Title
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Journal
Geometriae Dedicata, 143(1)

ISSN
1572-9168

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Publication Date
2009-12-01

DOI
10.1007/s10711-009-9368-z

Peer reviewed

Geom Dedicata (2009) 143:7–18
DOI 10.1007/s10711-009-9368-z

ORIGINAL PAPER

Knots and k-width

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Received: 29 April 2008 / Accepted: 1 April 2009 / Published online: 23 April 2009

Abstract We investigate several natural invariants of curves and knots in $\mathbb{R}^3$. These invariants generalize bridge number and width. As with bridge number, there are connections to the total curvature of a curve.

Keywords Knot theory · Three-dimensional topology · Thin position · 2-width · k-width

Mathematics Subject Classification (2000) 57M25

1 Introduction

In this article, we investigate the notions of k-bridge number and k-width for a knot or link in $\mathbb{R}^3$, where k is an integer between 1 and 4. These provide increasingly detailed information, as k grows, on the intersections of a curve with flat planes and round spheres in $\mathbb{R}^3$. We examine properties of curves that minimize k-bridge number or k-width within their isotopy class. These notions correspond to well known quantities for $k = 1$, but do not appear to have been studied before for $k > 1$.

We have two main motivations for these constructions. The first is a search for geometric interpretations of some of the new knot and 3-manifold invariants introduced in recent years. Recently introduced invariants associated to curves and knots in $\mathbb{R}^3$ come from many sources, such as topological quantum field theory and the categorification of previously known invariants. The geometric and topological interpretations of many of these invariants

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are not well understood. There appears to be a need for geometrically constructed knot and curve invariants which can then be compared to these recently discovered invariants.

A second motivation is to better understand geometric properties of knots associated to the motion of a knotted strand through a physical medium. Such properties are connected to the understanding of biological molecules such as DNA and RNA. The motion of loops of DNA through a gel appears to depend on the knot type of the loop. This allows the construction of experiments that provide information about the behavior of certain enzymes [12–14]. Experimental work and simulations indicate that the average crossing number of a knot is closely connected to its motion under gel electrophoresis. The average crossing number is a rough measure of how tightly compacted is a loop of given length. It may be that other measures of the cross-sectional size of a knot are more accurate predictors of its motion through a medium such as a liquid or a gel. The widths we study in this paper are natural quantities arising from this point of view.

In Sects. 3 and 4 we develop properties of 2-bridge number and 2-width in some detail. We show that 2-bridge number and 2-width are *proper invariants*, meaning that only finitely many knots have values less than a given constant. Like crossing number, 2-bridge number and 2-width give ways to order all knots in terms of increasing complexity. We also show that the unknot and the trefoil realize the two smallest values of the invariants for $k = 2$ and give explicit formulas for the value of these invariants on $(2, n)$ torus knots.

We then relate the 2-bridge number and 2-width of a curve to its curvature. Milnor and Fary showed that if a smooth curve γ has total curvature less than 4π then γ is unknotted [4,9]. Their arguments also show that if the total curvature of γ is less than $2n\pi$ then there is a direction relative to which γ has at most n maxima, and as a consequence, its bridge number is at most n. The converse is false. A curve of bridge number n can have arbitrarily large curvature. In contrast, we show that if a knot K in $\mathbb{R}^3$ has 2-bridge number n then some planar projection of n has total curvature at most $4\pi(n-1)^2$. Similarly if K has 2-width n, then some planar projection of γ has total curvature at most $2\pi n^{3/2}$. So if every planar projection has large total curvature then both the 2-width and 2-bridge number are large.

In Sect. 2 we introduce k-width for curves in $\mathbb{R}^3$. In Sect. 3 we show that there are only a finite number of knot types with 2-bridge number or 2-width bounded by a given constant and in Sect. 4 we look at the knots with small 2-bridge number or 2-width. In Sect. 5 we establish the connection between 2-bridge number or 2-width and the total curvature of a curve.

2 Definitions

Combinatorial knot invariants computed by counting intersections of a generic smooth curve γ with linear subspaces of $\mathbb{R}^3$ include Schubert's bridge number [11], Kuiper's superbridge number [7] and Gabai's thin position [6]. The bridge number of γ is the number of maxima of γ for the height function given by the z-coordinate function. It equals half the number of tangencies of γ to the 1-parameter family of planes parallel to the xy-plane in $\mathbb{R}^3$, and can therefore be regarded as a 1-parameter invariant. We present 2, 3 and 4-parameter families of planes and spheres in $\mathbb{R}^3$ that give natural generalizations of this notion. We will describe these in $\mathbb{R}^3$, but they are equally natural in $\mathbb{S}^3$.

