ON $n$-DEPENDENT GROUPS AND FIELDS II

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Abstract. We continue the study of $n$-dependent groups, fields and related structures. We demonstrate that $n$-dependence is witnessed by formulas with all but one variable singletons, provide a type-counting criterion for 2-dependence and use it to deduce 2-dependence for compositions of NIP relations with arbitrary binary functions. We prove a result on intersections of type-definable connected components over generic sets of parameters in $n$-dependent groups, generalizing Shelah’s results on absoluteness of $G_{\forall}$ in NIP theories and relative absoluteness of $G_{\forall}$ for 2-dependent theories. We show that Granger’s examples of non-degenerate bilinear forms over NIP fields are 2-dependent, and characterize preservation of $n$-dependence under expansion by generic relations for geometric theories in terms of disintegration of their algebraic closure. Finally, we show that every infinite $n$-dependent valued field of positive characteristic is henselian, generalizing a recent result of Johnson for NIP.

1. Introduction

A classical line of research in model theory aims to determine properties of algebraic structures such as groups and fields with additional structure that satisfy certain model-theoretic tameness assumptions, starting with Macintyre’s proof that all $\aleph_0$-stable fields are algebraically closed [26]. In this article we continue the study of groups, fields and related structures satisfying a model-theoretic tameness condition called $n$-dependence, for $n \in \mathbb{N}$, initiated in [19] and continued in [11]. The class of $n$-dependent theories was introduced by Shelah in [30], with the 1-dependent case corresponding to the class of NIP theories that has attracted a lot of attention recently (see e.g. [32] for an introduction to the area). Roughly speaking, $n$-dependence of a theory guarantees that the edge relation of an infinite generic $(n + 1)$-hypergraph is not definable in its models (see Definition 2.1). For $n \geq 2$, we say that a theory is strictly $n$-dependent if it is $n$-dependent, but not $(n - 1)$-dependent.

We give a brief overview of the most relevant literature. The initial work of Shelah introducing $n$-dependence and obtaining a chain condition for type-definable groups of bounded index in 2-dependent theories is contained in [30] and [31], respectively. Basic properties of $n$-dependence are investigated in [13], and some applications to hypergraph growth are studied in [33]. Groups and fields with $n$-dependent theories are further investigated in [19]. Moreover, [11] demonstrates preservation of $n$-dependence for Mekler’s construction and provides first examples of strictly $n$-dependent groups for arbitrary $n$. Finally, [15] establishes a strong regularity lemma for $n$-dependent hypergraphs demonstrating that every $n$-dependent relation of arbitrarily high arity can be approximated by relations of arity $n$ up to measure 0. Here we focus on the implications of the assumption of $n$-dependence for groups, valued fields and related structures, in particular bilinear forms and expansions by generic relations, obtaining some new results on general $n$-dependent theories on the way.

One of the results in [13] gives a characterization of $n$-dependence in terms of generalized indiscernibles (indexed by ordered random partite $n$-hypergraphs) and demonstrates, using this characterization, that in order to verify $n$-dependence of a theory it is enough
to check that every formula $\varphi(x; y_1, \ldots, y_n)$ with at least one of the tuples of variables $x, y_1, \ldots, y_n$ singleton is $n$-dependent (generalizing the well-known theorem of Shelah for NIP). In Section 2 we refine and generalize some of these results allowing indexing structures of larger cardinalities and obtaining a better reduction to singletons: a theory $T$ is $n$-dependent if and only if every formula $\varphi(x, y_1, \ldots, y_n)$ such that all but at most one of the tuples $x, y_1, \ldots, y_n$ are singletons is $n$-dependent (Theorem 2.12).

A crucial fact about (type-)definable groups in NIP theories is the absoluteness of their connected components. Namely, given a definable group $G$ and a small set of parameters $A$, we denote by $G^0_A$ the intersection of all subgroups of $G$ of bounded index type-definable over $A$ (see Section 3.1 for more details). Shelah shows that, assuming $T$ is NIP, for every small set $A$ one has $G^0_A = G^0_0$ 29. This can be viewed as an infinitary analog of the Baldwin-Saxl condition on intersections of uniformly definable families of subgroups in NIP 3. In 31, Shelah established the following relative absoluteness result for groups definable in 2-dependent theories: let $M$ be a sufficiently saturated model, and let $b$ be a finite tuple in $M$ (and not contained in $M$ in the case of interest), then $G^{00}_{M,b} = G^{00}_M \cap G^{00}_b$ for some small set $C \subseteq M$. In Section 3 we generalize this result from 2-dependent groups to $n$-dependent groups. Specifically, we show that if $T$ is $n$-dependent and $G = G(M)$ is a type-definable group (over $\emptyset$), then for any model $M$ and finite tuples $b_1, \ldots, b_{n-1}$ sufficiently independent over $M$ in an appropriate sense, we have that

$$G^{00}_{M_0 \cup b_1 \cup \cdots \cup b_{n-1}} = \bigcap_{i=1, \ldots, n-1} G^{00}_{M_0 \cup b_1 \cup \cdots \cup b_{n-1} \setminus b_i} \cap G^{00}_{C_0 \cup b_1 \cup \cdots \cup b_{n-1}}$$

for some $C \subseteq M$ of absolutely bounded size (Theorem 3.1 and its corollary). In other words, in the intersection on the left hand side we only need boundedly many groups whose definitions involve all $n - 1$ of the parameters $b_1, \ldots, b_{n-1}$ at the same time. The independence assumption on the parameters holds trivially in the cases $n = 1, 2$ giving the aforementioned results for NIP and 2-dependent groups, and in general can be achieved e.g. assuming that the $b$'s appear as the vertices of an amalgamation diagram, with respect to the independence relation of being a $\kappa$-coheir. We also observe that the chain condition on definable families of subgroups in $n$-dependent theories from 19 generalizes to simultaneous intersections of finitely many definable families of subgroups instead of just one (Proposition 3.18).

In 13 a generalization of the Sauer-Shelah lemma to $n$-dependent formulas is given, in particular demonstrating that a formula $\varphi(x; y_1, \ldots, y_n)$ is $n$-dependent if and only if the number of $\varphi$-types over an arbitrary large finite set $A$ of parameters is bounded by $2^{|A|^n - \varepsilon}$ for some $\varepsilon = \varepsilon(\varphi) \in \mathbb{R}_{>0}$. Concerning the number of types over infinite sets of parameters, a well-known result of Shelah 28 shows that if $\varphi(x, y)$ in NIP, then the number of $\varphi$-types over an infinite set of size $\kappa$ is at most $\text{ded}(\kappa)$, where $\text{ded}(\kappa)$ is the supremum over the number of Dedekind cuts in a linear order of cardinality $\kappa$. In Section 4 we characterize 2-dependence as bounding by $\text{ded}(\kappa)$ the number of types over a finite tuple and an indiscernible sequence of size $\kappa$ that are realized cofinally in a sequence mutually indiscernible to it — see Proposition 4.16 for the details.

In Section 5 this criterion is combined with set-theoretic absoluteness to obtain a more general version of the following finitary combinatorial statement of independent interest. Let $R \subseteq M^3$ be a ternary relation definable in an NIP structure, and let $f : M^3 \rightarrow M$ be an arbitrary (not necessarily definable) function. Then the ternary relation $R'(x, y, z) = R(f(x, y), f(x, z), f(y, z))$ is 2-dependent (Theorem 5.1). It is interesting to compare this to a line of results around Hilbert’s 13th problem demonstrating that a function of arbitrary arity can be expressed as a finite composition of binary functions (in the category of all functions, or of continuous functions on $\mathbb{R}$ — a celebrated theorem of Kolmogorov
and Arnold [11]). Our result can be viewed as saying that in such presentations, the outer relation is necessarily “fractal-like”.

In [19] it was observed that the theory of a bilinear form on an infinite dimensional vector space over a finite field is strictly 2-dependent. In Section 6, we investigate n-dependence for certain theories of bilinear forms on vector spaces with a separate sort for the field, finite or infinite, in the sense of Granger [18]. Using the result on 2-dependence of compositions of NIP relations and binary functions described above, we show that all such theories are 2-dependent assuming that the field is NIP, and that the NIP assumption is necessary (see Theorem 6.3).

Curiously, all of the “algebraic” examples of strictly n-dependent theories with n ≥ 2 that we are aware of tend to look like multi-linear forms over NIP fields. E.g., smoothly approximable structures are 2-dependent and coordinatizable via bilinear forms over finite fields [9]; and the strictly n-dependent pure groups constructed in [11] using Mekler’s construction are essentially of this form, using Baudisch’s interpretation of Mekler’s construction in alternating bilinear maps [4]. Additionally, the intersection conditions on the connected components discussed above resemble modular behavior in the 2-dependent case. This leads one to speculate that n-dependence might imply some form of “linearity relative to the NIP part”. While even formulating this precisely seems difficult, we state a specific instance of this principle as a conjecture.

**Conjecture 1.** There are no strictly n-dependent fields for n ≥ 2 (in the pure ring language).

The same should hold for fields expanded with natural operators such as derivations, valuations, etc. Some limited evidence towards this conjecture is given by the results in [19]: every infinite n-dependent field is Artin-Schreier closed (generalizing [24] for n = 1); every non-separable PAC (i.e. pseudo-algebraically closed) field is not n-dependent for any n (generalizing [16]). In particular, for fields with (super-)simple theories, our conjecture follows from the well-known conjecture that all such fields are (bounded) PAC (see e.g. [27]).

In Section 7 we consider n-dependence for expansions of geometric theories by generic predicates and relations of higher arity. In particular, we show that an expansion of a geometric theory T by a generic predicate is NIP if and only if it is n-dependent for some n, if and only if the algebraic closure in T is disintegrated (Corollary 7.11). This generalizes (and corrects) the corresponding result for NIP in [8]. This collapse of the n-dependence hierarchy caused by non-disintegrated algebraic closure could be viewed as a “toy example” of the conjectured situation for fields. Our proof for relations of higher arity relies on an infinitary generalization of Hrushovski’s observation [20] that the random n-ary hypergraph is not a finite Boolean combination of relations of arity n − 1.

In Section 8 we obtain further evidence towards the conjecture in the case of valued fields. A recent result of Johnson [22] shows that every infinite NIP valued field of positive characteristic is henselian. In Theorem 8.11 we generalize this to n-dependent fields, for arbitrary n. As in Johnson’s proof, the theorem is deduced by showing that any two valuations on an infinite n-dependent field of positive characteristic must be comparable. In the case of n = 1, this lemma can be quickly obtained using Artin-Schreier closeness of NIP fields and absoluteness of the connected component G^00 (in fact, an application of Baldwin-Saxl is sufficient). However, replacing the absolute connected component with a weaker condition for intersections of uniformly definable families of subgroups available in n-dependent theories (Proposition 3.18) requires a detailed analysis of the effect that the isomorphism for special linear groups from Kaplan-Scanlon-Wagner [24] has on multiple valuations. We are able to carry it out relying in particular on the explicit description
of this isomorphism given by Bays [9]. Finally, in Section 5.6 we observe that the model completions of multi-ordered and multi-valued fields with at least two orders (respectively, valuations) as studied in [34, 23] are not n-dependent for any n.

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2. Preliminaries and some general lemmas on n-dependence

2.1. Definition and basic properties. We begin by recalling the definition of n-dependence and some of the basic properties of n-dependent formulas.

Definition 2.1. A partitioned formula \( \varphi(x; y_0, \ldots, y_{n-1}) \) has the \( k \)-independence property (with respect to a theory \( T \)), if in some model of \( T \) there is a sequence of tuples \( (a_{0,i}, \ldots, a_{n-1,i}) \in \omega \) such that for every \( s \subseteq \omega \), there is a tuple \( b_s \) with the following property:

\[ \models \varphi(b_s; a_{0,i_0}, \ldots, a_{n-1,i_{n-1}}) \iff (i_0, \ldots, i_{n-1}) \in s. \]

Otherwise we say that \( \varphi(x, y_0, \ldots, y_{n-1}) \) is n-dependent. A theory is n-dependent if it implies that every formula is n-dependent.

Fact 2.2. [13] Proposition 6.5]

1. Let \( \varphi(x, y_1, \ldots, y_n) \) and \( \psi(x, y_1, \ldots, y_n) \) be n-dependent formulas. Then \( \neg\varphi \), \( \varphi \land \psi \) and \( \varphi \lor \psi \) are n-dependent.

2. Let \( \varphi(x, y_1, \ldots, y_n) \) be a formula. Suppose that \( (w, z_1, \ldots, z_n) \) is any permutation of the tuple \( (x, y_1, \ldots, y_n) \). Then \( \psi(w, z_1, \ldots, z_n) := \varphi(x, y_1, \ldots, y_n) \) is n-dependent if and only if \( \varphi(x, y_1, \ldots, y_n) \) is n-dependent.

3. A theory \( T \) is n-dependent if and only if every formula \( \varphi(x, y_1, \ldots, y_n) \) with \( |x| = 1 \) is n-dependent (see also Section 2.2).

2.2. Generalized indiscernibles. We will often use a characterizations of n-dependence from [13] in terms of generalized indiscernibles.

Definition 2.3. Fix a language \( L^n_{\text{opg}} = \{R_n(x_1, \ldots, x_n), <, P_1(x), \ldots, P_n(x)\} \). An ordered \( n \)-partite hypergraph is an \( L^n_{\text{opg}} \)-structure \( A = (A; <, R_n, P_1, \ldots, P_n) \) such that:

1. \( A \) is the disjoint union \( P_1^A \sqcup \ldots \sqcup P_n^A \),
2. \( R_n^A \) is a symmetric relation so that if \( (a_1, \ldots, a_n) \in R_n^A \) then \( P_i^A \cap \{a_1, \ldots, a_n\} \) is a singleton for every \( 1 \leq i \leq n \),
3. \( <^A \) is a linear ordering on \( A \) with \( P_1^A < \ldots < P_n^A \).

Fact 2.4. [13] Fact 4.4 + Remark 4.5] Let \( \mathcal{K} \) be the class of all finite ordered \( n \)-partite hypergraphs. Then \( \mathcal{K} \) is a Fraïssé class, and its limit is called the generic ordered \( n \)-partite hypergraph, denoted by \( G_{n,p} \). An ordered \( n \)-partite hypergraph \( A \) is a model of Th(\( G_{n,p} \)) if and only if:

- \( (P_i^A, <) \) is a dense linear order without endpoints for each \( 1 \leq i \leq n \),
- for every \( 1 \leq j \leq n \), finite disjoint sets \( A_0, A_1 \subset \prod_{1 \leq i \leq n, i \neq j} P_i^A \) and \( b_0 < b_1 \in P_j^A \), there is some \( b \in P_j^A \) such that \( b_0 < b < b_1 \) and: \( R_n(b, a) \) holds for every \( a \in A_0 \) and \( \neg R_n(b, a) \) holds for every \( a \in A_1 \).
We denote by $O_{n,p}$ the reduct of $G_{n,p}$ to the language $\mathcal{L}_{op}^n = \{<, P_1(x), \ldots, P_n(x)\}$.

**Remark 2.5.** It is easy to see from the axiomatization that given $G_{n,p}$ and any non-empty intervals $I_t \subseteq P_t$ for $t = 1, \ldots, n$, the set $I_1 \times \ldots \times I_n$ contains an induced copy of $G_{n,p}$.

**Definition 2.6.** Let $T$ be a theory in a language $\mathcal{L}$, and let $\mathbb{M}$ be a monster model of $T$.

1. Let $I$ be a structure in the language $\mathcal{L}_0$. We say that $\bar{a} = (a_i)_{i \in I}$, with $a_i$ a tuple in $\mathbb{M}$, is $I$-indiscernible over a set of parameters $C \subseteq \mathbb{M}$ if for all $n \in \omega$ and all $i_0, \ldots, i_n$ and $j_0, \ldots, j_n$ from $I$ we have:

$$qftp_{\mathcal{L}_0}(i_0, \ldots, i_n) = qftp_{\mathcal{L}_0}(j_0, \ldots, j_n) \Rightarrow$$

$$tp_{\mathcal{L}}(a_{i_0}, \ldots, a_{i_n}/C) = tp_{\mathcal{L}}(a_{j_0}, \ldots, a_{j_n}/C).$$

2. For $\mathcal{L}_0$-structures $I$ and $J$, we say that $(b_i)_{i \in I}$ is based on $(a_i)_{i \in I}$ over a set of parameters $C \subseteq \mathbb{M}$ if for any finite set $\Delta$ of $\mathcal{L}(C)$-formulas, and for any finite tuple $(j_0, \ldots, j_n)$ from $J$ there is a tuple $(i_0, \ldots, i_n)$ from $I$ such that:

- $qftp_{\mathcal{L}_0}(j_0, \ldots, j_n) = qftp_{\mathcal{L}_0}(i_0, \ldots, i_n)$ and

- $tp_{\Delta}(b_{j_0}, \ldots, b_{j_n}) = tp_{\Delta}(a_{i_0}, \ldots, a_{i_n}).$

The following fact gives a method for finding $G_{n,p}$-indiscernibles using structural Ramsey theory.

**Fact 2.7.** [13] Corollary 4.8] Let $C \subseteq \mathbb{M}$ be a small set of parameters.

1. For any $n \in \omega$ and $\bar{a} = (a_g)_{g \in O_{n,p}}$, there is some $(b_g)_{g \in O_{n,p}}$ which is $O_{n,p}$-indiscernible over $C$ and is based on $\bar{a}$ over $C$.

2. For any $n \in \omega$ and $\bar{a} = (a_g)_{g \in G_{n,p}}$, there is some $(b_g)_{g \in G_{n,p}}$ which is $G_{n,p}$-indiscernible over $C$ and is based on $\bar{a}$ over $C$.

**Fact 2.8.** The following are equivalent, in any theory $T$.

1. $\varphi(x; y_1, \ldots, y_n)$ is not $n$-dependent.

2. There are tuples $b$ and $(a_g)_{g \in G_{n,p}}$ such that

   - $(a_g)_{g \in G_{n,p}}$ is $O_{n,p}$-indiscernible over $\emptyset$ and $G_{n,p}$-indiscernible over $b$;
   - $\models \varphi(b; a_{g_1}, \ldots, a_{g_n}) \iff G_{n,p} \models R_n(g_1, \ldots, g_n)$, for all $g_i \in P_i$.

3. (2) holds for any small $G'_{n,p} \equiv G_{n,p}$ in the place of $G_{n,p}$.

**Proof.** The equivalence of (1) and (2) is from [13] Lemma 6.2] and (3) implies (2) is obvious. And given a witness to (2), we can find a witness to (3) by compactness as every finite substructure of $G'_{n,p}$ appears as a finite substructure of $G_{n,p}$. □

**Fact 2.9.** Let $T$ be a complete theory and let $\mathbb{M} \models T$ be a monster model. Then for any $n \in \mathbb{N}$, the following are equivalent:

1. $T$ is $n$-dependent.

2. For any $(a_g)_{g \in G_{n,p}}$ and $b$ with $a_g, b$ finite tuples in $\mathbb{M}$, if $(a_g)_{g \in G_{n,p}}$ is $G_{n,p}$-indiscernible over $b$ and $O_{n,p}$-indiscernible (over $\emptyset$), then it is $O_{n,p}$-indiscernible over $b$.

3. For any small $G'_{n,p} \equiv G_{n,p}$, $(a_g)_{g \in G_{n,p}}$ and $b$, if $(a_g)_{g \in G_{n,p}}$ is $G'_{n,p}$-indiscernible over $b$ and $O'_{n,p}$-indiscernible, then it is $O'_{n,p}$-indiscernible over $b$ (where $O'_{n,p}$ is the $\mathcal{L}_{op}^n$-reduct of $G'_{n,p}$).

**Proof.** The equivalence of (1) and (2) is from [13] Proposition 6.3], (3) implies (2) is obvious, and we show that (2) implies (3). Assume that (3) fails, i.e. there exist some
$G_{n,p}' \equiv G_{n,p}, (a_g)_{g \in G_{n,p}}$ and $b$ small tuples such that $(a_g)_{g \in G_{n,p}'}$ is $O_{n,p}'$-indiscernible, $G_{n,p}'$-indiscernible over $b$, but not $O_{n,p}$-indiscernible over $b$. By definition, this is witnessed by some finite set of formulas and some finite set of indices from $G_{n,p}'$. Restricting all of the $a_g$’s and $b$ to the corresponding subtuples appearing in those formulas, we may assume that (3) fails with all of $b$ and $a_g$ finite. Moreover, we can choose a countable elementary submodel of $G_{n,p}$ containing all of the indices witnessing failure of indiscernibility. It is isomorphic to $G_{n,p}$ by $\aleph_0$-categoricity of $\text{Th}(G_{n,p})$, hence restricting $(a_g)_{g \in G_{n,p}'}$ to the corresponding set of indices we get a failure of (2).

2.3. Improved reduction to singletons. We need a refinement of \cite{13} Proposition 6.3] showing that in the equivalence we can control not just one variable, but in fact all but at most one variables.

**Proposition 2.10.** Fix $n \geq 1$ and let $x, y_1, \ldots, y_{n-1}$ be some fixed finite tuples of variables. The following are equivalent:

1. There exists some $(a_g)_{g \in G_{n,p}}$ with $a_g \in M_{y_i}$ for all $g \in P_i$, $1 \leq i \leq n-1$ and $a_g \in M_{y_n}$ for some finite tuple of variables $y_n$ and all $g \in P_n$, and $b \in M_x$ such that $(a_g)_{g \in G_{n,p}}$ is $G_{n,p}$-indiscernible over $b$ and $O_{n,p}$-indiscernible over $\emptyset$, but is not $O_{n,p}$-indiscernible over $b$.

2. There exists some formula $\varphi(x, y_1, \ldots, y_{n-1}, y_n')$, with $y_n'$ some finite tuple of variables, which is not $n$-dependent.

**Proof.** (2) implies (1). Immediate from \cite{13} Lemma 6.2, with $y_n' = y_n''$.

(1) implies (2). We are following the proof in \cite{13} Lemma 6.3 with some modifications. Let $(a_g)_{g \in G_{n,p}}$ and $b$ be as given by (1). We define $a_g' := a_g$ for all $g \in \bigcup_{1 \leq i \leq n-1} P_i$ and $a_g' := a_g b$ for all $g \in P_n$. We have that $(a_g')_{g \in G_{n,p}}$ is not $O_{n,p}$-indiscernible, but is $G_{n,p}$-indiscernible (over $\emptyset$), from the corresponding properties of $(a_g)_{g \in G_{n,p}}$ over $b$ (namely, if there are some finite subsets $V, W \subseteq G_{n,p}$ with the $L_{op}$-isomorphic induced structures but such that $(a_g)_{g \in V} \not\equiv_b (a_g)_{g \in W}$, then taking some $h \in P_n$ above all of the elements of $V \cup W$ with respect to the order on $P_n$, we have that $V h \not\equiv_{L_{op}} W h$ and, since the tuple $a_h'$ contains $b$, we have $(a_g')_{g \in V h} \not\equiv (a_g')_{g \in W h}$. Then by \cite{13} Proposition 5.8] there is an $L_{op}$-substructure $G' \subseteq G_{n,p}$, a finite set $V \subseteq G_{n,p}$ and a formula $\psi(y_1, \ldots, y_{n-1}, y_n, z) \in L$ such that $y_n = y_n'' x$, $z$ is a finite tuple of variables corresponding to a fixed enumeration $(a_g)_{g \in V}$ of $V$, and

1. $G' \cong_{L_{op}} G_{n,p}$.
2. $R_n(g_1, \ldots, g_n)$ holds if and only if $\psi(a'_{g_1}, \ldots, a'_{g_n}, (a_g')_{g \in V})$ for every $g_i \in P_i(G')$.
3. For every finite $W, W' \subseteq G'$ we have $W V \equiv_{L_{op}} W' V$ whenever $W \equiv_{L_{op}} W'$.

