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MINIMAL BETTI NUMBERS

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Abstract. We give conditions for determining the extremal behavior for the (graded) Betti numbers of squarefree monomial ideals. For the case of non-unique minima, we give several conditions which we use to produce infinite families, exponentially growing with dimension, of Hilbert functions which have no smallest (graded) Betti numbers among squarefree monomial ideals and all ideals. For the case of unique minima, we give two families of Hilbert functions, one with exponential and one with linear growth as dimension grows, that have unique minimal Betti numbers among squarefree monomial ideals.

1. Introduction

Let R be a polynomial ring over a field K. Then given a Hilbert function H it is easy to see that there can be more than one ideal $I \subset R$ such that the Hilbert function of R/I is H. One can further distinguish such ideals by passing to a finer invariant, the graded Betti numbers, which gives rise to the question: given a particular Hilbert function H, what sets of graded Betti numbers actually occur? That this problem is bounded above, and hence finite, is due to an important result by Bigatti and Hulett [B, H] (independently in characteristic zero), and Pardue [P] (in characteristic p) which says that, given a Hilbert function H, the lexicographic ideal attaining H has everywhere largest graded Betti numbers. In fact, this says that the partial order on the set of sets of graded Betti numbers of ideals attaining a given Hilbert function has a unique maximum element. Shortly thereafter, it was shown by Charalambous and Evans [C-E] that this order need not have a unique minimal element. Examples of infinite families of Hilbert functions which did not support unique minimal elements were given by Richert [R].

The structure of the partial order on graded Betti numbers has also been considered on interesting subsets of the set of all ideals attaining a given Hilbert function. For instance, Geramita, Harima, and Shin [G-H-S] showed that if one restricts to the graded Betti numbers arising from a certain (dense) set

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of Gorenstein ideals, then the associated partial order has a unique maximal element which is constructed using a lexicographic ideal (Migliore and Nagel [M-N] were later able to extend this result to an even larger set of Gorenstein ideals attaining a given Hilbert function) while Richert [R] showed that a unique minimal element need not exist. For stable ideals in dimension at most three, Francisco [F] showed that, unlike the general and Gorenstein cases, there is always a unique smallest element.

It is a result of Aramova, Herzog, and Hibi [A-H-H1, A-H-H2] that if one restricts to the graded Betti numbers arising from squarefree monomial ideals attaining a given Hilbert function, then the associated partial order has a unique maximal element (which arises, not surprisingly, from the squarefree lexicographic ideal). Squarefree monomial ideals are particularly interesting (to algebraists, topologists, and combinatorists alike) because to each squarefree monomial ideal I in n variables can be associated a simplicial complex Δ_I on n vertices, while the Hilbert function of R/I is related to the face counts of Δ_I and the graded Betti numbers of R/I can be computed by considering sums of the ranks of the reduced homologies of subsets of Δ_I. It was known that (Gelvin, LaVictore, Reed, Richert [G-L-R-R]) for $n \leq 5$ variables (and after fixing a finite field), the partial orders arising from fixing a Hilbert function were totally ordered, but that this failed in six variables where, in fact, there is an example of a partial order which does not have a unique smallest element. In the current paper, we continue this line of inquiry. First, we generate an infinite family (the size of which grows exponentially with dimension) of Hilbert functions for which the partial order on the graded Betti numbers corresponding to squarefree monomial ideals fails to have a unique minimal element. We are able to show that this same family of Hilbert functions gives rise to partially ordered sets corresponding to all ideals (not only squarefree monomial ideals) which fail to have a unique minimal element. We then find an infinite family (again, growing exponentially) of Hilbert functions for which the partial order on the Betti numbers (not graded Betti numbers) of squarefree monomial ideals fails to have a unique smallest element (and are again able to show that this family gives posets without unique minimal elements in the case of all ideals). Next, we find an infinite family (growing exponentially) of Hilbert functions for which the partial order associated to the graded Betti numbers of squarefree monomial ideals has a single element, and thus a unique smallest element. We note an analogous family in the general case. Finally, we find a infinite family (growing linearly) of Hilbert functions for which the partial order associated to the graded Betti numbers of squarefree monomial ideals has a unique smallest element, and a nontrivial poset tree.

2. Background

Let $R = K[x_1, \ldots, x_n]$ where K is a field. In what follows, all ideals will be homogeneous. In this paper, we will be studying the graded Betti numbers of ideals with a fixed Hilbert function. Recall that given a homogeneous ideal $I \subset R$, the *Hilbert function* of R/I in degree d, denoted as $H(R/I, d)$ is given by

$$H(R/I, t) = \dim_K (R/I)_t.$$

Furthermore, we will write

$$0 \longrightarrow \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta^I_{h,j}} \longrightarrow \cdots \longrightarrow \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta^I_{1,j}} \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

to be a minimal free resolution of R/I, and $\beta^I = \{\beta^I_{i,j}\}$ to be the set of graded Betti numbers of R/I. It is useful to display graded Betti numbers in the following table known as a Betti Diagram (using the notation of Macaulay 2 [G-S]).

