UCLA

UCLA Previously Published Works

Title

On the creation of Wada basins in interval maps through fixed point tangent bifurcation

Permalink

<https://escholarship.org/uc/item/3j38277x>

Journal

Physica D-Nonlinear Phenomena, 207(1-2)

ISSN

0167-2789

Authors

Breban, Romulus
Nusse, H E

Publication Date

2005-07-01

DOI

10.1016/j.physd.2005.05.012

Peer reviewed

On the creation of Wada basins in interval maps through fixed point tangent bifurcation

Romulus Breban ^{a,*} Helena E. Nusse ^b

^a*Institute for Research in Electronics and Applied Physics, and Department of Physics, University of Maryland, College Park, Maryland 20742; Present Address: Department of Biomathematics at David Geffen School of Medicine, University of California at Los Angeles, Penthouse 2, 1100 Glendon Ave., Los Angeles, California 90024.*

^b*Institute for Physical Sciences and Technology, University of Maryland, College Park, Maryland 20742; Permanent Address: University of Groningen, Department of Econometrics, P.O. Box 800, NL-9700 AV, Groningen, The Netherlands.*

Abstract

Basin boundaries play an important role in the study of dynamics of nonlinear models in a variety of disciplines such as biology, chemistry, economics, engineering, and physics. One of the goals of nonlinear dynamics is to determine the global structure of the system such as boundaries of basins. A basin having the strange property that every point which is on the boundary of that basin is on the boundary of at least three different basins, is called a *Wada basin*, and its boundary is called a *Wada basin boundary*. Here we consider maps on the interval. We present a sufficient and necessary condition guaranteeing that three Wada basins are emerging from a tangent bifurcation for certain one dimensional maps having negative Schwarzian derivative, two fixed point attractors on one side of the tangent bifurcation, and three fixed point attractors on the other side of the tangent bifurcation. All the conditions involved are numerically verifiable.

Key words: Wada basin boundary, saddle-node bifurcation, interval map
PACS: 05.45.-a, 05.45.Df, 02.30.Oz; *MSC:* 37E05

---

doi:10.1016/j.physd.2005.05.012; published in Physica D **207** (2005) 52-63.

---

* Corresponding author.

Email address: breban@ucla.edu (Romulus Breban).

1 Introduction

In nonlinear dynamics, there are often two or more attractors. A *basin* of attraction is the collection of points whose trajectories approach a specified compact invariant set such as an attractor. A point x is a *boundary point* of a basin B if every open neighborhood of x has a nonempty intersection with basin B and at least one other basin. The *boundary* of a basin is the set of all boundary points of that basin. For a basin B , we write $\partial\overline{B}$ for the boundary of B . (Note that this definition is slightly different from the topological definition of boundary ∂B .) One of the goals of nonlinear dynamics is to determine the global structure of a system such as boundaries of basins.

One can imagine situations for which a boundary point of a basin is on the boundary of at least two other basins. Examples may suggest that there are only finitely many such points. Imagine trying to create a picture of three nonoverlapping regions in the plane, each connected, which have the property that every boundary point of any region is a boundary point of all three regions. Such regions can be created and a construction was given early on by the Dutch mathematician Brouwer [1]. Independently, Yoneyama gave in 1917 an example comparable to Brouwer's example leading to the result that every boundary point is a boundary point of three open sets. Yoneyama attributed the example to "Mr. Wada". In the book by Hocking and Young [2], this example is entitled "Lakes of Wada". As originally presented, these examples are not related to dynamical systems. It is hard to imagine that such a configuration of three basins could exist for simple dynamical processes. Kennedy and Yorke [3] discovered that such "Wada basins" might occur in some simple processes.

Indeed, a boundary point of a basin B may be a boundary point of at least two other basins. Following [4,5] we say that a point x is a *Wada point* if every open neighborhood of x has a nonempty intersection with at least three basins. A basin B is called a *Wada basin* if every boundary point of B is a Wada point; the boundary of a Wada basin is called a *Wada basin boundary*. In other words, if you zoom in on a boundary point, no matter how close, all three basins would be in the detailed picture. A system has the *Wada property* if the map has at least three basins of attraction and if all basin boundaries coincide. This definition was given by Kennedy and Yorke [3]. They were unable to prove that the Wada property occurs for basins except in rather special circumstances. However, based on pictures of basins, Kennedy and Yorke argued that the Wada property appears to exist even in the forced damped pendulum. General criteria guaranteeing the existence of Wada basins for two-dimensional systems are presented in [6]. Wada basins occur in a variety of systems and recent papers discussing Wada basins are [7–11].

In Refs. [12,10] it is argued why Wada basins emerge when a saddle-node bifurcation occurs on a fractal basin boundary in a variety of two-dimensional systems like the forced damped pendulum or the forced Duffing oscillator. In these references, it is assumed that the saddle-node bifurcation occurs on a fractal boundary. In this paper, we investigate whether Wada basins can emerge when a tangent bifurcation occurs for maps on a compact interval. We say basin B has a *fractal basin boundary* if its boundary is a Cantor set. Examples with fractal basin boundaries are common and have been studied extensively, see e.g., [13,14] and references therein. Note that a Wada basin boundary is a fractal basin boundary but a fractal basin boundary need not be a Wada basin boundary. Furthermore, we point out that if the map has the Wada property then every basin is a Wada basin, but the fact that every basin is a Wada basin does not imply that the map has the Wada property.

In the bifurcation theory literature, local bifurcations such as tangent bifurcations for one-dimensional maps are usually studied from a local point of view; see e.g., [15–19]. For example, in the tangent bifurcation theory for maps, attention has focused on the local stability properties of the two fixed points (or periodic points) of differentiable maps when a tangent bifurcation occurs. We say that a one parameter family of maps f_μ has at the critical parameter value μ_0 a *fixed point (periodic point) creating tangent bifurcation* at the location x_0 , if the map f_μ has no fixed points (periodic points) in a small neighborhood of x_0 for $\mu_0 - \epsilon < \mu < \mu_0$, and f_μ has two fixed points (periodic points) in a small neighborhood of x_0 for $\mu_0 < \mu < \mu_0 + \epsilon$, where $0 < \epsilon \ll 1$; see, for example, Fig. 3 below. Similarly, we say that a one parameter family of maps f_μ has at the critical parameter value μ_0 a *fixed point (periodic point) destroying tangent bifurcation* at the location x_0 if the map f_μ has two fixed points (periodic points) in a small neighborhood of x_0 for $\mu_0 - \epsilon < \mu < \mu_0$, and f_μ has no fixed points (periodic points) in a small neighborhood of x_0 for $\mu_0 < \mu < \mu_0 + \epsilon$, where $0 < \epsilon \ll 1$. We say that a one parameter family of maps f_μ has at the critical parameter value μ_0 a *tangent bifurcation* at the location x , if it is either a fixed point (periodic point) creating tangent bifurcation or a fixed point (periodic point) destroying tangent bifurcation.