Let $\varphi(x, y_1, \ldots, y_{n-1}, y_n'')$ be the formula $\psi(y_1, \ldots, y_{n-1}, y_n' x, z)$ with $y_n'' := y_n' z$, and let $a_g'' := a_g'$ for $g \in P_i(G')$, $1 \leq i \leq n-1$ and let $a_g'' := a_g(h \in V$ for $g \in P_n(G')$. Then $(a_g'')_{g \in G'}$ is $L_{op}$-indiscernible (by $L_{op}$-indiscernibility of $(a_g')_{g \in G_{n,p}}$ and the choice of $V$) and $\varphi(b, a_g''_1, \ldots, a_g''_n)$ holds if and only if $R(g_1, \ldots, g_n)$ does, for all $g_i \in P_i(G')$. Then $\varphi$ is not $n$-dependent by \cite{13} Lemma 6.2.}

**Lemma 2.11.** Let $y_1, \ldots, y_{n-1}$ be some fixed finite tuples of variables. If the condition (1) in Proposition 2.11 holds for some finite tuple of variables $x$, then it already holds with $x$ a single variable.

**Proof.** We assume that (1) fails for $|x| = 1$, and prove that then it fails for any tuple of variables $x$ by induction on $|x|$. So let $b \in M_x$ with $|b| > 1$ be given, say $b = b_1 b_2$ for some tuples $1 \leq |b_1|, |b_2| < n$. And assume that $(a_g)_{g \in G_{n,p}}$ with $a_g \in M_{y_i}$ for
\(g \in P_i, 1 \leq i \leq n - 1\) is such that \((a_g)_{g \in G_{n,p}}\) is \(G_{n,p}\)-indiscernible over \(b\) and \(O_{n,p}\)-indiscernible over \(\emptyset\). We need to show that \((a_g)_{g \in G_{n,p}}\) is \(O_{n,p}\)-indiscernible over \(b\).

In particular \((a_g)_{g \in G_{n,p}}\) is \(G_{n,p}\)-indiscernible over \(b_2\), hence it is \(O_{n,p}\)-indiscernible over \(b_2\) by the inductive assumption. Let \(a'_g := a_g\) for \(g \in P_i, 1 \leq i \leq n - 1\) and let \(a'_g := a_gb_2\) for \(g \in P_n\). Note that \((a'_g)_{g \in G_{n,p}}\) is \(G_{n,p}\)-indiscernible over \(b_1\), and is \(O_{n,p}\)-indiscernible over \(\emptyset\) by the previous sentence. Applying the inductive assumption again, we conclude that \((a'_g)_{g \in G_{n,p}}\) is \(O_{n,p}\)-indiscernible over \(b_1\), hence \((a_g)_{g \in G_{n,p}}\) is \(O_{n,p}\)-indiscernible over \(b = b_1b_2\).

**Corollary 2.12.** Assume that the formula \(\varphi(x, y_1, \ldots, y_n)\) is not \(n\)-dependent. Then there exists some formula \(\varphi'(x', y_1, \ldots, y_{n-1}, y'_n)\) which is not \(n\)-dependent, and such that \(x'\) is a single variable and \(y'_n\) is some finite tuple of variables extending \(y_n\).

**Proof.** By Lemma 2.11 and the equivalence of (1) and (2) in Proposition 2.10.

Using this, we can strengthen Fact 2.2.

**Theorem 2.13.** A theory \(T\) is \(n\)-dependent if and only if every formula \(\varphi(x, y_1, \ldots, y_n)\) such that all but at most one of the tuples \(x, y_1, \ldots, y_n\) are singletons is \(n\)-dependent.

**Proof.** Assume that some formula \(\varphi(x, y_1, \ldots, y_n)\) is not \(n\)-dependent. Applying Corollary 2.12, we find some formula \(\varphi'(x', y_1, \ldots, y_{n-1}, y'_n)\) which is not \(n\)-dependent, \(x'\) is a singleton and \(y'_n\) is a tuple of variables extending \(y_n\). Exchanging the roles of \(x'\) and \(y_1\) by Fact 2.2 we thus obtain a formula \(\varphi_1(y_1, y'_1, y_2, \ldots, y_{n-1}, y'_n)\) which is not \(n\)-dependent and \(|y'_1| = 1\). Repeating the same procedure recursively with \(y_i\) in the role of \(y_1\), for \(1 \leq i \leq n - 1\) we find formulas \(\varphi_i(y_1, y'_1, \ldots, y'_i, y_{i+1}, \ldots, y_{n-1}, y'_n)\) which are not \(n\)-dependent, \(|y'_i| = 1\) for \(1 \leq j \leq i\) and \(y_{i+1}^{n+1}\) extending \(y'_n\). Finally, taking \(\varphi_{n-1}(y_{n-1}, y'_1, \ldots, y_{n-1}^{n-1}, y_n^{n-1})\) and applying Corollary 2.12 one more time, we obtain the desired formula with all but the last variable singletons.

### 3. Connected components of \(n\)-dependent groups

#### 3.1. Connected components

We begin by recalling some facts about model-theoretic connected components and state the main theorem of the section.

**Definition 3.1.** Let \(T\) be an arbitrary theory, \(G = G(M)\) an \(\emptyset\)-definable group and \(A \subseteq M\) a small set of parameters. Then \(G^0_A\) is the intersection of all \(A\)-definable subgroups of \(G\) of finite index, and \(G^\emptyset_A\) is the intersection of all subgroups of \(G\) of bounded index that are type-definable over \(A\).

As \(A\) is small, we still have that \(G^0_A, G^\emptyset_A\) are type-definable subgroups of \(G\) of bounded index. The following lemma is standard.

**Lemma 3.2.** If \(H\) is a type-definable subgroup of \(G\) of bounded index, then it can be written as an intersection of groups of bounded index each of which is defined by a partial type consisting of countably many formulas.

A fundamental fact about NIP groups is the absoluteness of their connected components:

**Fact 3.3.** [29] Let \(T\) be NIP. Then \(G^0_A = G^0_{\emptyset}\) for every small set \(A\). In particular, the intersection of all type-definable subgroups of \(G\) of bounded index is a normal subgroup type-definable over \(\emptyset\) and is of index \(\leq 2^{21}\).
This fact doesn’t hold for 2-dependent groups:

**Example 1.** Let $G$ be the group $\mathbb{F}_2^{(\omega)}$, where $\mathbb{F}_2$ is the finite field with 2 elements. Let $M := (G, \mathbb{F}_2, 0, +, \cdot)$ be the structure with $+$ the addition in $G$ and $\cdot$ the bilinear form $(a_i) \cdot (b_i) = \sum a_i b_i$ from $G^2$ to $\mathbb{F}_2$. Then $\text{Th}(M)$ is simple and 2-dependent, and $G_A^{00} = \{ g \in G : g \cdot a = 0 \text{ for all } a \in A \}$ (see [19] Section 3 and Section 4), so the group $G_A^{00}$ gets smaller as $A$ grows.

However, in this example for any small sets $A, B$ we have $G_{A \cup B}^{00} = G_A^{00} \cap G_B^{00}$. The following theorem of Shelah shows that, up to a small error, this holds in an arbitrary 2-dependent group:

**Fact 3.4.** [31] Let $T$ be 2-dependent, $G = G(M)$ an $A$-type definable group, $\kappa := \beth_2(|A| + |T|^+)\cdot |M|$, $M \supseteq A$ a $\kappa$-saturated model, and $b$ an arbitrary finite tuple in $M$. Then

\[ G_{M,b}^{00} = G_M^{00} \cap G_C^{00} \]

for some $C \subseteq M$ with $|C| < \kappa$.

In this section, we generalize this result to $k$-dependent groups for arbitrary $k$. In order to state our generalization, we need to introduce an appropriate notion of independence.

**Definition 3.5.** ($\kappa$-coheirs) For any cardinal $\kappa$, any model $M$, and any tuple $\bar{a}$ we write

\[ a \downarrow_{M}^{u,\kappa} B \]

if for any set $C \subset B \cup M$ of size $< \kappa$, $\text{tp}(a/C)$ is realized in $M$.

Hence \( a \downarrow_{M}^{u,\kappa} B \iff a \downarrow_{M}^{u} B \), i.e. if and only if $\text{tp}(a/BM)$ is finitely satisfiable in $M$. Recall that for an infinite cardinal $\kappa$ and $n \in \omega$, the cardinal $\beth_n(\kappa)$ is defined inductively by $\beth_0(\kappa) = \kappa$ and $\beth_{n+1}(\kappa) = 2^{\beth_n(\kappa)}$. Then the Erdős-Rado theorem says that $(\beth_r(\kappa))^+ \rightarrow (\beth^+)^r_{\kappa}$ for all infinite $\kappa$ and $r \in \omega$.

**Definition 3.6.** (Generic position) Let $M$ be a small model, $A$ a subset of $M$, and $\bar{b}_1, \ldots, \bar{b}_{k-1}$ finite tuples in $M$. We say that $(M, A, \bar{b}_1, \ldots, \bar{b}_{k-1})$ are in a generic position if there exist regular cardinals $\kappa_1 < \kappa_2 < \ldots < \kappa_{k-1}$ and models $M_0 \subseteq M_1 \subseteq \ldots \subseteq M_{k-1} = M$ such that $A \subseteq M_0$, $\beth_2(|M_i|^+) \leq \kappa_{i+1}$ for $i = 0, \ldots, k-2$ and

\[ \bar{b}_i \downarrow_{M_i}^{u,\kappa_i} \bar{b}_i \circ M_{k-1} \]

for all $1 \leq i \leq k-1$.

Now we can state our main theorem of the section.

**Theorem 3.7.** Let $k \geq 1$, $T$ a $k$-dependent theory, $A \subseteq M \models T$ a small parameter set and $G = G(M)$ a type-definable group in $T$ over $A$ be given. Let $M \supseteq A$ be a model and $\bar{b}_1, \ldots, \bar{b}_{k-1}$ finite tuples in $M$ in a generic position.

Let $(H_\alpha : \alpha \in I)$ be any family of subgroups of $G$ of bounded index, with each $H_\alpha$ type-definable over $M\bar{b}_1 \ldots \bar{b}_{k-1}$. Then there is some $J \subseteq I$ with $|J| \leq \beth_2(|T| + |A|)$ such that

\[ \bigcap_{\alpha \in I} H_\alpha \cap \bigcap_{i=1,\ldots,k-1} G_{M,\bar{b}_1 \cup \ldots \cup \bar{b}_{k-1}}^{00} \bar{b}_i \bigcap \bigcap_{\alpha \in J} H_\alpha \cap \bigcap_{i=1,\ldots,k-1} G_{M,\bar{b}_1 \cup \ldots \cup \bar{b}_{k-1}}^{00} \bar{b}_i \]

**Corollary 3.8.** Let $T$ be a $k$-dependent theory, $A \subseteq M \models T$ a small parameter set and $G = G(M)$ a type-definable group in $T$ over $A$ be given. Let $M \supseteq A$ be a model and $\bar{b}_1, \ldots, \bar{b}_{k-1}$ finite tuples in $M$ in a generic position. Then there is some $C \subseteq M$ with $|C| \leq \beth_2(|T| + |A|)$ such that

\[ G_{M,\bar{b}_1 \cup \ldots \cup \bar{b}_{k-1}}^{00} = \bigcap_{i=1,\ldots,k-1} G_{M,\bar{b}_1 \cup \ldots \cup \bar{b}_{k-1}}^{00} \cap G_{C,\bar{b}_1 \cup \ldots \cup \bar{b}_{k-1}}^{00}. \]
By Theorem 3.7, it is already given by some sub-intersection of size at most $C$ so taking (1) For job.

\[ \text{Example 2.} \]

Let $h \in \mathcal{A}$ hence $\mathcal{L}'$. Problem 3.10. We don’t know if the independence assumptions on $\bar{b}_1, \ldots, \bar{b}_{k-1}$ in Theorem 3.4 are necessary or can be relaxed.

\[ \text{Remark 3.11.} \] Some variant of Corollary 3.8 is alluded to in [31, Discussion 2.14(2)], but we are not aware of any followup.

The rest of the section is organized as follows. In Section 3.2 we give a proof of Theorem 3.7, and in Section 3.3 we give a generalization of the chain condition for uniformly definable families of subgroups of $k$-dependent groups from [19], showing that it can be applied to finitely many families simultaneously (this will be applied to valued fields in Section 5).

3.2. Proof of Theorem 3.7. First a couple of folklore observations.

\[ \text{Lemma 3.12.} \]

Let $G = G(\mathcal{M})$ be a type definable group and let $H$ be a subgroup of $G$ of bounded index which is type-definable over a small set of parameters $C$. Then $[G : H] \leq 2^{[T] + |C|}$.

\[ \text{Proof.} \] Suppose not. Then there is a sequence $(g_\gamma)_{\gamma \in (2^{[T] + |C|})^+}$ of representative of cosets of $H$ in $G$, i.e. $\{g_\gamma H : \gamma \in (2^{[T] + |C|})^+\}$ is a list of distinct cosets of $H$ in $G$. Now, for
\[\gamma < \delta < (2^{|T|+|C|})^+\] we have that \(g_\gamma^{-1}g_\delta \notin H\). Thus there is a formula \(\psi_{\gamma,\delta} \in \mathcal{L}(C)\) in the partial type defining \(H\) such that \(\models \neg \psi_{\gamma,\delta}(g_\gamma^{-1} \cdot g_\delta)\). As there are at most \((|T| + |C|)^-\) many formulas over \(C\), by \(\text{Erdős-Rado}\) there exists an infinite subset \(I\) of \((2^{|T|+|C|})^+\) and a single formula \(\psi\) from the partial type defining \(H\) such that for all \(\gamma < \delta \in I\) we have that \(\psi_{\gamma,\delta} = \psi\). By compactness, for any small cardinal \(\kappa\) we can then find a sequence \((k_\gamma)_{\gamma \in \kappa}\) of elements of \(G\) such that for any \(\gamma < \delta < \kappa\), we have that \(\models \neg \psi(k_\gamma^{-1}k_\delta)\). This implies that \(k_\gamma^{-1}k_\delta \notin H\) for all \(\gamma < \delta < \kappa\), which contradicts that \(H\) has bounded index in \(G\).

Finally, we are ready to prove the main theorem.

**Proof of Theorem 3.7.** Assume that the conclusion fails, and let \(A \subseteq \mathcal{M}_0 \leq \mathcal{M}_1 \leq \cdots \leq \mathcal{M}_{k-1} = \mathcal{M}\) witness the generic position as in Definition 3.6. Then, using Lemma 3.2, we can find inductively a sequence of \((\mathcal{M}_{k-1} \cup \bar{b}_1 \cup \cdots \cup \bar{b}_{k-1})\)-type-definable subgroups \(H_\alpha, \alpha < \kappa := \beth_2(|T| + |A|)^+\) of \(G\) of bounded index such that \(H_\alpha = \bigcap_{n < \omega} \psi_n^\alpha(G; \bar{c}_\alpha, \bar{b}_1, \ldots, \bar{b}_{k-1})\) for some countable \(\bar{c}_\alpha\) from \(\mathcal{M}_{k-1}\), and elements \((d_\alpha)_{\alpha < \kappa}\) in \(G\) such that:

1. \(d_\alpha \in \bigcap_{n=1, \ldots, k-1} \psi_n^{\#}(\mathcal{M}_{k-1} \cup \bar{b}_1 \cup \cdots \cup \bar{b}_{k-1}) \setminus \bigcap_{\beta < \alpha} H_\beta\),
2. \(d_\alpha \notin H_\alpha\).

Using compactness, and possibly replacing each \(\psi_n^\alpha\) by a finite conjunction of \(\psi_i^\alpha\)'s, we may additionally assume that the following hold:

3. \(\models \neg \psi_0^\alpha(d_\alpha; \bar{c}_\alpha, \bar{b}_1, \ldots, \bar{b}_{k-1})\),
4. \(\psi_{n+1}^\alpha(x; \bar{c}_\alpha, \bar{b}_1, \ldots, \bar{b}_{k-1}), \psi_{n+1}^\alpha(y; \bar{c}_\alpha, \bar{b}_1, \ldots, \bar{b}_{k-1}) \models \psi_n^\alpha(xy; \bar{c}_\alpha, \bar{b}_1, \ldots, \bar{b}_{k-1}) \land \psi_n^\alpha(x^{-1}; \bar{c}_\alpha, \bar{b}_1, \ldots, \bar{b}_{k-1}) \land \psi_n^\alpha(y^{-1}; \bar{c}_\alpha, \bar{b}_1, \ldots, \bar{b}_{k-1})\).

As there are only \(|T|^{<\kappa_0}\) many formulas and \(\text{cf}(\kappa) > |T|^{\kappa_0}\), by pigeonhole and after dropping some of the \(H_\alpha\)'s, we can find \((\psi_n)_{n \in \omega}\) such that for all \(\alpha < \kappa\),

\[\psi_\alpha = \psi_n.\]

**Claim 3.13.** We may assume that for all \(i, j \in \omega\), we have that

\[d_i \in H_j \iff i \neq j.\]

**Proof.** By Lemma 3.12, for each \(\beta < \kappa\) there is a partition \(\{g_{\beta,\nu}H_\beta : \nu < \theta_\beta\}\) of \(G\), where \(\theta_\beta = |G : H_\beta| \leq 2^{\kappa_0}\). Moreover, considering \(d_\alpha\) for some \(\alpha < \kappa\), there is \(\nu_{\alpha,\beta} < \theta_\beta\) such that

\[d_\alpha \in g_{\beta,\nu_{\alpha,\beta}}H_\beta.\]

As \(\kappa = \beth_2(|T| + |A|)^+)\) by assumption, by \(\text{Erdős-Rado}\) we can find an infinite subset \(J\) of \(\kappa\) and \(\nu < 2^{\kappa_0}\), such that for all \(\alpha < \beta\) in \(J\), we have that

\[d_\alpha \in g_{\beta,\nu}H_\beta.\]

We may assume that \(J = \omega\). Now let \(i < j < k \in \omega\). Then by the above we have that

\[d_i \in g_{k,\nu}H_k \text{ and } d_j \in g_{k,\nu}H_k,\]

or in other words

\[d_i = g_{k,\mu}h_i \text{ and } d_j = g_{k,\mu}h_j\]

for some \(h_i, h_j \in H_k\). Thus

\[d_i^{-1}d_j = h_i^{-1}g_{k,\mu}h_j = h_i^{-1}h_j \in H_k.\]
Now for $i \in \omega$, let
\[ e_i = d_{2i}^{-1}d_{2i+1}, \quad K_i = H_{2i}, \quad \varphi_n = \psi_{n+1}, \text{ and } \bar{f}_i = \bar{c}_{2i}. \]
Then, after replacing $d_j$ by $e_i$, $H_i$ by $K_i$, $\psi_i$ by $\varphi_i$, and $c_i$ by $f_i$, the conditions (1) and (4) are obviously still satisfied. To show that condition (3) remains true, assume the opposite, i.e. $\models \varphi_0(e_i; f_i, b_1, \ldots, b_{k-1}) \iff \models \psi_1(e_i; c_{2i}, b_1, \ldots, b_{k-1})$. By (1), we know that $\models \psi_1((d_{2i+1}; c_{2i}, b_1, \ldots, b_{k-1})$. Now, using that $d_{2i} = d_{2i+1}e_i^{-1}$ and (4), we can conclude that $\models \psi_0(d_{2i}; c_{2i}, b_1, \ldots, b_{k-1})$ contradicting (3). Finally,

- if $i \neq j$, then
  \[ e_i = d_{2i}^{-1}d_{2i+1} \quad \text{by (1) if } j < i \quad \text{or by (e) if } i < j \quad H_{2j} = K_j; \]
- if $i = j$, then by (1) and (2), we have that $d_{2i} \notin H_{2i}$ but $d_{2i+1} \in H_{2i}$, so $e_i \notin H_{2i} = K_i$. This also shows that condition (2) is still satisfied.

This finishes the proof of the claim. \(\square\)

**Claim 3.14.** We may assume that there are sequences $(b_1, \gamma : \gamma < \omega), (b_2, \gamma : \gamma < \omega), \ldots, (b_k, \gamma : \gamma < \omega)$ in $\mathcal{M}_{k-1}$ and elements $(d_{i, \gamma_1, \ldots, \gamma_{k-1}} : (i, \gamma_1, \ldots, \gamma_{k-1}) \in \omega^n)$ in $G$ such that
\[
d_{i, \gamma_1, \ldots, \gamma_{k-1}} \in H_{j, \delta_1, \ldots, \delta_{k-1}} := \bigcap_{n < \omega} \psi_n(G; \bar{c}_n, \bar{b}_{1, \delta_1}, \ldots, \bar{b}_{k-1, \delta_{k-1}}) \ \iff (i, \gamma_1, \ldots, \gamma_{k-1}) \neq (j, \delta_1, \ldots, \delta_{k-1}) \ \iff \models \psi_{k-1}(d_{i, \gamma_1, \ldots, \gamma_{k-1}}; \bar{c}_j, \bar{b}_{1, \delta_1}, \ldots, \bar{b}_{k-1, \delta_{k-1}}).\]

**Proof.** To show the claim, we prove the following statement by reverse induction on $l = 1, \ldots, k$: there are sequences $(b_l, \gamma : \gamma < \omega), (b_{l+1}, \gamma : \gamma < \omega), \ldots, (b_k, \gamma : \gamma < \omega)$ in $\mathcal{M}_{k-1}$ and elements $(d_{i, \gamma_1, \ldots, \gamma_{k-1}} : (i, \gamma_1, \ldots, \gamma_{k-1}) \in \omega^{k-l+1})$ in $G$ such that $(\dagger_1)$–$(\dagger_4)$ below hold:

\[
d_{i, \gamma_1, \ldots, \gamma_{k-1}} \in H_{j, \delta_1, \ldots, \delta_{k-1}} := \bigcap_{n < \omega} \psi_n(G; \bar{c}_j, \bar{b}_{1, \delta_1}, \ldots, \bar{b}_{k-1, \delta_{k-1}}) \ \iff (i, \gamma_1, \ldots, \gamma_{k-1}) \neq (j, \delta_1, \ldots, \delta_{k-1}) \quad (\dagger_1) \]
\[
\models \psi_{k-l}(d_{i, \gamma_1, \ldots, \gamma_{k-1}}; \bar{c}_j, \bar{b}_{1, \delta_1}, \ldots, \bar{b}_{k-1, \delta_{k-1}}) \quad (\dagger_2) \]
\[
\text{and } \quad d_{i, \gamma_1, \ldots, \gamma_{k-1}} \in \bigcap_{l=1}^{k-1} G_{M_{l-1}}^0 \bar{e}_{b_{1, \delta_1}, \ldots, b_{k-1, \delta_{k-1}}} \quad (\dagger_3) \]

for all $\gamma_l, \ldots, \gamma_{k-1}, \delta_l, \ldots, \delta_{k-1}, i, j \in \omega$ (where $\mathcal{M}_0 \leq \cdots \leq \mathcal{M}_{k-1}$ are given by the assumptions), and

\[
\{ \psi_n(x; \bar{c}_i, \bar{b}_1, \ldots, \bar{b}_{l-1}, \bar{b}_{l, \gamma_1}, \ldots, \bar{b}_{k-1, \gamma_{k-1}}), \psi_{n+1}(y; \bar{c}_i, \bar{b}_1, \ldots, \bar{b}_{l-1}, \bar{b}_{l, \gamma_1}, \ldots, \bar{b}_{k-1, \gamma_{k-1}}) \} \]
\[
\models \psi_n(xy; \bar{c}_i, \bar{b}_1, \ldots, \bar{b}_{l-2}, \bar{b}_{l-1, \gamma_{l-1}}, \ldots, \bar{b}_{k-1, \gamma_{k-1}}) \quad \wedge \quad \psi_n(x^{-1}; \bar{c}_i, \bar{b}_1, \ldots, \bar{b}_{l-1}, \bar{b}_{l, \gamma_1}, \ldots, \bar{b}_{k-1, \gamma_{k-1}}) \quad \wedge \quad \psi_n(xy^{-1}; \bar{c}_i, \bar{b}_1, \ldots, \bar{b}_{l-1}, \bar{b}_{l, \gamma_1}, \ldots, \bar{b}_{k-1, \gamma_{k-1}}) \ \text{for all } i \in \omega. \]

Then letting $l = 1$, finishes the proof of the claim.
For $l = k$, this is Claim 3.13 together with (1), (3) and (4).

