Title
DIFFRACTION THEORY FOR VERY-HIGH-ENERGY SCATTERING

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Publication Date
1959-09-01

UCRL 8900

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BERKELEY, CALIFORNIA

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Berkeley, California

Contract No. W-7405-eng-48

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Printed for the U. S. Atomic Energy Commission

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ABSTRACT

In the simple diffraction theory, or black-sphere model, of Bethe and Placzcek, it is assumed that partial waves with $\ell \leqslant L$ are completely absorbed and partial waves with $\ell > L$ do not interact at all. (The projectile and target have no spin or charge, so that ℓ represents orbital angular momentum; L is a critical value of ℓ usually related to the radius of the black sphere.) We improve this primitive but useful model by taking into account (a) the gradual, rather than sharp, transition from maximum to zero absorption, (b) the generally small but important deviation from complete absorption, and (c) finite values for the real part of the scattering amplitude. By adoption of appropriate forms for these improvements, closed-form expressions for the various cross sections are obtained. Whenever necessary, systematic approximation methods are developed which allow estimates of errors to be made. The results are shown to be model-independent, i.e., independent of the detailed way in which the above generalizations are made. Further simple improvements for the Coulomb field and spin-$\frac{1}{2}$ projectiles are also discussed. Finally, these methods are applied to the scattering of neutrons from nuclei for neutron energies in the range from 0.3 to 4.5 Bev.

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* This work was performed under the auspices of the U.S. Atomic Energy Commission.

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I. INTRODUCTION

Most interpretations of scattering of strongly interacting particles from atomic nuclei (such as pions, nucleons, and collections of nucleons) are based on the optical or complex-potential model.[1] In this paper we propose to follow a different approach, which we shall call a generalized diffraction model, that emphasizes the scattering amplitude rather than the potential. The methods discussed here will find their most natural application to scattering processes in which many partial waves are absorbed.

Previous work with the optical model can be divided into two categories, the phenomenological and the basic optical-model approaches. We first note that it is usually possible to introduce an effective two-body interaction to describe entrance-channel phenomena for the scattering of many-particle systems. The phenomenological optical-model approach is simply the attempt to deduce this interaction from the observed scattering cross sections. The method employed is essentially trial and error. A potential is assumed, the cross sections are calculated in some approximation from the appropriate wave equation, comparison is made with experiment, and the entire process is repeated until satisfactory agreement is achieved. The entire procedure is quite tedious and requires modern automatic computing facilities. On the other hand, it is usually possible to obtain a high level of agreement with the experiments in this way. Because the assumed potentials are simple and reasonable one infers that there is a significance beyond pure phenomenology. For example, one cites the success of the optical model as evidence of independent particle motion in the nucleus.

The basic optical model approach involves attempts to solve the actual many-body scattering problem. These attempts have been restricted so far to the scattering of elementary particles from nuclei. In these cases the theory amounts essentially to the old index-of-refraction formula relating two-body forward scattering amplitudes to the many-body forward scattering. Such a theory can give quantitative results only in a restricted intermediate-energy region (if at all). The theory is not valid for very low energies, and for high energies it is difficult to obtain accurate information on the two-body scattering amplitudes. For nucleon-nucleus scattering, this energy range is expected to be roughly from 100 to 300 Mev.

Although there are no glaring discrepancies at present, the agreement between the basic and phenomenological optical-model descriptions is in an uncertain state.[2] Although a large number of phenomenological analyses have been carried out, the results are neither unique nor very precise. Uncertainty in the elementary scattering amplitudes has prevented accurate evaluation of even the simplest many-body theory mentioned above. The most that can be said is that a <u>qualitative</u> understanding of the optical model can be had in terms of the present information of the appropriate two-body interactions.

A major aim of this paper is to improve this situation by simplifying the phenomenological part of the analysis. This is done by working directly with the scattering amplitude. The cross sections are obtained by rather elementary analysis and the intermediary concept of a potential is avoided completely. This method is particularly attractive at extremely high energies, at which the measurements will probably remain relatively primitive for many years. For the scattering of 10-Bev nucleons on heavy nuclei, for example, several hundred partial waves are involved and even machine calculations become impractical.

The physical basis for our model is associated with the large absorption of strongly interacting particles by nuclear matter.[3] Except at very low energies, at which the effects of the exclusion principle are important, the absorption mean free path for any strongly interacting particle is at most of the same order as the nuclear radius. This gives rise to absorption cross sections close to "geometric" and opacities close to unity.[4] It suggests that it is a good initial approximation to regard the nucleus as "black" or completely opaque to strongly interacting particles. We shall adopt this point of view here, and then go beyond this and calculate the nonnegligible corrections to this simple model in a straightforward way. The evaluation of these corrections is facilitated by the fact that many partial waves are involved in the scattering processes of interest.

For most of this first discussion we shall assume that the collision partners are uncharged and have no spin. The inclusion of these effects is not difficult and will be discussed briefly in later sections. The scattering amplitude for this problem is then

$$f(\theta) = \frac{1}{2\,i\,k} \sum_{\ell=0} (\eta_\ell - 1)(2\ell + 1)P_\ell(\cos\theta) \,. \tag{I-1}$$

The amplitude η_ℓ of the ℓth scattered wave is related to the corresponding scattering phase shift by the equation

$$\eta_\ell = \exp(2i\,\delta_\ell) \,. \tag{I-2}$$

The phase shifts here will have both real and imaginary parts. The familiar formulae for the differential scattering cross section $\sigma(\theta)$, the total cross section $\sigma^{(t)}$, and the reaction cross section $\sigma^{(r)}$, are

$$\sigma(\theta) = |f(\theta)|^2 , \tag{I-3a}$$

$$\sigma^{(t)} = 2\pi \lambda\!\!\!^{-2} \sum_{\ell=0} (2\ell + 1)(1 - \mathrm{Re}\, \eta_\ell) , \tag{I-3b}$$

$$\sigma^{(r)} = \pi \lambda\!\!\!^{-2} \sum_{\ell=0} (2\ell + 1)(1 - |\eta_\ell|^2) . \tag{I-3c}$$

In these equations $k(= \lambda\!\!\!^{-\,-1})$ and θ have their usual meanings as wave number and scattering angle, respectively.

We first recall that our zero-order approximation, the black-sphere model, corresponds to the following assumption concerning the scattering coefficients η_ℓ :

$$\eta_\ell = \begin{cases} 0 & \ell \leqslant L \\ 1 & \ell > L . \end{cases} \tag{I-4}$$

There is only one parameter in this simple diffraction theory--L, which is the number of partial waves completely absorbed. For orbital angular momentum $\ell \leqslant L$ there is no outgoing wave and the incident (spherical) wave is completely absorbed. For $\ell > L$ the amplitude of the outgoing wave is the same as for the ingoing wave; hence there is no absorption and no scattering. The only scattering for this model is the shadow scattering associated with the complete absorption of the first $(L + 1)$ waves. The scattering amplitude and cross sections for this simple diffraction can be evaluated without any approximations:

$$f(\theta) = \frac{i}{2k} [P_L'(\cos\theta) + P_L'(\cos\theta)] , \qquad \text{(I-5a)}$$

$$\sigma^{(t)} = 2\pi \lambda\!\!\!^{-2}(L + 1)^2 , \qquad \text{(I-5b)}$$

$$\sigma^{(r)} = \pi \lambda\!\!\!^{-2}(L + 1)^2 . \qquad \text{(I-5c)}$$

Strangely enough, these exact expressions are almost never used in the literature, and are often replaced by approximate ones.

This simple diffraction theory was first introduced into nuclear physics by Bohr, Peierls, and Placzek and by Bethe and Placzek.[5] The effects of the Coulomb field were later studied by Akhieser and Pomeranchuk.[6] More recently, extensive applications of this model have been made to the scattering of alpha particles and heavier projectiles, particularly by Blair.[7] The unrealistic features of this simple model which we wish to improve on in this study are mainly these: (a) the change from complete absorption to no absorption does not occur suddenly, but over a large number of partial waves;[8] (b) the nucleus is not completely opaque, and small changes from complete opacity affect the cross sections considerably; (c) the real parts of the phase shifts are not exactly zero. All these points are quite important for accurate calculations of the elastic scattering and the total cross section.

The detailed way in which we generalize the simple diffraction theory is outlined in Section II. In Section III we derive the total and reaction cross sections, assuming the real part of the scattering amplitude to be zero, for a number of different assumptions about the transition region. These assumptions all lead to approximately the same cross sections if the "effective sizes" of the surface regions are chosen the same. Thus, in this sense, our treatment is essentially model-independent. A similar result holds if the real part of the scattering amplitude is not zero, as will be shown in Section IV. The differential cross section is calculated in Section V, and spin-orbit effects are considered in Section VII. As an example of these techniques, an analysis is carried out for high-energy neutron scattering (0.3 to 4.5 Bev) in Section VI. Further applications to elementary particle interactions will be discussed in a separate article.

II. DEFINITION OF THE MODEL

According to Equations (I-1) and (I-3), the cross sections are determined by two numbers for each partial wave, e.g., the real and imaginary parts of the phase shift (δ_ℓ) or the scattering coefficient (η_ℓ). In this discussion, however, we propose to emphasize the opacity, $(1 - |\eta_\ell|^2)$, and the phase of the scattering coefficient $\alpha_\ell = 2 \operatorname{Re} \delta_\ell$. Complete absorption of a partial wave corresponds to unit opacity with a contribution of $(2\ell + 1)\pi ƛ^2$ to both the reaction and scattering cross sections. The sets of all opacities and all phases for a given scattering process are referred to as the opacity and phase functions for that problem.

For cases of strong absorption of many partial waves, we now assume that the opacity and phase can be represented by smooth functions of ℓ. By assuming simple forms for these functions and by converting sums to integrals, closed-form expressions are obtained for the various cross sections. In addition to their obvious utility for discussing the experiments, these formulae are relatively model-independent.