We consider a condition for smooth maps which is called the “Schwarzian derivative”. The notion of Schwarzian derivative for one-dimensional maps has been introduced by Singer in 1978 [20]. The Schwarzian derivative is named for Hermann Schwarz, who in 1869 defined it and used it in the study of complex-valued functions. The importance of this notion for investigating one-dimensional maps has been early recognized by Guckenheimer [21], and Misiurewicz [22]. A related notion of the cross-ratio has been introduced by Allwright [23]. In Sec. 2 we present a selection of important properties for maps having a negative Schwarzian derivative.

The purpose of this paper is to investigate differentiable maps having negative

Schwarzian derivative (see Sec. 2 for its definition) having at least two but at most three fixed point attractors, and no other attractors. While a parameter is varied, a tangent bifurcation occurs at which a third fixed point attractor is created (or destroyed). We want to know whether Wada basins can emerge from this tangent bifurcation, and if so, under which conditions. For a discussion on the relevance of one-dimensional maps for higher dimensional systems such as the Lorentz system to model real fluid behavior and the experimental system being the Belousov-Zhabotinskii reaction in a well-stirred chemical reactor, see [14]

For two-dimensional systems, in order to have an emerging Wada basin from a saddle-node bifurcation, one of the assumptions in [12] is that a saddle-node bifurcation occurs on a common fractal boundary of two basins of attraction. A natural question is “Does there exist a numerically verifiable condition guaranteeing that the saddle-node bifurcation occurs on the common fractal basin boundary?” In the Refs. [12,10] no such condition is formulated. In other words, there is *no* criterion in the literature concerning the creation of Wada basins that is numerically verifiable. Since one-dimensional maps are easier to analyze, we formulate a numerically verifiable criterion on the creation of Wada basins in a certain class of one-dimensional maps satisfying some numerically verifiable conditions. This result can be considered as a first step in formulating numerically verifiable criteria on the creation of Wada basins in a more general approach such as in a general class of one-dimensional maps and higher dimensional systems.

As mentioned above, in the bifurcation theory literature, local bifurcations such as tangent bifurcations for one-dimensional maps are usually studied from a local point of view. In contrast, we investigate tangent bifurcations from a global point of view, or more precisely, we investigate the global consequences of tangent bifurcations. Let $F_\mu : I \rightarrow I$ be a one-parameter family of C^3 -maps that have negative Schwarzian derivative. Assume that (a) every attractor of F_μ is a fixed point, (b) F_μ has at least two fixed point attractors and at most three fixed point attractors, (c) at the parameter value μ_0 , F_μ has a fixed point creating tangent bifurcation at the location x_0 , and (d) for some $0 < \delta < 1$ (δ may be small) and for every $\mu_0 \leq \mu < \mu_0 + \delta$, all critical points of F_μ are contained in the basins of attraction. Note that all these assumptions on the maps F_μ are numerically verifiable. Our main result specifies explicitly the (sufficient and necessary) condition guaranteeing that the new emerging basin is a Wada basin and is the following.

Wada Property Criterion: F_μ has the Wada property for $\mu_0 \leq \mu < \mu_0 + \delta \Leftrightarrow$ The core matrix $A_{\text{core}}(F_{\mu_0})$ of F_{μ_0} is primitive.

In order to make the proof of the theorem more accessible, we present two main ingredients of the proof by discussing two specific examples. Concerning Wada

basins, we first present a well-studied example, namely the “Period 3 implies chaos” for the logistic map [24]. We show by elementary means that this map has the Wada property, and so Wada basins exist (Sec. 3). Understanding the dynamics of one-dimensional maps and knowing the definition of Wada property, it is straightforward to observe that this map has the Wada property. Therefore, we cannot claim that we are the first demonstrating the existence of the Wada property in one-dimensional maps. The main reason for its inclusion is for didactical merits, since its constructive proof is an important ingredient of the proof of our main result of this paper. Based on this example, we discuss in Sec. 4 a basin that emerges after a tangent bifurcation, and has a Wada basin boundary. Then, in Sec. 5 we study the situation in which two attracting fixed points exist and a third attracting fixed point emerges by a tangent bifurcation while a parameter is varied. More specifically, we assume that μ is the bifurcation parameter of a map, and that for μ in some parameter interval J , there are two fixed points attractors $A_1(\mu)$ and $A_2(\mu)$ and their corresponding basins are $B_1(\mu)$ and $B_2(\mu)$. We further assume that for every $\mu \in J$ and $\mu > \mu_0$, there exists a basin $B_3(\mu)$ which is the basin of an attracting fixed point $A_3(\mu)$ that emerges from a tangent bifurcation at $\mu = \mu_0$. The basin $B_3(\mu)$ does not exist for $\mu \in J$ and $\mu < \mu_0$. In this paper, we concentrate on properties of the boundary of the basin $B_3(\mu)$ for $\mu \in J$ and $\mu > \mu_0$. We explain that if all critical points of the map are contained in the basins of attraction, and if a second general condition (some well-defined matrix which will be specified later is primitive) is satisfied, then the boundary of $B_3(\mu)$ has the strange property that every point that is on the boundary of $B_3(\mu)$ is also on the boundary of at least two other basins. Both these conditions implying the surprising global phenomenon are numerically verifiable.

2 Schwarzian derivative

Let I be an interval, and let $F : I \rightarrow I$ be a C^3 -map with finitely many critical points. (A critical point of F is a point at which the derivative of F vanishes.) The *Schwarzian derivative* of F at $x \in I$ at which $F'(x) \neq 0$, denoted by $S_F(x)$, is defined by $S_F(x) = F'''(x)/F'(x) - (3/2)[F''(x)/F'(x)]^2$. The map F has a *negative Schwarzian derivative* if $S_F(x) < 0$ for every $x \in I$ with $F'(x) \neq 0$. For example, for every $r > 0$ the logistic map $F(x) = rx(1-x)$ has a negative Schwarzian derivative, since for $x \neq 0.5$, $S_F(x) = -(3/2)\{-2r/[r(1-2x)]\}^2 < 0$.

