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Extensive From Games in Continuous Time Part I: Pure Strategies



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Leo K. Simon and Maxwell Stinchcombe

July, 1986

Key words: Game theory, continuous time, subgame perfection

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This paper develops a new framework for modelling games in continuous time. We view continuous time as "discrete time, but with a grid that is arbitrarily fine." We define a class of continuous-time strategies and restrict them to an arbitrary, increasingly fine sequence of discrete-time grids. Our assumptions guarantee that this process generates a convergent sequence of outcomes, whose limit is independent of the sequence of grids. Our outcome function maps each strategy profile to its associated limit outcome.

We compare the perfect equilibria of our model to the approximate equilibria of "nearby" discrete-time games. If the restrictions to discrete-time grids of our continuous-time strategies are approximate equilibria, then the strategies themselves are exact equilibria. Moreover, under weak conditions, any perfect equilibrium of our model is close to an approximate perfect equilibrium for any "nearby" discrete-time model.

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# Extensive Form Games in Continuous Time.
# Part I: Pure Strategies.*

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## ABSTRACT

This paper develops a new framework for modelling games in continuous time. We view continuous time as "discrete time, but with a grid that is *arbitrarily* fine." We define a class of continuous-time strategies and restrict them to an arbitrary, increasingly fine sequence of discrete-time grids. Our assumptions guarantee that this process generates a convergent sequence of outcomes, whose limit is independent of the sequence of grids. Our outcome function maps each strategy profile to its associated limit outcome.

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* This research was originally motivated by Fudenberg-Tirole [2]. The authors have benefited greatly from discussions with numerous colleagues, but especially Robert Anderson, Karl Iorio, Ariel Rubenstein and Bill Zame.

## I. Introduction.

How to model time is an important question in game theory. The topic has arisen in recent years in many areas of economics, including bargaining theory, price-setting oligopoly models, patent-, innovation- and R&D-races, and models of capital augmentation. In such contexts, there are serious modelling problems both with the conventional discrete-time framework and also with existing continuous-time frameworks.

The problems with discrete time game theory are very familiar. In many games, the set of equilibria is extremely large. In others, backward induction plays a destructive role: intuitively appealing outcomes cannot be implemented as Nash or subgame perfect equilibria, even when the time horizon is infinite.[1] In still other games, discrete-time models yield striking results, but these are very sensitive to the particular specification of the time structure. For example, Rubinstein's [13] striking result in bargaining theory depends crucially on the particular, exogenous restrictions he imposes on the set of times at which agents can move.

In addition to these problems, discrete-time game theory has a particular bias: an agent can always obtain a one-period advantage by preempting other players.[2] The implications of this bias are pervasive. It is instructive, therefore, to explore the consequences of relaxing it. In our continuous-time setting, agents can react instantaneously to the actions taken by others. In some contexts, this difference has striking implications.

At present, there are two established frameworks for modelling games in continuous time: differential games and "c.d.f. games." In a differential game, an agent specifies a *rate of change* of a state variable, as a function of time and the current value of this variable.[3] This class of strategies is very restrictive: in particular, agents cannot induce discontinuous changes in the state variables. In many contexts, however, it is precisely these kinds of changes that

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1 Fudenberg-Tirole [2], Gul-Sonnenschein [5] and Gul-Sonnenschein-Wilson [6], and all discuss infinite horizon games in which the role of backward induction is disturbing.

2 For a discussion of the importance of this issue, see Anderson [1].

3 Two interesting examples of this approach are Judd [8] and Fershtman and Kamien [4].

one wishes to model (for example, in games of timing).

"C.d.f." games are better suited to such contexts. In a c.d.f. game, a strategy is a cumulative distribution function, that is, a right continuous, nondecreasing function, $F_i$, of time: $F_i(t)$ is the cumulative probability that agent $i$ will have moved by time $t$. This framework works well for certain kinds of timing games (when agents would rather follow than lead), but not for others (when they would rather lead than follow).[4] In the latter kind of game, there are intuitively natural outcomes that cannot be implemented by c.d.f. strategies.[5]

This paper is the first in a series that develops a new framework for modelling games in continuous time.[6] A problem arises at the outset: in continuous-time, there is no canonical way to associate outcomes to strategies (see section III below). Our response to this problem is to view continuous time as "discrete time, but with a grid that is *arbitrarily* fine." To formalize this idea, we specify a restricted class of continuous-time strategies. Formally, these strategies are functions defined on a large function space. They have, however, a simple heuristic interpretation as "master plans," instructing agents how to play the game on every conceivable discrete-time grid. Specifically, the restriction of a master plan to a finite grid is a well-defined discrete-time strategy. Thus any profile of master plans generates a well-defined discrete time outcome for every discrete-time grid. The conditions we impose on strategies guarantee that for every profile, there exists an outcome that is the limit of the outcomes generated by playing the profile on any sequence of increasingly fine grids. For this class of strategies, there is an natural way to define the outcome function: each strategy profile is mapped to its corresponding, uniquely defined limit outcome. Thus, there is a literal sense in which our continuous-time games are the limits of sequences of corresponding discrete-time games.

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[4] Wilson and Hendriks [14] study the c.d.f. equilibria of concession games. In this context, these strategies seem perfectly natural. On the other hand, Pitchik [13] shows that symmetric duels often have no equilibria, when strategies are c.d.f.'s.

[5] See Fudenberg and Tirole [2] for a thought-provoking discussion of these issues.

[6] In this paper, we consider only pure strategies. The model is extended to incorporate mixed-strategies in Part II. In Part III, we focus on applications.

The design of our model facilitates comparison between discrete- and continuous-time games. In particular, we can compare our continuous-time subgame perfect equilibria to the approximate equilibria of the corresponding discrete-time games.[7] There are sequences of discrete-time equilibria that have no analog in continuous time (e.g., equilibria in which agents move in alternate periods). However, under weak conditions on payoffs, if the restrictions of a given continuous-time profile to some sequence of grids are $\epsilon^n$-equilibria, with $\epsilon^n \underset{n}{\to} 0$ as the grids become finer, then the original profile will be an exact equilibrium for our model. On the other hand, under slightly stronger conditions, given any perfect equilibrium for our continuous-time model, there exists a sequence of approximate, discrete-time equilibria, implementing outcomes that become arbitrarily close to the original continuous-time equilibrium outcomes.

The paper is organized as follows. In section II, we specify the class of discrete-time games whose structure we mimic in continuous time. Section III shows why it is difficult to define an outcome function in continuous time. Section IV sets out the formal model. We specify a simple inductive procedure for computing continuous time outcomes, and verify that the outcomes defined in this way are indeed the limits of the corresponding discrete-time outcomes. In Section V, we present some examples. Our main example is an "irreversible" version of the repeated prisoners' dilemma: we show that the cooperative outcome can be implemented as a subgame perfect equilibrium, even though the time horizon is finite. Moreover, in this simple context, cooperation is the *unique* equilibrium that survives iterated elimination of dominated strategies. Section VI contains our results relating discrete- and continuous-time equilibria. The proofs are gathered together in the Appendix.

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[7] Our comparison parallels that of Fudenberg and Levine [3]. They compare the approximate perfect equilibria of finite-horizon discrete-time games with the exact equilibria of infinite-horizon games.

## II. From Discrete- to Continuous-Time .

We will construct the continuous-time analog of the following class of discrete-time, extensive-form games. Agents move simultaneously and information is complete. Let $A_i$ denote a finite set of actions for agent $i$ and let $A = \prod_{i \in I} A_i$ denote the set of action profiles. Let $R$ be a finite subset of [0, 1], containing 0 and 1. $R$ represents the set of times at which agents can act. The sets $A$ and $R$ define a game form, whose structure is described below.

For $r \in (0, 1]$, let $\lfloor r \rfloor^R$ denote the largest $s \in R$ strictly less than $r$. A history of the system up to time $r$ is a list of action profiles, $\alpha = (\alpha^0, \ldots, \alpha^{\lfloor r \rfloor^R})$, one for each point in $R$ before $r$. A decision node is a point in time, $r \in R$, paired with a history of the system up to time $r$. (There is, in addition, a distinguished node, $(0, \varnothing)$, representing the start of the game.) A pure strategy for $i$ is a function mapping decision nodes to points in $A_i$. A complete history is a list, $(\alpha(0), \ldots, \alpha(1))$, of action profiles, one for each time node in $R$. The subform beginning at $(r, \alpha)$ is the game form induced by starting the game at $(r, \alpha)$. The outcome function maps each subform (i.e., decision node), paired with a pure-strategy profile, to some complete history. This function is constructed in the obvious way by induction on $R$.

An extensive form game is a game-form, paired with a valuation function that assigns a vector of payoffs to each complete history of the game. The standard solution concepts for such games are Nash equilibrium (Nash [11]) and subgame perfect equilibrium (Selten [13]).

To construct a continuous-time analog of this class of games, we replace the finite set of times $R$ by the interval [0, 1]. With this change, the decision node is a pair $(0, \varnothing)$ or $(t, h)$, where $t \in (0, 1]$ and $h$ is a function from $[0, t)$ to $A$, representing the history of the game up to time $t$. A continuous-time strategy is now a function defined on the space of decision nodes. Since time is no longer well-ordered (with the conventional ordering), it is no longer possible to define the outcome function inductively. Indeed, we will show that for some seemingly well-behaved strategy profiles, there can be no sensible outcome whatsoever.

## III. The Technical Issues.

In this section, we illustrate some difficulties that arise in attempting to define a sensible outcome function in continuous time. The first example motivates our idea of viewing strategies as master-plans. The others show that this approach works well only for a restricted class of strategies.

We say that a history is a history is consistent with a strategy profile if there is no open interval on which the history is constant, but some agent's strategy calls for a change in the status quo.[8] A minimal property that an outcome function should satisfy is that is should map each strategy profiles to a history that is consistent with that profile. Our first example shows that in continuous time, a unique, consistent outcome cannot be identified by induction.[9] The problem arises because the continuum is not well-ordered: for $t > 0$, there is, simply, no "last $s$ before $t$."

Consider the following strategy, for a game played by one player with two choices, *left* and *right*.

$$\begin{cases} \text{play } \textit{left} & \text{if } t = 0 \\ \text{play } \textit{left} & \text{if } t > 0 \text{ and } \textit{left} \text{ was played at each } s < t \\ \text{play } \textit{right} & \text{otherwise} \end{cases} \qquad \text{Example III.1}$$

In discrete-time, this strategy generates the outcome: play *left* at every time node. In continuous time, there is a continuum of outcomes that are consistent with this strategy: for any $\bar{t} > 0$, play *left* on $[0, \bar{t}]$ and *right* thereafter. In this example, there is a natural way to resolve the nonuniqueness problem: define the outcome to be the limit of the outcomes generated by playing the strategy on discrete grids. The unique limit, obviously, is: choose *left* at every $t$. This outcome is certainly consistent with the strategy.

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[8] An alternative statement of this condition is: there is a dense set of time nodes on which the strategy profile agrees with the history. Note that this is an extremely weak requirement.

[9] A similar example appears in Anderson [1].

Our second example shows that for many strategies, the discrete-time outcomes may have no sensible limit. Accordingly, we must impose conditions on strategies that guarantee that a limit exists. The following example illustrates the kinds of strategies that we will exclude. It is a modification of a game discussed by Krishna [9].[10] For $i = \{1, 2\}$, let $A_i = \{x, y\}$. The functions $f_1$ and $f_2$ below are well-defined maps from the continuous-time decision nodes to $A_i$. For any history $h = (h_1, h_2)$, let $h_{|t}$ denote the restriction of $h$ to $[0, t)$. Now define $f_i$ by:

$$f_1(t, h_{|t}) = \begin{cases} x & \text{if } t = 0 \\ x & \text{if } \exists \delta > 0 \text{ s.t. } h_{2|t}([t - \delta, t)) = x \\ y & \text{otherwise} \end{cases}$$

$$f_2(t, h_{|t}) = \begin{cases} x & \text{if } t = 0 \\ x & \text{if } \exists \delta > 0 \text{ s.t. } h_{1|t}([t - \delta, t)) = y. \\ y & \text{otherwise} \end{cases}$$

Example III.2

When played on any discrete-time grid, this profile generates the cycle: $(xx, xy, yy, yx, xx, \cdots)$. It can be shown that there is no Hausdorff topology on outcomes such that this cyclic outcome converges.

In our last example, there is a well-defined limit of the discrete-time outcomes, but it is inconsistent with the original strategy profile. Once again, let $A_i = \{x, y\}$, for $i = \{1, 2\}$. Define $f_1$ and $f_2$ by:

$$f_1(t, h_{|t}) = \begin{cases} x & \text{if } t < \tfrac{1}{2} \\ y & \text{if } t \in [\tfrac{1}{2}, \tfrac{3}{4}) \\ y & \text{if } t \geq \tfrac{3}{4} \text{ and if discontinuity points of } h_{1|t} \text{ and } h_{2|t} \text{ agree} \\ x & \text{otherwise} \end{cases}$$

$$f_2(t, h_{|t}) = \begin{cases} x & \text{if } t = 0 \\ x & \text{if } t > 0 \text{ and } \exists \delta \text{ s.t. } h_{1|t}([t - \delta, t)) = x. \\ y & \text{otherwise} \end{cases}$$

Example III.3

[10] This example was brought to our attention by Karl Iorio.

If these strategies are played on any (fine enough) discrete-time grid, player 1 jumps at the first grid point after $t = ½$, and 2 follows suit at the next grid point. At $t \geq ¾$, since the players' discontinuity points do not agree before ¾, player 1 switches back to $x$ and 2 follows suit. The limit of these outcomes is that both 1 and 2 jump simultaneously from $x$ to $y$ exactly at ½, and back again at simultaneously at ¾. But this outcome is inconsistent with player 1's strategy: at any time $t > ¾$, 1 should be playing $y$ at $t$, since the discontinuity points of agents' individual histories agree!

## IV. The Model.

*Agents and Actions.*

Our continuous-time game will be played on the unit interval, [0, 1]. Let $I = \{1, \ldots, i, \ldots, \bar{i}\}$ denote the set of agents. For each $i$, let $A_i$ denote the agent's action set and let $A = \prod_{i \in I} A_i$. An element $a = (a_i)_{i \in I}$ will be called an action profile. An action profile $a \in A$ will frequently be written as the pair $(a_i, a_{-i})$, where $a_{-i} = (a_j)_{j \neq i}$. Play proceeds as follows. Each agent chooses an action at time $t = 0$. An agent can change his action at (essentially) any subsequent time in [0, 1], either unilaterally or in response to a change in another agent's action. (In fact, our restrictions will guarantee that an agent changes his action only finitely often.)

Formally, our agents' action sets are independent of time and the past history of the system. In some games, this is an inappropriate assumption. For example, in simple games of timing, if an agent once "moves" (say, fires his one bullet), he cannot ever move again. We can, however, incorporate such restrictions into our framework by a suitable choice of payoff function: we simply assign a prohibitively low payoff to any history that has an inadmissible string of moves.

*Histories.*

A history is a function from [0, 1] to $A$. The role of histories in our model parallels their role in the conventional discrete-time model. At each point in time, agents can condition their actions on the history of the system up to that time. Our outcome function then assigns a unique history to each profile of pure-strategies.

The universe of possible histories is unmanageably large. Accordingly, we restrict attention to a small subset of admissible histories, the set of step functions that are right-continuous on (0, 1].[11] Let $H$ denote the set of admissible histories:[12]

$$H = \{h \in A^{[0,1]}: \forall t \in (0, 1), h(t+) = h(t);\ h \text{ has a finite number of discontinuity points } \}$$

where $h(t+) = \lim_{\delta \downarrow\downarrow 0} h(t + \delta)$.[13] By restricting attention to step functions, we are implicitly assuming that agents make only a finite number of moves. For $h \in H$, let $h = (h_1, \ldots, h_i, \ldots, h_{\bar{I}})$. We shall say that $h_i$ is agent $i$'s individual history.

We endow $H$ with a topology we call the EPS-topology.[14] In this topology, $h^n \underset{n}{\overset{EPS}{\rightarrow}} h$ iff for sufficiently large $n$, (a) $h^n$ and $h$ differ on a set of small measure; and, for all $i$, (b) $h_i^n$ and $h_i$ have the same number of discontinuity points; (c) if $t^n$ and $t$ are corresponding discontinuity points of $h_i^n$ and $h_i$, then $h_i^n(t^n+) = h_i(t+)$. We now define a metric for the EPS topology.

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[11] We will explain why we treat zero specially after we have defined a metric on histories.

[12] Our restriction to this class of functions is a little arbitrary. Indeed, for two person games, a compelling case can be made for allowing left-continuous functions also. Since this argument is loses its validity when $\bar{I} > 2$, we have chosen to work with the smaller, simpler space.

[13] The symbol "$\downarrow\downarrow$" means converging *strictly* from above.

[14] The EPS topology is Essentially the Product Skorohod topology on the right continuous functions mapping $[-1, 2]$ to $A$. (Maisonneuve [10]). Precisely, we identify $h \in H$ with $\theta^h: [-1, 2] \rightarrow A$ as follows:

$$\theta^h(t) = \begin{cases} h(0) & \text{if } t < 0 \\ h(0+) & \text{if } t = 0 \\ h(t) & \text{if } t \in (0, 1) \\ h(1) & \text{if } t \geq 1 \end{cases}$$

Now $h^n \underset{n}{\overset{EPS}{\rightarrow}} h$ iff $\theta^{h^n} \underset{n}{\rightarrow} \theta^h$ in the product Skorohod topology.

We first need some notation. Let $\theta$ be a step function mapping some convex subset of $[0, 1]$ to some finite set. Let $D(\theta)$ denote the set of discontinuity points of $\theta$ and let $n(\theta) = \#D(\theta)$. Define $\underline{s}(\theta) = \begin{cases} s^{n(\theta)}(\theta) & \text{if } n(\theta) > 0 \\ 0 & \text{otherwise} \end{cases}$. If $n(\theta) > 0$, let $(s^1(\theta) < \cdots < \underline{s}(\theta))$ be an enumeration of $D(\theta)$.

A metric for the EPS-topology is given by $d^H = \max_{i \in I} d_i^H$, where $d_i^H : H \times H \to \mathbb{R}$ is defined as follows:

$$d_i^H(h, h') = \begin{cases} \lambda(\{t : h_i(t) \neq h_i'(t)\}) & \text{if } \begin{cases} n(h_i) = n(h_i'), \\ h_i(0) = h_i'(0) \text{and} \\ h_i(s^k(h_i)+) = h_i'(s^k(h_i')+), \forall 1 \leq k \leq n(h_i) \end{cases} \\ 1 & \text{otherwise} \end{cases}$$

where $\lambda$ denotes Lebesgue measure on $[0, 1]$.

