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Some results on the topology of quasitoric manifolds and their equivariant mapping spaces

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Some Results On The Topology Of Quasitoric Manifolds And Their Equivariant Mapping Spaces

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Michael Gurvich

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2008

The dissertation of Michael Gurvich is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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Chair

University of California, San Diego

2008

DEDICATION

To my parents.

EPIGRAPH

Там, за нигде, за его пределом
— черным, бесцветным, возможно, белым —
есть какая-то вещь, предмет.

—И. Бродский

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ABSTRACT OF THE DISSERTATION

Some Results On The Topology Of Quasitoric Manifolds And Their Equivariant Mapping Spaces

by

Michael Gurvich
Doctor of Philosophy in Mathematics

University of California San Diego, 2008

Professor Nitya Kitchloo, Chair

We apply methods of homotopy theory to the study of of quasitoric manifolds. More specifically, we determine a simple criterion for rational ellipticity of a quasitoric manifold based on the combinatorics of the orbit polytope. We also study the topology of some equivariant mapping spaces of the quasitoric manifolds and their associated moment angle complexes. In case the image polytope is a product of simplices we completely determine the homotopy type of the mapping spaces in question. We also suggest a way to study the topology of the equivariant mapping spaces for a general simple polytope using the Bousfield-Kan spectral sequence. As an application we derive some connectivity results for equivariant mapping spaces of manifolds over 2-dimensional polytopes.

Chapter 1

Introduction

Quasitoric manifolds are an algebraic topologist's version of an algebraic geometer's toric variety, or a symplectic geometer's toric manifold. Essentially, they are $2n$-dimension manifolds with a "nice" action of an n-dimensional torus, whose orbit spaces are n-dimensional simple polytopes. These objects first appeared as "toric manifolds" in the paper of Davis and Januszkiewicz [3] in 1991. In more recent times they have become known as "quasitoric manifolds", so as to avoid confusion with their symplectic cousins.

The definition of a quasitoric manifold lends itsef very well to the study by an algebraic topologist. In fact, the study of quasitoric manifolds provides a furtile playground where topology, combinatorics, homological algebra and homotopy theory are all active participants. The most basic example illustrating this point is the equivariant cohomology of a quasitoric manifold which is simply the Stanley-Reisner ring of the underlying polytope. The topological information (cohomology) is encoded in a combinatorial object (polytope) and is given by a structure well-established in the realm of homological algebra (Stanley-Reisner ring).

Another interesting feature illustrating the interplay between the manifold and its image polytope lies in the fact that the quasitoric manifold may be viewed as a homotopy colimit of a certain diagram of tori, the shape of which is the orbit polytope (See [8], [9], or [11] for the case of toric varieties). The "homotopy colimit" is a kind of a gluing construction akin to a usual colimit, but best suited for use in homotopy theory.

Let us briefly discuss the main results of this paper. The goal was two-fold: on the one hand, we wanted to keep with the theme of toric-topology, which seems to be "combinatorics of the polytope determines the topology of the manifold", and investigate what can be said about the homotopy of the manifold based on the image polytope. Of course, computing homotopy groups of even the simplest of quasitoric manifolds (which happens to be a 2-sphere), turns out to be a formidable task, so we focused on rational homotopy groups instead. Then, one of the natural questions becomes to determine whether a quasitoric manifold is rationally elliptic or rationally hyperbolic. In keeping with the above stated theme, it seemed desirable to devise a criterion based solely on the image polytope. We do develop such a criterion and compute the rational homotopy groups in the rationally elliptic case. It turns out that a quasitoric manifold is rationally elliptic if and only if the image polytope is a product of simplicies.

Theorem A. *A quasitoric manifold M over a simple n-dimensional polytope P with m facets is rationally elliptic if and only if P is a product of simplices. Moreover, the rational homotopy of M is given by*

$$\pi_j(M) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q}^{\mu_j} & \text{if } j > 2 \\ \mathbb{Q}^{m-n} & \text{if } j = 2 \end{cases}$$

where μ_j represents the number of generating monomials of the Stanley-Reisner ideal of P of length $(j+1)/2$. Equivalently, every i-simplex factor of P, contributes a generator to $\pi_{2i+1}(M) \otimes \mathbb{Q}$.

It should be pointed out that we do not claim originality for this result. It is implicit in the work of Notbohm and Ray [7] and in the literature on rational homotopy theory. What we have done is combined various results and stated a simple criterion, which does not seem to be explicit in the literature.

The second goal of this paper, which appears to be somewhat new, is to study the equivariant mapping spaces of quasitoric manifolds. They are the spaces of self-maps of a quasitoric manifolds which commute with the acting torus. The nature of these spaces appears to be quite interesting, and deserving of some investigation. For example, if we look at the subspace consisting of all equivariant self maps not permuting the fixed points, then it can be shown that every such map is a homotopy equivalence, so equivariance seems to impose a kind of rigidity on the maps in question.

The idea behind studying the topology of the mapping spaces is to use the homotopy colimit decomposition of the underlying manifold. This allows us to view the mapping space as a homotopy inverse limit over a P-shaped diagram. For a special case when P is a product of simplices we use this homotopy limit decomposition of the mapping spaces to determine their homotopy type which turns out to be a product of iterrated loop spaces on odd-dimensional spheres.

Theorem B. *Let* $P = \Delta^n, M = \mathbb{C}P^n, \mathcal{Z}_P = S^{2n+1}$. *Then*

(a) $\mathrm{map}_{T^{n+1}}(\mathcal{Z}_P, \mathcal{Z}_P) \simeq \prod_{i=0}^{n} \prod_{\sigma_k^i} \Omega^i S^{2i+1}$

(b) $\mathrm{map}^1_{T^n}(M, M) \simeq T^n \times \prod_{i=1}^{n} \prod_{\sigma_k^i} \Omega^i S^{2i+1}$

Above, the second product is taken over all k *such that* σ_k^i *is an* i*-dimensional face of* P

Note that the '1' which appears in the superscript of the mapping space above, indicates that we are only looking at the equivariant maps that do not move the fixed points.

Another benefit of using homotopy limit decomposition of the mapping spaces is that the homotopy groups may be computed by means of a certain kind of a spectral sequence, called the Bousfield-Kan spectral sequence. This involves computing higher derived functors of lim over P-shaped diagrams. We illustrate this approach in its entirety for the case when P is a product of simplices.

As an application of this machinery we show that in the case that P is 2-dimensional, the space of equivariant maps of M that do not move the fixed points is connected. That is, any equivariant map of a 4-dimensional quasitoric manifold not moving the fixed points of the action is equivariantly homotopic to the identity map.

Theorem C. *If* M *is a quasitoric manifold over a 2-dimensional polytope* P*, then the space* $\mathrm{map}^1_{T^n}(M, M)$ *is connected*

In general, there is no reason to believe that the spaces $\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ are connected (though we do not know of a counterexample). We do believe, however, that the set of components, $\pi_0(\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P))$, forms a group. This is because the diagonal of the Bousfield-Kan spectral sequence, which gives information about $\pi_0(\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P))$, consists of groups except in $E_2^{0,0}$ which happens to be trivial. It should then be possible to show that the group structure in $\pi_0(\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P))$ is the same as the one given by the spectral sequence, though we do not pursue this question here.

Let us briefly comment on the contents of the chapters:

In Chapter 2 we give the definition of a quasitoric manifold and show how they can be reconstructed from their image polytope and a $\mathbb{Z}^n$-valued function of the facets, called the characteristic function. This establishes a one-to-one correspondence between quasitoric manifolds and a set of characteristic pairs that consist of a simple polytope and a characteristic function of the facets.

In Chapter 3 we look at how the cohomology of quasitoric manifolds can be read-off from the polytope and the characteristic function. It turns out that the betti numbers and equivariant cohomology depend only on the polytope, while the ordinary cohomology also depends on the characteristic function. We also introduce the notion of a moment angle complex associated to a polytope. It is a "universal" toric space that sits over a given polytope. Universality is meant in the sense that it fibers principally over any quasitoric manifold over the given polytope. These spaces will be used later in our study of equivariant mapping spaces.

In Chapter 4 we state the criterion for rationaly ellipticity for quasitoric manifolds. In the process we also show that in the case when the equivariant cohomology ring is a complete intersection, the borel construction of the quasitoric manifold is, rationally, the unique space realizing this cohomology. This result was obtained previously by Notbohm and Ray in [7], though by different means.

Finally, in Chapter 5 we begin our study of equivariant mapping spaces. In the first two sections we discuss how on the one hand they arise as homotopy inverse limits of certain polytope-shaped diagrams of spaces, and on the other, as a moduli space of maps from the polytope into the manifold satisfying certain boundary conditions. In section 5.3 we prove that all equivariant maps not permuting the fixed points are homotopy equivalences. We also relate the space of equivariant maps of a quasitoric manifold with the equivariant mapping space of the corresponding moment angle complex by means of a fibration. This will allow us to later focus on the mapping space of the moment angle complex which turns out to be a bit easier. In 5.4 we discuss the equivariant mapping space of $\mathbb{C}P^n$. We prove that the space of equivariant maps not permuting the fixed pints is connected, and show how to compute its higher rational homotopy groups using the Bousfield-Kan spectral sequence. In 5.5 we extend this result to the case of an arbitrary rationally eliptic quasitoric manifold. In 5.6 we use the homotopy limit decomposition to determine the homotopy type of the mapping spaces when P is a simplex. Chapter 5 is concluded by applying the machinery of the Bousfield-Kan spectral sequence to show that when the polytope is 2-dimensional, the equivariant mapping space is connected.

We conclude this introduction by once again bringing up the question of connected components of $\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$. It appears that the set of components forms a group and we know that these spaces are connected when the image polytope is 2-dimensional or is a product of simplices. What happens in general, however, still eludes the author. It would be interesting to either see an example with multiple components or a way to prove that the space is connected.

Chapter 2

Definitions and Constructions

The study of quasitoric manifolds was initiated by Davis and Januszkiewicz in [3]. In that paper these objects appeared under the name of ”toric manifolds”. To my knowledge, it was Panov and Buchstaber, that have given them a name of ”quasitoric manifolds”, in order to avoid confusion with corresponding objects in algebraic/symplectic geometry. Here we will recall basic definitions and constructions. Our exposition follows very closely that of Buchstaber and Panov [2].

Henceforth, T^n will denote a standard n-dimensional torus, considered as the following subgroup of $(\mathbb{C}^*)^n$

$$T^n = \{(t_1, \ldots, t_n) : t_i \in \mathbb{C}, |t_i| = 1\}.$$

The standard action of T^n on $\mathbb{C}^n$ is given by

$$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (t_1 z_1, \ldots, t_n z_n)$$

It is easy to see that the orbit space for this action is

$$\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) : x_i \geqslant 0\}$$

This standard action will serve as the model for actions of a torus on quasitoric manifolds. The following definition makes this notion precise.

Definition 2.0.1. Let M^{2n} be a manifold with a T^n action. A *standard chart* on M is a triple (U, f, ψ), where U is a T^n-invariant open subset of M, ψ an automorphism of T^n, and $f : U \to W$ is a ψ-equivariant homeomorphism onto a T^n-invariant open subset $W \subseteq \mathbb{C}^n$. A T^n-action on M is called *locally standard*, if every point in M lies in the domain of some standard chart.

Remark 2.0.2. A homeomorphism $f : U \to W$ is ψ-*equivariant*, means that $f(t \cdot u) = \psi(t) \cdot f(u)$

Example 2.0.3. Consider $M = S^2$ with a T^1-action, where every $t \in T^1$ rotates S^2 through angle t around a fixed axis. Pictured in the figure are some locally standard charts.

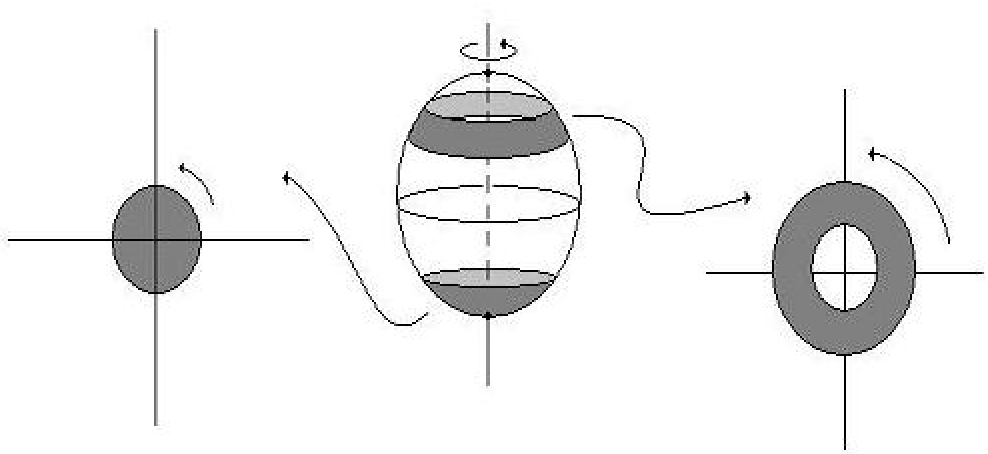


Figure 2.1: Locally standard charts on S^2

It follows from the definition, that the orbit space of a locally standard action of T^n on M^{2n} will be an n-dimensional manifold with corners. Quasitoric manifolds will correspond to the case when such an orbit space will be diffeomorphic (as a manifold with corners) to a simple polytope. (Recall that a simple n-dimensional polytope is one where exactly n codimension-1 faces meet at each vertex.)

Definition 2.0.4. A $2n$-dimensional manifold M^{2n} is *quasitoric over a simple polytope* P^n, if it is equipped with a locally standard action of T^n and a projection map $\pi : M^{2n} \to P^n$ such that every k-dimensional orbit is mapped to a point in the interior of a k-dimensional face of P^n

Note that the orbits over points in the interior of the polytope are principal, while the fixed points correspond to the vertices of the polytope.

Example 2.0.5. S^2 in the previous example is easily seen to be a quasitoric manifold over an interval. The endpoints of the segment correspond to the two fixed points at the poles, and the points in the interior correspond to the principal T^1 orbits.

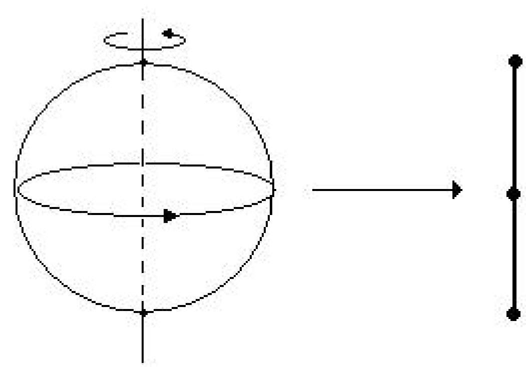


Figure 2.2: S^2 is a quasitoric manifold over a segment

Example 2.0.6. There is a locally standard action of $T^n \cong \frac{\overbrace{T^1 \times \ldots \times T^1}^{n+1}}{\Delta}$ on $\mathbb{C}P^n$ given by

$$(t_1, \ldots, t_{n+1}) \cdot [z_1 : z_2 : \ldots : z_{n+1}] = [t_1 z_1 : \ldots : t_{n+1} z_{n+1}]$$

It is easy to check that the orbit space is an n-simplex. (Above, $\Delta = \{(t, \ldots, t) : t \in T^1\}$ is the diagonal subgroup of $T^1 \times \ldots \times T^1$)

Example 2.0.7. If N^{2n} and M^{2m} are quasitoric manifolds over, respectively, P^n and Q^m, then $N \times M$ is a quasitoric T^{n+m}-manifold over $P \times Q$.

2.1 Quasitoric Manifolds from Characteristic Pairs

It would be very naive to hope for a bijective correspondence between ψ-equivariant homeomorphism classes of quasitoric manifolds and simple polytopes.

In fact, the quasitoric manifolds $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ are both manifolds over a combinatorial square, however, they are not diffeomorphic as their cohomology rings are different. To get a kind of a bijective correspondence, we will need to introduce additional combinatorial data associated to the quasitoric manifold and its orbit polytope.

Let M^{2n} be a quasitoric manifold over P^n. By a *facet* we will mean any codimension-1 face of P. Suppose $F_1, \ldots, F_m$ are facets of P. It can be shown that the preimages of any two points in the interior of F_j are codimension-1 orbits that have the same isotropy subgroup. We shall call this subgoup $T(F_j)$. This subgroup is rank one in the torus T^n, so we can write

$$T(F_j) = \{(e^{2\pi i \lambda_{1j} \theta}, \ldots, e^{2\pi i \lambda_{nj} \theta}) : \theta \in \mathbb{R}, \lambda_{sj} \in \mathbb{Z}\}$$

We may choose the vector $\boldsymbol{\lambda}_j = (\lambda_{1j}, \ldots, \lambda_{nj})$ to be primitive in $\mathbb{Z}^n$, and so it will be well defined up to a choice of the sign.

Definition 2.1.1. Let $\mathcal{F}$ be the set of facets of P^n. The function

$$\lambda : \mathcal{F} \longrightarrow \mathbb{Z}^n$$

given by $\lambda(F_j) = \boldsymbol{\lambda}_j$ is called the *characteristic function* of the quasitoric manifold M^{2n}.