Now suppose it is true for $1 < l < k$. We want to prove the statement for $l - 1$. First, as $\bar{b}_{l-1} \in M_{l-1}$, we can choose sequences $(\bar{b}_{l-1,\gamma} : \gamma < \kappa_{l-1})$ in $M_{l-1}$ and $(d_{i,\gamma_1,\ldots,\gamma_{k-1}} : (i, \gamma_1, \ldots, \gamma_{k-1}) \in \omega \times \kappa_{l-1} \times \omega^{k-l})$ in $G$ such that for any $\gamma < \kappa_{l-1}$ we have:

$$
\begin{align*}
(\bar{b}_{l-1,\gamma}, d_{i,\gamma_1,\ldots,\gamma_{k-1}} : (i, \gamma_1, \ldots, \gamma_{k-1}) \in \omega^{k-l+1}) & \text{ has the same type as } \\
(\bar{b}_{l-1,\gamma}, d_{i,\gamma_1,\ldots,\gamma_{k-1}} : (i, \gamma_1, \ldots, \gamma_{k-1}) \in \omega^{k-l+1}) & \text{ over } \\
M_{l-2}, c, \bar{b}_1, \ldots, \bar{b}_{l-2} & \cup \{\bar{b}_{l-1,\delta} : \delta < \gamma\} \cup \{\bar{b}_{j,\delta} : j = l, \ldots, k-1, \delta < \omega\}.
\end{align*}
$$

Then

(5) By (13) for $t = l - 1$ and (4), using that $\bar{b}_{l-1,\gamma} \in M_{l-1}$ for all $\gamma < \kappa_{l-1}$, we obtain

$$
d_{i,\gamma_1,\ldots,\gamma_{k-1}} \in \bigcap \{H_j, \delta_1, \ldots, \delta_{k-1} : (j, \delta_1, \ldots, \delta_{k-1}) \in \omega^{k-l+1}, \delta_{l-1} < \gamma_{l-1}\}
$$

for all $i \in \omega$.

(6) By (11) and (4)

$$
d_{i,\gamma_1,\ldots,\gamma_{k-1}} \in \bigcap \{H_j, \gamma_{l-1}, \delta_1, \ldots, \delta_{k-1} : (i, \gamma_1, \ldots, \gamma_{k-1}) \neq (j, \delta_1, \ldots, \delta_{k-1})\}.
$$

(7) By (11) and (4)

$$
d_{i,\gamma_1,\ldots,\gamma_{k-1}} \not\in H_i, \gamma_{l-1}, \ldots, \gamma_{k-1}:
$$

In particular by (12) and (4)

$$
\models \neg \psi_{l-1}(d_{i,\gamma_1,\ldots,\gamma_{k-1}}; \bar{c}_i, \bar{b}_1, \ldots, \bar{b}_{l-2}, \bar{b}_{l-1,\gamma_{l-1}}, \ldots, \bar{b}_{k-1,\gamma_{k-1}});
$$

(8) By (14) and (4) the formula

$$
\psi_{l+1} (x; \bar{c}_i, \bar{b}_1, \ldots, \bar{b}_{l-2}, \bar{b}_{l-1,\gamma_{l-1}}, \ldots, \bar{b}_{k-1,\gamma_{k-1}}) \land
\psi_{l+1} (y; \bar{c}_i, \bar{b}_1, \ldots, \bar{b}_{l-2}, \bar{b}_{l-1,\gamma_{l-1}}, \ldots, \bar{b}_{k-1,\gamma_{k-1}})
$$

implies

$$
\models \psi_{l}(xy; \bar{c}_i, \bar{b}_1, \ldots, \bar{b}_{l-2}, \bar{b}_{l-1,\gamma_{l-1}}, \ldots, \bar{b}_{k-1,\gamma_{k-1}}) \land
\psi_{l}(x^{-1}; \bar{c}_i, \bar{b}_1, \ldots, \bar{b}_{l-2}, \bar{b}_{l-1,\gamma_{l-1}}, \ldots, \bar{b}_{k-1,\gamma_{k-1}}) \land
\psi_{l}(xy^{-1}; \bar{c}_i, \bar{b}_1, \ldots, \bar{b}_{l-2}, \bar{b}_{l-1,\gamma_{l-1}}, \ldots, \bar{b}_{k-1,\gamma_{k-1}});
$$

(9) By (13) and (4), and as $\bar{b}_{l-1,\delta_{l-1}}$ are all in $M_{l-1}$

$$
d_{i,\gamma_1,\ldots,\gamma_{k-1}} \in \bigcap_{t=1,\ldots,l-2; \delta_{l-1} \leq \gamma_{l-1}, \delta_1, \ldots, \delta_{k-1} \in \omega^{k-l}} G_{M_{l-2}, \bar{c}, \bar{b}_1, \ldots, \bar{b}_{l-2}, \bar{b}_{l-1,\delta_{l-1}}, \ldots, \bar{b}_{k-1,\delta_{k-1}}}^{00} \bar{b}_i.
$$

Consider the sequence of countable tuples

$$
(\bar{b}_{l-1,\gamma}, (d_{i,\gamma_1,\ldots,\gamma_{k-1}} : (i, \gamma_1, \ldots, \gamma_{k-1}) \in \omega^{k-l+1}) \gamma \in \kappa_{l-1}.
$$

Note that for any $\delta < \kappa_{l-1}$, the group

$$
K_\delta := G_{M_{l-2}, \bar{c}, \bar{b}_1, \ldots, \bar{b}_{l-2}, \bar{b}_{l-1,\delta_{l-1}}, \ldots, \bar{b}_{k-1,\delta_{k-1}}}^{00} \{b_\delta, \delta_1, \ldots, \delta_{k-1} : (\delta_1, \ldots, \delta_{k-1}) \in \omega^{k-l}\}
$$

is type definable over a set of size $|M_{l-2}| + 2^0 = |M_{l-2}|$ and has bounded index in $G$, hence by Lemma 3.12 its index is at most $2^{k-2|M_{l-2}|}$. Let $(g_{i,\nu} : \nu < 2^{k-2|M_{l-2}|})$ be a set of representatives of its cosets in $G$. For each $\gamma < \delta < \kappa_{l-1}$, consider a countable tuple $\bar{v}_{\gamma,\delta} := (\nu_1, \delta_1, \ldots, \delta_{k-1} : i, \delta_1, \ldots, \delta_{k-1} \in \omega)$ listing cosets of the elements $(d_{i,\gamma_1,\ldots,\gamma_{k-1}} : (i, \gamma_1, \ldots, \gamma_{k-1}) \in \omega^{k-l+1})$ with respect to the group $K_\delta$. There are at most $(2^{k-2|M_{l-2}|})^{\delta_0} =...$
$2^{[\mathcal{M}_{l-2}]}$ possible choices for this tuple. As $\kappa_{l-1} \geq \varnothing_2([\mathcal{M}_{l-2}])^+$ by assumption, applying Erdős-Rado there is an infinite subsequence such that $\nu_{\gamma,\delta}$ is constant for all $\gamma < \delta$ from this subsequence. As in the proof of Claim 3.13 restricting to this subsequence we have that (5)–(9) still hold, and additionally for any fixed $(i, \gamma_1, \ldots, \gamma_{k-1}) \in \omega^{k-l+1}$ we have

$$d_{i,\gamma_1,\gamma_2,\ldots,\gamma_{k-1}} \in G_{\mathcal{M}_{l-2}}^{\mathcal{H}} \{b_1, \ldots, b_{k-1}, \delta_{k-1} = (\delta_{i}, \ldots, \delta_{k-1}) \in \omega^{k-l} \} \quad (\dagger\dagger)$$

for all $\gamma < \gamma' < \delta < \omega$.

Next, for $(i, \gamma_1, \ldots, \gamma_{k-1}) \in \omega^{k-l+1}$ and $\gamma_{l-1} < \kappa_{l-1}$, let

$$e_{i,\gamma_1,\gamma_2,\ldots,\gamma_{k-1}} = d_{i,\gamma_1,\gamma_2,\ldots,\gamma_{k-1}}^{-1} \in G_{\mathcal{M}_{l-2}}^{\mathcal{H}} \{b_1, \ldots, b_{k-1}, \delta_{k-1} = (\delta_{i}, \ldots, \delta_{k-1}) \in \omega^{k-l} \} \quad (\dagger\dagger)$$

by (9).

Now let $(j, \gamma_1, \ldots, \gamma_{k-1}) \neq (i, \delta_{i-1}, \ldots, \delta_{k-1})$. We consider two cases:

**Case 1:** $\gamma_{l-1} = \delta_{l-1}$.

Then $(j, \gamma_1, \ldots, \gamma_{k-1}) \neq (i, \delta_{i-1}, \ldots, \delta_{k-1})$. Thus

$$d_{i,\gamma_1,\gamma_2,\ldots,\gamma_{k-1}} \in H_{i,\gamma_1,\gamma_2,\ldots,\gamma_{k-1}} \delta_{t-1}, \ldots, \delta_{k-1} \quad (\dagger)$$

and

$$d_{j,\gamma_1,\gamma_2,\ldots,\gamma_{k-1}} \in H_{j,\gamma_1,\gamma_2,\ldots,\gamma_{k-1}} \delta_{t-1}, \ldots, \delta_{k-1} \quad (\dagger\dagger)$$

So

$$d_{i,\gamma_1,\gamma_2,\ldots,\gamma_{k-1}} \in H_{j,\gamma_1,\gamma_2,\ldots,\gamma_{k-1}} \delta_{t-1}, \ldots, \delta_{k-1} \quad (\dagger\dagger)$$

by (6)

and

$$d_{j,\gamma_1,\gamma_2,\ldots,\gamma_{k-1}} \in H_{j,\gamma_1,\gamma_2,\ldots,\gamma_{k-1}} \delta_{t-1}, \ldots, \delta_{k-1} \quad (\dagger\dagger)$$

So in particular

$$e_{i,\gamma_1,\gamma_2,\ldots,\gamma_{k-1}} = d_{i,\gamma_1,\gamma_2,\ldots,\gamma_{k-1}}^{-1} \in H_{j,\gamma_1,\gamma_2,\ldots,\gamma_{k-1}} \delta_{t-1}, \delta_{k-1} \quad (\dagger\dagger)$$

by (5).

So

$$e_{i,\gamma_1,\gamma_2,\ldots,\gamma_{k-1}} = d_{i,\gamma_1,\gamma_2,\ldots,\gamma_{k-1}}^{-1} \in H_{j,\gamma_1,\gamma_2,\ldots,\gamma_{k-1}} \delta_{t-1}, \delta_{k-1} \quad (\dagger\dagger)$$

by (5).

If $\delta_{l-1} < \gamma_{l-1}$, then $e_{i,\gamma_1,\gamma_2,\ldots,\gamma_{k-1}} = d_{i,\gamma_1,\gamma_2,\ldots,\gamma_{k-1}}^{-1} \in H_{j,\gamma_1,\gamma_2,\ldots,\gamma_{k-1}} \delta_{t-1}, \delta_{k-1} \quad (\dagger\dagger)$

and

$$e_{i,\gamma_1,\gamma_2,\ldots,\gamma_{k-1}} \in \bigcap_{\delta_{l-1} > \gamma_{l-1}, (\delta_{i}, \ldots, \delta_{k-1}) \in \omega^{k-l}} G_{\mathcal{M}_{l-2}}^{\mathcal{H}} \{b_1, \ldots, b_{k-1}, \delta_{k-1} \} \quad (\dagger\dagger)$$

Together with (\dagger\dagger), this gives

$$e_{i,\gamma_1,\gamma_2,\ldots,\gamma_{k-1}} \in \bigcap_{\delta_{l-1} > \gamma_{l-1}, (\delta_{i}, \ldots, \delta_{k-1}) \in \omega^{k-l}} G_{\mathcal{M}_{l-2}}^{\mathcal{H}} \{b_1, \ldots, b_{k-1}, \delta_{k-1} \} \quad (\dagger\dagger)$$
On the other hand for given \((j, \gamma_1, \ldots, \gamma_{k-1}) \in \omega^k - t + 2\), we have that
\[
d_i, 2\gamma_{j+1}, \ldots, \gamma_{k-1} \notin H_i, 2\gamma_{j+1}, \ldots, \gamma_{k-1} \text{ by (7)}
\]
but
\[
d_i, 2\gamma_{j+1}, \ldots, \gamma_{k-1} \in H_i, 2\gamma_{j+1}, \ldots, \gamma_{k-1} \text{ using (5)}.
\]
Hence
\[
ed_i, 2\gamma_{j+1}, \ldots, \gamma_{k-1} = d_i, 2\gamma_{j+1}, \ldots, \gamma_{k-1} \setminus \psi_{l+1}(e_i, \gamma_{l+1}, \ldots, \gamma_{k-1} - 1, b_i, l, \ldots, b_{k-1} - 1, l - 1).
\]

Now suppose towards a contradiction that
\[
\models \psi_{k+1}(e_i, \gamma_{l+1}, \ldots, \gamma_{k-1} - 1, b_i, l, \ldots, b_{k-1} - 1, l - 1).
\]
By (5), we also have that
\[
\models \psi_{k+1}(d_i, 2\gamma_{j+1}, \ldots, \gamma_{k-1} - 1, b_i, l, \ldots, b_{k-1} - 1, l - 1).
\]
Now, using that \(d_i, 2\gamma_{j+1}, \ldots, \gamma_{k-1} = d_i, 2\gamma_{j+1}, \ldots, \gamma_{k-1} - 1, b_i, l, \ldots, b_{k-1} - 1, l - 1\) and (8), we conclude that
\[
\models \psi_{k+1}(d_i, 2\gamma_{j+1}, \ldots, \gamma_{k-1} - 1, b_i, l, \ldots, b_{k-1} - 1, l - 1)\]
contradicting (7). Thus
\[
\models \psi_{k+1}(e_i, \gamma_{l+1}, \ldots, \gamma_{k-1} - 1, b_i, l, \ldots, b_{k-1} - 1, l - 1).
\]
Now, for \(l - 1\), we obtain \((t_1)\) from (10) and (13), \((t_2)\) from (11) and (14), \((t_3)\) from (12), and \((t_4)\) from (8) by replacing \(e_i, \gamma_{l+1}, \ldots, \gamma_{k-1} - 1, b_i, l, \ldots, b_{k-1} - 1, l - 1\) by \(d_i, 2\gamma_{j+1}, \ldots, \gamma_{k-1} - 1, b_i, l, \ldots, b_{k-1} - 1, l - 1\), and \(K_i, \gamma_{l+1}, \ldots, \gamma_{k-1} - 1, b_i, l, \ldots, b_{k-1} - 1, l - 1\). This finishes the proof of the claim.

**Claim 3.15.** For any \(I \subseteq (\omega)^k\) there is some \(d \in G\) such that \(\models \psi_{k+1}(d; e_i, l, \ldots, b_{k-1} - 1, l - 1)\) if and only if \((i, \gamma_1, \ldots, \gamma_{k-1}) \notin I\).

**Proof.** Suppose first that \(I\) is finite and let \(((i_1, \gamma_{1,1}, \ldots, \gamma_{1,k-1}), \ldots, (i_s, \gamma_{s,1}, \ldots, \gamma_{s,k-1})) : l \leq s\) be an enumeration of the elements in \(I\). Define \(d = d_{i_0, \gamma_{1,0}}, \ldots, \gamma_{k-1} - 1, l = d_{i_1, \gamma_{1,1}}, \ldots, \gamma_{k-1} - 1, l \setminus \psi_{k+1}(e_i, l, \ldots, b_{k-1} - 1, l - 1)\).

If \((j, \delta_1, \ldots, \delta_{k-1}) \notin I\), then \(d_{i_0, \gamma_{1,0}}, \ldots, \gamma_{k-1} - 1, l \in H_{j, \delta_1, \ldots, \delta_{k-1}}\) for all \(l \leq s\) by Claim 3.14 and hence \(d \in H_{j, \delta_1, \ldots, \delta_{k-1}}\) and in particular \(\models \psi_{k+1}(d; e_i, l, \ldots, b_{k-1} - 1, l - 1)\).

On the other hand, consider \((i_1, \gamma_{1,1}, \ldots, \gamma_{1,k-1}) \in I\). Let
\[
e_1 = d_{i_0, \gamma_{1,0}}, \ldots, \gamma_{k-1} - 1, l = d_{i_1, \gamma_{1,1}}, \ldots, \gamma_{k-1} - 1, l - 1,
\]
and
\[
e_2 = d_{i_1, \gamma_{1,1}}, \ldots, \gamma_{k-1} - 1, l = d_{i_2, \gamma_{1,2}}, \ldots, \gamma_{k-1} - 1, l - 1.
\]
Then \(d_{i_1, \gamma_{1,1}}, \ldots, \gamma_{k-1} - 1, l = e_1^{-1} e_2^{-1}\). Observe that \(\models -\psi_{k+1}(d_{i_1, \gamma_{1,1}}, \ldots, \gamma_{k-1} - 1, l, e_i, l, \ldots, b_{k-1} - 1, l - 1)\), \(\models -\psi_{k+1}(e_1, \gamma_{l+1}, \ldots, \gamma_{k-1} - 1, l)\) and \(\models -\psi_{k+1}(e_1, \gamma_{l+1}, \ldots, \gamma_{k-1} - 1, l)\) by Claim 3.13. Using (8), we conclude that
\[
\models -\psi_{k+1}(d_{i_1, \gamma_{1,1}}, \ldots, \gamma_{k-1} - 1, l, e_i, l, \ldots, b_{k-1} - 1, l - 1)\]

\[
\models -\psi_{k+1}(e_1, \gamma_{l+1}, \ldots, \gamma_{k-1} - 1, l)\]

\[
\models -\psi_{k+1}(d, e_i, l, \ldots, b_{k-1} - 1, l - 1)\]

Using compactness we get the claim. □

Finally, Claim 3.15 contradicts \(k\)-dependence of \(\psi_{k+1}\), which finishes the proof. □
3.3. Additional conditions on intersections of subgroups. Recall the “chain condition” for definable families of subgroups in \( n \)-dependent theories.

**Fact 3.16.** [19 Proposition 4.1] Let \( G \) be a definable group, and let \( \psi(x; y_0, \ldots, y_{n-1}) \) be an \( n \)-dependent formula such that \( \psi(G; b_0, \ldots, b_{n-1}) \) is a subgroup of \( G \) for any parameters \( b_0, \ldots, b_{n-1} \). Then there exists some \( m_\psi \in \omega \) such that for any \( d \geq m_\psi \) and any array of parameters \( (b_{i,j} : i < n, j \leq d) \) there is some \( \nu \in d^n \) such that

\[
\bigcap_{\eta \in d^n} H_\eta = \bigcap_{\eta \in d^n, \eta \neq \nu} H_\eta,
\]

where \( H_\eta := \psi(G; b_{0,i_0}, \ldots, b_{n-1,i_{n-1}}) \) for \( \eta = (i_0, \ldots, i_{n-1}) \).

We generalize it to simultaneous intersections of several definable families of subgroups. We will need the following version of Ramsey’s theorem.

**Fact 3.17.** For every \( l, m, n \in \omega \) there is some \( R = R(l, m, n) \in \omega \) such that: for any function \( f : R^n \to m \) there are some sets \( s_0, \ldots, s_{n-1} \subseteq R \) with \( |s_0|, \ldots, |s_{n-1}| \geq l \) and such that \( f \mid_{s_0 \times \cdots \times s_{n-1}} \) is constant.

**Proposition 3.18.** Let \( G \) be a definable group, and for \( t < k \) let \( \psi_t(x; y_0, \ldots, y_{n-1}) \) be an \( n \)-dependent formula such that \( \psi_t(G; b_0, \ldots, b_{n-1}) \) is a subgroup of \( G \) for any \( t < k \) and any parameters \( b_0, \ldots, b_{n-1} \). Then there exists some \( m = m(\psi_0, \ldots, \psi_{k-1}) \) in \( \omega \) such that: for any \( d \geq m \) and any array of parameters \( (b_{i,j} : i < n, j \leq d) \) there is some \( \nu \in d^n \) such that

\[
\bigcap_{\eta \in d^n} H^t_\eta = \bigcap_{\eta \in d^n, \eta \neq \nu} H^t_\eta \text{ for all } t < k \text{ simultaneously,}
\]

where \( H^t_\eta := \psi_t(G; b_{0,i_0}, \ldots, b_{n-1,i_{n-1}}) \) for \( \eta = (i_0, \ldots, i_{n-1}) \).

**Proof.** We argue by induction on \( k \), the base case \( k = 1 \) given by Fact 3.16. Let \( m_1 := m(\psi_0) \) and \( m_2 := m(\psi_1, \ldots, \psi_{k-1}) \) be given by the inductive hypothesis. Let \( R := R(m_2, m_1, n) \) be given by Fact 3.17. Finally, we take \( m := m(\psi_0, \ldots, \psi_{k-1}) := Rm_1 \).

Let an array \( B = (b_{i,j} : i < n, j \leq m) \) be given. For each \( \gamma = (\gamma_0, \ldots, \gamma_{n-1}) \in R^n \), consider the subarray

\[
B_\gamma = (b_{0,\gamma_0+1}, \ldots, b_{n-1,\gamma_{n-1}+1}) : \gamma = (\eta_0, \ldots, \eta_{n-1}) \in m_1^n.
\]

By the choice of \( m_1 \), for each \( \gamma \in R^n \) there is some \( \nu_\gamma \in m_1^n \) such that

\[
\bigcap_{\eta \in m_1^n} H_\eta^0 = \bigcap_{\eta \in m_1^n, \eta \neq \nu_\gamma} H_\eta^0 \quad (\ast)
\]

By the choice of \( R \), there are some sets \( s_0, \ldots, s_{n-1} \subseteq R \) with \( |s_0| = \ldots = |s_{n-1}| = m_2 \) such that \( \nu_\gamma \) is equal to some fixed \( \nu \in m_1^n \) for all \( \gamma \in s_0 \times \cdots \times s_{n-1} \). But then we consider the subarray

\[
B' = (b_{0,\gamma_0+1}, \ldots, b_{n-1,\gamma_{n-1}+1}) : \gamma \in s_0 \times \cdots \times s_{n-1}
\]

of \( B \). By the choice of \( m_2 \), there is some \( \gamma' \in s_0 \times \cdots \times s_{n-1} \) such that

\[
\bigcap_{\gamma' \in s_0 \times \cdots \times s_{n-1}} H_{\gamma'}^t = \bigcap_{\gamma' \in s_0 \times \cdots \times s_{n-1}, \gamma \neq \gamma'} H_{\gamma'}^t \text{ for all } 1 \leq t < k.
\]


Finally, let \( \nu := (\gamma'_0 m_1 + \nu'_0, \ldots, \gamma'_{n-1} m_1 + \nu'_{n-1}) \). By (**) we have
\[
\bigcap_{\eta \in m^n} H^t_\eta = \bigcap_{\eta \in m^n, \eta \notin \nu} H^t_\eta \text{ for all } 0 \leq t < k,
\]
as wanted. \( \square \)

4. Characterization of 2-dependence by a type-counting criterion

We first recall a type-counting criterion for NIP. For the following two facts see e.g. [14] or [12] Section 6] and references there.

**Fact 4.1.** (Shelah) Let \( T \) be a theory in a countable language, and for an infinite cardinal \( \kappa \), let \( f_T(\kappa) := \sup\{|S_1(M)| : |M| = \kappa, M \models T\} \).

1. If \( T \) is NIP, then \( f_T(\kappa) \leq (\text{ded} \kappa)^{\aleph_0} \) for all infinite cardinals \( \kappa \).
2. If \( T \) has IP, then \( f_T(\kappa) = 2^\kappa \) for all infinite cardinals \( \kappa \).

It is possible that in a model of ZFC, \( \text{ded} \kappa = 2^\kappa \) for all infinite cardinals \( \kappa \) (e.g. in a model of the Generalized Continuum Hypothesis). However, there are models of ZFC in which these two functions are different.

**Fact 4.2.** (Mitchell) For every cardinal \( \kappa \) of uncountable cofinality, there is a cardinal preserving Cohen extension such that \( (\text{ded} \kappa)^{\aleph_0} < 2^\kappa \).