Definition. Let $\mathcal{S}_1$ be the set of planes in $\mathbb{R}^3$ parallel to the xy-plane, $\mathcal{S}_2$ the set of planes in $\mathbb{R}^3$ parallel to the z-axis, $\mathcal{S}_3$ the set of all planes in $\mathbb{R}^3$ and $\mathcal{S}_4$ the set of planes and round

2-spheres in $\mathbb{R}^3$. For $1 \leq k \leq 4$, let $\mathcal{T}_k(\gamma)$ be the open subset of elements of $\mathcal{S}_k$ consisting of planes or spheres that are transverse to γ.

The points in $\mathcal{S}_k$ are in 1–1 correspondence with an open k-dimensional manifold. We will assume that the curve γ is in general position relative to the family of planes or spheres in $\mathcal{S}_k$. In particular, any plane or sphere in $\mathcal{S}_k$ has at most k tangencies with γ.

Definition. The *k-bridge number* $b_k(\gamma)$ is the number of path components in $\mathcal{T}_k(\gamma)$. A curve γ is in *k-bridge position* if it minimizes k-bridge number among curves in its isotopy class.

We now give the relationship between the *1-bridge number* $b_1(\gamma)$ and the usual notion of bridge number for curves in $\mathbb{R}^3$.

Lemma 1 *A curve γ is in bridge position if and only if it is in 1-bridge position.*

Proof The 1-bridge number of a curve equals twice its bridge number plus one, where the addition of one is caused by the planes not meeting the curve. Therefore standard bridge position also minimizes 1-bridge number. □

A related but distinct invariant is the *superbridge index* of a curve, introduced by N. Kuiper [7]. It is defined to be the maximum bridge number of the curve for all directions in $\mathbb{R}^3$. From our point of view, superbridge index is a 3-parameter invariant since it is obtained by considering intersections with planes in $\mathcal{S}_3$. In [7] it is shown that there are infinitely many knots with superbridge index bounded by 4. In fact, Kuiper showed that for all odd q, an appropriate embedding of the $(2, q)$-torus knot has superbridge index 4, realized by positioning the $(2, q)$ torus knot as a satellite of a "tennis ball" curve on the 2-sphere, a curve that intersects any plane in at most four points. When positioned to be almost parallel to the tennis ball curve, the $(2, q)$ torus knot intersects any flat plane in at most 8 points and thus has superbridge index 4. In contrast, the 3-bridge numbers of these knots do not appear to be bounded.

Definition. Let $\phi : \mathbb{S} \to \mathbb{Z}$ be a function from a set $\mathbb{S}$ to the integers. We say that ϕ is a *proper invariant* of $\mathbb{S}$ if for each $n \in \mathbb{Z}$, the number of elements $s \in \mathbb{S}$ with $\phi(s) \leq n$ is finite.

Proper invariants are useful for compiling lists of objects in organizing a classification. The integers can be replaced with a general ordered set in this definition. In knot theory, invariants such as crossing number and stick number are proper, while genus and bridge number are not. The above remark shows that the superbridge number is not a proper knot invariant.

A variation of the notion of k-bridge number that contains more information is the k-width.

Definition. The *k-width* $w_k(\gamma)$ is the total number of intersections of γ with a collection of planes, with one plane in each path component of $\mathcal{T}_k(\gamma)$:

$$w_k(\gamma) = \sum_{P_i \in \mathcal{T}_k(\gamma)} \#(P_i \cap \gamma).$$

A curve γ is *k-width minimizing* if it minimizes k-width among curves in its isotopy class.

The notion of 1-width coincides with the notion of width introduced by Gabai, and a 1-width minimizing curve is a curve in thin position [6].

The manifold $\mathcal{S}_k$ is separated by the planes and spheres with non-transverse intersection with γ into a collection of open k-dimensional components. For generic γ the non-transverse planes and spheres form a $(k-1)$-complex in $\mathcal{S}_k$ that is the image of a piecewise-smooth immersion in general position. The non-transverse points are the image of a curve in $\mathcal{S}_2$, of a torus in $\mathcal{S}_3$, and of a line bundle over a torus in $\mathcal{S}_4$. This point of view can be exploited to analyze 2-bridge number and 2-width.