Remark. If $F'(x) \neq 0$, then $S_F(x) < 0 \Leftrightarrow 2F'(x)F'''(x) - 3[F''(x)]^2 < 0$.

The Schwarzian derivative has several well-known important properties of which the ones needed for this exposition are presented below. For proofs,

see [20,25].

Property SD-1. The composition of two maps with negative Schwarzian derivative also has a negative Schwarzian derivative.

Property SD-2. If F has a negative Schwarzian derivative, then for each $n \in \mathbb{N}$, the n th iterate of F , denoted by F^n , has a negative Schwarzian derivative.

Property SD-3. If F has a negative Schwarzian derivative and the interval I is compact, then the basin of a stable periodic orbit contains a critical point of F or an end point of the interval I .

Property SD-4. Assume that $W \subset I$ is a compact interval such that the interior of W is not empty and does not contain any critical point of F . If F has a negative Schwarzian derivative, then $|F'|$ restricted to the interval W assumes its minimum value at a point of the boundary of W .

For any stable periodic orbit Q of F , we call the union of intervals in the basin of Q that contain some point of the orbit Q , the *immediate basin* of Q . Hence, the immediate basin of a stable period- m orbit consists of m intervals. In particular, the immediate basin of a stable fixed point q is one interval and will be denoted by $IB(q)$. We leave it to the reader to verify that F maps the immediate basin of q into itself, and F maps the boundary of the immediate basin of q into itself. The next property deals with the question whether the immediate basin of a stable period- m orbit contains a critical point of the map.

Property SD-5. Assume that F has a negative Schwarzian derivative. Let Q be a stable period- m orbit of F . Then: (a) If $m > 3$, then the immediate basin of Q contains at least one critical point of the map F ; (b) If the immediate basin of Q does not contain any critical point of F , then either $m = 1$ or $m = 2$.

If F does not have a negative Schwarzian derivative, then the conclusions of Property SD-5 are not true anymore. In particular, it is not guaranteed that the immediate basin of a stable periodic orbit contains a critical point of F . A generalization of Property SD-5(a) is formulated next and may be relevant for maps having multiple critical points.

Property SD-6. Assume that F has a negative Schwarzian derivative. Let q be a stable periodic point for F such that F has a periodic point to both sides of q . Then the immediate basin of the orbit of q contains at least one critical point of F .

Properties SD-5 and SD-6 imply that if a map with finitely many critical points has a negative Schwarzian derivative, then at most finitely many different stable periodic orbits can coexist.

3 Wada property in iterates of the logistic map

We first consider the well studied logistic map $g : [0, 1] \rightarrow [0, 1]$ defined by $g(x) = 3.84x(1 - x)$. The map g has a period-3 attracting orbit, and the third iterate of the map, denoted by G , has consequently three basins and their boundaries are fractal. See Fig. 1 for the graph of the map G . In fact, the boundaries of the three basins (of the third iterate) coincide, and so the map G has the Wada property. We now give a constructive proof of why each of the three basins of G is a Wada basin.

The map G has three attracting fixed points; call them A_1 , A_2 and A_3 from right to left. Denote the basin of A_k by B_k , ($1 \leq k \leq 3$). Each of the basins consists of infinitely many intervals; the interval in B_k that contains A_k is the immediate basin, denoted by $IB_k = (a_k, b_k)$, ($1 \leq k \leq 3$). Since $g(0) = 0$ and $g'(0) > 1$, by Property SD-3 the map g has exactly one attracting periodic orbit which is a period-3 orbit. Hence, the map G has three attracting fixed points and no other attractors.

We now explain why the three boundaries coincide. Coloring the basins, basin B_1 is colored blue, basin B_2 is colored green, and basin B_3 is colored red, so IB_1 is blue, IB_2 is green and IB_3 is red. All the critical values of G are contained in the union of the three immediate basins. Hence, if the critical points of G are denoted by $c_1 < c_2 < c_3 < c_4 < c_5 < c_6 < c_7$, then we have $G(c_1) \in IB_1$, $G(c_2) \in IB_3$, $G(c_3) \in IB_1$, $G(c_4) \in IB_2$, $G(c_5) \in IB_1$, $G(c_6) \in IB_3$, and $G(c_7) \in IB_1$. Since the intervals $[0, c_1]$, $[c_1, c_2]$ and $[c_7, 1]$ are mapped over the union of intervals $[c_2, c_7] \setminus (IB_1 \cup IB_2 \cup IB_3)$ when the map G is applied and the critical values of G are contained in $(IB_1 \cup IB_2 \cup IB_3)$, it is sufficient to consider the basin boundaries in the union of five intervals, denoted by J_i ($1 \leq i \leq 5$), as shown in Fig. 2.

Write b (blue) for the endpoint of IB_1 that is an unstable fixed point of G , write g (green) for the endpoint of IB_2 that is an unstable fixed point of G , and write r (red) for the endpoint of IB_3 that is an unstable fixed point of G . Hence, $r < g < b$. Select the compact intervals J_i ($1 \leq i \leq 5$) as indicated in Fig. 2. From Fig. 2 it is easily seen that $G(J_1) = [r, b]$, $G(J_2) = [g, b]$, $G(J_3) = [g, b]$, $G(J_4) = [r, b]$, and $G(J_5) = [r, b]$. This choice of the intervals J_i ($1 \leq i \leq 5$) and the continuity of G imply that $G(J_1)$ contains J_i ($1 \leq i \leq 5$), $G(J_2)$ contains J_i ($3 \leq i \leq 5$), $G(J_3)$ contains J_i ($3 \leq i \leq 5$), $G(J_4)$ contains J_i ($1 \leq i \leq 5$), and $G(J_5)$ contains J_i ($1 \leq i \leq 5$). Hence, $G(J_1) \supset J_1 \cup J_2 \cup J_3 \cup J_4 \cup J_5$, $G(J_2) \supset J_3 \cup J_4 \cup J_5$, $G(J_3) \supset J_3 \cup J_4 \cup J_5$, $G(J_4) \supset J_1 \cup J_2 \cup J_3 \cup J_4 \cup J_5$, and $G(J_5) \supset J_1 \cup J_2 \cup J_3 \cup J_4 \cup J_5$.

