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Teaching the Perceptual Structure of Algebraic Expressions: Preliminary Findings from the Pushing Symbols Intervention

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Abstract

We describe an intervention being developed by our research team, Pushing Symbols (PS). This intervention is designed to encourage learners to treat symbol systems as physical objects that move and change over time according to dynamic principles. We provide students with the opportunities to explore algebraic structure by physically manipulating and interacting with concrete and virtual symbolic systems that enforce rules through constraints on physical transformations.

Here we present an instantiation of this approach aimed at helping students learn the structure of algebraic notation in general, and in particular learn to simplify like terms. This instantiation combines colored symbol tiles with a new touchscreen software technology adapted from the commercial *Algebra Touch* software. We present preliminary findings from a study with 70 middle-school students who participated in the PS intervention over a three-hour period.

Keywords: Algebra education; learning; perception; mathematical cognition

Introduction

The core conceptual content of algebra is extraordinarily simple: it is largely exhausted by the properties of addition and multiplication over the real numbers, such as commutativity, associativity, and distributivity, together with basic properties of functions and equivalence relations over the same structure. This formal simplicity belies the great difficulty students have in mastering basic algebra content (NAEP, 2011) — and especially the notation universally used to express algebraic claims (McNeil, 2008; Koedinger & Alibali, 2008).

One way to explain the difficulty of algebra is that unlike number cognition, algebraic reasoning does not seem to fit neatly into a core conceptual domain (Dehaene, 1997; Carey, 2009). Children may then face the challenge of assembling new cognitive tools appropriate to algebraic interactions. This task is made more challenging because typical instruction in basic algebraic notation is often brief and involves an emphasis on memorization of abstract rules.

Algebraic literacy—the fluent construction, interpretation, and manipulation of algebraic notations—involves not just memorizing rules, but also learning appropriate perceptual processes (Goldstone, Landy, & Son, 2010; Kirshner, 1989; Landy & Goldstone, 2007, 2008, 2010; Kellman, Massey, & Son, 2010). Like other formal diagrammatic systems (such as, for example, Venn diagrams) algebraic notation aligns the structure of the content domain with automatic perceptual properties and necessary physical laws (Cheng, 1999; Landy, Allen, and Anderson, 2011; Landy, 2010). In this way reasoning that is properly cognitive can be accomplished by perceptual-motor systems such as attention (Patsenko & Altmann, 2010) or perceptual organization (Landy & Goldstone, 2007; Novick & Catley, 2008). Although such transformation of cognitive work into perceptual processing may the carry distinctive risk of mistaking perceptual properties of representations for content principles (Novick & Catley, 2007; Kirshner & Awtry, 2004), it may also be critical to reducing cognitive load in complex operations (Sweller, 1994).

Successful students often use perceptual and visual patterns available in notations to solve mathematical problems. Like many skills learned from long practices learning algebra involves perceptual training- learning to *see* equations as structured objects (Landy and Goldstone, 2007; Kellman et al., 2008; Kirshner & Awtry, 2004). For instance, people seem to group symbols into perceptual chunks and use these groups, rather than just calculation rules, to perform mathematics. Although in some cases the appropriate perceptual patterns are fairly easy to see (Kirshner & Awtry, 2004), in other cases understanding the visual forms requires that a learner internalize an appropriate way of seeing a piece of notation. Real-world motion, changes, and transformations are naturally memorable and easy to acquire, making these processes natural tools for helping students grapple with algebra (Landy, 2010). Some successful object-centered transformations, however, may not be as immediately obvious as others in traditional instruction. Therefore, training students to *see* the structure of algebra may be a promising approach to teaching algebraic ideas.