It is possible to define 2-bridge number and 2-width while working entirely with the projection $p(\gamma)$ of γ onto the xy-plane, since changes in its z-coordinate do not affect the intersections of γ with a plane in $\mathcal{S}_2$. However to define b_3, b_4, w_3, w_4 requires considering curves in $\mathbb{R}^3$. Thus a 3-bridge or 3-width minimizing curve gives one approach to finding an "optimal" imbedding of a knot into $\mathbb{R}^3$. Note however that these numbers are preserved by affine maps of $\mathbb{R}^3$, so 3-bridge or 3-width minimizing curves are far from unique. Similarly the 2-width of a curve is preserved by affine maps of $\mathbb{R}^2$. As with the knot energy studied by Freedman, He and Wang, b_4 and w_4 are conformal invariants of γ [5].

Lemma 2 *If γ' is the image of γ by a conformal map of $\mathbb{R}^3$, then $b_4(\gamma') = b_4(\gamma)$ and $w_4(\gamma') = w_4(\gamma)$.*

Proof There is a 1–1 correspondence between the planes and spheres transverse to γ in $\mathcal{T}_4$ and those for γ'. This correspondence preserves the number of intersections with the curve. □

We can define widths of still higher order by considering intersections with quartics and other families of surfaces. The main results of this article concern 2-width.

3 Knots with bounded 2-bridge number or 2-width

Knot tables are usually arranged to reflect increasing crossing number. This is feasible because for any constant n, only finitely many knots have crossing number less than n. This property does not hold for other common invariants, such as unknotting number, tunnel number and bridge number. While 1-width and 1-bridge number share this shortcoming, we prove below a finiteness result for 2-bridge number and for the 2-width invariant.

The planes in $\mathcal{S}_2$ are parallel to the z-axis, and are in 1–1 correspondence with the set $\mathcal{M}$ of lines in the xy-plane, homeomorphic to a Mobius band. The Mobius band $\mathcal{M}$ is double covered by the set of oriented lines in the plane, which is homeomorphic to an open annulus. At each point q on a generic γ there is a unique plane in $\mathcal{S}_2$ tangent to γ at q. The set of tangent planes describes a curve λ in the open Mobius band $\mathcal{S}_2$. Double tangencies of γ with planes in $\mathcal{S}_2$ give double points of λ and inflection points of the planar projection of γ give rise to cusps of λ. For generic γ the number of regions in $\mathcal{S}_2$ complementary to λ is finite and equal to the 2-bridge number. Two components of $\mathcal{S}_2$ with a common boundary segment correspond to planes whose intersection numbers with γ differ by two. One of the complementary components corresponds to planes that miss γ, leading to an upper bound on the 2-width of a generic curve. There are a total of $b_2(\gamma)$ components. One contains planes missing γ and components with a common segment have intersection numbers that differ by two, so

$$w_2(\gamma) \leq 0 + 2 + \cdots + 2(b_2(\gamma) - 1) = (b_2(\gamma))(b_2(\gamma) - 1)$$

We now show that for any constant n, only finitely many knots in $\mathbb{R}^3$ have 2-width or 2-bridge number less than or equal to n.

Theorem 1 *The 2-bridge number and the 2-width are proper invariants.*

Proof Let $\gamma \subset \mathbb{R}^3$ be a generic embedded curve. Using the projection $p : \mathbb{R}^3 \to \mathbb{R}^2$, we project γ to a planar curve $p(\gamma)$ in the *xy*-plane. To each point in $p(\gamma)$ we associate the tangent line at that point. The set of tangent lines gives a graph α in $\mathcal{S}_2$ which we call the *graphic* of γ. The 4-valent graph α is the image of a curve that is immersed in $\mathcal{S}_2$ except at a finite number of cusps that are the images of inflection points of $p(\gamma)$. Only finitely many planes in $\mathcal{S}_2$ are tangent to γ at two points. Such *double tangent planes* correspond to order four vertices in α.

Let v be the number of such vertices and r the number of regions of the complement of α in $\mathcal{S}_2$. These regions consist of f simply-connected faces, one non-compact annular region and possibly one Mobius band region. We can assign an integer to each region, equal to the number of times a representative plane intersects γ. The 2-width of γ is the sum of these integers over all the regions. The 2-bridge number is just the number of regions r.