Example 2.1.2. Let $M = \mathbb{C}P^2$. The image polytope is shown in the figure below. Let F_1, F_2, F_3 be the facets spanning, respectively, sides $[ab], [ac], [bc]$. The inverse image of each facet is a facial submanifold of codimension-2. The facial submanifolds corresponding to F_1, F_2, F_3 are $[z_1 : z_2 : 0], [z_1 : 0 : z_3], [0 : z_2 : z_3]$ respectively. Therefore the corresponding stabilizers are

$$T(F_1) = \{(1, 1, e^{2\pi i \theta})\}$$

$$T(F_2) = \{(1, e^{2\pi i \theta}, 1)\}$$

$$T(F_3) = \{(e^{2\pi i \theta}, 1, 1)\}$$

Under the identification $(T^1)^3/\Delta \cong T^1 \times T^1$, where $(t_1, t_2, t_3) \mapsto (\frac{t_2}{t_1}, \frac{t_3}{t_1})$ we get $\lambda(F_1) = (0, 1), \lambda(F_2) = (1, 0)$ and $\lambda(F_3) = (1, 1)$

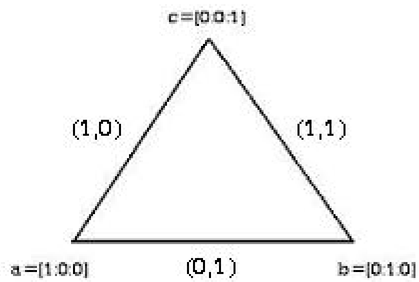


Figure 2.3: Characteristic function of $\mathbb{C}P^2$

We see that given a quasitoric manifold, we obtain a characteristic function on the facets of the image polytope. The question arises if given some function defined on the facets of a simple polytope $\lambda : \mathcal{F} \to \mathbb{Z}^n$, we can go in the opposite direction and find a quasitoric manifold with λ as the characteristic function. Certainly we cannot do that for an arbitrary λ. So what is the defining feature of a characteristic function?

To answer this question let us consider a quasitoric manifold M^{2n} over P^n. Let G be a codimension-k face of P. Since P is a simple polytope, we can express G as the intersection of all k facets containing it, that is, $G = F_{i_1} \cap \ldots \cap F_{i_k}$. The facial submanifolds $M_{i_1}, \ldots, M_{i_k}$ will intersect transversely in the submanifold M_G, called the facial submanifold corresponding to the face G. We note that the group $T(F_{i_1}) \times \ldots \times T(F_{i_k})$ can be identified with the isotropy subgroup of M_G, and so there is an injection

$$T(F_{i_1}) \times \ldots \times T(F_{i_k}) \hookrightarrow T^n$$

This means that the vectors $\lambda(F_{i_1}), \ldots, \lambda(F_{i_k})$ must form a part of a basis of $\mathbb{Z}^n$. This is the so called condition $(*)$ of Davis and Januszkiewicz [3, p423]. It turns out that this condition is also sufficient for a function λ to be a characteristic function of a quasitoric manifold.

Definition 2.1.3. Let P^n be a simple polytope and $\lambda : \mathcal{F} \to \mathbb{Z}^n$ be a mapping satisfying the following condition: $\{\lambda(F_{i_1}), \ldots, \lambda(F_{i_k})\}$ form a part of an integral basis of $\mathbb{Z}^n$ whenever $F_{i_1} \cap \ldots \cap F_{i_k} \neq \emptyset$. Then the pair (P^n, λ) is called a *characteristic pair*.

Note that given a characteristic pair, each facet F_j of P defines a rank one subgroup of T^n by

$$\lambda(F_j) = (\lambda_{1j}, \lambda_{2j}, \ldots, \lambda_{nj}) \mapsto \{(e^{2\pi i \lambda_{1j}\theta}, \ldots, e^{2\pi i \lambda_{nj}\theta}) : \theta \in \mathbb{R}\}$$

We will call this subgroup $T(F_j)$.

Similarly we can use λ to get a map from the facial poset of the polytope P to the poset of subtori of T^n. Namely, if $G = F_{j_1} \cap \ldots \cap F_{j_k}$ is a face of P, we define the corresponding subgroup of the torus T^n as the one corresponding to the subspace of $\mathbb{Z}^n$ spanned by $\lambda(F_{j_1}), \ldots, \lambda(F_{j_k})$, that is:

$$T(G) := \{\exp(2\pi i \sum_{r=1}^{k} \theta_r \lambda(F_{j_r})) : \theta_r \in \mathbb{R}\}$$

Construction 2.1.4. The idea behind reconstructing a quasitoric manifold from a characteristic pair is to first start with $T^n \times P^n$. (Here we think of each point in P^n as having a principal T^n orbit). Then we use the characteristic function λ to blow down the principal orbits, so as to force $T(G)$ to be the stabilizer of the face G. More precisely we define

$$M^{2n}(\lambda) := (T^n \times P^n)/\sim$$

where $(t,p) \sim (s,q)$ if and only if $p = q$ and $ts^{-1} \in T(G(p))$. Here $G(p)$ is the smallest face of P containing p in its relative interior. It is not hard to check that the resulting quotient is a quasitoric manifold over P [2, p65].

Thus far we saw that starting with a quasitoric manifold over a simple polytope P, we obtain a characteristic function λ, which together form a characterstic pair (P, λ). Conversely, we could also start with a characterstic pair, and recover a quasitoric manifold with the prescribed data. These two operations turn out to be mutually inverse, and we state this fact as a

Proposition 2.1.5. *There is a bijection between equivariant diffeomorphism classes of quasitoric manifolds and characteristic pairs.*

The proof can be found in [3]

Remark 2.1.6. An automorphism $\psi : T^n \to T^n$ induces a translation of the characteristic pair. The corresponding quasitoric manifold will be ψ-equivariantly diffeomorphic to the original one.

Chapter 3

Cohomology of Quasitoric Manifolds

The cohomology of quasitoric manifolds is completely determined by its image polytope and the characterstic function. As such it admits a completely combinatorial description.

3.1 A Perfect Cell Decomposition and Betti Numbers

We begin by determining the Betti numbers of a quasitoric manifold M^{2n} over P^n. We will do so by finding a perfect cell decomposition of M using a Morse theoretic argument. To this end, we position P^n in $\mathbb{R}^n$ and choose a linear height function $\phi : \mathbb{R}^n \to \mathbb{R}$ in such a way that ϕ is not constant along any of the edges of P^n. (This is always possible, simply define $\phi(x) = \langle x, v \rangle$ for a generic choice of vector v.) With such a choice of ϕ, each edge of P receives an orientation in the direction of increase of ϕ. Given a vertex $\nu \in P$, we may now define its index, $\text{ind}(\nu)$, as the number of incident edges that point toward ν. We are now ready to describe the cell structure of M.

Proposition 3.1.1. M *has a perfect cell structure with the number of cells of dimension* $2k$ *equal to the number of vertices of* P *with index* k. *Whence,* $b_{2k}(M) = \#\{\nu : \text{ind}(\nu) = k\}$.

For a proof see [2, p66]

Corollary 3.1.2. *Any quasitoric manifold is simply connected*

Remark 3.1.3. It follows from the proposition that the Betti numbers of a quasitoric manifold depend only on the combinatorics of the underlying polytope, and not on its characteristic function.

Example 3.1.4. *(Betti numbers of $\mathbb{C}P^n$)* $\mathbb{C}P^n$ is a quasitoric manifold over an n-simplex. Let $e_1, \ldots, e_n$ be the standard basis vectors of $\mathbb{R}^n$. Realize an n-simplex as the convex hull of $\{e_0 = 0, e_1, \ldots, e_n\}$. Define the height function ϕ by

$$\phi(x) = \langle x, e_1 + 2e_2 + \ldots + ne_n \rangle$$

Note that $\phi(e_i) - \phi(e_j) = i - j$, so if $i < j$ then ϕ increases along the edge from e_i to e_j. Therefore, $\mathrm{ind}(e_i) = i$ and so $\mathbb{C}P^n$ has exactly one $2i$-cell, for $0 \leqslant i \leqslant n$. We conclude that $b_{2i}(\mathbb{C}P^n) = 1$, for $0 \leqslant i \leqslant n$, as expected.

Example 3.1.5. Let M be a quasitoric manifold over a cube P. We position P in $\mathbb{R}^3$ as shown in the figure below. The height function is given by projection onto the z-coordinate. There is exactly one vertex for each index 0 and 3, and three vertices for each of the indexes 1 and 2. Thus $b_0(M) = b_6(M) = 1$ and $b_2(M) = b_4(M) = 3$.

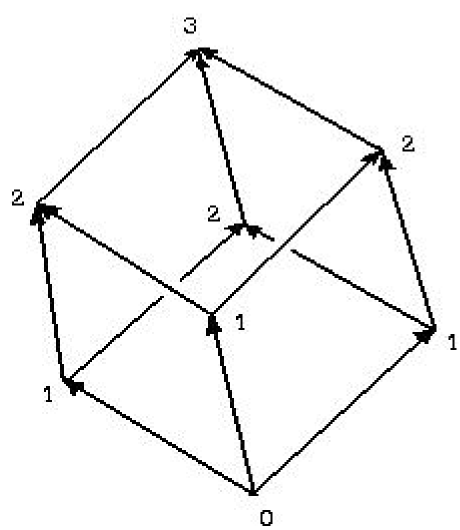


Figure 3.1: Betti numbers of a quasitoric manifold over a cube

The number of vertices of index i in a simple polytope P adimts a more combinatorial description as follows: Let $f_i(P)$ denote the number of i-dimensional faces of P. We define $h_i(P)$ by the equation

$$h_0t^n + h_1t^{n-1} + \ldots + h_{n-1}t + h_n = (t-1)^n + f_0(t-1)^{n-1} + \ldots + f_{n-1}$$

Then $b_{2i}(M) = \#\{v : \mathrm{ind}(v) = i\} = h_i(P)$. For details see [2, p66].

3.2 Moment Angle Complexes and Borel Constructions

In order to get a deeper insight into the structure of the cohomology of quasitoric manifolds, we will define a toric space $\mathcal{Z}_P$ corresponding to a simple polytope P. It will turn out to be a universal toric space over P in the sense that for any quasitoric manifold M over P, there is a principal torus bundle

$$T \to \mathcal{Z}_P \xrightarrow{p} M$$

such that if $\pi : M \to P$ is the orbit map then $p \circ \pi$ is the orbit map for $\mathcal{Z}_P$.

Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be the set of facets of a simple polytope P^n. We define

$$l : \mathcal{F} \to \mathbb{Z}^m$$

by $l(F_i) = e_i$, where e_i is the i^{th} standard basis vector.

As before, for each face $G \leqslant P$ we get a subtorus $T(G) \leqslant T^m$. In fact, if $G = F_{i_1} \cap \ldots \cap F_{i_k}$ then $T(G)$ is the product of the respective coordinate 1-tori in T^m. We then define

$$\mathcal{Z}_P := (T^m \times P^n)/\sim$$

where $(t, p) \sim (s, q)$ if and only if $p = q$ and $ts^{-1} \in T(G(p))$

It turns out that for a simple polytope P, $\mathcal{Z}_P$ is a smooth manifold of dimension $m+n$ [2, p85]. It is called the *moment angle complex* corresponding to P. We note that $\mathcal{Z}_P$ depends only on the combinatorial type of the polytope, and not on the characteristic function of the quasitoric manifold.

Suppose M^{2n} is a quasitoric manifold over P^n, with a characteristic function $\lambda : \mathcal{F} \to \mathbb{Z}^n$. We obtain a map $g : \mathbb{Z}^m \to \mathbb{Z}^n$ by setting $g(e_i) = \lambda(F_i)$. Let $H = \ker(g)$. Because of the condition imposed on λ in definition 2.1.3, H is an $m-n$ dimensional subspace of $\mathbb{Z}^m$. It is easy to check that the corresponding subtorus $T(H) \cong T^{m-n}$ will act freely on $\mathcal{Z}_P$. The map

$$\exp(g) \times \mathrm{id} : T^m \times P^n \to T^n \times P^n$$

will descend to the quotient

$$\frac{T^m \times P^n}{\sim} \to \frac{T^n \times P^n}{\sim}$$

and give a principal torus bundle:

$$T^{m-n} \to \mathcal{Z}_P \to M$$

We have shown the following:

Proposition 3.2.1. *There is a free action of* $T(H)$ *on* $\mathcal{Z}_P$*, giving a principal* T^{m-n}*-bundle* $T^{m-n} \to \mathcal{Z}_P \to M$

Example 3.2.2. Let $M = \mathbb{C}P^1$ with the standard T^1-action. The image polytope P is a 1-simplex. $\mathcal{Z}_P$ is obtained from $T^2 \times P$ by collapsing each boundary torus using the projections onto the coordinate circles. At this point we refer the reader to the figure below. If we pick a point on the interior of P and cut along the principal T^2-orbit that is above that point, we will get a union of two solid tori. We glue them together along their boundary via the diffeomorphism that exchanges longitudinal and meridional circles. The resulting space will be the three dimensional sphere, so $\mathcal{Z}_P = S^3$. The associated principal T^1-bundle is the Hopf fibration

$$S^1 \to S^3 \to \mathbb{C}P^1$$

Next, we consider the Borel constructions of M and $\mathcal{Z}_P$ with respect to the torus actions. We denote $ET^n \times_{T^n} M$ and $ET^m \times_{T^m} \mathcal{Z}_P$ by $B_T M$ and $B_T \mathcal{Z}_P$ respectively.

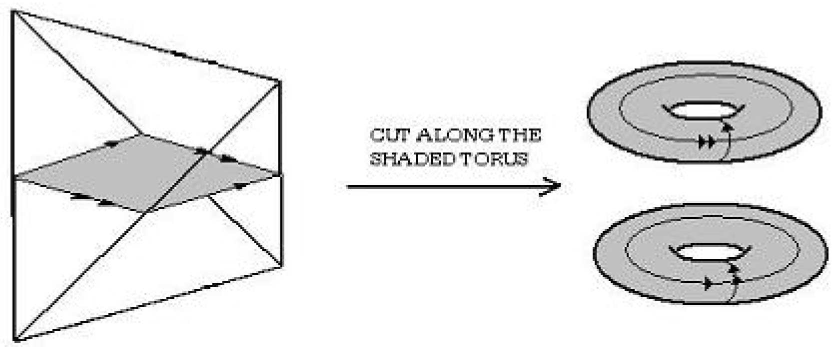


Figure 3.2: Moment angle complex of a 1-simplex

Proposition 3.2.3. $B_T M$ *is homotopy equivalent to* $B_T \mathcal{Z}_P$

Proof. The proof will rest on two basic facts:

- If G is a group and H is a subgroup, then there is a natural action of G on $E(G/H)$ such that the stabilizer of every point is H
- If G acts on a space X and H is a subgroup of G acting freely on X, then $EG \times_H X \simeq X/H$

Let $r = m - n$, and let T^r be the sbugroup of T^m acting freely on $\mathcal{Z}_P$ as in the previous proposition. By the first fact above, we may take $E(T^m/T^r) \times ET^m$ as a model for ET^m with T^m acting diagonally. We then have

$$\begin{aligned} ET^m \times_{T^m} \mathcal{Z}_P &\simeq \frac{E(T^m/T^r) \times ET^m \times \mathcal{Z}_P}{T^m} && (3.1) \\ &= \frac{E(T^m/T^r) \times (ET^m \times_{T^r} \mathcal{Z}_P)}{T^m/T^r} && (3.2) \\ &\simeq \frac{E(T^m/T^r) \times (\mathcal{Z}_P/T^r)}{T^m/T^r} && (3.3) \\ &= ET^n \times_{T^n} M && (3.4) \end{aligned}$$

□

Since $\mathcal{Z}_P$ does not depend on the characteristic function of the facets, the previous proposition says that the homotopy type of the Borel construction of M only depends on the combinatorics of P. In particular, this means that the T-equivariant cohomology of a quasitoric manifold is only a function of its image polytope.

3.3 Davis-Januszkiewicz Spaces and Equivariant Cohomology

Our aim at this point is to state what the T-equivariant cohomology of a quasitoric manifold is. In view of the homotopy equivalence given by Proposition 3.2.3, it is enough to determine the cohomology of $B_T\mathcal{Z}_P$. Before we state the result, however, some prelimenary constructions are needed.

Given a simple polytope P^n, we begin by constructing a dual $(n-1)$-dimensional simplicial complex K_P. This is done as follows: There is exactly one vertex v_i for each facet F_i of P. The set of k vertices $\{v_1, \ldots, v_k\}$ will span a $(k-1)$-simplex if and only if the intersection of the corresponding facets $F_1 \cap \ldots \cap F_k$ is not empty. This construction defines a simplicial complex K_P with one $(k-1)$-simplex for every codimension-k face of P. In fact, K_P is a simplicial $(n-1)$-sphere whenever P is a simple polytope.

Let R be a commutative ring with unit and P a simple polytope with facets $\{F_1, \ldots, F_m\}$.

Definition 3.3.1. The *Stanley-Reisner* algebra of P,(also known as the *face ring* of P), is a graded polynomial ring denoted by $R(P)$ and given by

$$R(P) = \frac{R[F_1, \ldots, F_m]}{I_P}$$

where I_P is the homogeneous ideal generated by all products $F_{i_1} \cdots F_{i_k}$ such that $F_{i_1} \cap \ldots \cap F_{i_k} = \emptyset$. The grading on $R(P)$ is given by declaring that $\deg(F_i) = 2$.

Likewise, given a simplicial complex K with vertex set $\{v_1, \ldots, v_m\}$, we define $R(K)$ as $R[v_1, \ldots, v_m]/I_K$, where I_K is the ideal generated by all products $v_{i_1} \cdots v_{i_k}$, such that the simplex $[v_{i_1} \ldots v_{i_k}]$ spanned by these vertices is not in K. We note that $R(P) \cong R(K_P)$.