While this perceptual-motor understanding of algebraic forms is a potentially rich and powerful source of student understanding, it also stands as a barrier to learning if visual patterning is not taught in a controlled manner. While some students learn easily, others latch on to incorrect perceptions and, consequently, generalizations (Marquis, 1988; Kirshner, 1989; Nogueira de Lima & Tall, 2007). Our goal is to find instructional and pedagogical paths through which students can make use of the strength of perceptual patterns in algebraic notation without falling prey to misleading visual structures or overly procedural, low-level understandings.

Pushing Symbols: Teaching the Structure of Algebraic Expressions

The purpose of the PS intervention is to explore an alternative method of algebra instruction that focuses

Perceptual Process	Formal Transformation	Illustration
Rigid Motion	Commutation Transposition Rearrangements	x+a = a+x
Splitting	Distribution	a(b+c) = (ab + ac)
Joining	Factoring, Canceling	t + 2x - 2x + q = t + q
Symmetric Creation / Destruction	Equation Transformation, Simplifying Fractions	x - 8 = 3 x - 8 + 8 = 3 + 8

Figure 1. Algebraic Transformation Visualizations

student efforts on the visual structure of formalisms, both by directly presenting those visual patterns and by challenging students to maintain and explain them. This method is being instantiated in a pedagogical intervention (PS) consisting of a set of in-class discussions, activities, and a dynamic computer-based visualization method. The intervention allows students to physically and dynamically interact with algebraic expression elements, providing a potentially powerful source of perceptual-motor experiences. Rather than simply rewriting different static expressions, in PS learners directly interact with expression objects and transform them using dynamical laws. Because rigid motion is a powerful perceptual grouping mechanism (Palmer, 1999) it is anticipated that training in which students see correct algebraic structures in dynamic transformations may lead to improved understanding of algebraic concepts.

The PS intervention has several specific aims. First we aim to increase fluency and accuracy by improving the alignment between students' visual-motor processes and proper formal operations and transformations. (Figure 1). Second the PS program is designed to be engaging for students, which is intended to build efficacy in students and develop the attitude that algebra can be intuitive, predictable, and even fun.

Algebra Structure Tiles

The Pushing Symbols manipulative system uses colored magnets and tiles to decompose the structure of algebraic expressions. There are 4 different colored tiles in a set (see Figure 2), and each color represents a specific mathematical object (number, variable, coefficient, symbol). Yellow tiles represent numbers (from ±1-9), blue tiles represent symbols or mathematical operations. (+), red tiles represent x variables and coefficients (from ±1-9), and green tiles represent y variables and coefficients (from ±1-9). After modeling an expression, the tiles can be rearranged and simplified into equivalent expressions.

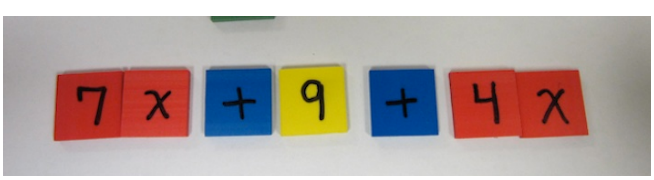

Figure 2: Algebra Structure Tiles

The *Algebra Touch Research* (ATR) Software

The PS system uses a computer application developed in collaboration with Regular Berry software to teach students basic algebraic principles while richly engaging perceptual-motor systems (Figure 3). We describe software developed by Regular Berry software based on the *Algebra Touch* system, which instantiates the transformations specified by the PS intervention (We will call this *Algebra Touch: Research,* or *ATR)* In ATR students perform arithmetic functions by tapping on a sign and algebraic rearrangements are carried out by touching appropriate symbols and moving them into the desired location. ATR provides dynamical models of basic algebraic properties and transformations such as distributivity of multiplication over addition, commutativity, simplification of like terms through addition, and reduction of fractions to lowest terms.

ATR does not allow students to make mistakes; if they attempt to do something against the laws of mathematics, a brief side-to-side motion (a "shake") provides immediate feedback that their desired action was illegal. As a result, students immediately see how the rules result in legal transformations or manipulations in a way that is impossible with a traditional blackboard or overhead projector lesson.