The example below illustrates some of the properties of $d^H$. Let $I = \{1, 2\}$ and $A_i = \{x, y\}$. Define $\bar{h}$, $h^n$ and $\eta^n$ by:

$$\bar{h}(t) = \begin{cases} xx & \text{if } t < \tfrac{1}{2} \\ yy & \text{if } t \geq \tfrac{1}{2} \end{cases} ; \quad h^n(t) = \begin{cases} xx & \text{if } t < \tfrac{1}{2} \\ yx & \text{if } t \in [\tfrac{1}{2}, \frac{n+1}{2n}) \\ yy & \text{if } t \geq \frac{n+1}{2n} \end{cases} ; \quad \eta^n(t) = \begin{cases} xx & \text{if } t < \tfrac{1}{2} \\ yx & \text{if } t \in [\tfrac{1}{2}, \frac{n+1}{2n}) \\ xy & \text{if } t \in [\frac{n+1}{2n}, \frac{n+2}{2n}) \\ yy & \text{if } t \geq \frac{n+2}{2n} \end{cases}$$

In the EPS topology, the sequence $(h^n)$ converges to $\bar{h}$ but $(\eta^n)$ does not. On the other hand, $(\eta^n)$ does converge *in measure* to $\bar{h}$.

In all of the applications we consider, we assume that agents' payoffs are continuous w.r.t. $d^H$. Had we chosen a coarser topology (say, convergence in measure), there would be many games (duels, etc) in which payoffs would not be continuous.

We now explain by example why we do not require that histories are right continuous at zero. Define the sequence $(h^n)$ by: $h^n(t) = \begin{cases} xx & \text{if } t < \frac{1}{n} \\ yy & \text{if } t \geq \frac{1}{n} \end{cases}$. It is important that se-

quences like this have limits in $H$. Moreover, it is important that this limit preserves the information that agents have played "$x$." In the EPS topology, $(h^n)$ converges to $\bar{h}(t) = \begin{cases} xx & \text{if } t = 0 \\ yy & \text{if } t > 0 \end{cases}$, which is an element of $H$, and indeed contains this information.

We conclude this section with some definitions that will be used later in the paper. We first define a truncation operation on histories. Given $h \in H$ and $t \in (0, 1]$, let $h_{|t} \in H$ denote the truncation of $h$ before $t$, defined by:

$$h_{|t}(s) = \begin{cases} h(s) & \text{if } s < t \\ h(t-) & \text{if } s \geq t. \end{cases}$$

where $h(t-) = \lim_{\delta \downarrow\downarrow 0} h(t - \delta)$. The truncation of agent $i$'s individual history, $h_{i|t}$, is defined similarly.

Next, given $(h, a, t) \in H \times A \times [0, 1]$, we define $h^{a,t}$ as follows: if $t = 0$, then $h^{a,t}$ agrees with $h$ at zero, and equals $a$, for positive $t$. Otherwise, $h^{a,t}$ agrees with $h$ up to $t$, and is constant at $a$ from $t$ on; Summarizing;

$$h^{a,0}(s) = \begin{cases} h(s) & \text{if } s = 0 \\ a & \text{if } s > t; \end{cases} \qquad h^{a,t}(s) = \begin{cases} h(s) & \text{if } 0 \leq s < t \\ a & \text{if } s > t \end{cases}$$

Finally, for $a \in A$, let $h^a$ denote the history that is constant at $a$.

*Decision Nodes.*

There is a distinguished node, called the initial decision node. This node is denoted $(0, \varnothing)$ and represents the start of the game. A regular decision node is a time $t > 0$, paired with some history whose last discontinuity point is strictly less than $t$.[15] Note that, for all $h$, $(t, h_{|t})$ is a regular decision node. Let $DN$ denote the set of all decision nodes:

$$DN = \bigcup_{h \in H} \{(t, h): t > \underline{s}(h)\} \cup \{(0, \varnothing)\} = \{(t, h_{|t}): h \in H, t > 0\} \cup \{(0, \varnothing)\}.$$

[15] This definition departs slightly from the conventions of discrete-time game theory (see p. 4). In discrete-time, a decision node is a point in time, paired with a history of the system up to time $t$. Our definition is effectively equivalent to this one and, for our purposes, notationally more efficient.

We denote the generic element of DN by $(t, h)$. The reader should be aware that whenever we write $(t, h) \in DN$, we are implicitly asserting that *either* $t > \underline{s}(h)$ *or* $(t, h) = (0, \varnothing)$.

For $\bar{t} < 1$ and $h \in H$, we will frequently refer to the subset, $(t, h_{|t})_{t>\bar{t}}$, of $DN$ as a branch of the decision tree. Branches of the form $(t, h)_{t>\underline{s}(h)}$ will be particularly important.

We will also use $DN$ to represent the set of subforms. Our outcome function will map each profile of strategies, paired with an element of $DN$, to some history in $H$.

*Pure Strategies.*

A pure strategy for $i$ is a function $f_i$ from $DN$ to $A_i$, satisfying the four restrictions set out below. Before proceeding, we alert the reader that we will treat zero in a very special way. If the following heuristic interpretation of our game is adopted, however, our treatment will seem quite natural: Imagine that agents' initial actions are "actually" implemented *before* the start of the game, say, at $t = -1$. (Think of athletes taking their positions before the start of the race.) Also, suppose that "in fact," the "real" game does not begin until *immediately after* zero. (The starting pistol is fired exactly at zero, and the race begins *immediately after* the pistol shot).[16]

*F1 Step functions with respect to time.*

We require that strategies are step functions with respect to time. That is, along any branch, $(t, h_{|t})_{t>0}$, of the decision tree an agent can change his action only finitely many times. Formally,

> *for each* $f_i$, *the function of time*, $f_i(\cdot, h_{|\cdot})$, *induced by restricting* $f_i$ *to the branch*, $(t, h_{|t})_{t>0}$, *has a finite number of discontinuity points.* (F1)

The following examples for a one-person game illustrate the kinds of strategies that satis-

[16] This story is clearly consistent with our identification of the EPS topology on [0, 1] with the Product Skorohod on [−1, 2]. Specifically, in footnote 14, we identified the point zero with the *interval* [−1, 0).

fy this condition. Let $I = \{1\}$ and let $A_1 = \{x, y\}$. Compare the two strategies, $f_1$ and $g_1$:

$$f_1(t, h) = \begin{cases} x & \text{if } t = 0 \\ x & \text{if } h = h^x \text{ and } t \neq \tfrac{1}{2} \\ y & \text{if } h = h^x \text{ and } t = \tfrac{1}{2} \\ y & \text{otherwise} \end{cases}; \quad g_1(t, h) = \begin{cases} x & \text{if } t = 0 \\ x & \text{if } h = h^x \text{ and } t \leq \tfrac{1}{2} \\ y & \text{if } h = h^x \text{ and } t > \tfrac{1}{2} \\ y & \text{otherwise} \end{cases} \qquad \text{Example IV.1}$$

(Recall that $h^x$ is the history that is constant at $x$.) Strategy $f_1$ has the following interpretation: starting from any subgame $(t, h^x)$, $t < \tfrac{1}{2}$, player 1 continues to play "$x$" until $t = \tfrac{1}{2}$, and then jumps to "$y$" at exactly $t = \tfrac{1}{2}$; if, however, play starts from a subgame $(t, h^x)$, $t > \tfrac{1}{2}$, $i$ continues to play "$x$" for the rest of the game. Strategy $g_1$ has player 1 waiting until *immediately after* $t = \tfrac{1}{2}$, before moving for the first time. If either strategy is played from the initial subgame, $(0, \varnothing)$, the outcome is the same: it is the history $\bar{h}_1 = \begin{cases} x & \text{if } t < \tfrac{1}{2} \\ y & \text{if } t \geq \tfrac{1}{2} \end{cases}$.

Restriction (F1) is not sufficient to guarantee that the outcomes generated by our strategies will have only finitely many discontinuity points. For example, consider a profile of strategies satisfying the following conditions: agent 1 initiates the first jump at $t = \tfrac{1}{4}$ and

$$\text{for } i, j = 1, 2, \text{ if the } n\text{'th jump in agent } j\text{'s history occurs at } t, \text{ then } i \text{ jumps at } t + \frac{1}{2^{n+2}} \qquad \text{Example IV.2}$$

Any history consistent with this profile must have an infinite number of discontinuity points between $\tfrac{1}{4}$ and $\tfrac{3}{4}$, i.e., at $\{1/4, 3/8, 1/2, 9/16, \ldots, \}$. (Note that the strategies do satisfy (F1): each $h \in H$ has finitely many discontinuity points, and $f_i(\cdot, h_{|\cdot})$ can have at most one more discontinuity point than $h_j$.) To exclude strategies such as these, we will require that there be a uniform upper bound to the number of jumps that an agent can initiate.

*F2 Uniformly Bounded Number of Jumps initiated by $i$.*

We will say that $f_i$ "initiates a jump at $(t, h_{|t})$" along the branch $(s, h_{|s})_{s>0}$ if the following conditions are satisfied: (a) $h_{|\cdot}$ is continuous at $t$; $f_i$ specifies that (b) $i$ should maintain the status quo until $t$; (c) change his action either at or immediately after $t$. Formally, $f_i$ initiates a jump at $(t, h_{|t})$ if

$$\lim_{\delta \downarrow\downarrow 0} h(t - \delta) = \lim_{\delta \downarrow\downarrow 0} f_i(t - \delta, h_{|t-\delta}) = h_i(t);$$
$$f_i(\cdot, h_{|\cdot}) \text{ is discontinuous at } t;$$

This definition distinguishes between jumps that $i$ initiates and instantaneous reactions to other agents' jumps. For example, both $f_1$ and $g_1$ in Example IV.1 above initiate jumps at $t = ½$. On the other hand, the strategy below--play the action you have just been playing--has many discontinuities, but initiates no jumps at all:[17]

$$\hat{f}_i(t, h) = \begin{cases} x & \text{if } t = 0 \\ x & \text{if } t > 0 \text{ and } h_i(t) = x \\ y & \text{if } t > 0 \text{ and } h_i(t) = y \end{cases}$$

To see this, consider, say, a history $\hat{h}$ that jumps from $x$ to $y$ at $t = ½$. $\hat{f}_i$ satisfies conditions (b) and (c) above, since $\lim_{\delta \downarrow\downarrow 0} \hat{f}_i(t - \delta, \hat{h}_{|t-\delta}) = x$ and $\hat{f}_i(\cdot, \hat{h}_{|\cdot})$ is discontinuous at $t$. However, condition (a) is not satisfied, since $\lim_{\delta \downarrow\downarrow 0} \hat{h}_i(t - \delta) \neq \hat{h}_i(t)$. Therefore, this jump in $\hat{h}_i$ was not initiated by $i$.

We can now state our second restriction:

*for each $f_i$, $\exists k$ s.t. along any branch $(t, h_{|t})_{t>0}$ of the decision tree, $f_i$ initiates at most $k$ jumps.* (F2)

This restriction clearly eliminates Example IV.2 above.

[17] Along any branch with $k$ discontinuities in $h_i$, however, $f_i(\cdot, h_{|\cdot})$ has $k$ discontinuity points.

*F3 Positive Refractory Time.*

Our third restriction prevents agents from making several jumps in instantaneous succession. In particular, agent $i$ is not permitted to react instantaneously to other agents' instantaneous reactions to $i$'s own jumps. (In applications, we will endogenize this and the previous restriction by assuming that it is prohibitively costly for agents to make several jumps in rapid succession (see assumption V2, section VI below)). Condition (F3) is neeeded to exclude strategies such as

$$f_i^*(t, h) = \begin{cases} x & \text{if } t = 0 \\ y & \text{if } t > 0 \text{ and } h_i(t) = x. \\ x & \text{if } t > 0 \text{ and } h_i(t) = y \end{cases} \qquad \text{Example IV.3}$$

$f_i^*$ requires that $i$ jump from $x$ to $y$ and back again, arbitrarily rapidly.[18] As we have observed above (p. 6), there is no simple way to assign an outcome to such a strategy.

We now explain how we formalize this restriction. Fix a decision node $(t, h)$, such that $t > 0$. Suppose that $i$ *either* initiates a jump to $a_i$ at $(t, h_{|t})$ *or* jumps to $a_i$ as an instantaneous response to jump(s) by other agent(s). Condition (F3) below requires that in either case, $i$ must play $a_i$ for an instant after $t$, in response to *any* history $h'$ that agrees with $h$ before $t$, and has $i$ playing $a_i$ at $t$. Formally,

$$\begin{aligned} &\text{for all } h \in H \text{ and all } t \in (0, 1], \text{ for all } a_i \neq h_i(t-), \text{ if} \\ &\qquad \textit{either} \quad f_i(t, h_{|t}) = a_i \\ &\qquad \textit{or} \quad \lim_{\delta \downarrow\downarrow 0} f_i(t + \delta, h_{|t+\delta}) = a_i \\ &\textit{then } \forall a_{-i} \in A_{-i}, \lim_{\delta \downarrow\downarrow 0} f_i(t + \delta, h^{(a_i, a_{-i}), t}) = a_i. \end{aligned} \qquad \text{(F3)}$$

Example IV.3 above clearly violates (F3). A more subtle kind of failure is illustrated by Example III.2 (p. 6). Consider the history, $\bar{h}$, defined by: $\bar{h}(t) = \begin{cases} yx & \text{if } t < \frac{1}{2} \\ xx & \text{if } t \geq \frac{1}{2} \end{cases}$. When player 1 switches from $y$ to $x$, player 2 is instructed to switch from $x$ to $y$, (i.e.,

[18] Note that $f_i^*$ satisfies restrictions F1 and F2.

$\lim_{\delta \downarrow\downarrow 0} f_2(½ + \delta, \bar{h}_{|½+\delta}) = y$.) Condition (F3) therefore requires that whatever 1 does in response to 2's jump, 2 should continue to play $y$ for an instant after ½. This condition is violated, however, since $f_2(½ + \delta, \bar{h}^{yy,½}) = x$.

An analogous condition is required at $t = 0$:

$$\begin{aligned} &\textit{for all } h \in H \textit{ and all } a_i \neq h_i(0), \text{ if} \\ &\qquad \lim_{\delta \downarrow\downarrow 0} f_i(\delta, h_{|\delta}) = a_i \\ &\textit{then } \forall a_{-i} \in A_{-i}, \lim_{\delta \downarrow\downarrow 0} f_i(\delta, h^{(a_i, a_{-i}),t}) = a_i. \end{aligned} \tag{F3}$$

*F4 Box Measurability with respect to h.*

Our final condition restricts the way agents can condition their actions on histories. Given any $f_i$, there is a partition of the branches of the decision tree into a countable collection of boxes, with the following property: along any two branches in the same box, an agent reacts in the same way.

For $h, h' \in H$, we will write $h' \succeq h$ if for each $i$, the first $n(h_i)$ discontinuity points of $h_i'$ weakly exceed the corresponding discontinuity points of $h_i$. Formally,

$$h' \succeq h \text{ iff } \forall i, \forall 1 \leq k \leq n(h_i), s^k(h_i') \geq s^k(h_i).$$

We now require that for every $h$, there exists a positive $\delta$ such that if $h'$ is within $\delta$ of $h$ and $h' \succeq h$, then along the two branches $(t, h)_{t>\underline{s}(h)}$ and $(t, h')_{t>\underline{s}(h')}$, of the decision tree, $i$'s decisions are *identical*.[19] Formally, we say that a strategy $f_i$ is <u>box measurable w.r.t. $h$</u> if

$$\begin{aligned} &\forall h \in H, \exists \delta > 0 \textit{ s.t. if } d^H(h', h) < \delta \text{ and } h' \succeq h, \text{ then} \\ &\qquad f_i(s, h) = f_i(s, h'), \forall s > \underline{s}(h'). \end{aligned} \tag{F4}$$

This assumption is significantly more restrictive than the others. However, it plays two essential roles. Without it, a unique limit of the discrete-time outcomes may not exist.

[19] Our outcome function will still be well-defined for strategies that satisfy a weaker condition: replace "identical" with "close." Moreover, strategy profiles satisfying this weaker condition are close, in a strong sense, to box measurable strategies. Details will be reported in the sequel to this paper.

Second, even if such a limit does exist, it may not be consistent with the limit strategy profile. The latter problem is illustrated by Example III.3 (p. 6). Player 1's strategy in that example violates condition (F4), since at $(t, h)$, $t \geq 3/4$, his decision depends critically on whether or not the previous discontinuity points of $h_1$ and $h_2$ agree *exactly*.

Let $F_i$ denote the set of pure strategies for $i$ satisfying (F1)-(F4). Let $F = \prod_{i \in I} F_i$. An vector $f = (f_1, \ldots, f_i, \ldots, f_{\bar{i}})$ in $F$ is called a pure strategy profile.

*Discrete-Time Game Forms.*

Every discrete-time game form has an equivalent representation in our framework. The *only* distinction between our representation and the conventional one (e.g. Fudenberg-Levine ([3]), described above [p. 4]) is that our histories are step functions, measurable with respect to the given discrete-time grid, rather than finite lists of action profiles. This difference is purely formal.

Let $X$ be a finite subset of $[0, 1]$. For each $t \in (0, 1]$, we let $\lfloor t \rfloor^X$ denote the predecessor of $t$ in $X$, that is, $\lfloor t \rfloor^X = \max\{s \in X: s < t\}$. Similarly, for $t \in [0, 1)$, $\lceil t \rceil^X$ denotes the successor of $t$ in $X$: $\lceil t \rceil^X = \min\{s \in X: s > t\}$.

We will say that a finite set $R \subset [0, 1]$ is a finite approximation to $[0, 1]$ if $R$ contains 0 and 1. Let $\mathbf{R}$ denote the set of all finite approximations to $[0, 1]$. For each $R \in \mathbf{R}$, we define an $R$-segment of $[0, 1]$ to be an interval of the form $[r, \lceil r \rceil^R)$, where $r \in R$, $r < 1$. Let $\delta(R)$ denote the length of the largest $R$-segment of $R$, i.e., $\delta(R) = \max_{\substack{r \in R \\ r < 1}} (\lceil r \rceil^R - r)$.

A history will be called $R$-measurable if it is constant on every $R$-segment of $[0, 1]$. Define $H^R$ by: $H^R = \{h \in H: h \text{ is } R\text{-measurable}\}$. A regular decision node, $(t_r, h_{\restriction t_r})$, will be called $R$-measurable if $r \in R$ and $h \in H^R$. By convention, the initial decision node, $(0, \varnothing)$, is taken be to $R$-measurable, for all $R$. Let $DN^R$ denote the set of $R$-measurable decision nodes. For any $R \in \mathbf{R}$, $H^R$ and hence $DN^R$ are finite sets. We will sometimes refer to an element of $DN^R$ as an $R$-measurable subform.

A strategy for $i$ defined on $DN^R$ is a function from $DN^R$ to $A_i$. Let $F_i^R$ denote the strategies for $i$ defined on $DN^R$ and $F^R$ the set of strategy profiles defined on $DN^R$. Since no *a priori* restrictions are imposed on discrete-time strategies, the restriction to $DN^R$ of any pure strategy in $F_i$ is a well-defined pure strategy on $DN^R$.

