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Fluid reasoning predicts future mathematical performance among children and adolescents

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ABSTRACT

The aim of this longitudinal study was to determine whether fluid reasoning (FR) plays a significant role in the acquisition of mathematics skills above and beyond the effects of other cognitive and numerical abilities. Using a longitudinal cohort sequential design, we examined how FR measured at three assessment occasions, spaced approximately 1.5 years apart, predicted math outcomes for a group of 69 participants between ages 6 and 21 years across all three assessment occasions. We used structural equation modeling (SEM) to examine the direct and indirect relations between children's previous cognitive abilities and their future math achievement. A model including age, FR, vocabulary, and spatial skills accounted for 90% of the variance in future math achievement. In this model, FR was the only significant predictor of future math achievement; age, vocabulary, and spatial skills were not significant predictors. Thus, FR was the only predictor of future math achievement across a wide age range that spanned primary school and secondary school. These findings build on Cattell's conceptualization of FR as a scaffold for learning, showing that this domain-general ability supports the acquisition of rudimentary math skills as well as the ability to solve more complex mathematical problems.

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Introduction

American educators face the tall order of improving outcomes in science, technology, engineering, and mathematics (STEM). To generate solutions to some of the globe's most pressing challenges, educators will need to teach children to become better problem solvers who can apply to their work the information learned in their STEM courses. Courses in mathematics are especially challenging for many students, and these courses have become a gatekeeper to higher education and job opportunities in technological fields (Moses & Cobb, 2001). Because math instruction builds on previously acquired knowledge and skills, it is difficult for children who fall behind early to catch up with their peers. In an effort to improve math and language outcomes across the nation, educators have recently released new national standards for math and language arts education called the Common Core State Standards (National Governors Association and Council of Chief State School Offices, 2014). The new standards lay out progressions of math skill building benchmarks that have opened up discussions about how teachers can provide better support for students in bolstering their math proficiency skills.

Complementary lines of research in psychology and education aim to identify which cognitive precursors lead to proficient acquisition of mathematics skills. A long-term aim of this line of research is to inform educators about the precursors to math development, so that they may create lesson plans that target not only specific math skills but also underlying domain-general cognitive processes. The cognitive abilities required to solve math problems have been difficult to isolate because mathematics is a heterogeneous subject matter (e.g., arithmetic, fractions, geometry, statistics) and problems within the same topic area require several different operations and computations (e.g., adding, subtracting, multiplying, dividing). Nevertheless, researchers have begun to identify common key cognitive functions that are critically important for disparate types of mathematical computations (Bisanz, Sherman, Rasmussen, & Ho, 2005; Desoete & Grégoire, 2007; Krajewski & Schneider, 2009).

Relationships between math and cognitive abilities are often studied within the framework of the Cattell–Horn–Carroll (CHC) theory, arguably the most comprehensive and empirically supported theory of cognitive abilities derived from more than 70 years of psychometric research using factor analytic theory (Keith & Reynolds, 2010). The utility of the theory is in clarifying the relations between cognitive and academic abilities to inform educational and psychological practices. The most recent revision of this model, by Schneider and McGrew (2012), includes 16 broad cognitive abilities, all of which contain more narrow cognitive abilities within them. This model does not include a general intelligence *g* factor; rather, it is based on accumulating evidence that broad and narrow CHC cognitive abilities explain more variance in specific academic abilities than *g* alone and that these specific relationships are more informative to educational practice than general intelligence (e.g., Floyd, McGrew, & Evans, 2008; McGrew, Flanagan, Keith, & Vanderwood, 1997; Vanderwood, McGrew, Flanagan, & Keith, 2002).

In a recent synthesis of studies investigating the concurrent relationships between CHC cognitive abilities and achievement measures (CHC–ACH) by McGrew and Wendling (2010), fluid reasoning (FR) was one of three broad cognitive abilities that was consistently related to mathematical performance in calculation and problem solving at all age ranges throughout development (the other two were verbal comprehension and processing speed). FR was consistently related to future math achievement above and beyond the contribution of general intelligence. FR has been defined by contemporary CHC theory as the ability to flexibly and deliberately solve novel problems without using previous information (Schneider & McGrew, 2012). More specifically, it is the ability to analyze novel problems, identify patterns and relationships that underpin these problems, and apply logic. On FR tests, one or both of the following logic abilities is required: (a) *induction*, the ability to discover an underlying characteristic (e.g., rule, concept, trend) that governs a set of materials, and (b) *general sequential reasoning* (*deduction*), the ability to start with stated rules or premises and engage in one or more steps to reach a solution to a novel problem (Schneider & McGrew, 2012). FR tests are commonly administered as part of IQ batteries that are administered to children in schools or in clinical settings. Whereas FR performance is strongly correlated to general intelligence (*g*), as is verbal comprehension, there is unique shared variance among tests of FR that cannot be accounted for by *g* alone (McGrew et al., 1997).

FR development

In typically developing children, FR begins to emerge during the first 2 years of life, increases rapidly during early and middle childhood, continues to increase at a slower rate during adolescence, and reaches asymptotic values at around age 25 years, after which it begins to decline (McArdle, Ferrer-Caja, Hamagami, & Woodcock, 2002).