Let $A_G = [a_{ik}]_{1 \leq i, k \leq 5}$ be the 5×5 matrix defined by a_{ik} is the number of times that G maps the interval J_i over J_k , $1 \leq i, k \leq 5$. Hence,

$$A_G = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The matrix A_G is primitive, that is, there exists $m \in \mathbb{N}$ such that A_G^m has only positive entries. (Note that $m = 2$ suffices.)

We now consider the basins of the three stable fixed points of the map G . We denote by D_0 the open set $[r, b] \setminus (\cup_{i=1}^5 J_i)$. The set D_0 consists of four open intervals, red, blue, green and blue (from right to left); see Fig. 2. We denote by C_1 the complement of D_0 in the interval $[r, b]$, so $C_1 = \cup_{i=1}^5 J_i$. By induction, we define for every positive integer n ,

$$\begin{aligned} D_n &= \{x \in C_n : G(x) \in D_{n-1}\}, \\ C_{n+1} &= \{x \in C_n : G(x) \in C_n\}. \end{aligned}$$

In other words, D_n is the set of points whose $(n-1)$ th iterates (when G is applied) are still in C_1 and whose n th iterates are in one of the four colored intervals, and C_{n+1} is the set of points whose n th iterates are still in C_1 . Note that D_n is the set of points whose $(n+1)$ th iterates are in the union of the three immediate basins of attraction IB_k ($1 \leq k \leq 3$). Finally, write C_∞ for the set of points whose trajectories will stay in C_1 under forward iteration, that is,

$$C_\infty = \{x \in [r, b] : G^k(x) \in C_1 \text{ for each integer } k \geq 0\}.$$

Hence, C_∞ is the set of points which are not contained in any of the three basins of attraction. (Applying a result from [25] it follows that C_∞ is a Cantor set of Lebesgue measure zero.)

We now explain that every $p \in C_\infty$ is a Wada point, that is, it is a boundary point of at least three basins. That is, we want to show that for every $\epsilon > 0$, the interval $(p - \epsilon, p + \epsilon)$ has a nonempty intersection with the red, green and blue basins. For convenience we write for every integer $n > 0$, $D_n^{\text{blue}} = D_n \cap B_1$, $D_n^{\text{green}} = D_n \cap B_2$, and $D_n^{\text{red}} = D_n \cap B_3$. From the continuity of G and the primitivity of the matrix A_G , we have the following.

Three Color Property. For every $n \in \mathbb{N}$, for each component K of C_n , there exist a component U^{blue} of $D_n^{\text{blue}} \cup D_{n+1}^{\text{blue}}$, a component U^{green} of $D_n^{\text{green}} \cup D_{n+1}^{\text{green}}$,

and a component U^{red} of $D_n^{\text{red}} \cup D_{n+1}^{\text{red}}$ such that $U^{\text{blue}} \subset K$, $U^{\text{green}} \subset K$, and $U^{\text{red}} \subset K$.

Since the map g has a negative Schwarzian derivative, by applying Property SD-2 we have that the map $G^2 = g^6$ has a negative Schwarzian derivative. Applying Property SD-4 and a simple computation yield that the map G^2 is expanding on C_2 , that is, there exists $\delta > 0$ such that $|(G^2)'(x)| \geq 1 + \delta$ for all $x \in C_2$. This implies that the lengths of the components of C_n go to zero as $n \rightarrow \infty$.

We now show that the boundaries of the basins coincide in the interval $[r, b]$. Let $p \in C_\infty$ be given. Let $\epsilon > 0$ be given. Since the lengths of the components of C_n go to zero as $n \rightarrow \infty$, let W be a component of C_n (for some $n \in \mathbb{N}$) such that $p \in W \subset (p - \epsilon, p + \epsilon)$. Apply the Three Color Property and it follows that p is a Wada point. Since there are exactly three basins of attraction, $\partial \overline{B_1} \cap [r, b] = \partial \overline{B_2} \cap [r, b] = \partial \overline{B_3} \cap [r, b]$.

Finally, we show that the three boundaries coincide. Let $p \in [0, 1]$ be an arbitrarily given boundary point of any of the basins. It is left to the reader to show that there exists an integer $m > 0$ such that $G^m(p) \in C_\infty$. Hence, the boundaries of the three basins of G coincide. Therefore, each of the three basins has a Wada basin boundary. We conclude that G has three Wada basins. Therefore, G has the Wada property. $\square$

Based on the idea of this example, many new examples of maps having the Wada property may be generated as follows. Let $m \geq 3$ be a natural number and assume that the logistic map $g_r : [0, 1] \rightarrow [0, 1]$ defined by $g_r(x) = rx(1-x)$ has an attracting periodic orbit of period m for some $3.8 < r < 4$, which has been created by a periodic point creating tangent bifurcation similar to the period-3 point creating tangent bifurcation above. Then the map g_r^m has the Wada property: the map g_r^m has m basins and their boundaries coincide. Hence, the logistic map may serve as a Wada property generating map.

4 On the creation of Wada basins in the perturbed logistic map

To have a first example of a one-parameter family of maps that has a tangent bifurcation at which the Wada property for this family is born, we modify the third iterate of the logistic map $g(x) = 3.84x(1-x)$ of Sec. 3. The idea is to introduce maps that modify g^3 such that the new maps have negative Schwarzian derivative and undergo a tangent bifurcation in the neighborhood of the left stable fixed point of g^3 . Thus, the number of attracting fixed points of the new maps jumps from two to three when the parameter is varied.