Let K be a simplicial complex with m vertices. Denote by BT_i the i^{th} component of the product

$$BT^m \cong BT_1 \times \ldots \times BT_m \cong (\mathbb{C}P^\infty)^m$$

For $\sigma \in K$ define BT_σ as the subcomplex of $\prod BT_i$ given by $\prod_{j \in \sigma} BT_j$.

Definition 3.3.2. For a simplicial complex K, the *Davis-Januszkiewicz space* $DJ(K)$ is defined as $DJ(K) = \bigcup_{\sigma \in K} BT_\sigma \subseteq BT^m$

Note that, by construction, the cellular cochain algebra $C^*(DJ(K))$ and hence $H^*(DJ(K))$, are isomorphic to $R(K)$. Moreover, if we identify $H^*(BT^m)$ with $R[v_1, \ldots, v_m]$, the inclusion $i : DJ(K) \hookrightarrow BT^m$ induces an epimorphism

$$i^* : R[v_1, \ldots, v_m] \to R(K)$$

with kernel I_K.

Remark 3.3.3. Note that $DJ(K)$ is defined for any simplicial complex. There is a generalization of the moment angle complex $\mathcal{Z}_P$ of a simple polytope P, to a moment angle complex $\mathcal{Z}_K$ corresponding to an arbitrary simplicial complex K. The next theorem, the proof of which may be found in [2], holds in this more general context, but we are only interested in the case when K is a simplicial sphere (i.e. K is the dual of a simple polytope).

Theorem 3.3.4. *There is a deformation retraction* $B_T\mathcal{Z}_K \to DJ(K)$ *making the following diagram commute*

$$\begin{array}{ccc} B_T\mathcal{Z}_K & \xrightarrow{p} & BT^m \\ \downarrow & & \| \\ DJ(K) & \xrightarrow{i} & BT^m \end{array}$$

Since p is a fiber bundle with fiber $\mathcal{Z}_K$, the theorem implies that $\mathcal{Z}_K$ is the homotopy fiber of the inclusion $DJ(K) \hookrightarrow BT^m$. As the corollary of this theorem we obtain the T^m-equivariant cohomology of $\mathcal{Z}_P$

Corollary 3.3.5. $H^*_{T^m}(\mathcal{Z}_P) = H^*(B_T\mathcal{Z}_P) \cong R(P)$

In view of Proposition 3.2.3, we also get

Corollary 3.3.6. $H^*_{T^n}(M) \cong R(P)$

In conclusion, we will describe the ordinary cohomology of a quasitoric manifold M over P. Consider the fibration

$$M \xrightarrow{i} B_{T^n}M \xrightarrow{p} BT^n$$

and let

$$H^*(BT^n) = \mathbb{Z}[c_1, \ldots, c_n]$$

By what was said above the equivariant cohomology of M is generated by degree 2 classes corresponding to the facets of P, so we put

$$H^*(B_{T^n}M) = \frac{\mathbb{Z}[v_1, \ldots, v_m]}{I}$$

As before, I is the Stanley-Reisner ideal of P. We also define linear forms $\theta_1, \ldots, \theta_n$ given by

$$\theta_i = \lambda_{i1}v_1 + \ldots + \lambda_{im}v_m$$

where the vectors $\boldsymbol{\lambda}_j = (\lambda_{1j}, \ldots, \lambda_{nj})$ are the characteristic vectors of M. The proof of the following theorem may be found in [2] or [3]

Theorem 3.3.7. *With the above notation, the cohomology of* M *is given by*

$$H^*(M) = \frac{\mathbb{Z}[v_1, \ldots, v_m]}{I + J}$$

where J *is the ideal generated by the linear forms* $\theta_1, \ldots, \theta_n$. *Moreover* $i^*(v_i) = v_i$ *and* $p^*(c_i) = \theta_i$

Chapter 4

A Criterion for Rational Ellipticity of Quasitoric Manifolds

In this section we will give a simple criterion for determining whether a quasitoric manifold is rationally elliptic or rationally hyperbolic. We will prove the following result

Theorem A. *A quasitoric manifold* M *over* P *is rationally elliptic if and only if* P *is a product of simplexes.*

The idea is to study the cohomology of $B_T M$, more precisely the ideal I_P. It turns out that when trying to build a minimal model for $H^*(B_T M; \mathbb{Q})$, one needs only a finite number of generators precisely when the generating monomials of I_P are, in some sense "independent". When there are certain relations between the monomials, the number of generators required quickly blows up. The algebraic formulation of "independence" is given by the notion of a *regular sequence*. That is, the generating monomials of I_P must form a regular sequence in $S = \mathbb{Q}[x_1, \ldots, x_m]$ in order for M to be rationally elliptic. Equivalently, the cohomology ring $H^*(B_T M) = \mathbb{Q}(P)$ must be a complete intersection.

We begin by determining when $\mathbb{Q}(P)$ is a complete intersection ring.

Definition 4.0.8. Two square free monomials $\sigma = x_{i_1} \cdots x_{i_r}$ and $\tau = x_{j_1} \cdots x_{j_s}$ are *disjoint*, if $i_p \neq j_q$ for all $1 \leqslant p \leqslant r$ and $1 \leqslant q \leqslant s$

Lemma 4.0.9. *Suppose* $I_P = (\sigma_1, \ldots, \sigma_k)$ *is a monomial ideal in* S *generated by square free monomials. Also assume that* $\sigma_1, \ldots, \sigma_k$ *is a minimal system of generators. Then* $\sigma_1, \ldots, \sigma_k$ *is a regular sequence if and only if* σ_i *are pairwise disjoint for* $1 \leqslant i \leqslant k$.

Proof. First suppose that $\{\sigma_i\}$ are pairwise disjoint, and assume that $\sigma_j f = 0$ in $S/(\sigma_1, \ldots, \sigma_{j-1})$ for some $f \in S/(\sigma_1, \ldots, \sigma_{j-1})$. That is, $\sigma_j f = f_1\sigma_1 + \ldots + f_{j-1}\sigma_{j-1}$. Let $\sigma_j = x_{j_1} \ldots x_{j_r}$. Since $x_{j_p} | \sigma_j f$, but $x_{j_p} \nmid \sigma_l$ for $1 \leqslant l \leqslant j-1$ by disjointness, we see that x_{j_p} must divide f_l for $1 \leqslant l \leqslant j-1$ and $1 \leqslant p \leqslant r$. Thus σ_j must divide f_l for all l between 1 and $j-1$. Cancelling σ_j from both sides we get $f = f'_1\sigma_1 + \ldots + f'_{j-1}\sigma_{j-1}$. Thus $f = 0$ in $S/(\sigma_1, \ldots, \sigma_{j-1})$ and therefore σ_j is not a zero-divisor.

Conversely, suppose $\sigma_1, \ldots, \sigma_k$ are not pairwise disjoint. That is, we can find σ_i and σ_j, with $i < j$, such that $\sigma_i = m\tau$ and $\sigma_j = m\tau'$, where τ and τ' are disjoint. We may also assume that $\tau \notin (\sigma_1, \ldots, \sigma_i)$ by minimality of the generating set of monomials. But then $\sigma_j\tau = m\tau\tau' = \sigma_i\tau' \in (\sigma_1, \ldots, \sigma_i)$, so σ_j is a zero divisor in $S/(\sigma_1, \ldots, \sigma_{j-1})$. □

We can also determine when a polytope is a product of simplices using the following

Proposition 4.0.10. *A simple polytope is a product of simplices if and only if its Stanley-Reisner ideal can be generated by disjoint monomials*

Proof. If P_1 and P_2 are simple polytopes, then so is $P_1 \times P_2$. The face ring of the product is given by [2]

$$\mathbb{Q}(P_1 \times P_2) \cong \mathbb{Q}(P_1) \otimes \mathbb{Q}(P_2)$$

Taking Δ^i and Δ^j to be respectively i and j-simplices we obtain

$$\begin{aligned} \mathbb{Q}(\Delta^i \times \Delta^j) &\cong \mathbb{Q}(\Delta^i) \otimes \mathbb{Q}(\Delta^j) \\ &\cong \frac{\mathbb{Q}[v_1, \ldots, v_{i+1}]}{(v_1 \cdots v_{i+1})} \otimes \frac{\mathbb{Q}[w_1, \ldots, w_{j+1}]}{(w_1 \cdots w_{j+1})} \\ &\cong \frac{\mathbb{Q}[F_1, \ldots, F_{i+j+2}]}{(F_1 \cdots F_{i+1}, F_{i+2} \cdots F_{i+j+2})} \end{aligned}$$

Inductively, we obtain the analogous result for a product of several simplices.

Conversely, if P is a simple polytope with

$$\mathbb{Q}(P) = \frac{\mathbb{Q}[F_1, \ldots, F_m]}{(m_1, \ldots, m_k)}$$

where m_i are disjoint monomials of length $|m_i|$, then we observe that

$$\mathbb{Q}(\Delta^{|m_1|-1} \times \ldots \times \Delta^{|m_k|-1}) \cong \mathbb{Q}(P)$$

Because the Stanley-Reisner ideals inside $\mathbb{Q}[F_1, \ldots, F_m]$ uniquely determine simplicial complexes on m vertices [6], the dual simplicial complexes of P and $\Delta^{|m_1|-1} \times \ldots \times \Delta^{|m_k|-1}$ must coincide, and therefore, the polytopes must coincide as well. □

We now get back to topology and consider the following construction:

Suppose P is a simple polytope and $\mathbb{Q}(P) = S/(\sigma_1, \ldots, \sigma_k)$ is its face ring. We identify $H^*((\mathbb{C}P^\infty)^m; \mathbb{Q})$ with $S = \mathbb{Q}[x_1, \ldots, x_m]$ and think of σ_i as elements in $H^{|\sigma_i|}((\mathbb{C}P^\infty)^m; \mathbb{Q})$. We then get the following map, classifying the cohomology classes σ_i:

$$(\mathbb{C}P^\infty)^m \xrightarrow{(\sigma_1,\ldots,\sigma_k)} \prod_{i=1}^{k} K(\mathbb{Q}, |\sigma_i|)$$

Let $\mathcal{F}$ be the homotopy fiber of the above map.

Proposition 4.0.11. *$\mathcal{F}$ is rationally elliptic.*

Proof. We loop back the fibration

$$\mathcal{F} \longrightarrow (\mathbb{C}P^\infty)^m \longrightarrow \prod_{i=1}^{k} K(\mathbb{Q}, |\sigma_i|)$$

to get a fibration

$$(S^1)^m \longrightarrow \prod_{i=1}^{k} K(\mathbb{Q}, |\sigma_i| - 1) \longrightarrow \mathcal{F}$$

Since $(S^1)^m$ and $\prod_{i=1}^{k} K(\mathbb{Q}, |\sigma_i| - 1)$ are rationally elliptic, the result follows from the homotopy long exact sequence. □

It turns out that in the case when $(\sigma_1, \ldots, \sigma_k)$ is a regular sequence, $\mathcal{F}$ is rationally homotopy equivalent to $DJ(K_P)$. To prove this, we will need this

Lemma 4.0.12. *If* $(\sigma_1, \ldots, \sigma_k)$ *is a regular sequence, then* $H^*(\mathcal{F}; \mathbb{Q}) \cong \mathbb{Q}(P)$

Proof. We will use the Eilenberg-Moore spectral sequence for the fibration

$$\mathcal{F} \longrightarrow (\mathbb{C}P^\infty)^m \longrightarrow \prod_{i=1}^{k} K(\mathbb{Q}, |\sigma_i|)$$

The E_2 page is given by

$$E_2^{**} = \mathrm{Tor}^{**}_{\mathbb{Q}[\sigma_1,\ldots,\sigma_k]}(\mathbb{Q}[x_1, \ldots, x_m], \mathbb{Q})$$

Since $\sigma_1, \ldots, \sigma_k$ is a regular sequence, it is well known that $\mathbb{Q}[x_1, \ldots, x_m]$ is then a free $\mathbb{Q}[\sigma_1, \ldots, \sigma_k]$-module. Therefore,

$$\begin{aligned} \mathrm{Tor}^{**}_{\mathbb{Q}[\sigma_1,\ldots,\sigma_k]}(\mathbb{Q}[x_1, \ldots, x_m], \mathbb{Q}) &= \mathrm{Tor}^{0,*}_{\mathbb{Q}[\sigma_1,\ldots,\sigma_k]}(\mathbb{Q}[x_1, \ldots, x_m], \mathbb{Q}) \\ &= \mathbb{Q}[x_1, \ldots, x_m] \otimes_{\mathbb{Q}[\sigma_1,\ldots,\sigma_k]} \mathbb{Q} \\ &= \frac{\mathbb{Q}[x_1, \ldots, x_m]}{(\sigma_1, \ldots, \sigma_k)} \end{aligned}$$

So

$$E_2^{0,*} = \frac{\mathbb{Q}[x_1, \ldots, x_m]}{(\sigma_1, \ldots, \sigma_k)} \text{ and } E_2^{-i,*} = 0 \text{ for } i > 0$$

Thus, the spectral sequence collapses at the E_2-page and

$$H^*(\mathcal{F}; \mathbb{Q}) \cong \frac{\mathbb{Q}[x_1, \ldots, x_m]}{(\sigma_1, \ldots, \sigma_k)}$$

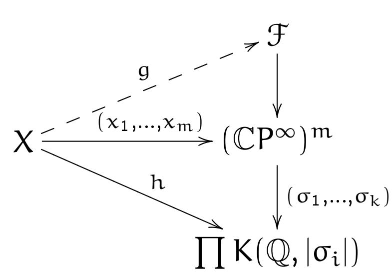
□

Proposition 4.0.13. *Suppose* $(\sigma_1, \ldots, \sigma_k)$ *is a regular sequence. If* X *is a simply connected space with*

$$H^*(X; \mathbb{Q}) \cong \frac{\mathbb{Q}[x_1, \ldots, x_m]}{(\sigma_1, \ldots, \sigma_k)}$$

then X *is rationally homotopy equivalent to* $\mathcal{F}$

Proof. Consider the following commutative diagram:



We first note that the map h represents the classes σ_j in the cohomology of X. Since these classes are zero, the map h must be null-homotopic. Because h is null homotopic we can find a lift

$$g : X \to \mathcal{F}$$

We claim that g induces an isomorphism on rationally cohomology groups. To this end, we observe that the map $(x_1, \dots, x_m)$ classifying the cohomology classes $x_1, \dots, x_m$, induces a surjection

$$H^2((\mathbb{C}P^\infty)^m; \mathbb{Q}) \to H^2(X; \mathbb{Q})$$

given simply by

$$x_i \mapsto x_i$$

Because the cohomology is generated by elements in degree two, it follows that g^* is also a surjection of $\mathbb{Q}$-vector spaces

$$g^* : H^*(\mathcal{F}; \mathbb{Q}) \to H^*(X; \mathbb{Q})$$

Because the vector spaces in each dimension are finite dimensional, the map g^* is an isomorphism in each dimension.

Finally, by the universal coeffiecients theorem the map g_* is an isomorphism on homology, and since X is simply connected, Whitehead's theorem implies that g is a rational homotopy equivalence, which is what we wanted to show. □

Corollary 4.0.14. *If $\mathbb{Q}(P)$ is a complete intersection ring, then, rationally, $DJ(K_P)$ is the unique space that realizes $\mathbb{Q}(P)$.*

Proof. In fact, any space that realizes $\mathbb{Q}(P)$ is rationally homotopy equivalent to $\mathcal{F}$ by the previous proposition. So $DJ(K_P) \simeq_{\mathbb{Q}} \mathcal{F}$ and the result follows. □

Corollary 4.0.15. *If M is a quasitoric manifold over P and $\mathbb{Q}(P)$ is a complete intersection, then $B_T M$ (and therefore M) are rationally elliptic.*

Proof. Since $\mathbb{Q}(P)$ is a complete intersection, $B_T M \simeq DJ(K_P) \simeq \mathcal{F}$ by proposition 4.0.13. But $\mathcal{F}$ is rationally elliptic by proposition 4.0.11. □

Let us summarize our discussion so far. If P is a simple polytope that is a product of simplices, then by proposition 4.0.10, its Stanley-Reisner ideal consists of disjoint monomials. In turn, by lemma 4.0.9, this means that it defines a regular sequence in the given polynomial ring. Finally, by the previous corollary we obtain one of the implications of Theorem A: if a quasitoric manifold lies over a product of simplices, then it is rationally elliptic.

Our path to proving the converse begins with computing the rational homotopy groups of rationally elliptic quasitoric manifolds.

Proposition 4.0.16. *If M is a rationally elliptic manifold over P, then*

1. $\pi_j(B_T M) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q}^{\mu_j} & \textit{if } j > 2 \\ \mathbb{Q}^{m} & \textit{if } j = 2. \end{cases}$

2. $\pi_j(M) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q}^{\mu_j} & \textit{if } j > 2 \\ \mathbb{Q}^{m-n} & \textit{if } j = 2 \end{cases}$

where μ_j represents the number of generating monomials of the Stanley-Reisner ideal of P of length $\dfrac{j+1}{2}$

Remark 4.0.17. It follows from the proposition that the even dimensional rational homotopy of rationally elliptic quasitoric manifolds is concentrated exclusively in dimension two. This is because $\frac{j+1}{2}$ is an integer if and only if j is odd.