Analyses of longitudinal data from large samples that were used to create norms for the standardized Woodcock–Johnson Cognitive Abilities testing battery (Schrank & Wendling, 2009; Woodcock, McGrew, & Mather, 2001) reveal that both FR performance (as measured by Analysis Synthesis and Concept Formation tests) and math achievement increase rapidly during childhood, peaking during late adolescence to age 24 years and beginning to decline during adulthood (Ferrer & McArdle, 2004). The trajectories of FR development and improvements in math abilities parallel one other throughout development—and more so between ages 5 and 10 years than between ages 11 and 24 years (Ferrer & McArdle, 2004). This observation hints at the possibility that FR plays a bigger role in early math skill development in kindergarten and elementary school than in higher levels of education. However, additional data are needed to examine the relationships between these skills over development.

Fluid reasoning and math achievement

Hypothesized link

One mechanism by which FR could support math skill acquisition is related to the fact that both FR and math problems engage a common underlying cognitive process called relational reasoning, or the ability to jointly consider multiple relationships between different components of a problem (Halford, Wilson, & Phillips, 1998; Miller Singley & Bunge, 2014). According to this framework, understanding mathematics requires the ability to form abstract representations of quantitative and qualitative relations between variables (Halford et al., 1998). For instance, when children first learn fractions, they must keep several numerical relationships in mind; whole unit integers need to be understood as sub-units, and children must learn to coordinate the value in the numerator and the value in the denominator (Saxe, Taylor, McIntosh, & Gearhart, 2005).

Furthermore, solving story word problems requires children to draw conceptual connections between real-world situations and analogous numerical symbols and operations to solve the problems (Clement, 1982). Another example of relational reasoning is evident in early algebra when students are asked to solve for one or more unknown numbers and must keep in mind the relationship between numbers on either side of the equal sign to determine which operand is required to solve for the missing variable. Empirical support for a link between FR and math achievement comes from both cross-sectional and longitudinal research.

Whereas multiple longitudinal studies have elucidated the importance of spatial skills in math development (for a review, see Cheng & Mix, 2014), only a few longitudinal studies have explored the unique developmental role of previous FR. Further research is needed to disentangle the role that each of these two cognitive abilities plays in math development because, although FR and spatial abilities are highly correlated (Fry & Hale, 1996), they rely on both overlapping and separable cognitive processes and brain regions. FR tests (e.g., Matrix Reasoning) not only require spatial skills (including visualization) but also require relational reasoning, or the ability to consider relationships between multiple pieces of information to detect the underlying conceptual relationship among visual objects and to use reasoning to identify and apply rules (Halford et al., 1998; Holyoak, 2012; Vendetti & Bunge, 2014). In regard to the neural correlates, a visuospatial skill that is commonly implicated in math achievement, visuospatial working memory, relies on the intraparietal and superior frontal regions (for a review, see Klingberg, 2006), whereas the relational reasoning component of FR relies on the rostrolateral prefrontal cortex and lateral parietal regions (for reviews, see Krawczyk, 2012; Vendetti & Bunge, 2014). Therefore, it is plausible that visuospatial abilities and FR make unique contributions to math achievement.

Longitudinal precursors to math achievement

Fluid reasoning

As mentioned, there are only a limited number of longitudinal studies that have examined the extent to which FR skills uniquely contribute to the development of math proficiency during childhood separately from general IQ and other domain-general cognitive abilities. In one such study (Fuchs et al., 2010), the authors compared the effect of basic numerical cognition and other domain-general cognitive abilities measured at the beginning of the school year on 280 first-grade students' development of math problem solving over the course of that academic year. They found that FR (measured by Matrix Reasoning) during the fall semester was just as predictive of children's gains in word problem solving over the course of the year as their basic numerical cognition skills. Primi, Ferrão, and Almeida (2010) showed that seventh- and eighth-grade students' initial level of FR (measured by tests of numerical, verbal, spatial, and abstract reasoning) was positively related to their subsequent growth in quantitative abilities over the course of the next 2 academic years, such that children with higher FR ability at the start of the year demonstrated more growth in math over the course of 2 academic years than children with lower FR scores.

Spatial skills

Several longitudinal studies have examined the robust role of spatial skills in the development of math proficiency during childhood (for a review, see Cheng & Mix, 2014). However, studies in this literature rely on different operational definitions of spatial ability such as visualization and spatial working memory.

Visualization. Visualization is the most commonly studied spatial ability related to mathematics (Cheng & Mix, 2014). Visualization is the ability to perceive visual patterns and mentally manipulate them to simulate how they might look when transformed (e.g., rotated, changed in size, partially obscured) (Flanagan, Ortiz, & Alfonso, 2013). Visualization is frequently measured using tests such as Block Design, which measures the ability use two-color cubes to construct replicas of two-dimensional geometric patterns under timed conditions. This test assesses the ability to mentally transform (or rotate) blocks. One such study by Zhang and colleagues (2014) found that spatial skills in kindergartners (measured by spatial visualization), along with verbal skills, predicted level of arithmetic in first grade as well as arithmetic growth through third grade. Another such study by Casey and colleagues (2015) examined the predictors in first grade of math problem solving in fifth grade, comparing the predictive power of performance on Block Design with the predictive power of verbal and arithmetic skills. They found that Block Design performance in first grade was just as predictive of fifth-grade math problem solving as early arithmetic skills.

Visuospatial working memory. More recently, studies have also found that visuospatial working memory, or the ability to temporarily store and process visual information to complete a task, is a robust predictor of math achievement. For example, Li and Geary (2013) showed that developmental gains in visuospatial working memory between first and fifth grades was a strong predictor of math achievement at the end of fifth grade, as was general intelligence (measured by Wechsler Abbreviated Scale of Intelligence [WASI] Matrix Reasoning, Block Design, Vocabulary, and Similarities tests) and in-class attentive behavior. Similarly, LeFevre and colleagues (2010) advanced a developmental theory suggesting that three key pathways contribute differentially to early math development: quantitative, linguistic, and spatial pathways. They found that at age 4 or 5 years, early spatial attention (measured by Spatial Span) significantly predicted both number naming and processing of numerical magnitude 2 years later.