Define the one-parameter family of maps F on the unit interval by $F_\mu(x) = G(x) + (1-\mu)x \exp(-7x) = g^3(x) + (1-\mu)x \exp(-7x)$. For three selected values of μ , the graphs of F_μ are shown in Figs. 3(a)-(c). The map F_μ is smooth and has a negative Schwarzian derivative, and the parameter μ belongs to the interval $[0.75, 1]$. For $0.75 \leq \mu < \mu_0 \approx 0.826$, the map F has two fixed point attractors $A_1(\mu)$ and $A_2(\mu)$ for which $A_2(\mu) < A_1(\mu)$. For $\mu_0 < \mu \leq 1$, the map F_μ has three fixed point attractors $A_1(\mu)$, $A_2(\mu)$, and $A_3(\mu)$ for which $A_3(\mu) < A_2(\mu) < A_1(\mu)$, and at $\mu = \mu_0$, there is a tangent bifurcation at x_0 . (Utilizing results in [25], we have that for $\mu_0 < \mu \leq 1$, Lebesgue almost every initial condition is in one of the three basins.)

Proposition. At $\mu = \mu_0 \approx 0.826$, the map F_μ has a fixed point creating tangent bifurcation and the map F_μ has the Wada property for $\mu_0 < \mu \leq 1$.

Proof. We leave it to the reader to verify that there is a tangent bifurcation at $\mu = \mu_0 \approx 0.826$. For every $0.75 \leq \mu \leq 1$, it is straightforward to verify that the map F_μ has negative Schwarzian derivative, and we write $c_{1+\mu}$ for the critical point of F_μ which equals the critical point c_2 of g^3 when $\mu = 1$. For $\mu_0 < \mu \leq 1$, write r_μ for the leftmost positive unstable fixed point, so r_μ equals the unstable fixed point r of g^3 when $\mu = 1$. With these notations, the proof that F_μ has the Wada property for $\mu_0 < \mu \leq 1$ is similar to the constructive proof of the example of the third iterate of the logistic map in Sec. 3. Therefore, we leave the details to the reader. $\square$

Another way to understand why Wada basins emerge after the tangent bifurcation in the example above, is the following reasoning. For $0.75 \leq \mu \leq 1$, write $B_i(\mu)$ for the basin of the fixed point attractor $A_i(\mu)$ ($1 \leq i \leq 2$), and for $\mu_0 < \mu \leq 1$, write $B_3(\mu)$ for the basin of the fixed point attractor $A_3(\mu)$. For $0.75 \leq \mu \leq 1$, the basins $B_1(\mu)$ and $B_2(\mu)$ have a common boundary, so $\partial B_1(\mu) = \partial B_2(\mu)$, and the boundary is a fractal basin boundary. At $\mu = \mu_0$, the tangent bifurcation occurs at x_0 which is on the fractal basin boundary $\partial B_1(\mu_0)$. This immediately implies that every open interval $(x_0 - \epsilon, x_0 + \epsilon)$ intersects the three basins, so x_0 is a Wada point. Utilizing methods from [25] one shows that every boundary point of $B_3(\mu_0)$ is a Wada point. Hence,

$B_3(\mu_0)$ is a Wada basin.

5 Wada tangent bifurcations in smooth one-dimensional maps

We consider one-dimensional maps with multiple critical points to explain whether and why Wada basin boundaries emerge from saddle node bifurcations. Basin boundaries for one-dimensional maps with multiple critical points that are hyperbolic on the basin boundary, are well-understood, see [25,26].

We first discuss a second example of a one-parameter family of maps F_μ with negative Schwarzian derivative. Assume that at μ_0 the family of maps F_μ has a tangent bifurcation at r as shown in Fig. 4 and that two fixed points are created as the parameter μ increases. All critical values of the map are contained in the immediate basins of attraction of the two attracting fixed points and the one one-sided attracting fixed point. By just looking at the graph, one observes that to the left of the middle attracting fixed point, there is a fractal boundary between the green and red basins and to the right of the middle attracting fixed point, there is a fractal boundary between the green and blue basins, but no basin is a Wada basin. Hence, there is no Wada basin emerging from the tangent bifurcation.

Using the matrix approach, and labeling the intervals as shown in Fig. 4, we get that the corresponding matrix $A_{F_\mu} = [a_{ik}]_{1 \leq i, k \leq 6}$ of F_μ (for $\mu_0 < \mu < \mu_0 + \epsilon$) is a 6×6 matrix. The matrix entry a_{ik} is the number of times that F_μ maps the interval J_i over J_k , $1 \leq i, k \leq 6$. Hence,

$$A_{F_\mu} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

The matrix A_{F_μ} is not primitive. In fact, it is not even irreducible.

Before we can formulate a theorem concerning “Wada tangent bifurcations”, we first define the notion of “core matrix”. Let $F : I \rightarrow I$ be a C^3 -map that has negative Schwarzian derivative, and assume that (a) every attractor of F is a fixed point, (b) every critical point of F is contained in a basin of attraction of a fixed point attractor, and (c) every periodic point of F is

hyperbolic (a periodic point $q \in I$ of period n is hyperbolic $\Leftrightarrow F^n(q) = q$ and $|(F^n)'(q)| \neq 1$).

We denote by D_0 the union of the immediate basins of the stable fixed points of F , and we denote by C_1 the complement of D_0 in the compact interval I , so $C_1 = I \setminus D_0$. By induction, we define for every positive integer n ,

$$\begin{aligned} D_n &= \{x \in C_n : F(x) \in D_{n-1}\}, \\ C_{n+1} &= \{x \in C_n : F(x) \in C_n\}. \end{aligned}$$

In other words, D_n is the set of points whose $(n-1)$ th iterates are still in C_1 and whose n th iterates are in the immediate basin of a stable fixed point of F . The set C_{n+1} is the set of points whose n th iterates are still in C_1 .

Finally, write C_∞ for the set of points whose trajectories will stay in C_1 under forward iteration, that is,

$$C_\infty = \{x \in I : F^k(x) \in C_1 \text{ for each integer } k \geq 0\}.$$

Hence, C_∞ is the set of points whose trajectories do not converge to a stable fixed point of F . The Capture Property, the Expanding Property, the Invading Property, and the Measure-zero Property below are properties in their own interest and are the key ingredients for the proof of the theorem below. For proofs of these properties, we refer the reader to [25] since they are special cases of results in that article.

Capture property. There exists $m \in \mathbb{N} \cup \{0\}$ such that every critical point of F is contained in the set $\cup_{i=0}^m D_i$, that is, if $F'(c) = 0$ then $c \in \cup_{i=0}^m D_i$.

Expanding property. Let m be the minimal number for which the Capture Property holds. If every periodic point of F is hyperbolic, then there exists $N \in \mathbb{N}$ such that $|(F^N)'(x)| > 1$ for all $x \in C_{N+m}$.