Example 4.0.18. Let M be a quasitoric manifold over $P = \Delta^1 \times \Delta^1 \times \Delta^2$. P is a 4-dimensional simple polytope with 7 facets. The Stanley-Reisner ideal of P is

$$I_P = (F_1F_2, F_3F_4, F_5F_6F_7)$$

M is rationally elliptic with the monomials of length 2 contributing to π_3 and the monomials of length 3 contributing to π_5. Thus

$$\begin{aligned} \pi_2(M) \otimes \mathbb{Q} &= \mathbb{Q}^{7-4} = \mathbb{Q}^3 \\ \pi_3(M) \otimes \mathbb{Q} &= \mathbb{Q}^2 \\ \pi_5(M) \otimes \mathbb{Q} &= \mathbb{Q} \end{aligned}$$

We now prove proposition 4.0.16

Proof. Let $\mathcal{F}$ be as above. Since $\mathcal{F} \simeq_{\mathbb{Q}} B_T M$, to prove (1) we consider the homotopy long exact sequence for the fibration

$$T^m \longrightarrow \prod_{i=1}^{k} K(\mathbb{Q}, |\sigma_i| - 1) \longrightarrow \mathcal{F}$$

From the long exact sequence it is clear that for $j > 2$

$$\pi_j(\mathcal{F}) \cong \prod_{i=1}^{k} \pi_j K(\mathbb{Q}, |\sigma_i| - 1)$$

Every i such that $|\sigma_i| - 1 = j$ contributes a factor of $\mathbb{Q}$ to $\pi_j(\mathcal{F})$. Thus we get as many factors as there are monomials of length $\frac{j+1}{2}$.

In lower degrees the homotopy long exact sequence looks like

$$0 \to \pi_2(\mathcal{F}) \to \pi_1(T^m) \to 0$$

proving (1) for $j = 2$

To prove (2) we use the homotopy long exact sequence for the fibration

$$M \to B_T M \to BT$$

For $j > 2$ we get

$$\pi_j(M) \cong \pi_j(B_T M)$$

The tail end of the homotopy exact sequence is

$$0 \to \pi_2(M) \to \pi_2(B_T M) \to \pi_1(T^n) \to 0$$

Since $\pi_1(T^n)$ is free abelian, this short exact sequence is split, so $\pi_2(M) \cong \mathbb{Z}^{m-n}$ □

The proof of the following proposition can be found in [4, p447].

Proposition 4.0.19. *Let X be a simply connected space with finite dimensional rational homology. Let $q = \dim \pi_{even}(X) \otimes \mathbb{Q}$. If X is rationally elliptic then the following are equivalent:*

1. $\chi(X) > 0$

2. $H^*(X;\mathbb{Q})$ *is the quotient of a polynomial algebra in* q *variables of even degree by an ideal generated by a regular sequence of length* q

3. $\dim \pi_{\text{even}}(X) \otimes \mathbb{Q} = \dim \pi_{\text{odd}}(X) \otimes \mathbb{Q}$

We can now prove the converse of the main theorem of this section.

Proposition 4.0.20. *If* M *is a rationally elliptic quasitoric manifold over* P *and*

$$H^*(B_T M;\mathbb{Q}) = \frac{\mathbb{Q}[v_1,\ldots,v_m]}{I_P}$$

where $I_P = (\sigma_1,\ldots,\sigma_k)$ *is the Stanley-Reisner ideal of* P*, then* $\sigma_1,\ldots,\sigma_k$ *forms a regular sequence in* $\mathbb{Q}[v_1,\ldots,v_m]$

Proof. We first note that M satisfies the hypotheses of proposition 4.0.19. Moreover, since M consists of only even dimensional cells, $\chi(M) > 0$.

Next, observe that

$$\pi_{\text{even}}(M) \otimes \mathbb{Q} = \pi_2(M) \otimes \mathbb{Q} = \mathbb{Q}^{m-n}$$

and by proposition 4.0.19

$$m - n = \dim \pi_{\text{even}}(M) \otimes \mathbb{Q} = \dim \pi_{\text{odd}}(M) \otimes \mathbb{Q}$$

Since every generating monomial of the Stanley-Reisner ideal contributes exactly one generator to the odd dimensional homotopy, we conclude that, in fact, $k = m - n$.

The cohomology of M is given by

$$H^*(M;\mathbb{Q}) = \frac{\mathbb{Q}[v_1,\ldots,v_m]}{(\sigma_1,\ldots,\sigma_{m-n},\theta_1,\ldots,\theta_n)}$$

Without loss of generality we may assume that $v_{m-n+1},\ldots,v_m$ correspond to some n facets of P that meet at a vertex. Since the characteristic function restricted to these facets gives a basis for $\mathbb{Z}^n$, we may use the relations $\theta_1,\ldots,\theta_n$ to express $v_{m-n+1},\ldots,v_m$ as linear combinations of $v_1,\ldots,v_{m-n}$. We can now use these linear combinations to eliminate the variables $v_{m-n+1}, \dots, v_m$ from the monomials $\sigma_1, \dots, \sigma_{m-n}$. Denote the resulting polynomials by

$$\tau_i = \sigma_i(v_1, \dots, v_{m-n})$$

We can now rewrite the cohomology of M as

$$H^*(M;\mathbb{Q}) = \frac{\mathbb{Q}[v_1, \dots, v_m]}{(\sigma_1, \dots, \sigma_{m-n}, \theta_1, \dots, \theta_n)} = \frac{\mathbb{Q}[v_1, \dots, v_{m-n}]}{(\tau_1, \dots, \tau_{m-n})}$$

By 4.0.19, $\tau_1, \dots, \tau_{m-n}$ is a regular sequence in $\mathbb{Q}[v_1, \dots, v_{m-n}]$. To complete the proof we argue by contradiction:

Suppose $\sigma_1, \dots, \sigma_{m-n}$ is not a regular sequence in $\mathbb{Q}[v_1, \dots, v_m]$. Then for some i, σ_i is a zero divisor in $\mathbb{Q}[v_1, \dots, v_m]/(\sigma_1, \dots, \sigma_{i-1})$. That is, there exists a non-zero p such that $\sigma_i p \in (\sigma_1, \dots, \sigma_{i-1})$ Eliminating extraneous variables in this relation, we see that $\tau_i p(v_1, \dots, v_{m-n}) \in (\tau_1, \dots, \tau_{i-1})$, so it is a zero-divizor. This contradicts the fact that $\tau_1, \dots, \tau_{m-n}$ is a regular sequence and completes the proof of the proposition □

We can now combine the results of this section in a

Theorem A. *A quasitoric manifold M over a simple n-dimensional polytope P with m facets is rationally elliptic if and only if P is a product of simplices. Moreover, the rational homotopy of M is given by*

$$\pi_j(M) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q}^{\mu_j} & \text{if } j > 2 \\ \mathbb{Q}^{m-n} & \text{if } j = 2 \end{cases}$$

where μ_j represents the number of generating monomials of the Stanley-Reisner ideal of P of length $(j+1)/2$. Equivalently, every i-simplex factor of P, contributes a generator to $\pi_{2i+1}(M) \otimes \mathbb{Q}$.

Chapter 5

Equivariant Mapping Spaces

Let $M = M^{2n}$ be a quasitoric manifold over P, and $\mathcal{Z}_P$ its associated moment angle complex. In what follows we will consider the mapping space $\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ of T^m-equivariant self maps of $\mathcal{Z}_P$. The natural structure of $\mathcal{Z}_P$ as a homotopy colimit allows us to study this mapping space by considering it as a homotopy inverse limit. One of the benefits of this approach lies in being able to use the Bousfield-Kan spectral sequence to compute the rational homotopy groups of $\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ when P a product of simplexes.

In taking a more geometric view, we will also interpret the spaces $\mathrm{map}_{T^n}(M, M)$ and $\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ as a space of maps out of the polytope P into M (resp $\mathcal{Z}_P$) satisfying certain boundary conditions. This will let us view $\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ as a tower of fibrations, and when P is a simplex, the fibers have the homotopy type of iterated loopspaces on odd dimensional spheres.

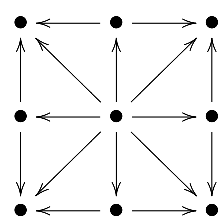
For the definitions of the homotopy limits and colimits and the results on the convergence of the Bousfield-Kan spectral sequence used in this chapter, the reader is directed to Appendix A. Those desiring a more detailed introduction to these topics are referred to [1] or [5].

5.1 Equivariant Mapping Spaces as Homotopy Inverse Limits

We first recall how to view M as a homotopy colimit. Let P be the category obtained from the polytope P as follows:

The objects of P are the faces of P. Thre is a morphism from a face σ to a face τ if $\tau \subset \sigma$. The category P can thus be viewed as simply the barycentric subdivision of P.

Example 5.1.1. If P is a square, then the category P can be viewed as this diagram:



For a quasitoric manifold M over P, and its moment angle complex $\mathcal{Z}_P$ we consider the following covariant functors taking values in the category of T-spaces:

$$\begin{aligned} \mathsf{Orb}_{M,P} &: P \rightarrow T\text{-Top} \\ \mathsf{Orb}_{P} &: P \rightarrow T\text{-Top} \\ \mathsf{Fix}_{P} &: P^{op} \rightarrow T\text{-Top} \end{aligned}$$

These functors are defined by:

$$\begin{aligned} \mathsf{Orb}_{M,P}(\sigma) &= \mathsf{T}^n/\mathsf{T}_\sigma \\ \mathsf{Orb}_{P}(\sigma) &= \mathsf{T}^m/\mathsf{T}_\sigma \\ \mathsf{Fix}_{P}(\sigma) &= \mathcal{Z}_P^{\mathsf{T}_\sigma} \end{aligned}$$

In the first equation above, T_σ is the stabilizier of the face σ in M, while in the last two equations T_σ is the stabilizer of σ in $\mathcal{Z}_P$.

The induced morphisms for $\mathsf{Orb}_{M,P}$ are just the projections

$$\mathsf{Orb}_{M,P}(\sigma \rightarrow \tau) = (\mathsf{T}^n/\mathsf{T}_\sigma \rightarrow \mathsf{T}^n/\mathsf{T}_\tau)$$

The induced morphisms for Orb_P are defined similarly, and

$$\mathsf{Fix}_P(\sigma \to \tau) = (\mathcal{Z}_P^{T_\tau} \hookrightarrow \mathcal{Z}_P^{T_\sigma})$$

is just the inclusion of subspaces.

The following proposition can be found in [8] or [9]

Proposition 5.1.2. *Let M be a quasitoric manifold over P and $\mathcal{Z}_P$ its moment angle complex. Then*

$$\begin{aligned} M &= \operatorname*{hocolim}_P \mathsf{Orb}_{M,P} \\ \mathcal{Z}_P &= \operatorname*{hocolim}_P \mathsf{Orb}_P \end{aligned}$$

Remark 5.1.3. As a consequence of the construction of the moment angle complex $\mathcal{Z}_P$, the stabilizers of the faces are simply the coordinate subtori of T^m, and in case of $M = \mathbb{C}P^n$, $P = \Delta^n$ and $\mathcal{Z}_P \approx S^{2n+1}$

We will need the following lemma before we can express the mapping space $\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ as a homotopy inverse limit.

Lemma 5.1.4. *Let G be a group and X, Y be G-spaces. Suppose P be a small category and F a functor from P to G-Top. If X can be written as a homotopy colimit*

$$X = \operatorname*{hocolim}_P F$$

then

$$\mathrm{map}_G(X, Y) = \mathrm{map}_G(\operatorname*{hocolim}_P F, Y) = \operatorname*{holim}_{P^{op}} \mathrm{map}_G(F, Y)$$

Proof. Consider the action of G on $\mathrm{map}(X, Y)$ given by

$$(g \cdot f)(x) = g^{-1} f(g \cdot x)$$

The fixed points of this action are precisely the G-equivariant maps in $\mathrm{map}(X, Y)$, so

$$(\mathrm{map}(X, Y))^G = \mathrm{map}_G(X, Y)$$

Now let G be a category with one object whose morphims are elements of G. Define a functor

$$M_{X,Y} : G \to G\text{-Top}$$

given by

$$M_{X,Y}(\star) = \mathrm{map}(X, Y)$$

The fixed point set of the G-action on map(X, Y) is the equalizer of the functor $M_{X,Y}$ over the diagram G, i.e.

$$\mathrm{map}_G(X, Y) = (\mathrm{map}(X, Y))^G = \lim_G M_{X,Y}$$

The result of the lemma can now be obtained as a formal consequence:

$$\begin{aligned} \mathrm{map}_G(\mathrm{hocolim}_P F, Y) &= \lim_G M_{\mathrm{hocolim}_P F, Y} = \\ \lim_G \mathrm{holim}_{P^{op}} M_{F,Y} &= \mathrm{holim}_{P^{op}} \lim_G M_{F,Y} \\ &= \mathrm{holim}_{P^{op}} \mathrm{map}_G(F, Y) \end{aligned}$$

The second equality holds because we can bring the homotopy direct limit out of the mapping space as the homotopy inverse limit, while the third equality holds because we can commute inverse limits with homotopy inverse limits. □

Proposition 5.1.5. $\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P) = \mathrm{holim}_{P^{op}} \mathrm{Fix}_P$

Proof.

$$\begin{aligned} \mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P) &= \mathrm{map}_{T^m}(\mathrm{hocolim}_{\sigma \in P} \mathrm{Orb}_P(\sigma), \mathcal{Z}_P) \\ &= \mathrm{map}_{T^m}(\mathrm{hocolim}_{\sigma \in P} T^m/T_\sigma, \mathcal{Z}_P) \\ &= \mathrm{holim}_{\sigma \in P^{op}} \mathrm{map}_{T^m}(T^m/T_\sigma, \mathcal{Z}_P) \\ &= \mathrm{holim}_{\sigma \in P^{op}} \mathcal{Z}_P^{T_\sigma} \\ &= \mathrm{holim}_{P^{op}} \mathrm{Fix}_P \end{aligned}$$

where the third equality is justified by the previous lemma. □

5.2 Equivariant Mapping Spaces as Moduli Space of Maps

An equivariant map is entirely determined by how it acts on each orbit representative. Our first task is to identify a convinient set of orbit representatives inside of $\mathcal{Z}_P$

Proposition 5.2.1. *The map*

$$\begin{aligned} s: \ P &\to \mathcal{Z}_P \\ p &\mapsto [(1,p)] \end{aligned}$$

defines a continuous section to the projection map $\mathcal{Z}_P \to P$. Moreover, it is a homeomorphism onto its image.

Proof. It is easily checked that s is a section. Since s is a composition of continuous maps

$$\begin{aligned} P \to \quad & T^m \times P & \to \mathcal{Z}_P \\ p \mapsto \quad & (1,p) & \mapsto [(1,p)] \end{aligned}$$

it is continuous. Finally, since P is compact and $\mathcal{Z}_P$ is hausdorff, s is a homeomorphism onto its image. □

From now on, we will identify the space of orbits P, with the embedded submanifold of $\mathcal{Z}_P$ constructed in the proposition.

Remark 5.2.2. Note that the same construction works also for M

Construction 5.2.3. We consider the space $\text{map}^{\partial}(P, \mathcal{Z}_P)$ (resp. $\text{map}^{\partial}(P, M)$) of maps $\{f : P \to \mathcal{Z}_P\}$ (resp. $\{f : P \to M\}$), subject to the following boundary conditions:

- The vertices of P are mapped to their corresponding tori in $\mathcal{Z}_P$ (resp. to the corresponding T^n-fixed points of M)
- A face $\sigma \subset P$ is mapped into the facial submanifold $\mathcal{Z}_\sigma$ of $\mathcal{Z}_P$ (resp. M)

Remark 5.2.4. Since every equivariant map is completely determined by its action on the orbit representatives, once we identify P with the embedded submanifold of $\mathcal{Z}_P$, every element $f \in \text{map}^{\partial}(P, \mathcal{Z}_P)$ gives rise to an equivariant map f which does not permute the tori lying over the vertices of P. Conversely, every equivariant map $f \in \text{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ not permuting the tori over the vertices, defines a map $f \in \text{map}^{\partial}(P, \mathcal{Z}_P)$.

Example 5.2.5. In this example we consider the space $\mathrm{map}_{T^1}(\mathbb{C}P^1, \mathbb{C}P^1)$. It turns out that it admits an especially nice description. To begin with, we can identify the orbit space of the T^1-action on $\mathbb{C}P^1$ with a great semi-circle connecting the north pole N and the south pole S inside of $\mathbb{C}P^1$. Let us call this segment I. Any T^1-equivariant map $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ is determined by its values on the orbit representatives inside of I. Thus we can think of f as a map from I to $\mathbb{C}P^1$. Equivariance forces f to map fixed points to fixed points, i.e. the poles must be sent to the poles. So

$$\mathrm{map}_{T^1}(\mathbb{C}P^1, \mathbb{C}P^1) = \left\{ f \in \mathrm{map}(I, \mathbb{C}P^1) : f(\partial I) \subset \partial I \right\}$$

This shows that $\mathrm{map}_{T^1}(\mathbb{C}P^1, \mathbb{C}P^1)$ consists of four connected components: two components where both poles $\{N, S\}$ are sent respectively to the north pole N or the south pole S; one component where N is sent to N and S is sent to S; and the last component where the poles are interchanged via an equatorial reflection. Each of these components clearly has the homotopy type of $\Omega\mathbb{C}P^1$. So we can conclude that

$$\mathrm{map}_{T^1}(\mathbb{C}P^1, \mathbb{C}P^1) \simeq \coprod_1^4 \Omega\mathbb{C}P^1$$

The component that interests us the most in what follows, is the one where the poles are fixed. We note that it is identity component of the topological monoid $\mathrm{map}_{T^1}(\mathbb{C}P^1, \mathbb{C}P^1)$.