We want to provide a formula-free characterization of \( n \)-dependence which doesn’t include any assumption of indiscernibility of the witnessing sequence over the additional parameters (unlike the characterization in [13] Proposition 6.3] where additional indiscernibility of the parameter needs to be assumed). We can do it for \( 2 \)-dependence under some set-theoretic assumption.

**Lemma 4.3.** Let \( \varphi(x; y_1, y_2) \) be 2-dependent. Then there is some \( n \in \mathbb{N} \) such that for any \( c \in \mathbb{M}_x \), any \( I \subseteq \mathbb{M}_{y_1}, J \subseteq \mathbb{M}_{y_2} \) endless mutually indiscernible sequences, and any \( A \subseteq I \) of size \( > n \) there is some \( b_A \in J \) such that \( A \) cannot be shattered by the family \( \{ \varphi(c, y_1, b) : b \in J, b > b_A \} \).

**Proof.** Assume that \( I, J \) are endless mutually indiscernible sequences and \( c \) is such that the conclusion is not satisfied for any \( n \in \omega \). Let \( D \subseteq I \times J \) be any finite set. Let \( a_1 < \ldots < a_n \) and \( b_1 < \ldots < b_m \) list the projections of \( D \) on \( I \) and \( J \), respectively.

By assumption, there is some \( A \subseteq I \) of size \( n \) such that for any \( b' \in B_A \), \( A \) is shattered by the family \( \{ \varphi(c, y_1, b) : b \in J, b > b' \} \). List \( A \) as \( a_1' < \ldots < a_n' \). Then we can choose some \( b_1' < \ldots < b_m' \in J \) such that \( (a_i, b_j) \in D \). As \( I, J \) are mutually indiscernible, taking an automorphism of \( \mathbb{M} \) sending \( a_i' \) to \( a_i \) and \( b_j' \) to \( b_j \), for all \( 1 \leq i \leq n, 1 \leq j \leq m, \) is sent to some \( c_D \) such that \( (a_i, b_j) \in D \). This implies that \( \varphi(x; y_1, y_2) \) is not 2-dependent, a contradiction. Hence the conclusion holds for \( c, I, J \) for some \( n \).

By compactness it is not hard to conclude that \( n \) can be chosen depending only on \( \varphi \) (and not on \( I, J, c \)). \( \square \)

We will need the following lemma (originally from Shelah, with simplifications by Adler and Casanovas, see e.g. [10] Lemma 2.7.1])

**Fact 4.4.** If \( \kappa \) is an infinite cardinal, \( \mathcal{F} \subseteq 2^\kappa \) and \( |\mathcal{F}| > \text{ded} \kappa \), then for each \( n \in \omega \) there is some \( S \subseteq \kappa \) such that \( |S| = n \) and \( \mathcal{F} \upharpoonright S = 2^S \).
Definition 4.5. Given sets $A \subseteq M_x$, $B \subseteq M_y$ and a formula $\varphi(x, y) \in \mathcal{L}$, we denote by $S_{\varphi,B}(A)$ the set of all $\varphi$-types over $A$ realized in $B$, where by a $\varphi$-type over $B$ we mean a maximal consistent collection of formulas of the form $\varphi(x, a)$, $\neg\varphi(x, a)$ with $a \in A$. And by $S_B(A)$ the set of all complete types over $A$ realized in $B$.

Proposition 4.6. Let $T$ be 2-dependent, let $\kappa \geq |T|$ be an infinite cardinal, and let $\lambda > \kappa$ be a regular cardinal. Then for any mutually indiscernible sequences $I = (a_i : i < \kappa), J = (b_j : j \in \lambda)$ of finite tuples and a finite tuple $c$, there is some $\beta \in \lambda$ such that $|S_{J, \beta}(Ic)| \leq (\text{ded } \kappa)^{|T|}$.

Proof. Let $I, J$ and $c$ be given. We will show that for each $\varphi(x, y_1, y_2) \in \mathcal{L}$ there is some $\beta_\varphi \in \lambda$ such that $|\mathcal{S}_{\varphi, J, \beta_\varphi}(Ic)| \leq \text{ded } \kappa$. This is enough, as then we can take any $\beta \in \lambda$ with $\beta > \beta_\varphi$ for all $\varphi \in \mathcal{L}$ (possible as $\lambda = \text{cof } (\lambda) > |T|$), and $|\mathcal{S}_{J, \beta}(Ic)| \leq |\Pi_{\varphi \in L} \mathcal{S}_{\varphi, J, \beta_\varphi}(Ic)| \leq (\text{ded } \kappa)^{|T|}$.

So let $\varphi \in \mathcal{L}$ be fixed, and assume that for any $\beta \in \lambda$, $|\mathcal{S}_{\varphi, J, \beta}(Ic)| > \text{ded } \kappa$. Then by Fact 4.3, considering $\mathcal{F} = \{f_p : p \in \mathcal{S}_{\varphi, J, \beta}(Ic)\}$ (where $f_p \in 2^\kappa$ is given by $f_p(\alpha) = 1 \iff \varphi(c, a_\alpha, y_2) \in p$, for all $\alpha \in \kappa$), for any $n \in \omega$ there is some $S \subseteq I, |S| = n$, such that $S$ is shattered by the family $\{\varphi(c, y_1, b_j) : j \in \lambda, j > \beta\}$. Using regularity of $\lambda$, by transfinite induction we can choose a strictly increasing sequence $\beta_\alpha : \alpha \in \lambda$ with $\beta_\alpha \in \lambda$ such that for each $\alpha \in \lambda$ there is some $S_\alpha \subseteq I, |S_\alpha| = n$ shattered by the family $\{\varphi(c, y_1, b_j) : j \in \lambda, \beta_\alpha < j < \beta_{\alpha+1}\}$. As $\lambda > \kappa = \kappa^n$ is regular, passing to a subsequence we may assume that there is some $S \subseteq I, |S| = n$ such that $S_\alpha = S$ for all $\alpha \in \lambda$, i.e., this set $S$ can be shattered arbitrarily far into the sequence. Now by Lemma 4.3, this contradicts 2-dependence of $\varphi$ if we take $n$ large enough.

Lemma 4.7. For any cardinal $\kappa$ and any regular cardinal $\lambda \geq 2^\kappa$ there is a bipartite graph $\mathcal{G}_{\kappa, \lambda} = (\kappa, \lambda, E)$ satisfying the following: for any sets $A, A' \subseteq \kappa$ with $A \cap A' = \emptyset$ and $b \in \lambda$ there is some $b^* \in \lambda$, $b^* > b$ satisfying $\bigwedge_{a \in A} E(a, b^*) \land \bigwedge_{a' \in A'} E(a', b^*)$.

Proof. Let $\lambda \geq 2^\kappa$ be any regular cardinal. Let $D := \{(A, A', b) : A, A' \subseteq \kappa, A \cap A' = \emptyset, b \in \lambda\}$. Then $|D| \leq \lambda$ by assumption, let’s enumerate it as $((A_{\alpha}, A'_{\alpha}, b_\alpha) : \alpha < \lambda)$. We define $E_{\alpha} \subseteq \kappa \times \lambda$ by transfinite induction on $\alpha < \lambda$. On step $\alpha$, we choose some $c_\alpha \in \lambda$ such that $c_\alpha > \{b_\beta, c_\beta : \beta < \alpha\}$ — possible by regularity of $\lambda$, and we take $E_{\alpha} := \{(a, c_\alpha) : a \in A_{\alpha}\}$. Let $E := \bigsqcup_{\alpha < \lambda} E_{\alpha}$ — it satisfies the requirement by construction.

Lemma 4.8. Let $\varphi(x, y, z)$ have IP$_2$. Then for every regular $\lambda > 2^\kappa$ there exist mutually indiscernible sequences $I = (a_i : i < \kappa)$ in $M_y$, $(b_j : j < \lambda)$ in $M_z$ and $c \in M_x$ such that for every $\beta < \lambda$ we have $|S_{\varphi, J, \beta}(Ic)| = 2^\kappa$.

Proof. For any $\kappa, \lambda$ we can find some mutually indiscernible sequences $I, J$ such that the family $\{\varphi(c, y_1, y_2) : c \in M_x\}$ shatters $I \times J$. In particular, we can find $c$ such that $M \models \varphi(c, a_i, b_j) \iff \mathcal{G}_{\kappa, \lambda} \models E(a_i, b_j)$, and we can conclude by Lemma 4.7.

Definition 4.9. We say that a theory $T$ is globally 2-dependent if there are cardinals $\kappa \leq \lambda$ as above such that the following holds. Given any mutually indiscernible sequences $I = (a_i : i \in \kappa), J = (b_j : j \in \lambda)$ of finite tuples and a finite tuple $c$, if $\mathcal{G}_{\kappa, \lambda}$ is as above then there are some $i \in \kappa$ and $j, j' \in \lambda$ such that $c_{a_i}b_j \equiv c_{a_i}b_{j'}$ but $E(i, j) \land \neg E(i, j')$ holds.
So the idea is that $T$ is globally 2-dependent if on mutually indiscernible sequences, we cannot distinguish the edges from the non-edges of a random graph not only by any single formula, but also by a complete type.

**Remark 4.10.** If $T$ is not 2-dependent, then it is not globally 2-dependent.

**Proof.** Let $\varphi(x, y_1, y_2)$ be a formula witnessing failure of 2-dependence, then as in Lemma 4.8.

**Proposition 4.11.** Let $T$ be a countable 2-dependent theory and assume that there is some cardinal $\kappa$ such that $(\text{ded} \kappa)^{2^{\kappa}} < 2^{\kappa}$. Then $T$ is globally 2-dependent.

**Proof.** Fix such a $\kappa$, and let $\lambda$ be any regular cardinal $\geq 2^\kappa$. Let $G_{\kappa, \lambda}$ be as given by Lemma 4.7. Moreover, let $I, J$ and $c$ be as in Definition 4.9. By Proposition 4.6, there is some $\beta < \lambda$ such that $|S_{I, J, \beta} (I)| \leq (\text{ded} \kappa)^{2^{\kappa}}$. On the other hand, by definition of $G_{\kappa, \lambda}$, we still have $|S_{E, \{ \alpha < \lambda, \alpha > \beta \}} (\kappa)| = 2^\kappa > (\text{ded} \kappa)^{2^{\kappa}}$ by assumption. Then we can find some $j, j' \in \lambda$ such that $\text{tp}_E (j/\kappa) \neq \text{tp}_E (j'/\kappa)$ but $\text{tp} (b_j/Ic) = \text{tp} (b_j/Ic)$. But then there is some $i \in \kappa$ such that $E(i, j) \leftrightarrow \neg E(i, j')$ and still $b_i a_i c \equiv b_j a_j c$, as wanted.

**Problem 4.12.** Is it true that $n$-dependent implies globally $n$-dependent (defined analogously), in ZFC, or at least consistently for $n > 2$?

**Remark 4.13.** Let $T$ be $n$-dependent and $\omega$-categorical. Then $T$ is globally $n$-dependent (since every type in finitely many variables is equivalent to a formula, hence $n$-dependent and can’t define the random $n$-hypergraph on mutually indiscernible sequences).

**5. 2-DEPENDENCE FOR COMPOSITIONS OF NIP RELATIONS AND BINARY FUNCTIONS**

All the variables below are allowed to be tuples of arbitrary finite length.

**Theorem 5.1.** Let $\mathcal{M}$ be a first-order $\mathcal{L}$-structure with an NIP sort $K$ and let $\varphi(x_0, \ldots, x_{d-1})$ be an $\mathcal{L}_K$-formula. For each $i \in \{0, \ldots, d-1\}$, fix some $s_i < t_i \in \{1, 2, 3\}$ and let $f_i : M_{y_i} \times M_{y_i} \rightarrow K_{x_i}$ be an $\mathcal{L}$-definable binary function. Then the $\mathcal{L}$-formula

$$\psi(y_1; y_2, y_3) = \varphi (f_1(y_{s_1}, y_{t_1}), \ldots, f_d(y_{s_{d-1}}, y_{t_{d-1}}))$$

is 2-dependent.

**Proof.** Assume $\psi(y_1; y_2, y_3)$ has IP$_2$, and let $I = (a_\alpha : \alpha < \kappa)$ with $a_\alpha \in M_{y_2}$, $J = (b_\beta : \beta < \lambda)$ with $b_\beta \in M_{y_3}$ and $c \in M_{y_1}$ be as given by Lemma 4.8 with $\lambda > 2^\kappa > |T|$, that is for every $\gamma < \lambda$ we have $|S_{G_{\kappa, \lambda}} (Ic)| = 2^\kappa$.

Let $V_{1, 2} := \{ i < d : (t_i, s_i) = (1, 2) \}$, $V_{1, 3} := \{ i < d : (t_i, s_i) = (1, 3) \}$, and $V_{2, 3} := \{ i < d : (t_i, s_i) = (2, 3) \}$, then $V_{1, 2}, V_{1, 3}, V_{2, 3}$ is a partition of $\{0, \ldots, d-1\}$. Let $f(c, a_\alpha) := (f_1(c, a_\alpha) : i \in V_{1, 2})$, $f(c, b_\beta) = (f_1(c, b_\beta) : i \in V_{1, 3})$ and $f(a_\alpha, b_\beta) := (f_1(a_\alpha, b_\beta) : i \in V_{2, 3})$. So we have

$$\models \psi(c; a_\alpha, b_\beta) \iff \varphi'(f(c, a_\alpha), f(c, b_\beta), f(a_\alpha, b_\beta)),$$

where $\varphi'$ is $\mathcal{L}_K$ is obtained from $\varphi$ by regrouping the variables accordingly.

Let $A := \{ f(c, a_\alpha) : \alpha < \kappa \} \subseteq K$. Consider the rectangular array $(f(a_\alpha, b_\beta) : \alpha < \kappa, \beta < \lambda)$. It is an indiscernible array by mutual indiscernibility of the sequences $(a_\alpha)$ and $(b_\beta)$. In particular, the sequence of columns $((f(a_\alpha, b_\beta) : \alpha < \kappa) : \beta < \lambda)$ is 0-indiscernible. As $T_K$ is NIP, $|A| \leq \kappa$ and $\lambda > (|T| + \kappa)$ is regular, there is some $\gamma < \lambda$ such that the sequence of columns $((f(a_\alpha, b_\beta) : \alpha < \kappa) : \gamma < \beta < \lambda)$ is $\mathcal{L}_K$-indiscernible over $A$.
Fix $\gamma < \beta < \lambda$. For any tuple $e \in M_{(x; i \in V_1, 3)}$, let

$$S^\beta_\gamma := \{ \alpha < \kappa : \psi'(f(c, a_\alpha), e, f(a_\alpha, b_\beta)) \} \subseteq \kappa,$$

and let $F_\beta := \{ S^\beta_\gamma : e \in M_{(x; i \in V_1, 3)} \}$, i.e. the collection of all such subsets of $\kappa$ that can be realized by some tuple.

We then have that $F_\beta = F_\beta'$ for any $\gamma < \beta, \beta' < \lambda$. Indeed, by the above there is some $\sigma \in \text{Aut}(M/A)$ sending $(f(a_\alpha, b_\beta) : \alpha < \kappa)$ to $(f(a_\alpha, b_\beta') : \alpha < \kappa)$. But then for any $e$ we have that $S^\beta_\epsilon = S^\beta_{\sigma(e)}$ (recalling that $f(c, a_\alpha) \in A$ for all $\alpha < \kappa$), hence $F_\beta \subseteq F_{\beta'}$, and vice versa exchanging the roles of $\beta$ and $\beta'$. So let $F := F_\beta$ for some (equivalently, any) $\beta > \gamma$. Note that $S^\beta_\epsilon$ is determined by the $\mathcal{L}_K$-type $tp_\epsilon(e/(f(a_\alpha, b_\beta) : \alpha < \kappa) A)$. As $|f(a_\alpha, b_\beta) : \alpha < \kappa) A| \leq \kappa$ and $\psi'$ is NIP, we get that $|F| \leq \text{ded}(\kappa)$ by Fact 4.1.

Now we estimate $|S_{\psi, J_{3, \gamma}(Ic)}|$ (see Definition 4.3). Given $\gamma < \beta < \lambda$, we have that $tp_\epsilon(b_\beta/Ic)$ is determined by the set

$$S^\beta_{f(c, b_\beta)} = \{ \alpha < \kappa : \psi'(f(c, a_\alpha), f(c, b_\beta), f(a_\alpha, b_\beta)) \} \subseteq F.$$

But by the previous paragraph, there are only $\text{ded}(\kappa)$ choices for this set, hence $|S_{\psi, J_{3, \gamma}(Ic)}| \leq \text{ded}(\kappa)$.

Carrying out this proof in Mitchell’s model (see Fact 4.2) we thus get a contradiction. But since the property of a formula $\psi$ being 2-dependent is arithmetic, hence set-theoretically absolute, we obtain the result in ZFC.

**Example 3.** Let $f : \mathbb{C}^2 \to \mathbb{C}$ be an arbitrary function, and let $p(x, y, z)$ be a polynomial over $\mathbb{C}$. Consider the relation $E \subseteq \mathbb{C}^3$ given by $E(x, y, z) \iff p(f(x, y), f(x, z), f(y, z)) = 0$. Then there is some finite tripartite 3-hypergraph $H$ such that $E$ doesn’t contain it as an induced tripartite hypergraph.

**Remark 5.2.** As the argument in Section 6.1 shows, we cannot relax the assumption that $K$ is NIP to just 2-dependent. Generalizations of Theorem 6.1 for $n$-dependence and functions of arbitrary arity will be investigated in future work.

### 6. 2-DEPENDENCE OF GRANGER’S EXAMPLES

We recall some definitions and results from [18]. We consider structures in the language $\mathcal{L}$ consisting of two sorts $V$ and $K$, the field language on $K$, the vector space language on $V$, scalar multiplication function $K \times V \to V$ and the bilinear form function $[x, y] : V \times V \to K$. The language $\mathcal{L}_\theta$ is obtained from $\mathcal{L}$ by adding for each $n \in \omega$ a (definable) $n$-ary predicate $\theta_n(x_1, \ldots, x_n)$ which holds if and only if $x_1, \ldots, x_n \in V$ are linearly independent over $K$. Finally, let $\mathcal{L}_\theta^K$ be a language expanding $\mathcal{L}_\theta$ by relations on $K^n$ definable in the language of rings such that $K$ eliminates quantifiers in $\mathcal{L}_\theta^K$ (e.g. we can always take Morleyization of $K$).

**Definition 6.1.** For $K$ a field, $m \in \mathbb{N} \cup \{ \infty \}$ and $F \in \{ A, S \}$, let $T_m^K_F$ denote the $\mathcal{L}$-theory expressing that sort corresponding to $K$ is a field which is moreover a model of $\text{Th}(K)$, $V$ a $K$-vector space of dimension $m$, $[x, y] : V \times V \to K$ is a non-degenerate bilinear form of type $F$, where a form of type $S$ is a symmetric form, and a form of type $A$ is an alternating form.

**Fact 6.2.** [18] Theorem 9.2.3] Let $K$ be a field with $\text{char}(K) \neq 2$ and let $m \in \mathbb{N} \cup \{ \infty \}$. Let $F \in \{ A, S \}$. Then the theory $T_m^K_F$ has elimination of quantifiers in the language $\mathcal{L}_\theta^K$.

**Theorem 6.3.** Let $T = T_m^K_F$. 

---

**Note:** This text is a natural reading representation of the document, focusing on the content and structure rather than on the formatting details. The symbols and notation used are consistent with standard mathematical and logical conventions. The text is optimized for readability and clarity, ensuring that the logic and mathematical content are accurately conveyed.
(1) If $K$ is NIP then $T$ is 2-dependent (and is strictly 2-dependent if $m = \infty$).
(2) If $K$ has $\text{IP}_n$ and $m = \infty$, then $T$ has $\text{IP}_{2n}$.

**Remark 6.4.** In fact, if $m < \infty$, then $K$ is $n$-dependent if and only if $T$ is $n$-dependent, for any $n \geq 1$. This can be seen as the above structure can be interpreted in $K$ using $K^m \cong \mathcal{V}$ for some $m \in \mathbb{N}$ as follows. Interpreting the vector space structure is obvious. Now, let $B = \{e_1, \ldots, e_m\}$ be the standard basis of $K^m$. Then the bilinear form is completely determined by fixing $k_{i,j} = [e_i, e_j]$ for all $1 \leq i, j \leq m$. Let $\pi_i : K^m \to K$ be the projection map onto the $i$-th coordinate. Then for $v, w \in K^m$, we have that $[v, w] = \sum_{i,j=1}^m \pi_i(v)\pi_j(w)k_{i,j}$, which is definable over $\{k_{i,j} : 1 \leq i, j \leq m\}$.

**Corollary 6.5.** (1) The case of a finite field $K$ corresponding to extra-special $p$-groups was treated in [19, Section 3].
(2) In [3], for each $n \in \mathbb{N}$ and $p$, Baudisch constructs a structure $D(n)$ in the language of groups with $n$ additional constant symbols, with $D(1)$ corresponding to extra-special $p$-groups. Since all these examples are interpretable in the bilinear form with additional constant symbols, they are all 2-dependent.

The rest of the section constitutes a proof of the theorem.

6.1. **Proof of Theorem 6.3(2).** Assume that $K$ has $\text{IP}_n$, then by Theorem 2.11, it must be witnessed by some $\mathcal{L}_K$-formula $\varphi(\bar{x}; y_1, \ldots, y_n)$ with each $y_i$ a single variable. Then by compactness for $1 \leq k \leq n$ we can find sequences $(c^{k}_{i,j,k}) : (i, j, k) \in \omega \times \omega$ with $\omega \times \omega$ ordered lexicographically and all $c^{k}_{i,j,k}$ pairwise distinct elements in $K$, such that for every $A \subseteq (\omega \times \omega)^n$ there is some $\bar{e}_A$ satisfying

$$\models \varphi(\bar{e}_A; c^{1}_{(i_1,j_1)}, \ldots, c^{n}_{(i_n,j_n)}) \iff (i_1, j_1, \ldots, (i_n, j_n)) \in A.$$ 

As $m = \infty$, we can choose $(a_k^n : 1 \leq k \leq n, i \in \omega)$ a tuple consisting of linearly independent elements in $V$. For each $1 \leq k \leq n$ and $j \in \omega$, let $f^k_j : V \to K$ be a linear function satisfying $f^k_j(a_k^n) = c^{k}_{i,j}$ for all $i \in \omega$. Since the bilinear form is non-degenerate, there exists some $b^k_j \in V$ such that $f^k_j(x) = [x, b^k_j]$ for all $x \in V$. But then, identifying $(\omega \times \omega)^n$ with $\omega^{2n}$, for any set $A \subseteq \omega^{2n}$, we have

$$\models \varphi(\bar{e}_A; a^{1}_{i_1,j_1}, \ldots, a^{n}_{i_n,j_n}) \iff (i_1, j_1, \ldots, (i_n, j_n)) \in A,$$

hence the formula $\psi(\bar{x}; y_1, y_2, \ldots, y_{2n-1}, y_{2n}) = \varphi(\bar{x}, [y_1, y_2], \ldots, [y_{2n-1}, y_{2n}])$ has $\text{IP}_{2n}$-witnessed by the sequences $(a^{1}_{i_1,j_1}, \ldots, a^{n}_{i_n,j_n})$.

