix approach also has its own drawback: While elastic scattering of two light hadrons at energy m_B has been well understood, we know less about their inelastic scattering. Nonetheless we would like to show here that our method may be useful in some of B decay modes.

It is believed that the decay $B \to K\pi$ occurs primarily with the penguin interaction $\sim [(\bar{b}s)(\bar{q}q) + h.c]$. In the penguin process the coupling to channels such as $K^*\rho$ need to be studied as a source of the strong phase of the $K\pi$ amplitude. However, it has been argued that $K\pi$ can be produced indirectly with the tree interaction $\sim [(\bar{b}c)(\bar{c}s) + h.c.]$ as well through the charmed-meson-pair states such as DD_s and $D^*D_s^*$[7]. The $K\pi$ amplitude of this process acquires a strong phase different from the direct penguin amplitude. What can our approach say about this problem ?

The branching fractions have been measured for the following two-body channels that couple to $K\pi$ in B^0 decay[23]:

$$
\begin{aligned}
B(K^+\pi^-) &= (1.88 \pm 0.07) \times 10^{-5}, \\
B(K^0\pi^0) &= (1.15 \pm 0.10) \times 10^{-5}, \\
B(K\eta) &< 2.0 \times 10^{-6}, \quad\quad\quad\quad\quad (52) \\
B(K\eta') &= (6.5 \pm 0.4) \times 10^{-5}, \quad\quad\quad (53) \\
B(K^{*0}\phi) &= (0.95 \pm 0.08) \times 10^{-5} \\
B(K^{*+}\rho^-) &< 1.20 \times 10^{-5} \\
B(K^{*0}\omega) &< 4.2 \times 10^{-6} \\
B(D^-D_s^+) &= (6.5 \pm 1.3) \times 10^{-3}, \\
B(D^{*-}D_s^{*+}) &= (1.77 \pm 0.14) \times 10^{-2}. \quad\quad (54)
\end{aligned}
$$

All of them can make $J^P = 0^+$, the spin-parity of $K\pi$. The states $K\pi$ and $K^*\rho$ can be either in $I = \frac{1}{2}$ or $I = \frac{3}{2}$ while all other modes are only in $I = \frac{1}{2}$.

From the Regge phenomenology on elastic $K\pi$ scattering, we have already estimated ϵ for the Pomeron contribution in Section III. The ρ and f_2 trajectories with exchange degeneracy allow us to estimate the small imaginary part of S_{11}. By adding the ρ and f_2 Regge contributions, we obtain numerically

$$
S_{11} \simeq \begin{cases} 0.39 \times e^{0.06i} & (I = \frac{1}{2}), \\ 0.46 \times e^{-0.10i} & (I = \frac{3}{2}). \end{cases} \quad\quad (55)
$$

The ρ/f_2 contributions to the scattering amplitudes T are dominantly imaginary for $I = \frac{1}{2}$ and real for $I = \frac{3}{2}$, as we expect from s to t-channel duality. Therefore they generate a larger imaginary part for $S = 1 + 2iT$ in the $I = \frac{3}{2}$ channel. This is very different from our intuitive picture in the single-channel case. Though we do not attach errors to our estimate, errors as large as factor two are possible for the phases in Eq. (55).

Estimate of S_{12} is less reliable than S_{11} owing to very indirect experimental information and to larger theoretical uncertainties. We eliminate the $K^*\phi$ channel from our consideration since the Regge residue or the coupling of ϕ to $\pi\rho$ is highly suppressed ("OZI-forbidden"). In contrast $K^*\rho$ can couple to $K\pi$ without such suppression. While a loose upper bound

has been set on $K^*\rho$ experimentally (see Eq. (54)), we believe that the $K^*\rho$ channel is more important than the $K^*\phi$ channel. We may use the Regge phenomenology to estimate S_{12} for the $K^*\rho$ state of longitudinal polarizations in the final state. For the scattering $K\pi \to K^*\rho$, the leading Regge poles are ω and a_2 which are exchange degenerate.[12] The couplings of $\pi\rho\omega$ and $\pi\rho a_2$ are known on the mass shells of ω and a_2, respectively, from low-energy spectroscopy. The corresponding KK^* couplings of ω and a_2 are obtained by an SU(3) rotation. But we need to extrapolate them to the off-shell ρ and a_2 to relate them to the Regge residues. This extrapolation is a major source of uncertainty. If we ignore the extrapolation, they are at the same level in magnitude as the nonleading Regge contributions in S_{11};

$$S_{12}^{K^*\rho} \approx \begin{cases} -0.07 + 0.02i & (I = \frac{1}{2}), \\ -0.05i & (I = \frac{3}{2}). \end{cases} \tag{56}$$