Taking the point of view of $\mathrm{map}_{T^1}(\mathbb{C}P^1, \mathbb{C}P^1)$ as a homotopy inverse limit, in view of proposition 5.1.5 we obtain a homotopy pullback diagram

$$\begin{array}{ccc} \mathrm{map}_{T^1}(\mathbb{C}P^1, \mathbb{C}P^1) & \longrightarrow & \{N, S\} \\ \downarrow & & \downarrow \\ \{N, S\} & \longrightarrow & \mathbb{C}P^1 \end{array}$$

Using the standard model of the homotopy pullback we get

$$\mathrm{map}_{T^1}(\mathbb{C}P^1, \mathbb{C}P^1) = \left\{ (a, \omega, b) \in \{N, S\} \times (\mathbb{C}P^1)^I \times \{N, S\} : \omega(N) = a, \omega(S) = b \right\}$$

This gives the same four connected components as described in the first half of this example.

Remark 5.2.6. Though the homotopy type of the identity component of the topological monoid $\mathrm{map}_{T^1}(\mathbb{C}P^1, \mathbb{C}P^1)$ is ΩS^2, it can be shown that as a loop space it is isomorphic to $S^1 \times \Omega S^3$

5.3 A Fibration

A torus equivariant self-map f of $\mathcal{Z}_P$ or M gives, after passing to the orbit space, a self map $\bar{f}$ of the image polytope. By equivariance, $\bar{f}$ must map vertices to vertices, thus giving a self-map of the vertex set of the polytope.

Definition 5.3.1. Define $\mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ to be the space of torus equivaraint self maps of $\mathcal{Z}_P$ which descend to the identity map of the vertex set of P. Likewise, the space $\mathrm{map}^1_{T^n}(M, M)$ is defined to be the space of torus equivariant self-maps of M which descend to the identity map of the vertex set of P.

In 5.2.5 we saw that an equivariant self map of a quasitoric manifold M may descend to map on a polytope P that permutes or collapses the vertices. This is not the case when we are looking at equivariant selfmaps of a moment angle complex $\mathcal{Z}_P$, as the following proposition shows.

Proposition 5.3.2. *Any T^m-equivariant map of $\mathcal{Z}_P$ descends to a vertex preserving map of the image polytope P, that is*

$$\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P) = \mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$$

Proof. Let a and b be two vertices of P. Assume, for a contradiction, that the map on P induced from f takes a to b. Then f must send the T^m-orbit over a to the T^m-orbit over b. Let x and y be two points in the T^m-orbits over a and b respectively such that $f(x) = y$. By construction of $\mathcal{Z}_P$ it is evident that the stabilizers T_a, T_b of the orbits over, respectively, a and b are distinct. Let t be an element of T_a that is not in T_b. Then

$$f(x) = f(t \cdot x) = t \cdot f(x) = t \cdot y$$

Since t must move y we obtain a contradiction to $f(x) = y$. Thus f must leave the vertices fixed. $\square$

It turns out that the equivariance of the maps in question together with prescribed behavior on the vertices forces these maps to be quite rigid. In proposition below we use the notation of Theorem 3.3.7:

$$H_T^*(M) = \frac{\mathbb{Z}[v_1, \ldots, v_m]}{I}$$

$$H^*(M) = \frac{\mathbb{Z}[v_1, \ldots, v_m]}{I + J}$$

$$H^*(BT) = \mathbb{Z}[c_1, \ldots, c_n]$$

where I is the Stanley-Reisner ideal of the image polytope, and J is the ideal generated by linear forms $\theta_1, \ldots, \theta_n$ given by

$$\theta_i = \lambda_{i1} v_1 + \ldots + \lambda_{im} v_m$$

The vectors $\boldsymbol{\lambda}_j = (\lambda_{1j}, \ldots, \lambda_{nj})$ are the characteristic vectors of the manifold M (c.f. definition 2.1.1). The maps i^* and p^* are given by

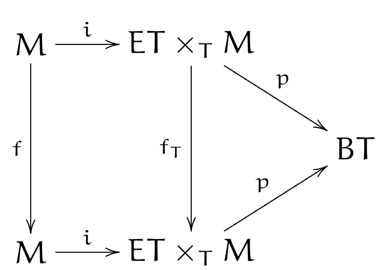
$$\begin{aligned} i^*(v_i) &= v_i \\ p^*(c_i) &= \theta_i \end{aligned}$$

Proposition 5.3.3. *If* $f \in \mathrm{map}^1_{T^n}(M, M)$*, then the induced map* $f^* : H^*(M) \to H^*(M)$ *is the identity map.*

Proof. Since the ring $H^*(M)$ is generated by degree two elements, it suffices to show that

$$f^* : H^2(M) \to H^2(M)$$

is the identity map. We consider the commutative diagram



where the maps i and p are, respectively, the inclusion of the fiber and the projection maps of the obvious fibration. Passing to cohomology we get the following diagram:

$$\begin{array}{ccccc} H^*(M) & \xleftarrow{i^*} & H_T^*(M) & & \\ \uparrow f^* & & \uparrow f_T^* & \nwarrow p^* & \\ & & & & H^*(BT) \\ & & & \swarrow p^* & \\ H^*(M) & \xleftarrow{i^*} & H_T^*(M) & & \end{array}$$

We first observe that the map

$$f^* : H^2(M) \to H^2(M)$$

must be a diagonal map, that is, its matrix A will be of the form

$$f^* = A = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_m \end{pmatrix}$$

This is because the generators $v_1, \ldots, v_m$ of $H^2(M)$ are cohomology classes dual to facial submanifolds M^{2n-2} of M. Since the map on the polytope induced from f preserves the vertices and hence the facets of P, f must map facial submanifolds, lying over the facets, to themselves. This forces the map

$$f_T^* : H_T^2(M) \to H_T^2(M)$$

to also be the diagonal map

$$f_T^* = A$$

The map

$$p^* : H^2(BT) \to H_T^2(M)$$

is given by the matrix

$$\Lambda = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{n1} \\ \vdots & & \vdots \\ \lambda_{1m} & \cdots & \lambda_{nm} \end{pmatrix}$$

Commutativity of the diagram then says that

$$A\Lambda = \Lambda$$

that is

$$\alpha_j \lambda_{ij} = \lambda_{ij}$$

for all i and j. Since $\lambda_{ij} \neq 0$ for at least one i, $\alpha_j = 1$ for all j, and hence $A = \mathrm{id}$ □

Corollary 5.3.4. *The maps* $f \in \mathrm{map}^1_{T^n}(M, M)$ *are self homotopy equivalences*

Proof. Given an equivariant self map of M fixing the vertices, the induced map on cohomology is identity. By the universal coefficient theorem, the induced map on homology is also identity. Since M is simply connected, Whitehead's theorem gives the result. □

Remark 5.3.5. The above result shows that equivariance and fixing of vertices imposes a kind of a rigidity on the mapping space in question. Namely, all the elements are forced to be homotopy equivalences. This makes the problem of counting connected components of $\mathrm{map}^1_{T^n}(M, M)$ more subtle, as we will not be able to use standard invariants such as homology and homotopy to distinguish homotopy classes of maps. Thus a more sensitive way of distinguishing connected components is desired.

In view of what was said in remark 5.2.4, we have:

Proposition 5.3.6. *There are homeomorphisms*

$$\mathrm{map}^{\partial}(P, \mathcal{Z}_P) \approx \mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$$

$$\mathrm{map}^{\partial}(P, M) \approx \mathrm{map}^1_{T^n}(M, M)$$

Remark 5.3.7. In the case when $M = \mathbb{C}P^n$, the space $\mathrm{map}^1_{T^n}(M, M)$ is 0-connected, as we will see later. It is interesting to see what can be said of connectivity of $\mathrm{map}^1_{T^n}(M, M)$ for general M.

We will now relate the spaces $\mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ and $\mathrm{map}^1_{T^n}(M, M)$ by means of a more or less obvious fibration. Namely, we can construct a map

$$\pi : \mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P) \to \mathrm{map}^1_{T^n}(M, M)$$

as follows:
Let $r = m - n$. We start with the principal T^r-bundle (See proposition 3.2.1)

$$T^r \to \mathcal{Z}_P \xrightarrow{p} M$$

For $f \in \mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ and $x \in M$, define

$$(\pi f)(x) := p(f(z))$$

where z is some point in $p^{-1}(x)$.

Proposition 5.3.8. $\pi : \mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P) \to \mathrm{map}^1_{T^n}(M, M)$ *is a well defined surjective monoid homomorphism with kernel homotopy equivalent to* T^r

Proof. (a) where we prove that π is well defined:
Let z and z' both lie in $p^{-1}(x)$. Since z and z' are in the same orbit over x, there exists $t \in T^r$ such that $z' = t \cdot z$. Thus,

$$\begin{aligned} (\pi f)(x) &= p(f(z')) \\ &= p(f(t \cdot z)) \\ &= p(t \cdot f(z)) \\ &= p(f(z)) \end{aligned}$$

(b) where we prove that πf is T^n-equivariant:
We view T^n as T^m/T^r, and the action of T^n on M as the residual action of T^m/T^r on $\mathcal{Z}_P/T^r$. Let x be a point in M and take z in $\mathcal{Z}_P$ such that $p(z) = x$. Let $[t] \in T^n = T^m/T^r$ be the image of $t \in T^m$ under the natural projection. We note that $p(t \cdot z) = [t] \cdot x$, so that

$$\begin{aligned} (\pi f)([t] \cdot x) &= p(f(t \cdot z)) \\ &= p(t \cdot f(z)) \\ &= [t] \cdot p(f(z)) \\ &= [t] \cdot (\pi f)(x) \end{aligned}$$

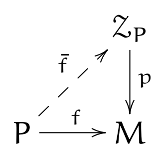
(c) where we prove that π is a monoid homomorphism:

Let $f, g \in \mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P), x \in M$ and $z \in p^{-1}(x)$.

$$\begin{aligned}(\pi f) \circ (\pi g)(x) &= (\pi f)(p(g(z))) \\ &= p(f(g(z))) \\ &= p(f \circ g(z)) \\ &= \pi(f \circ g)(x)\end{aligned}$$

(d) where we prove that π is surjective:

To prove this claim we consider an equivariant map $f \in \mathrm{map}^1_{T^n}(M, M)$ as an element of the space $\mathrm{map}^\partial(P, M)$. The question of surjectivity will reduce to finding a lift



and checking that it satisfies the boundary conditions.

The existence of the lift is established by noticing that $p : \mathcal{Z}_P \to M$ is a Serre fibration, so any map from a topological disk into M has a lift. To check that the boundary conditions are satisfied we must show that for a face $\sigma \in P$, $\bar{f}(\sigma)$ lies in the facial submanifold of $\mathcal{Z}_P$ over σ. To this end, we first note that $f(\sigma) \subset M_\sigma$, where M_σ is the facial sibmanifold of M over σ. Then, by the commutativity of the diagram, $p\bar{f}(\sigma) \subset M_\sigma$. This forces $\bar{f}(\sigma)$ to lie over M_σ, and hence over σ. Thus $\bar{f} \in \mathrm{map}^\partial(P, \mathcal{Z}_P)$, so it is T^m-equivariant. It remains to show that $\pi\bar{f} = f$
Let $x \in M$. We recall that the orbit space P can be viewed as a submanifold of M. There exists an orbit representative $a \in P \subset M$ of x. That is, there is some

$t \in T^n$ such that $t \cdot a = x$. Thus,

$$\begin{aligned}(\pi\bar{f})(x) &= (\pi\bar{f})(t \cdot a) \\ &= t \cdot (\pi\bar{f})(a) \\ &= t \cdot p\bar{f}(a) \\ &= t \cdot f(a) \\ &= f(t \cdot a) \\ &= f(x)\end{aligned}$$

(e)where we prove that $\ker \pi \simeq T^r$
Suppose $\pi f = \mathrm{id}_M$. Pick any $x \in M$ and take $z \in p^{-1}(x)$. Then

$$(\pi f)(x) = pf(z) = x$$

implies that

$$f(z) = t \cdot z$$

for some $t \in T^r$. This means that f maps every orbit $a \in P$ to itself, turning it by some element of the torus $t(a) \in T^r$. So f is completely determined by the map $t : P \to T^r$, and hence

$$\ker \pi \simeq \mathrm{map}(P, T^r) \simeq T^r$$

This completes the proof of the proposition.

$\square$

Proposition 5.3.9. *There is a retraction* η *for a short exact sequence of topological monoids*

$$1 \longrightarrow T^r \xrightarrow[i]{} \mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P) \xrightarrow{\pi} \mathrm{map}^1_{T^n}(M, M) \longrightarrow 1$$

Proof. Pick a vertex σ of P. Let T_σ be the stabilizer of the T^m-orbit over σ in $\mathcal{Z}_P$. The action of T^r on $\mathcal{Z}_P$ is free, so the intersection $T^r \cap T_\sigma$ consists of only the identity element. Because of this, there is a splitting

$$T^m = T_\sigma \times T^r$$

with

$$\rho : T^m \to T^r$$

the projection map.

Let

$$f \in \mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$$

be given. We define η as follows:

By construction of $\mathcal{Z}_P$ the orbit over σ is just the quotient T^m/T_σ. Since f fixes the orbits over the vertices, it must map T^m/T_σ to itself T^m-equivariantly. Take $1 \in T^m/T_\sigma$ and define

$$\eta(f) := \rho(f(1))$$

To show η is a retraction we must see that

$$\eta \circ i = \mathrm{id}_{T^r}$$

Take $t \in T^r$ and let $i(t) = h \in \mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$. Up to homotopy we may assume that for $z \in \mathcal{Z}_P$,

$$h(z) = t \cdot z$$

Then

$$(\eta \circ i)(t) = \eta(h) = \rho(h(1)) = \rho(t \cdot 1) = \rho(t) = t$$

This proves η is indeed a retraction.

Verifying that η is a monoid homomorphism is completely straight-forward: Take $f, g \in \mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$. Denote by t_f and t_g the images of 1 in T^m/T_σ under f and g respectively. By equivariance we then have

$$\begin{aligned}
\eta(fg) &= \rho(f(g(1))) \\
&= \rho(f(t_g)) \\
&= \rho(t_g \cdot f(1)) \\
&= \rho(t_g \cdot t_f) \\
&= t_f \cdot t_g \\
&= \rho(t_f) \cdot \rho(t_g) \\
&= \rho(f(1)) \cdot \rho(g(1)) \\
&= \eta(f) \cdot \eta(g)
\end{aligned}$$

This completes the proof of the proposition. □

Corollary 5.3.10. *There is an isomorphism of topological monoids*

$$\mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P) \cong T^r \times \mathrm{map}^1_{T^n}(M, M)$$

Proof. The isomorphism is given by the map

$$f \mapsto (\eta(f), \pi(f))$$

The proof that the map above is injective is the same as in the case of groups. The only subtle point appears in the proof of surjectivity which we now provide.

Fix $(t, g) \in T^r \times \mathrm{map}^1_{T^n}(M, M)$. Pick $f \in \mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ such that $\pi(f) = g$. It is easy to see that the element

$$f \circ i(\eta(f)^{-1} \cdot t) \in \mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$$

is mapped to (t, g) under (η, π).

We note that above we have taken an inverse of $\eta(f)$. This is perfectly fine as the kernel of π is an honest group, whereas the spaces $\mathrm{map}^1_{T^n}(M, M)$ and $\mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ are only monoids. □

Corollary 5.3.11. *Fixing compatible basepoints, the homotopy groups of* $\mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ *and* $\mathrm{map}^1_{T^n}(M, M)$ *are related as follows:*

$$\begin{aligned} \pi_1(\mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)) &\cong \mathbb{Z}^r \times \pi_1(\mathrm{map}^1_{T^n}(M, M)) \\ \pi_j(\mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)) &\cong \pi_j(\mathrm{map}^1_{T^n}(M, M)) \text{ for } j > 1 \end{aligned}$$

5.4 Equivariant Mapping Spaces of Complex Projective Spaces

In this section we use the homotopy inverse limit decomposition of the equivariant mapping spaces $\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ and $\mathrm{map}_{T^n}(M, M)$ to compute its rational homotopy groups using the Bousfield-Kan spectral sequence.

For the remainder of this section we let $M = \mathbb{C}P^n$ with the usual T^n-action. We let P be the orbit polytope which is just the n-simplex Δ^n. The following fact can be found in [2, p87]

Proposition 5.4.1. *The moment angle complex* $\mathcal{Z}_P$ *corresponding to* $P = \Delta^n$ *is homeomorphic to* S^{2n+1}.

The first order of business will be to settle the question of components of the space $\mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$. As the result of Proposition 5.3.6 we may as well work with the space $\mathrm{map}^\partial(P, \mathcal{Z}_P)$. Let P^i denote the i-skeleton of the n-simplex P. We can consider maps

$$\mathrm{res}_i : \mathrm{map}^\partial(P^{i+1}, \mathcal{Z}_P) \to \mathrm{map}^\partial(P^i, \mathcal{Z}_P)$$

which are simply the restrictions to the i-skeleton of P.