The double tangent planes of γ are divided into two types. A double tangent plane is said to be *exterior* if it does not separate small neighborhoods of the two tangent points and *interior* if it does separate them. Let t be the number of exterior double tangent planes of γ and s the number of interior double tangent planes. Let i be the number of inflection points and c the number of crossings of $p(\gamma)$. An inflection point corresponds to a point where γ is tangent to a vertical plane and this plane separates any neighborhood in γ of the point of tangency. A crossing point of $p(\gamma)$ is a point where a vertical line meets γ twice.

A theorem of Fabricius-Bjerre [3] provides the tool needed to link properties of γ to the graph α. Fabricius-Bjerre considered the relation between the numbers of double tangent lines, crossing points and inflection points of a plane curve δ. He established the following result [3].

Theorem 2 **(Fabricius-Bjerre)** *Let δ be a generic plane curve with t exterior double tangent lines, s interior double tangent lines, i inflection points and k crossings. Then $c + \frac{1}{2}i = t - s$.*

Lemma 3 *For a generic plane curve with c crossings and whose graphic has v valence-4 vertices, we have $c \le v$.*

Proof Double tangent planes in $\mathcal{S}_2$ correspond to valence-4 vertices of the graph α, so the number of such vertices v of α equals $s + t$. Applying Theorem 2, we obtain

$$c \le c + \frac{1}{2}i = t - s \le t + s = v.$$

□

The next lemma relates v to the number of simply connected faces f in the complement of α. While α may have cusps, they do not play a role in this computation.

Lemma 4 *For a generic plane curve whose graphic has v valence-4 vertices, f simply-connected faces, and r faces of all types, we have $v = f$ and either $r = f + 1$ or $r = f + 2$. The latter occurs when there is a region in the graphic homeomorphic to a Mobius band.*

Proof One region in the complement of α, corresponding to planes missing γ, is homeomorphic to a half-open annulus. There may be one region homeomorphic to a Mobius band. Such a region exists when there is a path of planes transverse to γ that starts at a plane and ends at the same plane with reversed orientation. When a curve is in braid position in $\mathbb{R}^3$, the set of planes through the braid axis gives such a Mobius band. Annular and Mobius band regions do not contribute to the Euler characteristic computation. Let e be the number of edges connecting valence four vertices. The Euler characteristic of $\mathcal{S}_2$ is zero, obtained by taking the sum, giving $v - e + f = 0$. Since vertices are of valence four, we have $e = 2v$, and so $f = v$ as claimed. Depending on whether a Mobius band component exists in the complement of α, we have $r = f + 1$ or $r = f + 2$. □

Now let $R_1, R_2, \ldots, R_r$ denote the regions in the complement of the graph α, with associated intersection numbers with γ given by $n_1, n_2, \ldots, n_r$ respectively. Each n_i is a non-negative even integer, and the sum of these integers is $w_2(\gamma)$. At most one of the n_i's equals zero, since there is a single equivalence class of planes in $\mathbb{R}^3$ disjoint from γ. Thus all the other integers contribute at least 2 to $w_2(\gamma)$. which satisfies

$$w_2(\gamma) \geq 2(r-1) \geq 2f = 2v \geq 2c.$$

Similarly, the 2-bridge number satisfies

$$b_2(\gamma) = r \geq f = v \geq c.$$

So the crossing number of γ is bounded above by half of its 2-width. The crossing number is also bounded above by the 2-bridge number. Since only finitely many knots have crossing number less than a given integer, the theorem follows. □

4 Bounds on 2-width

Definition. A knot projection is *positively curved* if it contains no inflection points.

Lemma 5 *Every curve in $\mathbb{R}^3$ can be isotoped to have a positively curved projection.*

Proof Every curve γ is isotopic to a curve with a braid presentation [2]. This representation of the curve can be given in cylindrical coordinates by the formula $\gamma(s) = (r(s), \theta(s), z(s))$ with $\theta'(s) > 0$ and $r(s) > 0$. By scaling we can assume $0 < r(s) < 1$. An isotopy $\gamma_t(s) = (t + (1-t)r(s), \theta(s), z(s)), 0 \leq t \leq 1 - \epsilon$ takes γ to a curve in a 2ϵ-neighborhood of the unit radius cylinder $\{r = 1\}$. As $\epsilon \to 0$, $p(\gamma)$ smoothly converges to a cover of the unit circle. Thus for ϵ sufficiently small, the projection $p(\gamma_{1-\epsilon})$ has positive curvature. □

We can apply this to get an upper bound on the 2-width and 2-bridge number of a knot.