Problems in ATR can be presented in either an untimed *list* mode or a *game* mode. In both modes the presentation and interaction with individual problems is identical. However, in the *game* mode problems are collected into *level,* and performance on any particular level is scored with a number of stars. Stars are based on the number of mistakes made during problem solution, and the speed with which a particular problem is solved. If too many mistakes are made or time runs out, the level is "failed" and must be restarted. At the end of each problem, the program provides immediate feedback to students about the number of errors they made and the speed to which they simplified the expression.

Study Details

The PS approach has been instantiated in a single trial lesson covering combination of like terms. This lesson lasts approximately 90 minutes, and involves a large set of symbol tiles for teacher demonstrations on a whiteboard, smaller tiles used by students in pairs, and the ATR software.

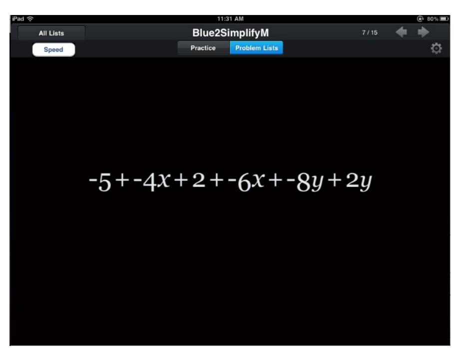

Figure 3: *Algebra Touch Research* software

We anticipated that the intervention would decrease the amount of structural errors that students made, and improve their overall understanding of simplifying expressions. Since the intervention did not explicitly address solving word problems, we did not anticipate a change in the number of word problems solved successfully. We also hypothesized that pre-test scores, self-efficacy, engagement, and performance on the iPad would positively contribute to post-test scores, while math anxiety would negatively contribute to post-test performance. We also predicted that the intervention would shift participation and engagement.

Participants

Seventy eighth-grade students from an urban public middle school in the mid-east United States participated in this study during their regular mathematics instruction time. These students had never received instruction on like-terms or simplifying expressions before this intervention. Student assent and parental consent were obtained prior to participation in this study, in accordance with the directions of the University of Richmond Institutional Review Board.

Study Procedures

The study took approximately 3 hours in total and occurred over three class periods. On the first day (90 minutes), students completed a pre-test on simplifying algebraic expressions and a Mathematics Self-Efficacy and Anxiety questionnaire. Next, students received a whole-group lesson on simplifying expressions. During this lesson, the teacher (the first author) led a series of discussions and used colored tiles to demonstrate algebraic structure. Students were then put into groups of 3 and used colored tiles to identify and combine like terms and simplify expressions. Third, students participated in a 20-minute exploration and training activity that provided students with an opportunity to learn how to use the iPad and *ATR* technology.

On the second day (90 minutes), students were each given an iPad, and were given 40 minutes to solve problems. Practice was divided into two phases. In the first phase, students simplified simple expressions involving no more than about 4 terms; in the second phase, more complex expressions involving up to 8 terms. Each 20 minutes phase was divided between an initial list of 10 untimed problems, followed by a set of 40 game problems.

Any pedagogical approach, especially those based on software interventions, must address the assistance dilemma (Aleven and Koedinger, 2002): how and how much help should be provided to learners, and when? ATR makes several fixed commitments: students cannot complete illegal transformations, for instance. In the current study, we also varied the amount of arithmetic support given to students. Participants were randomly assigned into 2 groups. In one group, students manually calculated the simple problems[1], but arithmetic in structurally more complex problems was calculated automatically by the software; in the second condition, assistance pattern was reversed. There were no differences in structural understanding or success in word problems between the two groups, and, this manipulation will not be discussed further.

At the end of the intervention, students completed a questionnaire about their engagement during the intervention and a post-test. We also conducted student focus groups to receive feedback on what aspects of the intervention were most helpful and enjoyable. 2 weeks after the intervention, students completed a retention test.