	s_0	s_1	s_2	s_3	$\ldots$
0	$\beta_{0,0}$	$\beta_{1,1}$	$\beta_{2,2}$	$\beta_{3,3}$	$\ldots$
1	$\beta_{0,1}$	$\beta_{1,2}$	$\beta_{2,3}$	$\beta_{3,4}$	$\ldots$
2	$\beta_{0,2}$	$\beta_{1,3}$	$\beta_{2,4}$	$\beta_{3,5}$	$\ldots$
3	$\beta_{0,3}$	$\beta_{1,4}$	$\beta_{2,5}$	$\beta_{3,6}$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

where s_i is the sum of the entries in the i^{th} column. Then $S = \{s_i\}$ is the set of Betti numbers. There is an obvious partial order on the Betti diagrams which arise for ideals with a given Hilbert function. If β^I and β^J are the graded Betti numbers of the ideals I and J, then we say that $\beta^I \geq \beta^J$ if $\beta^I_{i,j} \geq \beta^J_{i,j}$ for all i and j. Furthermore, $\beta^I > \beta^J$ if $\beta^I \geq \beta^J$ and there is a pair (i, j) such that $\beta^I_{i,j} > \beta^J_{i,j}$. We are interested in the extremal properties of this ordering. A useful definition in this regard is that of q-linearity: A minimal free resolution of an ideal I is called q-linear if I is minimally generated by q-forms and $\beta_{i,j} = 0$ for each $j \neq q + i - 1$ and $j \neq 0$.

It turns out that the graded Betti numbers are a finer invariant than Hilbert functions. They are related by the following useful equation.

Theorem 2.1. [S] *Given an ideal $I \subset R = K[x_1, \ldots, x_n]$ with graded Betti numbers β^I,*

$$\sum_{d=0}^{\infty} H(R/I, d) t^d = \frac{\sum_{j=0}^{\infty} \sum_{i=0}^{n} (-1)^i \beta^I_{i,j} t^j}{(1-t)^n}$$

It follows from this formula that the diagonal alternating sums of a Betti diagram are invariant for all Betti diagrams of ideals attaining a given Hilbert function. We define this alternating sum as follows:

Definition 2.2. Given an ideal I, the j^{th} diagonal alternating sum of the Betti diagram of I is:

$$d_j = \sum_{i=0}^{j} (-1)^i \beta_{i,j}$$

where the $\beta_{i,j}$ are the graded Betti numbers of I.

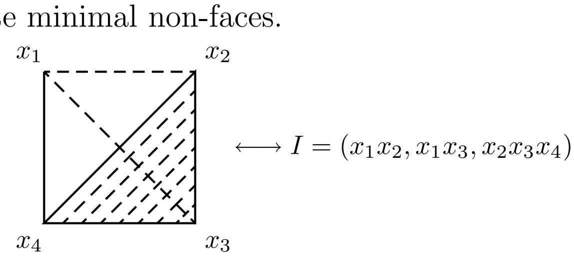
The partial ordering on Betti diagrams has a unique maximal Betti diagram [B, H, P]. In order to identify an ideal attaining the maximal graded Betti numbers, we need the following notation: Let $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ and $x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n}$ be monomials such that $\sum a_i = \sum b_i$ (i.e. they are of the same degree). We say $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n} >_{lex} x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n}$ if and only if the first nonzero entry of $(a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n)$ is positive. This is known as the *lexicographic* or *lex* ordering on monomials. An ideal L is a *lex ideal* if for all monomials $m \in L_d$ and $n \in R_d$, then $n \geq_{lex} m$ implies $n \in L_d$. A lex ideal achieves the maximal graded Betti numbers among all ideals attaining its Hilbert function.

In this paper we are mainly concerned with squarefree monomial ideals. A *squarefree monomial ideal* is an ideal in $R = K[x_1, \ldots, x_n]$ minimally generated by elements of the form $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$, where $a_i \in \{0, 1\}$.

We will make use of the fact that the squarefree monomial ideals are in one-to-one correspondence with simplicial complexes. Recall the definition of a simplicial complex: A *simplicial complex* Δ on the vertex set $\{1, \ldots, n\}$ is a collection of subsets called faces or simplices, closed under taking subsets; that is if $\sigma \in \Delta$ is a face and $\tau \subseteq \sigma$, then $\tau \in \Delta$. A simplex $\sigma \in \Delta$ of cardinality $|\sigma| = i + 1$ has dimension i and is called an i-face.

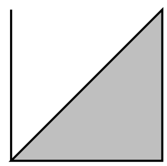
The (well known, see for instance [S]) procedure to pass from a simplicial complex Δ to a squarefree monomial ideal is to form the ideal generated by monomials corresponding to the minimal non-faces of Δ, after renaming the vertex set of Δ to be $\{x_1, \ldots, x_n\}$.

Example 2.3. The simplicial complex $\Delta = \big\{\{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\big\}$ corresponds to the ideal $I = (x_1x_2, x_1x_3, x_2x_3x_4)$. We use dashed lines in the picture to indicate minimal non-faces.