We assume that the opacity function, $(1 - |\eta_\ell|^2)$, (a) is a continuous, monotonic decreasing function of ℓ, (b) has approximately zero slope for small ℓ, and (c) has a relatively sharp transition region in which it falls from its "interior" value to zero. This type of function is shown in Fig. 1. It may be characterized by three (and not less than three) parameters:

$\beta (0 \leqslant \beta \leqslant 1)$, the opacity for small ℓ;

$L (>> 1)$, the number of partial waves strongly absorbed (determined by the half-value point of the distribution); and

$\Delta (1 << \Delta < L)$, a measure of the transition region.

There are, of course, many functions of this type, with the three parameters different for each case. However, in the present context--i.e., for scattering in which many partial waves are strongly absorbed--these sets of parameters are all related, so that one assumption about the opacity is as good as another within certain limits.

The actual coefficient of the ℓth partial wave is written

$$\eta_\ell = \eta(\ell)\, e^{i\alpha(\ell)} , \qquad \text{(II-1)}$$

where the real functions $\eta(\ell)$ and $\alpha(\ell)$ are the absolute value and the phase of η_ℓ . The above opacity function is then

$$1 - \eta^2(\ell) \ . \qquad \text{(II-2)}$$

We next assume that the phase function $\alpha(\ell)$, in addition to being a function of L and Δ (and perhaps β), depends on only one new parameter, denoted simply by α . Again the exact form or the phase function is not important. (A detailed discussion of the model independence of the phase function is given in Section IV.)

The above assumptions are quite reasonable when the de Broglie wave length is small compared to distances over which the properties of the scattering interaction change significantly. Examination of the results of exact calculations for potential models show just the described behavior of the opacity and phase.[9] The familiar formulae of the WKB approximation[10] also exhibit these properties very clearly. Finally we note again that our statements are also the natural generalization of the simple diffraction theory or black-sphere model.

It may be useful to define two lengths corresponding to the two angular momentum parameters, L and Δ . Because our model deals only with the scattering amplitude we have no fundamental way of introducing these lengths except on dimensional grounds. Hence we define an effective radius R and an effective surface thickness t as

$$kR = L , \tag{II-3a}$$

$$kt = 2\Delta . \tag{II-3b}$$

We insist that there may be no precise equality between these parameters and those used in other problems, e.g., the nucleon density or the optical potential for the same scattering process under discussion here. It is expected, of course, that all parameters referring to the size of the nucleus are of the same order of magnitude. For example, the main dependence of R on atomic weight is of the usual $A^{1/3}$ type, but the coefficient is not necessarily the same as used for electron scattering, and other powers of A may also be important. In a similar way, t probably does not have the same $A^{1/3}$ dependence as R but varies less rapidly, if at all.

The conversion of sums to integrals is carried out in a systematic way with the Euler-McLaurin formula:

$$\sum_{\ell=0}^{L} f_\ell = \int_{-1/2}^{L+1/2} f(\ell)d\ell + \frac{1}{4}[(f_0 - f_{1/2}) - (f_{L+1/2} - f_L)]$$

$$+ \frac{1}{48}[(3f'_L + f'_{L+1/2}) - (3f'_0 + f'_{-1/2})] + \ldots \quad . \tag{II-4}$$

Here $f(\ell)$ is the continuous function representing the set of physical quantities f_ℓ. In case it is necessary to use a set of piecewise continuous functions, we have

$$\sum_{\ell=0}^{L_1} f_\ell + \sum_{\ell=L_1+1}^{L_2} g_\ell + \ldots = \int_{-1/2}^{L_1+1/2} d\ell\, f(\ell) + \int_{L_1+1/2}^{L_2+1/2} d\ell\, g(\ell) + E ; \qquad \text{(II-5)}$$

the quantity E represents the error in converting sums to integrals:

$$E = \frac{1}{4} [(f_0 - f_{-1/2}) - (f_{L_1+1/2} - f_{L_1}) + (g_{L_1+1} - g_{L_1+1/2}) - (g_{L_2+1/2} - g_{L_2}) + \ldots]$$
$$+ \frac{1}{48} [(3f'_{L_1} + f'_{L_1+1/2}) - (3f'_0 + f'_{-1/2}) + (3g'_{L_2} + g'_{L_2+1/2}) - (3g'_{L_1+1} + g'_{L_1+1/2}) + \ldots] + \ldots . \qquad \text{(II-6)}$$

The limits on the integrals have been chosen to minimize the importance of E. For linear functions, for example, $E = 0$. The error term has been retained mainly to check the accuracy of our calculations. Of course the model itself is also approximate in this sense because the physical quantities do not vary in a perfectly smooth way.

In the evaluation of the scattering amplitude we use MacDonald's expansion for the Legendre polynomials,[11]

$$P_\ell(\cos\theta) = J_0(x) + \sin^2\frac{\theta}{2} \left[\frac{J_1(x)}{2x} - J_2(x) \frac{xJ_3(x)}{6} \right] + \sin^4\frac{\theta}{2} [\quad] + \ldots , \qquad \text{(II-7)}$$

with $x = (2\ell + 1) \sin\frac{\theta}{2}$. To obtain accuracy of order $1/L^2$ or $\sin^2\frac{\theta}{2}$, the second term must be kept. To retain an accuracy of order $1/L$, unity cannot be neglected in expressions such as $(2\ell + 1)$ or $(\ell + 1)$. Such approximations are often made in the literature under conditions for which they are not valid, both in the expansion of Eq.(II-7) and the simple diffraction formulae of Eq. (I-5).

III. TOTAL AND REACTION CROSS SECTIONS

In this section the total and reaction cross sections are derived on the basis of three models chosen for the opacity function of Eq. (II-2). The three forms all have the general shape of Fig. 1 but differ in the details of the transition region. In each case the phase $\alpha(\ell)$ is assumed a constant $(= \alpha)$ for $\eta(\ell) < 1$, and zero for $\eta(\ell) = 1$. More general assumptions concerning the phase function are made in later sections.

Case A. $1 - \eta^2(\ell)$ is a linear function of ℓ in the transition region.

The opacity function is plotted in Fig. 2 and defined by the equations

$$1 - \eta^2(\ell) = \begin{cases} \beta & \text{for } \ell \leqslant L - \Delta , \\ \beta \dfrac{L + \Delta - \ell}{2\Delta} & \text{for } L - \Delta < \ell \leqslant L + \Delta , \\ 0 & \text{for } L + \Delta < \ell . \end{cases} \qquad \text{(III-1)}$$

In terms of η_ℓ the model is

$$\eta_\ell = \begin{cases} \sqrt{1 - \beta}\ e^{i\alpha} & \text{for } \ell \leqslant L - \Delta , \\ \sqrt{1 - \beta\left(\dfrac{L + \Delta - \ell}{2\Delta}\right)}\ e^{i\alpha} & \text{for } L - \Delta < \ell \leqslant L + \Delta , \\ 1 & \text{for } L + \Delta < \ell . \end{cases} \qquad \text{(III-2)}$$

From Eq. (III-2) we see that the phase $\alpha(\ell)$ has been defined as

$$\alpha(\ell) = \begin{cases} \alpha & \text{for } \ell \leqslant L + \Delta , \\ 0 & \text{for } L + \Delta < \ell . \end{cases} \qquad \text{(III-3)}$$

If we substitute Eq. (III-1) into Eq. (I-3c) we obtain

$$\sigma^{(r)} = \pi ƛ^2 \sum_{\ell=0}^{L-\Delta} \beta(2\ell + 1) + \sum_{\ell=L-\Delta+1}^{L+\Delta} \beta \frac{L + \Delta - \ell}{2\Delta} (2\ell + 1) .$$

When Eq. (II-5), the rule for changing sums to integrals, is used the reaction cross section becomes

$$\sigma^{(r)} = \pi ƛ^2 \left\{ \int_{-1/2}^{L-\Delta+1/2} \beta(2\ell + 1)d\ell + \int_{L-\Delta+1/2}^{L+\Delta+1/2} \beta \frac{L + \Delta - \ell}{2\Delta} (2\ell + 1)d\ell + E \right\} , \qquad \text{(III-4)}$$

where E is given by Eq. (II-6) with the substitutions $L_1 = L - \Delta$ and $L_2 = L + \Delta$. The integrals of Eq. (III-4) are simplified by making the substitution $x = 2\ell + 1$, and they yield

$$\sigma^{(r)} = \beta \pi ƛ^2 [L^2 + L + \frac{\Delta^2}{3}] + \pi ƛ^2 E . \qquad \text{(III-5)}$$

The value of the correction term E from Eq. (III-3) is $E = \beta/6$. For large L, this is only of order $1/L^2$ compared to the leading term in Eq. (III-5). For the remainder of this work we assume that such terms are negligible, i.e., we neglect all terms of order unity compared with the L^2 . In this case the reaction cross section is given by Eq. (III-5) with $E = 0$.

The total cross section is evaluated in a similar manner by substituting Eq. (III-2) into Eq. (I-3b), and making use again of the relation of Eq. (II-5). To the same order of approximation used in obtaining the reaction cross section we find

$$\sigma^{(t)} = 2\pi \lambda\!\!\!^{-2} [(L^2 - \frac{2}{3}\Delta L + L + \frac{7}{15}\Delta^2 - \frac{\Delta}{3}) - \epsilon(L^2 - 2\Delta L + \Delta^2 + L - \Delta)$$

$$- \epsilon^2(\frac{8}{3}\Delta L - \frac{8}{5}\Delta^2 + \frac{4}{3}\Delta) + \ldots + \delta(\frac{8}{3}\Delta L + \frac{8}{15}\Delta^2 + L + \frac{7}{3}\Delta)$$

$$+ \delta\epsilon(L^2 - 2\Delta L + \Delta^2 + L - \Delta) + \ldots] , \qquad \text{(III-6)}$$

where we have changed parameters from α and β to δ and ϵ , respectively:

$$\epsilon = \sqrt{1 - \beta} ,$$

$$\delta = 1 - \cos\alpha . \qquad \text{(III-7)}$$

This change of parameters here is useful since both ϵ and δ are small quantities for most of the analysis and we can usually neglect the higher-order terms in Eq. (III-6). The reaction cross section does not depend on the phase α , whereas the total cross section contains a term proportional to $\cos \alpha$. The original expressions, Eqs. (I-3b) and (I-3c), have these properties, of course.

