Title
Confidence Limits from Experiments with Small Statistics

Author
Alston-Garnjost, M.

Publication Date
1995-09-01

LBL-37772
UC-414
Preprint

Lawrence Berkeley Laboratory

UNIVERSITY OF CALIFORNIA

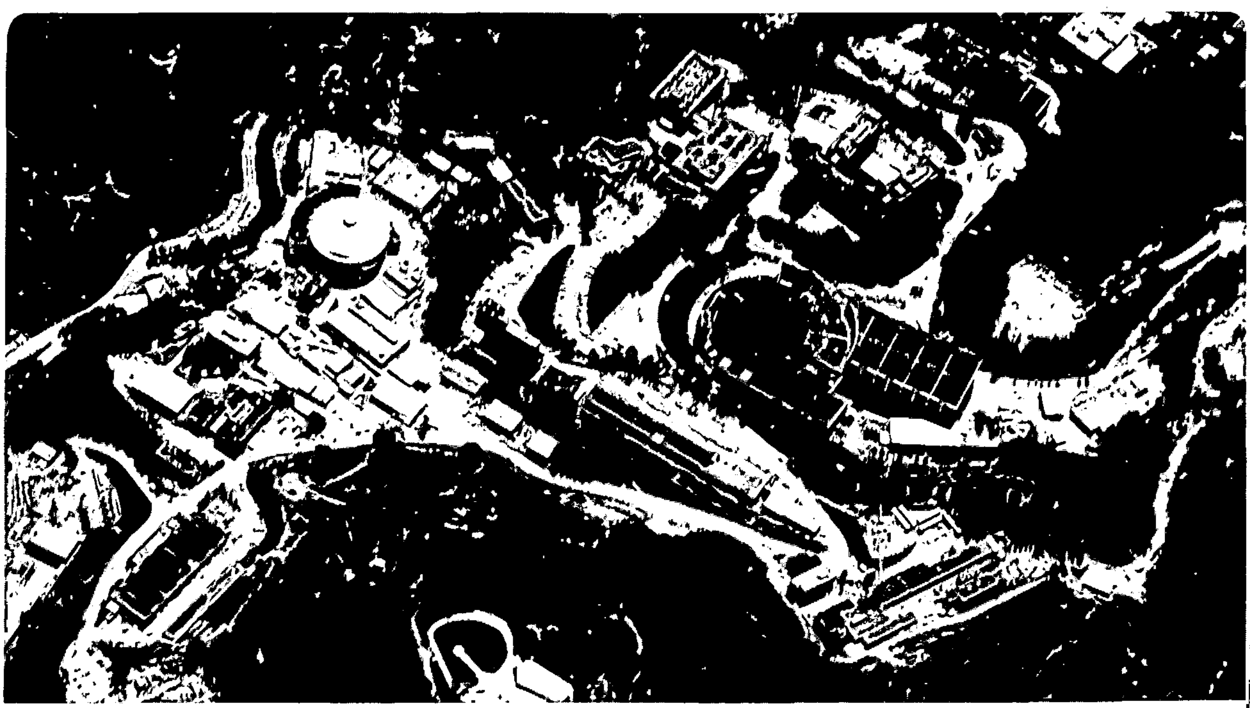
Physics Division

Submitted to Nuclear Instruments and Methods in Physics Research A

Confidence Limits from Experiments with Small Statistics

M. Alston-Garnjost and H.W. Nicholson

September 1995



Prepared for the U.S. Department of Energy under Contract Number DE-AC03-76SF00098

DISCLAIMER

LBL-37772
UC-414

Confidence Limits from Experiments with Small Statistics

M. Alston-Garnjost

Physics Division
Lawrence Berkeley National Laboratory
University of California
Berkeley, California 94720

H. W. Nicholson

Department of Physics
Mount Holyoke College
South Hadley, Massachusetts 01075

September 1995

This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics, of the U.S. Department of Energy under Contract No. DE-AC03-76SF00098 and by a National Aeronautics and Space Administration Research Training Grant (NGT-50783) and the University of California.

