Lawrence Berkeley Laboratory

UNIVERSITY OF CALIFORNIA

Accelerator & Fusion Research Division

Presented at the 3rd Summer School on High-Energy Particle Accelerators, Fermi National Accelerator Laboratory, Batavia, IL, July 1983

STATISTICAL PHENOMENA IN PARTICLE BEAMS

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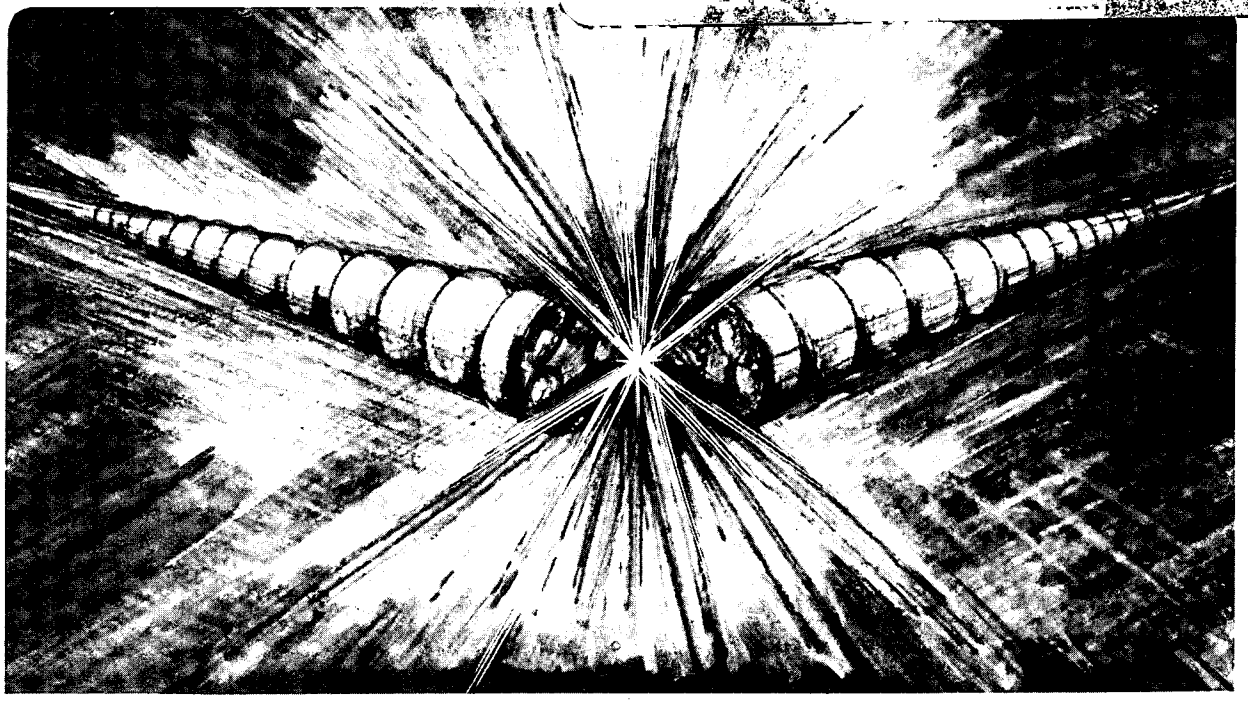
September 1984



Prepared for the U.S. Department of Energy under Contract DE-AC03-76SF00098

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LBL-17714

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*This work was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, High Energy Physics Division, U.S. Dept. of Energy, under Contract No. DE-AC03-76SF00098.

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1 INTRODUCTION

Particle beams are subject to a variety of apparently distinct statistical phenomena such as intrabeam scattering, stochastic cooling, electron cooling, coherent instabilities, and radio-frequency noise diffusion. In fact, both the physics and mathematical description of these mechanisms are quite similar, with the notion of correlation as a powerful unifying principle. Consider the following examples:

a. When two particles collide, their phase space coordinates determine the strength of the interaction. After collision these coordinates are no longer statistically independent.

b. The normal Schottky spectrum of a beam (which results from random phases of particle orbits) is suppressed by a stochastic cooling feedback system. Feedback has produced some "micro-ordering" of the beam.

c. In electron cooling the cold electron beam polarizes in response to the hot proton beam. This polarization provides the coupling necessary for temperature relaxation.

d. A coherent oscillation of a beam requires that particles move in unison. Statistical independence is lost.

e. The spectrum of radio-frequency noise is characterized by temporal coherence, and its effects on a particle beam can be reduced by interparticle screening.

In all the above cases the randomness of coordinates among beam particles is at least partially lost. Correlations have developed.

In this presentation we will attempt to provide both a physical and a mathematical basis for understanding the wide range of statistical phenoma that have been discussed. In the course of this study the tools of the trade will be introduced, e.g., the Vlasov and Fokker-Planck equations, noise theory, correlation functions, and beam transfer functions. Although a major concern will be to provide equations for analyzing machine design, the primary goal is to introduce a basic set of physical concepts having a very broad range of applicability.

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*This work was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, High Energy Physics Division, U.S. Dept. of Energy, under Contract No. DE-AC03-76SF00098.

2 CORRELATIONS IN PHYSICAL SYSTEMS

Consider a collection of N beam particles with random transverse positions x_i distributed normally with an rms deviation σ. The rms transverse position of the beam center is given by

$$\bar{x}_{rms} = \left\langle \bar{x}^2 \right\rangle^{1/2} \equiv \left\langle \left(\frac{1}{N} \sum_i x_i \right)^2 \right\rangle^{1/2} = \frac{\sigma}{\sqrt{N}} \tag{1}$$

if the transverse positions are statistically independent. Now consider the action of a feedback system which damps x to zero. We have

$$0 = \left\langle \bar{x}^2 \right\rangle = \frac{1}{N^2} \sum_i \left\langle x_i^2 \right\rangle + \left(\frac{1}{N^2} \sum_i \sum_{j \neq i} \left\langle x_i x_j \right\rangle \right) \tag{2}$$

and, since the first term on the right is positive, the second term must be negative. The feedback system has destroyed the statistical independence of the particle transverse positions, and in fact has introduced on the average a negative correlation.