The discrete-time outcome function, $o(R, \cdot, \cdot, \cdot)$: $F^R \times DN^R \to H^R$, maps each strategy profile on $DN^R$ and $R$-measurable subform to some $R$-measurable history. This history is defined by induction in exactly the conventional way. That is, for $g \in F^R$ and $(\bar{r}, \bar{h}) \in DN^R$, $o(R, g, \bar{r}, \bar{h})$ is the unique history, $h \in H^R$, that satisfies:

$$h(t) = \begin{cases} \bar{h}(t) & \text{if } t < \bar{r} \\ g(t, h_{|t}) & \text{if } t \geq \bar{r} \text{ and } t \in R \\ h(\lfloor t \rfloor^R) & \text{if } t \geq \bar{r} \text{ and } t \notin R \end{cases}.$$

*Continuous Time Outcomes as the limits of Discrete Time Outcomes.*

Our continuous time outcome function, $o([0, 1], \cdot, \cdot, \cdot)$: $F \times DN \to H$, maps each pair, $(f, (t, h))$, to the limit of the discrete-time outcomes generated by playing $f$ from the subgame beginning at $(t, h)$. To "play" a profile, $f$ on a particular grid, $R$, we first modify $f$ to take account of the particular structure of $R$, then restrict the modified profile to $DN^R$. (To see why the first step is necessary, recall Example IV.1 on p. 12 above. Unless $f_1$ is modified to take account of $R$, the discrete-time outcome, $o(R, f_{1|DN^R}, 0, \varnothing)$, will be very different, depending on whether or not $\frac{1}{2} \in R$!)

We now explain how $f$ is modified. For each $f_i \in F_i$, we define a family of discrete-time strategies, $(f_i^R)_{R \in \mathbf{R}}$. For each $R$, $f_i^R(\cdot, h)$ is a function defined on $R \cap (\underline{s}(h), 1]$ that "shifts to the right" the discontinuity points of $f_i(\cdot, h)$ in the following way: if the open interval between $r$ and its predecessor in $R$ contains one or more discontinuity points of $f_i(\cdot, h)$, then $f_i^R(r, h)$ is equated to the value of $f(\cdot, h)$ at $\lfloor r \rfloor^{D(f_i(\cdot, h))}$, the last discontinuity point before $r$. Otherwise, $f^R(r, h) = f(r, h)$. Note that for each $R$ and $h \in H^R$, if $\delta(R)$ is sufficiently small, then $f_i^R(\cdot, h)$ will have essentially the same graph as $f_i(\cdot, h)$. Formally, the

graph preserving restriction of $f_i$ to $DN^R$, $f_i^R$: $DN^R \to A_i$ is defined by, for $(r, h) \in DN^R$,

$$f_i^R(t_r, h) = \begin{cases} f_i(\lfloor r \rfloor^{D(f_i(\cdot, h))}, h) & \text{if } \lfloor r \rfloor^{D(f_i(\cdot, h))} \in (\lfloor r \rfloor^R, r) \\ f_i(t_r, h) & \text{otherwise} \end{cases}$$

For example, let $I = \{1, 2\}$, $A_i = \{x, y\}$ and let $\bar{h}$ be some constant history. Let $f_i$ be a strategy satisfying:

$$f_1(t, \bar{h}) = \begin{cases} x & \text{if } t \in (0, 1/3] \cup \{2/3\} \\ y & \text{otherwise} \end{cases}; \quad f_2(t, \bar{h}) = \begin{cases} x & \text{if } t \in (0, 1/3) \cup \{2/3\} \\ y & \text{otherwise.} \end{cases}$$

Let $R = \{0, 1/5, \ldots, 4/5, 1\}$. In this case, $f_i^R$ is defined as follows:

$$f_1^R(r, \bar{h}) = \begin{cases} x & \text{if } r \in \{1/5, 2/5, 4/5\} \\ y & \text{otherwise} \end{cases}; \quad f_2^R(r, \bar{h}) = \begin{cases} x & \text{if } r \in \{1/5, 4/5\} \\ y & \text{otherwise.} \end{cases}$$

Observe how the construction distinguishes between discontinuities from the left and from the right (e.g., compare $f_1$ and $f_2$ at $t = 1/3$).

We can now state more precisely the relationship between our continuous-time outcome function and the discrete-time outcomes generated by $f$. For each $f \in F$ and each $(\bar{t}, \bar{h}) \in DN$, there exists a unique $\eta \in H$ with the following property: if for $n$ sufficiently large, $(\bar{t}, \bar{h})$ is $R^n$-measurable, and if $(R^n)$ satisfies $\delta(R^n) \underset{n}{\to} 0$, then the sequence of outcomes generated by the $f^{R^n}$'s from $(\bar{t}, \bar{h})$ converges to $\eta$.

The following three examples illustrate the construction. Let $I = \{1, 2\}$, $A_i = \{x, y\}$ and fix $R \in \mathbf{R}$ such that $\delta(R)$ is small. First, consider the strategy profile $\bar{f}$ defined by:

$$\bar{f}_1(t, h) = \begin{cases} x & \text{if } t = 0 \\ x & \text{if } t < 1/2; \\ y & \text{if } t \geq 1/2 \end{cases} \quad \bar{f}_2(t, h) = \begin{cases} x & \text{if } t = 0 \\ x & \text{if } h(t) = xx \\ y & \text{if } h(t) \neq xx \end{cases}$$

The outcome $o(R, \bar{f}^R, 0, \varnothing)$ is that player 1 jumps at the first grid point weakly greater than $1/2$ and player 2 follows suit at the next grid point in $R$. Now let $\bar{\bar{f}}$ be identical to $\bar{f}$, *except* that $\bar{\bar{f}}$ has player 1 playing $x$ rather than $y$ at $t = 1/2$. The outcome $o(R, \bar{\bar{f}}^R, 0, \varnothing)$ has player

1 jumping at the *second* grid point (weakly) greater than ½ and player 2 following suit at the next opportunity. Finally, consider $\tilde{f}$, defined by:

$$\tilde{f}_1(t, h) = \begin{cases} x & \text{if } t = 0 \\ y & \text{if } t = \tfrac{1}{2} \text{ and } h_1(t) = x \\ x & \text{if } t \neq \tfrac{1}{2} \text{ and } h_1(t) = x \\ y & \text{if } h_1(t) = y \end{cases}; \quad \tilde{f}_2(t, h) = \begin{cases} x & \text{if } t = 0 \\ y & \text{if } t > \tfrac{1}{2} \text{ and } h(t) = xx \\ x & \text{otherwise} \end{cases}$$

The outcome $o(R, \tilde{f}^R, 0, \varnothing)$ is that player 1 jumps at the first grid point weakly greater than ½, while player 2 never jumps. Notice that in all three cases, the outcome is independent of whether or not $\tfrac{1}{2} \in R$.

In each of the above examples, there is a well-defined limit history. In each case, player 1 jumps exactly at ½. In the first two cases, player 2 also jumps at ½; in the third, he never jumps. In the following section, we show how to compute the limit outcome directly from the limit profile.

*The Continuous Time Outcome Function: a Explicit Construction.*

There is an inductive procedure for calculating continuous-time outcomes directly. This procedure is considerably simpler than the explicit construction of a sequence of discrete-time outcomes. Fix a profile $f$ and a regular decision node $(\bar{t}, \bar{h})$. Let $\eta = o(T, f, \bar{t}, \bar{h})$. We will explain how to construct $\eta$. If $f(s, \bar{h}) = \bar{h}(\bar{t})$, for all $s > \bar{t}$, then set $\eta = \bar{h}$. Otherwise, set $\tau = \inf\{s \geq \bar{t}: f(s, \bar{h}) \neq \bar{h}(\bar{t})\}$, i.e., $\tau$ is the earliest time along the branch $(s, \bar{h})_{s>\bar{t}}$ that some $i$ chooses an action *other than* the status quo, $\bar{h}_i(\bar{t})$. Now set $\eta = \bar{h}$ on $[0, \tau)$. We will explain informally how to determine $\eta(\tau)$. For any $a_i \in A_i$, $\eta_i(\tau)$ will be equal to $a_i$ iff one of the following conditions is satisfied: (a) player $i$ selects $a_i$ at $\tau$; (b) all agents (including $i$) select $\bar{h}(\tau-)$ at $\tau$ *and* $i$ selects $a_i$ immediately after $\tau$; (c) other agents jump to $a_{-i}$ either at or immediately after $\tau$, and, immediately afterwards, $i$ jumps to $a_i$ in response. Having determined $\eta(\tau)$ in this way, we are now guaranteed by condition (F3)--agents cannot move immediately after they have just moved--that $\eta(\cdot)$ will be constant at $\eta(\tau)$ on some open interval

after $\tau$. Now proceed to $\inf\{s \geq \tau: f(s, \bar{h}^{\eta(\tau),\tau}) \neq \eta(\tau)\}$, and compute the next value of $\eta$ in the same way.

Condition (c) above is clearly rather intricate. Player $i$ may end up choosing $a_i$ after a chain of instantaneous reactions to instantaneous reactions. To keep track of such chains, we introduce the idea of an "instantaneous response chain." For $f \in F$ and each $(t, h) \in DN$, the chain generated by $f$ at $(t, h)$ is an $(\bar{i} \times \bar{i}+1)$-matrix, $\beta(f, t, h)$, of action profiles, defined inductively as follows:

$$\beta^r(f, t, h) = \begin{cases} f(t, h) & \text{if } t < 1 \text{ and } r = 0 \\ \lim_{\delta \downarrow\downarrow 0} f(\delta, (h^{f(0,\varnothing)})^{\beta^{r-1}(f,t,h),0}) & \text{if } t = 0 \text{ and } 1 \leq r \leq \bar{i} \\ \lim_{\delta \downarrow\downarrow 0} f(t+\delta, h^{\beta^{r-1}(f,t,h),t}) & \text{if } 0 < t < 1 \text{ and } 1 \leq r \leq \bar{i} \\ f(1, h) & \text{if } t = 1 \end{cases}$$

(Recall that $(h^a)^{a',0}$ is the history that is "$a$" at zero, and "$a'$" thereafter.) Note that for all $(t, h)$, if $t$ is strictly less than $\inf\{s \geq \underline{s}(h): f(s, h) \neq h(1)\}$, then $\beta^r(f, t, h)$ is the constant chain, $(h(t), \ldots, h(t))$.

To illustrate the construction of $\beta$, we return to the three profiles defined above (pp. 18-19). We have: $\beta(\bar{f}, \frac{1}{2}, h^{xx}) = (yx, yy, yy)$, $\beta(\bar{\bar{f}}, \frac{1}{2}, h^{xx}) = (xx, yx, yy)$ and $\beta(\tilde{f}, \frac{1}{2}, h^{xx}) = (yx, yx, yx)$.

The relevant properties of the $\beta$ matrix are set out in Lemma 1 below. These properties are: (a) for $r \geq 1$, if the $r$'th column of $\beta$ agrees with the preceding column, then all subsequent columns agree with the $r$'th; (b) the $i$'th row of $(\beta^0(f, t, h), \ldots, \beta^{\bar{i}}(f, t, h))$ can have at most two distinct values; (c) *unless* some agent changes the status quo either at or immediately after $t$, all columns of the matrix agree with the "status quo" (that is, with $f(0, \varnothing)$, if $t = 0$; otherwise, with $h(t)$); (d) if $\beta^{\bar{i}}(f, t, h)$ differs from the status quo, then all agents play $\beta^{\bar{i}}(f, t, h)$ for an instant after $t$. Formally,

*Lemma 1.* Fix $f \in F$, $(t, h) \in DN$ s.t. $t < 1$ and let $\bar{a} = \begin{cases} f(0, \varnothing) & \text{if } t = 0 \\ h(t) & \text{if } t > 0 \end{cases}$. Then

$$\forall 1 \le r \le \bar{t},\ \beta^r(f,t,h) = \beta^{r-1}(f,t,h) \text{ implies } \beta^{r'}(f,t,h) = \beta^r(f,t,h),\ \forall r' \ge r\ ; \tag{a}$$

$$\forall i,\ \forall 1 \le r \le \bar{t},\ \beta_i^r(f,t,h) \ne \beta_i^{r-1}(f,t,h) \text{ implies } \beta_i^{r'}(f,t,h) = \begin{cases} \beta_i^{\bar{t}}(f,t,h) & \text{if } r' \ge r \\ \bar{a}_i & \text{if } r' < r \end{cases}; \tag{b}$$

$$\beta^{\bar{t}}(f,t,h) = \bar{a} \text{ iff } \lim_{\delta \downarrow\downarrow 0} f(t+\delta, h) = f(t,h) = \bar{a}; \tag{c}$$

$$\text{if } \beta^{\bar{t}}(f,t,h) \ne \bar{a} \text{ then } \exists \delta > 0 \text{ s.t. } f(s, h^{\beta^{\bar{t}}(f,t,h),t}) = \beta^{\bar{t}}(f,t,h),\ \forall s \in (t, t+\delta). \tag{d}$$

We can now give a simple, explicit definition of the continuous time outcome function, $o([0,1],\cdot,\cdot,\cdot)$. For $f \in F$ and a subgame $(\bar{t}, \bar{h}) \in DN$, the outcome $o([0,1], f, \bar{t}, \bar{h})$, is the history $\eta \in H$ defined by:

$$\begin{aligned} \text{when } \bar{t} = 0: \quad \eta(t) &= \begin{cases} f(0,\varnothing) & \text{if } t = 0 \\ \beta^{\bar{t}}(f,t,\eta_{|t}) & \text{if } t > 0 \end{cases} \\ \text{when } \bar{t} > 0: \quad \eta(t) &= \begin{cases} \bar{h}(t) & \text{if } t < \bar{t} \\ \beta^{\bar{t}}(f,t,\eta_{|t}) & \text{if } t \ge \bar{t} \end{cases} \end{aligned} \tag{4.1}$$

The following proposition verifies that the outcome function is well-defined. Moreover, we show that outcomes satisfy the minimal consistency property described on p. 5:

*Proposition I:* For each $f \in F$ and subgame $(\bar{t}, \bar{h}) \in DN$, there exists a unique $\bar{\eta} \in H$ satisfying (4.1). Moreover, $\bar{\eta}$ is consistent with $f$ beyond $\bar{t}$, in the following sense:

$$\text{for all } \bar{t} \le t < t' \le 1],\ \bar{\eta}\cdot \text{ is constant on } (t,t') \text{ iff } f(\cdot, \bar{\eta}_{|}\cdot) = \bar{\eta}(\cdot) \text{ on } (t,t').$$

We can now state formally the main result of this section: the history $\eta$ defined by $f$ from $(\bar{t}, \bar{h})$ is the $d^H$-limit of the histories generated by playing the graph-perturbed restrictions of $f$ from $(\bar{t}, \bar{h})$ on any sequence of increasingly fine grids.

*Theorem II:* Fix $f \in F$ and a subgame $(\bar{t}, \bar{h}) \in DN$. Then

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ s.t. } \forall R \in \mathbf{R}, \text{ if } (\bar{t}, \bar{h}) \in DN^R \text{ and } \delta(R) < \delta \text{ then}$$
$$d^H(o(R, f^R, \bar{t}, \bar{h}),\ o([0,1], f, \bar{t}, \bar{h})) < \epsilon.$$

*Payoff and Valuation Functions.*

The valuation function, $V: H \to \mathbb{R}^{\#I}$, assigns a payoff vector to each history. In most of the applications we consider, we will assume:

$$V_i \text{ is uniformly continuous with respect to } d^H; \qquad \text{(V1)}$$

For example, in many games, the valuation function is obtained by integrating some instantaneous flow payoff matrix, $u: A \times [0, 1] \to \mathbb{R}^{\#I}$, with respect to $t$, i.e., for each $i$,

$$V_i(h) = \int_{[0,1]} u_i(s, h(s))d\lambda(s).$$

(recall that $\lambda$ denotes Lebesgue measure on [0, 1].) Whenever $u_i$ is integrable, the $V_i$'s satisfy (V1). Continuous-time repeated games are special cases of this class of games. In a repeated game, $u$ is independent of time.

We will devote considerable attention to a simple but rich class of games called termination games. In a termination game, each agent has two strategies: *CONTINUE* ($C$) and *TERMINATE* ($T$): if an agent once plays $T$, he must play $T$ for the remainder of the game. For such games, the valuation function is defined by:[20]

$$V_i(h) = \begin{cases} \int_{[0,1]} u_i(s, h(s))d\lambda(s) & \text{if } \forall t,\ h_i(t) = T \text{ implies } h_i([t, 1]) = T \\ \underline{v} - 1 & \text{otherwise} \end{cases}$$

where $\underline{v} = \inf_{h \in H} V_i(h)$.

The continuous-time payoff function, $P: F \times DN \to \mathbb{R}^{\#I}$, assigns a payoff vector to each strategy profile and subgame. $P_i(f, t, h)$ is player $i$'s payoff if agents play $f$ from the subgame beginning at $(t, h)$. In most of our applications, the payoff function will be defined directly from $V$, i.e., for $f \in F$,

$$P_i(f, t, h) = V_i(o([0, 1], f, t, h)).$$

---

[20] Note that these payoffs would not be continuous, if our topology on histories had been convergence in measure.

In general, however, the relationship between $P$ and $V$ may be less simple. For example, we will consider the effect on the model of introducing various kinds of costs, such as "reaction" and "implementation" costs. To model these effects, we will add terms to the $P_i$'s, capturing the idea, say, that extremely fast reactions to moves by other players are extremely costly.

For discrete-time games, payoffs are defined in the obvious way. Given a grid, $R \in \mathbf{R}$, we define the R-measurable payoff function, $P^R: F^R \times DN^R \to \mathbb{R}^{\#I}$, by, for $g \in F^R$:

$$P_i^R(g, t, h) = V_i(o(R, g, t, h)).$$

$P_i^R$ assigns to each discrete-time strategy profile and $R$-measurable subgame the payoffs generated by playing the given profile on the discrete-time grid $R$.

*Equilibrium Notions.*[21]

Given a subgame $(t, h) \in DN$, we will say that profile $f \in F$ is a $\epsilon$-best reply from $(t, h)$ if for all $i$, and all $f_i' \in F_i$,

$$P_i(f, t, h) \geq P_i((f_i', f_{-i}), t, h) - \epsilon.$$

A profile $f \in F$ is an $\epsilon$-subgame perfect equilibrium ($\epsilon$-SGP equilibrium) if it is an $\epsilon$ best reply from every subgame. Finally, $f$ is a subgame perfect equilibrium if it is an $\epsilon$-SGP equilibrium, for every $\epsilon > 0$.

For certain kinds of games, the set of SGP equilibria may be extremely large. In some instances, this set can be reduced considerably by iterative elimination of dominated strategies. To formalize this idea, we define the notion of an IU-equilibrium.[22] For $i \in I$, set $F_i^0 = F$ and $F_{-i}^0 = \prod_{j \neq i} F_j^0$. For $k \in \mathbb{N}$, we will say that $f_i \in F_i$ is $k$-th order dominated if there exists $f_i' \in F_i^{k-1}$ and $(t, h) \in DN$ such that

[21] While definitions are specified in terms of continuous-time payoffs, they apply equally well to discrete-time payoffs.

[22] IU stands for Iteratively Undominated.

$$\forall f_{-i} \in F_{-i}^{k-1}, \quad \forall (t, h) \in DN, \quad P_i((f_i', f_{-i}), t, h) \geq P_i((f_i, f_{-i}), t, h);$$
$$\exists f_{-i} \in F_{-i}^{k-1} \text{ and } (t, h) \in DN \text{ s.t. } P_i((f_i', f_{-i}), t, h) > P_i((f_i, f_{-i}), t, h).$$

For $k \in \mathbb{N}$, define $F_i^k$ by:

$$F_i^k = F_i^{k-1} - \{f_i \in F_i^{k-1}: f_i \text{ is } k\text{-th order dominated}\}.$$

and define $F_{-i}^k = \prod_{j \neq i} F_j^k$. We now define a profile $f$ to be an IU equilibrium if $f$ is SGP and if for all $i$, $f_i \in \bigcap_{k=0}^{\infty} F_i^k$.