This correspondence gives a bijection between simplicial complexes and squarefree monomial ideals with no linear terms. In general, we will conflate simplicial complexes and ideals due to this correspondence, and so we are free to talk about the Betti numbers of a simplicial complex or the homology of an ideal.

Given a simplicial complex with f_i i-faces, its corresponding *f-vector* is the n-tuple $(f_0, f_1, \ldots, f_{n-1})$. For example, consider the simplicial complex



$(4, 4, 1, 0)$

which has f-vector $(4, 4, 1, 0)$. Here the f-vector counts the 4 0-faces (or points), 4 1-faces (or edges), 1 2-face, and 0 3-faces (or volume).

The f-vector is an important invariant of a simplicial complex because of the following:

Theorem 2.4. [S] *Let I be the Stanley-Reisner ideal of a simplicial complex with f-vector $(f_0, f_1, \ldots, f_{n-1})$. Then*

$$H(R/I, m) = \begin{cases} 1 & m = 0, \\ \sum_{i=0}^{n-1} f_i \binom{m-1}{i} & m > 0. \end{cases}$$

From the theorem, it follows that f-vectors and Hilbert functions are in one-to-one correspondence.

Given a Hilbert function which arises for squarefree monomial ideals, we will be particularly interested in the corresponding squarefree lex ideal (because it always exists and is known to exhibit the maximal Betti diagram [A-H-H1, A-H-H2]). An ideal L is a *squarefree lex ideal* if for squarefree monomials $m \in L_d$ and $n \in R_d$, $n \geq_{\text{lex}} m$ implies $n \in L_d$.

A useful tool in studying simplicial complexes (and therefore their associated ideals) is simplicial homology. If Δ is a simplicial complex, we let Δ^l be the set of all l-faces in Δ, and let $C_l(\Delta)$ denote the K vector space whose basis consists of Δ^l. We define the boundary map $\partial_l : C_l(\Delta) \to C_{l-1}(\Delta)$to be

$$\partial_l(\{i_1, \ldots, i_{l+1}\}) = \sum_{j=1}^{l+1} (-1)^{j+1} \{i_1, \ldots, \hat{i_j}, \ldots, i_{l+1}\}$$

where the hat indicates omission. We define the Δ^l subspaces $Z_l(\Delta) = \ker(\partial_l)$ and $B_l(\Delta) = \text{image}(\partial_{l+1})$. By standard algebraic topology, we have $B_l(\Delta) \subseteq Z_l(\Delta)$, and so we can define the *$l - homology$* of Δ, $H_l(\Delta)$, to be the K vector space $Z_l(\Delta)/B_l(\Delta)$. In addition, the *reduced $l - homology$* $\tilde{H}_l(\Delta)$ is given by $H_0(\Delta) \cong \tilde{H}_0(\Delta) \oplus K$, and $H_l(\Delta)$ when $l > 0$. We now present the connection between the simplicial homology of certain subsets of Δ and graded Betti numbers, in the form of Hochster's formula. To properly express this formula, we need a new notation. If $W \subseteq \{1, \dots n\}$, then given a simplicial complex Δ, let Δ_W be the simplicial complex defined by $\Delta_W = \Delta \cap P(W)$ where P denotes power set. Then we have

Theorem 2.5. [Ho] *Let $I \subseteq R$ be a squarefree monomial ideal, with Δ the associated simplicial complex. Then we have*

$$\beta^I_{i,j} = \sum_{W \subseteq \{1,\dots,n\}, |W|=j} \dim_K \tilde{H}_{j-i-1}(\Delta_W; K)$$

for all i and j.

3. Non-unique Minimal Graded and Nongraded Betti Numbers

3.1. **Graded Betti Numbers.** In this section we construct a fast-growing family of Hilbert functions which fail to have unique minimal graded Betti numbers among both squarefree monomial ideals, and all ideals. We proceed using simplicial complexes, giving a method for preserving parts of the Betti diagrams which guarantee incomparability. Our methods allow us to double the number of f-vectors in each dimension with this property, and so our family of f-vectors grows exponentially with the dimension of the polynomial ring.

Definition 3.1. Let Δ be a simplicial complex on n vertices. We define the j-cone of Δ on the vertex $\{n+1\}$, denoted $C_{(j)}\Delta$, to be the simplicial complex on $n+1$ vertices such that $\{i_1, \dots, i_k\} \in C_{(j)}\Delta$ if and only if either $\{i_1, \dots, i_k\} \in \Delta$ or $i_k = n+1$, $\{i_1, \dots, i_{k-1}\} \in \Delta$ and $k-1 \leq j$. As suggested by the definition, we define $C_{(\infty)}\Delta$ to be the n-cone of Δ, $C\Delta$.