Using the ratio $|A_2/A_1|$ computed with the measured branching fraction and the upper bound listed in Eq. (54), we reach the crude estimate,

$$|S_{12}^{K^*\rho}A_2^*| < 0.1 \times |S_{11}A_1^*|, \tag{57}$$

Despite large uncertainty of these numbers we may conclude with Eq. (56) that the inelastic term $S_{12}A_2^*$ of the $K^*\rho$ channel in the $K\pi$ mode is not significant relative to the elastic term $S_{11}A_1^*$. It is certainly not a major source of the strong phase for the $K\pi$ amplitude in the standard penguin decay (not through $c\bar{c}$). The same line of estimate suggests that the $K^*\omega$ state is neither important to the FSI phase of $K\pi$. Some may wonder about one-pion exchange in $K\pi \leftrightarrow K^*\rho$. The Reggeized pion exchange amplitude for $K\pi \to K^*\rho$ is down by another power of $s^{0.5}$ relative to those of ω and a_2 exchanges. The small denominator of the pion propagator does not enhance the amplitude near forward direction because the Lorentz structure requires the amplitude near the pion pole to be proportional to

$$\frac{(\epsilon^{(\rho)} \cdot p_\pi)(\epsilon^{(K^*)} \cdot p_K)}{m_\rho^2 - 2(p_\pi \cdot p_\rho)}. \tag{58}$$

The both factors in the numerator vanish up to $O(4m_\rho^2/m_B^2, 4m_{K^*}^2/m_B^2)$ for longitudinally polarized K^* and ρ in the forward direction ($\mathbf{p}_\rho \parallel \mathbf{p}_\pi$). They eliminate a forward peak from the pion pole. The Regge theory predicts that the rest of the scattering amplitude falls sharply ($\sim e^{\alpha'_\pi(0)t\ln s}$) off the forward direction. Therefore the pion exchange can be dismissed. Then we feel safe to conclude with Eq. (57) that the coupling of $K\pi$ to the $K^*\rho$ channel is not important in determining the FSI phases for the $K\pi$ modes.

The $K\eta$ and $K\eta'$ channels can couple to the $K\pi$ channel of $I = \frac{1}{2}$ through the a_2 Regge exchange. Although this contribution has the same m_B dependence as the ρ and f_2 Regge exchanges, the a_2 Regge residues with $\pi\eta(\eta')$ are most likely smaller than the residue with $\pi\rho$: We can estimate from the a_2 decay branching that, after the d-wave phase space factor is separated, the on-shell a_2 couplings to $\pi\eta$ and $\pi\eta'$ are about factor 20 smaller than those to $\pi\rho$. Therefore neither $K\eta$ nor $K\eta'$ compete with $K^*\rho$ in the final state. Therefore we may leave out $K\eta$ and $K\eta'$ from our consideration.

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[12] The K^* trajectory generates a backward peak is generated, but it is less important than the forward peak. We neglect the backward peak contribution to S_{12}.

The contributions from nonresonant three-body final states are harder to estimate since computation of S_{12} is next to impossible. There are many nonresonant multiparticle channels with relatively minor branching fractions. If rescattering of $K\pi$ to multibody channels is quasi-diffractive with no quantum number exchange, the Pomeron can contribute. In such scattering the final states consist of two lumps of relatively small invariant masses that carry the same flavors as K and π. They are likely to end up in two-body states of highly excited meson states, for instance, $K_2(1430)a_1$. While this is a possibility, none of such modes have been positively identified so far in measurement.

Genuine nonresonant three-body channels are probably not a major source of the FSI phases, unless their contributions add up by constructive interference to a large value. In fact, it is conceivable that they sum up in random phases into relatively a small number[4, 5]. That is one motivation when we have introduced the truncated approximation. Our tentative conclusion on the $K\pi$ amplitudes ($I = \frac{1}{2}, \frac{3}{2}$) of the light-quark penguin decay operators is that the FSI phase produced by coupling to the inelastic channels is insignificant. Analysis of the $K\pi$ amplitudes in search of the weak phases was started more than ten years ago. With little knowledge of the strong phases, however, the analysis could be carried out only by assuming that the long-distance strong phases be negligible[24].