## V. Examples.

We provide two examples, illustrating the solution concepts defined in the previous section. A more extensive discussion of applications is deferred to the third part of this paper. Both examples are two-person termination games. In each case, payoffs are determined by an instantaneous payoff matrix, $u^\gamma$. For $\gamma \in \{1, 2\}$, the valuation function, $V_i^\gamma$, is then defined by:

$$V_i^\gamma(h) = \begin{cases} \int_{[0,1]} u_i^\gamma(h(s))ds & \text{if } h_i(t) = T \text{ implies } h_i([t, 1]) = T \\ \underline{v}^\gamma - 1 & \text{otherwise} \end{cases}$$

We call our first example the "irreversible prisoners' dilemma." The instantaneous payoff matrix, $u^1$, is defined by:

| Example V.1: The Irreversible Prisoners' Dilemma: $u^1$ | | |
|---|---|---|
| | $C$ | $T$ |
| $C$ | (1, 1) | (-2, 2) |
| $T$ | (2, -2) | ( 0, 0) |

In discrete time, the unique Nash, and hence SGP, equilibrium is that agents play $T$ at every time node. In continuous time, there is a one-dimensional family of SGP equilibria. For $\tau \in [0, 1] \cup \infty$, the profile $f^\tau$ generates the outcome: cooperate until time $\tau$, and then defect (read "$\tau = \infty$" as "never terminate"):

$$f_i^\tau(t, h) = \begin{cases} C & \text{if } (t, h) = (0, \varnothing) \\ C & \text{if } t \in (0, \tau) \text{ and } h = h^{CC} \\ T & \text{otherwise} \end{cases}$$

In this example, only the (payoff equivalent) equilibria, $f^\infty$ and $f^1$, survive iterated elimination of dominated strategies. The *unique* IU equilibrium outcome for this game is: cooperate throughout the game. This outcome is implemented by the payoff-equivalent profiles, $f^\infty$ and $f^1$. To see that all other SGP profiles are eliminated, observe that once one player has terminated, the other player's best action is to terminate immediately. We can, therefore, eliminate any strategy that involves waiting a finite time before terminating, once the other player has terminated. The set of strategies that remains after this round of elimination is just: $\{f^\tau: \tau \in [0, 1] \cup \infty\}$. We now argue that for each $i$ and $\tau < 1$, the strategy $f_i^\tau$ is dominated by $f_i^\infty$. If player $j$ plays $f_j^s$, for some $s \leq \tau$, it makes no difference whether $i$ plays $f_i^\tau$ or $f_i^\infty$: in either case, the resulting outcome has both players terminating simultaneously at $s$. Now suppose $j$ plays $f_j^s$, for some $s > \tau$. If $i$ plays $f_i^\tau$, both players will terminate exactly at $\tau$; if he plays $f_i^\infty$, they will cooperate until $s$, and $i$'s payoff will be higher. Thus,

the only profiles that survive two rounds of elimination of dominated strategies are $f_i^1$ and $f_i^\infty$.

Our second example illustrates the flexibility of our model. In this game, player 1 wants player 2 to terminate. If player 2 continues until $t = ½$, however, #1 will then choose to terminate unilaterally, *provided* 2 does not intend to follow suit. Player 2 prefers that both players continue, but would rather be the first than the second to terminate. The instantaneous payoff matrix, $u^2$, is defined by:

Example V.2: Instantaneous Payoff matrix: $u^2$

| | $C$ | $T$ |
|---|---|---|
| $C$ | (0, 0) | ( 1, -1) |
| $T$ | $(4t - 2, -2)$ | (-4, -4) |

The profile $f$ below is the unique SGP-equilibrium for this game:

$$f_1(t, h) = \begin{cases} T & \text{if } t > ½ \text{ and } h(t) = CC \\ T & \text{if } h_1(t) = T \\ C & \text{otherwise} \end{cases}$$

$$f_2(t, h) = \begin{cases} T & \text{if } t = ½ \text{ and } h(½) = CC \\ T & \text{if } h_2(t) = T \\ C & \text{otherwise} \end{cases}$$

The outcome generated by this profile is that player 2 jumps at $t = ½$, while the player 1 never jumps. Player 1's payoff is ½, while 2's is $-½$. We now argue that $f$ is an equilibrium. Note that player 1 clearly has no incentive to preempt player 2. Also, player 2's best response to $f_1$ is to wait until "the last moment" before 1 would terminate (i.e., ½) and then preempt. This shows that $f$ is a Nash equilibrium. The only nontrivial subgames are the family $(t, h^{CC})_{t>½}$. Since player 2 does not plan to move along this branch, player 1's best action is certainly to terminate immediately. Since player 2 cannot preempt 1 in any such subgame,

his best response is to play "continue" until the end of the game. This verifies that $f$ is indeed an SGP-equilibrium.

## VI. Equilibria of Discrete- and Continuous Time Games.

In this section, we investigate the continuity properties of the SGP equilibrium correspondence. Our first result is a weak upper-hemi-continuity property: for $f \in F$, if agents' valuation functions are $d^H$-continuous and if the graph-preserving restrictions of $f$ to some sequence of grids are $\epsilon^n$-equilibria, with $\epsilon^n \underset{n}{\to} 0$ as the grids become finer, then $f$ will be an exact equilibrium in continuous time. A natural conjecture is the converse result: if $f$ is a SGP equilibrium in continuous time and $R$ is a very fine grid, then $f^R$ will be an approximate equilibrium for the corresponding game on $R$. This conjecture is false, but a slightly weaker statement is true: if $f$ is an SGP equilibrium, and $R$ is a very fine grid, there exists an approximate equilibrium, $g$, for the corresponding game on $R$ such that that from every subgame, the outcome generated by $g$ is close to the outcome generated by $f^R$ from a nearby subgame.

Restated formally, our first result is:

*Theorem III:* Assume that agents' valuation functions are $d^H$-continuous. Fix $f \in F$, and a sequence $(R^n)$ in $\mathbf{R}$ such that $\delta(R^n) \underset{n}{\to} 0$. Suppose that there exists a sequence $(\epsilon^n)$, $\epsilon^n \to 0$, such that for all $n$ sufficiently large, $f^{R^n}$ is an $\epsilon^n$-SGP equilibrium for the game played on $R$. Then $f$ is an SGP-equilibrium for the continuous-time game.

The following example shows that the converse of this Theorem is false. The example is a two-person termination game defined by the following instantaneous payoff matrix.

| Example VI.1: Instantaneous Payoff matrix. | | |
|---|---|---|
| | *CONTINUE* | *TERMINATE* |
| *CONTINUE* | (0, 0) | (0, $4t-2$) |
| *TERMINATE* | (1, $4t-3$) | (-1, 0) |

The profile $f$ defined below is an SGP equilibrium for this game.

$$f_1(t,h) = \begin{cases} C & \text{if } t = 0 \\ C & \text{if } h_1(t) = C \\ T & \text{if } h_1(t) = T \end{cases}; \quad f_2(t,h) = \begin{cases} C & \text{if } t = 0 \\ C & \text{if } t < \tfrac{1}{2} \text{ and } h(t) = CC \\ C & \text{if } t \geq \tfrac{1}{2} \text{ and } h(t) = TC \\ T & \text{otherwise} \end{cases}$$

The outcome generated by this profile is that player #2 jumps at $t = \frac{1}{2}$, while player #1 never jumps. Player #1's payoff is zero, while #2's is ½.

Player #1 is deterred from terminating before ½ by #2's (credible) threat to follow suit immediately afterwards. For any $R$, however, $f^R$ is not even an approximate equilibrium for the corresponding game played on $R$. Player #1 has a substantially better response than $f_1^R$ against $f_2$: he can terminate at the last grid point in $R$ strictly before ½. Player #2 has no opportunity to react to this deviation before ½, at $t \geq \frac{1}{2}$, however, $f_2^R$ instructs him to continue. The defection thus yields #1 a payoff slightly exceeding ½.

The above example shows that when a continuous-time profile is restricted to a discrete-time grid, the strategic opportunities available to agents may be significantly altered. Given any grid, $R$, however, there exists a profile, $g$, on $DN^R$, that does preserve the basic "strategic flavor" of $f$. From every $R$-measurable subgame, $g$ generates an outcome that is close to the outcome generated by $f$ from some nearby subgame. Moreover, if $R$ is a very fine grid, $g$ will be an approximate equilibrium for the game played on $R$. The key to the construction of $g$ is: if either agent deviates from the equilibrium path (which has #2 ter-

minating at ½), the other player follows the path *that he would have been following* , had the deviation occured in the continuous time game. In particular, suppose that when the game is played on $R$, player #1 terminates at some time $r < ½$; in continuous-time, #2 would have responded immediately; therefore, $g_2$ instructs him to play $T$ at $\lceil r \rceil^R$ (even if $\lceil r \rceil^R > ½$). If $R$ is a very fine grid, the cost to #2 of modifying his strategy in this way will be negligible: at $r \approx ½$, he is nearly indifferent between continuing and terminating. Precisely, the profile $g$ on $DN^R$, defined below is an $\epsilon$-SGP for the game played on $R$, where $\epsilon = \int_{\lceil ½ \rceil^R}^{1} (4s - 3)ds$:

$$g_1(t_r, h) = \begin{cases} C & \text{if } r = 0 \\ C & \text{if } h_1(r) = C; \\ T & \text{if } h_1(r) = T \end{cases} \quad g_2(t_r, h) = \begin{cases} C & \text{if } r = 0 \\ C & \text{if } r < ½ \text{ and } h(r) = CC \\ C & \text{if } r \geq \lceil ½ \rceil^R \text{ and } h(r) = TC \\ T & \text{otherwise} \end{cases}$$

If agents' valuation functions satisfy assumption, (V1) above, and two additional assumptions specified below, the technique just described can be generalized to obtain the following lower-hemi-continuity result. Given such a game, suppose that $f \in F$ is an SGP-equilibrium in continuous time. For every positive $\epsilon$, there exists $\delta$ such that if $\delta(R) < \delta$, an $\epsilon$-SGP equilibrium, $g$, can be constructed for the game played on $R$ such that from every $R$-measurable subgame, the outcome generated by $g$ is within $\epsilon$ of the outcome generated by $f$ from a "nearby subgame."

Formally, for $f \in F$ and $g \in F^R$, we will say that $g$ <u>$\epsilon$-approximates</u> $f$ if:

$$d^H(o(R, g, 0, \varnothing), o([0, 1], f, 0, \varnothing)) < \epsilon \quad \text{and}$$

$$\forall (t_r, h_{|t_r}) \in DN^R \sim \{(0, \varnothing)\}, \exists (\dot{t}, \eta) \in DN \text{ s.t. } \begin{cases} |t - r| < \epsilon, \\ d^H(h, \eta) < \epsilon \text{ and} \\ d^H(o(R, g, t_r, h), o([0, 1], f, t, \eta)) < \epsilon \end{cases}$$

We can now state the result:

*Theorem IV:* Suppose that for all $i$, $V_i$ satisfies V1, and V2 and V3 specified below. Let $f$ be a continuous-time SGP equilibrium. Then

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. if } \delta(R) < \delta, \exists g \in F^R \text{ s.t.:}$$
$$g \text{ is an } \epsilon\text{-SGP equilibrium for the game played on } R \text{ and}$$
$$g \ \epsilon\text{-approximates } f.$$

We now specify the additional restrictions on valuation functions. The first of these, (V2), endogenizes restrictions F2 and F3 on strategies. We assume that it is prohibitively costly for agents to make two or more jumps in rapid succession. We first define the function $\rho_i: H \times [0, 1) \to \mathbb{R}_+$ as follows: $\rho_i(h, s)$ is the smallest distance between two successive discontinuity points of $h_i$, at least one of which occurs after $s$. Formally,

$$\rho_i(h, s) = \begin{cases} \min\{(t - t'): t, t' \in D(h_i), t < t' \text{ and } t' \in [s, 1]\} & \text{if } \#D(h_i) \cap [s, 1] \geq 1 \\ 1 & \text{otherwise} \end{cases}$$

We can now state (V2):

$$\exists \epsilon^{IR} \in (0, 1) \text{ s.t. } \forall i, \forall (h, s) \in H \times [0, 1), \quad \rho_i(h, s) < \epsilon^{IR} \text{ implies } V_i(h) < V_i(h_{i|s}, h_{-i}) - 1. \tag{V2}$$

("IR" stands for "individually rational.") For example, the following valuation function fails (V2): $V_i(h) = \#D(h_i)$. (This function is uniformly continuous w.r.t. $d^H$.) On the other hand, let $V_i$ be any bounded function from $H$ to $\mathbb{R}$. The following function $W_i$, defined from $V_i$, satisfies the condition. For $h \in H$,

$$W_i(h) = \begin{cases} V_i(h) - \sum_{n=2}^{n(h_i)} \frac{1}{s^n(h_i) - s^{n-1}(h_i)} & \text{if } n(h_i) > 1 \\ V_i(h) & \text{if } n(h_i) \leq 1 \end{cases}$$

(Recall that $s^n(h)$ is the $n$'th discontinuity point of $h$.)

(V3) states that agents' payoffs are not too sensitive to other agents' actions at the very end of the game. Formally,

$$\lim_{\delta \downarrow\downarrow 0} \sup\{ \, | V_i(h) - V_i(h') | : h_i = h_i' \text{ and } h_{-i|1-\delta} = h'_{-i|1-\delta} \, \} = 0. \tag{V3}$$

If $V_i$ is determined by integrating an instantaneous flow payoff matrix, (V3) will clearly be satisfied. More generally, however, the assumption is nontrivial. For example, consider a timing game that determines the parameters for a second game, to be played at time 1. In this case, the terminal state of the system could be crucial, and (V3) would be very restrictive.

To see why some such condition is needed, suppose that (V3) is violated in a game with more than 2 players. In continuous time, $i$ might be deterred from initiating a jump just before the end of the game, because of the chain of instantaneous reactions that would follow. If this chain were to be truncated, say, before the $k$'th link, $i$ might have an incentive to deviate. In discrete time, if $i$ initiates his jump $k-1$ grid points before the end of the game, the $k$'th link in the chain of reactions to reactions will never be reached. A continuous time equilibrium might, therefore, have no discrete-time analog.

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## APPENDIX.

*Proof of Lemma 1.*

*Part (a).* Suppose $\beta^r(f, t, h) = \beta^{r-1}(f, t, h)$, for $1 \leq r \leq \bar{i}$. Therefore, $h^{\beta^r(f,t,h),t} = h^{\beta^{r-1}(f,t,h),t}$, so that $\beta^{r+1}(f, t, h) = \beta^r(f, t, h)$. Now proceed by induction in the obvious way.

*Part (b).* This follows immediately from condition F4 (positive refractory time).

*Part (c)* The " *if* " part follows immediately from Part (a) and the " *only if* " part from Part (b).

*Part (d)* By Part (b), the $i$'th component of the matrix $(\beta^0(f, t, h), \ldots, \beta^{\bar{i}}(f, t, h))$ can take at most 2 different values. Therefore, if every element of $(\beta^0(f, t, h), \ldots, \beta^{\bar{i}}(f, t, h))$ is different, then for each $i$, $\beta_i^{\bar{i}}(f, t, h) \neq \beta_i^0(f, t, h)$. In this case, by (F3):

$$\lim_{\delta \downarrow\downarrow 0} f(t + \delta, h^{\beta^{\bar{i}}(f,t,h),t}) = \beta^{\bar{i}}(f, t, h).$$

Suppose now that for some $r \leq \bar{i}$, $\beta^r(f, t, h) = \beta^{r-1}(f, t, h)$. If this equality holds for $r = \bar{i}$, we are done, since

$$\beta^{\bar{i}}(f, t, h) = \lim_{\delta \downarrow\downarrow 0} f(t + \delta, h^{\beta^{\bar{i}-1}(f,t,h),t}) = \lim_{\delta \downarrow\downarrow 0} f(t + \delta, h^{\beta^{\bar{i}}(f,t,h),t}).$$

If it holds for $r < \bar{i}$, then by Part (a), it also holds for $r = \bar{i}$. □

*Proof of Proposition I:*

We first construct the unique outcome $\bar{\eta}$ that satisfies condition (4.1). We then verify that it is consistent with $f$. Fix $\bar{k} \in \mathbb{N}$, such that for all $i$ and all $h$, $f_i(\cdot, h_{\lfloor \cdot})$ has no more than $\frac{\bar{k}}{\bar{i}}$ discontinuity points. Such a $\bar{k}$ exists, by condition (F3). If $(\bar{t}, \bar{h}) = (0, \varnothing)$, set $\eta^0 = h^{f(0,\varnothing)}$. Otherwise, define $\eta^0 = \bar{h}$. Define $\tau^1$ by

$$\tau^1 = \begin{cases} \infty & \text{if } f(s, \eta^0) = \eta^0(\bar{t}),\ \forall s \in [\bar{t}, 1] \\ \bar{t} & \text{if } \{s > \bar{t}: f(s, \eta^0) = \eta^0(\bar{t})\} = \varnothing \\ \inf\{s \geq \bar{t}: f(s, \eta^0) \neq \eta^0(\bar{t})\} & \text{otherwise} \end{cases}$$

If $\tau^1 = \infty$, define $\bar{\eta} = \eta^0$. Otherwise, define $\eta^1 = (\eta^0)^{\beta^{\bar{i}}(f, \tau^1, \eta^0), \tau^1}$. Note that $\eta^1 \in H$, since it is right continuous on (0, 1) and has a finite number of discontinuity points. If $\tau^1 = 1$, define $\bar{\eta} = \eta^1$.

Now fix $1 \leq k \leq \bar{k}$, assume that $\tau^k = \inf\{s \geq \tau^{k-1}: f(s, \eta^{k-1}) \neq \eta^{k-1}(\tau^{k-1})\} < 1$ and that $\eta^k = (\eta^{k-1})^{\beta^{\bar{i}}(f, \tau^k, \eta^{k-1}), \tau^k} \in H$. Let

$$\tau^{k+1} = \begin{cases} \infty & \text{if } f(s, \eta^k) = \eta^k(\tau^k),\ \forall s \in [\tau^k, 1] \\ \inf\{s \geq \tau^k: f(s, \eta^k) \neq \eta^k(\tau^k)\} & \text{otherwise} \end{cases}$$

By definition of $\tau^k$, *either* $f(\tau^k, \eta^{k-1}) \neq \eta^{k-1}(\tau^k)$ *or* $\lim_{\delta \downarrow\downarrow 0} f(\tau^k + \delta, \eta^{k-1}) \neq \eta^{k-1}(\tau^k)$. By part (c) of Lemma 1, therefore $\beta^{\bar{i}}(f, \tau^k, \eta^{k-1}) \neq \eta^{k-1}(\tau^k)$. By part (d) of Lemma 1, there exists $\delta > 0$ such that $\forall s \in (\tau^k, \tau^k + \delta)$,

$$f(s, \eta^k) = f(s, (\eta^{k-1})^{\beta^{\bar{i}}(f, \tau^k, \eta^{k-1}), \tau^k}) = \beta^{\bar{i}}(f, \tau^k, \eta^{k-1}) = \eta^k(\tau^k).$$

Therefore, $\tau^{k+1} > \tau^k$. If $\tau^{k+1} = 1$, define $\bar{\eta} = \eta^{k+1}$. Otherwise, define $\eta^{k+1} = (\eta^k)^{\beta^{\bar{i}}(f, \tau^{k+1}, \eta^k), \tau^{k+1}}$. Once again, $\eta^{k+1} \in H$, since it is right continuous on (0, 1) and has exactly one more discontinuity point than $\eta^k \in H$. If $\tau^k = 1$, define $\bar{\eta} = \eta^k$.