Confidence Limits from Experiments with Small Statistics

M. Alston-Garnjost and H. W. Nicholson

ABSTRACT

A general procedure is described for obtaining upper confidence limits in experiments producing data samples too small to be analyzed using Gaussian techniques. It is shown this procedure reproduces the well-known results for the special cases described in the literature.

Experiments which search for rare processes frequently obtain very small data samples upon which confidence level limits on signal rates must be based. Procedures exist for obtaining confidence limits on the signal in null experiments and in experiments in which the background is known [1]. In this short paper these procedures are extended to experiments in which information about the *shapes* of signal and background energy spectra over the region of interest are also known and the energy of each event is measured.

Consider an experiment of duration T in which a total of N events are observed in an energy range $E_1 \leq E \leq E_2$, and the energy E_i of each event i is measured. If this event sample consists of a mixture of signal and background, assume the mean values of the signal and background are s and b, and that s is unknown but b is known or its value can be assumed. Then in the experimental data sample, the expected fractions of signal and background events are $f_s = s/(s+b)$ and $f_b = b/(s+b)$. In addition, assume that the spectral shapes of the signal $F_s(E)$ and background $F_b(E)$ are known experimentally or from Monte Carlo simulations and have been normalized to unity over the energy range E_1 to E_2 so that $f_sF_s(E_i) + f_bF_b(E_i)$ is the weighting factor for observing the i^{th} event.

Then the likelihood function $\mathcal{L}$ is given by

$$\mathcal{L} = \mathcal{N}\frac{e^{-(b+s)}(b+s)^N}{N!}\Pi_{i=1}^{N}[f_bF_b(E_i) + f_sF_s(E_i)] \quad (1)$$

where $\mathcal{N}$ is the normalization constant defined such that

$$\int_0^\infty \mathcal{L}ds = 1 \quad (2)$$

This function is called the *extended likelihood function* [2], and is the product of the probability of observing N events from a Poisson distribution with mean $(b+s)$ and the likelihood of getting the energy distribution of the observed events.

The confidence level CL of an upper limit ℓ on s can then be defined to be the probability that $s < \ell$ and is given by

$$CL = \int_0^\ell \mathcal{L}ds \quad (3)$$

The remainder of this paper shows how a knowledge of the likelihood function can be used to obtain the upper confidence limit ℓ on the signal for various experimental situations.

Case I. No background and no signal shape information.

When the background b is zero and the spectral shape of the signal is constant over the energy range or is not measured, the normalization factor $\mathcal{N}$ is unity and the likelihood function is

$$\mathcal{L} = \frac{e^{-s} s^N}{N!}$$

When N is zero, Equation 3 reduces to

$$CL = 1 - e^{-\ell}$$

which gives

$$\ell = \ln\left(\frac{1}{1 - CL}\right) \tag{4}$$

Equation 4 gives a value of 1.14 events for ℓ with $CL = .68$ corresponding to the 68% confidence level, and a value of 2.30 for ℓ for the 90% confidence level. This agrees with Table 17.3 in ref. 1.

When N is greater than zero, Equation 3 can either be solved numerically to find values of ℓ for various confidence limits or evaluated using the following tabulated integral [3]

$$\int_0^1 x^m e^{-ax} dx = \frac{m!}{a^{m+1}}[1 - e^{-a} \sum_{r=0}^{m} \frac{a^r}{r!}]$$

A scale change gives

$$\int_0^\ell s^N e^{-s} ds = N!(1 - e^{-\ell} \sum_{n=0}^{N} \frac{\ell^n}{n!}) \tag{5}$$

which is used to obtain

$$CL = 1 - e^{-\ell} \sum_{n=0}^{N} \frac{\ell^n}{n!} \tag{6}$$

By making the variable changes $CL \rightarrow 1 - \epsilon$, $N \rightarrow n_0$, and $\ell \rightarrow N$, Equation 6 becomes Equation 17.32 in Reference [1]. Cousins has already pointed out this equality [4].