Lemma 5.4.2. *The maps* $\mathrm{res}_i : \mathrm{map}^\partial(P^{i+1}, \mathcal{Z}_P) \to \mathrm{map}^\partial(P^i, \mathcal{Z}_P)$ *are fibrations.*

Proof. Let σ be a maximal face of P^{i+1} and $\mathcal{Z}_\sigma$ the corresponding facial submanifold of $\mathcal{Z}_P$. The inclusion

$$P^i \cap \sigma \hookrightarrow P^{i+1} \cap \sigma$$

is a cofibration. Therefore the map

$$\mathrm{map}(P^{i+1} \cap \sigma, \mathcal{Z}_\sigma) \to \mathrm{map}(P^i \cap \sigma, \mathcal{Z}_\sigma)$$

is a fibration. The map res_i fits into the pullback diagram below, and is hence a fibration

$$\begin{array}{ccc} \mathrm{map}^\partial(P^{i+1} \cap \sigma, \mathcal{Z}_P) & \longrightarrow & \mathrm{map}(P^{i+1} \cap \sigma, \mathcal{Z}_\sigma) \\ \downarrow{\scriptstyle \mathrm{res}_i} & & \downarrow \\ \mathrm{map}^\partial(P^i \cap \sigma, \mathcal{Z}_P) & \hookrightarrow & \mathrm{map}(P^i \cap \sigma, \mathcal{Z}_\sigma) \end{array}$$

From here it is not hard to see that the map

$$\mathrm{res}_i : \mathrm{map}^\partial(P^{i+1}, \mathcal{Z}_P) \to \mathrm{map}^\partial(P^i, \mathcal{Z}_P)$$

is also a fibration.

$\square$

Proposition 5.4.3. *The spaces* $\mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ *and* $\mathrm{map}^1_{T^n}(M, M)$ *are 0-connected, and are therefore the identity components of the respective H-spaces* $\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ *and* $\mathrm{map}_{T^n}(M, M)$

Proof. The spaces $\mathrm{map}^\partial(P^i, \mathcal{Z}_P)$ fit into a tower of fibrations:

$$\begin{array}{ccc}
X_n & \longrightarrow & \mathrm{map}^\partial(P^n, \mathcal{Z}_P) \\
 & & \downarrow {\scriptstyle \mathrm{res}_{n-1}} \\
X_{n-1} & \longrightarrow & \mathrm{map}^\partial(P^{n-1}, \mathcal{Z}_P) \\
 & & \downarrow {\scriptstyle \mathrm{res}_{n-2}} \\
 & & \vdots \\
 & & \downarrow \\
X_2 & \longrightarrow & \mathrm{map}^\partial(P^2, \mathcal{Z}_P) \\
 & & \downarrow {\scriptstyle \mathrm{res}_{1}} \\
X_1 & \longrightarrow & \mathrm{map}^\partial(P^1, \mathcal{Z}_P) \\
 & & \downarrow {\scriptstyle \mathrm{res}_{0}} \\
 & & \mathrm{map}^\partial(P^0, \mathcal{Z}_P)
\end{array}$$

We proceed by induction on the skeleta. The space $\mathrm{map}^\partial(P^0, \mathcal{Z}_P)$ is homeomorphic to $(T^1)^{n+1}$, and so is connected. Let us assume that $\mathrm{map}^\partial(P^i, \mathcal{Z}_P)$ is connected. We examine the fiber X_{i+1} of the fibration

$$X_{i+1} \to \mathrm{map}^\partial(P^{i+1}, \mathcal{Z}_P) \to \mathrm{map}^\partial(P^i, \mathcal{Z}_P)$$

Fix a maximal face σ of P^{i+1}. Since σ is maximal it must be an $(i+1)$-simplex. Let f be a map in $\mathrm{map}^\partial(P^i \cap \sigma, \mathcal{Z}_P)$. The boundary conditions force f to map each i-simplex of $\partial\sigma$ to the corresponding facial submanifold of $\mathcal{Z}_P$. These facial submanifolds are just the coordinate spheres S^{2i+1}. The fiber over f of the map

$$\mathrm{map}^\partial(P^{i+1} \cap \sigma, \mathcal{Z}_P) \to \mathrm{map}^\partial(P^i \cap \sigma, \mathcal{Z}_P)$$

is the space of all maps from σ to $\mathcal{Z}_P$ that restrict to f and satisfy our boundary conditions. The boundary conditions require that the image of σ lie in its facial submanifold, which is homeomorphic to S^{2i+3}. Thus the fiber is the space of maps from an $(i+1)$-simplex to S^{2i+3} that restrict to a given map f on the boundary. This space is homotopy equivalent to $\Omega^{i+1}S^{2i+3}$.

Now, if we are given a map $g \in \mathrm{map}^{\partial}(P^i, \mathcal{Z}_P)$, we can extend it independently over each maximal face of P^{i+1}, and so the fiber is just the product of iterated loop spaces on spheres, with as many factors as there are $(i+1)$-simplices, i.e.

$$X_{i+1} \simeq \underbrace{\Omega^{i+1}S^{2i+3} \times \ldots \times \Omega^{i+1}S^{2i+3}}_{\binom{n}{i+2} times}$$

Since $\pi_{i+1}(S^{2i+3}) = 0$, the space X_{i+1} is connected. Applying the homotopy long exact sequence to

$$X_{i+1} \to \mathrm{map}^{\partial}(P^{i+1}, \mathcal{Z}_P) \to \mathrm{map}^{\partial}(P^i, \mathcal{Z}_P)$$

and looking at the π_0 terms, proves the proposition for $\mathrm{map}^{\partial}(P, \mathcal{Z}_P)$. The result for $\mathrm{map}^{\partial}(P, M)$ follows by applying the homotopy long exact sequence to the fibration in Proposition 5.3.8.

□

Remark 5.4.4. We note that extending a map from the boundary of a simplex to the whole simplex in the previous proposition is possible precisely because of high connectivity of S^{2n+1}.

A consequence of the above proposition is this

Corollary 5.4.5. *Any torus equivariant map of $\mathbb{C}P^n$ not permuting the fixed points is equivariantly homotopic to the identity.*

We recall from Proposition 5.1.5 that the equivariant mapping space $\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ is the homotopy inverse limit of the fixed point functor

$$\mathsf{Fix}_P : P \to \mathrm{Top}$$

We need to understand what the values of Fix_P are when P is the simplex category. Let σ be a k-dimensional face of P. The value of the functor Fix_P on σ is given by

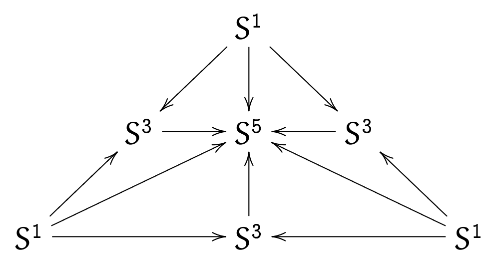
$$\mathsf{Fix}_P(\sigma) = \mathcal{Z}_P^{T_\sigma}$$

It is the fixed submanifold of $\mathcal{Z}_P$ under the action of the stabilizer torus of σ, T_σ. But T_σ is just the coordinate subtorus, with coordinates corresponding to the $n-k$ facets of P which intersect in σ. Thus, the fixed points of the $(2n+1)$-sphere $\mathcal{Z}_P$ under the action of T_σ is some coordinate $(2k+1)$-sphere. So in fact

$$\mathsf{Fix}_P(\sigma) \approx S^{2|\sigma|+1}$$

It is also not hard to see that if $\sigma \subset \tau$ is an inclusion of faces, then Fix_P induces the inclusion of fixed submanifolds $\mathcal{Z}_P^{T_\sigma} \subset \mathcal{Z}_P^{T_\tau}$

Example 5.4.6. When $P = \Delta^2$, Fix_P applied to the category P gives the following diagram in Top



Before we begin to apply the machinery of the Bousfield-Kan spectral sequence, we will need a prelimenary algebraic result. The set up is as follows:

Let $P = \Delta^n$, and P be the associated simplex category. Let R be a commutative ring with unity and $\Pi : P^{op} \to R\text{-Mod}$ be a functor into the category of R-modules defined by

$$\Pi(\sigma) = \begin{cases} R & \text{if } \sigma = [0, 1, \dots, n] \\ 0 & \text{else} \end{cases}$$

Lemma 5.4.7. *The* i*-th derived functor of the inverse limit of* Π *is given by:*

$$\lim^i_{P^{op}} \Pi = \begin{cases} R & \textit{if } i = n \\ 0 & \textit{else} \end{cases}$$

Proof. The idea behind the proof will be to construct an isomorphism

$$\lim^i_{P^{op}} \Pi \cong \widetilde{H}^{i-1}_{simp}(\mathrm{bs}(\partial P); R) \text{ for } i > 0$$

The term on the right hand side is the reduced simplicial cohomology of the barycentric subdivision of the boundary of P.

Let us describe the chain complex

$$0 \to C_0 \overset{d_0}{\to} C_1 \overset{d_1}{\to} C_2 \to \ldots \to C_n \to 0$$

whose i-th homology computes the groups $\lim^i_{\mathcal{P}^{op}} \Pi$. [1] The generators of the free R-module C_k are elements

$$(\sigma_k \subsetneq \ldots \subsetneq \sigma_0 = P)$$

with each σ_k a face of P.

For $k > 0$ the differential

$$d_k : C_k \to C_{k+1}$$

is given by

$$d_k(\sigma_{i_k} \subsetneq \ldots \subsetneq \sigma_{i_0} = P) = \sum_{j=1}^{k+1}(-1)^j \sum_{(r_{k+1},\ldots,r_0)} (\sigma_{r_{k+1}} \subsetneq \ldots \subsetneq \sigma_{r_j} \subsetneq \ldots \subsetneq \sigma_{r_0} = P) \tag{5.1}$$

where the second sum is taken over all tuples $(r_{k+1}, \ldots, r_j, \ldots, r_0)$ such that

$$(\sigma_{r_{k+1}} \subsetneq \ldots \widehat{\sigma}_{r_j} \subsetneq \ldots \subsetneq \sigma_{r_0}) = (\sigma_{i_k} \subsetneq \ldots \subsetneq \sigma_{i_0})$$

For $k = 0$,

$$d_0(\sigma_0 = P) = -\sum_{\sigma} (\sigma \subsetneq \sigma_0)$$

Since $\ker d_0 = 0$, this shows that $\lim^0_{\mathcal{P}^{op}} \Pi = 0$. In the future it will be helpful to think of $d_0 : C_0 \to C_1$ as a coaugmentation map.

For $k \geqslant 1$, the isomorphism

$$C_k \cong C^{k-1}(bs(\partial P); R)$$

is simply given by

$$(\sigma_{i_k} \subsetneq \ldots \subsetneq \sigma_{i_0}) \mapsto f^{[\sigma_{i_k},\ldots,\sigma_{i_1}]}$$

[1] For a brief discussion of higher derived functors of inverse limits and a way to compute them the reader is referred to [10, p86]

Here, $[\sigma_{i_k}, \dots, \sigma_{i_1}]$ is a $(k-1)$-face of $bs(\partial P)$, where we think of every face σ_{i_j} of ∂P as a vertex in $bs(\partial P)$. The cochain $f^{[\sigma_{i_k},\dots,\sigma_{i_1}]}$ takes the value 1 on the face $[\sigma_{i_k}, \dots, \sigma_{i_1}]$ and 0 elsewhere.

It remains to check that the differential d_k coincides with the simplicial coboundary operator δ^{k-1}. To this end we note that

$$\delta(f^{[\sigma_{i_k},\dots,\sigma_{i_1}]}) = \sum_{j=1}^{k+1}(-1)^{j-1} \sum_{(r_{k+1},\dots,r_1)} f^{[\sigma_{r_{k+1}},\dots,\sigma_{r_j},\dots,\sigma_{r_1}]}$$

where the second sum is taken over all $(r_{k+1}, \dots, r_1)$ such that

$$[\sigma_{i_k}, \dots, \sigma_{i_1}] = [\sigma_{r_{k+1}}, \dots, \widehat{\sigma}_{r_j}, \dots, \sigma_{r_1}]$$

But under our identification of C_k with $C^{k-1}(bs(\partial P); R)$, this is just the differential d_k from (5.1) but with opposite sign.

This shows that for $i \geqslant 1$ there are isomorphisms

$$\lim^i_{P^{op}} \Pi \cong \widetilde{H}^{i-1}_{simp}(bs(\partial P); R)$$

Since ∂P is a simplicial sphere, the result follows. □

Remark 5.4.8. We note that in the course of the proof of the previous proposition we have not used anywhere that P is an n-simplex. As long as the polytope P has a polyhedral sphere as the boundary, and the functor Π is modified accordingly, the lemma remains true.

In the next proposition $\pi_j^{\mathbb{Q}}$ will denote the j-th rational homotopy group, i.e. $\pi_j^{\mathbb{Q}} \cong \pi_j \otimes \mathbb{Q}$

Proposition 5.4.9. *The Bousfield-Kan spectral sequence for* $\operatorname{holim}_{P^{op}} \mathsf{Fix}_P$ *collapses at the* E_2*-page*

Proof. The E_2^{ij} term of the spectral sequence is given by

$$E_2^{ij} = \lim^i_{P^{op}} \pi_j^{\mathbb{Q}} \circ \mathsf{Fix}_P$$

If σ is a k-dimensional face of P, then

$$\pi_j^{\mathbb{Q}}(\mathsf{Fix}_P(\sigma)) = \pi_j^{\mathbb{Q}}(S^{2k+1}) = \begin{cases} \mathbb{Q} & \text{if } j = 2k+1 \\ 0 & \text{else} \end{cases}$$

Since there is no interaction between the faces, by Lemma 5.4.7, each k-face is going to contribute a factor of $\mathbb{Q}$ to the i-th derived inverse limit. That is:

$$\lim\nolimits^i_{P^{op}} \pi^{\mathbb{Q}}_{2k+1} \circ \mathsf{Fix}_P = \begin{cases} \mathbb{Q}^{\binom{n+1}{k+1}} & \text{if } i = k \\ 0 & \text{else} \end{cases}$$

When j is positive and even, the j-th rational homotopy groups of odd dimensional spheres vanish, therefore, the E_2^{ij} in this case will be zero. We will not concern ourselves with the case of $j = 0$, since the 0-th row of the E_2-page cannot be hit by any differentials, which have bidegree $(r+1, r)$. This omission is inconsequential, as we already know that the spaces $\mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ and $\mathrm{map}^1_{T^n}(M, M)$ are connected. In view of the above calculations, we can now fill in the relevant portions of the E_2 page of the Bousfield-Kan spectral sequence as done in Figure 6 below. Since the r-th differential has bidegree $(r, r-1)$, it must be trivial and

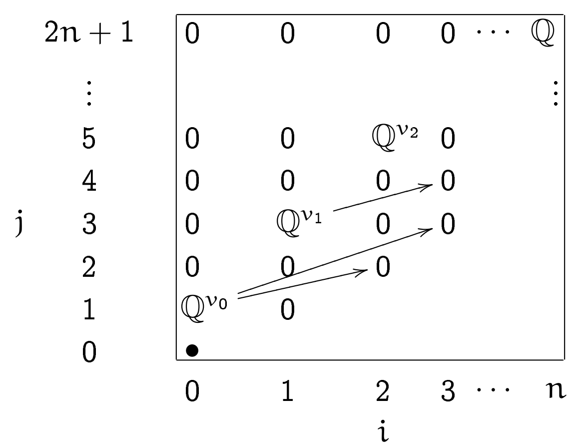


Figure 5.1: The E_2-page of the Bousfield-Kan spectral sequence. Here ν_k denote the number of k dimensional faces in the n-simplex P

hence the spectral sequence collapses at the E_2 page. □

Proposition 5.4.10. $\mathrm{rank}(\pi_k(\mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P))) = \binom{n+1}{k}$

Proof. Because the Bousfield-Kan spectral sequence collapses at the E_2-page, it satisfies the conditions of lemma A.4.2 with $N = 2$, and so

$$E_2^{ij} \Rightarrow \pi_{j-i}(\operatorname{holim}_{P^{op}} \mathsf{Fix}_P) \text{ for } j - i > 0$$

Recall that in view of proposition 5.1.5

$$\operatorname{holim}_{P^{op}} \mathsf{Fix}_P = \mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$$

and by proposition 5.3.2

$$\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P) = \mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$$

with $\mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ having exactly one component by 5.4.3. The E_∞-page of the Bousfield-Kan spectral sequence obtained in the previous proposition (see figure), then says that the homotopy groups of this component are given by

$$\pi_k^{\mathbb{Q}}(\mathrm{map}^1_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)) = \mathbb{Q}^{\binom{n+1}{k}}$$

which completes the proof of the proposition. □

We state our results so far as a

Proposition 5.4.11. *The spaces* $\mathrm{map}^1_{T^{n+1}}(S^{2n+1}, S^{2n+1})$ *and* $\mathrm{map}^1_{T^n}(\mathbb{C}P^n, \mathbb{C}P^n)$ *are connected. Their homotopy groups are related as follows:*

(a) $\pi_i(\mathrm{map}^1_{T^{n+1}}(S^{2n+1}, S^{2n+1})) \cong \pi_i(\mathrm{map}^1_{T^n}(\mathbb{C}P^n, \mathbb{C}P^n))$ *for* $i > 1$

(b) $\pi_1(\mathrm{map}^1_{T^{n+1}}(S^{2n+1}, S^{2n+1})) \cong \mathbb{Z} \oplus \pi_1(\mathrm{map}^1_{T^n}(\mathbb{C}P^n, \mathbb{C}P^n))$

Moreover,

$$\mathrm{rank}(\pi_i(\mathrm{map}^1_{T^{n+1}}(S^{2n+1}, S^{2n+1}))) = \begin{cases} \binom{n+1}{i} & \text{if } 1 \leqslant i \leqslant n+1 \\ 0 & \text{otherwise} \end{cases}$$