6.2. **Proof of Theorem 6.3(1).** Let $\mathbb{M} \models T$ be a monster model. If $T$ is not 2-dependent, by Fact 2.8 we can find tuples $\bar{a}_\alpha, \bar{b}_\beta$ witnessing this, with respect to some formula $\varphi(\bar{x}; \bar{y}, \bar{z})$ without parameters. That is, we have that $(\bar{a}_\alpha, \bar{b}_\beta : \alpha, \beta \in \mathbb{Q})$ is $O_{2,p}$-indiscernible over $\emptyset$ and it is shattered by $\varphi$, more precisely for every $A \subseteq \mathbb{Q} \times \mathbb{Q}$ there is some $\bar{e}_A$ such that

$$\models \varphi(\bar{e}_A; \bar{a}_\alpha, \bar{b}_\beta) \iff (\alpha, \beta) \in A.$$ 

We write $\bar{x} = \bar{x}^{K} \cap \bar{x}^{V}, \bar{y} = \bar{y}^{K} \cap \bar{y}^{V}, \bar{z} = \bar{z}^{K} \cap \bar{z}^{V}$ for the subtuples of the variables of the corresponding sorts, where $\bar{x}^{K} = (x^{K}_i : i \in X^K)$ and $\bar{x}^{V} = (x^{V}_i : i \in X^V)$ and $X^K \sqcup X^V$ is a partition of $\{1, \ldots, |\bar{x}|\}$, and similarly for $\bar{y}$ and $\bar{z}$. Let $a_\alpha = a^{K}_\alpha \cap a^{V}_\alpha, \bar{b}_\beta = b^{K}_\beta \cap b^{V}_\beta$ for the corresponding subtuples in the $K$-sort and the $V$-sort, respectively. Let $\bar{a}_\alpha = (a^{k}_{\alpha,i} : i \in Y^K), \bar{b}_\beta = (b^{k}_{\beta,i} : i \in Z^K), etc.$

**Claim 6.6.** We can find a finite tuple $\bar{e}$ in $V$, a formula $\varphi'(\bar{x}', \bar{w}, \bar{y}', \bar{z}') \in \mathcal{L}$ and sequences of tuples $(\bar{a}_\alpha, \bar{b}_\beta : \alpha, \beta \in \mathbb{Q})$ such that:
Let \( \bar{a} : \alpha, \beta \in \mathbb{Q} \) is \( O_{2,p} \)-indiscernible over \( \bar{e} \);
(2) for any \( \alpha^* \in \mathbb{Q} \) and \( i^* \in (Y')^V \), we have that \( (a')_{\alpha^*,i^*} \notin \text{Span} ((i')_{\alpha^*,i} : i \in (Y')^V \setminus \{i^*\}) \), \( (\bar{a})_{\alpha}^V : \alpha \in \mathbb{Q} \setminus \{\alpha^*\} \), \( (\bar{b})_{\beta}^V : \beta \in \mathbb{Q} \), \( \bar{e} \);
(3) for any \( \beta^* \in \mathbb{Q} \) and \( j^* \in (Z')^V \), we have that \( (b')_{\gamma^*,j^*} \notin \text{Span} ((j')_{\gamma,j} : j \in (Z')^V \setminus \{j^*\}) \), \( (\bar{b})_{\beta}^V : \beta \in \mathbb{Q} \setminus \{\beta^*\} \), \( (\bar{a})_{\alpha}^V : \alpha \in \mathbb{Q} \), \( \bar{e} \);
(4) \( \varphi'(\bar{x}, \bar{e}, \bar{g}', \bar{z}') \) shatters \( (\bar{a}, \bar{b}) : \alpha, \beta \in \mathbb{Q} \), i.e. for every \( A \subseteq \mathbb{Q}^2 \), there is some \( \bar{e}_A \).

Proof. Assume that \( (\bar{a}, \bar{b}) \) don’t satisfy (2) with \( \bar{e} = \emptyset \). Then there are some \( \alpha^* \in \mathbb{Q} \), \( i^* \in Y^V \) and finite sets \( I, J \subseteq \mathbb{Q} \), \( \alpha^* \notin I \) such that

\[
a_{\alpha^*,i^*} \in \text{Span} ((a')_{\alpha^*,i} : i \in Y^V \setminus \{i^*\}) \cup (\bar{a})_{\alpha}^V I (\bar{b})_{\beta}^V J \bar{e}.
\]

Then there is a \( \emptyset \)-definable function \( f \) and some finite tuple \( \bar{k} \) in \( K \) such that

\[
a_{\alpha^*,i^*}^V = f(\bar{k}; (a')_{\alpha^*,i} : i \in Y^V \setminus \{i^*\}), (\bar{a})_{\alpha}^V I (\bar{b})_{\beta}^V J \bar{e}, \bar{e}).
\]

Let \( \gamma^+ := \min(\alpha \in I : \alpha > \alpha^*) \) and \( \gamma^- := \max(\alpha \in I : \alpha < \alpha^*) \). Let \( \delta := \max(J) \). Let \( \bar{e}' := (\bar{a})_{\alpha^*,i}^V I (\bar{b})_{\beta}^V J \bar{e} \). Let \( \bar{a}' := (a')_{\alpha^*,i} : i \in Y^V \setminus \{i^*\} \). Then \( (a_{\alpha}^V, (a')_{\alpha^*,i} : i \in Y^V \setminus \{i^*\}) \)

\[
(\bar{a})_{\alpha^*}^V : \alpha \in (\gamma^-, \gamma^+) \text{ is } O_{2,p} \text{-indiscernible over } \bar{e}'. \text{ As } \alpha^* \in (\gamma^-, \gamma^+) \text{, it follows that for every } \alpha \in (\gamma^-, \gamma^+) \text{ there is some tuple } \bar{k}_\alpha \text{ in } K \text{ such that } \bar{a}_{\alpha}^V = f(\bar{k}_\alpha; (a')_{\alpha^*,i}, \bar{e}').
\]

\[
\varphi'(\bar{x}, \bar{w}, \bar{y}', \bar{z}) := \varphi(\bar{x}; \bar{g}, (y_i')_{i \in Y^V \setminus \{i^*\}}, f(g_1^K, (y_i')_{i \in Y^V \setminus \{i^*\}}, \bar{w})), (y_i')_{i \in Y^V \setminus \{i^*\}}, \bar{z}),
\]

where \( \bar{y}' = (g_1')^{\gamma^-} (\bar{g})^{\gamma'}, (\bar{g})^{\gamma} = \bar{g}^{\gamma^-} \bar{g}^{\gamma'} (\bar{g})^{\gamma} = (y_i') : i \in Y^V \setminus \{i^*\} \). \( \bar{a}' := \bar{a}_{\alpha}^V (a')_{\alpha^*,i} \text{ (so the tuple } k_\alpha^\gamma \text{ corresponds to the variables } y') \)

Restricting to the set \( (\gamma^-, \gamma^+) \times (\delta, \infty) \) we may thus assume:

1. \( (a_{\alpha}'_{\alpha}, \bar{b} : \alpha, \beta \in \mathbb{Q} \) is \( O_{2,p} \)-indiscernible over \( \bar{e}' \) (follows by the choice of \( \gamma^-, \gamma^+, \delta \), definition of \( \bar{a}' \) and \( O_{2,p} \)-indiscernibility of \( (a_{\alpha}, \bar{b} : \alpha, \beta \in \mathbb{Q} \)),
2. \( \varphi'(\bar{x}, \bar{e}', \bar{y}', \bar{z}) \) shatters \( (k_\alpha a_{\alpha}, \bar{b} : \alpha, \beta \in \mathbb{Q} \) (as for any \( \bar{c} \) and \( \alpha, \beta \in \mathbb{Q} \) by the above we have \( \models \varphi(\bar{c}, \bar{a}, \bar{b}) \iff \models \varphi(\bar{c}, \bar{e}', \bar{k}_\alpha a_{\alpha}, \bar{b}) \)).

By Fact 2.7 let \( (h_\alpha a_{\alpha}', \bar{b} : \alpha, \beta \in \mathbb{Q} \) be an \( O_{2,p} \)-indiscernible over \( \bar{e}' \), based on \( (h_\alpha a_{\alpha}, \bar{b} : \alpha, \beta \in \mathbb{Q} \) over \( \bar{e}' \). We still have that \( \varphi'(\bar{x}, \bar{e}', \bar{y}', \bar{z}) \) shatters \( (h_\alpha a_{\alpha}', \bar{b} : \alpha, \beta \in \mathbb{Q} \), replacing \( \varphi \) by \( \varphi' \), \( \bar{e} \) by \( \bar{e}' \) and the sequences \( (a_{\alpha}, \bar{b} : \alpha, \beta \in \mathbb{Q} \) by \((h_\alpha a_{\alpha}', \bar{b} : \alpha, \beta \in \mathbb{Q} \), we have thus reduced the length of the tuples \( a_{\alpha}^V \) (at the price of increasing the length of \( a_{\alpha}^V \)). Repeating this argument finitely many times if necessary (for both \( \bar{a}' \)’s and \( \bar{b}' \)’s), we obtain the conclusion of the claim.

By Claim 6.6(4), let \( \bar{c} \) be such that \( \models \varphi'(\bar{c}, \bar{e}, \bar{a}', \bar{b}) \iff G_{2,p} \models R_{2}(\alpha, \beta) \). We write \( \bar{e} = \bar{e}' \).

Claim 6.7. We may moreover assume that for any \( i \in X^V \),

\[
c_{\alpha}^V \notin \text{Span} ((c_{i}^V : j \neq i) (\bar{a})_{\alpha}^V (\bar{b})_{\beta}^V \bar{e}).
\]

Proof. Assume that there is some \( i^* \in X^V \) such that \( c_{\alpha}^V \) is in the span of \((c_{i}^V : i \in X^V \setminus \{i^*\})(\bar{a})_{\alpha}^V (\bar{b})_{\beta}^V \bar{e} \) for some finite sets \( I, J \subseteq \mathbb{Q} \). Then

\[
c_{\alpha}^V = f(\bar{c}_1^K, (c_{i}^V)_{i \in X^V \setminus \{i^*\}}, (\bar{a})_{\alpha}^V I (\bar{b})_{\beta}^V J \bar{e}, \bar{c})
\]
for some $\emptyset$-definable function $f$ and some tuple $\vec{c}_1^K$ in $K$. Let $\alpha^* := \max(I \cup J)$. We let $\vec{c}' := (\vec{c}'_1^K, (\vec{c}'_1)V)$, where $\vec{c}'_1^K := \vec{c}_1^K \sim \vec{c}_1^K$ and $(\vec{c}')_1^V := (\vec{c}_1^V : i \in X^V \setminus \{i^*\})$. Let $\vec{c}' := (\vec{a}'_1, \vec{b}'_1)_{i \neq j} \vec{e}$. Restricting to a copy of $G_{2,p}$ contained in $(\alpha^*, \infty) \times (\alpha^*, \infty)$ (Remark 2.53), we thus have:

$$G_{2,p} = R_2(\alpha, \beta) \iff \models \varphi(\vec{c}, \vec{e}, \vec{a}'_\alpha, \vec{b}'_\beta) \iff \models \varphi'(\vec{c}', \vec{e}', \vec{a}'_\alpha, \vec{b}'_\beta)$$

for an appropriate $\mathcal{L}$-formula $\varphi''$. Also (1), (2) and (3) in Claim 6.6 still hold, with respect to $\vec{e}'$ (follows as $\vec{e}$ satisfies (1), (2), (3) and all the new elements in $\vec{e}'$ are from $(\vec{a}'_\alpha, \vec{b}'_\beta : \alpha, \beta < \alpha^*)$). Repeating this argument finitely many times if necessary, we obtain the claim.

\[\square\]

**Claim 6.8.** We may moreover assume that for any $i \in |\vec{e}|$, $e_i \notin \text{Span}(e_j : j \neq i)$.

**Proof.** As in the previous two claims, if $e_i \in \text{Span}(e_i : i \neq i^*)$, then $e_i = f(\vec{k}(e_i)_{i \neq i^*})$ for some $\emptyset$-definable function $f$ and a tuple $\vec{k}$ in $K$. Replacing $\vec{e}$ by $\vec{e}(e_i)_{i \neq i^*}$ (hence (1),(2),(3) in Claim 6.6 and the condition in Claim 6.7 still hold as we pass to a subtuple), adding $\vec{k}$ to $\vec{c}^K$ and modifying the formula accordingly, in finitely many steps we obtain the claim.

These three claims together imply that all elements in the tuple $\vec{c}^V(\vec{a}')_\alpha \in Q(\vec{b}')_\beta \in Q \vec{e}$ are linearly independent. Hence adjoining $\vec{c}^V$ to $\vec{c}^V$ and replacing $\varphi$ by $\varphi'$ and $\vec{a}_\alpha, \vec{b}_\beta$ by $\vec{a}'_\alpha, \vec{b}'_\beta$ respectively, we get:

**Claim 6.9.** There is an $\mathcal{L}$-formula $\varphi(\vec{x}, \vec{y}, \vec{z})$ and tuples $\vec{c}, \vec{a}_\alpha, \vec{b}_\beta$ such that:

1. $(\vec{a}_\alpha, \vec{b}_\beta : \alpha, \beta \in \mathbb{Q})$ is $O_{2,p}$-indiscernible;
2. all elements in $\vec{c}^V(\vec{a}')_\alpha \in Q(\vec{b}')_\beta \in Q \vec{e}$ are linearly independent;
3. $\models \varphi(\vec{c}, \vec{a}_\alpha, \vec{b}_\beta) \iff G_{2,p} \models R_2(\alpha, \beta)$.

Every $\mathcal{L}$-formula is a boolean combination of formulas of the form treated in the following cases. So it is enough to show that each of those is 2-dependent by Fact 2.72.

**Case 1.** The formula is of the form $\psi(t_1(\vec{x}, \vec{y}, \vec{z}), \ldots, t_d(\vec{x}, \vec{y}, \vec{z}))$ for some $\psi \in \mathcal{L}_K$ and some terms $t_l(\vec{x}, \vec{y}, \vec{z})$ taking values in $K$ and $1 \leq l \leq d$. For each $l$ we have the following possibilities:

- $t_l(\vec{x}, \vec{y}, \vec{z})$ has height 1, i.e. it is one of the variables in $\vec{x}^K \sim \vec{y}^K \sim \vec{z}^K$;
- $t_l(\vec{x}, \vec{y}, \vec{z}) = t_{l_1}^1(\vec{x}, \vec{y}, \vec{z}) + t_{l_2}^2(\vec{x}, \vec{y}, \vec{z})$ or $t_l(\vec{x}, \vec{y}, \vec{z}) = t_{l_1}^1(\vec{x}, \vec{y}, \vec{z}) \cdot t_{l_2}^2(\vec{x}, \vec{y}, \vec{z})$, for some terms $t_{l_1}^1, t_{l_2}^2$ of smaller height taking values in $K$;
- $t_l(\vec{x}, \vec{y}, \vec{z}) = [t_{l_1}^1(\vec{x}, \vec{y}, \vec{z}), t_{l_2}^2(\vec{x}, \vec{y}, \vec{z})]$ for some terms $t_{l_1}^1, t_{l_2}^2$ of smaller height taking values in $V$, but then:
  - either $t_{l_1}^1$ is of height 1, i.e. one of the variables in $\vec{x}^V \sim \vec{y}^V \sim \vec{z}^V$;
  - or $t_{l_1}^1(\vec{x}, \vec{y}, \vec{z}) = s_{l_1}^1(\vec{x}, \vec{y}, \vec{z}) \cdot V s_{l_2}^2(\vec{x}, \vec{y}, \vec{z})$ for some terms $s_{l_1}^1, s_{l_2}^2$ of smaller height taking values in $K$ and $V$ respectively, in which case $t_l(\vec{x}, \vec{y}, \vec{z}) = s_{l_1}^1(\vec{x}, \vec{y}, \vec{z}) \cdot [s_{l_2}^2(\vec{x}, \vec{y}, \vec{z}), t_{l_2}^2(\vec{x}, \vec{y}, \vec{z})]$;
  - or $t_{l_1}^1(\vec{x}, \vec{y}, \vec{z}) = s_{l_1}^1(\vec{x}, \vec{y}, \vec{z}) + V s_{l_2}^2(\vec{x}, \vec{y}, \vec{z})$ for some terms $s_{l_1}^1, s_{l_2}^2$ of smaller height taking values in $V$, in which case $t_l(\vec{x}, \vec{y}, \vec{z}) = [s_{l_1}^1(\vec{x}, \vec{y}, \vec{z}), t_{l_2}^2(\vec{x}, \vec{y}, \vec{z})]$.

And similarly for $t_{l_2}^2$. 

\[\square\]
Applying this for each \( l \) and iterating recursively, we thus conclude that the formula 
\[ \psi(t_1(\vec{x}, \vec{y}, \vec{z}), \ldots, t_d(\vec{x}, \vec{y}, \vec{z})) \]  
is equivalent to 
\[ \psi'(\{([x^i_1, y^i_1])_{i \in X \cap j \in Y}, ([x^i_2, y^i_2])_{i \in X \cap j \in Y}, ([y^i_1, z^i_1])_{i \in Y \cap j \in Z}, ([z^i_1, z^i_2])_{i,j \in Z \cap j \in Z}\} \]
for some \( L_K \)-formula \( \psi' \). But then, as \( K \) is NIP, Theorem 5.1 implies that this formula is 2-dependent.

**Case 2.** The formula \( \varphi(\vec{x}; \vec{y}, \vec{z}) \) is given by 
\[ \theta_d(t_1(\vec{x}, \vec{y}, \vec{z}), \ldots, t_d(\vec{x}, \vec{y}, \vec{z})) \]
for some \( d \in \mathbb{N} \) and terms \( t_l(\vec{x}, \vec{y}, \vec{z}) \) taking values in \( V \).

By a simple recursion on the height of the terms, we see that for \( 1 \leq l \leq d \) the term \( t_l \) must be of the form 
\[ \sum_{i \in X} t^X_l(\vec{x}, \vec{y}, \vec{z})x^i_1 + \sum_{j \in Y} t^Y_l(\vec{x}, \vec{y}, \vec{z})y^j + \sum_{k \in Z} t^Z_l(\vec{x}, \vec{y}, \vec{z})z^k \]
for some terms \( t^X_l, t^Y_l, t^Z_l \) taking values in \( K \).

By Claim 6.9 we have that for any \( \alpha, \beta \in \mathbb{Q} \) the set of all elements in the tuple \( \vec{c}^{V} - \vec{a}^{V} - \vec{b}^{V}_\beta \) is linearly independent. Then for any \( \alpha, \beta \) we have 
\[ G_{2,p} \models R_2(\alpha, \beta) \iff -\theta_d(t_1(c, \bar{a}_\alpha, \bar{b}_\beta), \ldots, t_d(c, \bar{a}_\alpha, \bar{b}_\beta)) \iff \]
\[ \models (\exists e_1, \ldots, e_d \in K)(e_1, \ldots, e_d) \neq (0, \ldots, 0) \land \bigwedge_{i \in X} \left( \sum_{l=1}^d e_l \cdot t^X_l(c, \bar{a}_\alpha, \bar{b}_\beta) = 0 \right) \land \]
\[ \bigwedge_{j \in Y} \left( \sum_{l=1}^d e_l \cdot t^Y_l(c, \bar{a}_\alpha, \bar{b}_\beta) = 0 \right) \land \bigwedge_{k \in Z} \left( \sum_{l=1}^d e_l \cdot t^Z_l(c, \bar{a}_\alpha, \bar{b}_\beta) = 0 \right) \iff \]
\[ \models \psi(\{t^X_l(c, \bar{a}_\alpha, \bar{b}_\beta)\}_{1 \leq l \leq d, i \in X}^{}, \{t^Y_l(c, \bar{a}_\alpha, \bar{b}_\beta)\}_{1 \leq l \leq d, j \in Y}^{}, \{t^Z_l(c, \bar{a}_\alpha, \bar{b}_\beta)\}_{1 \leq l \leq d, k \in Z}^{}) \]
for an appropriate formula \( \psi \in L_K \). But this is impossible by Case 1.

**Case 3.** The formula is of the form \( t(\vec{x}, \vec{y}, \vec{z}) = 0 \). As in the previous case, then \( t \) must be of the form 
\[ \sum_{i \in X} t^X_l(\vec{x}, \vec{y}, \vec{z})x^i_1 + \sum_{j \in Y} t^Y_l(\vec{x}, \vec{y}, \vec{z})y^j + \sum_{k \in Z} t^Z_l(\vec{x}, \vec{y}, \vec{z})z^k \]
for some terms \( t^X_l, t^Y_l, t^Z_l \) taking values in \( K \). By Claim 6.9 we have that for any \( \alpha, \beta \in \mathbb{Q} \) the set of all elements in the tuple \( \vec{c}^{V} - \vec{a}^{V} - \vec{b}^{V}_\beta \) is linearly independent. But then for any \( \alpha, \beta \) we have:
\[ G_{2,p} \models R_2(\alpha, \beta) \iff t(c, \bar{a}_\alpha, \bar{b}_\beta) = 0 \iff \]
\[ \bigwedge_{i \in X} t^X_l(c, \bar{a}_\alpha, \bar{b}_\beta) = 0 \land \bigwedge_{j \in Y} t^Y_l(c, \bar{a}_\alpha, \bar{b}_\beta) = 0 \land \bigwedge_{k \in Z} t^Z_l(c, \bar{a}_\alpha, \bar{b}_\beta) = 0 \]
\[ \iff \psi(\{t^X_l(c, \bar{a}_\alpha, \bar{b}_\beta)\}_{i \in X}^{}, \{t^Y_l(c, \bar{a}_\alpha, \bar{b}_\beta)\}_{j \in Y}^{}, \{t^Z_l(c, \bar{a}_\alpha, \bar{b}_\beta)\}_{k \in Z}^{}) \]
for an appropriate \( L_K \)-formula \( \psi \) — which is impossible by Case 1.
7. The effects of adding a random predicate

Recall that a theory $T$ is geometric if it eliminates the $\exists^\infty$-quantifier and acl satisfies exchange. In this section we denote by $\downarrow_C$ the independence relation given by algebraic independence (i.e. $A \downarrow_C B \iff \text{acl}(AC) \cap \text{acl}(BC) \subseteq \text{acl}(C)$).

**Definition 7.1.** [7] Definition 2.6] Let $T$ be a geometric theory, $M \models T$ and $B \subseteq M$. We say that a tuple $\bar{a} = (a_0, \ldots, a_{n-1}) \in M^n$ is an algebraic $n$-gon over $B$ if $\dim(\bar{a}/B) = n - 1$, but any subset of $\{a_0, \ldots, a_{n-1}\}$ of size $n - 1$ is independent over $B$.

Recall that a theory $T$ has disintegrated algebraic closure if $\text{acl}(A) = \bigcup_{a \in A} \text{acl}(a)$ for any set $A$ in a model of $T$.

**Fact 7.2.** [8]Lemma 2.7] Suppose that $T$ is a geometric theory with non-disintegrated algebraic closure. Then for every $n \geq 3$ there exist some finite set $B$ and an algebraic $n$-gon over $B$.

We recall the setting and some results from [8]. Let $\text{acl}_T$ denote the algebraic closure in the sense of $T$, and let $S$ be a distinguished $\emptyset$-definable set in $T$. We denote by $T_{0,S}$ the theory in a language $\mathcal{L}_P := \mathcal{L} \cup \{P(x)\}$ given by $T \cup \{P(x) \rightarrow S(x)\}$, and write $\text{acl}_S$ to denote algebraic closure intersected with $S$.

**Fact 7.3.** (1) [8] Theorem 2.4] The theory $T_{0,S}$ has a model companion $T_{P,S}$ with the following axiomatization: $(M, P) \models T_{P,S} \iff$

(a) $M \models T$;

(b) for every $\mathcal{L}$-formula $\theta(\bar{x}, \bar{y})$ with $\bar{x} = (x_1, \ldots, x_n)$, for every subset $I \subseteq \{1, \ldots, n\}$, $(M, P)$ satisfies

$$\forall \bar{z} \left( \exists \bar{x} \theta(\bar{x}, \bar{z}) \land (\bar{x} \cap \text{acl}_T(\bar{z}) = \emptyset) \land \bigwedge_{i=1}^n S(x_i) \land \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \right)$$

$$\rightarrow \left( \exists \bar{x} \theta(\bar{x}, \bar{z}) \land \bigwedge_{i \in I} (x_i \in P) \land \bigwedge_{i \notin I} (x_i \in S \setminus P) \right).$$

(2) Let $(M, P) \models T_{P,S}$. Assume $\bar{a}, \bar{b}$ are small tuples from $M$, and $A \subseteq M$ is a small set of parameters. Then the following are equivalent:

(a) $\text{tp}(\bar{a}/A) = \text{tp}(\bar{b}/A)$;

(b) there is an $A$-isomorphism of $\mathcal{L}_P$-structures from $\text{acl}_T(A, \bar{a})$ to $\text{acl}_T(A, \bar{b})$ which carries $\bar{a}$ to $\bar{b}$.