FR tests may require visualization or spatial working memory, but they are distinguished from these purely spatial ability tests because they require inductive or general sequential (deductive) reasoning (Schneider & McGrew, 2012). Factor analysis contributing to CHC theory has demonstrated that FR measures tap into a separable construct than spatial abilities (Schneider & McGrew, 2012).

Quantitative skills

Although multiple studies have demonstrated the strong predictive power of early math skills on math achievement over and above reading, attentive behavior, and domain-general cognitive predictors, many of these studies have been conducted in populations of primary school-aged children and consequently are limited to more basic numerical competencies (e.g., magnitude comparisons, number naming, arithmetic, fractions) and do not incorporate measures of FR as a unique factor in their models (e.g., [Duncan et al., 2007](#); [Fuchs et al., 2012](#); [LeFevre et al., 2010](#); [Watts, Duncan, Siegler, & Davis-Kean, 2014](#)). Therefore, more research is needed to better understand the relative predictive power of FR, and other domain-general cognitive skills, in relation to previous numerical skills in predicting math achievement across primary and secondary school grades.

Study goals

In the current study, we sought to expand on previous research to evaluate the extent to which previous FR predicts later math outcomes in children between ages 6 and 21 years above and beyond other cognitive and numerical abilities that have previously been implicated in math development. Our aims were threefold: (a) to test a latent model of FR that combines three well-known psychometric tests, (b) to compare the contribution of previous FR to that of previous math reasoning in predicting future math achievement at Time 3 (T3), and (c) to compare the relative contribution of previous FR to spatial skills, verbal skills, and age in predicting future math achievement at T3. Each of these cognitive abilities has been shown to be a strong independent predictor of later math achievement (e.g., [Primi et al., 2010](#)).

To this end, we collected and analyzed data within the context of a larger longitudinal cohort sequential design study examining the neurodevelopment of FR. We administered a battery of age-normed neuropsychological tests to measure FR, as well as vocabulary and spatial skills, at three time points (~1.5 years apart) in a group of 69 children who ranged in age from 6 to 21 years at the first assessment. At the second assessment (T2), we assessed participants on a measure of math reasoning. At the final assessment (T3), we assessed participants on three different math achievement measures: math problem solving, arithmetic fluency, and math reasoning.

Method

Participants

Participants were individuals in a longitudinal study designed to examine the cognitive and neural factors that underlie the development of FR. All participants and their parents gave their informed assent or consent to participate in the study approved by the committee for protection of human subjects. In addition, all participants were screened for neurological impairment, psychiatric illness, and history of learning disabilities or developmental delays.

Understanding developmental processes requires longitudinal studies that focus on within-person changes over time. This study design involved a cohort-sequential design in which 201 participants, ranging between ages 5 and 15 years at the time of recruitment, were assessed at one to three time points with an average delay of 1.5 years between time points. This cohort-sequential design enabled us to examine both between-person differences and within-person changes over a 5-year span—the 5 years of the funded research program—and with fewer participants than a traditional longitudinal design. This approach provides insight into the interplay of factors underlying such within-person changes over time, that is, improvements in cognitive abilities over development ([Bell, 1953](#); [Ferrer & McArdle, 2004](#); [McArdle et al., 2002](#)).

Parents completed the Child Behavioral Checklist ([Achenbach, 1991](#)) on behalf of their children. Participants who scored in the clinical range for either externalizing or internalizing behaviors were excluded from further analyses. Of the 172 children and adolescents enrolled in the study who scored in the normal range on the Child Behavior Checklist, 69 successfully completed testing at three time points: T1, T2, and T3—a substantive time commitment, involving six long testing sessions (one

behavioral session and one brain imaging session at each of the three time points). There was no statistical difference in performance between children who participated at all three time points and those who did not.

The mean assessment ages for these 69 participants were 10.18 years ($SD = 3.32$) at T1, 11.67 years ($SD = 3.35$) at T2, and 13.45 years ($SD = 3.38$) at T3. Across all of these participants and time points, data were collected between ages 6 and 21 years. The ethnicity of the sample reflected the ethnic and racial diversity found in the local population (7% Hispanic/Latino, 56% White, 12% Asian, 10% Black/African American, 15% multiple ethnicities). Both genders were represented equally (48% male, 52% female). Most of these children came from middle-class homes, and the majority of families (85%) reported two adults living in their homes. All mothers in the study had completed high school, and the majority (84%) had completed some post-secondary education in the form of a bachelor's or associate's degree or a diploma from a vocational college. Most of the children spoke English at home.

Measures

The behavioral measures selected for our longitudinal study were standardized measures with very high internal consistency and test–retest reliability of .94 or .95 (McArdle et al., 2002; McGrew, Werder, & Woodcock, 1991).

Fluid reasoning

FR ability was assessed using three standardized measures: the Matrix Reasoning subtest of the WASI (Wechsler, 1999) and the Analysis Synthesis and Concept Formation subtests of the Woodcock–Johnson Tests of Cognitive Abilities–Revised (Woodcock et al., 2001). Although these three tests are quite different from one another, they were all designed as measures of FR that rely on one or more narrow FR abilities. As shown below, all three tests loaded onto a single factor “FR” in our sample, which is consistent with previous factor analytic work contributing to CHC theory (Schneider & McGrew, 2012). Thus, we used scores from this FR factor in all subsequent analyses (for an example of this approach, see Primi et al., 2010).