Some additional simplification can be achieved if we are dealing only with very large values of L . For $L \approx 100$, for instance, we may well wish to drop terms in Eqs. (III-5) and (III-6) that are on the order of L or smaller, i.e., L, Δ, and perhaps Δ^2. The neglect of these terms should lead to errors on the order of only a few percent for such large values of L. Although the above formulae are fairly simple, the differential cross sections in Section V are considerably more complicated. In that case the retention of such small terms becomes a luxury which is afforded only at the expense of quite cumbersome expressions.

<u>Case B. $1 - \eta^2(\ell)$ is a quadratic function of ℓ in the transition region.</u>

This second model, which is also shown in Fig. 2, is defined by the relations

$$1 - \eta^2(\ell) = \begin{cases} \beta & \text{for } \ell \leqslant L - \Delta , \\ \beta\left[1 - \frac{1}{2}\left(\frac{\ell - (L - \Delta)}{\Delta}\right)^2\right] & \text{for } L - \Delta < \ell \leqslant L , \\ \frac{\beta}{2}\left(\frac{L + \Delta - \ell}{\Delta}\right)^2 & \text{for } L < \ell \leqslant L + \Delta , \\ 0 & \text{for } L + \Delta < \ell , \end{cases} \tag{III-8}$$

or, for the coefficient η_ℓ ,

$$\eta(\ell) = \begin{cases} \sqrt{1 - \beta}\; e^{i\alpha} & \text{for } \ell \leqslant L - \Delta , \\ \sqrt{1 - \beta\left[1 - \frac{1}{2}\left(\frac{L - \Delta - \ell}{\Delta}\right)^2\right]}\; e^{i\alpha} & \text{for } L - \Delta < \ell \leqslant L , \\ \sqrt{1 - \frac{\beta}{2}\left(\frac{L + \Delta - \ell}{\Delta}\right)^2}\; e^{i\alpha} & \text{for } L < \ell \leqslant L + \Delta , \\ 1 & \text{for } L + \Delta < \ell . \end{cases} \tag{III-9}$$

Following the steps outlined in the linear model of Case A (above), we find, for the quadratic model,

$$\sigma^{(r)} = \beta \pi ƛ^2 [L^2 + L + \frac{\Delta^2}{6}] \tag{III-10}$$

and

$$\sigma^{(t)} = 2\pi\, \lambda\!\!\!^{-2}[(L^2 - 0.486\,\Delta L + L + 0.239\,\Delta^2 + 0.424\,\Delta)$$
$$- \epsilon(L^2 - 1.432\,\Delta L + 0.432\,\Delta^2 + L - 0.716\,\Delta)$$
$$+ \epsilon^2(-1.496\,\Delta L + 0.065\,\Delta^2 - 0.90\,\Delta)$$
$$+ \delta(2.49\,\Delta L + 0.76\,\Delta^2 + L + 1.58\,\Delta)$$
$$+ \delta\epsilon(L^2 - 1.43\,\Delta L + 0.43\,\Delta^2 + L - 0.72\,\Delta) + \ldots] . \qquad \text{(III-11)}$$

Case C. Quadratic in $\eta(\ell)$.

This last model is chosen specifically for the simple rational form for $\eta(\ell)$, rather than for $1 - \eta^2(\ell)$. It is defined by the relation

$$\eta(\ell) = \begin{cases} \sqrt{1-\beta}\;\; e^{i\alpha} & \text{for } \ell \leqslant L - \Delta , \\ [1 - \dfrac{1 - \sqrt{1-\beta}}{4\,\Delta^2}(\ell - L - \Delta)^2]e^{i\alpha} & \text{for } L - \Delta < \ell \leqslant L + \Delta , \\ 1 & \text{for } L + \Delta < \ell , \end{cases} \qquad \text{(III-12)}$$

and the opacity function $(1 - \eta^2(\ell))$ is shown in Fig. 2. The formulae for $\sigma^{(t)}$ and $\sigma^{(r)}$ can be written in closed form with no approximations because both $\eta(\ell)$ and $1 - \eta^2(\ell)$ are sums of powers of ℓ . Using the above techniques, we find to the same accuracy as the preceding results,

$$\sigma^{(r)} = \beta\,\pi\,\lambda\!\!\!^{-2}[L'^2 + L' + \frac{17}{75}\Delta^2] , \qquad \text{(III-13)}$$

$$\sigma^{(t)} = 2\pi\,\lambda\!\!\!^{-2}[L'^2 - \frac{4}{15}\Delta L' + \frac{6}{25}\Delta^2 + L' - \frac{2}{15}\Delta)(1 - \epsilon)$$
$$+ \delta(\frac{8}{3}\Delta L' + \frac{6}{5}\Delta^2 + L' + \frac{38}{15}\Delta) + \delta\epsilon(L'^2 - \frac{4}{15}\Delta L' + \frac{6}{25}\Delta^2 + L' - \frac{2}{15}\Delta)] , \qquad \text{(III-14)}$$

where we have defined $L' = L - \frac{\Delta}{5}$ to compensate for the fact that the transition region is not symmetric about L.

To exhibit the degree of model independence for this theory, we now show that, to some order of approximation, the results for the cross sections in cases A, B, and C are the same. If we assume that the only difference in the models is in the effective surface thickness, Δ , for the three cases, we can attempt to equate the three results for the total and reaction cross sections in the limit of large L, and for $\beta = 1$ $(\epsilon = 0)$, and $\alpha = 0$. Good agreement can indeed be obtained with the condition $\Delta_A = 0.728\ \Delta_B = 0.400\ \Delta_C$, where the subscripts refer to the particular model. Therefore in this limit of zero phase and complete absorption at $\ell = 0$, the only essential difference between the models is in the normalization of the surface parameter, Δ .

An exact numerical comparison for the case $\beta < 1$, and $\alpha \neq 0$ is quite difficult to carry out. However, since we are usually interested only in the ratio of $\sigma^{(t)}/\sigma^{(r)}$, we can compare this ratio to see if there are any large discrepancies between the three results. These ratios are plotted as functions of Δ/L and β in Fig. 3 for the case $\alpha = 0$, and in Fig. 4 for $\alpha = 90^{\circ}$. It is evident that the results are essentially equivalent for a wide range of values of α and β . We might point out here, incidentally, that a graph of the form of Figs. 3 and 4 is useful in obtaining phenomenological values of the parameter β from experimental data.

IV. EFFECT OF A NONCONSTANT $\alpha(\ell)$

In the preceding section the derivations of the formulas for $\sigma^{(t)}$ and $\sigma^{(r)}$ were based on the assumption that $\alpha(\ell)$ was constant $(= \alpha)$ for $\eta(\ell) < 1$, and zero for $\eta = 1$. [See Eq. (III-3).] We will now show the effect of a more realistic form for $\alpha(\ell)$. As in the case of the opacity function $[1 - \eta^2(\ell)]$, we assume here that $\alpha(\ell)$ is a real monotonically decreasing function of ℓ, and has the value α at $\ell = 0$. Because of the relative simplicity of Case C above, we choose it for the model to be considered both for $\eta(\ell)$ and for the nonconstant phase. We define therefore

$$\cos[\alpha(\ell)] = \begin{cases} \cos\alpha & \text{for } \ell \leqslant L - \Delta, \\ 1 - \dfrac{1 - \cos\alpha}{4\Delta^2}(L + \Delta - \ell)^2 & \text{for } L - \Delta < \ell \leqslant L + \Delta, \\ 1 & \text{for } L + \Delta < \ell. \end{cases} \tag{4-1}$$

Equation (IV-2) produces an approximately linear variation of $\alpha(\ell)$ with ℓ, and is compared with the phase resulting from an optical-model calculation in Fig. 5.

The formula for the reaction cross section is the same as given in (III-13). The new form for the total cross section is

$$\sigma^{(t)} = 2\pi\lambda\!\!\!^{-2}[(L^2 - \frac{2}{3}\Delta L + \frac{\Delta^2}{3} - \frac{\Delta}{3} + L)(1 - \epsilon) + \delta(\frac{8}{15}L\Delta - \frac{2}{15}\Delta^2 + \frac{4}{15}\Delta) + \delta\epsilon(L^2 + \frac{7}{15}\Delta^2 - \frac{6}{5}L\Delta - \frac{3}{5}\Delta + L)]. \tag{IV-2}$$

The difference between the results for constant α of Eqs. (III-14) and (IV-2) above must vanish if the two cross sections are to be equal. The condition for this is found to be

$$\cos \alpha_1 = a \cos \alpha_2 , \qquad \text{(IV-3)}$$

where α_1 is the phase constant of the preceding discussion (Section III) and α_2 is the phase in Eq. (IV-1); a is a function of Δ and L, but it depends very weakly on β for values of β nearly equal to unity. Thus, if the condition of Eq. (IV-3) is met the two models for $\alpha(\ell)$ give the same results, and we conclude that except for a linear scaling factor $a(\Delta, L)$, the form for the total cross section of Eq. (III-19) is independent of the form chosen for $\alpha(\ell)$. Although only the two forms for $\alpha(\ell)$ of Eqs. (III-3) and (IV-1) have been used, we may reasonably infer that similarly defined functions would lead to substantially the same results.

V. DIFFERENTIAL CROSS SECTION

In order to simplify the discussion of the scattering amplitude we adopt for $\eta(\ell)$ the form introduced in Case C of Section III, Eq. (III-12), and use a constant phase, Eq. (III-3). We are encouraged to rely on only one case here because of the model independence of our results for the total and reaction cross sections. It is important to investigate a number of terms beside the first one in Eq. (II-1) [the expansion of $P_\ell(\cos\theta)$ in terms of Bessel functions]. In the end, terms smaller than $L^2 \sin^2 \frac{\theta}{2}$ are neglected; Δ is also assumed small enough that terms involving both $\sin\frac{\theta}{2}$ and Δ can be dropped.