As a second example consider a collection of electronic oscillators with frequencies ω_i and random amplitudes and phase. The signal produced by this collection will be of the form

$$s(t) = \sum_i \left(a_i \cos \omega_i t + b_i \sin \omega_i t \right) \tag{3}$$

where a_i and b_i are real, independent random variables with rms values σ_i. This signal crudely models thermal noise, which arises from many randomly excited modes of a system (e.g., a resistor). Consider the average value of the product of this signal at two different times. In other words, calculate the autocorrection of $s(t)$. We have

$$\left\langle s(t_1)\, s(t_2) \right\rangle = \sum_i \sigma_i^2 \cos \omega_i (t_1 - t_2) \, . \tag{4}$$

Suppose the ω_i are uniformly distributed in the interval $(0, \Omega)$. Then, when $|t_1 - t_2| < \pi/2\Omega$, all the terms in the sum (4) are positive, and the noise has a positive autocorrelation. On the other hand, for $|t_1 - t_2| >> \pi/2\Omega$, the cosines will tend to average to zero, and the correlation will vanish. In the limit of infinite Ω, the correlation time tends to zero. Also note that the auto-correlation function's frequency spectrum coincides with the frequency distribution of the oscillators.

In what follows, the correlations illustrated above will be studied in more rigorous detail.

3 NOISE AND STOCHASTIC PROCESSES

Consider a collection of similar noise sources (labeled r) and their respective signals $s(t,r)$. For any fixed time t_0, $s(t_0,r)$ is a simple random variable with some distribution function. Statistical properties of the signal are determined by averaging over the collection of sources. Of fundamental importance is the average of the product of signals at different times, the autocorrelation function $R(t_1,t_2)$. We have

$$R(t_1,t_2) = \left\langle s(t_1,r)\, s(t_2,r) \right\rangle_r . \tag{5}$$

If R is a function of $t_1 - t_2$ only, the noise or stochastic process is said to be weakly stationary. Having established the nature of the averaging process, from here on we drop the r variable.

Given a noise signal $s(t)$ for a particular source, we can try to Fourier decompose it. We must expect some delta function character similar to plane waves since $s(t)$ on the average does not fall off and thus there is infinite energy when integrated for all time. Let $s(t)$ have the Fourier decomposition

$$s(t) = \int_{-\infty}^{+\infty} d\omega\, a(\omega)\, e^{i\omega t} \tag{6}$$

with $a(\omega) = a(-\omega)^*$ for $s(t)$ real. As in the earlier model of electronic noise, oscillators at different frequencies should be uncorrelated. For this continuous distribution of oscillators, this condition is summarized by the relation

$$\left\langle a(\omega')\, a(\omega'') \right\rangle = P(\omega')\, \delta(\omega + \omega') \tag{7}$$

where P is the distribution function for the oscillators.

The average power of the noise source is defined by

$$P_{av} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} dt\, |s(t)|^2 . \tag{8}$$

Using relation (7) we have

$$P_{av} = \lim_{T\to\infty} \iint d\omega\, d\omega' \left\langle a(\omega)\, a(\omega')^* \right\rangle \frac{\sin(\omega - \omega')T}{(\omega - \omega')T} = \int d\omega\, P(\omega) \qquad (9)$$

and the fundamental relation that P is the power spectrum of the noise process. [Problem: Using Eq. (7) show that the power spectrum is related to the auto-correlation through the Fourier transform.] For more details on random processes and noise see Papoulis.[8]

4 SCHOTTKY NOISE

Of particular interest to the accelerator physicist is the noise produced by a particle beam, the Schottky noise or signal (one person's noise is another's signal). The current of a single particle circulating in a storage ring is given by

$$I(t) = \sum_n \delta(t - nT_j) = f_j \sum_n e^{in\omega_j t}\, e^{in\phi_j} \qquad (10)$$

where f_j is the revolution frequency, $\omega_j = 2\pi f_j$, $T_j = 1/f_j$, and ϕ_j is a random phase. The last equivalence follows from a Fourier series decomposition of the periodic delta functions. The $a(\omega)$ decomposition for the beam Schottky noise is

$$a(\omega) = \sum_j \sum_n f_j\, e^{in\phi_j}\, \delta(\omega - n\omega_j) \qquad (11)$$

where we sum over all particles in the beam. If the particle frequency differencies are small, the spectrum reduces to separate frequency bands for each n. Within one band the $a(\omega)$ satisfy the condition

$$\left\langle a(\omega)\, a(\omega') \right\rangle = \sum_i \sum_j f_i f_j\, e^{in(\phi_i - \phi_j)} \cdot \delta(\omega - n\omega_i)\, \delta(\omega' + n\omega_j)\,. \qquad (12)$$

If ϕ_i is independent of ϕ_j, and $f(\omega)$ is the distribution of ω_i (normalized to unity),

$$P_{av} = \int d\omega\, \frac{N}{n}\, f\left(\frac{\omega}{n}\right) \left(\frac{\omega_0}{2\pi}\right)^2 \qquad (13)$$

and

$$P(\omega) = \left(\frac{\omega_0}{2\pi}\right)^2 \frac{N}{n}\, f\left(\frac{\omega}{n}\right) \qquad (14)$$

near $\omega = n\omega_0$, ω_0 = average revolution frequency. We have that the power spectrum of the coasting beam Schottky signal mirrors the frequency distribution of the beam. But what if the revolution frequencies and phases are correlated (from, for example, feedback through a damping system or machine impedance)? Let the correlation be described by a distribution f_2,

$$f_2(\omega_1,\omega_2,\phi_1,\phi_2) = \sum_n f_n(\omega_1,\omega_2)\, e^{in(\phi_1-\phi_2)} \tag{15}$$

where the decomposition follows from rotational invariance of the coasting beam. It can be shown that the power spectrum (14) is modified by an additional term

$$\left(\frac{\omega_0}{2\pi}\right)^2 \frac{N^2}{n^2} \int d\omega'\, f_n\left(\frac{\omega}{n}, \frac{\omega'}{n}\right) \tag{16}$$

for modest frequency resolutions. Correlations have deformed the Schottky signals, and the simple interpretation in terms of the frequency distribution of the beam is lost. Thus, when the self-interaction of the beam is strong, either from feedback or through space charge or other machine impedances, the interpretation of Schottky scans becomes subtle. [Problem: Derive Eq. (16) and the equivalent expression when the beam is macroscopically bunched starting from Eq. (12).]

