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## Author

Hiskes, John.

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# UCRL LECTURES ON NUMERICAL ANALYSIS AND APPLIED MATHEMATICS

Lecture XI

John Hiskes

December 2, 1952

Berkeley, California

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University of California
Radiation Laboratory
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# PARTIAL DIFFERENTIAL EQUATIONS

## 1. Introduction

The following material is devoted to the solution of partial differential equations by the iteration and relaxation methods. Specifically, the partial differential equations to be considered are

(a) Laplace's equation in two dimensions,

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \quad ;$$

(b) Laplace's equation in three dimensions,

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0 \quad ;$$

(c) Poisson's equation,

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \rho(x, y) \quad ;$$

(d) Laplace's equation in cylindrical coordinates,

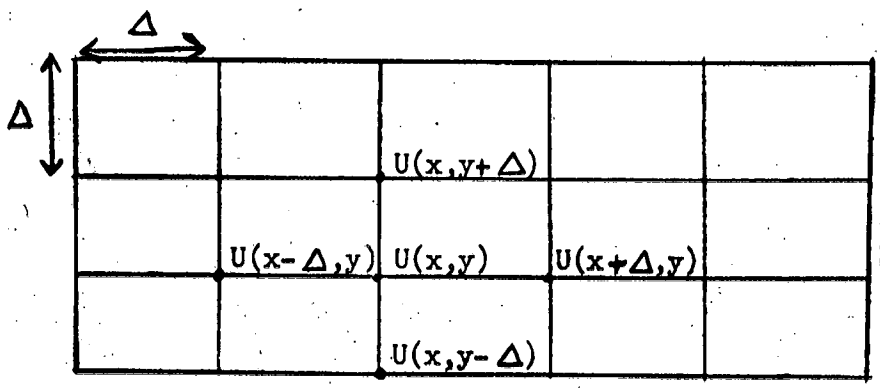
$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2} = 0$$

Other equations can be solved by these methods if it is possible to replace the derivatives in the equation by finite differences. The solution then will be immediately apparent from the special cases that are considered in this paper.

## 2. The Iteration Method

The iteration method will be explained by solving Laplace's equation in two dimensions, $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$.

First, a square mesh of increment size $\Delta$ is constructed.




Then the following differences are computed:

First forward difference: $U_x = \frac{U(x+\Delta,y) - U(x,y)}{\Delta}$

First backward difference: $U_{\bar{x}} = \frac{U(x,y) - U(x-\Delta,y)}{\Delta}$

Second difference: $U_{x\bar{x}} = \frac{U_x - U_{\bar{x}}}{\Delta}$

and similarly: $U_{y\bar{y}} = \frac{U_y - U_{\bar{y}}}{\Delta}$

Now the differential equation is replaced by the difference equation

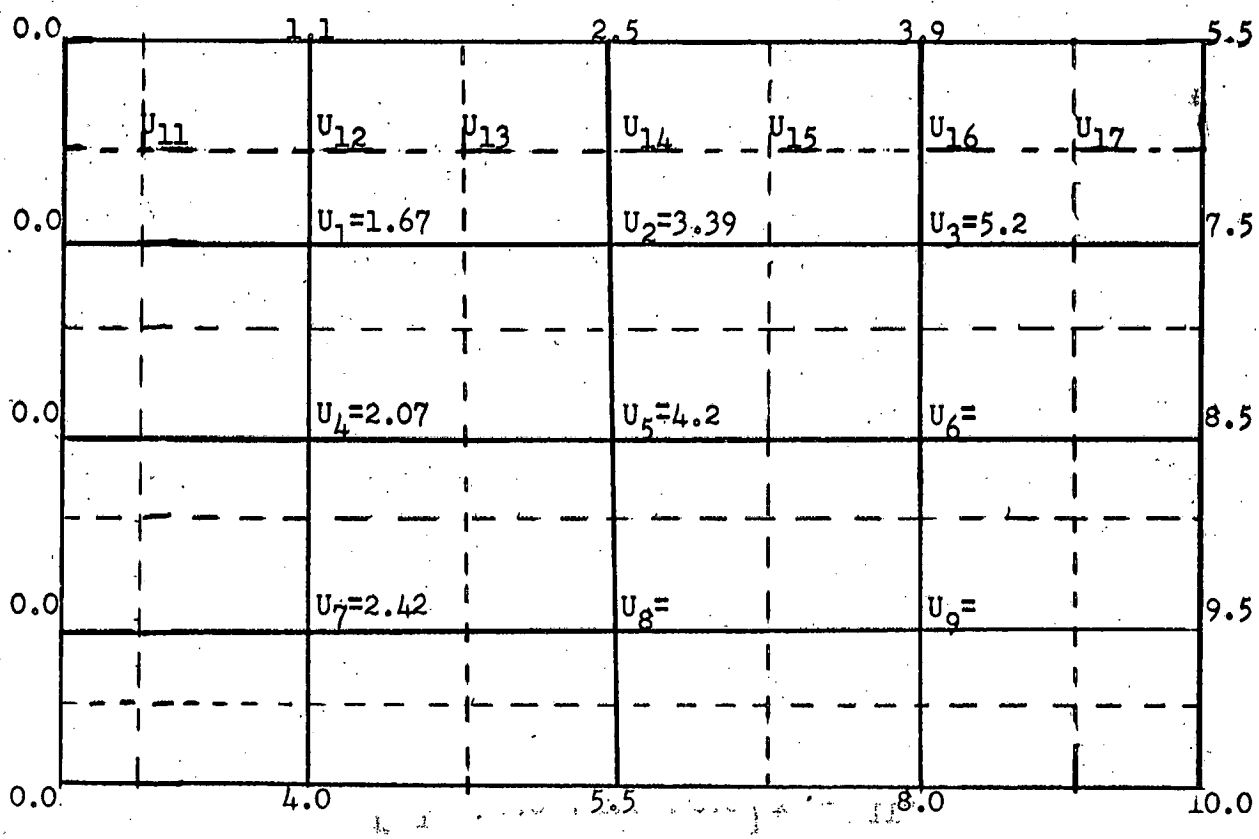
$$U_{x\bar{x}} + U_{y\bar{y}} = 0$$

Solving for $U(x, y)$,

$$U(x,y) = \tfrac{1}{4}\left[U(x+\Delta,y) + U(x,y+\Delta) + U(x-\Delta,y) + U(x,y-\Delta)\right] .$$

This equation, together with the specified boundary values, is used for determing $U(x, y)$ at interior points.

For example, consider a simple boundary with the following specified boundary values:



The center point is first determined by averaging the four boundary values on the corners of the square.

$$U_5 = \tfrac{1}{4}\left[0.0 + 8.5 + 2.5 + 5.5\right] = 4.2$$

The remaining first approximations are similarly determined.

$$U_1 = \tfrac{1}{4}\left[0.0 + 0.0 + 2.5 + 4.2\right] = 1.67$$

$$U_3 = \tfrac{1}{4}\left[4.2 + 8.5 + 5.5 + 7.5\right] = 3.39 \quad , \text{ etc.}$$

These values are then corrected by using the equation for $U(x, y)$ and the functional values most recently computed. This computation is called the <u>first iteration cycle</u>.

The First Iteration cycle:

$$U_1' = \tfrac{1}{4}(1.1 + 0.0 + U_4 + U_2)$$

$$U_2' = \tfrac{1}{4}(2.5 + U_1' + U_5 + U_3) \quad , \text{etc.}$$

It is to be noted that in case of error, the method is self-correcting.

As soon as the first iteration cycles produce no change in the functional values, the mesh size is refined, as is indicated in the diagram by the dashed lines.

Now the first approximations for the new values of the finer mesh are computed.

$$U_{11} = \tfrac{1}{4}\left[0.0 + 1.1 + 0.0 + U_1^f\right]$$

$$U_{13} = \tfrac{1}{4}\left[1.1 + U_1^f + 2.5 + U_2^f\right]$$

$$U_{12} = \tfrac{1}{4}\left[U_{11} + U_1^f + U_{13} + 1.1\right] \quad , \text{etc.,}$$

where $U_i^f$ indicates the final functional value obtained by the first iteration cycle.