Matrix Reasoning. This test was modeled after a traditional test of “fluid” or nonverbal reasoning—Raven's Progressive Matrix Reasoning (Raven, 1938)—and requires participants to examine an incomplete matrix, or geometric pattern, and then select the missing piece from five response options arranged according to one or more progression rules. The Matrix Reasoning subtest assesses FR induction skills, or the ability to identify an underlying characteristic (e.g., rule or trend) that governs the existing pattern, and the ability to choose a missing piece that contains this characteristic.

Analysis Synthesis. On this test, participants are asked to analyze an incomplete logic puzzle made up of colored squares and to use a key to determine the missing color in the puzzle. To complete this task successfully, participants must use general sequential (or deductive) reasoning skills to draw correct conclusions from a color combination key, with more difficult items requiring a series of sequential steps.

Concept Formation. On this test, participants are asked to view a complete puzzle made up of colored squares and to identify and state the “rules” (color and shape) when shown illustrations of both instances and non-instances of the concept (e.g., red square). The Concept Formation test requires frequent switching from one rule to another. To complete this task successfully, participants must use inductive reasoning skills to discover the rule that governs the puzzle.

Vocabulary

We used the WASI Vocabulary measure (Wechsler, 1999) to probe crystallized knowledge and, indirectly, semantic memory. This test is a norm-referenced measure of expressive vocabulary. On this test, the examiner presents stimulus words to participants and asks them to state each word's meaning.

Spatial skills

We administered two tests of spatial skills. Spatial Span is considered a measure of visual memory, or the ability to remember visual images over short periods of time (< 30 s) (Schneider & McGrew, 2012). Block Design is considered a measure of visualization, or the ability to mentally organize visual information by analyzing part-whole relationships when information is presented spatially (Schneider & McGrew, 2012). Based on previous factor analytic work contributing to CHC theory demonstrating that visual memory and visualization load onto a single factor, we created a factor score called spatial skills using multiple imputation in *Amos* (Schneider & McGrew, 2012). We used scores from this spatial skills factor in all subsequent analyses.

Spatial Span. The Spatial Span test in the fourth edition of the Wechsler Intelligence Scale for Children (WISC-IV) is a norm-referenced measure that requires participants to remember a sequence of spatial locations on a grid in forward and reverse order (Wechsler, 2003). The forward condition measures spatial attention and short-term visuospatial memory, whereas the backward condition also measures the ability to manipulate visuospatial representations in working memory. Participants' scores on each of the conditions are summed into a Spatial Span total score.

Block Design. The Block Design test in the WASI (Wechsler, 1999) is a norm-referenced measure that requires participants to perceive patterns and mentally stimulate how they might look when transformed (e.g., rotated). On the Block Design test, participants are asked to arrange a set of red and white blocks in such a way as to reproduce a two-dimensional visual pattern shown on a set of cards. The test is timed, and scoring is based on both efficiency and accuracy of the pattern reproduction.

Math achievement

All math measures came from the Woodcock–Johnson III Tests of Achievement and Cognitive Abilities (WJ III ACH & COG), designed for use across the lifespan (Woodcock et al., 2001). A math reasoning test was administered at the second time point, and three math measures were administered at the final time point.

Number Series. We administered the Number Series test from the WJ-III Cognitive Abilities testing battery at the second and third time points as a measure of mathematical reasoning. On this test, the examiner presents participants with a page of numerical sequences that contain a missing number. Participants are asked to complete each sequence by identifying and applying the rule that applies to the other numbers in the sequence. As the test advances, the underlying rules become more challenging (e.g., 2, 3, 4, __? as compared with 15, 18, 21, __?). Participants are awarded 1 point for each correct answer and 0 points for each incorrect answer. The examiner discontinues the test when participants either finish all items or miss six consecutive items.

Applied Problems. We administered this WJ-III subtest to measure participants' ability to solve practical math word problems using simple counting, addition, or subtraction operations at the third time point. On the Applied Problems test, participants are presented with a picture (e.g., a group of mixed coins) and asked to listen to a problem (e.g., "How much money is this?"). To solve a problem, children must recognize the mathematical procedure to be followed and perform the appropriate calculations. As the test advances, children must carry out more complex operations and have more advanced experience with each particular concept such as telling time or solving word problems. Participants are awarded 1 point for each correct answer and 0 for each incorrect answer. The examiner discontinues the test when children either finish all items or miss six consecutive items by the completion of the test page.

Math Fluency. To measure participants' ability to complete basic arithmetic problems, we administered the Math Fluency test on the WJ-III Tests of Achievement testing battery at the third time point. This test measures participants' ability to solve simple addition, subtraction, and multiplication facts within a 1-min time limit. At the beginning of the test, children are presented with a worksheet composed of simple arithmetic problems and asked to solve as many problems as they can in 1 min.

Hypotheses

Latent construct of FR

To examine whether the three tests represent a common construct, we used confirmatory factor analysis (CFA) to create a latent variable called *fluid reasoning (FR)* from participants' scores on three different tests at each time point: Matrix Reasoning, Analysis Synthesis, and Concept Formation. CFA procedures were conducted to test the fit of the data to the FR construct for each time point (Fig. 1).