The philosophy behind our expansion for the scattering amplitude is that, for small angles and large L, the leading term is the simple approximation to "black sphere" or simple diffraction scattering:

$$f_0(\theta) = \frac{i}{2k} \int_{-1/2}^{L+1/2} d\ell (2\ell + 1) J_0([2\ell + 1]\sin\frac{\theta}{2}) = \frac{i}{k}(L+1)^2 \left[\frac{J_1(a)}{a}\right], \qquad \text{(V-1)}$$

with

$$a = 2(L+1)\sin\frac{\theta}{2} . \qquad \text{(V-2)}$$

The corrections to this are of three kinds, which may be roughly described as:

(a) corrections due to the fact that our model allows for incomplete absorption, (β), nonzero phase (α), and a gradual transition region (Δ);

(b) corrections from the higher-order terms in Eq. (II-7) [the Bessel-function expansion for $P_\ell(\cos\theta)$]; and

(c) corrections for converting sums to integrals.

These corrections are not simply additive, and the three terms now introduced correspond only approximately to the above three effects:

$$f(\theta) = f_0(\theta) + f_1(\theta) + f_2(\theta) + f_3(\theta) . \qquad \text{(V-3)}$$

We will describe these amplitudes more precisely but without giving the details of the derivations.

The term $f_1(\theta)$ arises from the finite values of Δ, β, and α in our model, which was defined by Eqs. (III-3) and (III-12). At the same time only the leading terms are kept in Eqs. (II-5) and (II-7). We write, for $f_1(\theta)$,

$$f_1(\theta) = \frac{i}{k}\Bigg\{(L + \Delta + 1)^2 \frac{J_1(a + b)}{a + b} - (L + 1)^2 \frac{J_1(a)}{a}$$

$$- e^{i\alpha}\Bigg[(L + \Delta + 1)^2 \frac{J_1(a + b)}{a + b} + (\epsilon - 1)\Big\langle(L - \Delta + 1)^2 \frac{J_1(a - b)}{a - b}$$

$$+ \frac{\Delta L - L + \Delta}{2} \cos(b)\, J_0(a) + (L^2 - \frac{\Delta L}{\Delta}) \frac{J_1(a)}{a} \Big\rangle\Bigg]\Bigg\} , \qquad \text{(V-4)}$$

where a is defined by Eq. (V-3), and we set

$$b = 2\,\Delta \sin\frac{\theta}{2} . \qquad \text{(V-5)}$$

An alternative expression for $f_1(\theta)$ may be obtained by expanding the Bessel function of arguments $(a - b)$ and $(a + b)$ about the point a. The scattering amplitude can then be expressed in terms of functions of the same argument, a; however, the resulting algebraic expression is considerably more cumbersome than that of Eq. (V-4).

For $f_2(\theta)$, we use the higher-order terms proportional to $\sin^2 \frac{\theta}{2}$ of Eq. (II-7), and only the first-order terms (the integrals) of Eq. (II-5), and the main term ("sharp refracting, gray sphere") of Eq. (III-12). The higher-order terms in these last two equations are not included, since they yield results considerably smaller than $L^2 \sin^2 \frac{\theta}{2}$. To obtain expressions accurate at larger angles, or at lower L values, the products of the higher-order terms in Eqs. (II-5), (II-7) and (III-12) must be included. We write, then, for $f_2(\theta)$,

$$f_2(\theta) = \frac{i}{2k} \int_{-1/2}^{L+1/2} (2\ell + 1)(1 - \sqrt{1 - \beta}\; e^{i\alpha})[\sin^2 \frac{\theta}{2} \{ \frac{J_1(x)}{2x} - J_2(x) + \frac{xJ_3(x)}{6} \} + \ldots d\ell , \qquad \text{(V-6)}$$

where

$$x = (2\ell + 1) \sin \frac{\theta}{2} . \qquad \text{(V-7)}$$

If we write the Bessel functions of Eq. (V-6) in terms of $J_0(x)$ and $J_1(x)$, and integrate, we find

$$f_2(\theta) = \frac{i(1 - \epsilon\; e^{i\alpha})}{6k} L^2 \sin^2 \frac{\theta}{2} J_0(a) . \qquad \text{(V-8)}$$

The usual treatments of diffraction scattering omit terms of this type and consequently are valid only for very large L, or very small θ. At the first diffraction minimum, for instance, $L^2 \sin^2 \frac{\theta}{2} = (\frac{1.22\,\pi}{2})^2 \simeq 3.7$, which is negligible compared with the leading L^2 term only if L is quite large. At angles beyond the first minimum, the neglect of the $L^2 \sin^2 \frac{\theta}{2}$ term can be serious for even large L values.

The third correction, $f_3(\theta)$, is obtained from the higher-order terms of Eq. (II-5), which are the corrections contained in the E term of Eq. (II-6), and from Eq. (III-12) with the Δ-, β-, and α-dependent quantities considered exactly. However, only the leading $J_0(x)$ term of Eq. (II-7) is considered to this order of approximation. The algebra is involved, but quite straightforward. We substitute the relations

$$f(\ell) = (2\ell + 1)(1 - \epsilon\ e^{i\alpha})J_0[(2\ell + 1)\sin \frac{\theta}{2}]$$

and

$$g(\ell) = (2\ell + 1)[1 - \{ 1 - \frac{1 - \epsilon}{4\Delta^2} (\ell - L - \Delta)^2\}e^{i\alpha}]J_0[(2\ell + 1)\sin \frac{\theta}{2}]$$

into Eq. (II-4) and find

$$f_3(\theta) = \frac{i}{2k} E$$

$$= \frac{i}{k} [-(1 - \epsilon\ e^{i\alpha})L^2 \sin^2 \frac{\theta}{2} \frac{J_1(a - b)}{a - b} + \frac{1 - e^{i\alpha}}{6} L^2 \sin^2 \frac{\theta}{2} \frac{J_1(a + b)}{a + b}] . \tag{V-9}$$

Finally, collecting the results in Eqs. (V-9), (V-8), (V-4), and (V-2), we obtain for the scattering amplitude (V-10)

$$f(\theta) = \frac{i}{k} \Bigg\{(L + \Delta + 1)^2 \frac{J_1(a + b)}{a + b} + L^2 \sin^2 \frac{\theta}{2} [\frac{J_0(a)}{6} + \frac{J_1(a + b)}{6(a + b)} - \frac{J_1(a - b)}{a - b}]$$
$$- e^{i\alpha}\Bigg[(L + \Delta + 1)^2 \frac{J_1(a + b)}{a + b} + L^2 \sin^2 \frac{\theta}{2} \Bigg\{\frac{J_1(a + b)}{6(a + b)} + \epsilon \frac{J_0(a)}{6} - \epsilon \frac{J_1(a-b)}{a-b}\Bigg\}$$
$$+ (\epsilon - 1)\Bigg\{(L - \Delta + 1)^2 \frac{J_1(a - b)}{a - b} + \frac{\Delta L - L + \Delta}{2} \cos b\ J_0(a) + (L^2 - \frac{1}{4} \Delta L) \frac{J_1(a)}{a}\Bigg\}\Bigg]\Bigg\}$$

The case of zero phase is often of physical interest (e.g., high-energy nucleon-nucleus scattering, as we shall see in the next section), and here the scattering amplitude assumes the simple form

$$f(\theta)\Big|_{\alpha=0} = \frac{i}{k}(1 - \epsilon)\Big[\Big(L^2 - \frac{\Delta L}{4}\Big)\frac{J_1(a)}{a} + (L^2 + \Delta^2 - 2\Delta L - 2\Delta + 2L)\frac{J_1(a-b)}{a-b)}$$

$$+ \frac{\Delta L - L + \Delta}{2}\cos(b)\, J_0(a)$$

$$+ L^2 \sin^2\frac{\theta}{2}\Big(\frac{J_0(a)}{6} - \frac{J_1(a-b)}{a-b}\Big)\Big] . \qquad \text{(V-11)}$$

VI. PHENOMENOLOGICAL ANALYSIS OF NUCLEON-NUCLEUS SCATTERING

As an application of this generalized diffraction model, we consider the total and reaction cross sections for neutrons incident on various nuclei. The experimental data have been plotted in Fig. 6 for four elements--C, Al, Cu, and Pb--for neutron energies of 270 Mev and greater.[12] There have been isolated attempts[13] to discuss experiments at particular energies with the method of Fernbach, Sèrber, and Taylor.[10] No attempt has been made, however, to discuss all the high-energy data at once. We are able to do this here because of the simplicity of the formulae we have derived for the various cross sections. A brief report of this treatment for three nuclei has already been published.[14]

One conclusion can be drawn immediately from the data for the four cases: a successful analysis based on the black-sphere one-parameter (L) analysis of Eq. (I-5) is impossible. In each case the total cross section exhibits a large energy variation, whereas the black-sphere model implies constant cross sections at these energies. Thus we expect the observed energy variation to be associated with one or more of the other three parameters introduced in this generalized theory. For the present we assume that the major part of this energy dependence is contained in the opacity parameter β .

To proceed with the analysis, we assume first $L \propto kA^{1/3}$, and we obtain the proportionality constant by fitting the almost constant reaction cross section for lead to the formulas for one of the models in Section III, say, Eq. (III-13). We use lead in the beginning of the analysis for two reasons: (a) Because of the large number of interacting partial waves for such a heavy nucleus, we expect $L \gg \Delta$, so that terms of the order of Δ^2 may be neglected compared with L^2. Thus the reaction cross section for a heavy nucleus is essentially independent of Δ. (b) Since the mean free path for neutrons in nuclear matter is much smaller than the dimensions of the lead nucleus, we may also assume in this case $\beta \approx 1$, i.e., there is almost complete absorption for all energies. In fact, we arbitrarily set $\beta = 1$ at 1 Bev, the energy at which the mean free path is a minimum. We then find $L = k \times 1.26 \times 10^{-13}$ cm $A^{1/3}$.