Example 3.2. Let Δ be the simplicial complex

$$\Delta = \{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,4\},\{2,4\},\{3,4\},\{1,2,4\}\}\} \subset \{1,2,3,4\},$$

with f-vector $(4,4,1,0)$. Then

$$\begin{aligned}
C_{(0)}(\Delta) &= \{\{1\},\{2\},\{3\},\{4\},\{5\},\{1,2\},\{1,4\},\{2,4\},\{3,4\},\{1,2,4\}\},\\
C_{(1)}(\Delta) &= \{\{1\},\{2\},\{3\},\{4\},\{5\},\{1,2\},\{1,4\},\{1,5\},\{2,4\},\{2,5\},\{3,4\},\\
&\quad \{3,5\},\{4,5\},\{1,2,4\}\},\\
C_{(2)}(\Delta) &= \{\{1\},\{2\},\{3\},\{4\},\{5\},\{1,2\},\{1,4\},\{1,5\},\{2,4\},\{2,5\},\{3,4\},\\
&\quad \{3,5\},\{4,5\},\{1,2,4\},\{1,2,5\},\{1,4,5\},\{2,4,5\},\{3,4,5\}\},\\
C_{(3)}(\Delta) &= \{\{1\},\{2\},\{3\},\{4\},\{5\},\{1,2\},\{1,4\},\{1,5\},\{2,4\},\{2,5\},\{3,4\},\\
&\quad \{3,5\},\{4,5\},\{1,2,4\},\{1,2,5\},\{1,4,5\},\{2,4,5\},\{3,4,5\},\\
&\quad \{1,2,4,5\}\},
\end{aligned}$$

and, of course, $C_{(3)}(\Delta) = C_{(\infty)}(\Delta)$. Here $C_{(0)}$ has f-vector $(5,4,1,0,0)$, $C_{(1)}$ has f-vector $(5,8,1,0,0)$, $C_{(2)}$ has f-vector $(5,8,5,0,0)$, and $C_{(3)}$ has f-vector $(5,8,5,1,0)$.

On of the nice things about coning a simplicial complex is that the graded Betti numbers do not change. We show below that we can similarly preserve certain of the graded Betti numbers of a simplicial complex after j-coning.

Lemma 3.3. *Given a simplicial complex Δ, the Betti diagram of the simplicial complex $C_{(j)}\Delta$ will be identical to the Betti diagram of Δ for the first $j+1$ diagonals.*

Proof. We employ Hochster's formula. Let $k \leq j+1$. Then we have

$$\beta_{i,k}^{C_{(j)}(\Delta)} = \sum_{W \subseteq \{1,\ldots,n+1\}, |W|=k} \dim_K \tilde{H}_{k-i-1}(\Delta_W; K)$$

$$= \sum_{W \subseteq \{1,\ldots n\}, |W|=k} \dim_K \tilde{H}_{k-i-1}(\Delta_W; K) + \sum_{\{n+1\} \in W, |W|=k} \dim_K \tilde{H}_{k-i-1}(\Delta_W; K)$$

However, from the definition of j-coning and the complex Δ_W, we have that for any W with $\{n+1\} \in W$ and at most j vertices in $\{1,\ldots n\}$, Δ_W is a cone. By standard algebraic topology, a cone is contractible and so all of its reduced homology spaces are zero. Therefore, the second term in the above sum vanishes, and as the first term is $\beta_{i,k}^{\Delta}$ by Hochster's formula, the lemma is proved. □

Definition 3.4. Given j, and an f-vector $(f_0, f_1, \ldots, f_{n-1})$ of a simplicial complex Δ, define $f_m^{[k]}$ to be the m^{th} entry of the f-vector $(C_{(j)})^k\Delta$. Also, for $m > j$, define

$$f_m^{(k)} = \begin{cases} f_m & \text{for } m = j+1 \text{ or } k = 0, \\ f_{m-1}^{(k-1)} + f_m^{(k-1)} & \text{otherwise.} \end{cases}$$

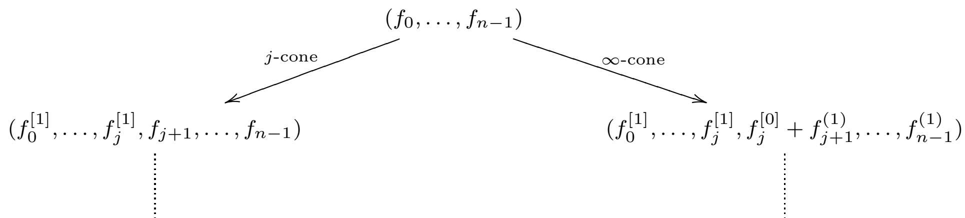
Remark 3.5. If the f-vector of Δ is $(f_0, f_1, \ldots, f_{n-1})$, then it is easy to show that the f-vector of $C_{(j)}\Delta$ is:

$$\begin{cases} (1+f_0, f_0+f_1, f_1+f_2, \ldots, f_{j-1}+f_j, f_{j+1}, \ldots, f_{n-1}, 0) & j < n \\ (1+f_0, f_0+f_1, f_1+f_2, \ldots, f_{n-2}+f_{n-1}, f_{n-1}) & j \geq n \end{cases}$$

or, in the notation of Definition 3.4,

$$\begin{cases} (f_0^{[1]}, f_1^{[1]}, \ldots, f_j^{[1]}, f_{j+1}, \ldots, f_{n-1}, 0) & j < n \\ (f_0^{[1]}, f_1^{[1]}, \ldots, f_{n-1}^{[1]}, f_{n-1}) & j \geq n \end{cases}$$

We speak of this f-vector as the one generated by j-coning the initial f-vector. Consider the family of f-vectors derived by starting with an initial f-vector and at each stage, both j-coning and ∞-coning it. The first iteration of such a tree is shown below, assuming $j < n$:



We will use this technique to generate families of f-vectors which grow exponentially with dimension.