5 STOCHASTIC DIFFERENTIAL EQUATIONS AND SAMPLING

In studying stochastic cooling and radio-frequency noise phenomena, one often encounters differential equations of the form

$$\frac{dx}{dt} = s(t) \tag{17}$$

where $s(t)$ is a random process (i.e., noise) and x is some beam parameter (betatron amplitude or action, energy error). The basic approach to solution of Eq. (17) is to treat $s(t)$ as a simple function (after all, once the experiment is done we know what $s(t)$ is) throughout the calculation, and take expectations values at the last moment. The formal solution to Eq. (17) is simply

$$x(T) = \int_0^T dt\, s(t) \tag{18}$$

If we ask for the rms increase in x per unit time, we need to calculate

$$\frac{1}{T}\,|x(T)|^2 \tag{19}$$

for large T. Using the Fourier expansion for s, Eq. (6), we obtain

$$\frac{|X(T)|^2}{T} = \frac{1}{T}\int_0^T dt' \int_0^T dt'' \int_{-\infty}^{+\infty} d\omega' \int_{-\infty}^{+\infty} d\omega'' \left\langle a(\omega')\, a(\omega'')\right\rangle e^{i\omega' t'}\, e^{i\omega'' t''} . \tag{20}$$

The limit

$$\lim_{T\to\infty} \; T\,\frac{\sin^2 \omega T/2}{(\omega T/2)^2} = 2\pi\,\delta(\omega) \tag{21}$$

yields the final result

$$\lim_{T\to\infty} \frac{|X(T)|^2}{T} = 2\pi\, P(0) \tag{22}$$

that is, after long times, only the spectrum in the neighborhood of zero frequency matters.

Now consider a beam in a ring with a noise source localized at one particular azimuth (e.g., an radio-frequency cavity). The signal at the cavity is $s(t)$, but a given particle samples this signal only once per revolution. The signal "seen" by the particle is

$$g(t) = \sum_n \frac{2\pi}{\omega_0}\,\delta\left(t - \frac{2\pi n}{\omega}\right) s(t)$$

$$= \sum_n e^{in\omega t}\, s(t) . \tag{23}$$

It is left to the reader to show that Fourier transform g is given by

$$\tilde{g}(\omega) = \sum_n a(\omega - n\omega_0) . \tag{24}$$

Again, if we are interested in the rms behavior of x,

$$\frac{|x(T)|^2}{T} \to 2\pi \sum_n P(n\omega_0) . \tag{25}$$

Thus the particle's rms parameter is affected by noise power only at harmonics of its revolution frequency.

6 CUMULANT EXPANSION

In solving stochastic differential equations we are often in the position of taking expectation values of functions of a random variable. One particularly important function is the exponential and its expectation value

$$\left\langle e^{\int dt\, A(t)} \right\rangle \tag{26}$$

where $A(t)$ is some random process. Although there is a general relation (the cumulant expansion) for this expectation value, we will derive here an expression for a simple but extremely valuable special case. First, consider the correlation of four time signals of the noise source defined in Eq. (3). It can be shown [Problem: Show] that for N oscillators

$$\left\langle s(t_1)s(t_2)s(t_3)s(t_4) \right\rangle = \sum_{\text{perm}} R(t_i - t_j)R(t_k - t_\ell) + O\left(\frac{1}{N}\right). \tag{27}$$

In the limit of an infinite number of oscillators this yields the basic relation

$$\left\langle a(\omega_1)a(\omega_2)a(\omega_3)a(\omega_4) \right\rangle = \sum_{\text{perm}} \left\langle a(\omega_i)a(\omega_j) \right\rangle \left\langle a(\omega_k)a(\omega_\ell) \right\rangle \tag{28}$$

where $a(\omega)$ is as defined in Eq. (6). This property is typically true of electronic noise, with the generalization that the expectation of the product of $2M$ signals is the sum over all permutations of the two by two autocorrelations. Now consider the series expansion of (26). We have (assuming $\langle A \rangle = 0$)

$$\begin{aligned} \left\langle e^{\int dt\, A(t)} \right\rangle &= 1 + \frac{1}{2} \int dt_1 dt_2 \left\langle A(t_1)A(t_2) \right\rangle \\ &\quad + \frac{1}{4!} \int dt_1 dt_2 dt_3 dt_4 \sum_{\text{perm}} \left\langle A(t_i)A(t_j) \right\rangle \left\langle A(t_k)A(t_\ell) \right\rangle \\ &\quad + \ldots \quad . \end{aligned} \tag{29}$$

After some combinatoric gymnastics [which are left to the reader] we have

$$\left\langle e^{\int dt\, A(t)} \right\rangle = e^{\frac{1}{2} \int \langle A(t_1)A(t_2) \rangle dt_1 dt_2} \quad . \tag{30}$$

The fully general derivation of the cumulant expansion may be found in Van Kampen.[10]

7 LIOUVILLE'S EQUATION AND CONSEQUENCES

Consider a system of N particles with 6N canonical coordinates. We can represent the state of this system at some fixed time by a single vector in 6N-dimensional space. Points in this space represent possible states for these N particles. A distribution of such systems (representing our incomplete knowledge of all the particle coordinates) is described by a density function $D(q,p,t)$ where q and p are the 6N canonical variables and $D(q,p,t)dqdp$ gives the fraction of systems in the phase space volume $dqdp$. Let $u = (\dot{q},\dot{p})$. Then conservation of the number of systems is expressed by the continuity condition

$$\frac{\partial D}{\partial t} + \nabla \cdot (uD) = 0 \ . \tag{31}$$

For Hamiltonian systems

$$\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} = 0 \tag{32}$$

and (31) reduces to the condition of incompressible fluid flow

$$\frac{\partial D}{\partial t} + u \cdot \nabla D = 0 \tag{33}$$

which is Liouville's theorem.[7] A few comments are in order. First, Liouville's theorem is a statement on the 6N-dimensional ensemble space, not on the 6-dimensional phase space of the one-particle distribution of the beam. As we shall see, if interparticle correlations are negligible (the Vlasov equation regime), then the one-particle distribution also behaves as an incompressible fluid. But this situation is only approximate. Secondly, not all particle interactions in an accelerator are Hamiltonian (when attention is restricted to the beam). Wall resistance is one example of a dissipative force. Of more interest are feedback systems which if properly phased produce a dissipative, velocity-dependent self-interaction.