Invading property. Let m and N be as in the Expanding property. For every integer $n \geq N+m$ and for every component C of C_n , there exists a component D of $\cup_{i=n}^{n+m} D_i$ such that $D \subset C$.

Measure-zero property. The Lebesgue measure of the set C_∞ is zero.

We refer to the minimum number m for which the Capture property holds as the *capture number* of F . Write r for the right endpoint of the leftmost immediate basin of attraction; write b for the left endpoint of the rightmost immediate basin of attraction (for an example, see the proof of the Wada property for the third iterate of the logistic map in Sec. 3). Let m be the capture number of F , let N be a positive integer as in the Expanding property,

and consider the map F on C_{N+m} . Note that C_{N+m} does not contain critical points of F , since F is expanding on C_{N+m} . Denote the number of intervals in $C_{N+m} \cap [r, b]$ by N_m and denote the intervals of $C_{N+m} \cap [r, b]$ by J_i ($1 \leq i \leq N_m$) with the right endpoint of J_i smaller than the left endpoint of J_{i+1} ($1 \leq i \leq N_m - 1$). The *core matrix* of F is the $N_m \times N_m$ matrix $A_{\text{core}}[F] = [a_{ik}]_{1 \leq i, k \leq N_m}$ defined by a_{ik} is the number of times that F maps the interval J_i over J_k , $1 \leq i, k \leq N_m$. Hence, the core matrix of F is defined by $a_{ik} = 1 \Leftrightarrow F(J_i) \cap \text{Int}(J_k) \neq \emptyset$ and $a_{ik} = 0 \Leftrightarrow F(J_i) \cap \text{Int}(J_k) = \emptyset$, $1 \leq i, k \leq N_m$. Note that if $F(J_i) \cap \text{Int}(J_k) \neq \emptyset$ then $F(J_i) \supset J_k$. Note further that the matrix of the map G defined in Sec. 3 is slightly different from the core matrix. The core matrix of G is an 8×8 matrix and is given by

$$A_{\text{core}}[G] = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Note that the matrix $A_{\text{core}}[G]$ is primitive, that is, there exists $1 \leq k \leq 8$ such that each entry of the k -th power of $A_{\text{core}}[G]$ is positive. A handy way to check that the matrix is primitive is to first to establish that the matrix is irreducible by verifying that the corresponding directed graph is strongly connected. Then the primitivity follows from the facts that the matrix is irreducible and the trace is positive. Now we are able to state a theorem.

Theorem. Let $F_\mu : I \rightarrow I$ be a one-parameter family of C^3 -maps that have negative Schwarzian derivative. Assume that (a) every attractor of F_μ is a fixed point, (b) F_μ has at least two fixed point attractors and at most three fixed point attractors, (c) at the parameter value μ_0 , F_μ has a fixed point creating tangent bifurcation at the location x_0 , and (d) for some $0 < \delta < 1$ (δ may be small) and for every $\mu_0 \leq \mu < \mu_0 + \delta$, all critical points of F_μ are contained in the basins of attraction. Then: F_μ has the Wada property for $\mu_0 < \mu < \mu_0 + \delta \Leftrightarrow$ The core matrix $A_{\text{core}}[F_{\mu_0}]$ of F_{μ_0} is primitive.

Corollary. Let the one parameter family of maps F_μ be as in the Theorem, and denote the basins of attraction of the two fixed point attractors that exist before μ_0 by $B_1(\mu)$ and $B_2(\mu)$. Then: F_μ has the Wada property for $\mu_0 < \mu < \mu_0 + \delta \Leftrightarrow x_0$ is on the common boundary of $B_1(\mu_0)$ and $B_2(\mu_0)$, that

is, $x_0 \in \overline{\partial B_1(\mu_0)} = \overline{\partial B_2(\mu_0)}$.

Proof. Let the one parameter family of maps F_μ be as in the Theorem and assume it satisfies conditions (a)-(d). For $\mu_0 - \delta < \mu < \mu_0 + \delta$, denote the basins of the two fixed point attractors that exist and persist by $B_1(\mu)$ and $B_2(\mu)$, and for $\mu_0 \leq \mu < \mu_0 + \delta$, denote the basin of attraction that is created at μ_0 by $B_3(\mu)$. For $\mu_0 - \delta < \mu < \mu_0 + \delta$, we denote by $D_0(\mu)$ the union of the immediate basins of the stable fixed points of F_μ , and we denote by $C_1(\mu)$ the complement of $D_0(\mu)$ in the compact interval I , so $C_1(\mu) = I \setminus D_0(\mu)$. By induction, we define for every $n \in \mathbb{N}$, $D_n(\mu) = \{x \in C_n(\mu) : F_\mu(x) \in D_{n-1}(\mu)\}$ and $C_{n+1}(\mu) = \{x \in C_n(\mu) : F_\mu(x) \in C_n(\mu)\}$. In other words, $D_n(\mu)$ is the set of points whose $(n-1)$ th iterates are still in $C_1(\mu)$ and whose n th iterates are in the immediate basin of some stable fixed point of F_μ , and $C_{n+1}(\mu)$ is the set of points whose n th iterates are still in $C_1(\mu)$. Finally, write $C_\infty(\mu)$ for the set of points whose forward iterates stay in $C_1(\mu)$, so $C_\infty(\mu) = \{x \in I : F_\mu^k(x) \in C_1(\mu) \text{ for each integer } k \geq 0\}$.

For convenience, we write for every integer $n \geq 0$, $D_n^{[1]}(\mu) = D_n(\mu) \cap B_1(\mu)$ and $D_n^{[2]}(\mu) = D_n(\mu) \cap B_2(\mu)$ if $\mu_0 - \delta < \mu < \mu_0 + \delta$, and $D_n^{[3]}(\mu) = D_n(\mu) \cap B_3(\mu)$ if $\mu_0 \leq \mu < \mu_0 + \delta$. For $\mu_0 \leq \mu < \mu_0 + \delta$, all the critical points of F_μ are contained in the three basins of attraction. Let $\delta > 0$ such that the capture number of F_μ is constant and $A_{\text{core}}[F_\mu] = A_{\text{core}}[F_{\mu_0}]$. Denote the capture number by m .