Now suppose that $\tau^{\bar{k}} < 1$. By the way the $\tau^k$'s are defined, for each $k$, $\tau^k$ must be a discontinuity point of $f_i(\cdot, \eta^k_{\uparrow\cdot})$, for some $i$. By our choice of $\bar{k}$, therefore, each $f_i(\cdot, \eta^{\bar{k}}_{\uparrow\cdot})$ must have exactly $\frac{\bar{k}}{\bar{i}}$ discontinuity points. Therefore, for each $i$, $f_i(\cdot, \eta^{\bar{k}}_{\uparrow\cdot})$ must be constant on $[\tau^{\bar{k}}, 1]$. Set $\bar{\eta} = \eta^{\bar{k}}$.

We now establish that $\bar{\eta}$ satisfies condition (4.1). Pick $t, t' \in [\bar{t}, 1]$, $t < t'$. If $t = 0$, set $k = 0$, otherwise let $\tau^k$ be the last discontinuity point of $\bar{\eta}$ before $t$. By our construction of $\bar{\eta}$, $\bar{\eta}$ is constant on $(t, t')$ iff $t' \leq \tau^{k+1}$. Suppose $t' \leq \tau^{k+1}$. By definition of the $\tau$'s, we have $f(\cdot, \eta^k) = \eta^k(\tau^k)$ on $(t, t')$. Also, by our construction of $\eta^k$, $\eta^k(\cdot) = \eta^k(\tau^k)$ on $(\tau^k, \tau^{k+1})$. Finally, $\bar{\eta}_{\uparrow\cdot}$, $\bar{\eta}$ and $\eta^k$ agree on $(\tau^k, \tau^{k+1})$. Therefore, substituting in the expression above, we

have: $f(\cdot, \bar{\eta}_{|s}) = \bar{\eta}(\cdot)$ on $(t, t')$. If $t' > \tau^{k+1}$, then by definition of $\tau^{k+1}$, $f(\cdot, \bar{\eta}_{|s})$ is not constant on $(t, t')$. □

*Proof of Theorem II:*

The proof is by induction. Let $\tilde{\eta} = o([0, 1], f, \bar{t}, \bar{h})$ and let $\tilde{n} = \#(D(\tilde{\eta}) \cap (\bar{t}, 1])$, the number of discontinuity points of $\tilde{\eta}$ weakly larger than $\bar{t}$. If $\tilde{n} = 0$, the result is trivial. Assume, therefore, that $\tilde{n} > 0$. Enumerate $\bar{t} \cup \{D(\tilde{\eta}) \cap [\bar{t}, 1]$ as $(\tau^0 < \tau^1 < \cdots < \tau^k < \cdots < \tau^{\tilde{n}})$. Pick $\tilde{\epsilon} \in (0, 1)$ and $\bar{\delta} \in (0, \frac{\tilde{\epsilon}}{\bar{t}^2(\tilde{n}+1)})$ small enough that $\tau^{\tilde{n}} \neq 1$ implies $\tau^{\tilde{n}} < 1 - \bar{t}\bar{\delta}$ and:

$$f(\cdot, \tilde{\eta}) \text{ is constant on } (0 \vee (\bar{t} - \delta), \bar{t}) \cup (1 - \delta, 1); \tag{A.1a}$$

and, for $1 \leq k \leq \tilde{n}$ and $a \in A$,

$$f(\cdot, \bar{\eta}^{a,\tau^k}) \text{ is constant on } (\tau^k, \tau^k + \delta(\tilde{n}+1)\bar{t}^2); \tag{A.1b}$$

$$\forall h' \geq \bar{\eta}^{a,\tau^k} \text{ s.t. } d^H(h', \bar{\eta}^{a,\tau^k}) < \delta(\tilde{n}+1)\bar{t}^2, \forall s \in (\underline{s}(h'), 1], f(s\ h') = f(s, \bar{\eta}^{a,\tau^k}). \tag{A.1c}$$

Such a $\delta$ exists, by assumption (F2), part (d) of Lemma 1 and assumption (F4). An implication of (A.1c) is that for $1 \leq k \leq \tilde{n}$, $\tau^k > \tau^{k-1} + \delta(\tilde{n}+1)\bar{t}^2$.

Pick $R \in \mathbf{R}$ such that $\delta(R) < \bar{\delta}$ and $(\bar{t}, \bar{h}) \in DN^R$. Let $\tilde{h} = o(R, f^R, \bar{t}, \bar{h})$. We will show that $d^H(\tilde{h}, \tilde{\eta}) < \tilde{\epsilon}$. Define $t^{0,\bar{t}} = \bar{t}$, $\beta^{0,\bar{t}} = \begin{cases} f(0, \varnothing) & \text{if } \bar{t} = 0 \\ \bar{h}(\bar{t}) & \text{if } \bar{t} > 0 \end{cases}$ and $\eta^{0,\bar{t}} = \begin{cases} h^{f(0,\varnothing)} & \text{if } \tau^1 = 0 \\ \bar{h} & \text{if } \tau^1 > 0 \end{cases}$. Now, for $1 \leq k \leq \tilde{n}$ and $0 \leq r \leq \bar{t}$, define:

$$(k,r)+1 = \begin{cases}(k,r+1) & \text{if } r < \bar{i}\\ (k+1,0) & \text{if } r = \bar{i}\end{cases}$$

$$(k,r)-1 = \begin{cases}(k,r-1) & \text{if } r > 0\\ (k-1,\bar{i}) & \text{if } r = 0\end{cases}$$

$$t^{(k,r)} = \begin{cases}\tau^k & \text{if } r = 0 \text{ and } \tau^k \in R\\ \lceil \tau^k \rceil^R & \text{if } r = 0 \text{ and } \tau^k \notin R\\ \lceil t^{(k,r)-1} \rceil^R & \text{if } r > 0 \text{ and } \lceil t^{(k,r)-1} \rceil^R < 1\\ 1 & \text{if } r > 0 \text{ and } \lceil t^{(k,r)-1} \rceil^R = 1\end{cases}$$

$$\beta^{(k,r)} = \begin{cases} f(0,\varnothing) & \text{if } \tau^k = 0 \text{ and } r = 0\\ f(\tau^k, \tilde{\eta}_{|\tau^k}) & \text{if } 0 < \tau^k < 1 \text{ and } r = 0\\ \lim\limits_{\delta \downarrow\downarrow 0} f(\tau^k + \delta, \tilde{\eta}^{\beta^{r-1}(f,\tau^k,\tilde{\eta}_{|\tau^k}),\tau^k}) & \text{if } \tau^k < 1 \text{ and } 1 \le r \le \bar{i}\\ f(1, \tilde{\eta}_{|1}) & \text{if } \tau^k = 1\end{cases}$$

$$\eta^{(k,r)} = \tilde{\eta}^{\beta^{k,r},\tau^k}.$$

We first prove that for $(k,r) = \begin{cases}(1,0) & \text{if } \tau^1 = 0\\ (0,\bar{i}) & \text{if } \tau^1 > 0\end{cases}$,

$$\tilde{h}(t^{(k,r)}) = \beta^{(k,r)};$$
$$\tilde{h}_{|t^{(k,r)+1}} = \eta^{(k,r)}.$$

If $(k,r) = (1,0)$, then $\tau^1 = 0$. We have: $\tilde{h}_{|t^{(1,1)}} = \tilde{h}_{|\lceil 0 \rceil^R} = h^{f(0,\varnothing)}$. Also $\eta^{(0,\bar{i})} = h^{f(0,\varnothing)}$. Therefore, both conditions are satisfied by definition. If $(k,r) = (0,\bar{i})$, then $\tilde{h}(t^{(0,\bar{i})}) = \tilde{h}(\bar{t}) = \beta^{(0,\bar{i})}$. Also $\tilde{h}_{|t^{(0,\bar{i})+1}} = \tilde{h}_{|t^{(1,0)}} = \bar{h}_{|\bar{t}} = \eta^{(0,\bar{i})}$. The second equality holds because by definition of $\tau^1$, $f(\cdot, \bar{h})$ is constant on $[\bar{t}, \tau^1)$, so that, by definition of $t^{(1,0)}$, $f^R(\cdot, \bar{h})$ is constant on $[\bar{t}, t^{(1,0)})$. Once again, therefore, both conditions are satisfied.

*The Inductive Step:* Fix $(\tilde{n},\bar{i}) \geq (k,r) \geq \begin{cases}(1,1) & \text{if } \tau^1 = 0\\ (1,0) & \text{if } \tau^1 > 0\end{cases}$. Suppose that

$$\tilde{h}(t^{(k,r)-1}) = \beta^{(k,r)-1}; \tag{a}$$

$$\tilde{h}_{\,|\,t^{(k,r)}} \geq \eta^{(k,r)-1}; \tag{b}$$

$$d^H(\eta^{(k,r)-1}, \tilde{h}_{\,|\,t^{(k,r)}}) < \delta\bar{i}(k\bar{i} + r - 1). \tag{c}$$

Then

$$\tilde{h}(t^{(k,r)}) = \beta^{(k,r)}; \tag{i}$$

$$\tilde{h}_{\,|\,t^{(k,r)+1}} \geq \eta^{(k,r)}; \tag{ii}$$

$$d^H(\eta^{(k,r)}, \tilde{h}_{\,|\,t^{(k,r)+1}}) < \delta\bar{i}(k\bar{i} + r). \tag{iii}$$

(If $t^{(k,r)} = 1$, or $(k,r) = (\tilde{n},\bar{i})$, set $\tilde{h}_{\,|\,t^{(k,r)+1}} = \tilde{h}$.)

Since $\eta^{\tilde{n},\bar{i}} = \tilde{\eta}$, $\tilde{h}_{\,|\,t^{(\tilde{n},\bar{i})}} = \tilde{h}$, and $\delta(\tilde{n}+1)\bar{i}^2 \quad < \epsilon$, once we have proved the inductive step we have proved that $d^H(\tilde{h}, \tilde{\eta}) < \tilde{\epsilon}$ and so proved the theorem.

*Proof of the Inductive Step:* We first verify conclusion (i) of the inductive step. We have

$$\beta^{(k,r)} = \begin{cases} f(\tau^k, \eta_{\,|\,\tau^k}) = f(\tau^k, \eta^{(k-1,\bar{i})}) = f(t^{(k,0)}, \tilde{h}_{\,|\,t^{(k,0)}}) = f^R(t^{(k,0)}, \tilde{h}_{\,|\,t^{(k,0)}}) & \text{if } r = 0\\ \lim\limits_{\delta \downarrow\downarrow 0} f(\tau^k + \delta, \eta^{k,r-1}) \quad = f(t^{(k,r)}, \tilde{h}_{\,|\,t^{(k,r)}}) = f^R(t^{(k,r)}, \tilde{h}_{\,|\,t^{(k,r)}}) & \text{if } r > 0 \end{cases}.$$

If $r = 0$, the first equality follows from the definition of $\beta$; the second from observation (A.) and the fact that $\tilde{\eta}$ is consistent with $f$ (Proposition I); the third from (A.1c) and premises (b) and (c); the last from (A.1b) and the definition of $f^R$. If $r > 0$, the explanations are similar. By the definition of $\tilde{h}$, therefore, $\tilde{h}(t^{(k,r)}) = \beta^{(k,r)}$. This completes the verification of conclusion (i).

We now show that

$$\tilde{h}([t^{(k,r)}, t^{(k,r)+1})) = \beta^{(k,r)}. \tag{A.2}$$

If $r < \bar{i}$, then $t^{(k,r)+1} = \lceil t^{(k,r)} \rceil^R$, so that (A.2) follows trivially from the $R$-measurability of $\tilde{h}$. Suppose now that $r = \bar{i}$. It follows from Proposition I that $f(\cdot, \eta^{(k,r)}) = \beta^{(k,r)}$ on

$[\lceil t^{k,\bar{i}} \rceil^R, \tau^{k+1})$. Therefore, using premises (b), (c), (F4) and (A.1c), we have, for all $t \in [\lceil t^{k,\bar{i}} \rceil^R, \tau^{k+1}) \cap R$,

$$\tilde{h}(t) = f^R(t, \tilde{h}_{\mid t^{(k,r)}}) = f(t, \tilde{h}_{\mid t^{(k,r)}}) = f(t, \eta^{(k,r)}) = \beta^{(k,r)}.$$

We have verified, therefore, that (A.2) is true for all $0 \le r \le \bar{i}$.

Finally, we verify conclusions (ii) and (iii) of the inductive step. Now fix $i \in I$. If $\beta_i^{(k,r)} = \beta_i^{(k,r)-1}$, then, by (A.2), $\tilde{h}([t^{(k,r)}, t^{(k,r)+1})) = \tilde{h}_i(t^{(k,r)}-)$, so that $\tilde{h}_{i \mid t^{(k,r)+1}} = \tilde{h}_{i \mid t^{(k,r)}}$. Similarly, $\eta_i^{(k,r)} = \eta_i^{(k,r)-1}$. In this case, it follows immediately that conclusions (ii) and (iii) hold for the $i$'th components of $\tilde{h}_{\mid t^{(k,r)+1}}$ and $\eta^{(k,r)}$.

Now suppose that $\beta_i^{(k,r)} \neq \beta_i^{(k,r)-1}$. If $r > 0$, then by part (b) of Lemma 1, $\beta_i^{(k,r)-1} = \begin{cases} f_i(0, \varnothing) & \text{if } \tau^k = 0 \\ \tilde{\eta}_{i \mid \tau^k}(\tau^k) & \text{if } \tau^k > 0 \end{cases} = \beta_i^{(k-1,\bar{i})}$. For all $r$, therefore, $\beta_i^{(k,r)} \neq \beta_i^{(k,r)-1}$ implies

$$\beta_i^{(k,r)-1} = \beta_i^{(k-1,\bar{i})}. \tag{A.3}$$

Combining (A.3) with premises (b) and (c), we have

$$\tilde{h}_{i \mid t^{(k,r)}} \ge \eta^{(k-1,\bar{i})}; \tag{b'}$$

$$d_i^H(\eta^{(k-1,\bar{i})}, , \tilde{h}_{i \mid t^{(k,r)}}) < \delta\bar{i}(k\bar{i} + r - 1). \tag{c'}$$

To complete the proof of the inductive step, we need to show that

$$\tilde{h}_{i \mid t^{(k,r)+1}} \ge \eta_i^{(k,r)}; \tag{ii'}$$

$$d^H(\eta_i^{(k,r)}, \tilde{h}_{i \mid t^{(k,r)+1}}) < \delta\bar{i}(k\bar{i} + r). \tag{iii'}$$

By (A.2), $\tilde{h}_{i \mid t^{(k,r)+1}}$ has exactly one more discontinuity point than $\tilde{h}_{i \mid t^{(k,r)}}$. Also $\eta_i^{(k,r)}$ has exactly one more discontinuity point than $\eta^{(k-1,\bar{i})}$. Conclusions (ii) and (iii) now follow, because $0 \le t^{k,r} - \tau^k < \delta\bar{i}$ and, by conclusion (i), $\eta_i^{(k,r)}$ and $\tilde{h}_{i \mid t^{(k,r)+1}})$ agree on the last of their constant segments. This completes the proof of the inductive step and hence the theorem. □

*Proof of Theorem III.*

For $R \in \mathbf{R}$ and $\bar{t} \in [0, 1]$, define $[\bar{t}]^R = \begin{cases} \bar{t} & \text{if } \bar{t} \in R \\ !s+Rtbar & \text{if } \bar{t} \notin R \end{cases}$. The theorem is an immediate corollary of the following:

Lemma: $\forall \epsilon > 0$, $\forall f \in F$, $\forall (\bar{t}, \bar{h}) \in DN$, $\exists \delta > 0$ s.t. $\forall h' \in H$ and $R \in \mathbf{R}$,

$$\text{if } \begin{cases} h' \geq \bar{h} \\ d(h', \bar{h}) < \delta, \text{ then } d^H(o([0, 1], f, \bar{t}, \bar{h}), o(R, f^R, [\bar{t}]^R, h')) < \epsilon. \\ \delta(R) < \delta \end{cases}$$

It follows immediately from assumption (F4) and the definition of $o([0, 1],\cdot,\cdot,\cdot)$ that if $\delta$ is sufficiently small, if $h' \geq \bar{h}$ and $d(h', \bar{h}) < \delta$, then

$$d^H(o([0, 1], f, \bar{t}, \bar{h}), o([0, 1], f, \bar{t}, h') = d^H(\bar{h}, h').$$

To prove the Lemma, therefore, we need only prove that:

$$\forall \epsilon > 0, \forall f \in F, \forall (\bar{t}, \bar{h}) \in DN, \exists \delta > 0 \text{ s.t. } \forall R \in \mathbf{R}, \text{ if } \delta(R) < \delta, \text{ then} \quad d^H(o([0, 1], f, \bar{t}, \bar{h}), o(R, f^R, [\bar{t}]^R, \bar{h})) < \epsilon. \tag{A.4}$$

If $[\bar{t}]^R = \bar{t}$, there is, obviously, nothing to prove. Also, if $\inf\{s \geq \bar{t}: f(s, \bar{h}) \neq \bar{h}(\bar{t})\} > \bar{t}$, then the proof of (A.4) is identical to the proof of Theorem III. Assume, therefore, that $\inf\{s \geq \bar{t}: f(s, \bar{h}) \neq \bar{h}(\bar{t})\} = \bar{t}$ and that $[\bar{t}]^R > \bar{t}$. (Since $0 \in R$, we have $\bar{t} > 0$.) Let $\tilde{\eta} = o([0, 1], f, \bar{t}, \bar{h})$ and $\tilde{h} = o(R, f^R, [\bar{t}]^R, \bar{h}))$. Pick $R \in \mathbf{R}$ and let $\bar{\tau}^R = [[\bar{t}]^R]^R$, ($[[\bar{t}]^R]^R$ is the successor in $R$ of the successor in $R$ of $\bar{t}$.) Observe that by definition of $f^R$, if $\delta(R)$ is sufficiently small, then

$$\tilde{\eta}^{f(\bar{t},\bar{h}),\bar{t}} \approx \tilde{h}_{|\bar{\tau}^R} = o(R, f^R, \bar{t}, \bar{h})_{|\bar{\tau}^R}.$$

But we can now proceed to the inductive step of Theorem III. since, in the terminology of that theorem, we have established that for $(k,r) = (1,0)$,

$$\tilde{h}(t^{(k,r)}) = \beta^{(k,r)};$$
$$\tilde{h}_{|t^{(k,r)+1}} = \bar{\eta}^{(k,r)}.$$

This completes the proof of the Lemma.