The next parameter, α, may be deduced in this energy region by considering the large variation in $\sigma^{(t)}$ from Fig. 6. Now the contribution of just the ℓth partial wave is $\sigma_\ell^{(t)} \propto 1 - \mathrm{Re}(\eta_\ell) = 1 - \eta(\ell)\cos\alpha$. For a given change in $\eta(\ell)$, the maximum change in $\sigma_\ell^{(t)}$ is obviously obtained for η_ℓ pure real, i.e. $\alpha = 0$. The experimental values of $\sigma^{(t)}$ are all less than $2\pi \lambda\!\!\!^{-2}(L + 1)^2$, which limits α to the range $0 < \alpha < \frac{\pi}{2}$. In addition the experimental variations are so large that the only possible choice of α is $\alpha = 0$. For instance, if η_ℓ is pure imaginary, $\sigma^{(t)}$ has a constant value of $2\pi \lambda\!\!\!^{-2}(L + 1)^2$ and is independent of the energy-dependent parameters, β. For η_ℓ real, but $\alpha = \pi$, then $\sigma^{(t)} > 2\pi \lambda\!\!\!^{-2}(L + 1)^2$, which clearly does not agree with the experiments under consideration. Thus the experimental energy variation of $\sigma^{(t)}$ requires that the scattering amplitude be essentially pure imaginary, or $\alpha \approx 0$.

Now that we have determined the magnitude of the cross sections by fixing the dependence of L, and have concluded that $\alpha = 0$, we may greatly simplify the subsequent analysis by comparing the experimental ratios $\sigma^{(t)}/\sigma^{(r)}$ with those given in Fig. 4. We make the reasonable assumption that $\Delta \propto k$ (or $\frac{\Delta}{L} \propto A^{-1/3}$) and simultaneously determine Δ and β from Fig. 4. The value of Δ/L depends only on A, or the nuclear size, and can be adjusted for the best fit of the opacity β for all four elements considered.

In the present analysis we find the proportionality constant for Δ is 0.61×10^{-13} cm. The best fit for β at various energies is given in the table.

Experimental Values of the Opacity

	0.30 Bev	0.70 Bev	1.40 Bev	5.0 Bev
Pb	0.97	0.99	1.00	0.94
Cu	0.93	0.97	0.99	0.94
Al	0.88	0.95	0.97	0.93
C	0.82	0.89	0.94	0.89

It is worth while at this point to connect this phenomenological approach with some physical properties of the nucleus and of the incident nucleon. Although some quantitative results can be obtained along these lines, most of what follows is qualitative.

One of the primary features of the data in Fig. 6 is the large energy variation in $\sigma^{(t)}$ for almost constant $\sigma^{(r)}$. The explanation of this is simple when one considers the individual partial-wave cross sections for the case of almost total absorption [$\eta(\ell)$small]. For $\eta_\ell$real , we see that $\sigma_\ell^{(r)} \propto 1 - \eta_\ell^2$ varies quadratically with η_ℓ , and for small η_ℓ it changes very little. However, $\sigma^{(t)} \sim 1 - \eta_\ell$ varies linearly with η_ℓ , and reflects the changes in η_ℓ more sensitively than $\sigma^{(r)}$. Figure 7 shows the variation in $\sigma^{(t)}$ and $\sigma^{(r)}$ as a function of the opacity β for various values of α and for $\Delta = 0$.

We may next ask why the opacity β increases as it does in the vicinity of 1 Bev. For this purpose we should consider the observed neutron-neutron (or proton-proton) and neutron-proton total cross sections shown in Figs. 8 and 9.[15] We see that some definite correlation exists between these basic two-body cross sections and the neutron-nucleus cross sections of Fig. 6. On a simple classical model the incident neutrons are absorbed exponentially along their trajectory in nuclear matter. The mean free path is taken as $\lambda = 1/\rho\,\overline{\sigma_t}$, where ρ is the density of the nuclear medium (2×10^{38} cm^{-3}) and $\overline{\sigma_t}$ is the "effective" two-body total cross section. By "effective" we mean a simple average over neutrons and protons; other effects such as the Pauli principle have been assumed to be negligible at these high energies. Then the expression for β is just

$$\beta = 1 - \exp(-2R/\lambda) , \qquad \text{(VI-1)}$$

where R is the nuclear radius. Thus in this simple approximation the energy dependence of β is a function of the energy dependence of λ, or of the nucleon-nucleon total cross section. The values of $\overline{\sigma_t}$ averaged over neutrons and protons are plotted in Fig. 10 along with the mean free path, λ. The values of β deduced from Eq. (VI-1) are given in the table below, and it can be seen by comparison with the phenomenological values of β that this model gives the correct qualitative results, but does not afford a satisfactory quantitative theory in this simple form.

Theoretical Estimates of Opacity

	0.30 Bev	0.70 Bev	1.40 Bev	5.0 Bev
Pb	1.00	1.00	1.00	1.00
Cu	0.99	1.00	1.00	0.99
Al	0.97	0.99	0.99	0.98
C	0.93	0.97	0.98	0.95

However, the situation may be considerably improved by considering two effects hitherto neglected. First, in the three models chosen for the opacity function $[1 - \eta^2(\ell)]$ in this paper, the opacity was constant for $\ell < L - \Delta$, i.e., we adopted a constant value of β for $\ell \leqslant L$, and then set it equal to zero for $\ell > L$. It would be more realistic to use a slowly decreasing opacity here. We have not considered this improvement here because it would considerably complicate the simple formulae of Section III. However, the qualitative effects of such an opacity function are the following:

(a) We can no longer assume $L \propto kA^{1/3}$, since the effective L value for light nuclei is additionally decreased because there is less absorption and hence very small opacity in the region $\ell \lesssim L$. Thus the theoretical curves of Fig. 6 for C and Al would agree better with experiment.

(b) The value of β from Eq. (VI-1), averaged over all trajectories, would be lower than the β obtained at $\ell = 0$. This effect would be increased in light nuclei, in which absorption effects are smaller, and would bring the theoretical value of β in the table above in to better agreement with the phenomenological values.

Second, there are several factors that tend to reduce the "effective" two-body cross sections, $\overline{\sigma_t}$ and thus lower the theoretical value of β .

(a) The effect of the Pauli exclusion principle eliminates part of the two-body scattering in which the nucleons are not excited above the nuclear Fermi level. This effect should be small, however, at high energies.

(b) At high incident energies there may be a certain amount of shadowing of target nucleons by other nucleons, which decreases the average contribution of the nucleons to $\overline{\sigma_t}$. Such an effect would be most important for heavy nuclei, and would tend to lower both $\overline{\sigma_t}$ and the opacity, β .

VII. SPIN-ORBIT EFFECTS

The scattering amplitude in the case of spin-1/2 particles incident on spin-zero targets may always be written as

$$f(\theta) = A(\theta) + B(\theta)\,\vec{\sigma}\cdot\hat{n} , \qquad \text{(VII-1)}$$

where $\vec{\sigma}$ is the Pauli spin vector of the incident particle, and $\hat{n}$ is a unit vector normal to the plane of scattering:

$$\hat{n} = \frac{\vec{k}_f \times \vec{k}_i}{|\vec{k}_f \times \vec{k}_i|} . \qquad \text{(VII-2)}$$

The polarization obtained from initially unpolarized particles is just

$$P = \frac{2\,\mathrm{Re}(A\,B^*)}{|A|^2 + |B|^2} . \qquad \text{(VII-3)}$$

The terms of Eq. (VII-1) can be written as

$$A(\theta) = \frac{i}{2k} \sum_{\ell=0}^{\infty} [(\ell+1)(1-\eta_\ell^{+}) + \ell(1-\eta^{-})]P_\ell(\cos\theta)$$

$$= \frac{i}{2k} \sum_{\ell=0}^{\infty} [(2\ell+1)(1-\bar{\eta}_\ell) - \frac{\xi_\ell}{2}]P_\ell(\cos\theta) \qquad \text{(VII-4)}$$

and

$$B(\theta) = \frac{\sin\theta}{2k} \frac{d}{d(\cos\theta)} \sum_{\ell=0}^{\infty} (\eta_\ell^{-} - \eta_\ell^{+})P_\ell(\cos\theta)$$

$$= \frac{\sin\theta}{2k} \frac{d}{d(\cos\theta)} \sum_{\ell=0}^{\infty} \xi_\ell\, P_\ell(\cos\theta) , \qquad \text{(VII-5)}$$

where we have set

$$\bar{\eta}_\ell = \frac{\eta_\ell^+ + \eta_\ell^-}{2} ,$$

$$\xi_\ell = \eta_\ell^+ - \eta_\ell^- . \qquad \text{(VII-6)}$$

Thus the two functions η_ℓ^+ and η_ℓ^-, which correspond to the angular momentum states of $J = \ell + \frac{1}{2}$ and $J = \ell - \frac{1}{2}$ respectively, can be written in terms of two new functions, $\bar{\eta}_\ell$ and ξ_ℓ. It can be seen that for no spin dependence, $\eta_\ell^+ = \eta_\ell^-$, and $\xi_\ell = 0$, so that Eq. (VII-4) reduces to Eq. (I-1), and Eq. (VII-5) vanishes.

We wish only to outline a simple calculation for the spin-orbit effects; therefore we choose $\alpha = \Delta = 0$. In addition we assume small $\eta_\ell^\pm$ so that the average, $\bar{\eta}_\ell$, of Eq. (VII-6) is of the same form as the η's of Section III. We let

$$\bar{\eta}(\ell) = \frac{\eta_\ell^+ + \eta_\ell^-}{2} = \sqrt{1 - \beta} \qquad \ell \leqslant L$$

$$= 1 \qquad \ell > L . \qquad \text{(VII-7)}$$

Since the spin-dependent term $B(\theta)$ of Eq. (VII-5) arises from an interaction of the form $\vec{\sigma}\cdot\vec{L}$, it is not unreasonable to assume that the partial-wave function, $\xi(\ell)$, is a monotonically increasing function of ℓ from $\ell = 0$ to $\ell = L$, where it drops to zero. For $\Delta \neq 0$, $\xi(\ell)$ would have a maximum in the region $\ell \simeq L$, and fall off smoothly to zero over a region Δ.