For $\mu_0 \leq \mu < \mu_0 + \delta$, write r_μ for the right endpoint of the leftmost immediate basin of attraction; write b_μ for the left endpoint of the rightmost immediate basin of attraction.

“ $\Rightarrow$ ” Assume that F_μ has the Wada property for $\mu_0 < \mu < \mu_0 + \delta$. This implies that for $\mu_0 < \mu < \mu_0 + \delta$, $\overline{\partial B_1(\mu)} = \overline{\partial B_2(\mu)} = \overline{\partial B_3(\mu)}$.

Let $\mu_0 < \mu < \mu_0 + \delta$ be given. Suppose that the matrix $A_{\text{core}}[F_{\mu_0}]$ is not primitive. Then the matrix $A_{\text{core}}[F_\mu]$ is not primitive, since $A_{\text{core}}[F_\mu] = A_{\text{core}}[F_{\mu_0}]$. Since the map F_μ has three stable fixed points and is continuous, there exists at least one interval J_r that contains a fixed point of F_μ , so $F_\mu(J_r) \supset J_r$. It follows that the trace of $A_{\text{core}}[F_\mu]$ is positive. We now have that $A_{\text{core}}[F_\mu]$ is reducible, since $A_{\text{core}}[F_\mu]$ is not primitive and the trace of $A_{\text{core}}[F_\mu]$ is positive. Hence, there exist intervals J_i and J_k such that $F_\mu^n(J_i) \cap \text{Int}(J_k) = \emptyset$ for every $n \in \mathbb{N}$, where $1 \leq i, k \leq N_m$. By the Invading property, there is a component D of $\cup_{i=0}^m D_{N+m+i}$ such that $D \subset J_k$. This implies that the three boundaries do not coincide. This contradicts the assumption that $\overline{\partial B_1(\mu)} = \overline{\partial B_2(\mu)} = \overline{\partial B_3(\mu)}$. Hence, $A_{\text{core}}[F_\mu]$ is primitive. Since $A_{\text{core}}[F_\mu] = A_{\text{core}}[F_{\mu_0}]$, the conclusion is that $A_{\text{core}}[F_{\mu_0}]$ is primitive.

“ $\Leftarrow$ ” Assume that the core matrix $A_{\text{core}}[F_{\mu_0}]$ of F_{μ_0} is primitive. Then every entry of the matrix N_m th power of the matrix $A_{\text{core}}[F_{\mu_0}]$ is positive.

We now explain that every $p \in C_\infty$ is a Wada point, that is, it is a boundary point of at least three basins. In other words, we want to show that for every $\epsilon > 0$, the interval $(p - \epsilon, p + \epsilon)$ has a nonempty intersection with the $B_1(\mu)$, $B_2(\mu)$ and $B_3(\mu)$ basins. From the continuity of the map F_μ and the primitivity of the matrix $A_{\text{core}}[F_\mu] = A_{\text{core}}[F_{\mu_0}]$ for $\mu_0 < \mu < \mu_0 + \delta$, we have the following.

Three Basin Property. Let $\mu_0 < \mu < \mu_0 + \delta$ be arbitrarily fixed. For every $n \in \mathbb{N}$, for each component K of $C_n(\mu) \cap [r_\mu, b_\mu]$, there exist components U^1 of $\cup_{i=n}^{n+m}(D_i^{[1]}(\mu) \cap [r_\mu, b_\mu])$, U^2 of $\cup_{i=n}^{n+m}(D_i^{[2]}(\mu) \cap [r_\mu, b_\mu])$, and U^3 of $\cup_{i=n}^{n+m}(D_i^{[3]}(\mu) \cap [r_\mu, b_\mu])$ such that $U^1 \subset K$, $U^2 \subset K$, and $U^3 \subset K$.

Since the map F_μ (for all μ) has a negative Schwarzian derivative, by applying Property SD-2 we have that all iterates of this map have a negative Schwarzian derivative. Let $\mu_0 < \mu < \mu_0 + \delta$ be arbitrarily fixed. Applying the Expanding Property we have that the $N[\mu]$ th iterate of F_μ , denoted by G , is expanding on $C_{N[\mu]}(\mu)$; that is, there exists $\xi > 1$ such that $|(G)'(x)| \geq \xi$ for all $x \in C_{N[\mu]}(\mu)$. This implies that the lengths of the components of $C_n(\mu)$ go to zero as $n \rightarrow \infty$. Note that this also follows from the Measure-zero Property.

We now show that the boundaries of the basins coincide in the interval $[r_\mu, b_\mu]$. Let $p \in C_\infty(\mu) \cap [r_\mu, b_\mu]$ be given. Let $\epsilon > 0$ be given. Since the lengths of the components of $C_n(\mu)$ go to zero as $n \rightarrow \infty$, let W be a component of $C_n(\mu)$ for some appropriately chosen $n \in \mathbb{N}$, such that $p \in W \subset (p - \epsilon, p + \epsilon)$. Apply the Three Basin Property and it follows that p is a Wada point. Since there are exactly three basins of attraction, $\partial \overline{B_1(\mu)} \cap [r_\mu, b_\mu] = \partial \overline{B_2(\mu)} \cap [r_\mu, b_\mu] = \partial \overline{B_3(\mu)} \cap [r_\mu, b_\mu]$.

Finally, we show that the three boundaries coincide. Let $p \in I$ be an arbitrarily given boundary point of any of the basins. It is left to the reader to show that there exists an $s \in \mathbb{N} \cup \{0\}$ such that $F_\mu^s(p) \in C_\infty(\mu) \cap [r_\mu, b_\mu]$. Hence, the boundaries of the three basins of F_μ coincide. Therefore, each of the three basins has a Wada basin boundary. We conclude that F_μ has three Wada basins. Hence, F_μ has the Wada property for every $\mu_0 < \mu < \mu_0 + \delta$. $\square$

6 Discussion and Conclusions

One of the goals of nonlinear dynamics is to determine the global structure of a system such as boundaries of basins. In Refs. [12,10] it is argued why Wada basins emerge when a saddle-node bifurcation occurs in a variety of two-dimensional systems like the forced damped pendulum or the forced Duffing oscillator. In these references, it is assumed that the saddle-node bifurcation occurs on a fractal basin boundary. However, in the literature no numerically verifiable condition guaranteeing that the saddle-node bifurcation occurs on

the common fractal basin boundary is formulated.