The charmed meson-pair channels are very different. Since they are the CKM dominant tree-decay final states, their branching fractions are a few orders of magnitude larger than that of the penguin-dominated decay. They can annihilate into $K\pi$. In the quark picture this process can be viewed as the on-shell contribution of the $c\bar{c}$ penguin to $K\pi$. Some call this process as "charming penguin"[7]. Among the charmed meson pairs, the most prominent decay channel of $J^P = 0^+$ is $D^*D_s^*$. Its amplitude can be estimated with Eq. (54) as

$$|A_{D^{*-}D_s^{*+}}/A_{K\pi(I=1/2)}| \simeq 25. \tag{59}$$

One important question here is how much of the observed total $A_{K\pi(I=1/2)}$ is the "charming penguin" contribution. Since we expect that long-distance physics enters the on-shell process of charmed-meson pairs at energy m_B, it is not easy to evaluate its magnitude. Some argue that it can be very large[17, 18]. However, a counter argument was made to advocate the short-distance argument[19]. Theorists have not come to consensus on magnitude of this contribution. Therefore we insert one fudge factor r here for this contribution to A_1 as

$$|A_{D^{*-}D_s^{*+}}/A_{K\pi(I=1/2 \ via \ c\bar{c})}| \simeq 25 \times \frac{1}{r}, \ \ (r < 1) \tag{60}$$

where r, the fraction of the $c\bar{c}$ contribution, may be as large as a half or even more[17]. It must be settled by theory rather than by experiment. With this fudge factor Eq. (47) turns into

$$\sin^2(\Delta_1 - \chi_1) = \frac{1}{4(1-\epsilon)}\left(\kappa'^2\frac{|A_2|^2}{|A_1|^2} - \epsilon^2\right), \tag{61}$$

where $\kappa' = \kappa/r = |S_{12}/rS_{11}|$. Even if the spill-over of $D^*D_s^*$ into $K\pi$ is as tiny as one tenth of percent ($\kappa^2 = |S_{12}/S_{11}|^2 \simeq 10^{-3}$), the $D^*D_s^*$ channel may control the FSI phase of the $K\pi$ amplitude that comes through $c\bar{c}$. To proceed further, we must look into the transition $K\pi \leftrightarrow D^*D_s^*$. (Fig.4)

Application of the Regge theory is questionable to the charm-pair channels since the charmed meson masses are around 2 GeV and the total energy is a little above 5 GeV. Departing from the Regge theory, let us make the Born approximation in t-channel exchange. The exchanged mesons are the charmed mesons D, $D^*, \cdots$. For D-exchange, we know
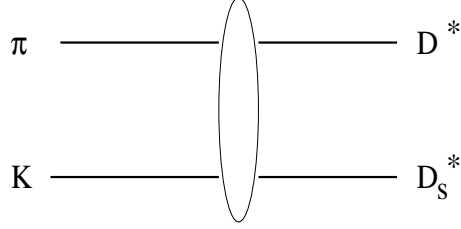
FIG. 4: The dominant inelastic scattering $K\pi \to D^*D_s^*$.

the $D^*\overline{D}\pi$ coupling from the decay $D^* \to D\pi$ and can compute the D^*D_sK coupling by an SU(3) rotation of the $D^*\overline{D}\pi$ coupling. The differential cross section rises toward the forward direction when the D meson is exchanged. For D^* exchange, we can deduce the $D^*\overline{D}^*\pi$ coupling from the $D^*\overline{D}\pi$ coupling with the heavy quark spin symmetry and rotate it into the $D^*D_s^*K$ coupling by SU(3). In contrast to the D exchange, the Lorentz structure of the vertex $\sim \varepsilon^{\mu\nu\kappa\lambda}\epsilon_\nu p_{1\kappa}p_{2\lambda}$ of D^* exchange cancels a forward peak that would be otherwise generated by the D^* propagator. For this reason the D^*-exchange is less important. When we compute the D contribution in the Feynman diagram with the on-shell couplings, we obtain $|S_{12}| \approx 0.5$. But this is obviously a nonsense. The reason is that we have ignored the form-factor damping effect of the exchanged off-shell D. For an order-of-magnitude estimate we may multiply a factor of m_*^2/m_D^2 as a form-factor effect where $m_* \simeq 0.3$ GeV is the binding scale of the charmed mesons. Then our estimate goes down by more than an order of magnitude from $|S_{12}| \approx 0.5$ to $|S_{12}| \approx 0.014$. This latter value is probably closer to reality. It is roughly in line with the rule of thumb; in the quark picture a pair creation probability of $c\bar{c}$ is suppressed by about $(m_q/m_c)^2$ relative to light-quark pair creation of $q\bar{q}$, where the quark masses are the constituent masses. This rule works roughly for $s\bar{s}$ and $c\bar{c}$ production in high-energy collision. If we use this rule, we obtain $|S_{12}| \simeq 0.01$ from Eq. (56) with $m_s/m_c \simeq 1/3$. Therefore we choose as our best guess