Two forms for $\xi(\ell)$ are investigated to show that, aside from a normalization constant, the results for $B(\theta)$ are relatively independent of the details of the function $\xi(\ell)$. The first form is linear in ℓ, and is defined by

$$\xi(\ell) = 2\gamma\ell \simeq \gamma(2\ell + 1) . \qquad \text{(VII-8)}$$

Then, considering only first-order effects, we find for $A(\theta)$ and $B(\theta)$

$$A(\theta) = \frac{i}{2k} \int_{-1/2}^{L+1/2} d\ell (2\ell + 1)\left(1 - \sqrt{1 - \beta} - \frac{\gamma}{2}\right) J_0 \left[(2\ell + 1)\sin\frac{\theta}{2}\right]$$

$$= \frac{i}{k}\left(1 - \sqrt{1 - \beta} - \frac{\gamma}{2}\right)(L)^2 \frac{J_1(a)}{a} \qquad \text{(VII-9)}$$

and

$$B(\theta) = \gamma \frac{\sin\theta}{2k} \frac{\partial}{\partial(\cos\theta)} \int_{-1/2}^{L+1/2} d\ell (2\ell + 1) J_0 \left[(2\ell + 1)\sin\frac{\theta}{2}\right]$$

$$= \frac{\gamma}{2k} \sin\theta \; L^3 \frac{J_2(a)}{a^2} . \qquad \text{(VII-10)}$$

The second case chosen is that of a cubic form for $\xi(\ell)$,

$$\xi(\ell) = \frac{\gamma'}{L^3} (2\ell + 1)^3 . \qquad \text{(VII-11)}$$

Then we get, for the scattering amplitudes,

$$A(\theta) = \frac{i}{k}\left[\left(1 - \epsilon - \frac{\gamma'}{4L}\right) L^2 \frac{J_1(a)}{a} + \frac{\gamma L}{2k} \frac{J_2(a)}{a^2}\right] , \qquad \text{(VII-12)}$$

and

$$B(\theta) = -\frac{\gamma' \sin\theta \, L^3}{2k}\left[\frac{J_2(a)}{a^2} - \frac{2 J_3(a)}{a^3}\right] . \qquad \text{(VII-13)}$$

From Eqs. (VII-9), (VII-10), (VII-12), and (VII-13), it can be seen that for two quite dissimilar models for $\xi(\ell)$, $A(\theta)$ is characterized mainly by $J_1(a)/a$, and $B(\theta)$ by $J_2(a)/a^2$. These results are quite similar to the spherical Bessel functions j_1 and j_2 obtained for $A(\theta)$ and $B(\theta)$ respectively in the closed-form WKB analysis of scattering from a square-well potential.[16]

VIII. CONCLUSION

The formulae derived here should be extremely useful in discussing scattering processes at very high energy in the presence of strong absorption. They involve only well-known functions for which tabulations are easily accessible. Thus the significance of scattering measurements at high energies may be deduced directly by experimentalists without recourse to excessive calculation.

This simple picture was obtained by constructing a model for the scattering amplitude rather than by introducing an abstract potential picture. The parameters that occur here describe such properties of the scattering amplitude as its phase, the number of partial waves strongly absorbed, and the strength of the absorption. They clearly have very direct physical significance.

It is important, of course, to have additional ways of interpreting the parameters that occur in this model. For example, assuming the validity of the Serber point of view, one would like detailed connections with the properties of the two-nucleon system. We have not investigated this problem here at all. There exists already, of course, a qualitative correction in terms of the index-of-refraction relation familiar from optics. More detailed conclusions must await more refined solutions of the quantum-mechanical many-body problem. On the other hand, our model probably has the advantage that it avoids the question of an intermediary effective potential, i.e., the optical model. The introduction of the optical model simply adds one more complication to an already difficult problem.

Most of the results given here refer to the scattering of spin-zero neutral particles from spherical spin-zero targets. The brief discussion of polarization showed, however, that spin effects are simple to treat. The results also show great promise for explaining some of the complicated observations on the effects of spin in scattering problems. Finally, the addition of the Coulomb scattering amplitude involves little difficulty, although the formulae become somewhat more complicated.

FOOTNOTES AND REFERENCES

1. More information on the optical model may be found in the recent reviews by H. Feshbach, Annual Reviews of Nuclear Science (Stanford University Press, Stanford, 1959) and by A. E. Glassgold, Progress in Nuclear Physics (Pergamon Press, 1959).

2. A similar remark applies to direct calculations of the scattering from the two-body amplitudes.

3. By absorption is meant any transition out of the entrance channel as well as true particle creation and destruction.

4. The nucleus is supposed to be transparent to neutrons with energies between 100 and 300 Mev, but the opacities range from 0.8 to 1.0.

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8. This defect has, of course, been mentioned previously in the literature but no systematic treatment has been given.

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16. K. R. Greider, University of California Radiation Laboratory Report UCRL-8836, July 16, 1959 (unpublished).

FIGURE CAPTIONS

<u>Fig. 1.</u> Typical shape for the opacity function $1 - \eta^2(\ell)$. The opacity is constant $(= \beta)$ from $\ell = 0$ to $\ell = L - \Delta$, and falls off smoothly to zero in the transition region from $L - \Delta$ to $L + \Delta$. Beyond $\ell = L + \Delta$, the opacity is zero.

<u>Fig. 2.</u> Comparison of three specific models for the opacity function. The analytic forms of these functions are given in Eqs. (III-1), (III-8), and (III-12) for Cases A, B, and C, respectively.

<u>Fig. 3.</u> The ratio $\sigma^{(t)}/\sigma^{(r)}$ as a function of the two parameters Δ/L and β for $\alpha = 0$. The dash line (- - - - -) is for Case A, a linear model for $1 - |\eta|^2$. The dash-dot line (-.-.-.-.) is for Case B, a quadratic model for $1 - |\eta|^2$. The solid line (————) is for Case C, a quadratic model for η.

<u>Fig. 4.</u> The ratio $\sigma^{(t)}/\sigma^{(r)}$ as a function of the two parameters Δ/L and β for $\alpha = 90^\circ$. The dash line (- - - - -) is for Case A, a linear model for $1 - |\eta|^2$. The dash-dot line (-.-.-.-.) is for Case B, a quadratic model for $1 - |\eta|^2$. The solid line (————) is for Case C, a quadratic model for η.

<u>Fig. 5.</u> Comparison of the phase function $\alpha(\ell)$ for two models. The dot-dash line (-.-.-.-.-.) is for the case of <u>constant</u> phase introduced in Eq.(III-3). The **dash** line (- - - -) is for the case of nonconstant α as introduced in Eq. (IV-1). The solid line (————) is the phase function obtained by numerical integration of the Schrodinger for a typical complex potential used for high energy scattering.

<u>Fig. 6.</u> Cross sections for the scattering of neutrons by C, Al, Cu, and Pb. The solid and open circles are the experimental measurements of the total and reaction cross sections, respectively. The solid (————) and dash (- - - - - -) curves are the corresponding theoretical fits obtained in this work.

<u>Fig. 7.</u> Total and reaction cross sections as a function of the opacity parameter β for three values of the phase α . The transition region Δ was set equal to zero in this analysis.

<u>Fig. 8.</u> Total and reaction cross sections for high energy proton-proton scattering. The solid and dash curves here are simply smooth fits to the experimental points.

<u>Fig. 9.</u> Total cross sections for high-energy neutron-proton scattering. The solid curve here is simply a smooth fit to the experimental points.

<u>Fig. 10.</u> The effective nucleon-nucleon total cross section (left scale in mb) and the mean free path of nucleons in nuclear matter (right scale in 10^{-13} cm) as a function of energy.