The next step in the analysis is to integrate Eq. (31) over 6N - 6 variables to derive an expression for the evolution of the one-particle distribution. Define the one- and two-particle distributions by

$$f_1(q_1,p_1,t) = \int dq_2 \ldots dp_N \, D(q,p,t) \ , \tag{34}$$

$$f_2(q_1,p_1,q_2,p_2,t) = \int dq_3 \ldots dp_N \, D(q,p,t) \, . \tag{35}$$

We have on integrating Eq. (31)

$$\frac{\partial f_1}{\partial t} = - \int dq_2 \ldots dp_N \left[\frac{\partial}{\partial q_1} (\dot{q}_1 D) + \frac{\partial}{\partial p_1} (\dot{p}_1 D) \right] . \tag{36}$$

We take a general interaction of the form

$$q_1 = X(1) + \sum_j F(i,j) \, , \tag{37}$$

$$p_1 = Y(1) + \sum_j G(i,j) \, , \tag{38}$$

where X,Y represent external forces and F,G interparticle forces with the auxiliary condition

$$\frac{\partial}{\partial q_i} \left(\dot{q}_i - F(ii) \right) = \frac{\partial}{\partial p_i} \left(\dot{p}_i - G(ii) \right) = 0 \tag{39}$$

which allows for possible self-interaction. Let

$$f_2(q_1,p_1,q_2,p_2,t) = f_1(q_1,p_1,t) \, f_1(q_2,p_2,t) + g(q_1,p_1,q_2,p_2,t) \, .$$

Then we finally (!) have

$$\frac{\partial f_1}{\partial t} = -\frac{\partial}{\partial q_1}\left(X(1)\ f_1\right) - \frac{\partial}{\partial p_1}\left(Y(1)\ f_1\right)$$

$$-\frac{\partial}{\partial q_1}\left(F(1,1)\ f_1\right) - \frac{\partial}{\partial p_1}\left(G(1,1)\ f_1\right)$$

$$-N\frac{\partial f_1}{\partial q_1}\int dq_2 dp_2\ F(1,2)\ f_1(q_2 p_2)$$

$$-N\frac{\partial f}{\partial p_1}\int dq_2 dp_2\ G(1,2)\ f_1(q_2 p_2)$$

$$-N\int dq_2 dp_2\ F(1,2)\ \frac{\partial}{\partial q_1}\ g(q_1,p_1,q_2,p_2)$$

$$-N\int dq_2 dp_2\ G(1,2)\ \frac{\partial}{\partial q_2}\ g(q_1,p_1,q_2,p_2)\ . \qquad (40)$$

An itemization of each term on the right side in this complicated equation is in order. The first two terms describe the action of an external force (which may be noise). The second two terms describe self-interaction (for stochastic cooling), which may be dissipative. The third two terms describes the average forces generated by the beam as if it were a continuous charge distribution. Integration is over the one-particle distribution, which gives the gross density of beam particles. These are the terms of the Vlasov equation used to derive instability thresholds. Finally, the last two terms describe micro-correlations of the beam particle's phase space coordinates. It is this set of terms which includes scattering and polarization effects. Note that without the self-interaction and correlation terms, the above equation is consistent with incompressible fluid flow. With the additional terms, it is only a continuity relation (the time derivative equal to a divergence of a current) which conserves unit normalization of particle number.

On integrating the basic Liouville relation (31) over 6N - 12, etc., variables equations for the two-particle correlation g and higher-order correlations can be derived. All such equations have the property of including the next higher order term, as the one-particle equation derived above required the two-particle correlation. The resulting nested set is the so-called BBGKY hierarchy. It is solved only approximately by truncation of the hierarchy at some level, with the introduction of some physical

reason why higher-order correlational effects are small. This will be discussed in greater detail in later sections.

We now have developed both noise and many-particle analytic tools. The next step is to combine them. The result will be our first encounter with the Fokker-Planck equation. Later on we will show that Eq. (40) with correlations can also be cast in the form of a Fokker-Planck with the addition of polarization or Debye screening effects. The final section of this paper will apply this analysis to the variety of phenomena discussed in the introduction.

So far we have looked at the action of noise on a single-particle coordinate. Of more interest to the accelerator scientist is the evolution of the beam distribution function which describes the phase space occupied by the entire beam. In the next sections, therefore, we seek equations for the time variation of the beam distribution functions in the presence of an external noise excitation. In addition, we will find a description of intrabeam effects such as scattering and polarization. In particular, the response of a beam to its own Schottky noise (scattering) will be remarkably similar to its response to external noise.

8 FOKKER-PLANCK EQUATION FROM NOISE CONSIDERATIONS

Diffusion due to noise (primarily from the radio-frequency system) limited the lifetime of stored beams at the SPS during early operation. Lifetimes were improved dramatically with the introduction of low noise oscillators and beam feedback, which provides screening of the noise. This phenomenon is best described by a Fokker-Planck equation,[4] and this section is devoted to a derivation of the same. We will follow the analysis of Van Kampen.[10] Alternative derivations from a "Markov process" point of view can be found almost everywhere.

Consider the longitudinal motion of beam particles undergoing (possibly nonlinear) synchrotron oscillations. The evolution of the one-particle distribution is most easily described in terms of action-angle variables, and it follows from Eq. (40) that for simple oscillatory motion the one-particle distribution evolves according to

$$\frac{\partial f}{\partial t} = - \nu(I) \frac{\partial f}{\partial \phi} \tag{41}$$

where I and ϕ are the action-angle variables and $\nu(I)$ is the oscillation frequency. Suppose a noise source $\xi(t)$ acts through the radio-frequency system to perturb the energy of a particle undergoing synchrotron oscillations. Then the action-angle equations become

$$\dot{I} = \sqrt{2I} \sin \phi \, \xi(t) \equiv a_I ,$$

$$\dot{\phi} = \nu + \frac{1}{\sqrt{2I}} \cos \phi \, \xi(t) \equiv a_\phi , \tag{42}$$

with the Hamiltonian condition

$$\frac{\partial \dot{I}}{\partial I} + \frac{\partial \dot{\phi}}{\partial \phi} = 0 \tag{43}$$

and Eq. (41) is modified to

$$\frac{\partial f}{\partial t} + \nu \frac{\partial f}{\partial \phi} = - \frac{\partial}{\partial \phi} (a_\phi f) - \frac{\partial}{\partial I} (a_I f) \ . \tag{44}$$