$$S_{12}^{D^*D_s^*} \approx 0.01i. \tag{62}$$

As we have noted earlier, S_{12} is purely imaginary for real $a_0(s)$. The number of Eq. (62) is obviously an order-of-magnitude estimate at best. With $\epsilon \simeq 0.5$ and tentatively $r \approx 0.5$, a crude central value of our estimate for κ' is

$$\kappa' \approx 0.01/[0.5 \times (1 - 0.5)] = 0.04. \tag{63}$$

We now substitute all these numbers in Eq. (61) of the truncated approximation. We take the number of Eq. (63) as a ballpark figure and sweep the value of κ'^2 by a factor two across this value to see what FSI phase can be generated for $K\pi$ of $I = \frac{1}{2}$ by the channel coupling to the $D^*D_s^*$ channel. The result is plotted in Fig. 5. The value of $\kappa'|A_2/A_1|$ is constrained between 0.5 and 1.5 by Eq. (61) and sweeps in the region between two vertical broken lines.

Within the uncertainty of κ', the FSI phase Δ_1 of the $K\pi$ amplitude through $c\bar{c}$ can be any value between 21° and 69°. Eq. (61) does not determine the sign of Δ_1 since it does not contain full information of A_2. Because of the large uncertainty of κ', we cannot constrain Δ_1 meaningfully at present. Even $\Delta_1 \simeq 90°$ is not reliably excluded. Keeping the uncertainty of our estimate in mind, we should state here only that the channel coupling to $D^*D_s^*$ is capable of producing a very large strong phase for the charming penguin $K\pi$ amplitude, in particular in the case that the "charming penguin amplitude" is a sizable fraction of the
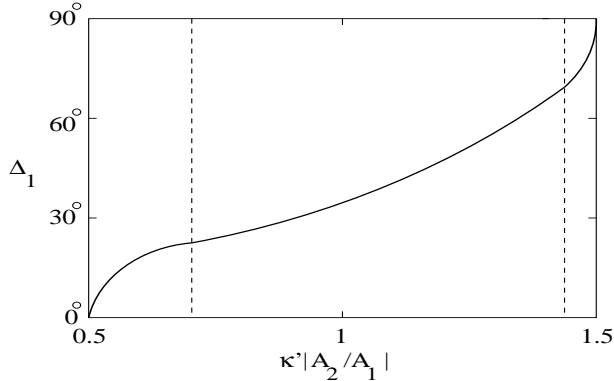
FIG. 5: The FSI (strong) phase of the $K\pi$ decay channel of $I = \frac{1}{2}$ as the $K\pi \to D^*D_s^*$ transition amplitude is varied in magnitude.

total amplitude. Quantitatively reliable computation of the FSI phase will be possible after we have obtained a better theoretical estimate of S_{12} for $K\pi \leftrightarrow D^*D_s^*$ as well as magnitude of the decay amplitude through $c\bar{c}$. Until then we must not set this strong phase to zero but leave it as an unknown parameter to be determined by fit to experimental data. If such experimental fit clearly requires a large FSI angle for the $K\pi$ mode of $I = \frac{1}{2}$ but not of $I = \frac{3}{2}$, we shall be able to assert that the on-shell $c\bar{c}$ intermediate state plays an important role in the decay $B \to K\pi$.