5.5 Equivariant Mapping Spaces of Rationally Elliptic Quasitoric Manifolds

In this section we will extend proposition 5.4.11 to the case of an arbitrary rationally elliptic manifold. We begin with the following

Proposition 5.5.1. *Let $\mathcal{Z}_P$ and $\mathcal{Z}_Q$ be moment angle complexes over P an Q, respectively. Let T_P and T_Q be the tori acting, respectively, on $\mathcal{Z}_P$ and $\mathcal{Z}_Q$. Then*

$$\mathrm{map}_{T_P \times T_Q}(\mathcal{Z}_P \times \mathcal{Z}_Q, \mathcal{Z}_P \times \mathcal{Z}_Q) \simeq \mathrm{map}_{T_P}(\mathcal{Z}_P, \mathcal{Z}_P) \times \mathrm{map}_{T_Q}(\mathcal{Z}_Q, \mathcal{Z}_Q)$$

Proof. The space $\mathcal{Z}_P \times \mathcal{Z}_Q$ is a moment angle complex corresponding to the polytope $P \times Q$. The faces of $P \times Q$ are simply the products of the faces of P and Q. Given such a face $\sigma \times \tau \subset P \times Q$, there is this identity between the stabilizers

$$T_{\sigma \times \tau} = T_\sigma \times T_\tau$$

By 5.1.2, we may view the product of moment angle complexes as a homotopy colimit over the product category

$$\mathcal{Z}_P \times \mathcal{Z}_Q = \underset{\sigma \times \tau \in P \times Q}{\mathrm{hocolim}}\, \mathrm{Orb}_{P \times Q}(\sigma \times \tau)$$

with functor $\mathrm{Orb}_{P \times Q}$ given by

$$\mathrm{Orb}_{P \times Q}(\sigma \times \tau) = \frac{T_P \times T_Q}{T_\sigma \times T_\tau}$$

The result of the proposition now follows formally:

$$
\begin{aligned}
&\operatorname{map}_{T_P\times T_Q}(\mathcal{Z}_P\times\mathcal{Z}_Q,\mathcal{Z}_P\times\mathcal{Z}_Q) &=\\
&\operatorname{map}_{T_P\times T_Q}(\operatorname*{hocolim}_{\sigma\times\tau\in P\times Q}\frac{T_P\times T_Q}{T_\sigma\times T_\tau},\mathcal{Z}_P\times\mathcal{Z}_Q) &=\\
&\operatorname*{holim}_{\sigma\times\tau\in P^{op}\times Q^{op}}\operatorname{map}_{T_P\times T_Q}(\frac{T_P\times T_Q}{T_\sigma\times T_\tau},\mathcal{Z}_P\times\mathcal{Z}_Q) &=\\
&\operatorname*{holim}_{\sigma\times\tau\in P^{op}\times Q^{op}}(\mathcal{Z}_P\times\mathcal{Z}_Q)^{T_\sigma\times T_\tau} &=\\
&\operatorname*{holim}_{\sigma\times\tau\in P^{op}\times Q^{op}}\mathcal{Z}_P^{T_\sigma}\times\mathcal{Z}_Q^{T_\tau} &=\\
&\operatorname*{holim}_{\sigma\in P^{op}}\mathcal{Z}_P^{T_\sigma}\times\operatorname*{holim}_{\tau\in Q^{op}}\mathcal{Z}_Q^{T_\tau} &=\\
&\operatorname{map}_{T_P}(\mathcal{Z}_P,\mathcal{Z}_P)\times\operatorname{map}_{T_Q}(\mathcal{Z}_Q,\mathcal{Z}_Q)
\end{aligned}
$$

□

Combining the result of the above proposition with 5.3.11 and proposition 5.4.11 we obtain:

Proposition 5.5.2. *Let* M *be a quasitoric manifold over* $P=\Delta^{i_1}\times\ldots\times\Delta^{i_k}$ *and* $\mathcal{Z}_P$ *be the corresponding moment angle complex and* $n=i_1+\ldots+i_k$*. The number of facets* m *of* P *is then* $i_1+\ldots+i_k+k$*. The spaces* $\operatorname{map}_{T^m}(\mathcal{Z}_P,\mathcal{Z}_P)$ *and* $\operatorname{map}^1_{T^n}(M,M)$ *are connected. Their homotopy groups are related as follows:*

(a) $\pi_j(\operatorname{map}_{T^m}(\mathcal{Z}_P,\mathcal{Z}_P))\cong\pi_j(\operatorname{map}^1_{T^n}(M,M))$ *for* $j>1$

(b) $\pi_1(\operatorname{map}_{T^m}(\mathcal{Z}_P,\mathcal{Z}_P))\cong\mathbb{Z}^k\oplus\pi_1(\operatorname{map}^1_{T^n}(M,M))$

Moreover,

$$\operatorname{rank}\ \pi_j(\operatorname{map}_{T^m}(\mathcal{Z}_P,\mathcal{Z}_P))=\sum_{r=1}^{k}\binom{i_r+1}{j}$$

Remark 5.5.3. In the above proposition, we use the convention that the binomial coefficient $\binom{a}{b}=0$, whenever $b>a$.

5.6 Homotopy Type of Equivariant Mapping Spaces of Complex Projective Spaces

In this section we will determine the homotopy type of the equivariant mapping spaces of complex projective spaces and their moment angle complexes. It turns out that these spaces are just the products of iterrated loop spaces on odd dimensional spheres. In view of the results of the previous section, this will also give us the homotopy type of the equivariant mapping spaces of quasitoric manifolds and the corresponding moment angle complexes over products of simplices. It will be seen that our calculations of the ranks of the homotopy groups using the machinery of the Bousfield-Kan spectral sequence are consistent with our results in this section. As a corollary we also deduce that when P is a product of simplices, the homotopy type of the space $\mathrm{map}^1_{T^n}(M, M)$ depends only on P. We begin by proving two simple lemmas.

Lemma 5.6.1. *If a fiber sequence*

$$\Omega B \longrightarrow F \underset{\curvearrowleft\, s}{\overset{\pi}{\longrightarrow}} E \longrightarrow B$$

possesses a section s*, so that* $\pi \circ s = \mathrm{id}$*, then there is a fiber homotopy equivalence* $F \simeq E \times \Omega B$

Proof. We form a pullback diagram of the path-loop fibration

$$\begin{array}{ccc} \Omega B & = & \Omega B \\ \downarrow & & \downarrow \\ F & \longrightarrow & PB \\ s \uparrow\downarrow \pi & & \downarrow \\ E & \longrightarrow & B \end{array}$$

Because $\pi \circ s = \mathrm{id}$, and PB is contractible, the map $E \to B$ must be null-homotopic. Therefore, the pull-back fibration is trivial, i.e. $F \simeq E \times \Omega B$. □

Lemma 5.6.2. *If* $X \hookrightarrow Y$ *is inclusion of spaces that is null-homotopic, then the homotopy fiber* F *of this inclusion is* $X \times \Omega Y$

Proof. We loop back the fibration

$$\Omega Y \longrightarrow F \longrightarrow X \longrightarrow Y, \qquad s : X \dashrightarrow F$$

Because $X \hookrightarrow Y$ is null-homotopic, there exists a section $s : X \to F$. By the previous lemma, $F \simeq X \times \Omega Y$ □

Theorem B. *Let* $P = \Delta^n, M = \mathbb{C}P^n, \mathcal{Z}_P = S^{2n+1}$. *Then*

(a) $\mathrm{map}_{T^{n+1}}(\mathcal{Z}_P, \mathcal{Z}_P) \simeq \prod_{i=0}^{n} \prod_{\sigma_k^i} \Omega^i S^{2i+1}$

(b) $\mathrm{map}^1_{T^n}(M, M) \simeq T^n \times \prod_{i=1}^{n} \prod_{\sigma_k^i} \Omega^i S^{2i+1}$

Above, the second product is taken over all k *such that* σ_k^i *is an* i*-dimensional face of* P

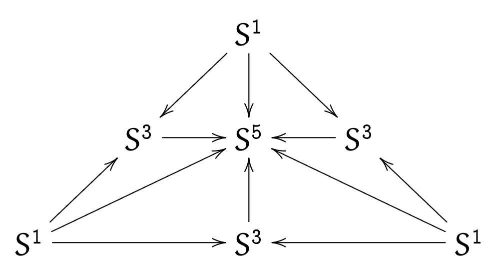
Proof. For the sake of clarity we prove part (a) of the theorem for $n = 1$ and 2. The reader should have no difficulties in using induction to prove this claim for all n.

When $n = 1$ proposition 5.1.5 implies

$$\mathrm{map}_{T^2}(\mathcal{Z}_P, \mathcal{Z}_P) = \mathrm{holim}(S^1 \hookrightarrow S^3 \hookleftarrow S^1)$$

By lemma 5.6.2, the homotopy pullback is $S^1 \times S^1 \times \Omega S^3$. This proves the claim for $n = 1$.

To prove the claim for $n = 2$ we need to determine the homotopy limit of the diagram



To this end, we replace it by a slightly fatter diagram whose homotopy limit is the same. The idea is to pick a vertex, and fatten it up without affecting the homotopy limit.

$$\begin{array}{ccccc} S^1 & \longrightarrow & S^3 & \longleftarrow & S^1 \\ \downarrow & & \downarrow & & \| \\ S^3 & \longrightarrow & S^5 & \longleftarrow & S^1 \\ \uparrow & & \uparrow & & \| \\ S^1 & \longrightarrow & S^3 & \longleftarrow & S^1 \end{array}$$

We can now take the homotopy limit of the vertical columns obtaining a pullback dagram

$$S^1 \times S^1 \times \Omega S^3 \to S^3 \times S^3 \times \Omega S^5 \leftarrow S^1$$

Since the map $S^1 \to S^3 \times S^3 \times \Omega S^5$ is null-homotopic, the homotopy pullback is given by

$$S^1 \times S^1 \times S^1 \times \Omega S^3 \times \Omega S^3 \times \Omega S^3 \times \Omega^2 S^5$$

This proves the claim for $n = 2$.

To prove part (b), recall that there is a split exact sequence of topological monoids:

$$T^1 \underset{i}{\longrightarrow} \mathrm{map}_{T^{n+1}}(\mathcal{Z}_P, \mathcal{Z}_P) \longrightarrow \mathrm{map}^1_{T^n}(M, M) \qquad (\eta : \mathrm{map}_{T^{n+1}}(\mathcal{Z}_P, \mathcal{Z}_P) \dashrightarrow T^1)$$

By part (a) we have

$$\mathrm{map}_{T^{n+1}}(\mathcal{Z}_P, \mathcal{Z}_P) \simeq T^{n+1} \times \prod_{i=1}^{n} \prod_{\sigma_i} \Omega^i S^{2i+1}$$

Since η is a retraction it must induce a surjection on π_1. This implies T^1 must sit inside T^{n+1}, because the second factor in the product is simply connected. Thus we have

$$\mathrm{map}_{T^{n+1}}(\mathcal{Z}_P, \mathcal{Z}_P) \simeq T^{n+1} \times \prod_{i=1}^{n} \prod_{\sigma_i} \Omega^i S^{2i+1} \simeq T^1 \times \mathrm{map}^1_{T^n}(M, M)$$

Because the circle T^1 is inside T^{n+1}, the space $\mathrm{map}^1_{T^n}(M, M)$ is just the product

$$\mathrm{map}^1_{T^n}(M, M) = T^{n+1}/T^1 \times \prod_{i=1}^{n} \prod_{\sigma_i} \Omega^i S^{2i+1}$$

This concludes the proof of the theorem. $\square$

The next corollary is immediate from the previous theorem and our discussion in the last section.

Corollary 5.6.3. *If* P *is a product of simplices and* M *is a quasitoric manifold over* P*, then the homotopy type of* $\mathrm{map}^1_{T^n}(M, M)$ *depends only on* P.

5.7 Equivariant Mapping Spaces of General Quasitoric Manifolds

Computing rational homotopy groups of $\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ when P is an arbitrary polytope presents a challenge, as the rational homotopy groups of $\mathcal{Z}_P$ quickly blow up if $\mathcal{Z}_P$ is rationally hyperbolic. What we can say, however, is that the Bousfield-Kan spectral sequence can still be used to compute the desired homotopy groups. In fact, it collapses at $(n+1)^{st}$-page, where n is the dimension of the polytope. In this section we also touch briefly upon the question of components of $\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$. We learn that all the components have isomorphic homotopy groups, and that in the case when the polytope is 2-dimensional, the space $\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ is connected.

We begin by proving a couple of lemmas.

Lemma 5.7.1. *Let* $\mathcal{Z}_P$ *be a moment angle complex corresponding to a simple polytope* P*, and* σ*,* τ *be two faces of* P*. If* T_σ *and* T_τ *are stabilizers of these faces, and* $T_\sigma \subset T_\tau$ *then* $\tau \subset \sigma$

Proof. We express σ and τ as the intersection of facets of P:

$$\sigma = \bigcap_{F \supset \sigma} F \text{ and } \tau = \bigcap_{G \supset \tau} G$$

By construction of $\mathcal{Z}_P$, the stabilizer of any facet F is just the coordinate subtorus $T_F \subset T^{\mathcal{F}}$, where $\mathcal{F}$ is the set of facets of P. Consequently, the stabilizer of any face is the product of the stabilizers of all the facets containing that face. In our case,

$$T_\sigma = \prod_{F \supset \sigma} T_F \text{ and } T_\tau = \prod_{G \supset \tau} T_G$$

Now, if $T_\sigma \subset T_\tau$, then

$$\{F|F \supset \sigma\} \subset \{G|G \supset \tau\}$$

and so $\tau \subset \sigma$ □

Lemma 5.7.2. *If $\mathcal{Z}_P$ is a moment angle complex corresponding to a simple polytope P, σ a face of P and $\mathcal{Z}_\sigma$ a facial submanifold lying over σ, then $\mathsf{Fix}_P(\sigma) = \mathcal{Z}_P^{T_\sigma} = \mathcal{Z}_\sigma$.*

Proof. Clearly $\mathcal{Z}_\sigma \subset \mathcal{Z}_P^{T_\sigma}$. Conversely, suppose there is a point $z \in \mathcal{Z}_P^{T_\sigma}$ that is contained in some other face, τ. Since T_σ fixes z we see that $T_\sigma \subset T_\tau$. But by the preceeding lemma this can only happen if $\tau \subset \sigma$, so z lies over σ. □

Proposition 5.7.3. *For every face σ of P, the fixed point spaces $\mathcal{Z}_P^{T_\sigma}$ are connected.*

Proof. Because of the principal T^r-bundle $\mathcal{Z}_P \to M$, we know that $\mathcal{Z}_\sigma$ fibers over M_σ with connected fiber T^r. Since both the base M_σ and the fiber T^r are connected, then so is the total space $\mathcal{Z}_\sigma$. The previous lemma implies that $\mathcal{Z}_P^{T_\sigma}$ is connected. □

We can now describe what the Bousfield-Kan spectral sequence for $\operatorname{holim}_P \mathsf{Fix}_P$ computes for any simple polytope P.

Proposition 5.7.4. *If P is an n-dimensional simple polytope, then the Bousfield-Kan spectral sequence for $\operatorname{holim}_P \mathsf{Fix}_P$ collapses at the $(n+1)^{st}$-page.*

Proof. Since the category P has no more than n composible arrows, the chain groups C_k in the complex computing the higher inverse limits [c.f. 5.4.7], $\lim^i \pi_j \circ \mathsf{Fix}_P$, will be zero for $k > n$, and so

$$E_2^{ij} = 0 \text{ for } i > n$$

Thus, the differentials d_r will be zero for $r > n$, and the spectral sequence will collapse at the $(n+1)^{st}$-page. □

Proposition 5.7.5. *The connected components of* $\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ *have isomorphic homotopy groups. Moreover,*

$$E_2^{ij} \Rightarrow \pi_{j-i}\mathrm{map}^{id}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$$

where the term on the left is the second page of the Bousfield-Kan spectral sequence, and the terms on the right are the homotopy groups of the identity component of $\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$*.*

Proof. A choice of a basepoint in $\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ descends to a set of compatible basepoints in $\{\mathsf{Fix}_P(\sigma)|\sigma \in P\}$, because $\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ is the homotopy inverse limit of these spaces over the diagram P. Since each $\mathsf{Fix}_P(\sigma)$ is connected by 5.7.3, the E_2 page of the spectral sequence, given by

$$E_2^{ij} = \lim{}^i \pi_j(\mathsf{Fix}_P(\sigma))$$

is independent of the initial choice of basepoint in $\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$. Choosing a basepoint from any of the components gives the same terminal page of the spectral sequence, and hence the homotopy groups of different components are isomorphic. □

The number of components of $\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ is unknown to the author in general. We do know, from previous discussion, that the space $\mathrm{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ is connected when P is a product of simplexes. Below we will prove this also for the case when P is 2-dimensional.

Lemma 5.7.6. *If* P *is a 2-dimensional polytope, then the facial submanifold of* $\mathcal{Z}_P$ *lying over a one dimensional face* σ *is homeomorphic to* $S^3 \times T^{r-1}$*.*

Proof. Consider the part of the diagram P corresponding to the face σ:

$$\nu_1 \longleftarrow \sigma \longrightarrow \nu_2$$

Applying the functor Orb_P, we get

$$T^m/T_{\nu_1} \longleftarrow T^m/T_\sigma \longrightarrow T^m/T_{\nu_2}$$

So $\mathcal{Z}_\sigma$ can be viewed as the homotopy pushout of

$$T^r \longleftarrow T^{r+1} \longrightarrow T^r$$

where the maps are projections onto two different r-dimensional coordinate subtori. We may think of this as

$$T^{r-1} \times S^1_1 \stackrel{\text{id}\times\text{proj}}{\longleftarrow} T^{r-1} \times (S^1_1 \times S^1_2) \stackrel{\text{id}\times\text{proj}}{\longrightarrow} T^{r-1} \times S^1_2 \tag{5.2}$$

which is the homotopy pushout

$$S^1_1 \longleftarrow S^1_1 \times S^1_2 \longrightarrow S^1_2 \tag{5.3}$$

crossed with the torus T^{r-1}.