Definition 3.6. Given a simplicial complex Δ and a k-tuple $(m_0, m_1, \dots, m_k)$ where each m is either a nonnegative integer or ∞, define $C_{(m_0,m_1,\dots,m_k)}\Delta$ to be the simplicial complex generated by m_0-coning, then m_1-coning, and so on. Similarly, if I is the associated ideal of Δ, define $C_{(m_0,m_1,\dots,m_k)}I$ to be the ideal associated to $C_{(m_0,m_1,\dots,m_k)}\Delta$.

Lemma 3.7. *Given an f-vector $(f_0, f_1, \dots, f_j, \dots, f_c, \dots)$ of a simplicial complex Δ where f_c is the last nonzero element of the f-vector and $j \leq c$, the f-vectors of the simplicial complexes in the tree generated by repeatedly j-coning and ∞-coning Δ are distinct.*

Proof. We can index each f-vector in the tree with the k-tuple $(m_0, m_1, \dots, m_k)$ where each m is either j or ∞ and $C_{(m_0,m_1,\dots,m_k)}\Delta$ is the corresponding simplicial complex. Let $\{t_i\}$, $0 \leq i < r$ be the t such that $m_{t_i} = \infty$ so that r is the number of times we ∞-cone. Then the f-vector of $C_{(m_0,m_1,\dots,m_k)}\Delta$ is:

$$(f_0^{[k]}, f_1^{[k]}, \dots, f_j^{[k]}, \sum_{i=0}^{r-1} f_j^{[t_i]} + f_{j+1}^{(r)}, \sum_{i=0}^{r-2} f_j^{[t_i]} + f_{j+2}^{(r)},$$
$$\dots, f_j^{[t_0]} + f_{j+r}^{(r)}, f_{j+r+1}^{(r)}, \dots, f_{c+r}^{(r)}, \dots).$$

From this f-vector we can read r from the index of the last nonzero entry minus c. Further, as the f_j^p are monotonically increasing and thus distinct, the elements at position $j+1$ through $j+r$ determine the $\{t_i\}$, and as the j^{th} element gives k, the f-vector uniquely determines $(m_0, m_1, \dots, m_k)$. □

Remark 3.8. This Lemma ensures that j-coning and ∞-coning a suitable f-vector gives a family of Hilbert functions which grows as 2^n with dimension. It can be shown that this growth is not necessarily achieved if we j-cone and l-cone for arbitrary j and l.

Theorem 3.9. *Given a set of squarefree monomial ideals I_k corresponding to an f-vector $\overrightarrow{v}$, if for some j their Betti diagrams have j^{th} diagonal such that* *the alternating sum d_j is nonzero and, for each i, $\min_k \beta^{I_k}_{i,j} = 0$, then each f-vector in the tree created by $(j-1)$-coning and ∞-coning $\overrightarrow{v}$ has an associated Hilbert function such that the poset tree of graded Betti numbers of all ideals attaining this Hilbert function will fail to have a unique minimum. The poset tree of each Hilbert function also fails to have a unique minimum if we restrict to the graded Betti numbers of all squarefree monomial ideals.*

Proof. We show that this condition on the j^{th} diagonal guarantees that a unique minimum cannot exist among the graded Betti numbers. As this diagonal is preserved under both $(j-1)$-coning and ∞-coning and the $C_{(m_0,\ldots,m_k)}I_k$ are squarefree monomial ideals, the theorem follows. As $\min\beta^{I_k}_{i,j} = 0$ for all i, any diagram that is less than all β^{I_k} in the partial order has all zeros on its j^{th} diagonal. However, as d_j is nonzero, no Betti diagram can have all zeros along the j^{th} diagonal and attain d_j. Thus, no ideal can be less than all β^{I_k} and attain this Hilbert function, and as the conditions on the I_k imply that at least two of the I_k are incomparable, a unique minimum cannot exist. □

Remark 3.10. The number of (not necessarily distinct) f-vectors in the tree constructed in Theorem 3.9 grows as 2^c where c is the number of times the initial f-vector has been coned. If the initial f-vector satisfies the hypothesis of Lemma 3.7, then we are guaranteed that these f-vectors will be distinct, and so this family grows exponentially with the dimension of the polynomial ring. We can enlarge this family by $(j-1), \ldots, (j+c-1)$-coning as well as ∞-coning where c is the number of times we have coned already, however, our computational evidence suggests that this family still grows exponentially.