We generalize [8] Proposition 2.10] from $n = 1$ to arbitrary $n \in \omega$.

**Proposition 7.4.** Assume that $T$ is a geometric theory and $\text{acl}_S$ is not disintegrated. Then $T_P$ is not $n$-dependent for any $n \geq 1$.

**Proof.** Fix $n \geq 1$. By Fact 7.2 let $a_1, \ldots, a_{n+2}$ be an algebraic $(n + 2)$-gon over a finite set $B$. Naming $B$ by constants, without loss of generality we may assume that $B = \emptyset$. Then $\{a_1, \ldots, a_{n+1}\}$ is an independent set. Using extension, symmetry and transitivity we can choose inductively sequences $\bar{a}_i = (a_i^j : j \in \omega)$ for $1 \leq i \leq n$ such that:

1. $a_i^j \downarrow \bar{a}_{i+1} a_{i+1} \ldots a_{n+1}$ for all $1 \leq i \leq n$ and all $j \in \omega$;
2. $a_i^j = a_1 \ldots a_{i-1} a_{i+1} \ldots a_{n+1} a_{i}$ for all $1 \leq i \leq n$ and all $j \in \omega$. 

We then choose a sequence $\bar{a}_0$ of the form $a_0, a_0, a_1, \ldots, a_{n+2}$ and define $\bar{a}_i$ inductively as $\bar{a}_i = \bar{a}_{i+1} a_{i+1} \ldots a_{n+1} a_{i}$ for $1 \leq i \leq n$. 

We then claim that $\bar{a}_i$ is an algebraic $n$-gon over $B$.
In particular, by forking calculus (1) implies that \( \{a_i^j : 1 \leq i \leq n, j \in \omega \}\cup \{a_{n+1}\} \) is an \( \omega \)-independent set, and (2) implies that \( a_1^{i_1} \cdots a_n^{i_n} a_{n+1} \equiv a_1 \cdots a_n a_{n+1} \) for all \( j_1, \ldots, j_n \in \omega \).

Recall that in particular \( a_{n+2} \in acl_s(a_1 \ldots a_{n+1}), a_{n+1} \in acl_s(a_1 \ldots a_n a_{n+2}). \) In particular, there exists a formula \( \varphi(x_1, \ldots, x_{n+2}) \) such that \( \varphi(a_1, \ldots, a_{n+1}, x_{n+2}) \) isolates \( tp(a_{n+2}/a_1 \ldots a_{n+1}) \) and there exists some \( k \in \omega \) such that \( |\varphi(a_1^1, \ldots, a_n^m, a_{n+1}, M)| \leq k \) for any \( a_1^1, \ldots, a_{n+1}^m \in M \), and \( \varphi(a_1, \ldots, a_n, x_{n+1}, a_{n+2}) \) isolates \( tp(a_{n+1}/a_1 \ldots a_n a_{n+2}) \).

**Claim 7.5.** (a) \( a_{n+1} \notin acl(\bar{a}_1 \ldots \bar{a}_n) \) and \( a_{n+1} \notin \varphi(a_1^1, \ldots, a_n^m, a_{n+1}, M) \) for any \( (i_1, \ldots, i_n) \in \omega^n \),

(b) For any \( (i_1, \ldots, i_n) \in \omega^n \) the set \( \varphi(a_1^{i_1}, \ldots, a_n^{i_n}, a_{n+1}, M) \cap acl(\bar{a}_1 \ldots \bar{a}_n) = \emptyset \),

(c) For any \( (i_1, \ldots, i_n) \neq (j_1, \ldots, j_n) \in \omega^n \) we have

\[
\varphi(a_1^{i_1}, \ldots, a_n^{i_n}, a_{n+1}, M) \cap \varphi(a_1^{j_1}, \ldots, a_n^{j_n}, a_{n+1}, M) = \emptyset;
\]

**Proof.** (a) By (1) above and forking calculus we have \( a_{n+1} \downarrow \bar{a}_1 \cdots \bar{a}_n \). And \( a_{n+1} \notin \varphi(a_1^{i_1}, \ldots, a_n^{i_n}, a_{n+1}, M) \) since \( a_1^{i_1} \ldots a_n^{i_n} a_{n+1} \equiv a_1 \ldots a_n a_{n+1} \).

(b) Taking any \( b \) such that \( a_1^{j_1} \ldots a_n^{j_n} a_{n+1} b \equiv a_1 \ldots a_n a_{n+1} a_{n+2} \) we see that the set \( \varphi(a_1, \ldots, a_{n+1}, M) \) is non-empty.

Assume that \( b \in \varphi(a_1^{i_1}, \ldots, a_n^{i_n}, a_{n+1}, M) \) is arbitrary. By (1) above and forking calculus we have \( a_{n+1} \downarrow \bigwedge_{i \in \omega^n} a_1^{i_1} \cdots a_n^{i_n} \). Hence if \( b \in acl(\bar{a}_1 \ldots \bar{a}_n) \), then already \( b \in acl(a_1^{i_1} \ldots a_n^{i_n}) \). But \( a_1^{i_1} \ldots a_n^{i_n} a_{n+1} b \equiv a_1 \ldots a_n a_{n+1} a_{n+2} \) by the choice of \( \varphi \) and (2) above, and \( a_{n+2} \notin acl(a_1 \ldots a_n) \) since \( (a_1, \ldots, a_{n+2}) \) is an algebraic \( n \)-gon — a contradiction.

(c) Let \( I := \{1 \leq t \leq n : i_t = j_t \} \), by assumption \( |I| < n \). By (1) above and forking calculus we have

\[
(a_i^t : t \notin I) \downarrow a_{n+1}(a_t^t : t \notin I).
\]

Assume that \( b \in \varphi(a_1^{i_1}, \ldots, a_n^{i_n}, a_{n+1}, M) \cap \varphi(a_1^{j_1}, \ldots, a_n^{j_n}, a_{n+1}, M) \). Then \( b \in acl((a_t^t : t \in I) a_{n+1}) \), and by the choice of \( \varphi \) and (2) above we have \( a_1^{i_1} \cdots a_n^{i_n} a_{n+1} b \equiv a_1 \ldots a_n a_{n+1} a_{n+2} \) and hence \( a_{n+2} \in acl((a_t^t : t \in I) a_{n+1}) \). But this is a contradiction since \( (a_1, \ldots, a_{n+2}) \) is an \( n \)-gon and \( |I| < n \).

Let \( \psi(x_1, \ldots, x_{n+2}) := \exists x_{n+2} \in P \varphi(x_1, \ldots, x_{n+1}, x_{n+2}) \). Let \( m \in \omega \) and \( I \subseteq m^n \) be arbitrary. Set \( \bar{a} := (a_i^1 : 1 \leq i \leq n, 1 \leq j \leq m) \) (with \( a_i^j \) as chosen in the beginning of the proof), and consider the formula

\[
\theta(\bar{x}, \bar{a}) = \theta(x_{n+1}, (\bar{x}_i : \bar{i} \in m^n), \bar{a}) := \bigwedge_{i \in m^n} \bigwedge_{1 \leq t \leq k} \varphi(a_1^{i_1}, \ldots, a_n^{i_n}, x_{n+1}, x_{i_1, \ldots, i_n}),
\]

where for all \( \bar{i} = (i_1, \ldots, i_n) \) we have \( \bar{x}_i = (x_1^{i_1}, \ldots, x_k^{i_k}) \). For each \( \bar{i} \in m^n \), let \( \bar{b}_i \) be a tuple of length \( k \) enumerating the set \( \varphi(a_1^{i_1}, \ldots, a_n^{i_n}, a_{n+1}, M) \), and let \( b := a_{n+1} \sim (\bar{b}_i : \bar{i} \in m^n) \). Then, by (a)–(c) of the Claim we have that all elements in \( b \subseteq S \) are pairwise-distinct, \( b \cap acl(\bar{a}) = \emptyset \) and \( |= \theta(\bar{b}, \bar{a}) \). Hence, applying Fact 7.3, there exists some \( b' = a_1^{i_1} \cdots a_n^{i_n} \) such that \( |= \theta(b', a) \), \( b' \subseteq P \) for all \( \bar{i} \in I \) and \( b_i \cap P = \emptyset \) for all \( \bar{i} \in m^n \setminus I \). But then, by the choice of \( k \), for each \( \bar{i} \in m^n \) we have

\[
|= \psi(a_1^{i_1}, \ldots, a_n^{i_n}, a_{n+1}^I) \iff \bar{i} \in I,
\]

hence \( \psi \) is not \( n \)-dependent.
Remark 7.6. The case of $n = 1$ in [3] Proposition 2.10] is claimed without the assumption that $T$ is geometric. However, their proof contains a gap and the claim is false as witnessed by the following example. Let $T$ be the theory of the infinite branching tree, i.e. the theory of an infinite graph $(G, R)$ such that

1. for every vertex $a \in G$ there are infinitely many $b$ such that $aRb$,
2. there are no cycles.

It is not hard to see by back-and-forth that $T$ is complete and admits quantifier elimination after adding distance predicates. Then $T_P$ is stable, e.g. since by [21] Theorem 1.4 every expansion of a planar graph by unary predicates is stable. However, acl is not disintegrated (for any $a \in G$ and two elements $b, c$ connected to it, we have that $a \in \text{dcl}(bc)$, but $a \not\in \text{acl}(b) \cup \text{acl}(c)$). Note that acl doesn’t satisfy exchange in this example since $b \not\in \text{acl}(ac)$.

Remark 7.7. Same argument as in the proof of Proposition 7.4 applies to any theory eliminating $\exists^\infty$ and such that $S$ contains an infinite definable subgroup.

Next we show that if the algebraic closure is disintegrated, then $n$-dependence is preserved after adding a random predicate, and more generally for predicates of arity at most $n$. The next fact follows from [25] Proposition 6.11] (using the notation there, applied with $T_0 := T$ the theory that we are expanding, $T_1$ the model companion of the empty theory in the language $\mathcal{L}' \setminus \mathcal{L}$ with the relations that we are expanding by, and $T_2$ the theory of equality in the empty language; note that acl$_{T_2} = \text{acl}_T$ by [25] Proposition 6.3].

Fact 7.8. Let $T$ be a theory in the language $\mathcal{L}$ eliminating $\exists^\infty$ on the predicate $S$, and assume that acl is disintegrated on $S$. Let $T'$ be a generic expansion of $T$ in a language $\mathcal{L}'$ such that $\mathcal{L}' \setminus \mathcal{L}$ only contains relational symbols living on $S$. Then:

$$\text{tp}_{\mathcal{L}'}(\bar{a}/A) = \text{tp}_{\mathcal{L}'}(\bar{b}/A) \text{ if and only if there is an } A\text{-isomorphism of } \mathcal{L}'\text{-structures from } \text{acl}_{\mathcal{L}}(A, \bar{a}) \text{ to } \text{acl}_{\mathcal{L}}(A, \bar{b}) \text{ which carries } \bar{a} \text{ to } \bar{b}$$

(†)

In [20] Lemma 2.1] Hrushovski observes that the random $n$-ary hypergraph is not a finite Boolean combination of relations of arity $n - 1$. We will need the following infinitary generalization of this fact.

Proposition 7.9. For each $n \in \omega, n \geq 1$ and an infinite cardinal $\kappa$ there exists some cardinal $\lambda \geq \kappa$ satisfying the following. Let $G_{n,p}$ be a $\lambda$-saturated model of $\text{Th}(G_{n,p})$, let $\hat{\mathcal{L}}$ be an arbitrary relational language with $|\hat{\mathcal{L}}| \leq \kappa$ containing only relations of arity at most $n - 1$, and let $\hat{O}_{n,p}$ be an expansion of $O_{n,p}$ obtained by adding arbitrary interpretations for all the relations in $\hat{\mathcal{L}}$. Then the following cannot hold:

for all $g_i, h_i \in P_{t}^{G_{n,p}}, 1 \leq i \leq n$, if $\text{qftp}_{\hat{\mathcal{L}}}(g_1, \ldots, g_n) = \text{qftp}_{\hat{\mathcal{L}}}(h_1, \ldots, h_n)$ then

$$G_{n,p}' R_n(g_1, \ldots, g_n) \text{ if and only if } G_{n,p}' R_n(h_1, \ldots, h_n).$$

(*)

Proof. By induction on $n$, the base case $n = 1$ obviously holds with $\lambda := \kappa$. Now fix $n \geq 2$ and $\kappa$, and let $\lambda = \lambda_{n-1}$ satisfy the proposition for $n - 1$ and $\kappa$. We will show that $\lambda = \lambda_n := \exists_{n-1} (2^{\lambda_{n-1}})^+$ satisfies the proposition for $n$. Assume that some $\lambda_n$-saturated $G_{n,p}' \equiv G_{n,p}$; some language $|\hat{\mathcal{L}}| \leq \kappa$ and some expansion $\hat{O}_{n,p}'$ satisfy (*).

By the choice of $\lambda_n$ and Erdős-Rado we have $\lambda_n \to (2^{\lambda_{n-1}})^+ \, n \!\!\!\!\!\!\downarrow _{2^{\lambda_{n-1}}}$, hence we can find some sets $A_i \subseteq P_{t}^{G_{n,p}}, 1 \leq i \leq n - 1$ such that $|A_i| \geq (2^{\lambda_{n-1}})^+$ and $\text{qftp}_{\hat{\mathcal{L}}}(g_1, \ldots, g_{n-1}) = \text{qftp}_{\hat{\mathcal{L}}}(h_1, \ldots, h_{n-1})$ for all $g_i, h_i \in A_i, 1 \leq i \leq n - 1$. Next, we can find a $\lambda_{n-1}$-saturated
structure $G'_{n-1,p} \equiv G_{n-1,p}$ with $|G'_{n-1,p}| \leq 2^{\lambda_{n-1}}$ and such that $P^G_{n-1,p} \subseteq A_i$, $1 \leq i \leq n - 1$. As $G'_{n,p}$ is $\lambda_n$-saturated and $\lambda_n > 2^{\lambda_{n-1}}$, by the axioms of $\text{Th}(G_{n,p})$ there exists some $c \in P^G_{n,p}$ such that for all $g_i \in P^G_{n-1,p}, 1 \leq i \leq n - 1$ we have

$$G'_{n-1,p} \models R_{n-1}(g_1, \ldots, g_{n-1}) \iff G'_{n-1,p} \models R_n(g_1, \ldots, g_{n-1}, c).$$

Now, without loss of generality, we may assume that all relations in $\hat{\mathcal{L}}$ are of arity $n - 1$. We consider the language $\hat{\mathcal{L}}_{n-1}$ containing an $(n - 2)$-ary relational symbol $F'_i$ for each relation symbol $F \in \hat{\mathcal{L}}$ and $1 \leq i \leq n - 1$, and an expansion $\hat{\mathcal{O}}'_{n-1,p}$ of $O'_{n-1,p}$ obtained by interpreting each such $F'_i \in \hat{\mathcal{L}}_{n-1}$ as $F(x_1, \ldots, x_{i-1}, c, x_{i+1}, \ldots, x_{n-1})$ restricted to the universe of $G'_{n-1,p}$. Hence $|\hat{\mathcal{L}}_{n-1}| \leq \kappa$ and all relations in $\hat{\mathcal{L}}_{n-1}$ have arity $n - 2$. Note that by the choice of $A_i$, $1 \leq i \leq n - 1$ we automatically have that for any $F \in \hat{\mathcal{L}}$ and any $g_i, h_i \in P^G_{n-1,p}, 1 \leq i \leq n - 1$,

$$\hat{\mathcal{O}}'_{n,p} \models F(g_1, \ldots, g_{n-1}) \iff \hat{\mathcal{O}}'_{n,p} \models F(h_1, \ldots, h_{n-1}).$$

We then have that for any $g_i, h_i \in P^G_{n-1,p}, 1 \leq i \leq n - 1$,

$$\text{qftp}_{\hat{\mathcal{L}}_{n-1}}(g_1, \ldots, g_{n-1}) = \text{qftp}_{\hat{\mathcal{L}}_{n-1}}(h_1, \ldots, h_{n-1}) \implies \text{qftp}_{\hat{\mathcal{L}}_{n-1}}(g_1, \ldots, g_{n-1}, c) = \text{qftp}_{\hat{\mathcal{L}}_{n-1}}(h_1, \ldots, h_{n-1}, c),$$

and since (*) holds for $\hat{\mathcal{O}}'_{n,p}$, by the choice of $c$ this implies

$$\text{qftp}_{\hat{\mathcal{L}}_{n-1}}(g_1, \ldots, g_{n-1}) = \text{qftp}_{\hat{\mathcal{L}}_{n-1}}(h_1, \ldots, h_{n-1}) \wedge G'_{n-1,p} \models R_{n-1}(g_1, \ldots, g_{n-1}) \implies G'_{n-1,p} \models R_{n-1}(h_1, \ldots, h_{n-1}).$$

That is, $\hat{\mathcal{O}}'_{n-1,p}$ and $\hat{\mathcal{L}}_{n-1}$ satisfy (*) — contradicting the induction hypothesis. \qed

On the other hand we have:

**Proposition 7.10.** Let $T$ be a theory in the language $\mathcal{L}$ eliminating $\exists^\infty$ on the predicate $S$, and assume that acl is disintegrated on $S$. Fix $n \geq 1$, and let $T'$ be a generic expansion of $T$ in a language $\mathcal{L}'$ such that $\mathcal{L}' \setminus \mathcal{L}$ only contains relational symbols of arity at most $n$ living on $S$. Then $T'$ is $n$-dependent if and only if $T$ is $n$-dependent.

**Proof.** Assume that $T$ is $n$-dependent, but that there is some formula $\varphi(x; y_1, \ldots, y_n) \in \mathcal{L}'$ which is not $n$-dependent. Let $T^{\text{Sk}}$ be a Skolemization of $T'$, in the language $\mathcal{L}^{\text{Sk}} \supseteq \mathcal{L}'$, $|\mathcal{L}^{\text{Sk}}| = |\mathcal{L}'|$. Let $\kappa := |\mathcal{L}'|$, and let $\lambda$ be as given by Proposition 7.9 for $n$ and $\kappa$.

Let $G_{n,p}^{\prime}$ be a $\lambda$-saturated model of $\text{Th}(G_{n,p})$ of size $\geq \left(2^{|\mathcal{L}^{\text{Sk}}|}\right)^+$. By Fact 2.8(3), there exist $(a_g)_{g \in G_{n,p}'}$ and $b$ such that $(a_g)_{g \in G_{n,p}'}$ is $O'_{n,p}$-indiscernible over $\emptyset$ and $G_{n,p}'$-indiscernible over $b$, both in the sense of $T^{\text{Sk}}$, and $|\varphi(b; a_{g_1}, \ldots, a_{g_n}) \iff G_{n,p}' \models R_n(g_1, \ldots, g_n)$, for all $g_i \in P_i$. For each $g \in G_{n,p}'$, let $a'_g$ be the tuple enumerating acl$_{\mathcal{L}'}(a_g)$ beginning with $a_g$, and let $b'$ be a tuple enumerating acl$_{\mathcal{L}'}(b)$. Then, as acl = dcl in $T^{\text{Sk}}$, we get that also $(a'_g)_{g \in G_{n,p}'}$ is $O'_{n,p}$-indiscernible over $\emptyset$ and $G_{n,p}'$-indiscernible over $b'$ in $T^{\text{Sk}}$ (where each tuple $a'_g$ is enumerated in the corresponding order), and hence in $T$ as well. As $T$ is $n$-dependent, it follows by Fact 2.9(3) that $(a'_g)_{g \in G_{n,p}'}$ is $O'_{n,p}$-indiscernible over $b'$. In particular, for any $g_i, h_i \in P_i, 1 \leq i \leq n$ there is an $\mathcal{L}$-automorphism $\sigma$ fixing $b'$ pointwise and such that $\sigma(a'_i) = a'_{hi}$ for all $1 \leq i \leq n$. As $T$ has disintegrated algebraic closure on $S$, we have acl$_{\mathcal{L}'}(b_{a_{g_1}} \ldots a_{g_n}) = b'a_{g_1} \ldots a'_{g_n}$, hence $\sigma$ sends acl$_{\mathcal{L}'}(b_{a_{g_1}} \ldots a_{g_n})$ to acl$_{\mathcal{L}'}(b_{a_{h_1}} \ldots a_{h_n})$ carrying $b_{a_{g_1}} \ldots a_{g_n}$ to $b_{a_{h_1}} \ldots a_{h_n}$. 


Now we consider the isomorphism type of the $L'$-induced structure on $b'a_{g_1} \ldots a_{g_n}$ as $\bar{g}$ varies. By $O_{n,p}$-indiscernibility in $T'$, for any $F \in L' \setminus L$ we have $\models F(a_{g_1}, \ldots, a_{g_n}) \iff \models F(a_{h_1}, \ldots, a_{h_n})$ for all $g_i, h_i \in \bar{P}_{i}^{G'_{n,p}}, 1 \leq i \leq n$. For $1 \leq t \leq n$, let $\kappa_t := |a_{g_t}|$ for some/any $g_t \in \bar{P}_{i}^{G'_{n,p}}$, and let $\kappa_0 := |b'|$. Note that $\kappa_t \leq \kappa$ for all $0 \leq t \leq n$. We consider an expansion $\bar{G}_{n,p}$ of $G_{n,p}$ where for each $F(x_1, \ldots, x_n) \in L' \setminus L$, each $1 \leq i \leq n$ and each $j_i \in \kappa_t, t \in \{0, \ldots, n\} \setminus \{i\}$ we add a new $(n-1)$-ary relation $R_{F,i,j} \subseteq \prod_{t \in \{1,\ldots,n\} \setminus \{i\}} \bar{P}_{t}^{G'_{n,p}}$ defined as follows: for any $(g_t)_{t \in \{1,\ldots,n\} \setminus \{i\}} \in \prod_{t \in \{1,\ldots,n\} \setminus \{i\}} \bar{P}_{t}^{G'_{n,p}}$ we have
\[(g_t)_{t \in \{1,\ldots,n\} \setminus \{i\}} \in R_{F,i,j} \iff \models F(a'_{g_{t,j_1}}, \ldots, a'_{g_{t,j_{t-1}}}, b', a'_{g_{t,j_{t+1}}}, \ldots, a'_{g_{t,n}}),\]
where $a'_{g_t} = (a'_{g_t,j} : j < \kappa_0)$ for $1 \leq t \leq n$ and $b' = (b'_j : j < \kappa_0)$. Note that taking $\bar{L} := \{R_{F,i,j} : F \in L' \setminus L, 1 \leq i \leq n, j_i \leq \kappa_t\} \cup \mathcal{L}$ we have $|\bar{L}| \leq \kappa$. Then, by Proposition 7.9 there exist some $g_i, h_i \in \bar{P}_{i}^{G'_{n,p}}, 1 \leq i \leq n$ such that:

1. $\models \text{qftp}_{L}(g_1, \ldots, g_n) = \text{qftp}_{\bar{L}}(h_1, \ldots, h_n),$
2. $\bar{G}_{n,p} \models R_n(g_1, \ldots, g_n),$
3. $\bar{G}_{n,p} \models \neg R_n(h_1, \ldots, h_n).$

Given an $L$-automorphism $\sigma$ as above, it then follows from the definition of $\bar{L}$ and (1) that $\sigma$ is an automorphism of $L'$-structures sending $b'a_{g_1} \ldots a_{g_n}$ to $b'a_{h_1} \ldots a_{h_n}$. Hence by the previous discussion and Fact 7.3\footnote{This example was suggested to us by Erik Walsberg.} we have
\[\text{tp}_{L'}(a_{g_1} \ldots a_{g_n}/b) = \text{tp}_{\bar{L}}(a_{h_1} \ldots a_{h_n}/b).\]
But this contradicts the choice of $\varphi$ in view of (2) and (3).

Combining Propositions 7.4 and 7.10 we thus have the following “baby case” of the relationship of the collapse of $n$-dependence to NIP and complicated geometry of algebraic closure that we expect to happen for fields.