Proposition 1 *Let c be the minimal crossing number of a braid projection of a knot K. Then*

$$w_2(K) \leq (c+1)(c+2)$$

and

$$b_2(K) \leq c + 2.$$

Proof Choose γ representing K so that $p(\gamma)$ has minimal crossing number over all braid representations of K. Further isotopy γ so that its projection $p(\gamma)$ is positively curved as in Lemma 5. This isotopy does not change its crossing number. After the isotopy $p(\gamma)$ has no inflection points and no internal double tangencies. So its crossing number c equals the number of external double tangencies t, which in turn equals the number of vertices v in the graphic of $p(\gamma)$. From Lemma 4 we know that the number of regions in the graphic r satisfies $c + 1 \leq r \leq c + 2$. In particular $b_2(K) = r \leq c + 2$.

Each region has width a non-negative even integer, and one region has width zero. Regions in the graphic with a common edge have width that differs by 2. So the sum of the widths is at most $2 + 4 + 6 + \cdots + 2(c + 1) = (c + 2)(c + 1)$. □

We can obtain a lower bound for 2-width and 2-bridge number in terms of the number of intersections between the curve and a plane in $\mathcal{S}_2$.

Proposition 2 *If a plane in $\mathcal{S}_2$ intersects γ in $2n$ points then $b_2(\gamma) \geq n + 1$ and $w_2(\gamma) \geq n(n + 1)$.*

Proof Some plane in $\mathcal{S}_2$ intersects γ in zero points. Each region in the graphic has width a positive even integer, and at least one region has width $2n$. Regions in the graphic with a common edge have width that differs by 2, so there are at least $n + 1$ regions in the graphic and $b_2(\gamma) \geq n + 1$. The regions have representative planes whose widths include the values $0, 2, 4, \ldots 2n$ and the total width is at least $2 + 4 + 6 + \cdots + 2n = n(n + 1)$. □

Theorem 3 *The only knots with 2-width less than or equal to 10 or 2-bridge number less than 7 are the trefoil and the unknot.*

Proof Suppose $w_2(\gamma) \leq 10$. If γ meets a plane in $\mathcal{S}_2$ in at least 6 points then Lemma 2 implies that its width is at least 12. So we can assume that γ transversely intersects any plane in $\mathcal{S}_2$ in at most 4 points.

Call a crossing of $p(\gamma)$ *exterior* if one of the complementary regions of the projection of the curve to the xy-plane whose closures contains the crossing is the unbounded region. Otherwise call it *interior*. A ray in the xy-plane from a point near an interior crossing to infinity must cross $p(\gamma)$ at least once. If γ is knotted then $p(\gamma)$ has an exterior crossing. Suppose $p(\gamma)$ also has an interior crossing. Then a line in the xy-plane passing very close to both an interior and an exterior crossing meets $p(\gamma)$ twice near each crossing and at least once more in passing from a point near the interior crossing to infinity. Since the intersection number with $p(\gamma)$ is even, the intersection consists of at least six points. So γ meets a plane in $\mathcal{S}_2$ transversely in at least six points and $w_2(\gamma) \geq 12$. So we can assume that all crossings of $p(\gamma)$ are exterior.

Color the complementary regions to $p(\gamma)$ black and white, with the unbounded region white and regions with a common edge having different colors. Rays based at points in a white region intersect $p(\gamma)$ in an even number of points, while rays based in a black region intersect it an odd number of times. If there are no bounded white regions then a small neighborhood of any crossing meets the unbounded region in two components, the knot is composite, and a simple induction argument shows that γ is unknotted, satisfying the conclusion of the Theorem. Otherwise there is at least one bounded white region. A ray from a bounded white region to infinity meets $p(\gamma)$ in at least two points. If there are two distinct bounded white regions, then a line segment connecting a point in each intersects $p(\gamma)$ in at least 2 points. Extending this segment to a line by adding two rays gives a line intersecting $p(\gamma)$ in at least six points, contradicting our width assumption. So we can assume that $p(\gamma)$ has exactly one bounded white region, W.