The statement above can be made for many other two-body light-hadron modes. The $\pi\pi$ mode couples to $\rho\rho$ whose branching fraction is nearly an order of magnitude larger than that of $\pi\pi$. But perturbative calculation of the $\pi\pi \leftrightarrow \rho\rho$ transition[16] by the Born diagrams without off-shell damping can easily overestimate it. With an estimate of S_{12} along the same line as for $K\pi \leftrightarrow K^*\rho$, the right-hand side of Eq. (47) comes out to be negative in the case of $\pi\pi \to \rho\rho$, It implies that the branching fraction of $\rho\rho$ is not large enough to affect the FSI phase for the $\pi\pi$ amplitudes of the tree decay $\sim (\bar{b}u)(\bar{q}q) + \text{h.c.}$ (q = light quarks). The $\pi\pi$ of $I = 0$ can be fed also by the charmed hadron channel $D^*\overline{D}^*$ whose branching fraction is two orders of magnitude larger than that of $\pi\pi$. For the $\pi\pi$ amplitude through $\bar{c}c$, coupling to $D^*\overline{D}^*$ is the most important source of the strong phase. To obtain the strong phases and to compare with experiment, we again need to know about relative importance of the two classes of decay; $b \to u\bar{u}d$ and $b \to c\bar{c}d$ for $B \to \pi\pi$.

In contrast, the tree-dominated CKM-favored decay modes such as $\overline{B} \to D\pi$ have no wide open inelastic channels. Transition to the $D^*\rho$ channel is as insignificant as the transition to $K\pi \to K^*\rho$ in the $K\pi$ mode. Since the $D^*\rho$ decay branching fraction is comparable to that of $D\pi$ and S_{12} is much smaller than S_{11} (cf Eqs. (56) and (57)), of its elastic scattering, we expect that $|S_{12}A_2^*| \ll |S_{11}A_1^*|$ and therefore the channel coupling contribution is unimportant. It means that the FSI phases are small in these modes and that simple short-distance calculation of the phases can produce an answer not far from reality.

VII.   COMMENT AND DISCUSSION

One of our purposes is to solve the most general two-channel model exactly and to clarify the mechanism of generating strong phases in this toy model. We have seen above that existence of competing channels completely changes the strong phase from that of

the "elastic rescattering phase" in all cases of two-channel S-matrix. Unitarity plays an important role here. The other purpose is to introduce a feasible approximation scheme which may be applicable to cases of special interest in B decay. We have applied this method to the decay $B \to K\pi$ and have made semiquantitative analysis. But its outcome is not numerically satisfying because of limitation in available knowledge about off-diagonal scattering. We have extended our analysis to the general multichannel case and come to one simple interesting observation: Though it may sound odd, inelastic scattering tend to enhance the strong phase of the elastic channel most when inelastic amplitudes are real (*i.e.,* $S_{1j} = iT_{1j} =$ imaginary) rather than imaginary, if the strong phases of the inelastic decay amplitudes are small. (cf Eq. (51).)

It is a big challenge to go beyond the two-channel approximation. As the number of channels n increases, the number of parameters in the FSI relation increases as $\frac{1}{2}n(n+1)$ for strong interaction. In addition we need piece of information from weak interaction. In the truncated approximation we have kept only a single dominant one among the inelastic channels. Our assumption is that all other inelastic channels are less important or else sum up in random phases to become numerically insignificant. If the higher inelastic channels do not sum randomly, there must be some underlying dynamical reason for it. In such a case some approach orthogonal to ours may have advantage. For instance, it is an approach based on quarks and gluons instead of hadrons.

The major source of the strong (FSI) phase in the multichannel case is the transition to inelastic channels. Consequently accurate computation of the FSI phases depends on knowledge of the transition S-matrix at energy m_B between a channel of our interest and dominant inelastic channels. That is, we need to know dynamics of long and intermediate distances at this energy. The present author is of the opinion that quantitatively we have a little better handle on hadron physics numerically than on the soft and collinear quarks and gluons in this territory. But opinions probably divide among physicists of different generations.