It is often easier to remove the zeroth-order oscillation by the method of characteristics. In action-angle variables this takes the form of the simple transformation

$$\phi = \theta + \nu t$$

with the transformed equations of motion

$$\dot{\theta} = \frac{1}{\sqrt{2J}} \cos(\theta + \nu t)\ \xi(t) + \nu' \sqrt{2I}\ \sin(\theta + \nu t)\ \xi(t) \equiv a_\theta \ ,$$

$$\frac{\partial f}{\partial t} = - \frac{\partial}{\partial \theta} (a_\theta f) - \frac{\partial}{\partial I} (a_I f) \ . \tag{45}$$

Equation (45) has the simple form of an operator differential equation

$$\frac{\partial f}{\partial t} = A(t)\ f \ . \tag{46}$$

which has the formal solution

$$f = e^{\int_0^t A(t)\ dt}\ f(0) \ . \tag{47}$$

Some care must be taken in defining the time ordering of terms in the series expansion of the exponential function. The reader is referred to Van Kampen[10] for a thorough discussion of this issue. The situation simplifies considerably if the correlation time of the noise source is short compared with the relaxation time of the system. That is, if the gross variation of f either due to oscillation frequency spread or due to the diffusion by the noise is slow relative to the correlation time of the noise source, subtleties of time ordering become unimportant. This is the case for radio-frequency noise diffusion, where the correlation time is determined by the bandwidth of the radio-frequency system (kilohertz to megahertz) with corresponding correlation times typically less than a millisecond, and diffusion times are minutes or hours.

Returning to Eq. (47), we need to evaluate the expectation value of the exponential operator. From the cumulant expansion (30) for Gaussian noise, we immediately have

$$f(t) = e^{\frac{1}{2}\int_0^t dt_1 \int_0^t dt_2 \langle A(t_1)A(t_2)\rangle} f(0)$$

$$= e^{\int_0^t dt_1 \int_0^{t_1} dt_2 \langle A(t_1)A(t_2)\rangle} f(0) \qquad (48)$$

or the equivalent integral equation, on differentiating (48),

$$\frac{\partial f}{\partial t} = \int_0^\infty d\tau \left\{ \frac{\partial}{\partial I} a_I(t)\, a_I(t+\tau) \frac{\partial f}{\partial I} + \frac{\partial}{\partial I} a_I(t)\, a_\theta(t+\tau) \frac{\partial f}{\partial \theta} \right.$$

$$\left. + \frac{\partial}{\partial \theta} a_\theta(t)\, a_I(t+\tau) + \frac{\partial}{\partial \theta} a_\theta(t)\, a_\theta(t+\tau) \frac{\partial f}{\partial \theta} \right\} \qquad (49)$$

where the limits of the integral have been extended to infinity by the assumption of short correlation time.

For the specific example defined by Eq. (42), Eq. (49) reduces to

$$\frac{\partial f}{\partial t} = \int_0^\infty \left\langle \frac{\partial}{\partial I} \left\{ \sqrt{2I} \sin\theta\, \xi(t) \sqrt{2I} \sin(\theta + \nu\tau)\, \xi(t+\tau) \right\} \right\rangle \frac{\partial f}{\partial I} \qquad (50)$$

for solutions with no θ dependence. If the noise source has a power spectrum $P(\omega)$, we have, on averaging over θ,

$$\frac{\partial f}{\partial t} = \int_0^\infty d\tau \frac{\partial}{\partial I} \left[I \cos \nu\tau\, \xi(t)\, \xi(t+\tau) \frac{\partial f}{\partial I} \right], \qquad (51)$$

$$\frac{\partial f}{\partial t} = \pi P(\nu) \frac{\partial}{\partial I} \left(I \frac{\partial f}{\partial I} \right), \qquad (52)$$

which is the promised Fokker-Planck equation. [Problem: Show that $(1/t)\exp(-I/Dt)$ is a solution of Eq. (52) which describes diffusion, with the rms spread in I increasing linearly with time.]

9 TWO-PARTICLE CORRELATION EQUATIONS

In the previous section we derived the basic Fokker-Planck equation for diffusion due to an external noise source. In the following we extended this analysis to include diffusion due to interparticle interaction, where the other beam particles act very much as an additional noise source. In addition, beam particles can polarize and shield a given particle both from the random fields generated by the beam and from external noise.

Integrating the Liouville relation (31) over 6N - 6 variables results in an equation for the two-particle correlation. Define

$$g(q_1,p_1,q_2p_2) = f_1(q_1p_1)\, f(q_2p_2) + f_2(q_1,p_1,q_2,p_2,t) ,$$

$$f \equiv f_1 . \tag{53}$$

Here we have subtracted from the two-particle distribution macroscopic effects deriving from possible lack of uniformity of the single-particle distribution.

For a coasting beam we have that $\dot{\theta} = \omega(p)$ is independent of θ. In the present notation this corresponds to $F(q,p) = F(p)$. With this assumption the correlation function g satisfies the equation

$$\frac{\partial g}{\partial t} + \dot{q}_1 \frac{\partial g}{\partial q_1} + \dot{q}_2 \frac{\partial g}{\partial q_2} + N \frac{\partial g}{\partial p_1} \int dq_3dp_3\, G(q_1,q_3,p_3)\, f_1(q_3,p_3)$$

$$+ N \frac{\partial g}{\partial p_2} \int dq_3dp_3\, G(q_2,q_3,p_3)\, f_1(q_3,p_3) =$$

$$- N \frac{\partial f_1}{\partial p_1} \int dq_3dp_3\, G(q_1,q_3,p_3)\, g(q_2,p_2,q_3,p_3)$$

$$- N \frac{\partial f_1}{\partial p_2} \int dq_3dp_3\, G(q_2,q_3,p_3)\, g(q_3,p_3,q_1,p_1)$$

$$- \frac{\partial}{\partial p_1} \left[G(q_1,q_2,p_2)\, f_1(q_1,p_1)\, f_1(q_2,p_2) \right]$$

$$- \frac{\partial}{\partial p_2} \left[G(q_2,q_1,p_1)\, f_1(q_1,p_1)\, f_1(q_2,p_2) \right]$$

$$- \left\{ \frac{\partial}{\partial p_1} G(q_1,q_2,p_3)g + \frac{\partial}{\partial p_2} G(q_2q_1q_3)g \right.$$

$$\left. + \frac{\partial}{\partial p_1} G(q_1q_1p_1)g + \frac{\partial}{\partial p_2} G(q_2q_2q_3)g \right\}$$

$$+ 0 \quad \text{(3-particle correlations)} . \tag{54}$$

For the most general equation with $F(q,p)$ see Chattopadhyay.[3] Again some description is necessary. First note that three-particle effects appear as the two-particle correlations appeared in the one-particle equation (40). This is a general phenomenon which continues in all orders of distribution equations, the so-called BBGKY hierarchy. This infinite set of equations must be truncated at some level to obtain closed-form solutions, and therefore some physical rationale must be introduced to justify such a step. In a plasma physics context this hierarchy is viewed as an expansion in a small parameter which is the ratio of the interaction energy to thermal energy. For feedback systems, this expansion parameter is more naturally expressed as a ratio of coherent damping or growth rate to various frequency spreads (revolution, synchrotron, betatron) of the beam. In any case it is generally true that this expansion parameter is less than unity. One primary exception is large-angle Coulomb scattering where the two-particle interaction can be quite large.