Measures

Simplifying Expressions Assessments. Each child completed an 18-item pre, post, and retention test on paper involving expression simplification. These tests assessed two major types of expression-related problem-solving skills: procedural facility with simplification (10-items), and expression construction and evaluation (word problems) (6 items). The problems on the pre, post, and retention tests were similar in form and difficulty.

We followed several steps to code the assessments. First, we coded each item on the assessment as incorrect, correct, or did not attempt. Next, to understand the source of the errors, we conducted error analyses on each item. Four error codes were used: 1) no error, 2) structural error; 3) addition or negative error; and 4) did not attempt. Structural errors include combining unlike terms, over-combination (simplifying the expression correctly and then combining un-like terms) or partial structural errors (moving around like terms but not completely simplifying the problem). Since the PS framework is designed to make structure concrete, naturally structural errors are particularly interesting for analysis. Addition and negative errors were coded when students used correct structure, but made an arithmetic error when combining terms. When a problem was left blank, we coded it as "did not attempt". On average, students did not attempt to solve 25% of the pre-test problems, 16% of the problems on the post-test, and 20% on the retention test.

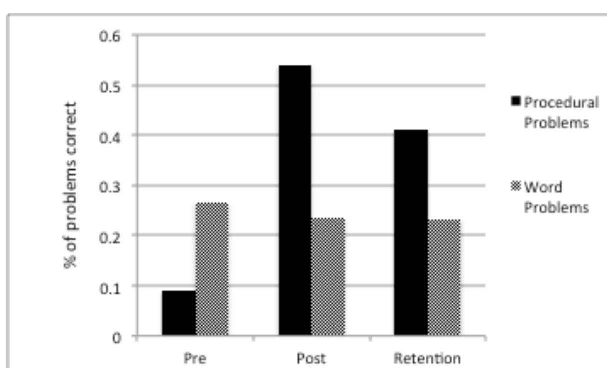

Figure 4: Proportion of Attempted Problems Solved Correctly (Free of structural errors)

[1] An example of manual and automatic calculation modes can be seen at http://davidlandy.net/PushingSymbols/RPS--12-1-11-Like-Terms-Manual-1.mov and http://davidlandy.net/PushingSymbols/RPS--12-1-11-Like-Terms-Automatic-1.mov

Third, for each assessment (pre, post, and retention), we calculated 2 composite scores. 1) proportion of attempted procedural problems that were free of structural errors. 2) proportion of attempted word problems that were solved correctly. These two scores were used to measure student understanding of algebraic expressions in the analyses.

ATR Performance. iPad Performance was measured at 2 different levels, using the *Algebra Speed* game. Level 1 asked students to simplify a series of 36 *simple* expressions (ex. 5+7+3; x+2+6). Level 2 asked students to simplify a series of 40 *complex* expressions (ex. 7+2x+5x+4y+1+-2y). Students could receive a maximum of 3 points for each problem solved. The points system accounted both the number of errors that they made and the speed to which they simplified the expression. At the end of each level students received a level performance score, which represented the total number of points received on the level. Total points on Level 1 (*simple)* and Level 2 (*complex*) were used as 2 measures of ATR performance.

Mathematics Self-Efficacy and Anxiety Questionnaire. Students were administered a set of 10-items pertaining to their self-efficacy and anxiety in mathematics. All 10 items were on a uniform 4-point scale (1=almost never, 2=sometimes, 3=most of the time, 4=almost all of the time). To assess students' math self- efficacy beliefs, 5 items were adapted from the Academic Efficacy subscale of the Patterns of Adaptive Learning Scales (Midgley et al., 2000) (e.g. "I know I can learn the skills taught in math this year") (α=.82). To measure students' feelings of math anxiety, 5 items were adapted from the Student Beliefs about Mathematics Survey (Kaya, 2008) (e.g. "I feel nervous when I do math because I think it's too hard") (α=.61). Scores for each construct were then averaged to create a mean math self-efficacy and mean mathematics anxiety composite.