Since one-dimensional maps are easier to analyze, we investigate whether Wada basins can emerge when a tangent bifurcation occurs for certain one-dimensional maps without assuming that this bifurcation occurs on a fractal basin boundary. We formulate a numerically verifiable criterion on the creation of Wada basins in a certain class of one-dimensional maps satisfying some hypotheses which are numerically verifiable. Our main result specifies explicitly the (sufficient and necessary) condition guaranteeing that the new emerging basin is a Wada basin.

In this paper, we present a sufficient and necessary condition guaranteeing three Wada basins that are emerging from a tangent bifurcation for certain one-dimensional maps (maps having at least two but at most three fixed point attractors). We provide a theorem guaranteeing these emerging Wada basins with hypotheses which are numerically verifiable. In a general approach, one eliminates the constraints that every attractor is a fixed point attractor and that there are at most three attractors. The result in this paper can be considered as a first step in formulating numerically verifiable criteria on the creation of Wada basins in a more general approach such as in a general class of one-dimensional maps and higher dimensional systems. Such a general approach is beyond the scope of this paper and a more general approach in one-dimensional maps is momentarily work in progress.

Acknowledgments

This work was in part supported by the National Sciences Foundation (Division of Mathematical Sciences and Physics, Grant No. 0104087).

References

- [1] L.E.J. Brouwer, Zur Analysis Situs, Math. Annalen 68 (1910), 422-434.
- [2] J. Hocking and G. Young, Topology, Addison-Wesley, Reading, Mass., 1961.
- [3] J. Kennedy and J.A. Yorke, Basins of Wada, Physica D 51 (1991), 213-225.
- [4] H.E. Nusse and J.A. Yorke, Basins of Attraction, Science 271 (1996), 1376-1380.
- [5] H.E. Nusse and J.A. Yorke, Wada basin boundaries and basin cells, Physica D 90 (1996), 242-261.
- [6] H.E. Nusse and J.A. Yorke, The structure of basins of attraction and their trapping regions, Ergodic Theory & Dynam. Systems 17 (1997), 463-481.

- [7] U. Feudel, A. Witt, Y.-C. Lai, and C. Grebogi, Basin bifurcation in quasiperiodically forced systems, *Phys. Rev. E* 58 (1998), 3060-3066.
- [8] J.H. Hubbard, The forced damped pendulum: Chaos, Complication and Control, *Amer. Math. Monthly* 106 (1999), 741-758.
- [9] J. Aguirre, J.C. Vallejo and M.A.F. Sanjuan, Wada basins and chaotic invariant sets in the Henon-Heiles system, *Phys. Rev. E* 64 (2003), 066208.
- [10] J. Aguirre and M.A.F. Sanjuan, Unpredictable behavior in the Duffing oscillator: Wada basins, *Physica D* 171 (2002), 41-51.
- [11] R. Breban, H.E. Nusse and E. Ott, Scaling properties of saddle-node bifurcations on fractal basin boundaries, *Phys. Rev. E* 68 (2003), 066213.
- [12] H.E. Nusse, E. Ott and J.A. Yorke, Saddle-node bifurcations on fractal basin boundaries, *Phys. Rev. Lett.* 75 (1995), 2482-2485.
- [13] S.W. McDonald, C. Grebogi, E. Ott and J.A. Yorke, Fractal basin boundaries, *Physica D* 17 (1985), 125-153.
- [14] E. Ott, *Chaos in dynamical systems*, Second edition, Cambridge University Press, Cambridge, 2002.
- [15] J. Guckenheimer, On the bifurcations of maps of the interval. *Invent. Math.* 39 (1977), 165-178.
- [16] D. Whitley, Discrete dynamical systems in dimension one and two, *Bull. Lond. Math. Soc.* 15 (1983), 177-217.
- [17] R.L. Devaney, *An introduction to Chaotic Dynamical Systems*, Second edition, Addison-Wesley Publishing Company, Redwood City, 1989.
- [18] K.T. Alligood, T.D. Sauer and J.A. Yorke, *Chaos - An introduction to dynamical systems*, Springer-Verlag, New York, 1997.
- [19] Y.A. Kuznetsov, *Elements of Applied Bifurcation Theory*, Second Edition, Springer-Verlag, New York, 1998.
- [20] D. Singer, Stable orbits and bifurcation of maps of the interval, *SIAM J. Appl. Math.* 78 (1978), 260-267.
- [21] J. Guckenheimer, Sensitive dependence on initial conditions for one-dimensional maps, *Commun. Math. Phys.* 70 (1979), 133-160.
- [22] M. Misiurewicz, Absolutely continuous measures for maps of an interval, *I.H.E.S. Publ. Math.* 53 (1981), 17-51.
- [23] D.J. Allwright, Hypergraphic functions and bifurcations in recurrence relations, *SIAM J. Appl. Math.* 34 (1978), 687-691.
- [24] T.-Y. Li and J.A. Yorke, Period three implies chaos, *Amer. Math. Monthly* 82 (1975), 985-992.

- [25] H.E. Nusse, Asymptotically periodic behaviour in the dynamics of chaotic mappings, SIAM J. Appl. Math. 47 (1987), 498-515.
- [26] H.E. Nusse, Qualitative analysis of the dynamics and stability properties for Axiom A maps, J. Math. Analysis and Applications, 136 (1988), 74-106.

Fig. 1. Graph of the third iterate of the logistic map, $G(x) = g^3(x)$, where $g(x) = 3.84x(1 - x)$. The immediate basins of attraction of the three fixed points of G are IB_1 , IB_2 , and IB_3 .

Fig. 2. The choice of the intervals J_i ($1 < i < 5$) for the map G .

Fig. 3. Panels (a), (b) and (c) show graphs of $F_\mu(x)$ for three different values of μ : $\mu = 0.95$, $\mu \approx \mu_0 \approx 0.826$, and $\mu = 0.75$, respectively. Hence, if one increases μ from 0.75 to 0.95, then at $\mu = \mu_0 \approx 0.826$ there is a fixed point creating tangent bifurcation. Panels (d), (e) and (f) are, respectively, blow-ups in a neighborhood of the location where the tangent bifurcation occurs.

Fig. 4. The choice of the intervals J_i ($1 < i < 6$) for the map F for which no Wada basins emerge after the fixed point creating tangent bifurcation.