This iteration technique is continued until a specified degree of accuracy is obtained.

## 3. The Relaxation Method

A somewhat faster method of solving this type of problem is provided by the relaxation method. This method involves calculations with residuals in the mesh rather than with the functional values.

The residual at a mesh point $U_o$ is defined as follows:

$$R_o = U_1 + U_2 + U_3 + U_4 - 4 U_o \quad .$$

The procedure is outlined as follows:

3.1 Construct a mesh with specified boundary values and initial interior functional values as before.

3.2 Compute residuals at every point in the mesh according to the above formula.

3.3 Starting at the largest residual, change the functional value by $h$, add $h$ to all four neighboring residuals, add $-4h$ to the residual in question.

This last calculation may be verified by considering a mesh with specified functional values.

| | | | | | |
|---|---|---|---|---|---|
| | $U_1$ | $U_2$ | $U_3$ | $U_4$ | $U_5$ |
| | $U_6$ | $U_7$ | $U_8$ | $U_9$ | $U_{10}$ |
| | $U_{11}$ | $U_{12}$ | $U_{13}$ | $U_{14}$ | $U_{15}$ |
| | $U_{16}$ | $U_{17}$ | $U_{18}$ | $U_{19}$ | $U_{20}$ |
| | | | | | |

$$R_8 = U_3 + U_7 + U_{13} + U_9 - 4\,U_8 .$$

Increasing $U_8$ by $\Delta U_8$ increases the residual by $\Delta R$.

$$R_8 + \Delta R_8 = U_3 + U_7 + U_{13} + U_9 - 4(U_8 + \Delta U_8)$$

$$\therefore \quad \Delta R_8 = -4\,\Delta U_8$$

$$R_7 = U_6 + U_{12} + U_8 + U_2 - 4\,U_7$$

$$R_7 + \Delta R_7 = U_6 + U_{12} + (U_8 + \Delta U_8) + U_2 - 4\,U_7$$

$$\therefore \quad \Delta R_7 = \Delta U_8 .$$

This process of modifying the functional values in order to make the residuals as small as possible is called relaxation.

The relaxation is continued until there ceases to be a considerable change in the residuals. Then the mesh is refined and the relaxation continued.

When the relaxation procedure is finished, all mesh points must be

checked by the finite difference equation approximating the partial differential equation since the method is not self-correcting.

## 4. Block Relaxation

If in a certain region of the lattice all the residuals are of the same sign, then the residuals can be made to converge much more rapidly by resorting to block relaxation. This technique may be outlined as follows:

4.1 All functional values within the block are changed by an amount $h$.

4.2 Residuals within the block are unchanged.

4.3 Residuals at border points are changed by $-n\,h$, where $n$ is the number of adjacent points outside the block.

4.4 Residuals at points just outside the block are changed by an amount $h$.

4.5 A good value of $h$ may be obtained from the formula,

$$h = \frac{\sum R}{n} = \frac{\text{Sum of block residuals}}{\text{number of adjacent points}}$$

As an example, consider the following where the block consists of points on the dashed line.

| 0.0 | | | 4.00 | | 8.00 | | 12.00 | | 16.00 | | 20.00 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | +2.00 | | +2.00 | | +2.00 | | +2.00 | | | |
| | | R ↓ | | U ↓ | | | | | | | |
| 0.0 | | +2.00 | 2.00 | +8.00 | 4.00 | +6.00 | 8.00 | +4.00 | 14.00 | | 20.00 |
| | +2.00 | -4.00 | 4.00 | +4.00 | 6.00 | +2.00 | 10.00 | -2.00 | 16.00 | +2.00 | |
| | | (a) | | (b) | | (c) | | (d) | | | |
| 0.0 | | +4.00 | 2.00 | +1.00 | 6.00 | +1.00 | 10.00 | +4.00 | 14.00 | | 20.00 |
| | | +6.00 | | +3.00 | | +3.00 | | +6.00 | | +2.00 | |

In this case,

$$h = \frac{\sum R}{n} = \frac{+20}{10} = +2.00$$

Points (a) and (d) each have three adjacent outside points. Their residuals are changed by -6.00. Points (b) and (c) each have two adjacent outside points. Their residuals are changed by -4.00. Here we have no points within the block, i.e., every point of this block has adjacent points outside the block.