Acknowledgments

APPENDIX A: REAL ORTHOGONALITY OF TRANSFORMATION

Expansion of the observable state $|j^{\text{out}}\rangle$ in the S-matrix eigenstates $|a^{\text{out}}\rangle$ is defined by

$$|j^{\text{out}}\rangle = \sum_a O_{ja}|a^{\text{out}}\rangle. \tag{A1}$$

At this stage the matrix $\mathcal{O}$ is assumed to be only unitary, not necessarily orthogonal. Make time-reversal on the scattering amplitude from eigenstate $|a\rangle$ to observable state $|j\rangle$ in our phase convention of states under time reversal:

$$\langle j^{\text{out}}|a^{\text{in}}\rangle = \langle j|S|a\rangle \overset{T}{=} \langle a|S|j\rangle = \langle a^{\text{out}}|j^{\text{in}}\rangle. \tag{A2}$$

Operating $S^\dagger$ on Eq. (A1), we obtain

$$|j^{\mathrm{in}}\rangle = \sum_a O_{ja}|a^{\mathrm{in}}\rangle. \tag{A3}$$

Substitution of Eqs. (A1) and (A3) in Eq. (A2) gives us

$$O_{ja}^* e^{2i\delta_a} = O_{ja} e^{2i\delta_a}, \tag{A4}$$

which proves that $\mathcal{O}$ is orthogonal:

$$O_{ja}^* = O_{ja}. \tag{A5}$$

[1] K. Watson, *Phys. Rev.* **95**, 228 (1954)).
    K. Aizu, Proc. Int. Conf. Theor. Phys. 1953 (Kyoto-Tokyo Science Council, 1954), p.200. E. Fermi, Nuovo Cim. **2**, 17 (1955). See also *Elementary Particles*, (Yale University Press, New Haven, CT, 1951), pp. 58-64.

[2] For a critical review of such earlier literature on the FSI phases, see M. Suzuki, *Proceedings of the third International Conference on B physics and CP Violation* (Taipei, Taiwan, December 1999), edited by H. -Y. Cheng and W. -S. Hou, World Scientific, pp. 259-265, and literature therein.

[3] J.F. Donoghue, E. Golowich, A.A. Petrov, and J.M. Soares, Phys. Rev. Lett. **77**, 2178 (1996).

[4] R. N. Cahn and M. Suzuki, hep-ph/9708208.

[5] L. L. Wolfenstein and M. Suzuki, Phye. Rev. D **60**, 074019 (1999).

[6] L. Wolfenstein, private communications.

[7] M. Chiuchini, E. Franco, G. Martinelli, and L. Silvestrini, Nucl. Phys. **B501**, 271 (1997).

[8] P. D. B. Collins and E. J. Squires, *Regge Poles in Particle Physics* (Springer-Verlag, Berlin, 1968).

[9] W.-M. Yao et al., *Review of Particle Physics*, J. Phys. G **33**, 1 (2006).

[10] E. M. Levin and L. L. Frankfurt, JETP Lett. **2**, 65 (1965).

[11] P. D. B. Collins and E. J. Squires, ibid, pp. 193-250.

[12] R. J. N. Phillips and W. Rarita, Phys. Rev. **139**, B1336 (1965).

[13] J.D. Hansen *et al.,* CERN/HERA 70-2; E. Flaminio *et al.,,* CERN/HERA 70-3.

[14] E. Bracci *et al.,* High-Energy Reactions Analysis Group, CERN/HERA-72-1.

[15] J. Gillespie, Final State Interactions (Holden-Day, San Francisco, CA, 1964).

[16] A. B. Kaidalov and M. I. Vysotsky, hep-ph/0704.0404

[17] C. W. Bauer, D. Pirjol, I. Z. Rothstein, and I. W. Stewart, Phys. Rev. D **70**, 054015 (2004);

[18] C. W. Bauer, I. Z. Rothstein, and I. W. Stewart, Phys. Rev. D **74**, 034010 (2006).

[19] M. Beneke, G. Buchalla, M. Neubert, and C. T. Sachrajda, Phys. Rev. D **72**, 098501 (2005).

[20] M. Beneke, G. Buchalla, M. Neubert, and C. T. Sachrajda, Phys. Rev. Lett. **83**, 1914 (1999).

[21] For a short review, see for instance S. Fleming, hep-ph/0601257 and Th. Feldmann, hep-ph/0610192 and references therein.

[22] J. Chay, H-n. Li, and S. Mishima, arXiv:0711.2953 and references therein.

[23] W.-M. Yao et al., *Review of Particle Physics*, J. Phys. G **33**, 1 (2006) and 2007 partial update for 2008.

[24] A.F. Falk, A.L. Kagan, Y. Nir, and A.A. Petrov, Phys. Rev. D **57**, 4290 (1998); A.A. Petrov, hep-ph/9903366.