Now assume that all the conditions of the Theorem are satisfied, but for some $(\bar{t}, \bar{h}) \in DN$, there exists $f_i' \in F_i$, such that

$$P_i(f, t, h) < P_i((f_i', f_{-i}), \bar{t}, \bar{h}) - 2\bar{\epsilon}.$$

A contradiction now follows immediately from the Lemma above. Pick $n$ large enough that $\epsilon^n < \epsilon$ and let $R = R^n$. Since payoffs are $d^H$-continuous, there is a positive $\delta$, and an $h' \in H^R$ such that: $\delta(R) < \delta$, $h' \geq \bar{h}$ and $d^H(\bar{h}, h') < \delta$ implies

$$P_i^R(f^R, [\bar{t}]^R, h')) < P_i^R((f_i^R, f_{-i}^R), [\bar{t}]^R, h') - \bar{\epsilon}.$$

Therefore, $f^R$ cannot be an $\epsilon^n$-equilibrium for the game played on $R$. □

*Proof of Theorem III.*

We begin with a Lemma.

*Lemma AIII.2:* Suppose that $V_i$ satisfies assumptions V1 and V3. Then

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall t, t' \in (1 - \delta, 1], \forall h, h' \in H,$$
$$\text{if } d^H((h_i, h_{-i\,|\,t}), (h_i', h'_{-i\,|\,t'}) < \delta, \text{ then } |V_i(h) - V_i(h')| < \epsilon.$$

*Proof of Lemma AIII.2:* Pick $\epsilon > 0$. By assumption V1, there exists $\delta^1 > 0$ such that $\forall$ $h, h' \in H$, if $d^H((h, h') < \delta^1$, then $|V_i(h) - V_i(h')| < \frac{\epsilon}{3}$. By assumption V3, there exists $\delta^2 > 0$ such that $\forall h, h' \in H$, $\forall t \in (1 - \delta, 1]$, if $h = h'$ and $h_{-i\,|\,t} = h'_{-i\,|\,t}$, then $|V_i(h) - V_i(h')| < \frac{\epsilon}{3}$. Set $\delta = \delta^1 \wedge \delta^2$ and pick $t$, $t'$, $h$ and $h'$ satisfying the assumptions of the Lemma. Then:

$$|V_i(h_i, h_{-i}) - V_i(h_i, h_{-i\,|\,t})| < \frac{\epsilon}{3} \quad \text{since } \delta \leq \delta^2;$$
$$|V_i(h_i, h_{-i\,|\,t}) - V_i(h_i', h'_{-i\,|\,t'})| < \frac{\epsilon}{3} \quad \text{since } \delta \leq \delta^1;$$
$$|V_i(h_i', h'_{-i\,|\,t'}) - V_i(h_i', h'_{-i})| < \frac{\epsilon}{3} \quad \text{since } \delta \leq \delta^2.$$

Therefore,

$$|V_i(h_i, h_{-i}) - V_i(h_i', h'_{-i})| = |V_i(h) - V_i(h')| < 3\frac{\epsilon}{3} = \epsilon. \qquad \square$$

Fix $\epsilon* > 0$. Assume w.l.o.g. that $\epsilon* < \frac{1}{2}$. Now choose $\delta^{V1}$ and $\delta^{LIII.2}$ such that

$$\forall h, h' \in H, \text{ if } d^H(h, h') < \delta^{V1}, \text{ then } \forall i, \ |V_i(h) - V_i(h')| < \frac{\epsilon*}{3}.$$

$$\forall t, t' \in (1 - \delta^{LIII.2}, 1], \forall h, h' \in H, \forall i,$$

$$\text{if } d^H((h_i, h_{-i\,|\,t}), (h_i', h_{-i\,|\,t'}) < \delta^{LIII.2}, \text{ then } |V_i(h) - V_i(h')| < \frac{\epsilon*}{3}.$$

Set $\tilde{\delta} = \delta^{V1} \wedge \delta^{LIII.2} \wedge \epsilon*$. Now choose $R \in \mathbf{R}$ such that $\delta(R) < \dfrac{\tilde{\delta}\epsilon^{IR}}{3\bar{i}^2(\bar{i} + 1)(\frac{1}{\epsilon^{IR}} + 1)}$. We will now define $\delta* = \delta*(R, \bar{f}) > 0$.

Define $S_R^1$: $2^H \to 2^H$ by:

$$S_R^1(B) = B \cup \{h' \in H: h' = \lim_{s \downarrow\downarrow t} o([0, 1], \bar{f}, s, h^{a,t}) \text{ where}$$

$$h \in B, t \in \{D(B) \cup R\} \sim \{1\} \text{ and } a \in A\}$$

$$\cup \{h' \in H: h' = o([0, 1], \bar{f}, t, h) \text{ where } h \in B, t \in R\}.$$

Inductively define the $k$-fold composition of $S_R^1$ by:

$$S_R^k(B) = S_R^1(S_R^{k-1}(B))$$

If $B$ is a finite set and $\bar{f}$ is an SGP strategy profile of a game in which payoffs satisfy assumption V2, then

$$\forall k > \bar{k} = \bar{i}(\frac{1}{\epsilon^{IR}} + 1) + \bar{\rho} + 1, , S_R^k(B) = S_R^{\bar{k}}(B).$$

We set $S = S_R^{\bar{k}}(H^R)$ $S' = \{(t, h_{\,|\,t}): t \in D(S) \text{ and } h \in S\}$. We will now define $\delta* = \delta*(S) > 0$.

First choose $\delta' > 0$ such that the three conditions below hold:

(*i*) $\forall (t, h) \in S'$ s.t. $t < 1$, $\forall 0 < \delta < \delta'$, $\forall a \in A$,
$o([0, 1], \bar{f}, t + \delta, h^{a,t}) = \lim_{s \downarrow\downarrow t} o([0, 1], \bar{f}, s, h^{a,t})$ on $(t + \delta, 1]$

(*ii*) $\forall h \in S$, $\forall t \in D(S)$, if $d^H(h_{\,|\,t}, h') < \delta'$ and $h' \geq h_{\,|\,t}$, then
$o([0, 1], \bar{f}, t, h_{\,|\,t}) = o([0, 1], \bar{f}, t, h')$ on $(\underline{s}(h') \wedge t, 1]$
$\bar{f}_i(s, h) = \bar{f}_i(s, h'), \forall s > \underline{s}(h')$.

(*iii*) $\delta' < \min\{|s - t|: s, t \in D(S) \text{ and } s \neq t\}$

We can pick $\delta'$ to satisfy:

(i) by the finiteness of $S$, assumptions F1 and F3 on strategies;

(ii) by Assumption F3 and the finiteness of $S$ and the obvious inductive argument;

(iii) by finiteness of $S$ and the fact that each $h \in S$ has only finitely many discontinuities.

Now define $\delta*$ by:

$$\delta* = \frac{\delta'}{3\bar{i}(\bar{i}+1)(\frac{1}{\epsilon^{IR}}+1)}$$

We now specify an algorithm that associates to the SGP profile $\bar{f}$ an $R$-measurable strategy $\tilde{f}$ that $\epsilon^*$-approximates $\bar{f}$ on $DN^R$ and is an $\epsilon*$-SGP equilibrium for the game played on $R$. The reader will notice that the verbal description of the steps in the algorithm often deal with fewer special cases than the numbered steps. This discrepancy arises because the point $1 \in [0, 1]$ is handled anomalously. The numbered steps deal with 1 as a special case, but these details are omitted from the written descriptions.

We first define two operations on discrete-time decision nodes, "succ" and "pred." For $R = \{0 = t_1, t_2, \ldots, t_{\bar{\rho}-1}, t_{\bar{\rho}} = 1\} \in \mathbf{R}$ and $(t_r, h_{|t_r}) \in DN^R$, we define:

$\text{succ}(t_r, h_{|t_r}) = \{(t_{r+1}, h'_{|t_{r+1}}): h'_{|t_r} = h_{|t_r}\}$.
$\text{pred}(t_{r+1}, h'_{|t_{r+1}}) = (t_r, h_{|t_r})$

Also, for each $i \in I$ and $(t, h) \in DN$, define $\tau_{-i}(t, h) = \inf\{s \geq t: f_{-i}(s, h) \neq h_{-i}(t)\}$.

We begin by calculating $\eta = o([0, 1], \bar{f}, 0, \varnothing)$. We then construct an $R$-measurable history $h \in H^R$ that is $d^H$-close to $\eta$ and "build" $\tilde{f}$ so that $h = o(R, \tilde{f}, 0, \varnothing)$. To do this, we define $\tau = \{\tau^1 < \cdots < \tau^j < \cdots < \tau^{last}\}$ to be the set of discontinuity points of $\eta$ (i.e., $D(\eta)$) union $\{0, 1\}$. Step 1 (lines 60-120) and step 2 (lines 200-250) are a two-stage procedure to locate an $h \in H^R$ very close to $\eta$.

In the first step, we define a mapping $G$ from $R$ to [0, 1], with the interpretation that for $t \in [0, 1]$, $G^{-1}(t)$ is the set of discrete-time nodes in $R$ corresponding to the continuous-time node $t$. If $t = \tau^j$, for some $\tau^j \in \tau$, then $G^{-1}$ associates $(\bar{i} + 1)$ distinct discrete-time nodes to $t$; otherwise, $G^{-1}(t)$ is a singleton set.

Next, $h$ is defined by induction. We first "expand" the chain generated by $\bar{f}$ at $(\tau^1, \eta_{|\tau^1})$, $\beta(\bar{f}, \tau^1, \eta_{|\tau^1})$, as follows: let $\beta^{1,r}$ denote the $r$'th entry in this matrix and set $h$ equal to $\beta^{1,r}$ between the $r$'th and $r+1$'th time nodes in $G^{-1}(\tau^1)$. (This technique is used throughout the algorithm: a chain of actions and reactions that follow each other instantaneously in continuous time is "stretched out" into a matching "cascade" of discrete-time actions and reactions, that follow each other as quickly as the discrete-time grid permits.) Next, set $h = \beta^{1,\bar{i}}$ between $G^{-1}(\tau^1)$ and $G^{-1}(\tau^2)$. Now proceed inductively from $\tau^2$ to $\tau^{last}$. Since the grid is extremely fine and $\eta$ can have at most $\bar{i}(\frac{1}{\epsilon^{IR}} + 1)$ discontinuity points (otherwise, if payoffs satisfy assumption V2, $\bar{f}$ could not be an SGP), it will be straightforward to verify that $\eta \approx h$ (see below).

Step 3 of the algorithm (lines 300-370) simultaneously builds a map "$m$" from $DN^R$ to $DN$, and the discrete-time profile, $\tilde{f}$. For each $t_k \in R$, $m(t_k, h_{|t_k})$ is a continuous-time decision node that is very close to $(t_k, h_{|t_k})$. Specifically, if $t_k$ is the $r$'th element of $G^{-1}(\tau^j)$, for $r \geq 1$, then $m(t_k, h_{|t_k}) = (\tau^j + \delta*, \eta^{\beta^{j,r-1},\tau^j})$; otherwise, $m(t_k, h_{|t_k}) = (t_k, \eta_{|t_k})$. Also, for each $t_k \in R$, we set $\tilde{f}(t_k, h_{|t_k}) = h(t_k)$. Clearly, for all $t_k \in R$, $h = o(R, \tilde{f}, t_k, h_{|t_k}) = o(R, \tilde{f}, 0, \varnothing)$. Also, by the definition of $\delta*$, we have, for all $i$: $d^H(\eta, o([0, 1], \tilde{f}, m(t_k, h_{|t_k})) < \delta*$ (recall that $\eta = o([0, 1], \tilde{f}, 0, \varnothing)$. Since $\eta$ and $h$ differ on at most $\bar{i}(\bar{i} + 1)(\frac{1}{\epsilon^{IR}} + 1)$ grid points of $R$, $\tilde{f}$ is indeed a good approximation to $\bar{f}$ at each $(t_k, h_{|t_k})$.

*The Algorithm: defining $G$, $h$, $\tilde{f}$ and $m$.*

Steps 1 and 2 use temporary constructions that will be used to define $\tilde{f}$ and $m$. In the later, inductive steps of the algorithm, $G$ and $h$ will be redefined, relative to new $\eta$'s.

05 $[\cdot]\colon [0, 1] \to R$ is defined by: $[t] := \min\{r \in R\colon t \leq r\}$.

10 $\eta \triangleq o([0, 1], \bar{f}, 0, \varnothing)$.

20 Temporarily define $\{\tau^1 < \cdots < \tau^j < \cdots < \tau^{last}\}$ by:

$$\{0 = \tau^1 < \cdots < \tau^j < \cdots < \tau^{last} = 1\} = D(\eta) \cup \{0, 1\}$$

30 For $0 \leq r \leq \bar{i}$ and $1 \leq j \leq last$, $\beta^{j,r} \triangleq \begin{cases} \beta^r(\bar{f}, 0, \varnothing) & \text{if } \tau^j = 0 \\ \beta^r(\bar{f}, \tau^j, \eta_{|\tau^j}) & \text{otherwise} \end{cases}$

55 $\alpha \triangleq 0$ | $\alpha$ counts branches;

56 $\gamma \triangleq 0$ | $\gamma$ counts $Dev_\alpha^{Lit}$;

57 $\sigma \triangleq 0$ | $\sigma$ counts $Dev_\alpha \sim Dev_\alpha^{Lit}$;

*Step 1:* Defining $G$ (this is a temporary construction). $G$ will define a correspondence from $R$ to $[0, 1]$. It will be defined by doing an induction through $R$. For all $t_r < 1$, $G(t_r)$ will be a singleton set. At various steps in the construction, $t_\kappa$ will denote the point in $R$ from which the construction of $G$ will proceed in later steps.

60 $\kappa \triangleq 1$

70 $j \triangleq 1$ | $j$ is a counter in the definition of $G$;

71 $G(t_{\bar{\rho}}) \triangleq \varnothing$

80 If $\tau^j = 1$, then

($i$) $G(t_{\bar{\rho}}) \triangleq G(t_{\bar{\rho}}) \cup \{\tau^j\}$

($ii$) go to 200 .

When the algorithm goes to 200, the process of defining $G$ will be done.

83 If $\kappa + \bar{i} \geq \bar{\rho}$, then

($i$) for $\kappa \leq k < \bar{\rho}$, $G(t_k) \triangleq \tau^j$;

($ii$) $G(t_{\bar{\rho}}) \triangleq G(t_{\bar{\rho}}) \cup \{\tau^j\}$;

($iii$) $\kappa \triangleq \bar{\rho}$;

($iv$) go to 110 .

85 If $\kappa + \bar{i} < \bar{\rho}$, then

($i$) $G(\{t_\kappa, t_{\kappa+1}, \ldots, t_{\kappa+\bar{i}}\}) \triangleq \tau^j$.

($ii$) $\kappa \triangleq \kappa + \bar{i} + 1$.

90 If $\tau^j < 1$ and $(t_{\kappa-1}, \tau^{j+1}) \cap R = \emptyset$, go to 110.

100 *If* $\tau^j < 1$ and $(t_{\kappa-1}, \tau^{j+1}) \cap R \neq \emptyset$, then

($i$) for $t_r \in (t_{\kappa-1}, \tau^{j+1}) \cap R$, $G(t_r) \triangleq t_r$

($ii$) temporarily define $\kappa$ *by*: $t_\kappa \triangleq [\tau^{j+1}]$

110 $j \triangleq j + 1$

120 Go to 80.

*Step 2:* Defining $h$ (another temporary construction).

200 $j \triangleq 1$

205 Let the temporary indices $k, k+1, \ldots, k+n$ be defined by:

$$G^{-1}\{\tau^j\} = \{t_k, t_{k+1}, \ldots, t_{k+n}\}.$$

210 If $k + \bar{i} \geq \bar{\rho}$, then

($i$) for integer $r$ s.t. $k \leq k + r < \bar{\rho}$, $h([t_{k+r}, t_{k+r+1})) \triangleq \beta^{j,r}$;

($ii$) $h(t_{\bar{\rho}}) \triangleq \beta^{last,\bar{i}}$;

($iii$) go to 300

215 If $k + \bar{i} < \bar{\rho}$, then for integer $r$ s.t. $k \leq k + r < k + n$, $h([t_{k+r}, t_{k+r+1})) \triangleq \beta^{j,r}$;

220 *If* $G^{-1}(\tau^j, \tau^{j+1}) \neq \emptyset$, then

($i$) define the temporary indices $k, k+1, \ldots, k+n$ by:

$$G^{-1}(\tau^j, \tau^{j+1}) = \{t_k, t_{k+1}, \ldots, t_{k+n}\}.$$

($ii$) $h([t_k, t_{k+n+1})) \triangleq \beta^{j,\bar{i}}$;

240 $j \triangleq j + 1$

250 Go to 205.

*Step 3:* Defining $\tilde{f}$ and $m$.

300 $j \geq 1$

305 Let the temporary indices $k, k+1, \ldots, k+r, \ldots, k+n$ be defined by:

$$G^{-1}\{\tau^j\} = \{t_k, t_{k+1}, \ldots, t_{k+r}, \ldots, t_{k+n}\}.$$

330 *If* $t_k = 0$, then

(*i*) $m(t_k, \varnothing) := (0, \varnothing)$ and $\tilde{f}(t_k, \varnothing) := h(0) = \eta(0) = \bar{f}(m(t_k, \varnothing))$.

(*ii*) for $t_k < t_{k+r} \leq t_{k+n}$,

$m(t_{k+r}, h_{|t_{k+r}}) := (\tau^j + \delta *, \eta^{\beta^{j,r-1},\tau^j})$

and $\tilde{f}(t_{k+r}, h_{|t_{k+r}}) := h(t_{k+r}) = \bar{f}(m(t_{k+r}, h_{|t_{k+r}})) = \beta^{j,r}$

(*iii*) go to 360

340 *If* $0 < t_k < t_{k+n} < t_{\bar{p}} (= 1)$, then

(*i*) $m(t_k, h_{|t_k}) := (\tau^j, \eta_{|\tau^j})$ and $\tilde{f}(t_k, h_{|t_k}) := h(t_k) = \bar{f}(m(t_k, h_{|t_k})) = \beta^{j,0}$

for $t_k < t_{k+r} \leq t_{k+n}$,

(*ii*) $m(t_{k+r}, h_{|t_{k+r}}) := (\tau^j + \delta *, \eta^{\beta^{j,r-1},\tau^j})$

and $\tilde{f}(t_{k+r}, h_{|t_{k+r}}) := h(t_{k+r}) = \bar{f}(m(t_{k+r}, h_{|t_{k+r}})) = \beta^{j,r}$

(*iii*) go to 360

350 *If* $0 < t_k < t_{k+n} = t_{\bar{p}} (= 1)$, then

(*i*) $m(t_k, h_{|t_k}) := (\tau^j, \eta_{|\tau^j})$ and $\tilde{f}(t_k, h_{|t_k}) := h(t_k) = \bar{f}(m(t_k, h_{|t_k})) = \beta^{j,0}$

(*ii*) for $t_k < t_{k+r} < t_{\bar{p}}$,

$m(t_{k+r}, h_{|t_{k+r}}) := (\tau^j + \delta *, \eta^{\beta^{j,r-1},\tau^j})$ and

$\tilde{f}(t_{k+r}, h_{|t_{k+r}}) := h(t_{k+r}) = \bar{f}(m(t_{k+r}, h_{|t_{k+r}})) = \beta^{j,r}$

(*iii*) $m(t_{\bar{p}}, h_{|t_{\bar{p}}}) := (\tau^j + \delta *, \eta^{\beta^{j,n-1},\tau^j})$ and $\tilde{f}(t_{\bar{p}}, h_{|t_{\bar{p}}}) := h(t_{\bar{p}})$

$(h(t_{\bar{p}})$ not neccesarily $= \bar{f}(m(t_{\bar{p}}, h_{|t_{\bar{p}}}))$

(*iv*) go to 394

355 *If* $0 < t_k = t_{k+n} = t_{\bar{p}} (= 1)$ and $j > 1$, then

(*i*) $m(t_{\bar{p}}, h_{|t_{\bar{p}}}) := (\tau^j, \eta_{|\tau^j})$ and

$\tilde{f}(t_{\bar{p}}, h_{|t_{\bar{p}}}) := h(t_{\bar{p}}) = \beta^{last,\bar{i}}$ (not necessarily$= \bar{f}(m(t_{\bar{p}}, h_{|t_{\bar{p}}}))$

(*ii*) go to 394

356 *If* $0 < t_k = t_{k+n} = t_{\bar{p}} (= 1)$ and $j = 1$, then

(*i*) $m(t_{\bar{p}}, h_{|t_{\bar{p}}})$ was defined on either line 1310 or 1530 and

$\tilde{f}(t_{\bar{p}}, h_{|t_{\bar{p}}}) := h(t_{\bar{p}}) = \beta^{last,\bar{i}}$ (not necessarily = $\bar{f}(m(t_{\bar{p}}, h_{|t_{\bar{p}}}))$)

(*ii*) go to 394

360 *If* $G^{-1}(\tau^j, \tau^{j+1}) = \varnothing$, then

(*i*) $j \triangleq j + 1$

(*ii*) go to 305

370 *If* $G^{-1}(\tau^j, \tau^{j+1}) \neq \varnothing$, then for all $t_r \in G^{-1}(\tau^j, \tau^{j+1})$,

(*i*) $m(t_r, h_{|t_r}) := (G(t_r), \eta_{|G(t_r)} = (t_r, \eta_{|t_r})$;

(*ii*) $\tilde{f}(t_r, h_{|t_r}) := h(t_r) = \bar{f}(m(t_r, h_{|t_r}))$.