Let W be an interior point of W. Every line through W intersects $p(\gamma)$ in exactly 4 points. So $p(\gamma)$ gives a 2-braid projection of γ, with axis the vertical line over W. The number of crossings of a 2-braid knot is odd, and since we assume that γ is not a trefoil or unknot we have that the number of crossings is at least 5. Applying Lemma 3 shows that the number of regions in the graphic is at least 7, with one annular and one Mobius band region. Thus the 2-bridge number is at least 7. One region has width 4 and at least five other regions have width at least two, giving that the 2-width of γ is no less than 14, contradicting our assumption.

We conclude that γ is a trefoil or an unknot. □

Theorem 4 *The* $(2, n)$*-torus knot* K *has 2-width* $2n + 4$.

Proof A positively curved 2-braid projection of K has a graphic with $r = n + 2$. One region has width 0, one has width 4 and n have width 2, giving a total width of $2n + 4$. It remains to show no representative of K has smaller 2-width.

Let γ be any representative of K. Since K has n crossings in an alternating projection, the number of crossings of $p(\gamma)$ is at least n [10]. Applying Lemmas 3 and 4 shows that the number of regions in the graphic of $p(\gamma)$ is $n + 1$ or $n + 2$. As in the proof of Theorem 3 there is at least one interior white region for $p(\gamma)$. An interior white region has width at least 4. If there is exactly one then the graphic has a Mobius band region of width 4 and $r = n+2$. In that case the width is at least $4 + 2n$ as claimed. If there is more than one interior white region, then the total width is at least $4 + 4 + 2(n - 2) = 2n + 4$. The result follows. □

Remark Width can be defined and computed for links in the same way. The number of zero-width regions in the graphic associated to a link can be greater than one if the link is split. The Hopf link and the 2-component unlink each have $w_2(L) = 8$ and the (2, 4)-torus link has $w_2(L) = 12$.

5 Curvature and 2-width

The *total curvature* of a curve in $\mathbb{R}^3$ is obtained by integrating the absolute value of the curvature function. Milnor and Fary showed that if a smooth curve γ has total curvature less than 4π then γ is unknotted [4,9]. Milnor's proof proceeds by finding a direction in $\mathbb{R}^3$ relative to which the curve has only one maximum and one minimum. The same argument shows that if the total curvature of γ is less than $2n\pi$ then there is a direction relative to which γ has fewer than n maxima. As a consequence, its bridge number is less than n. The converse is false. Curves of bridge number two can have unbounded curvature. A curve can be deformed to have arbitrarily large total curvature without changing its bridge number, so small bridge number does not imply small total curvature. In contrast, we show that if a curve has small 2-width then its projection onto the associated 2-plane has small total curvature.

Lemma 6 *Suppose* β *is a positively curved arc properly embedded in the upper half-plane that begins at the origin, curves counterclockwise and meets the x-axis precisely at its two endpoints. Then* β *ends on the non-positive x-axis.*

Proof If a neighborhood of its endpoints are fixed, the total curvature of β depends only on its isotopy class, being equal to the total angle turned. This total curvature is positive for curves that terminate on the negative x-axis and negative for those that terminate on the positive x-axis. □

Lemma 7 *Suppose a properly embedded planar curve* β *has a subarc* δ *that has non-zero curvature and total curvature* k_0. *Then* δ *intersects the line through its two endpoints in at least* $[k_0/\pi]$ *points and* β *intersects some line in at least* k_0/π *points.*

Proof By an isometry and scaling that doesn't change total curvature, we can assume that δ runs from the origin to the point $(1, 0)$ and is positively curved, so that it rotates monotonically counterclockwise. By adding small twists near the initial and final points, possibly increasing the total curvature to a new value k_0' but without changing the number of intersections with the x-axis, we can arrange that the initial and terminal tangent vectors are horizontal. The total curvature of δ is now a multiple of π. Consider a subarc α of δ of total curvature π whose initial and final points are horizontal. We show that such a subarc α must meet the x-axis.

Suppose that α runs from an initial point A to a terminal point B while staying below the x-axis and that B is closer to the x-axis. See Fig. 1.