Now we will try to give some physical description of the various terms of Eq. (54). The second two terms on the right side describe the direct effects of beam particles perturbing each other; note that the single-particle distribution provides the weighting for interaction. These terms, therefore, describe scattering of beam particles, each of which is assumed to be distributed independently. Alternatively, this term pictures the beam particles as independent noise sources. The first two terms describe how existing correlations modify this picture (e.g. whether there is polarization which screens particles that are distant from each other). The velocity terms on left side produce enhancement of interaction of particles close in velocity (just as for external noise, where only the power spectrum of the noise in the vicinity of some natural frequency of an oscillator can cause long-term diffusion). The last two terms on the right describe coherent forces due to the gross beam distribution. The last four terms on the right are of order $1/N$ relative to the others terms and can be dropped along with the three-particle correlation effects.

Equation (54) can be solved under the assumption that the relaxation time of the correlation g is fast on the scale of variation of f_1. Again, we will find, as was the case for external noise, that this disparity of time scales allows for a Fokker-Planck approximation to the problem. For typical feedback systems, g describes the buildup of coherence; e.g., screening or instability growth which have rise times of a few millisecond, whereas cooling or diffusion occurs with time scales of several seconds.

For simplicity consider a longitudinal interaction only and take as our variables (approximately canonical) azimuthal angle θ and energy error $x = E - E_0$. Such an interaction models a coasting beam, longitudinal stochastic cooling feedback system.[1] We assume the interaction can be expanded in the form

$$G(\theta_i,\theta_j,x_j) = \sum_n G_n(x_j)\, e^{in(\theta_i-\theta_j)} . \tag{55}$$

For a uniform beam the correlation can be expanded as

$$g(\theta_1,\theta_2,x_1,x_1,t) = \sum_\ell g_\ell(x_1,x_2,t)\, e^{i\ell(\theta_1-\theta_2)} \tag{56}$$

and we define

$$H_\ell(x_1) = N \int dx_2\, G_{-\ell}(x_2)\, g_\ell(x_1,x_2) + f(x_1)\, G_{-\ell}(x_1) . \tag{57}$$

For a uniform beam the one-particle distribution equation may be written as

$$\frac{\partial f}{\partial t}(x,t) = -\frac{\partial}{\partial x}\left[\sum_\ell H(x)\right] . \tag{58}$$

After Laplace transforming with f considered constant, we have from Eqs. (54) and (57) the integral equation

$$H_{\pm|\ell|}(x_1) = G_{\mp|\ell|}(x_1)\, f(x_1)/\epsilon_{\pm|\ell|} - \frac{1}{\epsilon_{\pm|\ell|}} \frac{N}{|\ell|} \frac{\partial f_1}{\partial x_1} \int_{\eta>0_+} dx_2 \frac{H_{\mp\ell}(x_2)}{\eta \pm i(\omega_1 - \omega_2)}\, G_{\mp|\ell|}(x_2) \tag{59}$$

where

$$\epsilon_{\pm|\ell|}(x_1) = 1 + \frac{N}{|\ell|} \int_{\eta>0_+} dx_2 \frac{\frac{\partial f}{\partial x_2}\, G_{\mp|\ell|}(x_2)}{\eta \pm i(\omega_1 - \omega_2)} \tag{60}$$

The ϵ-function is referred to as the dielectric function in the plasma physics literature, and, as we shall see, describes the polarization of the beam in response to a source.

Except for the details of the gain G, this is the integral equation of Lenard-Balescu. The solution requires some complex plane gymnastics. However, an iterative solution, assuming the second term on the right is small, yields on insertion into Eq. (58) the Fokker-Planck equation

$$\frac{\partial f}{\partial t} = - \sum_{\ell} \left\{ \frac{\partial}{\partial x} \frac{G_{\ell}(x)\ f(x,t)}{\epsilon_{-\ell}(x)} - \frac{\partial}{\partial x} \left[\frac{N\pi}{|\ell|} \left| \frac{\partial x}{\partial \omega} \right| \left| \frac{G_{\ell}}{\epsilon_{-\ell}} \right|^2 \frac{\partial f}{\partial x} f \right] \right\}$$

$$+ \text{(principle value integrals)} . \qquad (61)$$

The exact solution yields the same result without the principle value integrals. The second-derivative term is very much like that encountered earlier for external noise and will produce diffusion. Note that the distribution f appears twice in this term -- once as the evolving distribution function $(\partial f/\partial x)$ and once as the source of the effective noise $(f(x))$. The ϵ-function describes shielding. Note that for large gain G, $|\epsilon_{\ell}| >> 1$, and the beam noise is effectively reduced. External noise will be similarly shielded.

The first term on the right is new and can introduce damping of phase space. For example, consider the simple partial differential equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x} (gxf) \qquad (62)$$

with the solution

$$f(x,t) = e^{gt} f_0(e^{gt}x) \qquad (63)$$

where f is an arbitrary initial distribution function. Note that the width of the distribution decreases exponentially while the peak value increases exponentially. Normalization is preserved. For stochastic cooling systems, the gain G is chosen to yield a damping self-interaction as in (62) and (63) with

$$G(\theta,\theta) = \sum_{\ell=-\infty}^{+\infty} G_{\ell} \neq 0 .$$

The dielectric function acts to reduce this self-interaction. Note that, from Eq. (60), whether $|\epsilon_{\ell}| >> 1$ and there is strong shielding or signal suppression depends on the product

$$\frac{N}{|\ell|} \frac{\partial f}{\partial x} G_{\mp|\ell|}(x) . \qquad (64)$$

Thus, the optimal value of G_{ℓ} scales as $1/N$, with more effective cooling for large harmonic number ℓ and small particle number N. Also note that as the density derivative $\partial f/\partial x$ increases, smaller gains are required for optimal cooling.