Corollary 7.11. Let $T$ be a geometric NIP theory. The following are equivalent:

1. $T_P$ is NIP.
2. $T_P$ is $n$-dependent for some $n \in \omega$.
3. $T$ has disintegrated algebraic closure.

Example 4. Fix $n \in \omega$. Let $K \models \text{ACF}_p$, for $p = 0$ or prime, be $\aleph_1$-saturated, and let $I = (a_i)_{i \in \omega}$ be a countable set of elements in $K$ algebraically independent over the prime field. By quantifier elimination, $I$ is an $\theta$-indiscernible sequence, and $I$ is small in $K$. Let $T$ be an expansion of $\text{ACF}_p$ obtained by naming $I$ by a new predicate symbol $S$. Now, recall the following result of Baldwin and Benedict.

Fact 7.12. Corollary 8.4\footnote{Corollary 8.4} Let $T$ be a stable theory, $M \models T$ and let $I \subseteq M$ be an indiscernible set which is small in $M$ (that is, every type $p \in S_{<\omega}(I)$ is realized in $M$). Then $\text{Th}(M, I)$ is stable, and the structure induced on $I$ (in the pair) is just that of equality.

In particular $T$ defined above is stable, and $\text{acl}_S$ is disintegrated. Then the expansion $T'$ obtained from $T$ by naming a random $n$-ary relation on $S$ is strictly $n$-dependent, by Proposition 7.10.
8. $K$-DEPENDENT VALUED FIELDS

The main result of this section is the following theorem generalizing a recent result of Johnson [22] from $k = 1$ to all $k \in \mathbb{N}$.

**Theorem 8.1.** If $(K, \mathcal{O})$ is an infinite valued field of positive characteristic and $\text{Th}(K)$ is $k$-dependent for some $k \in \mathbb{N}$, then $K$ is henselian.

From now on, let $K$ be an infinite field of characteristic $p > 0$ and $\mathcal{O}_i$ a valuation ring on $K$ for $i = 1, 2$. We additionally fix the following notation.

- For $i = 1, 2$, let $m_i$ be the maximal ideal of $\mathcal{O}_i$;
- let $J := m_1 \cap m_2$.

**Fact 8.2.** [22, Remark 2.1] Assume $\mathcal{O}_1$ and $\mathcal{O}_2$ are incomparable (i.e. none of them is contained in the other). Then
\[(a + m_1) \cap (b + m_2) \neq \emptyset\]
for any $a \in \mathcal{O}_1$ and $b \in \mathcal{O}_2$.

We say that $b \in K$ is an Artin-Schreier root of $a \in K$ if $a = b^p - b$. We say that $K$ is Artin-Schreier closed if every element of $K$ has an Artin-Schreier root in $K$. Recall the following.

**Fact 8.3.** ([24] for $k = 1$, [19] for arbitrary $k \in \mathbb{N}$) Let $K$ be an infinite field of positive characteristic, such that $\text{Th}(K)$ is $k$-dependent. Then $K$ is Artin-Schreier closed.

Our main contribution is the following result.

**Proposition 8.4.** Suppose that $(K, \mathcal{O}_1, \mathcal{O}_2)$ is $k$-dependent and $\text{char}(K) = p > 0$. Then every element in $J$ has an Artin-Schreier root in $J$.

Being able to find an Artin-Schreier in both maximal ideals simultaneously forces the corresponding valuations to be comparable:

**Corollary 8.5.** If the structure $(K, \mathcal{O}_1, \mathcal{O}_2)$ is $k$-dependent and $\text{char}(K) = p > 0$, then $\mathcal{O}_1$ and $\mathcal{O}_2$ are comparable.

**Proof.** Assume not, then by Fact 8.2 with $a = 0$ and $b = 1$ there exists some $w \in m_1 \cap (1 + m_2)$. Let $y := w^p - w$. Now, as $\text{val}_1(w) > 0$, we have that
\[\text{val}_1(y) = \text{val}_1(w) > 0.\]
Secondly, let $z \in m_2$, i.e. $\text{val}_2(z) > 0$, be such that $w = 1 + z$. Then
\[\text{val}_2(y) = \text{val}_2(w^p - w) = \text{val}_2((1 + z)^p - (1 + z)) = \text{val}_2(z^p - z) = \text{val}_2(z) > 0.\]
Thus $y \in J$. However, the Artin-Schreier roots of $y$ are exactly $w, w + 1, \ldots, w + p - 1$ — none of which can lie in $m_1 \cap m_2 = J$. This contradicts Proposition 8.4. □

Then Theorem 8.1 follows from Corollary 8.5 exactly as in the proof of [22, Theorem 2.8] using that $k$-dependence is preserved under interpretations. Before proving Proposition 8.4 we need to introduce some notions and recall various facts.

**Definition 8.6.** (1) Let $K$ be a field of characteristic $p > 0$. Given $x_1, \ldots, x_m \in K$, the Moore matrix is the $m \times m$ matrix
\[
M(x_1, \ldots, x_m) = \begin{pmatrix}
x_1 & \cdots & x_m \\
x_1^p & \cdots & x_m^p \\
\vdots & \ddots & \vdots \\
x_1^{p^{m-1}} & \cdots & x_m^{p^{m-1}}
\end{pmatrix}.
\]

(2) The Moore determinant is $\Delta(x_1, \ldots, x_m) := \det M(x_1, \ldots, x_m)$.

**Fact 8.7.** [17] Lemma 1.3.3 The set $\{x_1, \ldots, x_m\}$ is linearly independent over $\mathbb{F}_p$ if and only if $\Delta(x_1, \ldots, x_m) \neq 0$.

8.2. A special vector group. Let $\mathbb{K}$ be the algebraic closure of $K$, $\mathcal{K}$ a perfect subfield of $K$, and let $\varphi(x)$ be the additive homomorphism $x \mapsto x^p - x$ on $\mathbb{K}$. We consider the following algebraic subgroups of $(\mathbb{K}, +)^n$:

**Definition 8.8.** For a singleton $a$ in $\mathbb{K}$, we let $G_a$ be equal to $(\mathbb{K}, +)$, and for a tuple $\bar{a} = (a_0, \ldots, a_{n-1}) \in \mathbb{K}^n$ with $n > 1$ we define:
\[
G_{\bar{a}} = \{ (x_0, \ldots, x_{n-1}) \in \mathbb{K}^n : a_0 \cdot \varphi(x_0) = a_i \cdot \varphi(x_i) \text{ for } 0 \leq i < n \}.
\]

Recall that for an algebraic group $G$, we denote by $G^0$ the connected component of the unit element of $G$ (in the Zariski topology). Note that if $G$ is definable over some parameter set $A$, its connected component $G^0$ coincides with the smallest $A$-definable subgroup of $G$ of finite index (in $\mathbb{K}$).

**Fact 8.9.** [19] Lemma 5.3 Let $\bar{a} = (a_0, \ldots, a_{n-1})$ be a tuple in $\mathbb{K}^n$ for which the set $\{ \frac{1}{a_0}, \ldots, \frac{1}{a_{n-1}} \}$ is linearly $\mathbb{F}_p$-independent. Then $G_{\bar{a}}$ is connected.

8.3. An explicit isomorphism $f$ for $G_{\bar{a}}$.

**Fact 8.10.** [19] Corollary 5.4 Let $\mathcal{K}$ be a perfect subfield of an algebraically closed field $\mathbb{K}$, and let $\bar{a} \in \mathbb{K}^n$ be such that the set $\{ \frac{1}{a_0}, \ldots, \frac{1}{a_{n-1}} \}$ is linearly $\mathbb{F}_p$-independent. Then $G_{\bar{a}}$ is isomorphic over $\mathcal{K}$ to $(\mathbb{K}, +)$. In particular, for any field $\mathcal{K} \supseteq K$ with $K \subseteq \mathbb{K}$, the group $G_{\bar{a}}(\mathcal{K})$ is isomorphic to $(K, +)$.

Now, fix $\bar{a} = (a_0, \ldots, a_m) \in \mathbb{K}^{m+1}$ such that $\{ \frac{1}{a_0}, \ldots, \frac{1}{a_{m-1}} \}$ is linearly $\mathbb{F}_p$-independent. Fact 8.10 yields the existence of an isomorphism $f : G_{\bar{a}}(\mathcal{K}) \to (K, +)$. In [6], Bays provides an explicit description of this isomorphism $f$. Namely, let
\[
A := M\left( a_0^{-\frac{1}{p^m}}, \ldots, a_m^{-\frac{1}{p^m}} \right).
\]

Note that $\{ a_0^{-\frac{1}{p^m}}, \ldots, a_m^{-\frac{1}{p^m}} \}$ are still in $\mathcal{K}$ and $\mathbb{F}_p$-linearly independent, hence $A$ is invertible by Fact 8.7. Let
\[
\bar{\alpha} = \begin{pmatrix}
\alpha_0 \\
\vdots \\
\alpha_{m-1} \\
\alpha_m
\end{pmatrix} := A^{-1} \begin{pmatrix}
0 \\
\vdots \\
0 \\
1
\end{pmatrix},
\]
that is $\alpha_i = (A^{-1})_{i,m} \in \mathcal{K}$ for $0 \leq i \leq m$. One still has that $(\alpha_0, \ldots, \alpha_m)$ are linearly $\mathbb{F}_p$-independent (see [6] Claim 0.2)], hence $M(\alpha_0, \ldots, \alpha_m)$ is invertible. Let $\beta_{i,j} \in \mathcal{K}$ be
the entries of the inverse matrix of $M(\alpha_0, \ldots, \alpha_m)$. Then the isomorphism $f$ and its inverse are given by:

$$f: G_\mathfrak{a}(K) \to (K, +), \bar{x} = (x_0, \ldots, x_m) \mapsto \sum_{j=0}^{m} \alpha_j x_j,$$

$$f^{-1}: (K, +) \to G_\mathfrak{a}(K), t \mapsto \left( \sum_{j=0}^{m} \beta_{i,j} t^{p^j} : 0 \leq i \leq m \right).$$

8.4. The effect of $f$ on the valuation. Assume now that $\mathcal{O}$ is a valuation ring on $K$, $m$ is its maximal ideal, and $\text{val}$ is the corresponding valuation.

For the rest of this subsection, we fix some $\alpha_0, \ldots, \alpha_m \in \mathcal{K}$ such that $\text{val}(\alpha_i) \neq \text{val}(\alpha_j)$ for all $0 \leq i \neq j \leq m$.

This implies in particular that $\left\{ \frac{1}{\alpha_0}, \ldots, \frac{1}{\alpha_m} \right\}$ are $\mathbb{F}_p$-linearly independent (as the valuation of any $\mathbb{F}_p$-linear combination is $\neq \infty$). Let $\alpha_0, \ldots, \alpha_m \in \mathcal{K}$ be as defined in Subsection 8.3.

Remark 8.11. We will use the following identity multiple times and thus we give it a name. Assume $(x_0, \ldots, x_m) \in G_\mathfrak{a}$, then

$$\text{val}(\alpha_i) + \text{val}(x_i^p - x_i) = \text{val}(\alpha_j) + \text{val}(x_j^p - x_j) \quad \text{(OFE)}$$

for all $0 \leq i, j \leq m$. Additionally note that

$$\text{val}(x_i^p - x_i) = \begin{cases} \text{val}(x_i) & \text{if } \text{val}(x_i) > 0 \\ p\text{val}(x_i) & \text{if } \text{val}(x_i) < 0 \end{cases}$$

and

$$\text{val}(x_i^p - x_i) \geq 0 \text{ if } \text{val}(x_i) = 0.$$

Lemma 8.12. Suppose that $0 < \text{val}(\alpha_0) < \cdots < \text{val}(\alpha_m)$. Then for any $0 \leq i \leq m$ we have

$$\text{val}(\alpha_i) = \frac{1}{p^{m-i}} \text{val}(\alpha_i) + \sum_{j=i}^{m-1} \frac{p-1}{p^{m-j}} \text{val}(\alpha_{j+1}) > 0.$$

In particular, the sequence $\{\text{val}(\alpha_i) : i \in \{0, \ldots, m\}\}$ is strictly increasing.

Proof. Recall that, by basic linear algebra, for each $0 \leq i \leq m$ we have:

$$\alpha_i = (A^{-1})_{i,m} = \frac{1}{\det(A)} C_{m,i},$$

where $C_{m,i}$ is the corresponding cofactor of $A$. That is,

$$\alpha_i = \frac{(-1)^{m+i} \Delta \left( a_0^{-\frac{1}{p^m}}, \ldots, a_{i-1}^{-\frac{1}{p^m}}, a_{i+1}^{-\frac{1}{p^m}}, \ldots, a_m^{-\frac{1}{p^m}} \right)}{\Delta \left( a_0^{-\frac{1}{p^m}}, \ldots, a_m^{-\frac{1}{p^m}} \right)}.$$

Now, we compute the valuation of the numerator and denominator separately. First,

$$\Delta \left( a_0^{-\frac{1}{p^m}}, \ldots, a_m^{-\frac{1}{p^m}} \right) = \sum_{\pi \in \text{Sym}(\{0, \ldots, m\})} \text{sign}(\pi) \left( a_{\pi(0)}^{-\frac{1}{p^m}} \right)^{p^0} \cdots \left( a_{\pi(m)}^{-\frac{1}{p^m}} \right)^{p^m}.$$

Let $i < j$, then

$$\text{val} \left( a_i^{-\frac{1}{p^m}} \right) = -\frac{1}{p^m} \text{val}(\alpha_i) > -\frac{1}{p^m} \text{val}(\alpha_j) = \text{val} \left( a_j^{-\frac{1}{p^m}} \right).$$
Thus \( \text{val}\left(a_i^{-\frac{1}{p^m}} : 0 \leq i \leq m \right) \) is strictly decreasing. Using this, we see that

\[
\text{val}\left(\left(a_{\pi(0)}^{-\frac{1}{p^0}} \cdots a_{\pi(m)}^{-\frac{1}{p^m}}\right)^{p^0} \cdots \left(a_{\pi(0)}^{-\frac{1}{p^0}} \cdots a_{\pi(m)}^{-\frac{1}{p^m}}\right)^{p^m}\right)
\]

is strictly minimal when \( \pi \) is the identity. Thus

\[
\text{val}\left(\Delta\left(a_{0}^{-\frac{1}{p^0}}, \ldots, a_{m}^{-\frac{1}{p^m}}\right)\right) = \text{val}\left(\left(a_{0}^{-\frac{1}{p^0}} \cdots a_{i}^{-\frac{1}{p^i}} \cdots a_{m}^{-\frac{1}{p^m}}\right)^{p^0} \cdots \left(a_{0}^{-\frac{1}{p^0}} \cdots a_{i}^{-\frac{1}{p^i}} \cdots a_{m}^{-\frac{1}{p^m}}\right)^{p^m}\right) = -\sum_{j=0}^{m} \frac{1}{p^{m-j}} \text{val}(a_j).
\]

Now we turn to the numerator:

\[
\Delta\left(a_{0}^{-\frac{1}{p^0}}, \ldots, a_{i-1}^{-\frac{1}{p^{i-1}}}, a_{i+1}^{-\frac{1}{p^{i+1}}}, \ldots, a_{m}^{-\frac{1}{p^m}}\right) = \sum_{\pi \in \text{Sym}\left(\{0,\ldots,i-1,i+1,\ldots,m\}\right)} \text{sign}(\pi) \prod_{0 \leq j \leq i-1} a_{\pi(j)}^{-\frac{1}{p^j}} \cdot \prod_{i+1 \leq j \leq m} a_{\pi(j)}^{-\frac{1}{p^{j-1}}}.
\]

Again,

\[
\text{val}\left(\prod_{0 \leq j \leq i-1} a_{\pi(j)}^{-\frac{1}{p^j}} \cdot \prod_{i+1 \leq j \leq m} a_{\pi(j)}^{-\frac{1}{p^{j-1}}}\right)
\]

is strictly minimal if \( \pi \) is the identity. Thus

\[
\text{val}\left(\Delta\left(a_{0}^{-\frac{1}{p^0}}, \ldots, a_{i-1}^{-\frac{1}{p^{i-1}}}, a_{i+1}^{-\frac{1}{p^{i+1}}}, \ldots, a_{m}^{-\frac{1}{p^m}}\right)\right) = \text{val}\left(\prod_{0 \leq j \leq i-1} a_{j}^{-\frac{1}{p^j}} \cdot \prod_{i+1 \leq j \leq m} a_{j}^{-\frac{1}{p^{j-1}}}\right) = -\sum_{j=0}^{i-1} \frac{1}{p^{m-j}} \text{val}(a_j) - \sum_{j=i+1}^{m} \frac{1}{p^{m-j+1}} \text{val}(a_j).
\]
Corollary 8.14. Let \( \alpha \leq \gamma \). Suppose that \( \gamma \leq \alpha \). Then \( \gamma = \alpha \). □

Lemma 8.15. Suppose that \( \alpha < \beta \) and let \( y \in K \) be such that \( \gamma \leq \beta \). Let \( (x_0, \ldots, x_m) \in G_\alpha(K) \) be such that \( f^{-1}(y) = (x_0, \ldots, x_m) \). Then \( \gamma = \beta \). In particular, \( \gamma > 0 \).

Proof. By the formula for \( f^{-1} \) in Subsection 8.3, we have that \( \beta = \sum_{j=0}^{m} \beta_{m,j} y^{p^j} \), (*)
where $\beta_{m,j}$ is the $(m,j)$-entry of the inverse of the matrix

$$D = (\alpha^p_j)_{i,j}, \quad 0 \leq i,j \leq m.$$  

So $\beta_{m,j} = \frac{1}{\det D} C_{j,m}$, where $C_{j,m}$ is the $(j,m)$-cofactor of $D$. We determine the valuation of each of the summands in (*) separately. Fix $0 \leq j \leq m$, then

$$\text{val}(\beta_{m,j} y^p) = \text{val}(\beta_{m,j}) + p^i \text{val}(y),$$

and

$$\text{val}(\beta_{m,j}) = \text{val}(C_{j,m}) - \text{val}(\det D).$$

Again let us compute the valuations on the right hand side of the above equation separately. First,

$$\text{val}(\det D) = \text{val} \left( \sum_{\pi \in \text{Sym}(\{0,\ldots,m\})} \text{sign}(\pi) \prod_{i=0}^m \alpha^p_{\pi(i)} \right).$$

As $\text{val}(\alpha_i)$ is strictly increasing with $i$ by Lemma 8.12, note that $\text{val} \left( \prod_{i=0}^m \alpha^p_{\pi(i)} \right)$ is strictly minimal if $\pi$ maps $i$ to $m - i$ for all $0 \leq i \leq m$. So

$$\text{val}(\det D) = \text{val} \left( \prod_{i=0}^m \alpha^p_{m-i} \right) = \sum_{i=0}^m p^i \text{val}(\alpha_{m-i}).$$

Similarly,

$$\text{val}(C_{j,m}) = \sum_{\pi \in \text{Sym}(\{0,\ldots,m-1\})} \text{sign}(\pi) \prod_{i=0}^{j-1} \alpha^p_{\pi(i)} \cdot \prod_{i=j+1}^m \alpha^p_{\pi(i-1)}. $$

Again, as $\text{val}(\alpha_i)$ is strictly increasing with $i$, we conclude that

$$\text{val} \left( \prod_{i=0}^{j-1} \alpha^p_{\pi(i)} \cdot \prod_{i=j+1}^m \alpha^p_{\pi(i-1)} \right)$$

is strictly minimal if $\pi(i) = (m - 1) - i$ for all $0 \leq i \leq m - 1$. Hence

$$\text{val}(C_{j,m}) = \text{val} \left( \prod_{i=0}^{j-1} \alpha^p_{m-1-i} \cdot \prod_{i=j+1}^m \alpha^p_{m-i} \right) = \sum_{i=0}^{j-1} p^i \text{val}(\alpha_{m-1-i}) + \sum_{i=j+1}^m p^i \text{val}(\alpha_{m-i}).$$
Thus

\[
\begin{align*}
\text{val}(\beta_{m,j}) &= \text{val}(C_{j,m}) - \text{val}(\det D) \\
&= \sum_{i=0}^{j-1} p^i \text{val}(\alpha_{m-1-i}) + \sum_{i=0}^j p^i \text{val}(\alpha_{m-i}) - \sum_{i=0}^j p^i \text{val}(\alpha_{m-i}) \\
&= \sum_{i=0}^{j-1} p^i \text{val}(\alpha_{m-1-i}) - \left(\text{val}(\alpha_m) + \sum_{i=1}^j p^i \text{val}(\alpha_{m-1})\right) \\
&= \sum_{i=0}^{j-1} p^i \text{val}(\alpha_{m-1-i}) - \text{val}(\alpha_m) - \sum_{i=0}^{j-1} p^{i+1} \text{val}(\alpha_{m-1-i}) \\
&= -\text{val}(\alpha_m) - \sum_{i=0}^{j-1} (p^{i+1} - p^i) \text{val}(\alpha_{m-1-i}) \\
&= -\text{val}(\alpha_m) - \sum_{i=1}^j p^{i-1} (p - 1) \text{val}(\alpha_{m-i}).
\end{align*}
\]

Note that for any \(1 \leq j \leq m\) we have

\[
\text{val}(\beta_{m,j-1}y^{p-1}) = \text{val}(\beta_{m,j-1}) + p^{j-1} \text{val}(y) \\
= \text{val}(\beta_{m,j}) + p^{j-1}(p - 1) \text{val}(\alpha_{m-j}) + p^{j-1} \text{val}(y) \\
< \text{val}(\beta_{m,j}) + p^j \text{val}(y) \\
= \text{val}(\beta_{m,j}y^{p^j}).
\]

Thus \(\text{val}(\beta_{m,0}y)\) is strictly minimal amongst them, so

\[
\text{val}(x_m) = \text{val}(\beta_{m,0}y) = -\text{val}(\alpha_m) + \text{val}(y) = -\text{val}(a_m) + \text{val}(y)
\]

using Lemma \(8.12\) \qedhere

Lemma 8.16. Let \((x_0, \ldots, x_m) \in G_0(K)\) be arbitrary.

1. Assume that for a fixed \(0 \leq l \leq m\) we have that \(\text{val}(a_l) > \text{val}(a_s)\) for all \(0 \leq s \neq l \leq m\) and \(\text{val}(x_l) \geq 0\). Then for any \(0 \leq s \neq l \leq m\), if \(\text{val}(x_s) > 0\) we obtain \(\text{val}(\alpha_s x_s) > \text{val}(\alpha_s x_l)\).
2. Suppose that \(0 \leq s \neq t \leq m\) are such that \(\text{val}(a_s) < \text{val}(a_t)\), \(\text{val}(x_s) = 0\) and \(\text{val}(x_t) \geq 0\). Then \(\text{val}(\alpha_s x_s) < \text{val}(\alpha_t x_t)\).

Proof. Reordering the \(a_i\)'s if necessary (relying on Remark \(8.13\)), we may assume that \(0 < \text{val}(a_0) < \cdots < \text{val}(a_m)\), so in particular \(l = m, s < t\) and \(\text{val}(x_m) \geq 0\) by assumption.

1. By (OFE) and assumption \(\text{val}(x_s) > 0\), we have

\[
\text{val}(x_s) = \text{val}(a_m) - \text{val}(a_s) + \text{val}(x_m^p - x_m). \quad (\dagger)
\]

Moreover, we have

\[
\frac{1}{p^{m-s}} + \sum_{s+1 \leq j \leq m} \frac{p-1}{p^{m+1-j}} = 1. \quad (*)
\]
Then
\[ \text{val}(\alpha_s x_s) = \text{val}(\alpha_s) + \text{val}(x_s) \]
\[
\begin{align*}
8.12 & & \frac{1}{p^{m-s}} \text{val}(\alpha_s) + \sum_{j=s+1}^{m} \frac{p-1}{p^{m+1-j}} \text{val}(a_j) + \text{val}(a_m) - \text{val}(\alpha_s) + \text{val}(x_m^p - x_m) \\
8.14 & & \text{val}(\alpha_m) + \text{val}(x_m^p - x_m) + \sum_{j=s+1}^{m} \frac{p-1}{p^{m+1-j}} (\text{val}(a_j) - \text{val}(\alpha_s)) \geq \text{val}(x_m) \\
& & > \text{val}(\alpha_m) + \text{val}(x_m) \\
& & = \text{val}(\alpha_m x_m).
\end{align*}
\]

(2) By Lemma 8.12 the sequence \( \text{val}(\alpha_i)'s \) is strictly increasing. Thus, if \( \text{val}(x_s) = 0 \), then
\[
\begin{align*}
\text{val}(\alpha_s x_s) = \text{val}(\alpha_s) < \text{val}(\alpha_i) + \text{val}(x_i) = \text{val}(\alpha_i x_i).
\end{align*}
\]

Lemma 8.17. Let \( y \in K \) be such that \( \text{val}(a_j) < \text{val}(y) \) for all \( 0 \leq j \leq m \). Now let \( (x_0, \ldots, x_m) \in G_a(K) \) be equal to \( f^{-1}(y) \). Then \( \text{val}(x_j) > 0 \) for all \( 0 \leq j \leq m \).