Example 3.11. From an exhaustive computational search, several examples were found that satisfy the hypothesis of the above theorem. The programs written to find these examples are available from the authors on request. The first example occurring in lex order was with the f-vector $(6, 8, 4, 0, 0, 0)$, and had previously been noted by Gelvin, LaVictore, Reed, Richert [G-L-R-R]. Two ideals attaining this f-vector are:

$$I = (x_1x_2, x_1x_3, x_2x_3, x_3x_4, x_3x_5, x_3x_6, x_4x_5)$$

$$J = (x_1x_2, x_1x_4, x_2x_3, x_2x_5, x_3x_4, x_4x_5, x_4x_6, x_1x_3x_5x_6)$$

with graded Betti numbers (calculated over $\mathbb{Z}_{101}$ with Macaulay 2):

$$\beta^I = \begin{array}{c|cccccc} & 1 & 7 & 13 & 11 & 5 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 7 & 12 & 10 & 5 & 1 \\ 2 & 0 & 0 & 1 & 1 & 0 & 0 \end{array} \qquad \beta^J = \begin{array}{c|ccccc} & 1 & 8 & 14 & 9 & 2 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 7 & 12 & 8 & 2 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 2 & 1 & 0 \end{array}$$

The 6^{th} diagonal of these Betti diagrams satisfies the condition that their alternating sums are not zero, but the minimum in each position on the diagonal is zero. Furthermore, $(C_{(\infty)})^3(6,8,4,0,0,0)$ has a nonzero entry in the 5^{th} position of its f-vector. Thus, the family of f-vectors created by 5-coning and ∞-coning $(C_{(\infty)})^3(6,8,4,0,0,0)$ has associated Hilbert functions with non-unique minimal graded Betti numbers for both the squarefree and general case and grows exponentially with dimension.

3.2. **Betti Numbers.** In this section we provide a method for constructing families of Hilbert functions without unique minimal Betti numbers among both squarefree monomial ideals, and all ideals. Again, we use methods for preserving the properties of the Betti diagrams which guarantee incomparability. Our family of f-vectors grows exponentially with the dimension of the polynomial ring.

Lemma 3.12. *If an ideal I has the Betti diagram:*

$$\beta^I = \begin{array}{c|cccc} & s_0 & s_1 & s_2 & \dots \\ \hline 0 & \beta_{0,0} & \beta_{1,1} & \beta_{2,2} & \dots \\ 1 & \beta_{0,1} & \beta_{1,2} & \beta_{2,3} & \dots \\ 2 & \beta_{0,2} & \beta_{1,3} & \beta_{2,4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

such that $\sum_{i=0}^{j} \beta_{i,j} = \mid d_j \mid$ *for all j, then if L is an ideal with the same Hilbert function,* $\sum_i s_i^L \geq \sum_i s_i$.

Proof. Given an ideal L with the same Hilbert function as I, $\sum_j s_j^L = \sum_{i,j} \beta_{i,j}^L \geq \sum_j \mid d_j \mid$. As $\sum_j s_j = \sum_j \mid d_j \mid$ for I, if L had Betti numbers s_i^L such that $\sum_i s_i^L < \sum_i s_i$, then $\sum_i s_i^L < \sum_j \mid d_j \mid$, a contradiction. □

Lemma 3.13. *Let Δ be a simplicial complex, and let I be its associated ideal. The graded Betti numbers of $C_{(0)}\Delta$ can be computed as:*

$$\beta_{i,j}^{C_{(0)}\Delta} = \begin{cases} \beta_{i,j}^I & \textit{for } i = 0, \\ \beta_{i-1,j-1}^I + \beta_{i,j}^I + \binom{n}{i} & \textit{for } i = j-1, \\ \beta_{i-1,j-1}^I + \beta_{i,j}^I & \textit{otherwise.} \end{cases}$$

where n is the number of vertices of Δ.

Proof. Recall that $C_{(0)}\Delta$ is the simplicial complex obtained by adding a point to Δ. By definition, we have that $\tilde{H}_l(C_{(0)}\Delta) = \tilde{H}_l(\Delta)$ if $l > 0$ and $\tilde{H}_0(C_{(0)}\Delta) = \tilde{H}_0(\Delta) + 1$. By Hochster's formula:

$$\begin{aligned}
\beta_{i,j}^{C_{(0)}\Delta} &= \sum_{W\subseteq\{1,\ldots n\},|W|=j} \dim_K \tilde{H}_{j-i-1}(\Delta_W;K) \\
&\quad + \sum_{\{n+1\}\in W,|W|=j} \dim_K \tilde{H}_{j-i-1}(\Delta_W;K) \\
&= \beta_{i,j}^I + \sum_{\{n+1\}\in W,|W|=j} \dim_K \tilde{H}_{(j-1)-i}(\Delta_W;K)
\end{aligned}$$