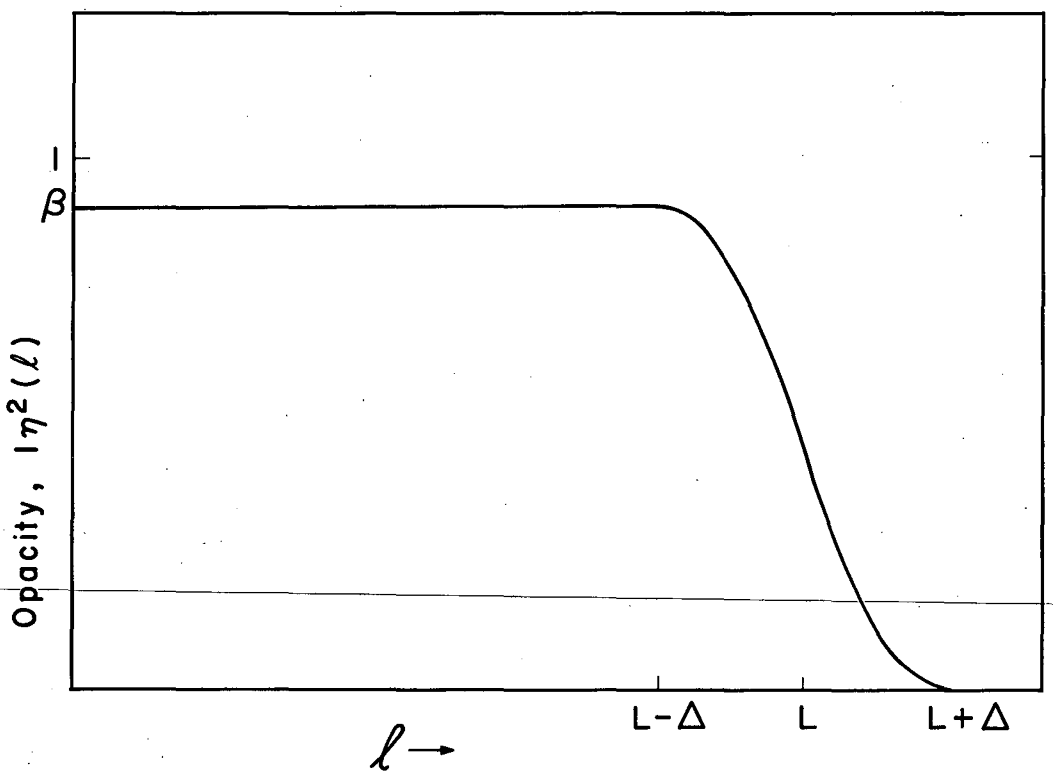


Fig. 1

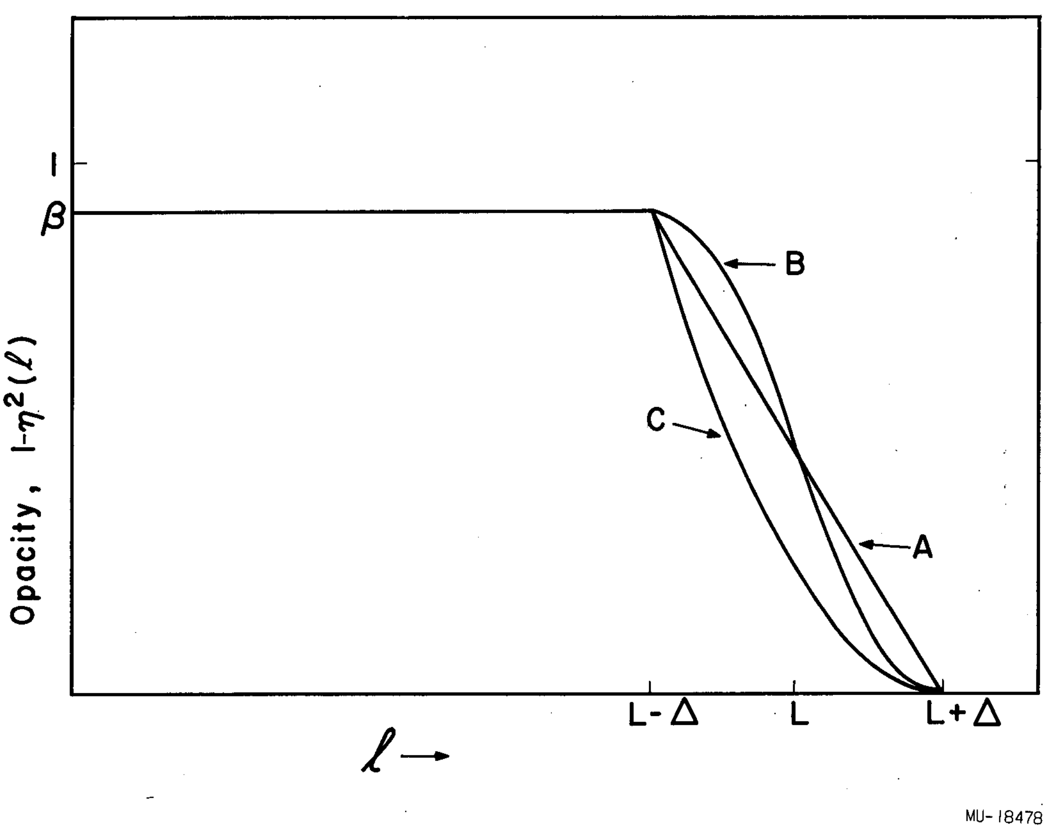


Fig. 2

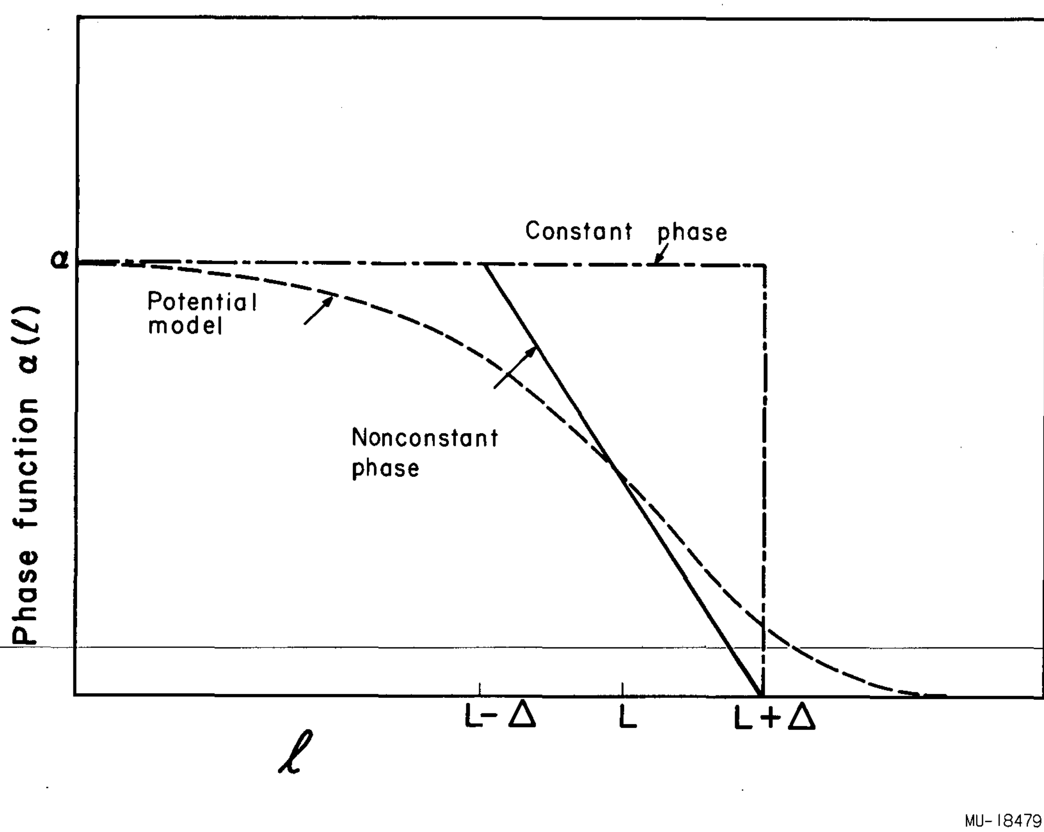
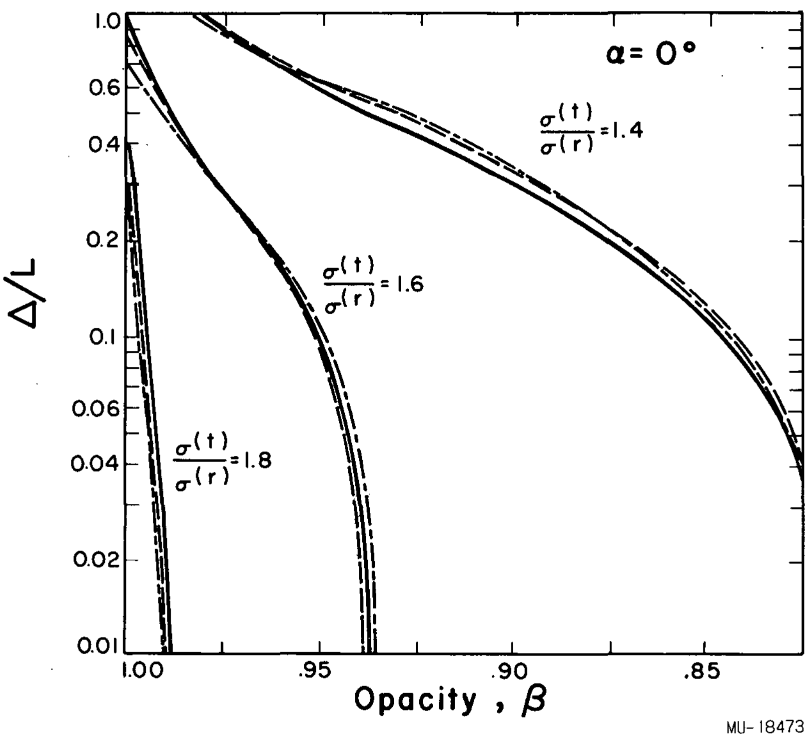
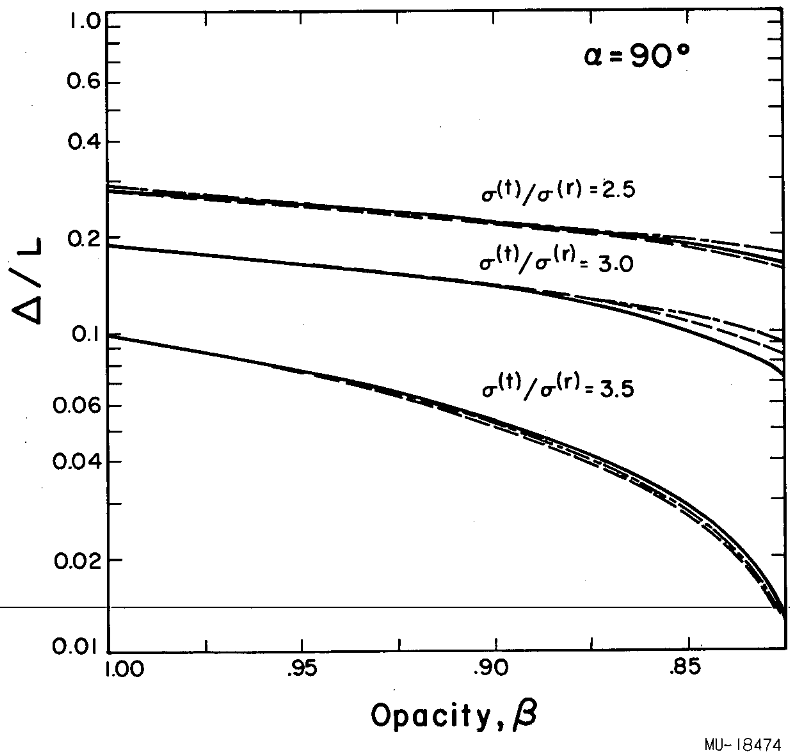