Comparing FR and math reasoning as predictors of later math achievement

Second, we tested a model including relations from FR skills and math reasoning skills to a diverse set of math skills at a future time point (see Fig. 2). To this end, we created a math latent variable called *math achievement* using three different math tests at T3, each measuring different math skills: math reasoning, Applied Problems, and Math Fluency. As shown in Fig. 2, we hypothesized that FR at T1 and T2 would be the strongest predictors of math achievement at T3 after accounting for previous math reasoning at T2. In the next analysis, we added age to the model.

Comparing FR with verbal and spatial abilities as predictors of later math achievement

Third, we compared the relative contributions between FR and future math achievement in relation to other cognitive abilities that have previously been implicated in math development: verbal reasoning (Vocabulary) and spatial skills (Spatial Span and Block Design). We tested the model hypothesizing

Fig. 1. Standardized parameter estimates from the CFA of FR for each measurement occasion. All three indicators loaded on significantly to the FR constructs at each time point.

Fig. 2. Longitudinal models predicting math achievement from previous FR and math reasoning. In Model 1, math achievement is predicted from previous assessments of FR and math reasoning. In Model 2, age is included at the two previous occasions. Circles represent latent variables of FR composed of three observed variables. One-headed arrows represent regressions, and two-headed arrows represent covariance or correlations.

that FR skills would remain a strong predictor of future math achievement after accounting for verbal and spatial skills (see Fig. 3).

Results

Missing value analysis

Because the percentage of missing values for four of the variables was above 5% (refer to Appendix for the percentages of missing values for the variables), the pattern of missingness was assessed via Little's MCAR (missing completely at random) procedure (Tabachnick & Fidell, 2007). This procedure revealed that the data were missing at random: $\text{MCAR}, \chi^2(336) = 360.32, p = .173$. Because of this, we used the *expectation maximization* algorithm to estimate the model parameters (Tabachnick & Fidell, 2007).

Descriptive statistics

Means and standard deviations for the study variables are shown in Table 1. Both the raw scores and T-scores are presented. The T-scores are standardized scores wherein the mean is 50 and the stan-

Fig. 3. Longitudinal model predicting math achievement from previous FR, spatial skills, vocabulary, and age.

dard deviation is 10. Factor scores for FR and spatial skills were derived using multiple imputation in Amos. As shown in Table 1, all mean raw scores and T-scores increased numerically over time, with the exception of the FR factor score, which decreased from T2 to T3 because the standardized loadings for the FR factor score in T3 are smaller than the standardized loadings in T2. When composite (or average) FR scores are generated, it is evident that performance increases across time. Pearson correlations between study variables are shown in Table 2.

The structure of FR

To test whether Matrix Reasoning, Analysis Synthesis, and Concept Formation could be combined into a latent factor of FR, we conducted CFA for each of the time points using Amos 23 software (Arbuckle, 2015). All factor loadings were statistically significant ($p < .001$), with standardized loadings above .65, thereby supporting a latent factor. Therefore, the FR construct was supported, and we computed one latent factor for each time period using the three psychometric tests.

Comparing FR and math reasoning as predictors of later math achievement

We employed structural equation modeling (SEM) to test our second hypothesis that previous FR at T1 and T2 would be stronger predictors of T3 math achievement than of T2 math reasoning. This approach also allowed us to examine the effects of the predictor variables simultaneously on a latent dependent measure. As shown in Table 3, our hypothesis was supported; previous FR was the strongest predictor of math achievement at T3. Any model that did not involve FR as a predictor of T3 math achievement fit significantly worse and decreased the amount of explained variance. By contrast, removing the path from T2 math reasoning to T3 math achievement did not worsen the fit or decrease the amount of explained variance of T3 math achievement. Results are reported in Table 4. This pattern was similar when using T2 and T3 data only (i.e., eliminating FR T1 from the model). Including age at T1 and T2 in the model depicted in Fig. 2 did not change the results. Indeed, removing all regression paths from age to the variables of interest (i.e., leaving age in the model but eliminating its effects) did not worsen the fit.

Table 1
Descriptive statistics for the study variables ($N = 69$).

Variable	Time 1					Time 2					Time 3				
	Raw score			T-score		Raw score			T-score		Raw score			T-score	
	Range	<i>M</i>	<i>SD</i>	<i>M</i>	<i>SD</i>	Range	<i>M</i>	<i>SD</i>	<i>M</i>	<i>SD</i>	Range	<i>M</i>	<i>SD</i>	<i>M</i>	<i>SD</i>
Assessment age (years)	5–16	10.18	3.32	–	–	6–19	11.67	3.35	–	–	8–21	13.45	3.85	–	–
Vocabulary	11–66	41.62	14.49	56.02	11.44	23–71	48.46	12.21	59.61	10.48	30–74	54.36	10.38	54.36	8.21
Matrix Reasoning	3–32	22.05	7.47	54.96	9.61	5–35	26.38	5.56	59.13	7.81	8–34	28.70	4.06	59.98	5.89
Analysis Synthesis	1–37	23.60	8.35	60.90	13.17	12–34	27.41	4.56	65.43	12.16	20–34	29.49	3.22	68.64	12.26
Concept Formation	1–35	23.63	10.03	58.51	10.42	7–35	28.49	7.32	64.39	10.67	7–35	31.69	4.86	69.37	8.56
Fluid reasoning	4–31	21.64	7.17	–	–	11–39	32.31	5.91	–	–	14–30	26.45	2.88	–	–
Number Series	–	–	–	–	–	7–22	15.77	3.42	–	–	11–23	17.43	2.81	–	–
Spatial skills	–	–	–	–	–	21–83	56.28	17.81	–	–	30–90	68.60	13.57	–	–
Applied Problems	–	–	–	–	–	–	–	–	–	–	29–60	46.24	7.60	65.90	12.74
Math Fluency	–	–	–	–	–	–	–	–	–	–	29–160	95.66	30.73	–	–

Table 2

Pearson correlations between the study variables.