Proof. Up to reordering the \( a_j \)'s (using Remark 8.13), we may assume that \( 0 < \text{val}(a_0) < \cdots < \text{val}(a_m) \). Then, \( \text{val}(x_m) > 0 \) by Lemma 8.15 and \( (\text{val}(\alpha_j) : 0 \leq j \leq m) \) is strictly increasing by Lemma 8.12. For any \( 0 \leq j \leq m \), by (OFE):
\[
\text{val}(x_j^p - x_j) = \text{val}(a_m) - \text{val}(a_j) + \text{val}(x_m^p - x_m) > 0
\]
Thus \( \text{val}(x_j) \geq 0 \).

If \( \text{val}(x_j) > 0 \) for all \( j \), we are done. Otherwise, let \( I = \{1 \leq j \leq m : \text{val}(x_j) = 0\} \neq \emptyset \) and let \( j_0 := \min\{I\} \).

Claim. \( \text{val}(\alpha_j x_{j_0}) \) is strictly minimal in \( \{\text{val}(\alpha_j x_j) : 1 \leq j \leq m\} \).

Proof. Assume first \( j \in I \setminus \{j_0\} \), then \( \text{val}(x_j) = 0 \) and \( j > j_0 \). Hence
\[
\text{val}(\alpha_j x_{j_0}) = \text{val}(\alpha_j) < \text{val}(\alpha_j x_j).
\]
Otherwise \( j \notin I \), i.e. \( \text{val}(x_j) > 0 \). Then
\[
\text{val}(\alpha_j x_j) > \text{val}(\alpha_m x_m) > \text{val}(\alpha_j x_{j_0}).
\]

Thus \( \text{val}(y) = \text{val} \left( \sum_{j=0}^{m} \alpha_j x_j \right) = \text{val}(\alpha_j x_{j_0}) = \text{val}(\alpha_j) < \text{val}(a_m) < \text{val}(y) \), which yields a contradiction. Hence \( \text{val}(x_j) > 0 \) for all \( 0 \leq j \leq m \).
8.5. Proof of Proposition 8.4. We may assume that \((K, \mathcal{O}_1, \mathcal{O}_2)\) is \(\aleph_0\)-saturated. Let \(K\) be the algebraic closure of \(K\) and let \(K := \bigcap_{n \in \mathbb{N}} K^{1/p^n}\) be the largest perfect subfield of \(K\). Let \(\ell\) be the natural number given by Proposition 8.18 for the uniformly defined subgroups \(x_0 \cdot \ldots \cdot x_{k-1} \cdot \varphi(K)\) and \(x_0 \cdot \ldots \cdot x_{k-1} \cdot \varphi(J)\) of \((K, +)\).

Let \(y \in J\) be arbitrary, we will show that it has an Artin-Schreier root in \(J\).

Claim. There exists an infinite sequence \((a_i)_{i \in \mathbb{N}}\) of elements of \(K\) such that \(0 < n \cdot \text{val}_t(a_{i+1}) < \text{val}_t(a_i) < \text{val}_t(y)\) holds for all \(i, n \in \mathbb{N}\) and both \(t \in \{1, 2\}\) simultaneously.

Proof. The field \(K\) is Artin-Schreier closed by Fact 8.3. Let \(b \in K\) be an Artin-Schreier root of \(\frac{1}{y}\), i.e., \(\frac{1}{y} = b^p - b\). For any \(t \in \{1, 2\}\) we have: \(\text{val}_t(y) > 0\) by assumption, hence \(\text{val}_t \left( b^p - b \right) < 0\). Consequently \(\text{val}_t \left( b^p - b \right) < 0\) as well as \(\text{val}_t(b) < 0\) and so \(\text{val}_t \left( b^p - b \right) = p \text{val}_t(b)\). Hence, letting \(b_1 = \frac{1}{y}\), we obtain that

\[
\text{val}_t(y) = -\text{val}_t \left( \frac{1}{y} \right) = -p \text{val}_t(b) = p \text{val}_t \left( \frac{1}{b} \right) = p \text{val}_t(b_1).
\]

Repeating this procedure with \(b_1\) instead of \(y\), etc., for each \(n \in \mathbb{N}\) we can find some element \(b_n \in K\) such that \(0 < \text{val}_t(b_n) = \frac{1}{p^n} \text{val}_t(y) < \text{val}_t(y)\) for both \(t \in \{1, 2\}\) simultaneously. Let now \(m, n \in \mathbb{N}\) be arbitrary, and we define \(a_i := b_{pm+m}^n\) for \(i \in \mathbb{N}\). Note that

\[
\text{val}_t(a_i) = \text{val}_t \left( b_{pm+m}^n \right) = p^m \text{val}_t(b_{n+m}) = p^m \frac{1}{p^{m+n}} \text{val}_t(y)
\]

and \(p^{n-1} \text{val}_t(a_i) = \frac{p^{n-1}}{p^m} \text{val}_t(y) < \text{val}_t(y)\) for all \(i \in \mathbb{N}\). Then:

- \(a_i \in K^{1/p^n}\) for all \(i \in \mathbb{N}\);
- for each \(t \in \{1, 2\}\) and all \(i \in \mathbb{N}\) we have: \(0 < p^{n-1} \cdot \text{val}_t(a_{i+1}) < \text{val}_t(a_i) < \text{val}_t(y)\).

As \(m, n\) were arbitrary and \(K\) is type-definable over \(\emptyset\), the claim follows by saturation of \(K\).

Now let \((a_i)_{i \in \mathbb{N}}\) be a sequence in \(K\) given by the claim. Then we can choose from it elements \(\{b_{j,t} : j < k, l < \ell\}\) in \(K\) such that: for all \(j < k, l < \ell\) we have

- \(\text{val}_t(b_{k-1,l}) < \text{val}_t(b_{0,l+1})\),
- \(0 < (j+1) \cdot \text{val}_t(b_{j,l}) < \text{val}_t(b_{j,l+1})\),
- \(k \cdot \text{val}_t(b_{k-1,l+1}) < \text{val}_t(y)\)

for both \(t \in \{1, 2\}\) simultaneously. For each \((l_0, \ldots, l_{k-1}) \in \ell^k\) we define

\[
b_{l_0,\ldots,l_{k-1}} := \prod_{j=0}^{k-1} b_{j,l_j} \in K.
\]

Claim. For each \(t \in \{1, 2\}\) we have:

- \(0 < \text{val}_t(b_{l_0,\ldots,l_{k-1}}) < \text{val}_t(y)\) for all \((l_0, \ldots, l_{k-1}) \in \ell^k\);
- \(\text{val}_t(b_{l_0,\ldots,l_{k-1}}) < \text{val}_t(b_{p_0,\ldots,p_{k-1}})\) if and only if \((l_{k-1}, \ldots, l_0) <_{\text{lex}} (p_{k-1}, \ldots, p_0)\).
Proof. The first item is clear by the choice of $b_{j,t}$, and we check the second one. Assume $(l_{k-1}, \ldots, l_0) <_{\text{lex}} (p_{k-1}, \ldots, p_0)$, and let $0 \leq j^* < k$ be maximal such that $l_{j^*} < p_{j^*}$. Then by the choice of $b_{j,t}$ we have

\[
\text{val}_t(b_{l_0, \ldots, l_{k-1}}) = \sum_{j=0}^{k-1} \text{val}_t(b_{j,t}) \leq \sum_{j=0}^{j^*} \text{val}_t(b_{j,t}) + \sum_{j=j^*+1}^{k-1} \text{val}_t(b_{j,t}) \\
\leq (j^* + 1) \cdot \text{val}_t(b_{j^*,t}) + \sum_{j=j^*+1}^{k-1} \text{val}_t(b_{j,t}) < \text{val}_t(b_{j^*,p_{j^*}}) + \sum_{j=j^*+1}^{k-1} \text{val}_t(b_{j,t}) \\
\leq \sum_{j=0}^{k-1} \text{val}_t(b_{j,p_{j^*}}) = \text{val}_t(b_{p_0, \ldots, p_{k-1}}).
\]

By the choice of $\ell$ and Proposition 3.17, there must exist some $(l'_0, \ldots, l'_{k-1}) \in \ell^k$ such that

\[
\bigcap_{(l_0, \ldots, l_{k-1}) \in \ell^k} b_{l_0, \ldots, l_{k-1}} \cdot \wp(J) = \bigcap_{(l_0, \ldots, l_{k-1}) \in \ell^k \setminus \{(l'_0, \ldots, l'_{k-1})\}} b_{l_0, \ldots, l_{k-1}} \cdot \wp(J),
\]

\[
\bigcap_{(l_0, \ldots, l_{k-1}) \in \ell^k} b_{l_0, \ldots, l_{k-1}} \cdot \wp(K) = \bigcap_{(l_0, \ldots, l_{k-1}) \in \ell^k \setminus \{(l'_0, \ldots, l'_{k-1})\}} b_{l_0, \ldots, l_{k-1}} \cdot \wp(K).
\]

Let now $m := \ell^k - 1$. We can choose an enumeration $(a_j : 0 \leq j \leq m)$ of the set \{b_{l_0, \ldots, l_{k-1}} : (l_0, \ldots, l_{k-1}) \in \ell^k\} so that:

1. $\bigcap_{j=0}^{m-1} a_j \wp(J) = \bigcap_{j=0}^{m-1} a_j \wp(J)$;
2. $\bigcap_{j=0}^{m-1} a_j \wp(K) = \bigcap_{j=0}^{m-1} a_j \wp(K)$;
3. $0 < \text{val}_t(a_0) < \cdots < \text{val}_t(a_{m-1})$ and $(\text{val}_t(a_j))_{0 \leq j \leq m}$ are pairwise distinct, for both $t \in \{1, 2\}$.

Note that we still have $a_j \in K$ and $0 < \text{val}_t(a_j) < \text{val}_t(y)$ for any $0 \leq j \leq m$ and $t \in \{1, 2\}$. Let $\bar{a} := (a_0, \ldots, a_m)$ and $\bar{a}' := (a_0, \ldots, a_{m-1})$, both tuples satisfy the assumption of Subsection 5.3. Now, we consider the following commutative diagram:

\[
G_{\bar{a}}(K) \xrightarrow{\pi} G_{\bar{a}'}(K) \\
\downarrow f \simeq \downarrow f' \simeq \\
(K, +) \xrightarrow{\wp} (K, +) \\
\uparrow \simeq \uparrow \rho' \\
(K, +)
\]

where $\pi$ is the natural projection $(x_0, \ldots, x_m) \mapsto (x_0, \ldots, x_{m-1})$, $f$ and $f'$ are isomorphisms defined over $K$ as in Subsection 5.3 for $\bar{a}$ and $\bar{a}'$ respectively, i.e.

\[
f : G_{\bar{a}}(K) \to (K, +), \bar{x} = (x_0, \ldots, x_m) \mapsto \sum_{j=0}^{m} a_j x_j, \text{ and}
\]

\[
f' : G_{\bar{a}'}(K) \to (K, +), \bar{x} = (x_0, \ldots, x_{m-1}) \mapsto \sum_{j=0}^{m-1} a'_j x_j
\]
for the appropriate choice of $\bar{\alpha}, \bar{\alpha}'$, and $\rho$ is the morphism that makes the diagram commute. Finally, let

$$\rho'(t) := \rho(\alpha mt).$$

Note that $0 \neq \alpha_m \in \ker(\rho)$ as $f^{-1}(\alpha_m) = (0, \ldots, 0, 1) \in K^{m+1}$, and $\pi((0, \ldots, 0, 1)) = (0, \ldots, 0) \in K^m$. As in the end of the proof of [24, Theorem 4.3], we get that

$$\rho'(t) = c(t^p - t)$$

for some $c \in K$.

**Claim.** Let $u \in K$ be arbitrary with $\val_t(u) > \max\{\val_t(a_{m-1}), \val_t(a_m)\}$ for both $t \in \{1, 2\}$. Then there is some $w \in J$ with $\val_t(w) < \val_t(u)$ for both $t \in \{1, 2\}$ and such that $\rho'(w) = u$.

**Proof.** Fix $(x_0, \ldots, x_{m-1}) \in G_{\overline{a}'}$ such that $(f')^{-1}(u) = (x_0, \ldots, x_{m-1})$. Note that $0 < \val_t(a_0) < \ldots < \val_t(a_{m-1}) < \val_t(u)$ for both $t \in \{1, 2\}$ by (3) and assumption, hence by Lemma 8.17 [24] we have that $\val_t(x_j) > 0$ for all $t \in \{1, 2\}$ and $0 \leq j \leq m - 1$. Whence $x_j \in J$ for all $0 \leq j \leq m - 1$. By (1) above there is some $x_m \in J$ such that $a_m(x_m^p - x_m) = a_j(x_j^p - x_j)$ for all $0 \leq j < m$. So there is a preimage of $(x_0, \ldots, x_{m-1})$ under $\pi$, namely $(x_0, \ldots, x_{m-1}, x_m)$, which lies in $J^{m+1}$. Now let

$$w := a_m^{-1}f(x_0, \ldots, x_{m-1}, x_m) = a_m^{-1}\sum_{j=0}^{m} a_j x_j,$$

thus $w$ is a preimage of $u$ under $\rho'$. Then for each $t \in \{1, 2\}$ we have

$$\val_t(w) = \val_t\left(a_m^{-1}\sum_{j=0}^{m} a_j x_j\right)$$

$$= -\val_t(a_m) + \val_t\left(\sum_{j=0}^{m} a_j x_j\right)$$

$$\geq -\val_t(a_m) + \val_t(a_m) + \val_t(x_m)$$

$$= \val_t(a_m) + \val_t(x_m) \quad \text{(OFFE)}$$

where the last inequality is by Remark 8.13 and Lemma 8.12. Also by Lemma 8.13 with respect to $\overline{a}'$, we have $\val_t(x_{m-1}) = \val_t(u) - \val_t(a_{m-1})$, hence

$$\val_t(u) = \val_t(a_{m-1}) + \val_t(x_{m-1}) = \val_t(a_m) + \val_t(x_m).$$

If $\val_t(a_{m-1}) < \val_t(a_m)$, then we have

$$\val_t(u) = \val_t(a_m) + \val_t(x_m) \geq \val_t(a_m) + \val_t(x_m)$$

$$> -\val_t(a_m) + \val_t(a_m) + \val_t(x_m) = \val_t(w).$$

If $\val_t(a_m) < \val_t(a_{m-1})$, then we have

$$\val_t(u) = \val_t(a_{m-1}) + \val_t(x_{m-1}) \geq \val_t(a_{m-1}) + \val_t(x_{m-1})$$

$$> -\val_t(a_m) + \val_t(a_{m-1}) + \val_t(x_{m-1}) = \val_t(w).$$

In either case, we obtain $\val_t(u) > \val_t(w)$. \qed

Let $w$ be as given by the claim for $u := y$. Then for both $t \in \{1, 2\}$ we have

$$\val_t(c) = \val_t(y) - \val_t(w^p - w) = \val_t(y) - \val_t(w) > 0.$$
Then \( \operatorname{val}_i(cy) > \max\{\operatorname{val}_i(a_{m-1}), \operatorname{val}_i(a_m)\} \). Thus by the claim applied to \( u := cy \), there is some \( w' \in J \) such that

\[
   cy = \rho'(w') = c((w')^p - w'),
\]

e.i. \( y = (w')^p - w' \). Thus \( w' \) is an Artin-Schreier root of \( y \) in \( J \). As \( y \in J \) was arbitrary, this finishes the proof.

8.6. **Generic multi-ordered/multi-valued fields.** We consider the model-companion of the theory of fields with several valuations and orderings, introduced in the thesis of van den Dries [34, Chapter III]. We use Johnson’s thesis [23, Chapter 11] as our reference.

Fix \( k \in \mathbb{N} \). For each \( 1 \leq i \leq k \), let \( T_i \) be one of the theories \( \text{ACVF}, \text{RCF} \) or \( \text{pCF} \), and let \( \mathcal{L}_i \) denote the language of \( T_i \) and \( \mathcal{L}_i \cap \mathcal{L}_j = \mathcal{L}_{\text{rings}} \) (i.e. \( \mathcal{L}_i \) additionally contains a binary predicate \( x < y \) if \( T_i \) is \( \text{RCF} \), or \( \operatorname{val}_i(x) < \operatorname{val}_i(y) \) if \( T_i \) is \( \text{ACVF} \) or \( \text{pCF} \)). Let \( \mathcal{L} := \bigcup_{i=1}^{k} \mathcal{L}_i \), and let \( T_0 := \bigcup_{i=1}^{k} T_i \).

**Fact 8.18.** [23 Theorem 11.2.3] The theory \( T^0 \) has a model companion \( T \), and \( K \models T_0 \) is a model of \( T \) if:

1. \( K \) is existentially closed with respect to finite extensions, i.e. if \( L/K \) is a finite algebraic extension and \( L \models T_0 \), then \( L = K \);
2. For any \( m \), let \( V \) be an \( m \)-dimensional absolutely irreducible variety over \( K \). For \( 1 \leq i \leq k \), let \( \varphi_i(x) \) be a \( V \)-dense quantifier-free \( \mathcal{L}_i \)-formula with parameters from \( K \). Then \( \bigcap_{i=1}^{k} \varphi_i(K) \neq \emptyset \).

(Where \( V \)-dense means that \( \varphi_i(K) \) is Zariski-dense in \( V(K^{\text{alg}}) \), see [23 Section 11.1.1].)

We use the following result established in the proof of [23 Claim 11.5.2].

**Fact 8.19.** Let \( K \models T \). For each \( i \), let \( \chi_i(y) \) be the formula:

1. \( y > 0 \) if \( T_i \) is \( \text{RCF} \);
2. \( \operatorname{val}_i(y - \frac{1}{q}) > 0 \) if \( T_i \) is \( \text{ACVF} \) or \( \text{pCF} \).

Let \( \chi(y) := \bigwedge_{i=1}^{k} \chi_i(y) \). Then \( \chi(K) \) is infinite, and there exists some \( \mathcal{L} \)-formula \( \psi(x, y) \) such that: for any \( m \in \mathbb{N} \), any \( a_1, \ldots, a_m \in \chi(K) \) pairwise distinct and any \( A \subseteq \{1, \ldots, n\} \) there exists some \( b \) such that \( \models \psi(b, a_j) \iff j \in A \).

This immediately implies that \( T \) has \( \text{IP} \), and the argument generalizes to get \( \text{IP}_n \) as follows.

**Proposition 8.20.** \( T \) has \( \text{IP}_n \), for all \( n \geq 1 \).

**Proof.** Let \( K \models T \) be a saturated model. Fix \( n \in \mathbb{N} \). Note that it is enough to find some sequences \((c_{\alpha_1}^{1}, \ldots, c_{\alpha_n}^{n} : \alpha_1, \ldots, \alpha_n \in \omega)\) of elements in \( K \) and \( e \in K \) such that all elements in the set \( \{c_{\alpha_1}^{1}, \ldots, c_{\alpha_n}^{n} + e : \alpha_1, \ldots, \alpha_n \in \omega\} \) are pairwise distinct and satisfy \( \chi(y) \), as then the formula \( \psi'(x; y_1, \ldots, y_n) = \psi(x; y_1 \cdot \cdots \cdot y_n + e) \) with \( \psi \) given by Fact 8.19 has \( \text{IP}_n \).

First, let \( e \in K \) be such that \( 0 < \operatorname{val}_i(\frac{1}{q} - e) < \infty \) for all \( 1 \leq i \leq k \) such that \( T_i \) is \( \text{ACVF} \) or \( \text{pCF} \), and \( e > 0 \) for all \( i \) such that \( T_i \) is \( \text{RCF} \). This is possible by Fact 8.18(2) since the formulas \( \operatorname{val}_i(\frac{1}{q} - x) > 0 \) and \( x > 0 \) are \( V \)-dense for \( V = \mathbb{A}^1 \). Let \( \gamma_i := \operatorname{val}_i(\frac{1}{q} - e) \).

By induction on \( 1 \leq t \leq n \) we choose sequences \((c_{\alpha}^{t} : \alpha \in \omega) \) in \( K \) such that the following holds for each \( 1 \leq i \leq k \).

1. If \( T_i \) is \( \text{ACVF} \) or \( \text{pCF} \):
(a) \( \text{val}_i(c^t_{\alpha+1}) > n \cdot \text{val}_i(c^t_\alpha) > \gamma_i \) for all \( 1 \leq t \leq n \) and \( \alpha \in \omega \);
(b) \( \text{val}_i(c^{t+1}_\alpha) > \text{val}_i(c^t_\beta) \) for all \( 1 \leq t \leq n - 1 \) and \( \alpha, \beta \in \omega \).

(2) If \( T_i \) is RCF:
(a) \( c^t_{\alpha+1} >_i (c^t_\alpha)^n > 0 \) and for all \( 1 \leq t \leq n \) and \( \alpha, \beta \in \omega \);
(b) \( c^{t+1}_\alpha >_i c^t_\beta \) for all \( 1 \leq t \leq n - 1 \) and \( \alpha, \beta \in \omega \).

This is possible by saturation of \( K \), as in order to chose an element \( c^t_\alpha \in K \) we only need to satisfy finitely many quantifier-free formulas with parameters from \( \{ c^s_\beta : s < t \lor (s = t \land \beta < \alpha) \} \subseteq K \), all of which are implied by a single condition of the form \( \text{val}_i(x) > \text{val}_i(c) \) or \( x >_i c \) for each \( i \) and some \( c \in K \) — which can be satisfied in \( K \) by Fact 8.18(2) as these formulas are \( V \)-dense for \( V = A^1 \).

Assume first that \( T_i \) is ACVF or pCF for some \( 1 \leq i \leq k \). Note that then for any \( (\alpha_1, \ldots, \alpha_n) \in \omega^n \) we have \( \text{val}_i(\prod_{t=1}^n c^t_{\alpha_t}) = \sum_{t=1}^n \text{val}_i(c^t_{\alpha_t}) > \gamma = \text{val}_i(\frac{1}{e} - e) \) by 1(a). Then \( \text{val}_i(\prod_{t=1}^n c^t_{\alpha_t} + e - \frac{1}{e}) = \text{val}_i(\prod_{t=1}^n c^t_{\alpha_t} - (\frac{1}{e} - e)) = \text{val}_i(\frac{1}{e} - e) = \gamma_i > 0 \), hence \( \models \chi_i(\prod_{t=1}^n c^t_{\alpha_t} + e) \).

As in the proof of the second claim in Subsection 8.3 we get that \( (\alpha_1, \ldots, \alpha_n) < (\beta_1, \ldots, \beta_n) \) in the lexicographic ordering on \( \omega^n \) if and only if
\[
\text{val}_i\left(\prod_{t=1}^n c^t_{\alpha_t}\right) < \text{val}_i\left(\prod_{t=1}^n c^t_{\alpha_t}\right),
\]
so in particular \( \prod_{t=1}^n c^t_{\alpha_t} + e \neq \prod_{t=1}^n c^t_{\beta_t} + e \), hence all these elements are pairwise distinct.

Otherwise \( T_i \) must be RCF for all \( 1 \leq i \leq k \). Then a similar calculation using 2(a) and 2(b) shows that \( \prod_{t=1}^n c^t_{\alpha_t} + e \models \chi_i \), and that all these elements are pairwise distinct. \( \square \)

References


