Common zeroes of families of smooth vector fields on surfaces

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Abstract

Let Y and X denote C^k vector fields on a possibly noncompact surface with empty boundary, $1 \leq k < \infty$. Say that Y *tracks* X if the dynamical system it generates locally permutes integral curves of X. Let K be a locally maximal compact set of zeroes of X.

Theorem. Assume the Poincaré-Hopf index of X at K is nonzero, and the k-jet of X at each point of K is nontrivial. If $\mathfrak{g}$ is a supersolvable Lie algebra of C^k vector fields that track X, then the elements of $\mathfrak{g}$ have a common zero in K.

Applications are made to attractors and transformation groups.

Contents

1 Introduction

M denotes a metrizable real analytic surface with empty boundary. The vector space of vector fields on M is $\mathcal{V}(M)$, topologized by uniform convergence on compact sets. The subspace of C^r vector fields is $\mathcal{V}^r(M)$. Here $r \in \mathbb{N}_+$ (the set of positive integers), or $r = \infty$ (infinitely differentiable), or $r = \omega$ (analytic).

X denotes a vector field on M, with zero set $\mathsf{Z}(X)$. A compact set $K \subset \mathsf{Z}(X)$ is a *block* of zeroes for X, or an *X-block*, if it has a precompact open neighborhood $U \subset M$ such that $\mathsf{Z}(X) \cap \overline{U} = K$. We say U is *isolating* for X and for (X, K).

The *index* of the X-block K is the integer $\mathsf{i}_K(X) := \mathsf{i}(X, U)$ defined as the Poincaré-Hopf index [20, 13] of any sufficiently close approximation to X having only finitely many zeroes in U. This number is independent of U, and is stable under perturbations of X : If $Y \in \mathcal{V}M$ is sufficiently close to X then U is isolating for Y, and $\mathsf{i}(Y, U) = \mathsf{i}(X, U)$(Proposition 2.2).[1]

K is *essential* if $\mathsf{i}_K(X) \neq 0$, which every isolating neighborhood of K meets $\mathsf{Z}(X)$. This powerful condition implies that if Y is sufficiently close to X then $\mathsf{Z}(Y) \cap U \neq \varnothing$.

We say *Y tracks X* provided $X, Y \in \mathcal{V}^1(M)$, and the corresponding local flows $\Phi^Y = \{\Phi^Y_t\}_{t\in\mathbb{R}}$ and $\Phi^X = \{\Phi^X_t\}_{t\in\mathbb{R}}$ have the following property: For each $t \in \mathbb{R}$ the C^1 diffeomorphism $\Phi^Y_t\colon\ \mathcal{D}_t \approx \mathcal{R}_t$ maps orbits of $X|\mathcal{D}_t$ to orbits of $X|\mathcal{R}_t$. Equivalently: There exists $f\colon\ M \setminus \mathsf{Z}(X) \to \mathbb{R}$ such that $X_p \neq 0 \implies [Y, X] = f(p)X_p$ (see Hirsch, [12, Prop. 2.4] or [10, Prop. 2.3]).[2]

If X spans an ideal in a Lie algebra $\mathfrak{g} \subset \mathcal{V}^k(M)$, every element of $\mathfrak{g}$ tracks X. When X is C^∞, the set of $Y \in \mathcal{V}^\infty(M)$ that track X is an infinite dimensional Lie algebra.

Statement of the main results

Throughout the rest of this article we assume:

- $X \in \mathcal{V}^k(M),\ k \in \mathbb{N}_+$.

X has *order* $\mathsf{ord}_p(X) := j \in \{1, \dots, k\}$ at $p \in \mathsf{Z}(X)$ if j is the smallest number in $\{1, \dots, k\}$ such that the j-jet of X at p is nontrivial. In other words: Some (and hence every) C^{k+1} chart $M \supset W' \approx W \subset \mathbb{R}^2$ on M, centered at p, represents $X|W$ by a C^k map $F\colon\ W \to \mathbb{R}^2$ whose partial derivatives at the orgin satisfy

$$F^{(i)}(0) = 0 \text{ for } i = 0, \dots, j-1, \qquad F^{(j)}(0) \neq 0. \tag{1}$$

If no integer j has this property, X is *k-flat* at p.

When X is analytic and nontrivial on a neighborhood of a continuum $L \subset \mathsf{Z}(X)$, the order of X is constant on L.

Theorem 1.1. *Assume $X, Y \in \mathcal{V}^k(M)$ have the following properties:*

(a) *$K \subset \mathsf{Z}(X)$ is an essential X-block,*

(b) *X is not k-flat at any point of K,*

(c) *Y tracks X.*

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[1] Equivalently: $\mathsf{i}(X, U)$ is the intersection number of $X|U$ with the zero section of the tangent bundle (C. Bonatti [3]). If X is generated by a smooth local flow ϕ then $\mathsf{i}(X, U)$ equals the fixed-point index $I(\phi_t|U)$ of A. Dold [5] for sufficiently small $t > 0$.