380 $j \triangleq j + 1$

390 Go to 310.

So far, we have defined $\tilde{f}$ on only a small subset of $DN^R$, the branch $BR_0 = (t_k, h_{|t_k})_{t_k \in R}$. We now expand the domain of definition of $\tilde{f}$ inductively. We first (lines 1000-1020) consider the set of discrete-time decision nodes, $Dev_1$, that are "one step removed" from $BR_0$, that is, the set of nodes $(t_k, h_{|t_k})$ such that $pred(t_k, h_{|t_k}) = (t_{k-1}, h_{|t_{k-1}}) \in BR_0$. We divide these nodes into two subsets. The first subset, called $Dev_1^{LIT}$, contains nodes reached either by a simultaneous deviation by two or more agents from $\tilde{f}$ along $BR_0$, or by some agent moving twice in succession at an individually irrational rate (i.e., faster than $\epsilon^{IR}$). Any $(t_r, h) \in Dev_1^{LIT}$ is "treated literally," i.e., we set $m(t_r, h) = (t_r, h)$. For each such node, we repeat the procedure described above, i.e., Steps 1-3 (see lines 1040 and 1280-1350).

For the second subset, we proceed in a more delicate fashion. The basic procedure is similar to the one described above: we map each $(t_r, h_{|t_r})$ to a close continuous-time node

$m(t_r, h_{|t_r})$; set $\eta = o([0, 1], \bar{f}, m(t_r, h))$; define $\tau$ and the $\beta$'s as above; proceed to generate an $h$ close to $\eta$ and specify that $\tilde{f}$ "follow" $h$ as above. The delicate step is the choice of $m(t_r, h_{|t_r})$. We distinguish between two kinds of deviations, "passive" (line 1525) and "all others" (line 1530).

Consider a node $(t_k, h_{|t_k})$, reached from $pred(t_k, h_{|t_k}) = (t_{k-1}, h_{|t_{k-1}}) \in BR_0$, in the following way: at $(t_{k-1}, h_{|t_{k-1}})$, $\tilde{f}$ called for $i$ and no other agent to jump and $(t_k, h_{|t_k})$ was reached because no agent moved. We call this a "passive" deviation by agent $i$. To define $m(t_k, h_{|t_k})$, we define $(\hat{t}, \hat{\eta}) = m(t_{k-1}, h_{|t_{k-1}})$ and calculate what would have happened in continuous time, had agent $i$ failed to jump at any point in $[\hat{t}, t_k]$. If no other agent moves in this interval (i.e., if $\tau_{-i}(\hat{t}, \hat{\eta}) \geq t_k$, then we set $m(t_k, h_{|t_k}) = (t_k, \hat{\eta})$. Otherwise, $(\tau_{-i}(\hat{t}, \hat{\eta}) < t_k)$, we set $m(t_k, h_{|t_k}) = (\tau_{-i}(\hat{t}, \hat{\eta}), \hat{\eta})$.

All other non-literal deviations are handled on line 1530. These involve agent $i$ jumping at $t_{k-1}$, when $\tilde{f}_i$ specified either that $i$ not jump, or jump to some other action. In these instances, $m(t_k, h_{|t_k})$ is chosen to ensure that when $\eta = o([0, 1], \bar{f}, m(t_k, h_{|t_k}))$ is calculated, it is as if all the agents reacted instantaneously to "the corresponding" deviation in continuous time at $(\hat{t}, \hat{\eta}) = m(t_{k-1}, h_{|t_{k-1}})$. (For details, see line 1530; to understand why we can use $\hat{t} + \delta*$, see lines (i) and (ii) above when we chose $\delta*$.)

We define $BR_1$ denote the set of decision nodes for which $\tilde{f}$ has been defined so far. Having defined $\tilde{f}$ on $BR_{\alpha-1}$, we can then define it as above on $BR_\alpha$, the set of nodes reached by single step deviations from $BR_{\alpha-1}$. This completes the algorithm.

*The Algorithm: the inductive step.*

Lines 394-399 define counters that control the process of working through $Dev_1$, and later $Dev_\alpha$, as just described.

394 If $\gamma = 0$ then go to 1000 .

395 If $0 < \gamma < \bar{\gamma}$, then go to 1300 .

396 If $\gamma = \bar{\gamma}$, then go to 1500 .

398 If $0 < \sigma < \bar{\sigma}$, then go to 1515 .

399 If $\sigma = \bar{\sigma}$, then go to 1000 .

1000 $Br_\alpha := \{(t_k, h_{|t_k}) \in DN^R : \tilde{f}(t_k, h_{|t_k})$ has been defined so far in the construction$\}$.

1010 $\alpha \triangleq \alpha + 1$

1020 $Dev_\alpha := \text{succ}(Br_{\alpha-1}) \sim Br_{\alpha-1}$

1030 If $Dev_\alpha = \varnothing$, then go to 2000

When the algorithm goes to 2000, the construction will be complete.

1040 $Dev_\alpha^{Lit} := \{(t_k, h_{|t_k}) \in DN^R$: either $(i)$ or $(ii)$ below hold:

$(i)\ \#\{i \in I : \tilde{f}_i(t_{k-1}, h_{|t_{k-1}}) \neq h_i(t_{k-1})\} > 1$

$(ii)\ \exists i \in I$ s.t. $t_{k-1} \in D(h_{i|t_k})$ and the distance between the last two jumps of $h_{i|t_k} < \epsilon^{IR}\}$

1050 $Dev_\alpha^{Pass,i} := \{(t_k, h_{|t_k}) \in Dev_\alpha \sim Dev_\alpha^{Lit}$: both $(i)$ and $(ii)$ below hold:

$(i)\ \tilde{f}_{-i}(t_{k-1}, h_{|t_{k-1}}) = h_{-i}(t_{k-1}) = h_{-i}(t_{k-2})$;

$(ii)\ \tilde{f}_i(t_{k-1}, h_{|t_{k-1}}) \neq h_i(t_{k-1}) = h_i(t_{k-2})\}$.

1280 $\gamma \triangleq 0$

1290 Enumerate $Dev_\alpha^{Lit}$ as $\{(t_{r_1}, h\restriction_{t_{r_1}}), \ldots, (t_{r_\gamma}, h\restriction_{t_{r_\gamma}}), \ldots, (t_{r_{\bar{\gamma}}}, h\restriction_{t_{r_{\bar{\gamma}}}})\}$

This temporarily defines $\bar{\gamma}$.

1300 $\gamma \triangleq \gamma + 1$

1315 $\kappa \triangleq r_\gamma$

1310 $m((t_{r_\gamma}, h\restriction_{t_{r_\gamma}})) := (t_{r_\gamma}, h\restriction_{t_{r_\gamma}})$

1320 $\eta := o([0, 1], \bar{f}, t_{r_\gamma}, h\restriction_{t_{r_\gamma}})$

1330 If $t_{r_\gamma} < 1$, then temporarily define $\{\tau^1 < \cdots \tau^j < \cdots \tau^{last}\}$ by:

$$\{t_{r_\gamma} = \tau^1 < \cdots \tau^j < \cdots \tau^{last} = 1\} \triangleq \{D(\eta) \cup \{t_{r_\gamma}, 1\}\} \cap [t_{r_\gamma}, 1]$$

1335 If $t_{r_\gamma} = 1$, then $\tau^1 \triangleq \tau^{last} \triangleq 1$

1340 For $0 \leq r \leq \bar{i}$ and $1 \leq j \leq last$, $\beta^{j,r} \triangleq \beta^r(\bar{f}, \tau^j, \eta_{|\tau^j})$

1350 Go to 70.

1500 $\sigma \triangleq 0$.

1510 Enumerate $Dev_\alpha \sim Dev_\alpha^{Lit}$ as $\{(t_{r_1}, h\restriction_{t_{r_1}}), \ldots, (t_{r_\sigma}, h\restriction_{t_{r_\sigma}}), \ldots, (t_{r_{\bar{\sigma}}}, h\restriction_{t_{r_{\bar{\sigma}}}})\}$

This temporarily defines $\bar{\sigma}$.

1515 $\sigma \triangleq \sigma + 1$

1520 Let $(\hat{t}, \hat{\eta})$ denote $m(pred(t_{r_\sigma}, h^{\frown}_{|t_{r_\sigma}}))$ | note that $\hat{\eta} = \hat{\eta}_{|\hat{t}}$.

1521 $\kappa \triangleq r_\sigma$

1525 If $\exists i \in I$ s.t. $(t_{r_\sigma}, h^{\frown}_{|t_{r_\sigma}}) \in Dev_\alpha^{Pass,i}$

(*i*) $m(t_{r_\sigma}, h^{\frown}_{|t_{r_\sigma}}) := (\tau_{-i}(\hat{t}, \hat{\eta})\wedge t_{r_\sigma}, \hat{\eta})$

(*ii*) $\eta \triangleq o([0, 1], \bar{f}, m(t_{r_\sigma}, h^{\frown}_{|t_{r_\sigma}}))$;

(*iii*) if $\tau_{-i}(\hat{t}, \hat{\eta})\wedge t_{r_\sigma} < 1$, then temporarily define $\{\tau^1 < \cdots \tau^j < \cdots \tau^{last}\}$ *by*:

$\{\tau_{-i}(\hat{t}, \hat{\eta})\wedge t_{r_\sigma} = \tau^1 < \cdots \tau^j < \cdots \tau^{last} = 1\} \triangleq \{D(\eta) \cup \{\tau_{-i}(\hat{t}, \hat{\eta})\wedge t_{r_\sigma}, 1\}\} \cap [\tau_{-i}(\hat{t}, \hat{\eta})\wedge t_{r_\sigma}, 1]$

(*iv*) go to 1555

1530 Otherwise

(*i*) $m(t_{r_\sigma}, h^{\frown}_{|t_{r_\sigma}}) := (\hat{t} + \delta*, \hat{\eta}^{h(t_{r_\sigma}-1),\hat{t}+\delta*})$

(*ii*) $\eta \triangleq o([0, 1], \bar{f}, m(t_{r_\sigma}, h^{\frown}_{|t_{r_\sigma}}))$;

(*iii*) if $t_{r_\sigma} < 1$, then temporarily define $\{\tau^1 < \cdots \tau^j < \cdots \tau^{last}\}$ *by*:

$\{\hat{t} + \delta = \tau^1 < \cdots \tau^j < \cdots \tau^{last} = 1\} \triangleq \{D(\eta) \cup \{\hat{t} + \delta, 1\}\} \cap [\hat{t} + \delta, 1]$

(*iv*) go to 1555

1555 If $t_{r_\sigma} = 1$, then $\tau^1 \triangleq \tau^{last} \triangleq 1$

1560 For $0 \leq r \leq \bar{l}$ and $1 \leq j \leq last$, $\beta^{j,r} \triangleq \beta^r(\bar{f}, \tau^j, \eta_{|\tau^j})$

1570 Go to 70.

2000 The construction is complete.

The following lemma establishes that the $\tilde{f}$ just constructed is in fact an $\epsilon$*-approximation to $\bar{f}$.

*Lemma AIII.1:* For all $(t_k, h_{|t_k}) \in DN^R$,

$$d^{DN}((t_k, h_{|t_k}), m(t_k, h_{|t_k})) < \epsilon* \tag{i}$$

$$d^H(o(R, \tilde{f}, t_k, h_{|t_k}), o([0, 1], \bar{f}, m(t_k, h_{|t_k}))) < \epsilon* \tag{ii}$$

It will follow immediately from the proof of this Lemma and our choice of $\tilde{\delta}$ that for all $i$ and for all $(t_k, h_{|t_k}) \in DN^R$, $|V_i(o(R, \tilde{f}, t_k, h_{|t_k})) - V_i(o([0, 1], \bar{f}, m(t_k, h_{|t_k})))| < \frac{\epsilon*}{3}$.

*Proof of Lemma AIII.1:* Fix $h \in H^R$. We will show that for all $t_k \in R$, inequalities (i) and (ii) hold for the decision node $(t_k, h_{|t_k})$. Since $h$ was chosen arbitrarily, this will be sufficient to establish the Lemma.

We first need another construction, one that will be used again in the proof that $\tilde{f}$ is an $\epsilon*$-equilibrium. For each $(t_k, h_{|t_k})$, let $(\hat{t}_k, \hat{\eta}_k) = m(t_k, h_{|t_k})$. Set $\tilde{G}(t_k) = \max\{s \leq \hat{t}_k : s \in D(S)\}$.

We now prove that $\tilde{G}$ just defined have a useful property. Define $B = \left\{\{t_k \in R : m(t_k, h_{|t_k}) \neq (t_k, h_{|t_k})\} \cap D(h)\right\}$ and $B' = \left\{ \bigcup_{s\in[0,1]} \{\tilde{G}^{-1}(s) : \#\tilde{G}^{-1}(s) > 1\} \cup \{1\} \cup \{t_k : m(t_k, h) \text{ was defined on 1525 }\}\right\}$. We will show that

$$B \subset B' \tag{1}$$

To establish (1), pick $t_k \in B$. Since $m(t_k, h_{|t_k}) \neq (t_k, h_{|t_k})$, we know that $m(t_k, h_{|t_k})$ was defined on one of the following lines: 330, 340, 350, 355, 356, 370, 1525 and 1530. If $m(t_k, h_{|t_k})$ was defined on one of lines 350, 355 or 356, then either $t_k = 1$ or $t_k \in \tilde{G}^{-1}(s)$, for some $s$ such that $\#\tilde{G}^{-1}(s) > 1$. In each of the remaining cases except 1525, line (iii) in the definition of $\delta*$ implies that $t_k \in \tilde{G}^{-1}(s)$, for some $s < 1$ such that $\#\tilde{G}^{-1}(s) > 1$. This establishes (1).

Now, it follows immediately from line 1530, the definition of $Dev_\alpha^{LIT}$ and $\bar{i}(\bar{i}+1)\delta(R) < \epsilon^{IR}$ that $\forall s$, $\#\tilde{G}^{-1}(s) < \bar{i}(\bar{i}+1)$. Also, at each line of the algorithm at which $m$ is defined (lines 330, 340, 350, 355, 356, 370, 1310, 1525 and 1530), $h_{i|t_k}$ and $\hat{\eta}_{ik}$ have the same sequence of jumps. Thus, by definition of $d^H$, to show that $d^H((t_k, h_{|t_k}), m(t_k, h_{|t_k})) < \epsilon*$, we must show that for all $i$, the set of times at which $h_{i|t_k}$ and $\hat{\eta}_{ik}$ disagree has Lebesgue measure less than $\epsilon*$.

Let $t_{k'} = \max\{t_r \in R: t_r \leq t_k \text{ and } m(t_r, h_{|t_r}) = (t_r, h_{|t_r})\}$. Since $m(0, \varnothing) = (0, \varnothing)$ (line 330 (i)), $t_{k'}$ is well-defined and nonnegative. First, if $t_{k'} = t_k$, then $d^{DN}((t_k, h_{|t_k}), m((t_k, h_{|t_k})) = 0 < \epsilon*$. Now suppose that $t_{k'} < t_k$. Observe that (a) $h_{i|t_k}$ and $\hat{\eta}_{ik}$ agree on $[0, t_{k'})$; (b) by definition of $Dev_{\alpha}^{LIT}$, $h_{|t_k}$ has at most $(t_k - t_{k'})\bar{i}(\frac{1}{\epsilon^{IR}} + 1) \leq t_k\bar{i}(\frac{1}{\epsilon^{IR}} + 1)$ discontinuity points on $[t_{k'}, t_k)$; and (c), each one of these discontinuity points, $t_r$, is either contained in a set of the form $\#\hat{G}^{-1}(s)$, or else $m(t_r, h_{|t_r})$ was defined on 1525. It can be shown that if $m(t_r, h_{|t_r})$ was defined on 1525, then $t_r - \hat{t}_r \leq \bar{i}(\bar{i} + 1)\delta(R)$. Moreover, as argued above, each such set has a length less than $\bar{i}(\bar{i} + 1)\delta(R)$. Therefore, $d^{DN}((t_k, h_{|t_k}), m(t_k, h_{|t_k})) < Q$, where $Q = t_k\bar{i}(\frac{1}{\epsilon^{IR}} + 1)\bar{i}(\bar{i} + 1)\delta(R)$. Because we chose $\delta(R)$ sufficiently small, we have $Q < \tilde{\delta} \leq \epsilon*$, thus establishing (i).

Finally, recall from the construction of $\tilde{f}$ that between $\hat{t}_k$ and 1, the outcome $h = o(R, \tilde{f}, t_k, h_{|t_k})$ differs from $\eta = o([0, 1], \bar{f}, m(t_k, h_{|t_k}))$ at most $(1 - \hat{t}_k)\bar{i}(\frac{1}{\epsilon^{IR}} + 1)(\bar{i} + 1) \leq (1 - t_k)\bar{i}(\frac{1}{\epsilon^{IR}} + 1)(\bar{i} + 1) + \bar{i}(\bar{i} + 1)$ nodes of $R$. Set $Q' = [(1 - t_k)\bar{i}(\frac{1}{\epsilon^{IR}} + 1)(\bar{i} + 1) + \bar{i}(\bar{i} + 1)]\delta(R)$. Since $Q + Q' < \tilde{\delta} \leq \epsilon*$, it follows that $d^H(o(R, \tilde{f}, t_k, h_{|t_k}), o([0, 1], \bar{f}, m(t_k, h_{|t_k}))) < \epsilon*$, thus establishing (ii). Since $\tilde{\delta} \leq \delta^{V1}$, it also follows that $|P_i(\bar{f}, m(t_r, h_{|t_r}) - P_i^R(\tilde{f}, t_r, h_{|t_r})| < \frac{\epsilon*}{3}$.