For multidimensions or multispecies the analysis proceeds analogously to the above, with the integrals becoming the product of multiple integrals and sums over species. For such multidimensional-species systems, the first-derivative term can be nonzero even if there is no self-interaction; in other words

$$\sum_{\ell} G_{\ell}/\epsilon_{\ell} \neq 0 \qquad \text{even if} \qquad \sum_{\ell} G_{\ell} = 0 \ .$$

(For one-dimensional systems with no self-interaction, various residues in the second derivative term cancel this effect of the dielectric function.) This so-called frictional term drives relaxation of temperatures between different dimensions or species.

Finally, we recall that diffusive effects of beam particles appear as a noise source with power spectrum proportional to $f(x)$, just as in our earlier derivation of the Schottky noise spectrum. In the Fokker-Planck equation, however, we find these effects modified by beam shielding through the ϵ-function. It is left to the reader to show from the expression (16) and Eq. (59) that the Schottky noise seen through a pickup is also deformed by a factor of $|\epsilon|^{-2}$.

10 RELAXATION PROCESSES WITH THE COULOMB INTERACTION

In the previous section we have seen that correlational effects lead both to diffusion (through fluctuating fields) and polarization (signal suppression) phenomena. In stochastic cooling systems, because of the single-particle dissipative interaction, there is a net frictional term which causes damping of phase space and an increase in the single-particle density. Schottky noise provides a diffusive mechanism, and polarization acts to shield particles, diminishing both damping introduced by the self-interaction and the diffusion introduced by Schottky noise. However, when multidimensional and multispecies effects are included, the induced polarization can also lead to frictional phenomena even if there is no dissipative self-interaction. For example, for the conservative Coulomb interaction these polarization induced frictional force can drive temperature relaxation between two species at differing temperatures (electron cooling). Also, both frictional effects and diffusive effects can lead to relaxation of phase space density between transverse and longitudinal dimensions of a particle beam (intrabeam scattering). Azimuthal variations in lattice functions of a strong focusing accelerator can also allow coupling to the gross beam kinetic energy, which can then be thermalized by this scattering mechanism.

The equation for a multicomponent plasma interacting through the Coulomb force follows from arguments analogous to those used in deriving Eq. (61) for feedback systems with the substitution of the simple Fourier kernel

$$\frac{4\pi}{k^2}\,\omega_p^{\,2} \tag{65}$$

of the Laplace equation, where k is the Fourier conjugate variable (2π/wavelength) to the spatial coordinate and ω_p is the plasma frequency

$$\omega_p^2 = 4\pi n q^2/m \tag{66}$$

where n = particle density, q = charge, and m = mass. For example, the evolution for the single-particle density for an unconfined plasma of infinite extent is[6]

$$\frac{\partial f(v_1)}{\partial t} = \frac{\omega_p^2}{n} \sum_k \frac{k}{k^2} \cdot \frac{\partial}{\partial v_1} f(v_1) \frac{\text{Im } \varepsilon(k, k\cdot v_1)}{|\varepsilon(k, k\cdot v_1)|^2}$$

$$+ \frac{\pi \omega_p^4}{n} \sum_k \frac{k}{k^2} \cdot \frac{\partial}{\partial v_1} \int dv_2 \left[\frac{k}{k^2} \cdot \frac{\partial}{\partial v_1} f(v_1)\, f(v_2) \right] \frac{\delta(k\cdot v_1 - k\cdot v_2)}{|\varepsilon(k, k\cdot v_1)|^2} \tag{67}$$

where v is the particle velocity and

$$\varepsilon(k,\omega) = 1 + \frac{\omega_p^2}{k^2} \int dv \frac{1}{\omega - k\cdot v} k \cdot \frac{\partial}{\partial v} f(v) \ . \tag{68}$$

Note the similarities to Eqs. (60) and (67). If there are more than one species, multidimensional integrals include summations over particle species as another (integer) coordinate.

Equation (67) can be rewritten (with the help of some Cauchy integral identities) in an alternative form which emphasizes two-particle scattering effects:

$$\frac{\partial f(v)}{\partial t} =$$

$$\frac{\pi \omega_p^2}{n} \sum_k \omega_p^2 \frac{k}{k^2} \cdot \frac{\partial}{\partial v} \int dv' \frac{k}{k^2} \left[\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial v'} \right) f(v)\, f(v) \right] \frac{\delta(k\cdot v - k\cdot v')}{|\varepsilon(k, k\cdot v)|^2} \tag{69}$$

The resulting equation is the small-angle approximation to the Boltzmann equation[11] with the addition of screening provided by the ε function. Although Eq. (69) follows from the basic phase space continuity condition (31) and the notions of polarization and fluctuation, the Boltzmann perspective is that of two-particle scattering, and Eq. (69) (without ε) is the small momentum transfer per collision limit (Taylor's expansion) of the fundamental Boltzmann equation

$$\frac{\partial f(v_1)}{\partial t} = \int dv_2 \; f(v_1') \; f(v_2') - f(v_1) \; f(v_2) \; |v_1 - v_2| d\sigma\left(v_1, v_2 \rightarrow v_1' v_2'\right) . \tag{70}$$

With this viewpoint the relaxation of the particle distribution is driven by two-particle scattering through the differential cross section $d\sigma$. Note that Eq. (67) with ε equal to unity contains both frictional (first-derivative) and diffusion (second-derivative) terms, and therefore, by equivalence to Eq. (69) implies that two-particle scattering can produce both frictional forces and diffusion. Lewis[11] details this decomposition. From Eq. (68) we have that $\varepsilon(k)$ can be large for small k (or long wavelength). Thus, long-range fields can be screened (Debye shielding), whereas short-range fields are not screened. Since Eq. (70) does not include screening, it must be inserted by hand in cutting off the infinite Coulomb cross section. On the other hand, large-angle scattering is not included in Eqs. (67) and (69).