But the homotopy pushout (5.3) is just the join of two circles $S^1_1 * S^1_2 \approx S^3$, so the homotopy pushout (5.2) is homeomorphic to $S^3 \times T^{r-1}$ □

Theorem C. *Let* P *be a simple 2-dimensional polytope, and* $\mathcal{Z}_P$ *the corresponding moment angle complex. Then the space* $\text{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ *is connected.*

Proof. We will prove this proposition by showing that the diagonal of the Bousfield-Kan spectral sequence for $\text{map}_{T^m}(\mathcal{Z}_P, \mathcal{Z}_P)$ consists of zeros, that is, we must show that

$$\lim{}^i \pi_i \circ \mathsf{Fix}_P = 0$$

We begin by observing that for any face σ, the space $\mathsf{Fix}_P(\sigma)$ is connected, so the inverse limit of directed sets

$$\lim{}^0 \pi_0 \circ \mathsf{Fix}_P = \lim \pi_0 \circ \mathsf{Fix}_P = \star$$

Next, we note that since the polytope P is 2-dimensional, the category P will have no more than two non-identity composible arrows, and hence

$$\lim{}^j \pi_j \circ \mathsf{Fix}_P = 0 \text{ for } j \geqslant 3$$

Since any moment angle complex is 2-connected [2], any sequence of two non-identity composible arrows will map into the zero group

$$\pi_2(\mathsf{Fix}_P(P)) = \pi_2(\mathcal{Z}_P) = 0$$

so

$$\lim^2 \pi_2 \circ \mathsf{Fix}_P = 0$$

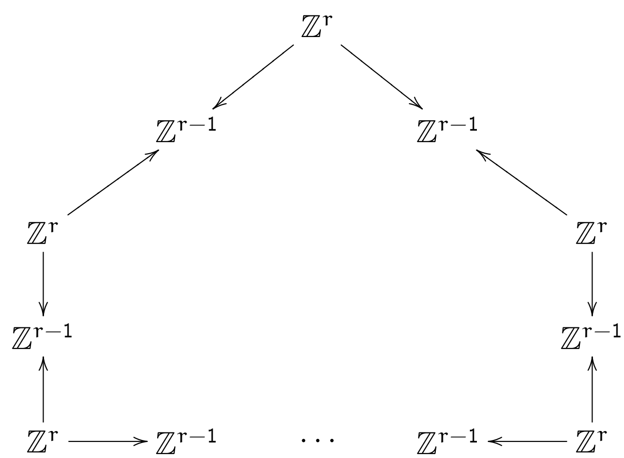
It remains to show that

$$\lim^1 \pi_1 \circ \mathsf{Fix}_P = 0$$

First note that if ν, σ and P are respectively a vertex, a facet and the 2-face of P, then by the previous lemma, the values of the functor Fix_P are given by

$$\begin{aligned} \mathsf{Fix}_P(\nu) &= \mathcal{Z}_\nu \approx T^r \\ \mathsf{Fix}_P(\sigma) &= \mathcal{Z}_\sigma \approx S^3 \times T^{r-1} \\ \mathsf{Fix}_P(P) &= \mathcal{Z}_P \end{aligned}$$

Since $\mathcal{Z}_P$ is simply-connected, the diagram that arises from the composition of functors and computes $\lim^1 \pi_1 \circ \mathsf{Fix}_P$ is



In the above diagram all the maps in question are projections, as they are induced by the maps

$$T^r = T^1 \times T^{r-1} \hookrightarrow S^3 \times T^{r-1}$$

which are inclusions in the first coordinate and identity in the second.

It is easy to see that

$$\lim^1 \pi_1 \circ \mathsf{Fix}_P = \bigoplus_{facets} \lim^1(\mathbb{Z}^r \to \mathbb{Z}^{r-1} \leftarrow \mathbb{Z}^r)$$

so to complete the prove we must show that

$$\lim^1(\mathbb{Z}^r \xrightarrow{p_1} \mathbb{Z}^{r-1} \xleftarrow{p_2} \mathbb{Z}^r) = 0$$

But [10]

$$\lim^1(\mathbb{Z}^r \xrightarrow{p_1} \mathbb{Z}^{r-1} \xleftarrow{p_2} \mathbb{Z}^r) \;\cong\; \mathrm{coker}(\mathbb{Z}^r \times \mathbb{Z}^r \xrightarrow{\alpha} \mathbb{Z}^{r-1})$$

where

$$\alpha(a, b) = p_1(a) - p_2(b)$$

Since p_1 and p_2 are projection maps, α is surjective, so the cokernel is zero, proving the claim. □

In view of the fibration in 5.3.8, we obtain the following:

Corollary 5.7.7. *Let* M *be a quasitoric manifold over a 2-dimensional polytope* P. *Then the space* $\mathrm{map}^1_{T^n}(M, M)$ *is connected.*

Appendix A

Homotopy Limits, Homotopy Colimits and Bousfield-Kan Spectral Sequence

In this section we review the requisite material concerning homotopy limits and homotopy colimits. For more details, the interested reader may consult [1] or [5]

A.1 Simplicial and Cosimplicial Objects

Let Δ be a category whose objects are finite ordinals $n = \{0 \leqslant 1 \leqslant \ldots \leqslant n\}$ and whose morphisms are non-decreasing functions $f : n \to m$. A *simplicial object* $X_\bullet$ over a category C is a contravariant functor from Δ to C, i.e. a covariant functor

$$X : \Delta^{op} \to C$$

Similarly, a *cosimplicial object* $X^\bullet$ over C is a covariant functor

$$X : \Delta \to C$$

Remark A.1.1. Recall that to define a simplicial object $X_\bullet$ over C it is enough to specify objects X_n of C together with face and degeneracy maps

$$\begin{aligned} d_i &: X_n \to X_{n-1}, \quad 0 \leqslant i \leqslant n \\ s_i &: X_n \to X_{n+1}, \quad 0 \leqslant i \leqslant n \end{aligned}$$

Likewise, to define a cosimplicial object $X_\bullet$ over C it suffices to define objects X^n in C together with co-face and co-degeneracy maps

$$\begin{aligned} d^i &: X_n \to X_{n-1}, \quad 0 \leqslant i \leqslant n \\ s^i &: X_n \to X_{n+1}, \quad 0 \leqslant i \leqslant n \end{aligned}$$

In the case of both simplicial and cosimplicial objects, these maps must satisfy a list of simplicial and cosimplicial identities. For details see [1] or [5]

A *(co)simplicial map* between (co)simplicial objects X and Y is a natural transformation of functors. Equivalently, a (co)simplicial map can be given as a sequence of maps between simplicies of (co)dimension n, which commute with (co)face and (co)degeneracy operators.

From what was said above, we see that given a category C we can form new categories sC and cC whose objects are, respectively, simplicial and cosimplicial objects over C and whose morphisms are respectively simplicial and cosimplicial maps. The objects of the category sSet are called *simplicial sets*, and the objects of sTop are called *simplicial spaces*. Likewise, we have *cosimplicial sets* and *cosimplicial spaces*.

Example A.1.2. The simplicial standard n-simplex is a simplicial set defined (as a contravariant functor from Δ to Set) by

$$\Delta[n] := \hom_\Delta(\ , \mathrm{n})$$

Equivalently, a k-simplex in $\Delta[n]$ is any string of the form

$$(0 \leqslant a_0 \leqslant \ldots \leqslant a_k \leqslant n)$$

with face and degeneracy maps given by

$$\begin{aligned} d_0(a_0 \leqslant \ldots \leqslant a_k) &= (a_1 \leqslant \ldots \leqslant a_k) \\ d_i(a_0 \leqslant \ldots \leqslant a_k) &= (a_0 \leqslant \ldots \leqslant a_{i-1} \leqslant a_{i+1} \leqslant \ldots \leqslant a_k) \\ s_i(a_0 \leqslant \ldots \leqslant a_k) &= (a_0 \leqslant \ldots \leqslant a_i \leqslant a_i \leqslant \ldots \leqslant a_k) \end{aligned}$$

Example A.1.3. Given a small cateory C we can form a simplicial set $BC_\bullet$ by declaring any string of n composible arrows in C

$$\sigma = c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} c_n$$

to be an n-simplex. The face and degeneracy maps are given by

$$\begin{aligned} d_0(\sigma) &= c_1 \to c_2 \to \dots \to c_n \\ d_i(\sigma) &= c_0 \to \dots \to c_{i-1} \stackrel{\alpha_i \circ \alpha_{i+1}}{\to} c_{i+1} \dots \to c_n \\ s_i(\sigma) &= c_0 \to \dots \to c_j \xrightarrow{\text{id}} c_j \to \dots \to c_n \end{aligned}$$

The simplicial set $BC_\bullet$ is called the *nerve of* C

Example A.1.4. A geometric n-simplex Δ^n is a subspace of $\mathbb{R}^{n+1}$ given by

$$\Delta^n = \{(x_0, \dots, x_n) : 0 \leqslant x_i \leqslant 1,\ x_0 + \dots + x_n = 1\}$$

The sequence of spaces $\Delta^0, \Delta^1, \dots$ can be made into a cosimplicial space $\Delta^\bullet$ by defining the co-face map δ^i to be the inclusion of Δ^{n-1} into Δ^n as the i^{th} face, and the co-degeneracy map σ^i to be the linear projection $\Delta^{n+1} \to \Delta^n$ onto the j^{th} face, identifyin the j^{th} and the $(j+1)^{st}$ vertices.

A.2 Geometric Realization and the Total Space

In this section we construct some functors that will be used in defining homotopy limits and colimits. We begin with the realization functors

$$\begin{aligned} |\cdot| &: \text{sSet} \to \text{Top} \\ |\cdot| &: s\text{Top} \to \text{Top} \end{aligned}$$

For a simplicial set $X_\bullet$, the realization $|X_\bullet|$ is obtained by taking a disjoint union

$$\coprod_{n \geqslant 0} X_n \times \Delta^n$$

and making identifications

$$\begin{aligned} (d_i x, u) &\sim (x, \delta^i u) \\ (s_i x, u) &\sim (x, \sigma^i u) \end{aligned}$$

with $\Delta^n, \delta^i, \sigma^i$ as in Example A.1.4. We note that the sets X_n are endowed with discrete topology.

If $X_\bullet$ is a simplicial space, then the construction differes only in that the spaces X_n now come equipped with a topology, and so the spaces $X_n \times \Delta^n$ are given the product topology.

Example A.2.1. If C is a small category then $|BC_\bullet|$ is the classifying space of C

Example A.2.2. The realization $|\Delta[n]|$ of the simplicial n-simplex is homeomorphic to Δ^n

In conclusion, we define the Tot functor from cTop to Top that is used to define homotopy limits. It is given by the space of cosimplicial maps

$$\mathsf{Tot}(X^\bullet) = \hom_{c\mathrm{Top}}(\Delta^\bullet, X^\bullet)$$

A.3 Simplicial and Cosimplicial Replacements

In this section we construct some simplicial and cosimplicial spaces that arise from diagrams of spaces. These constructions are then used to define homotopy limits and colimits.

We start with a finite category I, and a functor

$$\mathsf{F} : \mathrm{I} \to \mathrm{Top}$$

Such a functor is called a diagram of spaces. We may associate to this data a simplicial space $\coprod_\bullet \mathsf{F}$, called the *simplicial replacement of* F constructed as follows:

The n-simplices are disjoint unions taken over strings of n composible arrows in I

$$\coprod_{i_0 \xleftarrow{\alpha_1} \cdots \xleftarrow{\alpha_n} i_n} \mathsf{F}(i_n)$$

The face and degenracy maps are given by

$$\begin{aligned} d_j : \mathsf{F}(i_n) &\xrightarrow{\mathrm{id}} \mathsf{F}(i_n), \quad 0 \leqslant j < n \\ d_n : \mathsf{F}(i_n) &\xrightarrow{\mathsf{F}(\alpha_n)} \mathsf{F}(i_{n-1}) \\ s_j : \mathsf{F}(i_n) &\xrightarrow{\mathrm{id}} \mathsf{F}(i_n), \quad 0 \leqslant j \leqslant n \end{aligned}$$

With I and F be as above, the *homotopy colimit*, $\operatorname{hocolim}_I F$, is defined as the composition of functors:

$$\operatorname{hocolim} : \mathrm{Top}^I \xrightarrow{\coprod_\bullet} s\mathrm{Top} \xrightarrow{|\cdot|} \mathrm{Top}$$

With I and F as before, we can also construct a cosimplicial space $\prod^\bullet F$, called the *cosimplicial replacement of* F. The simplicies in codimension n are given by products taken over strings of n composible arrows in I:

$$\prod_{i_0 \xleftarrow{\alpha_1} \cdots \xleftarrow{\alpha_n} i_n} F(i_0)$$

The coface and codegeneracy maps are given by

$$\begin{aligned} d^0 : F(i_1) &\xrightarrow{F(\alpha_1)} F(i_0) \\ d^j : F(i_0) &\xrightarrow{\mathrm{id}} F(i_0), \quad 0 < j \leqslant n \\ s^j : F(i_0) &\xrightarrow{\mathrm{id}} F(i_0), \quad 0 \leqslant j \leqslant n \end{aligned}$$

The *homotopy limit*, $\operatorname{holim}_I F$, is defined as the composition of functors:

$$\operatorname{holim} : \mathrm{Top}^I \xrightarrow{\prod^\bullet} c\mathrm{Top} \xrightarrow{\mathrm{Tot}} \mathrm{Top}$$

A.4 The Bousfield-Kan Spectral Sequence

The Bousfield-Kan spectral sequence arises from an exact couple obtained from a tower of fibrations. Under certain conditions it converges to homotopy groups of the inverse limit of the tower. In the case of homotopy limits of small diagrams of spaces, a tower of fibrations can be gotten by restricting the maps out of the infinite simplex to successive skeleta. In this way one gets a Bousefield-Kan spectral sequence converging, under favorable conditions, to the homotopy limit of the diagram. The construction of the Bousefield-Kan spectral sequence can be found in [1] or [5], however we will only be needing these results:

Theorem A.4.1. *For a small category* I*, let* $D : I \to \mathrm{Top}_*$ *be a diagram of pointed spaces. Then there exists a spectral sequence,* $\{E_r D\}$ *called the* Bousfield-Kan spectral sequence *with*

$$E_2^{i,j} = \lim{}^i \pi_j D \textit{ for } 0 \leqslant i \leqslant j$$

and differentials

$$d_r : E_r^{i,j} \to E_{r+1}^{i+r,j+r-1}$$

Moreover, under the conditions of the following lemma, this spectral sequence converges completely to $\pi_{j-i}\text{holim}D$ *for* $j - i \geqslant 1$

The following lemma can be found in [5, p321]

Lemma A.4.2. *Suppose* D *and* I *satisfy the hypotheses of the above theorem. If for any* $k \geqslant 0$ *and any integer* s *there exists and integer* N *such that*

$$E_\infty^{s,s+k} \cong E_N^{s,s+k}$$

then for $j - i \geqslant 1$ *the Bousfield-Kan spectral sequence converges completely to* $\pi_{j-i}\text{holim}D$

Bibliography

[1] M. Bousfield and D.M. Kan, *Homotopy Limits, Completions and Localizations* , volume 304 of *Lecture Notes in Math.*, Springer, Berlin, Heidelberg, New York, 1972

[2] V. Buchstaber and T. Panov: Torus Actions and Their Applications in Topology and Combinatorics, University Lecture Series **24**, Amer. Math. Soc., Providence, R.I., 2002

[3] M. Davis and T. Januszkiewicz, *Convex Polytopes, Coxeter Orbifolds and Torus Actions.* Duke Math. J. **62** (1991), no. 2, 417-451

[4] Y. Felix, S. Halperin and J.-C. Thomas: Rational Homotopy Theory, GTM **205**, Springer-Verlag, 2001

[5] P. Goerss and J. Jardine: Simplicial Homotopy Theory, Birkhauser, 1999

[6] E. Miller and B. Sturmfels: Combinatorial Commutative Algebra, GTM **227**, Springer-Verlag, 2004

[7] D. Notbohm and N. Ray, *On Davis-Januszkiewicz Homotopy Types. I. Formality and Rationalisation* Algebr. Geom. Topol. **5** (2005), 31-51.

[8] T. Panov, N. Ray. *Categorical Aspects of Toric Topology*, arXiv:0707.0300v1

[9] T. Panov, N. Ray and R. Vogt. *Colimits, Stanley-Reisner algebras, and loop spaces.* In *Categorical Decomposition Techniques in Algebraic Topology* , volume 215 of *Porgress in Mathematics* . Birkhauser Verlag, Basel, Switzerland, 2004, pp. 261-291

[10] C. Weibel: An Introduction to Homological Algebra, Cambridge Studies in Advanced Mathematics **38**, Cambridge University Press, 1994

[11] V. Welker, G. Ziegler and R. Zivaljevic, *Homotopy Colimits - comparison lemmas for combinatorial applications* , J. Reine Angew. Math. **509** (1999), 117 - 149