We will now show that if $\tilde{f}$ is not an $\epsilon^*$-SGP for the game played on $R$ then $\bar{f}$ is not a SGP equilibrium for the continuous time game. Suppose that $\tilde{f}$ is not an $\epsilon*$-SGP equilibrium. Then there exists $(t_r, h_{|t_r}) \in DN^R$ and $\tilde{g}_i \in F_i^R$ such that $P_i^R((\tilde{g}_i, \tilde{f}_{-i}), t_r, h_{|t_r}) > P_i^R(\tilde{f}, t_r, h_{|t_r}) + \epsilon*$. We can assume w.l.o.g. that $\tilde{g}_i(t_r, h_{|t_r}) \neq \tilde{f}_i(t_r, h_{|t_r})$ and that for all $(t_k, h') \in DN^R$, $t_k < t_r$, $\tilde{g}_i(t_k, h') = \tilde{f}_i(t_k, h')$. Define $\tilde{h} = o(R, (\tilde{g}_i, \tilde{f}_{-i}), t_r, h_{|t_r})$ and, for $t_k \in R$, define $(\hat{t}_k, \hat{\eta}_k)$ by: $(\hat{t}_k, \hat{\eta}_k) = m(t_k, \tilde{h})$. We now define a continuous time strategy, $\bar{g}_i$, such that

$| P_i((\bar{g}_i, \bar{f}_{-i}), m(t_r, h_{|t_r})) - P_i((\tilde{g}_i, \tilde{f}_{-i}), t_r, h_{|t_r}) | < \frac{\epsilon^*}{3}$. Since by Lemma AIII.1, $| P_i(\bar{f}, m(t_r, h_{|t_r}) - P_i^R(\tilde{f}, t_r, h_{|t_r}) | < \frac{\epsilon^*}{3}$, this contradicts the assumption that $\bar{f}$ is a SGP equilibrium.

Case 1: If $t_r \geq t_{\bar{p}-\bar{i}(\bar{i}+1)-1}$, then we can construct $\bar{g}_i$ very simply. For all $(t, h) \in DN$ such that $t < \hat{t}_r$, set $\bar{g}_i(t, h) = \bar{f}_i(t, h)$; otherwise, set $\bar{g}_i(t, h) = \hat{h}_i(t_{\bar{p}})$. Define $\bar{h} = o([0, 1], (\bar{g}_i, \bar{f}_{-i}), \hat{t}_r, \hat{\eta}_r)$. The following three statements imply $| V_i(\bar{h}) - V_i(\tilde{h}) | < \frac{\epsilon^*}{3}$, which establishes a contradiction:

$$d_i^H(\bar{h}_i, \tilde{h}_i) < 2\delta(R)\bar{i}^2(\bar{i} + 1)(\frac{1}{\epsilon^{IR}} + 1) < \delta^{lemma}. \quad 1)$$

$$d_{-i}^H(\bar{h}_{-i\,|\,\hat{t}_r}, \tilde{h}_{-i\,|\,\hat{t}_r}) < \delta(R)\bar{i}^2(\bar{i} + 1)(\frac{1}{\epsilon^{IR}} + 1) < \delta^{lemma}. \quad 2)$$

$$\hat{t}_r \leq t_r \text{ and } 1 - \hat{t}_r \leq \delta(R)\bar{i}^2(\bar{i} + 1)(\frac{1}{\epsilon^{IR}} + 1) < \delta^{lemma}. \quad 3)$$

2) and 3) follow from the definition of $\delta(R)$ and Lemma AIII.1, which established that for all $(t_r, h_{|t_r}) \in DN^R$, $d^{DN}((t_r, h_{|t_r}), m(t_r, h_{|t_r})) < \delta(R)\bar{i}^2(\bar{i} + 1)(\frac{1}{\epsilon^{IR}} + 1)$. 1) follows from the same calculations and the fact that $t_r \geq t_{\bar{p}-\bar{i}(\bar{i}+1)-1}$.

Case 2: If $t_r < t_{\bar{p}-\bar{i}(\bar{i}+1)-1}$, then we define $\bar{g}_i$ inductively. First, let $\tilde{G}_i$ be defined as in Lemma AIII.1, that is, $\tilde{G}(t_k) = \max\{s \leq \hat{t}_k : s \in D(S)\}$. Let $\{s_1 < \cdots < s_m < \cdots < s_{\bar{m}}\} = \tilde{G}(\{t_r, \ldots, t_{\bar{p}-\bar{i}(\bar{i}+1)-1}\})$ and let $S_m = \{s_{m,1}, s_{m,2}, \ldots, s_{m,n(m)}\} = \tilde{G}^{-1}(s_m)$.

Define $g_i^0$ by:

$$g_i^0(t, h) = \begin{cases} \bar{f}_i(t, h) & \text{if } t < \tilde{G}(t_r) \\ \tilde{h}_i(t_{\bar{p}}). & \text{if } t \geq \tilde{G}(t_r) \end{cases}$$

Now assume that for $m \leq \bar{m}$, assume that $g_i^{m-1}$ has been defined. Define $g_i^m$ by

$$g_i^m(t, h) = \begin{cases} g_i^{m-1}(t, h) & \text{if } t < s_m \\ \tilde{h}_i(s_{m,1}) & \text{if } t = s_m \\ \tilde{h}_i(s_{m,1}) & \text{if } t > s_m \text{ and } \textit{either } s_{m,1} \in D_i(\tilde{h}) \textit{ or } S_m \cap D_i(\tilde{h}) = \varnothing \\ \tilde{h}_i(s_{m,k}) & \text{if } t > s_m,\ \{s_{m,k}\} = D_i(\tilde{h}) \cap S_m \text{ and } h(t) = \tilde{h}(s_{m,k-1}) \\ \tilde{h}_i(s_{m,1}) & \text{if } t > s_m,\ \{s_{m,k}\} = D_i(\tilde{h}) \cap S_m \text{ and } h(t) \neq \tilde{h}(s_{m,k-1}) \end{cases}$$

Finally, define

$$\bar{g}_i(t, h) = \begin{cases} g_i^{\bar{m}}(t, h) & \text{if } t < s_{\bar{m}} \\ \tilde{h}_i(t_{\bar{p}}). & \text{if } t \geq s_{\bar{m}}. \end{cases}$$

Set $\bar{\eta} = o([0, 1], (\bar{g}_i, \bar{f}_{-i}), m(t_r, h_{|t_r}))$. Before we can state the claim that will complete the proof, we need one more definition. Let $\mathbf{A} = \bigcup_{n=1}^{\bar{l}(\bar{l}+1)} A^n$. We define a mapping, $b$, from a subset of A to $A^{\bar{l}+1}$. We represent each matrix $M \in \mathbf{A}$ as

$$M = (m_{\alpha\gamma})_{\substack{\alpha \in I \\ 1 \leq \gamma \leq n}} = (m_{\cdot 1}, m_{\cdot 2}, \ldots, m_{\cdot n})$$

where each $m_{\cdot k} \in A$ is a column vector. Define dom($k$) by:

$$\text{dom}(b) = \{M \in \mathbf{A}: \forall \alpha,\ m_{\alpha\gamma'} \neq m_{\alpha 1} \text{ implies } m_{\alpha\gamma} = m_{\alpha\gamma'}, \forall \gamma' \leq \gamma \leq n\}.$$

For $M = m_{\alpha\gamma} \in \text{dom}(b)$ and $1 \leq k \leq \bar{l} + 1$, define $\gamma(k)$ inductively as follows: set $\gamma(1) = 1$; For $1 \leq k \leq \bar{l}$, given that $\gamma(k)$ has been defined, define $\Gamma(k + 1) = \{\gamma(k) < \gamma \leq \bar{l} + 1: m_{\cdot\gamma} \neq m_{\cdot\gamma(k)}\}$. Now define

$$\gamma(k + 1) = \begin{cases} \min\{\gamma \in \Gamma(k + 1) & \text{if } \Gamma(k + 1) \neq \varnothing \\ \gamma(k) & \text{otherwise} \end{cases}$$

Now define $b(M) = (m_{\cdot\gamma(1)}, m_{\cdot\gamma(2)}, \ldots, m_{\cdot\gamma(\bar{l}+1)})$. We now establish claims (i)-(iv) below:

$$\beta((\bar{g}_i, \bar{f}_{-i}), (\hat{t}_r, \hat{\eta}_r)) = b(\tilde{h}(S_1)); \quad \text{(i)}$$

$$\text{For } m \geq 2,\ \beta((\bar{g}_i, \bar{f}_{-i}), (s_m, \bar{\eta}_{|s_m})) = b(\tilde{h}(S_m)); \quad \text{(ii)}$$

$$\beta((\bar{g}_i, \bar{f}_{-i}), \cdot, \bar{\eta}_{|\cdot}) \text{ is constant on } (t_r, s_2); \quad \text{(iii)}$$

$$\text{For } m \geq 2,\ \beta((\bar{g}_i, \bar{f}_{-i}), \cdot, \bar{\eta}_{|\cdot}) \text{ is constant on } (s_m, s_{m+1}); \quad \text{(iv)}$$

Having established (i)-(iv), we will then argue that $d_i^H(\tilde{h}_i, \bar{\eta}_i) < 3^{LIII.2}$ and

$d^H_{-i}(\tilde{h}_{-i\,|\,s_{\bar{m},1}}, \bar{\eta}_{-i\,|\,s_{\bar{m}}}) < \delta^{LIII.2}$. Since $(1 - s_{\bar{m},1})$ and $(1 - s_{\bar{m}})$ are both less than $\delta^{LIII.2}$, we can then conclude that $|\, V_i(\bar{h}) - V_i(\tilde{h})\,| < \frac{\epsilon*}{3}$. This will establish a contradiction and complete the proof.

To see that $d^H_i(\tilde{h}_i, \bar{\eta}_i) < \delta^{LIII.2}$, note that:

(a) claims (i)-(iv) guarantee that $\tilde{h}_i$ and $\hat{\eta}_i$ have the same sequence of jumps;

(b) before $\hat{t}_r$, $\tilde{h}_i$ and $\bar{\eta}_i$ can differ on at most a set of time nodes with measure less than $\hat{t}_r \cdot \bar{i}(\bar{i} + 1)(\frac{1}{\epsilon^{JR}} + 1)\delta(R)$.

(c) after $\hat{t}_r$, $\tilde{h}_i$ and $\bar{\eta}_i$ can differ on at most a set of time nodes with measure less than $(1 - \hat{t}_r)\cdot \bar{i}(\bar{i} + 1)(\frac{1}{\epsilon^{JR}} + 1)(\delta(R) + \delta*) + \bar{i}(\bar{i} + 1)\delta(R)$.

(b) was established when we proved that $\tilde{f}$ was an $\epsilon^*$-approximation to $\bar{f}$. To establish (c), make the same arguments again, but include the extra fact that when $\hat{t}_k \neq t_k$, we have $|\, \hat{t}_k - \tilde{G}(t_k)\,| < \bar{i}\delta*$. By the definitions of $\delta(R)$ and $\delta*$, we can conclude that $d^H_i(\tilde{h}_i, \bar{\eta}_i) < \tilde{\delta} < \delta^{LIII.2}$.

To establish that $d^H_{-i}(\tilde{h}_{-i\,|\,s_{\bar{m},1}}, \bar{\eta}_{-i\,|\,s_{\bar{m}}}) < \delta^{LIII.2}$, proceed as in the previous paragraph, substituting $\tilde{h}_{-i\,|\,s_{\bar{m},1}}$ for $\tilde{h}_i$ and $\bar{\eta}_{-i\,|\,s_{\bar{m}}}$ for $\bar{\eta}_i$. [in the analog to (b), the addition of $\bar{i}(\bar{i} + 1)\delta(R)$ is not needed].

We will now prove (i) and (iii) above. The proofs of (ii) and (iv) are virtually identical. We begin with a claim.

Claim: For $t_r < t_k \leq t_{\bar{p}-\bar{i}(\bar{i}+1)-1}$, $t_k \in D(\tilde{h}_i)$ implies $\#\tilde{G}^{-1}(\tilde{G}(t_k)) > 0$.

Proof of the Claim: Pick $t_k \in D(\tilde{h}_i)$. There are three cases to consider:

Case (a): $\tilde{g}_i$ and $\tilde{f}_i$ call for the same jumps at $(t_k, h_{\,|\,t_k})$.

Case (b): $\tilde{g}_i$ and $\tilde{f}_i$ call for different jumps at $(t_k, h_{\,|\,t_k})$.

Case (c): $\tilde{g}_i$ calls for a jump at $(t_k, h_{\,|\,t_k})$ that $\tilde{f}_i$ does not call for.

Since we know that $|\, P_i(\bar{f}, m(t_r, h_{\,|\,t_r}) - P^R_i(\tilde{f}, t_r, h_{\,|\,t_r})\,| < \frac{\epsilon*}{3}$, and we are assuming

that $P_i((\tilde{g}_i, \tilde{f}_{-i}), t_r, h_{|t_r}) > P_i^R(\tilde{f}, t_r, h_{|t_r}) + \epsilon$, we can invoke assumption V2 to conclude that there is at most one jump in $\tilde{h}_i$ on the interval $(t_k, t_k + \epsilon^{IR})$. This interval contains at least $\frac{3\bar{l}^2(\bar{l}+1)(\frac{1}{\epsilon^{IR}}+1)}{\tilde{\delta}}$ grid points. From this an the assumption that each $j \neq i$ is playing $\tilde{f}_j$, we can conclude that $(t_k, \tilde{t}_{|t_k})$ is not handled literally (i.e., on line 1310) by the algorithm. Therefore, examining lines 1530, 85, 215, 340 and (iii) in the definition of $\delta*$ ( cases (b) and (c)) or lines 85, 215, 340 and line (iii) in the definition of $\delta*$ ( case (a)), we conclude that $\#\tilde{G}^{-1}(\tilde{G}(t_k)) > 0$.

We now establish (i) and (iii). There are eight cases to consider, depending on whether or not

a) $m(t_r, h_{|t_r})$ was defined on line 1525 (written $(t_r, h_{|t_r}) \in 1525$);

b) $\#S_m = 1$

c) $s_1 < \hat{t}_r$.

$$(t_r, h_{|t_r}) \in 1525) \text{ and } s_1 < \hat{t}_r. \qquad \text{(Cases 1 and 2)}$$

This cannot happen since by line 1525, (iii) in the definition of $\delta*$ and the definition of $s_1$, $(t_r, h_{|t_r}) \in 1525)$ implies $s_1 = \hat{t}_r$.

$$(t_r, h_{|t_r}) \in 1525), \#S_m = 1 \text{ and } s_1 = \hat{t}_r. \qquad \text{(Case 3)}$$

Using the last claim, we see that there is only one sequence of moves at $(t_r, h_{|t_r})$ by $\tilde{f}_i$ and $\tilde{g}_i$ that is compatible with this case: $\tilde{f}_{-i}$ calls for at least one $j \neq i$ to move at $t_r$; $\tilde{f}_{-i}$ calls for no agent to respond with a move and $\tilde{g}_i$ calls for $i$ not to move, that is $(t_{r+1}, \tilde{h}_{|t_{r+1}}) \in 1525$. This $\tilde{f}$ will happen only if in continuous time, $\bar{f}_{-i}$ calls for a right continuous jump at $\hat{t}_r = \tau^{-i}(m(pred(t_r, \tilde{h}_{|t_r}))) \wedge t_r$. The construction of $\bar{g}_i$ now guarantees that

$$\beta((\bar{g}_i, \bar{f}_{-i}), m(t_r, \tilde{h}_{|t_r}) = ((\tilde{h}_i(t_r), \bar{f}_{-i}(\tau^{-i}, \hat{\eta}), \ldots, )$$

which is in fact equal to $b(\tilde{h}(S_1))$. Conclusion (iii) is an immediate consequence of $(t_{r+1}, \tilde{h}_{|t_{r+1}}) \in 1525$.

$(t_r, h_{|t_r}) \notin 1525)$, $\#S_1 = 1$ and $s_1 = \hat{t}_r$. (Case 4)

Since $\#S_1 = 1$, the Claim above lets us conclude that $t_r \notin D(\tilde{h}_i)$. Therefore,

$$\tilde{g}_i(t_r, \tilde{h}_{|t_r}) = \tilde{h}_i(t_{r-1}) \neq \tilde{f}_i(t_r, \tilde{h}_{|t_r}),$$

so that either $(t_{r+1}, \tilde{h}_{|t_{r+1}}) \in 1525$ or $(t_{r+1}, \tilde{h}_{|t_{r+1}}) \in 1530$. If $(t_{r+1}, \tilde{h}_{|t_{r+1}}) \in 1525$, then $\tilde{f}_{-i}(t_r, \tilde{h}_{|t_r}) = \tilde{h}_{-i}(t_{r-1})$ and $\bar{f}_{-i}$ and $\bar{g}_{-i}$ call for no jumps on $[\hat{t}_r, s_2)$, giving us conclusions (i) and (iii) above. If $(t_{r+1}, \tilde{h}_{|t_{r+1}}) \in 1530$, then we know $\tilde{f}_{-i}(t_r, \tilde{h}_{|t_r}) \neq \tilde{h}_{-i}(t_{r-1})$. This and the fact that $\#S_1 = 1$ implies that the discontinuity was induced by a right continuous jump by $\bar{f}_{-i}$ at $m(t_r, \tilde{h}_{|t_r})$, that no $j \neq i$ responds instantaneously to this jump and that $\tilde{g}_i$ deviates passively at $(t_{r+1}, \tilde{h}_{|t_{r+1}})$. By the definition of $\bar{g}_i$, this establishes that for $0 \leq r \leq \bar{l}$, $\beta^r((\bar{g}_i, \bar{f}_{-i}), m(t_r, \tilde{h}_{|t_r})) = (\tilde{h}_i(t_r), \bar{f}_{-i}(\tau^{-i}, \hat{\eta}))$. which establishes conclusion (i). Conclusion (iii) now follows immediately from line 1525, and the definitions of $\tau^{-i}$ and $\bar{g}_i$.

$(t_r, h_{|t_r}) \notin 1525)$, $\#S_1 = 1$ and $s_1 < \hat{t}_r$. (Case 5)

We show that this case cannot happen, by exhaustively searching the algorithm at the places where $m(t_r, \tilde{h}_{|t_r})$ was defined.

$(t_r, h_{|t_r}) \notin 350, 355, 356$, since $t_r < t_{\bar{p}-\bar{l}(\bar{l}+1)-1}$.

$(t_r, h_{|t_r}) \notin 330, 340$, since $s_1 < \hat{t}_r$ would then imply that $\#S_m > 1$.

$(t_r, h_{|t_r}) \notin 360, 1310$, since $s_1 < \hat{t}_r$.

$(t_r, h_{|t_r}) \notin 1530$, since $\#S_1 = 1$.

In all of the remaining cases -- 6, 7, and 8 -- $\#S_1 > 1$. The proofs of conclusions (i) and (iii) are virtually identical: use lines 1530 1525, 330 and 340, (i) and (ii) in the definition of $\delta*$. □

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