As δ continues past B it falls, turning counterclockwise, and eventually becomes horizontal once again at a point C. Suppose that the y-coordinate of A is less than that of C. Extend a horizontal line segment S from C to the arc α. S cuts off a disk in the plane (shaded in Fig. 1) into which the continuation of δ enters after passing through C. By Lemma 6 δ cannot escape from this disk, since it cannot cross S. The cases where A is closer to the x-axis than C, or where α lies above the x-axis, are similar.

A curve with endpoints on the x-axis that turns through precisely 4π achieves equality, and therefore the estimate on δ is sharp. However it is sharp only if the arc δ turns through an angle that is an even multiple of π. In the case where the estimate on δ is sharp, the intersection of the x-axis with δ is not transverse, and a small perturbation of the axis yields a line that intersects β in an additional point. Thus in all cases we have that β intersects some line in at least k_0/π points. □

It follows from the arguments above that a simple arc of non-zero curvature can be decomposed into at most two spirals, as in Fig. 2. We will not need this fact here.

Lemma 8 *Suppose that γ projects to a C^2 curve $p(\gamma)$ immersed in the xy-plane and that $p(\gamma)$ contains a simple positively curved subarc δ with total curvature k_0. Then*

$$b_2(\gamma) \geq \frac{k_0}{2\pi} + 1$$

and

$$w_2(\gamma) \geq \frac{{k_0}^2}{4\pi^2}.$$

Proof Lemma 7 implies that $p(\gamma)$ has at least k_0/π intersection points with some line. Since adjacent regions in the graphic have representative planes whose intersection with γ differ by two, there are at least $(k_0/2\pi) + 1$ such regions, giving the stated lower bound for b_2.

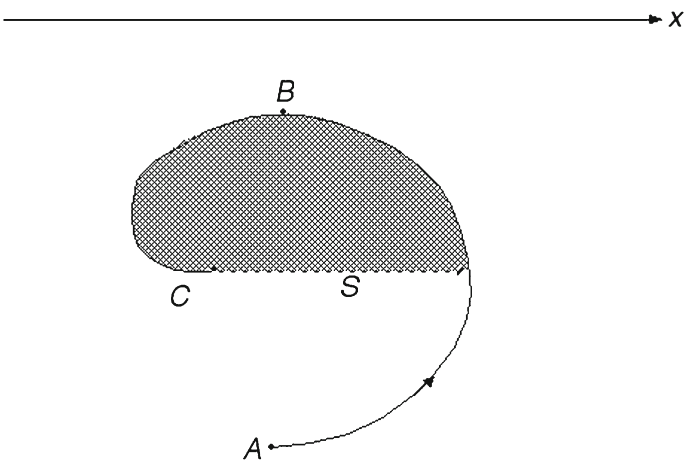

Fig. 1 A positively curved arc turning through π between A and B

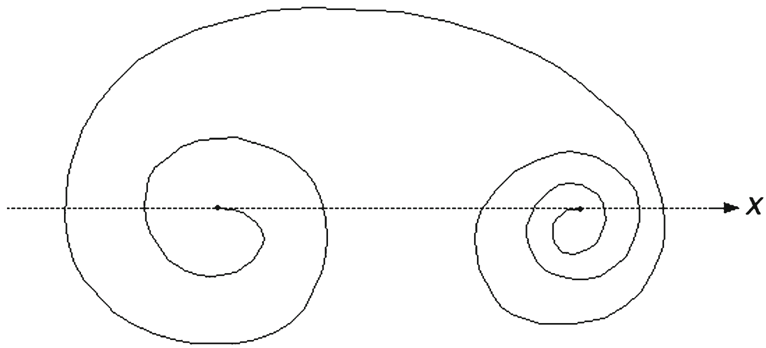

Fig. 2 A positively curved simple arc

For a bound on 2-width, assume first that k_0/π is an even integer. Then the 2-width of γ is at least

$$k_0/\pi + (k_0/\pi - 2) + \cdots + 2 = \frac{(k_0/\pi)(k_0/\pi + 2)}{4} > \frac{k_0{}^2}{4\pi^2}.$$

The intersection numbers of transverse lines with the projection $p(\gamma)$ are even. If a line intersects $p(\delta)$ non-transversely in an odd number of points, then a small translation results in a line intersecting $p(\delta)$ transversely in a larger, even number of points. So the number of intersection points of $p(\gamma)$ and some transverse line is an even integer that, by Lemma 7 is no less than k_0/π. The above argument then applies to give a lower bound on $w_2(\gamma)$. □