Both intrabeam scattering and electron cooling of heavy particle beams derive from Coulomb scattering, which causes coupling between degrees of freedom at different temperatures. For a typical storage ring beam the transverse temperature exceeds the longitudinal temperature in the beam rest frame, and scattering can lead to longitudinal emittance growth and associated damping of transverse emittance. This simplest intrabeam scattering mechanism can be modified by machine dispersion, with both longitudinal and transverse emittance growth. In electron cooling a cold electron beam interacts with a hotter heavy particle beam, with Coulomb scattering inducing temperature relaxation and an increased phase space density of the heavy particle beam.

The theory of intrabeam scattering as it exists today does not make full use of the correlational formalism that has been presented in this paper. Rather, moment equations are derived by averaging multiple Coulomb scattering over beam parameters. The approximations involved are very much in the spirit of the Boltzmann equation as discussed above. Single, large-angle scattering events which can lead to particle loss are not included in the analysis. (This loss mechanism, primarily important in low energy electron rings, is known as the Touschek effect.) Multiple intrabeam scattering has been successfully treated by Piwinski,[12] and by Bjorken and Mtingwa, who present a thorough exposition. Instead of duplicating their mathematical analysis, attention, here, will focus on the basic physical principles involved in multiple intrabeam scattering.

Consider a stored beam with nominal energy and momentum E and p, respectively. A particle with momentum $p + \Delta p$ in the laboratory frame will have a Lorentz transformed momentum in the beam frame given by

$$\Delta p' = \frac{E}{m} p + \frac{E}{m} \Delta p - \frac{p}{m} \sqrt{(p+\Delta p)^2 + m^2} . \tag{71}$$

For small Δp, the roots may be expanded to yield

$$\Delta p' = \Delta p/\gamma \ . \tag{72}$$

[Problem: Compare the longitudinal and transverse momenta in the beam frame of your favorite storage ring.]

To get a simple picture of the effect of scattering on beam emittance, consider the rather special case of elastic scattering of two particles with equal longitudinal momentum, $p_1^{\ell} = p_2^{\ell} = p$, opposite horizontal momentum $p_1^t = -p_2^t = x'p$, and horizontal position $x_1 = x_2 = 0$. Let the scattering event transfer the entire horizontal momentum into longitudinal momentum. From Eq. (72) we have in the laboratory frame that the longitudinal momentum of each particle has changed in absolute value by

$$|x'p\gamma| \ . \tag{73}$$

The scattering event has acted to increase the longitudinal momentum spread of the beam at the expense of transverse emittance. However, dispersion provides a mechanism which can simultaneously excite transverse emittance growth. The horizontal emittance w of an individual particle is defined by

$$\beta w = x_\beta^2 + (\alpha x_\beta + \beta x_\beta')^2 \tag{74}$$

where β and α are the usual lattice functions, and x_β and x_β are the betatron phase coordinates relative to the closed-orbit. Because each particle initially has zero momentum error, the horizontal emittance before collision is

$$\beta w = \beta^2 x'^2 \ . \tag{75}$$

Momentum error develops after the collision which moves the closed-orbit by $D\,\Delta p/p$, where D is the dispersion. Although the particles have not changed their horizontal position during the brief collision, relative to the closed-orbit we have

$$x_\beta \gtrless - D\frac{\Delta p}{} = \mp D\gamma x' \ ,$$

$$x_\beta' \gtrless - D'\frac{\Delta p}{} = \mp D'\gamma x' \ . \tag{76}$$

The emittance after collision is, from Eq. (74), equal to

$$\beta w = D^2\gamma^2 x'^2 + (\alpha D\gamma + \beta D'\gamma)^2 x'^2 \ . \tag{77}$$

The net change in emittance is

$$\Delta(\beta w) = -(\beta^2 - \tilde{D}^2\gamma^2)\, x'^2 \tag{78}$$

where
$$\tilde{D}^2 = D^2 + (\alpha D + \beta D')^2 \,. \tag{79}$$

If
$$\tilde{D}^2 > \beta^2 \tag{80}$$

both the horizontal and longitudinal emittances are increased by intrabeam scattering. The exact calculation of growth rates[2] requires averaging over the Coulomb cross section weighted by the beam phase space distribution; however, the basic requirement for damping or growth remains approximately correct. In the smooth lattice approximation for a ring of radius R, tune ν, and transition $\gamma_t = 1/\nu^2$, condition (80) reduces to

$$\gamma > \gamma_t \tag{81}$$

or that both transverse and longitudinal coordinates will grow because of intrabeam scattering when the beam energy is above the transition energy. For an actual alternating gradient machine there is no simple relationship between local dispersion as averaged by the intrabeam scattering and the transition energy, and this simple condition for transverse growth may be violated. [Problem: Show that a change in the transverse momentum by Δp_t induces a longitudinal momentum change of the order of only $\Delta p_t^2/2p$ in the laboratory frame. Also, show in the laboratory frame that the energy exchanged between the colliding particles discussed above comes primarily from longitudinal momentum transfer.] At first glance it may appear that growth of both transverse and longitudinal emittance may violate energy conservation. A betatron oscillation, however, is a continual exchange of energy between transverse and longitudinal coordinates in the machine frame (since with a pure magnetic lattice there is no potential energy). The change in longitudinal momentum as a betatron oscillation transfers transverse momentum (as calculated in the above problem) is second order in the transverse momentum, and is therefore negligible. The dispersion-induced betatron oscillation will also derive energy from the longitudinal kinetic energy of the beam. In the beam frame there is both an electric field (the Lorentz transformed magnetic field) and the angular momentum effective potential to ensure energy conservation.

For electron cooling of heavy particles, an electron beam moving parallel to a heavy particle beam at the same velocity increases the phase space density of the heavy particle beam through Coulomb scattering. The relaxation process is a competition between polarization induced friction and diffusion from field fluctuations, with equilibrium reached when these two effects cancel. Since for electron cooling it is usually the case that the temperature of the electron and heavy particle beams are disparate, the total system is far from equilibrium and the frictional term dominates.

The earlier work on electron cooling calculated the drag force from a simple binary collision model similar to that discussed for intrabeam scattering, with the effects of Debye screening crudely introduced as a long range cutoff of the Coulomb cross section. In recent efforts, Sorensen and Bonderup[9] have made use of the dielectric description of polarization as has been highlighted in this paper. This approach has clear advantages when densities are high enough and temperatures low enough that collective phenomena cannot be neglected. In such cases the exact nature of interparticle screening, as best described by the dielectric function ε, can no longer be ignored.

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