Case II. Known background but no shape information.

In this case the likelihood function is

$$\mathcal{L} = \mathcal{N}\frac{e^{-(b+s)}(b+s)^N}{N!}$$

Using Equation 2 the normalization constant $\mathcal{N}$ is given by

$$\mathcal{N} = \frac{N!}{\int_0^\infty e^{-(b+s)}(b+s)^N ds} .$$

Following Lavine [5] the confidence level CL when b is known is given by

$$CL = \int_0^\ell \mathcal{L}(s)ds = \frac{\int_0^\ell e^{-(b+s)}(b+s)^N ds}{\int_0^\infty e^{-(b+s)}(b+s)^N ds}$$

$$= \frac{\int_b^{b+\ell} e^{-x}x^N dx}{\int_b^\infty e^{-x}x^N dx} = 1 - \frac{e^{-(b+\ell)}\sum_{n=0}^N (b+l)^n/n!}{e^{-b}\sum_{n=0}^N b^n/n!} \qquad (7)$$

The integrals are incomplete Gamma functions and are evaluated with the help of Equation 5 above. Equation 7 with the appropriate change of variables becomes Equation 17.35 in Reference [1].

If the background has more than one component, then $b = b_1 + b_2 + ...$

Case III. Background and shape information.

The more general situation occurs when the signal and background spectral shapes are a function of energy over the energy range of interest. The full extended likelihood function must now be used to calculate confidence limits.

By moving the factor $(b+s)^N$ into the product in Equation 1 the likelihood function becomes

$$\mathcal{L} = \mathcal{N}\frac{e^{-(b+s)}}{N!}\Pi_{i=1}^N[bF_b(E_i) + sF_s(E_i)]$$

When b is known and by using Equation 2 to evaluate $\mathcal{N}$, the signal ℓ corresponding to the confidence level CL can be obtained from the equation

$$CL = \frac{\int_{s=0}^{s=\ell} e^{-s}\Pi_{i=1}^{N}[bF_b(E_i) + sF_s(E_i)]ds}{\int_{s=0}^{\infty} e^{-s}\Pi_{i=1}^{N}[bF_b(E_i) + sF_s(E_i)]ds} \quad (8)$$

In general, Equation 8 must be integrated numerically for various values of ℓ until a value of ℓ is found which gives the desired confidence level.

If the background has more than one component, then Equation 1 can be generalized by defining $f_b = f_{b_1} + f_{b_2} + ...$ where

$$f_{b_n} = \frac{b_n}{b+s}$$

with

$$b = b_1 + b_2 + ...$$

In this case, each background spectral shape must be separately normalized to one over the energy interval of interest.

In this short paper we have described a straightforward procedure which can be used to obtain upper confidence limits from an extended likelihood function and which yields the accepted formulas for the more specialized cases given in reference [1]. Extended likelihood functions can also be defined to include other information [2] and should be a useful way of obtaining confidence limits in many experiments with small statistics.

We would like to thank R.W. Kenney, G.R. Lynch and D.E. Groom for very helpful discussions. This work is supported by U.S. Department of Energy Grants No. DE-AC03-76SF00098 and DE-FG02-90ER40553.

[1] Review of Particle Properties, Physical Review D, 50 (3) 1994. pp. 1278-1282

[2] L.Lyons, Statistics for nuclear and particle physicists, Cambridge Univ. Press, 1986. pp. 98-102

[3] Handbook of Chemistry and Physics, 76th edition, CRC Press, 1995-1996. p. A-62, formula 668.

[4] R.D.Cousins, Am.J.Phys. 63, 398, 1995.

[5] T.L.Lavine, Search for Single-Photons from Radiative Production of Supersymmetric Particles or Neutrinos, PhD thesis, University of Wisconsin-Madison 1986. Appendix.

LAWRENCE BERKELEY NATIONAL LABORATORY
UNIVERSITY OF CALIFORNIA
TECHNICAL & ELECTRONIC INFORMATION DEPARTMENT
BERKELEY, CALIFORNIA 94720