Student Engagement in Mathematics Questionnaire. Student engagement during the lesson was measured using 18 items that were adapted from the Student Engagement in Mathematics Questionnaire (Kong, Wong, & Lam, 2003): (e.g. "Today I only paid attention in math when it was interesting."). All 18 items were on a 4-point scale (1=no, not at all true, 2=a little true, 3=often true, 4=yes, very true).

Results

Analysis 1: Does the Pushing Symbols Intervention improve student understanding of algebraic structure?

Procedural Problems. On average the intervention increased students' knowledge of algebraic structure (Figure 4). At pretest only 9.4% of problems were solved without structural errors. At post-test 54% of problems attempted were solved without structural errors (Improvement of 44.6%, t=10.48, p<0.01). At retention 41.4% of the problems were solved without structural errors (overall improvement of 32%, t=6.81, p<0.01). After 2 weeks students retained 72% of their structural learning.

Word Problems. As expected, the intervention did not appear to improve student understanding of word problems at post-test (t=-0.87, p>0.05) or retention (t=-0.07, p>0.05).

Analysis 2: Relations between structural performance, efficacy, anxiety, engagement, and performance on *ATR*.

We conducted regression analyses to examine potential predictors of structural performance on the post-test. We included the following variables in the analysis: gender, math self-efficacy, math anxiety, engagement, pre-test performance, and iPad performance.

Correlations and descriptive statistics are reported in Table 1 and the regression results are presented in Table 2. Three main effects were found. First, results indicate that math efficacy was related to higher performance on the post-test (a 1 point increase in efficacy was related to a 1.27 point increase in performance). Second, successfully completing more problems (both simple and complex) on ATR was related to higher scores on the post-test. Further, students who reported being more engaged during the PS intervention performed higher on the post-test (for every 1 point increase in engagement, students performed 1.80 points higher on the post test). Interestingly, students' performance on the pre-test or levels of math anxiety did not predict performance at post-test.

Table 1: Means, Standard Deviations, and Correlations for Measures of Performance, Beliefs, and Engagement

Variable	Mean	SD	1	2	3	4	5	6	7	8	9	10
1. Performance on Post-test	5.40	3.70	-									
2. Gender	0.55	0.51	-0.10	-								
3. Math Self-Efficacy	2.95	0.58	0.29*	-0.12	-							
4. Math Anxiety	2.01	0.61	-0.27*	0.04	-0.37**	-						
5. Performance on Pre-test	3.09	0.58	0.45**	-0.25	0.19	-0.16	-					
6. *AT* Level 1- Simple expressions	0.94	1.57	0.30*	-0.21	0.02	-0.21	0.12	-				
7. *AT* Level 2- Complex expressions	63.36	37.85	0.14	-0.19	-0.14	0.12	0.01	0.14	-			
8. Math Engagement	51.44	46.50	0.47**	0.15	0.08	-0.32**	0.22	0.31*	-0.30**	-		
9. Scaffold Group	0.59	0.50	-0.10	-0.16	-0.06	0.18	-0.13	-0.15	.65**	-0.68**	-	
10. Performance on Retention test	4.14	4.02	0.69**	-0.09	0.01	-0.30*	0.25	0.25*	0.22	0.33**	-0.01	-

Table 2: Predictors of Algebraic Structure Performance on Post-Test

Variable	β	SE	t
Intercept	-7.04	3.76	-1.87
Gender	0.06	0.79	0.57
Math Self-Efficacy	0.21**	0.68	1.85
Math Anxiety	-0.07	0.69	-0.52
Performance on Pre-test	0.15	0.26	1.30
AT Level 1- Simple expressions	0.29**	0.01	2.73
AT Level 2- Complex expressions	0.44**	0.01	2.06
Math Engagement	0.30**	0.66	2.75
Scaffold Group	0.09	1.24	0.50

Discussion

We have described an approach to algebra instruction that emphasizes perceptual and manual interactions with dynamically realized models of algebraic notation, as a vehicle for helping students become fluent with algebraic structure. Although our current results are quite preliminary and not experimental, they do demonstrate that a short intervention based on this framework may substantially improve student performance at simplifying expressions. Furthermore, this work adds to a small literature suggesting that touchscreen-based learning tools can successfully lead to student learning.