Variable	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
<i>First assessment</i>																				
1. Matrix Reasoning																				
2. Analysis Synthesis	0.68																			
3. Concept Formation	0.71	0.51																		
4. Vocabulary	0.81	0.63	0.69																	
5. Fluid reasoning	0.99	0.72	0.75	0.84																
<i>Second assessment</i>																				
6. Matrix Reasoning	0.69	0.77	0.60	0.77	0.82															
7. Analysis Synthesis	0.60	0.77	0.59	0.65	0.75	0.82														
8. Concept Formation	0.62	0.6	0.74	0.73	0.75	0.76	0.77													
9. Spatial skills	0.73	0.68	0.67	0.80	0.88	0.77	0.68	0.78												
10. Number Series	0.68	0.78	0.55	0.77	0.80	0.76	0.77	0.54	0.75											
11. Vocabulary	0.78	0.66	0.73	0.87	0.84	0.77	0.66	0.64	0.86	0.74										
12. Fluid reasoning	0.69	0.79	0.77	0.77	0.73	0.94	0.95	0.88	0.97	0.82	0.78									
<i>Third assessment</i>																				
13. Matrix Reasoning	0.60	0.61	0.50	0.67	0.76	0.79	0.71	0.34	0.64	0.63	0.58	0.80								
14. Analysis Synthesis	0.55	0.77	0.48	0.65	0.70	0.72	0.75	0.38	0.65	0.65	0.62	0.77	0.65							
15. Concept Formation	0.57	0.47	0.63	0.47	0.62	0.57	0.54	0.52	0.53	0.51	0.49	0.63	0.52	0.54						
16. Spatial skills	0.76	0.75	0.80	0.88	0.93	0.89	0.83	0.87	0.91	0.83	0.88	0.96	0.78	0.78	0.64					
17. Number Series	0.39	0.53	0.63	0.55	0.54	0.57	0.58	0.48	0.59	0.59	0.56	0.65	0.56	0.71	0.45	0.73				
18. Vocabulary	0.80	0.65	0.63	0.82	0.83	0.73	0.66	0.56	0.73	0.64	0.81	0.75	0.64	0.56	0.35	0.81	0.56			
19. Applied Problems	0.60	0.66	0.69	0.79	0.76	0.72	0.66	0.54	0.72	0.74	0.75	0.77	0.69	0.74	0.47	0.86	0.80	0.71		
20. Math Fluency	0.67	0.62	0.65	0.74	0.77	0.71	0.67	0.52	0.86	0.69	0.81	0.79	0.61	0.65	0.37	0.89	0.60	0.67	0.71	
21. Fluid reasoning	0.66	0.76	0.78	0.72	0.71	0.83	0.81	0.78	0.83	0.72	0.67	0.87	0.87	0.91	0.72	0.90	0.71	0.72	0.79	0.70

Note. All correlations were statistically significant at .05.

Table 3

Fit statistics for the structural models comparing FR with math reasoning in predicting future math achievement.

Variable	χ^2	df	$\Delta\chi^2/df$	R ²
Model 1 full (Fig. 2 without age)	61.02	19		0.94
Model 1A (FR _{T1} → Math _{T3} = 0)	62.63	20	1.61/1	0.90
Model 1B (FR _{T1} → Math _{T3} = 0) (FR _{T2} → Math _{T3} = 0)	85.03	21	22.40/1***	0.71
Model 1C (FR _{T1} → Math _{T3} = 0) (Math _{T2} → Math _{T3} = 0)	63.30	21	.67/1	0.91
Model 2 full (Fig. 2 with age)	139.71	31		0.92
Model 2A (age _{T1} → vars _{T2} = 0) (age _{T2} → vars _{T3} = 0)	155.23	34	15.52/3	0.90

Note. vars, simultaneous variables.

*** $p < .001$.

Comparing previous FR with other cognitive abilities

To test our third hypothesis, we carried out analyses to measure the relative predictive power of FR and other domain-general cognitive abilities indexed by vocabulary and spatial skills (Spatial Span and Block Design) on math achievement measured at T3. We tested the model depicted in Fig. 3. Lines in Fig. 3 represent hypothesized pathways for the longitudinal predictors of math outcomes. As shown in the figure, all possible pathways between the three types of cognitive skills and the three types of math outcomes were included in the model. This approach is based on previous research supporting these early cognitive skills as possible predictors of later math achievement (e.g., LeFevre et al., 2010; McGrew & Wendling, 2010).

Results from these analyses showed that both of the hypothesized models in Fig. 2 fit the data well (Tables 5 and 6). For the sake of simplicity, we feature here the results of the structural model with age having a direct effect on math achievement at T3 (Fig. 3). Specifically, FR at T2 significantly predicted math achievement at T3 ($\beta = .52, p < .001$). By contrast, spatial skills at T2 did not significantly predict math achievement at T3 ($\beta = .19, p = .205$). Vocabulary at T2 also did not significantly predict math achievement at T3 ($\beta = .15, p = .205$). Similarly, age at T2 did not significantly predict math achievement at T3 ($\beta = .16, p = .150$). The T2 predictors (age, spatial skills, vocabulary, and FR) accounted for 90.2% of the variance in math achievement at T3. In summary, this analysis shows that FR at T2 was a strong unique predictor of math achievement approximately 1.5 years later.