Theorem 5 *If $p(\gamma)$ is a C^2 curve immersed in the plane with total curvature k_1 then*

$$w_2(\gamma) > k_1{}^{2/3}/(2\pi)^{2/3}$$

and

$$b_2(\gamma) > \sqrt{k_1/4\pi} + 1.$$

Proof Let $m = 2c + i$ where c is the number of crossings of $p(\gamma)$ and i is the number of inflection points. By Theorem 2 we have that $2c + i = 2t - 2s \leq 2t + 2s = 2v$ where v counts the vertices in the graphic of $p(\gamma)$. Lemma 4 implies that $v = f$ and either $r = f + 1$ or $r = f + 2$. So $w_2(\gamma) \geq 2f \geq m$. If $m > k_1{}^{2/3}/(2\pi)^{2/3}$ then the Theorem follows. So assume $m \leq k_1{}^{2/3}/(2\pi)^{2/3}$.

The crossing and inflection points of γ divide γ into $m = 2c + i$ subarcs, each with no crossing or inflection points. Since γ has total curvature k_1, one of the subarcs of $p(\gamma)$ disjoint from its inflection and crossing points has non-zero curvature. We can assume it is positively curved with total curvature greater than k_1/m, by reflecting if necessary. By Lemma 8 we have that

$$w_2(\gamma) > \frac{(k_1/m)^2}{4\pi^2} = \frac{k_1{}^2}{(2m\pi)^2} > \frac{(2\pi)^{4/3} k_1{}^2}{(2\pi k_1{}^{2/3})^2} = \frac{k_1{}^{2/3}}{(2\pi)^{2/3}}.$$

A similar argument applies to 2-bridge number. We have

$$b_2(\gamma) \geq m/2 + 1.$$

If $b_2(\gamma) \leq \sqrt{k_1/4\pi} + 1$ then $m \leq \sqrt{k_1/\pi}$. Then a subarc of γ disjoint from its inflection and crossing points has non-zero curvature. Its total curvature is no less than k_1/m and the resulting lower bound on $b_2(\gamma)$ from Lemma 8 is

$$b_2(\gamma) \geq \frac{k_1/m}{2\pi} + 1 \geq \frac{k_1}{2\pi\sqrt{k_1/\pi}} + 1 = \sqrt{k_1/4\pi} + 1.$$

□

Theorem 5 has the following implication for knots in $\mathbb{R}^3$.

Corollary 1 *If $b_2(K) \leq n$ then some planar projection of K has total curvature at most $4\pi(n-1)^2$. If $w_2(K) \leq n$ then some planar projection of K has total curvature at most $2\pi n^{3/2}$.*

Proof If every planar projection of K has curvature greater than $4\pi(n-1)^2$ then Theorem 5 implies that $b_2(K) > n$. Similarly if every planar projection of K has curvature greater than $2\pi n^{3/2}$ then Theorem 5 implies that $w_2(K) > n$. □

6 Generic curves

In order to compute the width invariants discussed in this paper we need to assume that γ has certain generic properties. We state here the required conditions. For k-parameter invariants, the definition of genericity is based on the singular values of the associated graphic. In each case, the space $\mathcal{S}_k$ contains a singular subset corresponding to planes and spheres tangent to γ, possibly at multiple points. The condition we set for a curve γ to be generic is that these singular planes give rise to a singular set that is a piecewise immersed, smooth submanifold in general position. In the case of 1-width and 1-bridge numbers, we require that the height function on γ is a Morse function. This ensures that there are finitely many critical points and that critical values are distinct. For $k = 2$, the graphic is the image of a circle, formed by one vertical tangent plane in $\mathcal{S}_2$ for each point on a generic curve. This graphic is required to be a piecewise smooth immersed curve in general position. Note that the graphic has cusps corresponding to finitely many inflection points on γ, and therefore is not a regular immersion. For $k = 3$, the graphic is the image of a torus, formed by a circle's worth of tangent planes for each point of γ. For $k = 4$, the graphic is the image of a line bundle over a torus. We again require that these graphics are immersed piecewise smooth submanifolds in general position.

Acknowledgements Research of Joel Hass was supported in part by the NSF. Research of J. Hyam Rubinstein was supported in part by the Australian Research Council. Research of Abigail Thompson was supported in part by the NSF.

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