Although our results suggest that, on average, student performance increased substantially after receiving the intervention, not all students mastered the material. Many students still struggled with simplifying expressions or did not attempt many of the problems. It will be important to compare motion-based interventions such as this one with other methods of instruction in algebra notation in the future, to better understand the relative value of the AT system

The current system contrasts with many popular algebra manipulative systems, such as Algebra Tiles and Hands-on Equations (Foster, 2007), in its emphasis on the structure of mathematical expressions rather than models of the concepts referred to by them. We certainly believe that connecting algebraic structure to relevant and intuitive examples has an important place in the teaching of algebra. However, given the clear demonstrations that students struggle to understand basic algebraic notation (Koedinger, Alibali, & Nathan, 2008), that closely connecting structure to content can impede learning (Kaminski, Sloutsky, and Heckler, 2006), and existing evidence linking teaching algebraic structure to improved student understanding of algebraic expressions (Banerjee & Subramaniam, 2011), we believe that there is good reason to pursue manipulative systems that expressly communicate algebraic structure through engaging perceptual and motor interactions.

The current findings also suggest that a hands-on approach to teaching the structure of algebra may benefit students. Students reported that physically moving objects (the tiles and ATR) around helped them focus on the steps necessary to simplify expressions. There is also anecdotal evidence that students were applying the ideas of perceptual motion when solving problems on paper. We often observed students gesturing and moving the terms around with their fingers, as well as drawing lines or arrows to represent the legal moves and actions. These observations are consistent with research suggesting that gesturing or alternative ways to represent new ideas may improve student learning (Cook, Mitchell, & Goldin-Meadow, 2008). They also reported that this approach seemed to help them better understand previously taught concepts (such as commutative property, order of operations).

It is also worth noting that the intervention seemed to increase student interest, participation, and interactions. Both observational and student reported engagement during this intervention was high. Virtually all students reported that the intervention was engaging, and fun. In addition, virtually all students reported liking to solve algebra problems more in ATR than in more traditional approaches.

This study has several limitations that limit the conclusions that can be drawn from it. Although beyond the scope of this current study, future work utilizing a control group involving more traditional instruction and practice will better examine the efficacy of this intervention. It is also unclear how the learning from this intervention differs from learning that would occur from typical classroom instruction, and how such differences may impact learning of future topics (Schwartz and Black, 1996). The design of the study also does not allow us to tease apart which components of the intervention (classroom instruction, manipulatives, and/or practice on the iPad) are most useful in building student understanding. Although each of these components implemented the general framework and underlying cognitive principles, given the large current interest in technological interventions, it will be important in future work to distinguish the particular contributions of each of these components and their interaction.

The value of this research at its current stage lies in pointing the direction to a complex of ideas and practices that connect education, cognitive science, and interface design. As designed experiences become more ubiquitous and richly featured, it becomes increasingly possible to construct novel experiences that evoke abstract content in powerful, perceptually specific ways. The limit point of the approach we are pursuing is not just one in which problem solving is fun, game like, and perceptually powerful. Instead, this research represents a starting point toward a conception of formal learning in which the structures of mathematics are directly explorable—in which the abstract is rendered consistently concrete.

Acknowledgments

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The *Algebra Touch* software is owned and designed by Regular Berry software, and none of the authors is involved with the production or sale of *AT*. However, features of *AT* have been implemented in collaboration with the second author, with the express purpose of instantiating the *Pushing Symbols* framework.

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