However, subsets of size j containing $\{n+1\}$ are in bijective correspondence with subsets of $\{1,\ldots,n\}$ of size $j-1$. If $j > i+1$, then by the above formula for adding a point we have that each of the vector spaces in the last term above are isomorphic to the corresponding spaces $\tilde{H}_{(j-1)-i}(\Delta_{W'};K)$ where $W' = W \setminus \{n+1\}$, and so the last sum is just $\beta_{i-1,j-1}^I$. Finally, if $j = i+1$, then in addition to $\beta_{i-1,j-1}^I$, we add 1 for each subset of $\{1,\ldots,n\}$ of size $j-1 = i$, so this gives the second line of the formula. □

Lemma 3.14. *Suppose that two squarefree monomial ideals I and J have the same f-vector $\overrightarrow{v}$ and have Betti numbers s^I and s^J such that for some k, $s_0^I = s_0^J$, $s_1^I > s_1^J$, $s_k^I < s_k^J$, $s_{k+i}^I \le s_{k+i}^J$ for all positive i, and $\beta_{i,j}^I = 0$ when $j-i$ is even and $j > 0$. Then each f-vector in the tree created by 0-coning and ∞-coning $\overrightarrow{v}$ has a Hilbert function such that the poset tree of all Betti numbers of ideals with this Hilbert function will not have a unique minimum. This will remain true if we restrict to the Betti numbers of squarefree monomial ideals.*

Proof. We first observe that the conditions of the theorem still hold under 0-coning and ∞-coning. Graded Betti numbers are unchanged under ∞-coning, and using Lemma 3.13, it is easy to show that the conditions still hold under 0-coning. Thus, for all choices $m_i \in \{0,\infty\}$, the Betti numbers of $C_{(m_0,\ldots,m_k)}I$ and $C_{(m_0,\ldots,m_k)}J$ will be incomparable and $C_{(m_0,\ldots,m_k)}I$ will always satisfy $\sum_j s_j^{C_{(m_0,\ldots,m_k)}I} = \sum_j |\, d_j^{C_{(m_0,\ldots,m_k)}I}|$. As Lemma 3.12 proves that no ideal can have smaller Betti numbers than $C_{(m_0,\ldots,m_k)}I$, unique minimal Betti numbers cannot exist for the family of f-vectors created by 0-coning and ∞-coning $\overrightarrow{v}$. □

Remark 3.15. The theorem remains true with a nearly identical proof if we replace the clause $\beta_{i,j}^I = 0$ if $j-i$ is an even number and $j > 0$ with $\beta_{i,j}^J = 0$ if $j-i$ is an even number and $j > 0$.

Example 3.16. Recall from example 3.11 that the ideals

$$I = (x_1x_2, x_1x_3, x_2x_3, x_3x_4, x_3x_5, x_3x_6, x_4x_5), \text{ and}$$

$$J = (x_1x_2, x_1x_4, x_2x_3, x_2x_5, x_3x_4, x_4x_5, x_4x_6, x_1x_3x_5x_6),$$

corresponded to the f-vector $(6, 8, 4, 0, 0, 0)$ and have graded Betti numbers (over $\mathbb{Z}_{101}$)

$$\beta^I = \begin{array}{c|cccccc} & 1 & 7 & 13 & 11 & 5 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 7 & 12 & 10 & 5 & 1 \\ 2 & 0 & 0 & 1 & 1 & 0 & 0 \end{array} \qquad \beta^J = \begin{array}{c|ccccc} & 1 & 8 & 14 & 9 & 2 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 7 & 12 & 8 & 2 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 2 & 1 & 0 \end{array}$$

These Betti numbers satisfying the conditions laid out in Lemma 3.14 (with the roles of I and J reversed). Thus, the family generated by 0-coning and ∞-coning $(6, 8, 4, 0, 0, 0)$ consists of f-vectors corresponding to Hilbert functions whose poset trees of Betti numbers fail to have unique minimums. This family grows exponentially with dimension by Lemma 3.7.

Remark 3.17. The infinite family given above is also an infinite family of Hilbert functions with non-unique minimal *graded* Betti numbers, as incomparable Betti numbers implies incomparable graded Betti numbers.

4. Unique Minimal Betti Numbers

4.1. **Squarefree Monomial Ideals.** One of the simplest ways to construct Hilbert functions for squarefree monomial ideals with unique minimal graded Betti numbers is to find those for which the graded Betti numbers of the squarefree lex ideal are minimal. As the squarefree lex ideal always gives maximal graded Betti numbers (among squarefree monomial ideals), uniqueness follows immediately.

To proceed, we observe that a minimal free resolution of a squarefree lex ideal generated in a single degree d is d-linear (this was proved by Aramova, Hibi, and Herzog [A-H-H1]). In fact, it is true (see Herzog, Reiner, and Welker in [H-R-W]) that squarefree lex ideals are componentwise linear (an ideal I is componentwise linear if I_d is d-linear for all d). In particular, this implies that if L is a squarefree lex ideal with no minimal generators in degree t, then $\beta^L_{i,j} = 0$ for $j = t + i - 1$—or in words, the Betti diagram of a squarefree lex ideal L may contain a nonzero entry in row i only if $i = 0$ or L has a minimal generator in degree $i + 1$.

We now give a family, growing exponentially with dimension, of f-vectors for which the corresponding poset tree of graded Betti numbers (for squarefree monomial ideals) has a unique (and hence a unique minimal) element.