These analyses also enabled us to test the mediating effect of FR between age and math achievement at T3. Age significantly predicted FR, and FR significantly predicted math achievement at T3. Therefore, the first two criteria of mediation were met. As shown in Table 7, the indirect effect was statistically significant ($p < .001$), but the direct effect was not ($p = .198$). Therefore, the third and fourth criteria for mediation were met. As such, FR significantly mediated the relationship between

Table 4

Unstandardized and standardized path coefficients for the structural models comparing previous FR with math reasoning in predicting future math achievement.

Variable	B	SE	β
Model 1C			
FR _{T1} → FR _{T2}	0.77	0.09	0.98***
FR _{T1} → Math _{T2}	0.39	0.05	0.90***
FR _{T2} → Math _{T3}	0.99	0.12	0.96***
Model 2			
FR _{T1} → FR _{T2}	0.74	0.09	0.97***
FR _{T1} → Math _{T2}	0.34	0.04	0.86***
FR _{T1} → Math _{T3}	-.10	0.43	-.34
FR _{T2} → Math _{T3}	0.72	0.49	0.85

*** $p < .001$.

Table 5

Fit indexes for the structural model predicting math achievement at T3 from previous FR, spatial skills, vocabulary, and age.

Index	Indirect effects	Direct effect
Chi-square	67.68	65.69
Degrees of freedom	28	27
Probability level	.00	.00
Normed chi-square	2.42	2.43
Goodness of fit index (GFI)	.84	.85
Comparative fit index (CFI)	.94	.94
Root mean square error of approximation (RMSEA)	.14	.15
Lower bound 90% confidence interval	.1	.1
Upper bound 90% confidence interval	.19	.19
P-close	.10	.10
Standardized root mean square residual (SRMR)	0.04	.04

Table 6

Unstandardized and standardized path coefficients for the structural model predicting math achievement at T3 (with direct effect from age to math achievement).

Variable	<i>B</i>	<i>SE</i>	β
Age to:			
Vocabulary	2.96	.26	.81***
Spatial skills	4.29	.45	.78***
Fluid reasoning	1.37	.20	.71***
Math achievement	.32	.22	.16
Vocabulary → math achievement	.08	.08	.15
Spatial skills → math achievement	.07	.05	.19
Fluid reasoning → math achievement	.52	.16	.52***

*** $p < .001$.**Table 7**

Standardized direct and indirect effects of age on math reasoning at Time 3.

Effect	Model without direct effect	Model with direct effect
Total effect	.76***	.80***
Direct effect	–	.16
Indirect effect	.76***	.64***

*** $p < .001$.

age and math achievement at T3. By contrast, vocabulary and spatial skills did not significantly predict math achievement; thus, these factors did not significantly mediate the relationship between age and math achievement at T3.

Discussion

Summary of results

In this study, we sought to test whether FR, or the ability to analyze novel problems, identify patterns and relationships, and apply logic, contributes to future math achievement throughout primary and secondary schooling. We were particularly interested in comparing FR with other cognitive precursors (verbal and spatial skills) that have been previously linked to math development (e.g., McGrew & Wendling, 2010). Most previous developmental math research studies were conducted in populations of school-aged children in the primary grades and did not incorporate measures of FR into their predictive models. Therefore, the current research provides a necessary extension to the existing developmental math literature by examining the role of FR and its relation to other pertinent cognitive

precursors in predicting future math achievement across a wide age range of children, providing a more comprehensive model of math development.

To this end, we first created a latent factor score of FR from three psychometric tests designed to measure FR (Matrix Reasoning, Analysis Synthesis, and Concept Formation) using CFA. Second, we compared the strength of the associations between previous FR and previous math reasoning on later math achievement at T3 (measured by math reasoning, Applied Problems, and Math Fluency) using SEM. Results showed that across all three time points, previous FR was the strongest predictor of later math achievement at T3 after accounting for previous math reasoning and age. Once we had determined that FR was a better predictor of later math achievement than previous numerical reasoning, we compared the relative contribution of previous FR to other important cognitive abilities associated with math, indexed by verbal reasoning (measured by Vocabulary) and spatial skills (measured by Spatial Span and Block Design), with future math achievement at T3. This model accounted for more than 90% of the variance in math achievement. In this model, FR was the strongest cognitive predictor of future math achievement measured approximately 1.5 years later. Notably, spatial skills, vocabulary, and age were not significant predictors in this model.

Although some studies have shown that spatial skills and verbal comprehension are also robust precursors to future math achievement (e.g., [LeFevre et al., 2010](#); [Li & Geary, 2013](#)), many previous studies have not incorporated measures of FR into their predictive models. Thus, we interpret the current findings as support for the notion that FR is a foundational skill that influences future development of numerical reasoning and potentiates math problem-solving skills. Thus, the findings indicate that FR should be incorporated into future developmental models. These results support and extend [Cattell's \(1971, 1987\)](#) notion that FR development is an important cognitive precursor for even the most basic math skill development, including timed arithmetic as well as more complex equations and word problems.

Study limitations

A limitation of the study is that we did not administer math measures at the first assessment, rendering us unable to control for the initial effect of these domain-specific precursors on future math outcomes. However, we were able to include math reasoning at T2 in our model, which enabled us to compare the relative contribution of previous math reasoning with previous FR in predicting future math achievement. Previous FR emerged as a better predictor of future math achievement than previous math reasoning. This finding builds on previous studies showing that domain-general FR is as good a predictor of later math skills as previous numerical reasoning skills (e.g., [Fuchs et al., 2010](#)). Another limitation of the study is the relatively small sample size. However, the results are statistically reliable, and the wide age range enables us to make a novel contribution to the literature.