[2] Article [10] is a preliminary version of [12].

Then $\mathsf{Z}(Y) \cap K \neq \varnothing$.

When X and Y are commuting analytic vector fields, this is a special case of a remarkable theorem of C. Bonatti [3]— the inspiration for the present paper.

Theorem 1.1 is proved by demonstrating the strong form of the contrapositive stated below.

A *line field* Λ on a set $N \subset M$ is a (continuous) section $p \mapsto \Lambda_p$ of the fibre bundle over N whose fibre over $p \in N$ is the circle of unoriented lines through the origin in the tangent space $T_p(M)$. If $Y_p \in \Lambda_p$ for all $p \in N$ then Λ *controls* Y *in* N.

Theorem 1.2. *Assume:*

(a) $X \in \mathcal{V}^k(M)$ *is not k-flat at any point of the X-block K,*

(b) $Y \in \mathcal{V}^k(M)$ *tracks X,*

(c) $\mathsf{Z}(Y) \cap K = \varnothing$,

(d) $U \subset M$ *is an isolating neighborhood for* (X, K).

Then:

(i) $\mathsf{i}_K(X) = 0$.

(ii) *K has only finitely many components, and each component is a* C^k*-embedded circle.*

(iii) *X is controlled by a unique line field in U.*

(iv) *X has index zero at each component of K.*

(v) *X can be* C^k*-approximated by vector fields that have no zeroes in U and agree with X outside U.*

The is in Section 3.

Applications

Let $\mathfrak{g} \subset \mathcal{V}^k(M)$ denote a Lie algebra— a linear subspace closed under Lie brackets. The zero set of $\mathfrak{g}$ is defined as $\mathsf{Z}(\mathfrak{g}) := \bigcap_{Y \in \mathfrak{g}} \mathsf{Z}(Y)$. If every $Y \in \mathfrak{g}$ tracks X, then $\mathfrak{g}$ *tracks X*. We call $\mathfrak{g}$ *supersolvable* if it is faithfully represented by upper triangular real matrices.

Theorem 1.3. *Assume:*

(a) *K is an essential X-block,*

(b) *X is not k-flat at any point of K,*

(c) $\mathfrak{g} \subset \mathcal{V}^k(M)$ *is a supersolvable Lie algebra tracking X.*

Then $\mathsf{Z}(\mathfrak{g}) \cap K \neq \varnothing$.

Related theorems and counterexamples are discussed in Hirsch [12].

Theorem 1.4. *Suppose the local flow of* $X \in \mathcal{V}^\omega(M)$ *has a compact attractor* $P \subset M$ *with Euler characteristic* $\chi(P) \neq 0$. *Then there exists* $k \in \mathbb{N}_+$ *with the following property: If* $\mathfrak{h} \subset \mathcal{V}^k(M)$ *is a supersolvable Lie algebra that tracks* X, *then* $\mathsf{Z}(\mathfrak{h}) \cap \mathsf{Z}(X) \cap P \neq \varnothing$.

Proof. If $P = M$ then M is a closed surface and $\chi(M) \neq 0$. Therefore $\mathsf{Z}(X)$ is an essential X-block by Poincaré's Theorem [20], and the conclusion follows from Theorem 1.2.

Suppose $P \neq M$. The basin of attraction of P contains a smooth compact surface N with boundary such that Y is inwardly transverse to ∂N. The interior $N_0 := N \setminus \partial N$ is a precompact open set which is positively invariant under Φ^Y (F. Wilson [25, Th. 2.2]). Therefore

$$t > s \geq 0 \implies \Phi^Y_t(N_0) \subset \Phi^Y_s(N_0), \qquad \bigcap_{t\geq 0} \Phi^Y_t(N_0) = P.$$

The inclusion maps $P \hookrightarrow N_0 \hookrightarrow N$ induce isomorphisms of Čech cohomology groups, hence $\chi(N_0) \neq 0$. The conclusion follows from Theorem 1.3 appplied to $X := Y|N_0$ and the Lie algebra $\mathfrak{g} \subset \mathcal{V}^k(N_0)$ comprising the restrictions of the vector fields in $\mathfrak{h}$ to N_0. ∎

Example 1.5. Assume $P \subset \mathbb{R}^2$ is a compact global attractor for $X \in \mathcal{V}^\omega(\mathbb{R}^2)$ and $\mathfrak{h} \subset \mathcal{V}^k(\mathbb{R}^2)$ is a supersolvable Lie algebra tracking Y. Then:

- $\mathsf{Z}(\mathfrak{h}) \cap \mathsf{Z}(X) \cap P \neq \varnothing$.