Theorem 4.1. *Suppose that L is a squarefree lex ideal generated in a single degree. Then the poset tree of graded Betti numbers of squarefree monomial ideals with the same Hilbert function as R/L consists of a unique element (which is thus uniquely minimal). The family of f-vectors which give rise to such L grows exponentially with dimension.*

Proof. We know that a squarefree monomial lex ideal, L, generated in a single degree has a d-linear resolution. Thus, by Lemma 3.12, no other ideal with the same f-vector can have smaller Betti numbers than L and so it is the unique minimum. For a polynomial ring with n variables, there are $\binom{n}{i}$ squarefree monomials of degree i and thus $2^n - n - 1$ distinct squarefree monomial lex ideals generated in a single degree. This gives the exponential growth with dimension. □

Remark 4.2. The same techniques can be used to show that Hilbert functions containing lex ideals generated in a single degree will have unique minimal graded Betti numbers among all ideals.

4.2. **Unique Mins via Hochster's formula.** We now present a new method for finding unique minima among squarefree ideals using Hochster's formula.

Lemma 4.3. *Suppose that L is a squarefree lex ideal with $\beta^L_{i,j} \neq 0$ iff $i, j = 0$ or $j - i = 1$ or 2. Then if an ideal exists that attains the same f-vector as L, and satisfies the hypothesis of Lemma 3.12, it will have the unique minimal graded Betti numbers among all squarefree monomial ideals with this f-vector.*

Proof. The conditions on L imply that any ideal with the same f-vector will have at most two nonzero Betti numbers in each of its diagonal alternating sums, and these will have opposite signs in the summation. Thus, if an ideal I satisfies the hypothesis of Lemma 3.12, it will have at most one nonzero graded Betti number on each diagonal and so no other ideal can have a smaller graded Betti number than I while preserving the d_j. □

Theorem 4.4. *In the polynomial ring of dimension n, the Hilbert function corresponding to the f-vector $(n, k, 0, \ldots, 0)$ for $0 \leq k \leq n$ has an ideal which attains minimal graded Betti numbers among all squarefree monomial ideals with the same Hilbert function.*

Proof. If $k = 0$ we have that the given simplicial complex is the only one with the corresponding f-vector, and hence is minimal. For $k > 0$, we note that the squarefree lex ideal for this f-vector can only be generated in degree 2 and 3, as a generator in degree 4 would correspond to a minimal non-3-face, which would certainly imply that this simplicial complex has 2-faces; but the above f-vector has none. By Lemma 4.3, it remains to show that there exist ideals attaining $(n, k, 0, \ldots, 0)$ for each $1 \leq k \leq n$ which satisfy the hypothesis of Lemma 3.12. We do this in the following two results. □

In this first lemma, we show that for each $1 \leq k \leq n-1$, there is an ideal attaining $(n, k, 0, \ldots, 0)$ which satisfies the hypothesis of Lemma 3.12.

Lemma 4.5. *Let $1 \leq k \leq n-1$. Then the simplicial complex*

$$\Delta = \{\{1\}, \ldots, \{n\}, \{1,2\}, \{2,3\}, \ldots, \{k, k+1\}\}$$

has a 2-linear resolution.

Proof. By Hochster's formula, a resolution will be 2-linear if there is no reduced l-homology for $l \geq 1$, for any sub-complex Δ_W (because in that case $\beta_{i,j} = 0$ for $i = j$ and $\beta_{i,j}$, which for $j > i+1$ depends only on the spaces $\tilde{H}_{j-i-1}$, are all zero). But note that for any $W \subseteq \{1, \ldots n\}$ the simplicial complex Δ_W is homotopy equivalent to a finite set of points. Since a finite set of points never has reduced l-homology for $l \geq 1$, the lemma is proved. □

We now demonstrate that there is an ideal attaining $(n, n, 0, \ldots, 0)$ which satisfies the hypothesis of Lemma 3.12.

Lemma 4.6. *The only graded Betti numbers of the simplicial complex*

$$\{\{1\}, \ldots, \{n\}, \{1,2\}, \{2,3\}, \ldots, \{n,1\}\}$$

which can be nonzero are $\beta_{0,0}$, $\beta_{i,i+1}$ for $1 \leq i \leq n-2$ and $\beta_{n-2,n}$.

Proof. The case $\beta_{i,j}$ for $j < n$ is done above as if $W \subset \{1, \ldots, n\}$ (strict containment) then Δ_W is a simplicial complex in the form of the above lemma. If $j = n$, then $\beta_{i,n}$ depends only the space $\tilde{H}_{n-i-1}(\Delta, K)$. However, this simplicial complex is homotopy equivalent to a circle, so its only nonzero reduced homology space is $\tilde{H}_1$, which occurs in Hochster's formula when $i = n-2$. □

Remark 4.7. This family of Hilbert functions with unique minimal graded Betti numbers grows linearly with dimension; from Theorem 4.4, we have $n+1$ Hilbert functions in each dimension.

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