Theoretical implications

This work demonstrates that FR and mathematics achievement are linked throughout development and that FR supports mathematical thinking and reasoning throughout the school years. One account for the strong relation between FR and math assessments is that both engage a common underlying cognitive ability called relational reasoning, or the ability to jointly consider multiple relations between different components of a problem ([Carpenter, Fennema, & Franke, 2013](#); [Halford et al., 1998](#); [Miller Singley & Bunge, 2014](#); [Richland, Holyoak, & Stigler, 2004](#); [White, Alexander, & Daugherty, 1998](#)). The emerging ability to reason relationally may form the foundation for mathematical conceptual development, from the time children learn to compare the value of one number with that of another, to the time they learn to extract the value of a fraction by comparing the value of the numerator with the value of the denominator, to the time they learn algebra and need to solve for an unknown variable by keeping in mind the relationship between numbers on both sides of the equal sign, and so on ([Miller Singley & Bunge, 2014](#)).

Demonstrating that FR predicts future math achievement across ages, above and beyond the effects of age, math reasoning, and other cognitive factors correlated with math proficiency—vocabulary and

spatial skills—advances existing developmental theories of mathematics. Whereas many existing developmental theories were formed based on studies involving younger children (~ages 4–9 years) (e.g., [LeFevre et al., 2010](#)), the current sample spans a broader age range of 6 to 21 years. This work also replicates and extends the findings in previous longitudinal research conducted by [Fuchs and colleagues \(2010\)](#) and [Primi and colleagues \(2010\)](#), who found that FR was a robust cognitive predictor of future math achievement over the course of 1 to 2 academic years in children in Grades 1, 7, and 8. Our study included a wider age range of children and adolescents between 6 and 21 years and adopted an analytic approach that enabled us to look at sequential influences of FR and later math proficiency measured by three specific math achievement domains.

These findings expand on an existing developmental framework proposed by [LeFevre and colleagues \(2010\)](#), who hypothesized that there are three different pathways that contribute to early math development in children ages 4 to 7 years: previous linguistic (or verbal) skills, spatial skills, and quantitative skills. In the current analyses, we included these same pathways as well as a fourth pathway, fluid reasoning. Our findings indicate that FR is a robust pathway that may be even more influential to math development than linguistic and spatial skills, although the relative contributions of these predictors should be systematically compared in future research.

Broader implications

More generally, this line of research has possible relevance to school classroom settings. Fluid reasoning is thought to support all forms of new learning for which individuals need to problem solve (by integrating new information) without relying solely on previous knowledge; therefore, FR could be applicable to many subject areas. However, we posit that FR is particularly helpful for learning math, which is hierarchical in nature and requires individuals to solve novel problems as each new level advances. Currently, educators often focus on building computational proficiency as a means to improving mathematical achievement without much consideration of the cognitive precursors that underpin these skills or students' strengths and weaknesses ([Boaler, 1998](#)). Theories such as CHC, as well as longitudinal studies such as this one, provide insights into the links between specific cognitive abilities and math achievement that can inform educational practices. Although some new math curricula do incorporate spatial rotation or block construction exercises, FR has not typically been emphasized in current math curricula. However, even students with strong basic numerical skills and spatial skills might not be proficient in applying logical reasoning techniques to solve novel problems.

We argue that math curriculum should incorporate opportunities for students to practice a core aspect of FR known as relational thinking, or the ability to jointly consider several relations among mental representations ([Miller Singley & Bunge, 2014](#)). One example of a curriculum that incorporates relational thinking practice into math exercises is called early algebra ([Carpenter, Franke, & Levi, 2003](#)). This curriculum involves teaching children as young as 6 years to view the equal sign as a form of equivalency using nontraditional number sentences. For example, children solve equations such as " $5 + 3 = 6 + \underline{\quad}$ " and explain their thinking aloud. By solving these types of equations and having students explain their thinking, students come to understand the component relationship between numbers on opposite sides of the equal sign and can often identify the correct answer without doing any calculations ([Carpenter et al., 2003](#)). Another effective approach involves practicing early abstract reasoning skills with kindergartners and preschoolers to improve early numeracy skills ([Ciancio, Rojas, McMahon, & Pasnak, 2001](#); [Kidd, Pasnak, Gadzichowski, Ferral-Like, & Gallington, 2008](#)). Many other approaches can be used to incorporate FR skill-building opportunities into math curriculum (e.g., [Miller Singley & Bunge, 2014](#)).

Finally, the assessment of FR in elementary school could serve to identify students who are likely to have difficulty in learning math. This information could help guide teachers to better understand which interventions may be most fruitful for individual students at different developmental levels of FR and math achievement skills.

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Appendix A

Frequencies and percentages for missing data

Variable	Frequency	Percentage
Time 1		
Block Design	1	1.4
Matrix Reasoning	4	5.8
Analysis Synthesis	35	50.7
Concept Formation	2	2.9
Vocabulary	3	4.3
Digit Span	38	55.1
Time 2		
Block Design	1	1.4
Matrix Reasoning	0	0.0
Analysis Synthesis	1	1.4
Concept Formation	0	0.0
Vocabulary	10	14.5
Digit Span	0	0.0
Coding	16	23.2
Number Series	0	0.0

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