Proof. $\chi(P) \neq 0$ because $\mathbb{R}^2$ is contractible and P is a global attractor, so Theorem 1.4 yields the conclusion. ∎

Corollary 1.6. *Let* G *be a connected Lie group whose Lie algebra is supersolvable, with an effective* C^∞ *action on a compact surface* M. *If* $\chi(M) \neq 0$ *and the action is analytic on a normal 1-dimensional Lie subgroup, then* G *fixes a point of* M.

The special case in which M is compact and G acts analytically is due to Hirsch & Weinstein [7].

Proof. The action of G on M induces an isomorphism θ from the Lie algebra $\mathfrak{a}$ of G onto a subalgebra $\mathfrak{g} \subset \mathcal{V}^\infty(M)$. Let $Y \in \mathfrak{a}$ span the Lie algebra of H and set $\theta(Y) = X \in \mathfrak{g}$. Then $X \in \mathcal{V}^\omega(M)$ and $\mathfrak{g}$ tracks X, whence the conclusion from Theorem 1.4. ∎

Related results on Lie group actions and Lie algebras of vector fields can be found in the articles [1, 2, 4, 8, 9, 11, 15, 17, 18, 19, 22, 23, 24].

2 Index calculations

Proposition 2.1. *Assume $Y, X \in \mathcal{V}^k(M)$ and Y tracks X.*

(i) $\mathsf{Z}(X)$ *is invariant under* Φ^Y.

(ii) *If $p, q \in \mathsf{Z}(X)$ are in the same orbit of Φ^Y, then* $\mathsf{ord}_p(X) = \mathsf{ord}_q(X)$. ∎

Proof. Follows from the definition of tracking. ∎

These properties of the index function are crucial:

Proposition 2.2 (Stability)**.** *Let $U \subset M$ be isolating for X.*

(a) *If* $\mathsf{i}(X, U) \neq 0$ *then* $\mathsf{Z}(X) \cap U \neq \varnothing$.

(b) *If Y is sufficiently close to X then* $\mathsf{i}(Y, U) = \mathsf{i}(X, U)$.

(c) *Let $\{X^t\}_{t\in[0,1]}$ be a deformation of X. If each X^t is nonsingular on the frontier of U, then* $\mathsf{i}(X^t, U) = \mathsf{i}(X, U)$.

Proof. This is Theorem 3.9 of [12]. ∎

Proposition 2.3. *Assume $Y, Y' \in \mathcal{V}(M)$ and $U \subset M$ is isolating for both Y and Y'. Assume $N := \overline{U}$ is a compact C^1 surface such that Y_p and Y'_p are linearly dependent at all $p \in \partial N$. Then* $\mathsf{i}(Y, U) = \mathsf{i}(Y', U)$.

Proof. This consequence of Proposition 2.2 is a special case of [10, Prop. 3.12] or [12, Prop. 3.11]. ∎

Proposition 2.4. *Let $K \subset M$ be a block of zeroes for $X \in \mathcal{V}^k(M)$, and $U \subset M$ an isolating neighborhood for (X, K). If $X|U$ is controlled by a line field Λ on U, then* $\mathsf{i}_K(X) = 0$.

Proof. By shrinking U slightly, we assume $N := \overline{U}$ is a compact C^1 surface.

Suppose Λ is an orientable line field. Then Λ controls a nonsingular vector field Y on N, and $\mathsf{i}(Y, U) = 0$ because $\mathsf{Z}(Y) = \varnothing$. Evidently U is isolating for both X and Y, and X_p, Y_p are linearly dependent at all $p \in N$. Proposition 2.3 implies $\mathsf{i}(X, U) = \mathsf{i}(Y, U) = 0$, hence $\mathsf{i}_K(X) = 0$.

Now suppose Λ is nonorientable. There is a double covering $\pi\colon \tilde{V} \to V$ of an open neighborhood $V \subset M$ of N, isolating for (X, K), such that Λ lifts to an orientable line field on $\tilde{V}$. The orientable case shows that the vector field $\tilde{X}$ on $\tilde{V}$ that projects to $X|V$ under π has index zero in $\tilde{V}$.