## 5. Special Equations.

### 5.1 Poisson's Equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \rho(x, y) .$$

Replacing this by a difference form and solving for $U(x, y)$,

$$U(x, y) = \tfrac{1}{4} \left[ U(x + \Delta, y) + U(x - \Delta, y) + U(x, y + \Delta) + U(x, y - \Delta) - \Delta^2 \rho(x, y) \right] .$$

If the value of $\rho(x, y)$ is known at each point, then this problem can be solved by the previous methods.

It is to be noted that the value of $U(x, y)$ is not independent of the mesh parameter $\Delta$ .

### 5.2 Laplace's Equation in three dimensions

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0 .$$

In difference form this becomes

$$U(x, y, z) = \frac{1}{6}\Big[U(x+\Delta, y, z) + U(x, y+\Delta, z) + U(x, y, z+\Delta) + U(x-\Delta, y, z) + U(x, y-\Delta, z) + U(x, y, z-\Delta)\Big]$$

This case requires a three-dimensional mesh; or it can be handled by starting at an initial value of z and working across the corresponding x - y plane, then taking the next value of z and working across the next x - y plane, etc., until the entire three-dimensional mesh has been covered. This technique is repeated until the functional values converge as before

5.3 Laplace's Equation in cylindrical coordinates

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \phi^2} = 0 \quad .$$

This form can sometimes be treated in two dimensions. An interesting application is the determination of the electric potential distribution in the region between two coaxial cylinders. In this case there is no $\phi$ variation and the equation becomes

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2} = 0$$

In difference form,

$$\frac{1}{\Delta^2}\Big[U(r+\Delta, z) + U(r-\Delta, z) - 2U(r, z)\Big] + \frac{1}{2r\Delta}\Big[U(r+\Delta, z) - U(r-\Delta, z)\Big] + \frac{1}{\Delta^2}\Big[U(r, z+\Delta) + U(r, z-\Delta) - 2U(r, z)\Big] = 0 \quad .$$

Solving for $U(r, z)$,

$$U(r, z) = \frac{1}{4}\left\{\left[1 + \frac{\Delta}{2r}\right] U(r + \Delta, z) + \left[1 - \frac{\Delta}{2r}\right] U(r - \Delta, z) + U(r, z + \Delta) + U(r, z - \Delta)\right\}$$

This expression is similar to the two-dimensional Laplacian except for the coefficients $1 \pm \frac{\Delta}{2r}$ appearing on the right. Thus, this cylindrical form of the Laplacian can be treated as a two-dimensional problem provided the coefficients can be evaluated at the particular value of $r$ at which the iteration or relaxation is being performed. Moreover, the residuals obtained when relaxing a mesh are also functions of the coefficients $1 \pm \frac{\Delta}{2r}$.

## 6. Modifications for Special Boundaries.

If the boundary is not rectangular or if the various sides of the boundary are not commensurable, then the functional values near the boundary can not be obtained by the previous procedures. To obviate this difficulty, fictitious points are introduced outside the boundary in such a manner that the distance from a fictitious point to the nearest point of the mesh is equal to the mesh parameter $\Delta$. The functional value at a fictitious point is computed by linear interpolation with respect to the distance from the nearby boundary value and the nearest point of the mesh. Iteration or relaxation can then be continued in the same manner as before.

## 7. Errors.

No systematic method has been developed for estimating the errors which arise by approximating the partial differential equation by the finite difference equation. However, the computed functional values come closer and closer to the true values as the mesh parameter $\Delta$ is reduced in size. Thus, refinement of the mesh is continued until no change occurs in the functional values within the degree of accuracy required.

## 8. Bibliography.

8.1 Scarborough, J. B., Numerical Mathematical Analysis, 2nd edition.

8.2 Southwell, R. V., Relaxation Methods.