Fix $X_1 \in \mathcal{V}(V)$ with $\mathsf{Z}(X_1)$ finite, and such that the sum of the Poincaré-Hopf indices of the zeroes of X_1 equals $\mathsf{i}(X, V)$. Define $\tilde{X}_1 \in \mathcal{V}(\tilde{V})$ to be the vector field projecting to X_1 under π. Each zero p of X_1 is the image under π of exactly two zeroes q_1, q_2 of $\tilde{X}_1$, and the Poincaré-Hopf indices of $\tilde{X}$ at q_1 and q_2 both equal the Poincaré-Hopf index of X at p. Therefore

$$0 = \mathsf{i}(\tilde{X}_1, \tilde{V}) = 2\mathsf{i}(X_1, V) = 2\mathsf{i}(X, V) = 2\mathsf{i}_K(X),$$

completing the proof. ∎

Other calculations of indices can be found in [5, 6, 14, 16, 21].

3 Proof of Theorem 1.2

Lemma 3.1. *Let $L \subset K$ be a component. Then X has the same order at each point of L.*

Proof. The sets $L_j = \{p \in L\colon \operatorname{ord}_p(X) = j\}$, $j \in \{1, \dots, k\}$ are relatively open in L and mutually disjoint. As L is connected and covered by the L_j, it coincides with one of them. ∎

Let $p \in K$ be arbitrary. Choose a C^{k+1} *flowbox* h_p for Y centered at p:

$$h_p\colon\ W'_p \approx W_p = J_p \times J'_p \subset \mathbb{R}^2, \quad h(p) = (0,0). \tag{2}$$

This means W'_p is open in M, $J_p, J'_p \subset \mathbb{R}$ are open intervals around 0, and the C^{k+1} diffeomorphism h_p transforms $Y|W'_p$ to the constant vector field $\frac{\partial}{\partial x}\Big| W_p$, where x, y are the usual planar coordinates. Notice that $h_p(K \cap W'_p) = J_p \times \{0\}$ because K is Y-invariant.

The transform of $X|W'_p$ by h_p is a C^k vector field

$$\hat{X}(p) \in \mathcal{V}^k(W_p), \quad (x,y) \mapsto F_p(x,y),$$

where $F_p\colon\ W_p \to \mathbb{R}^2$ is C^k.

Set $\operatorname{ord}_p(X) = l \in \{1, \dots, k\}$. The partials of F_p satisfy

$$F_p^{(i)}(0) = 0 \text{ for } i = 0, \dots, l-1, \qquad F_p^{(l)}(0) \neq 0.$$

In a sufficiently small open disk

$$D := D_p \subset \mathbb{R}^2, \quad (0,0) \in D_p, \tag{3}$$

the l'th order Taylor expansion of F_p about $(0,0)$ takes the form

$$F_p(x,y) = y^l g(x,y), \qquad g(x,y) \neq (0,0) \tag{4}$$

with $g\colon\ D_p \to \mathbb{R}^2$ continuous. Therefore

$$\mathsf{Z}(\hat{X}(p)) = F_p^{-1}(0,0) = J_p \times \{0\},$$

whence $K \cap W'_p$ is an open arc, relatively closed in W'_p.

It follows that K has an open cover by open arcs. Thus K is a compact 1-manifold having only finitely many components, each of which is a topological circle. The restriction of Φ^Y to any component $L \subset K$ is a smooth flow with no fixed points. Therefore L is a periodic orbit of Φ^Y, and is thus a smooth submanifold. This proves Theorem 1.2(ii).

Lemma 3.2. *$\hat{X}$ is controlled by a unique line field $\Lambda(p)$ on D_p.*

Proof. Consider the unit vector field $\hat{F}$ on $D' := D \setminus J \times \{0\}$, as

$$\hat{F}(x, y) := \mathsf{sign}(y)\frac{F(x, y)}{\|F(x, y)\|}$$

where $\|\cdot\|$ denotes the Euclidean norm. Equation (4) implies

$$\lim_{y \to 0} \hat{F}(x, y) = \frac{g(x, 0)}{\|g(x, 0)\|} \text{ uniformly in } D'.$$

Therefore $\hat{F}$ extends to a unique continuous map $\tilde{F}\colon\ D \to \mathbf{S}^1$ (the unit circle). The desired line field sends $(x, y) \in D$ to the line through $(0, 0)$ spanned by $\tilde{F}_{(x,y)}$. ∎

Next we prove 1.2(iii). For each $p \in K$ let $V_p := h_p^{-1}(D_p)$, with notation as in Equations (2), (3). Define $V := \bigcap_{p \in K} V_p$. Let $\Lambda(p)$ be the line field defined in Lemma 3.2. The pullback of $\Lambda(p)$ by h_p is the unique line field $\Gamma(p)$ on V_p controlling $X|V_p$. The unique line field Γ on V that restricts to $\Gamma(p)$ for each $p \in K$ has the required properties.

Parts (i) and (iv) of 1.2 are consequences of (iii) and Proposition 2.4.

Part (v) follows from Propositions 3.13 and 3.14 of [12] when U is connected, and this implies the general case because the compact set K is covered by finitely many components of U.

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