Constant phase
a
Potential
model
Nonconstant
phase
Phase function a(ℓ)
L- Δ
L
L+Δ
ℓ
MU-18479

Fig. 3



α = 0°
σ(t)/σ(r) = 1.4
σ(t)/σ(r) = 1.6
σ(t)/σ(r) = 1.8
Δ/L
1.0
0.6
0.4
0.2
0.1
0.06
0.04
0.02
0.01
1.00
.95
.90
.85
Opacity , β
MU-18473

Fig. 4



α = 90°
σ(t)/σ(r) = 2.5
σ(t)/σ(r) = 3.0
σ(t)/σ(r) = 3.5
Δ / L
1.0
0.6
0.4
0.2
0.1
0.06
0.04
0.02
0.01
1.00
.95
.90
.85
Opacity, β
MU-18474

Fig. 5

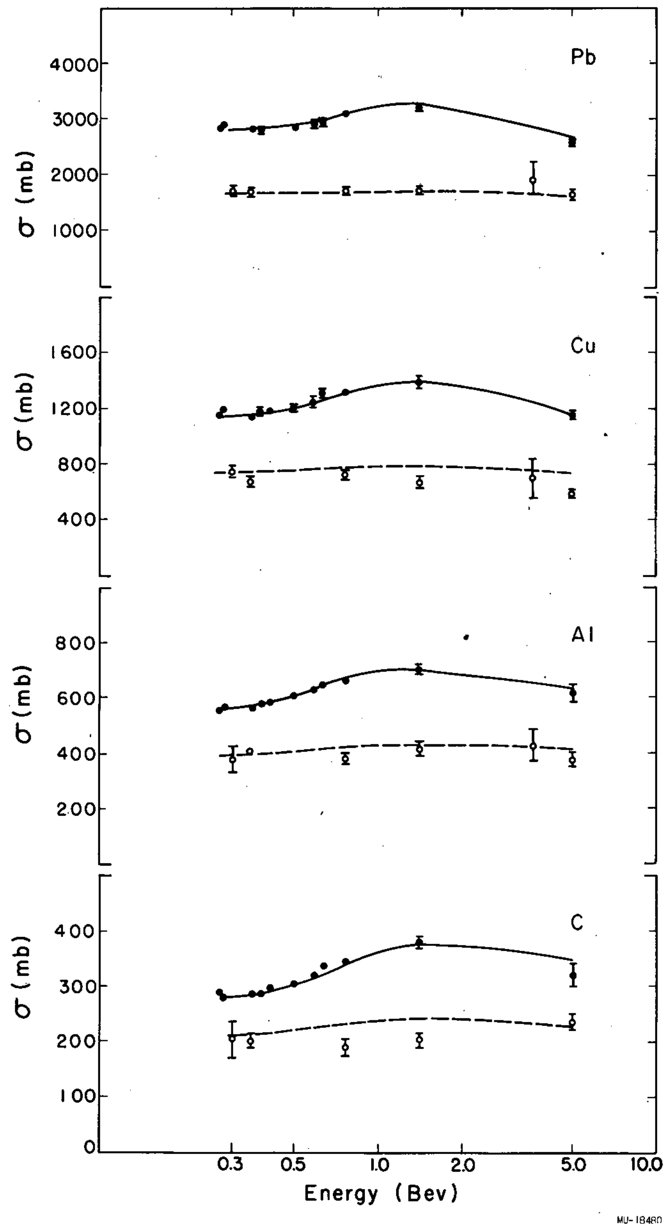
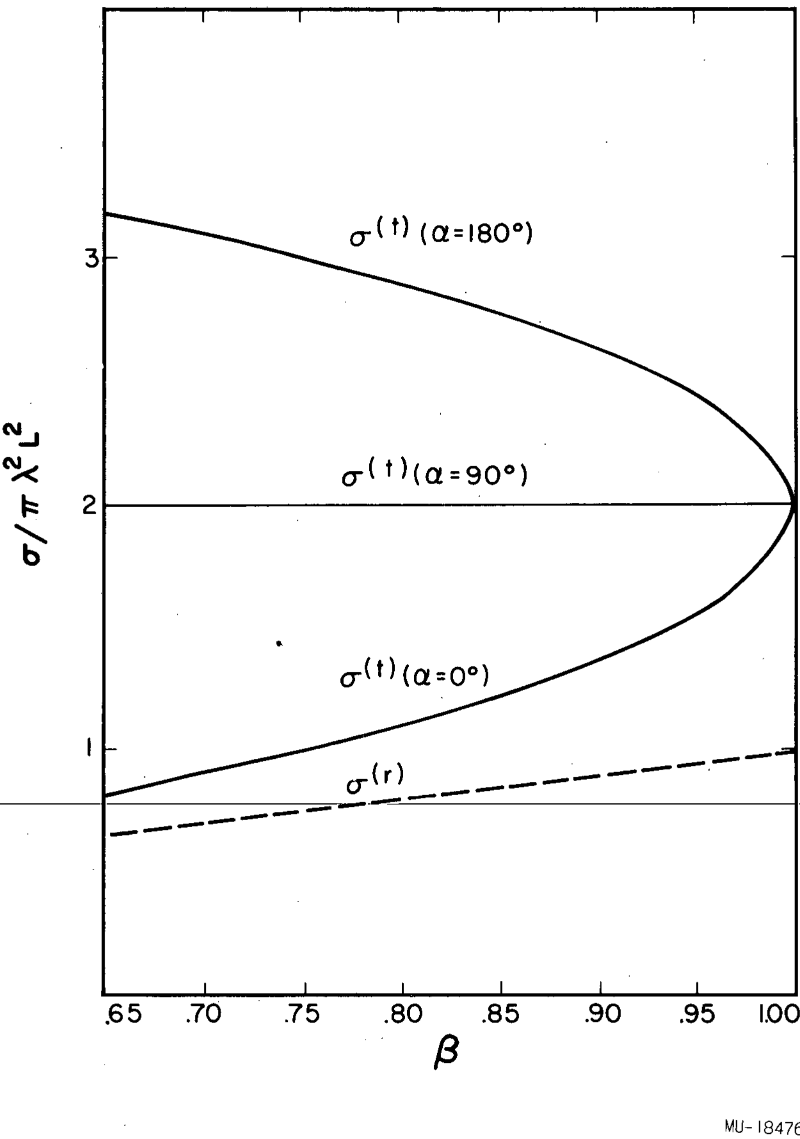

Pb
Cu
Al
C
σ (mb)
4000
3000
2000
1000
1600
1200
800
400
800
600
400
200
400
300
200
100
0
0.3
0.5
1.0
2.0
5.0
10.0
Energy (Bev)
MU-18480

Fig. 6



$\sigma^{(t)}(\alpha=180°)$
$\sigma^{(t)}(\alpha=90°)$
$\sigma^{(t)}(\alpha=0°)$
$\sigma^{(r)}$
$\sigma/\pi\lambda^2L^2$
3
2
1
.65
.70
.75
.80
.85
.90
.95
1.00
β
MU-18476

Fig. 7

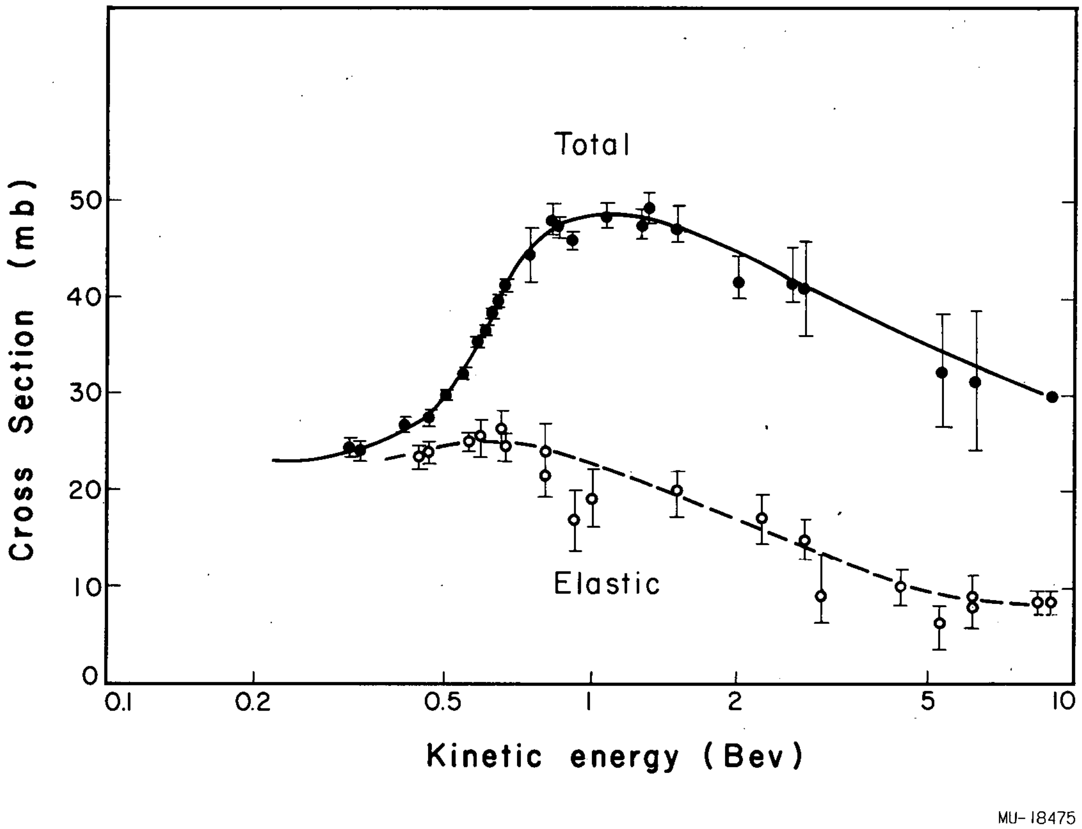

Total
Elastic
Cross Section (mb)
0
10
20
30
40
50
0.1
0.2
0.5
1
2
5
10
Kinetic energy (Bev)
MU-18475

Fig. 8

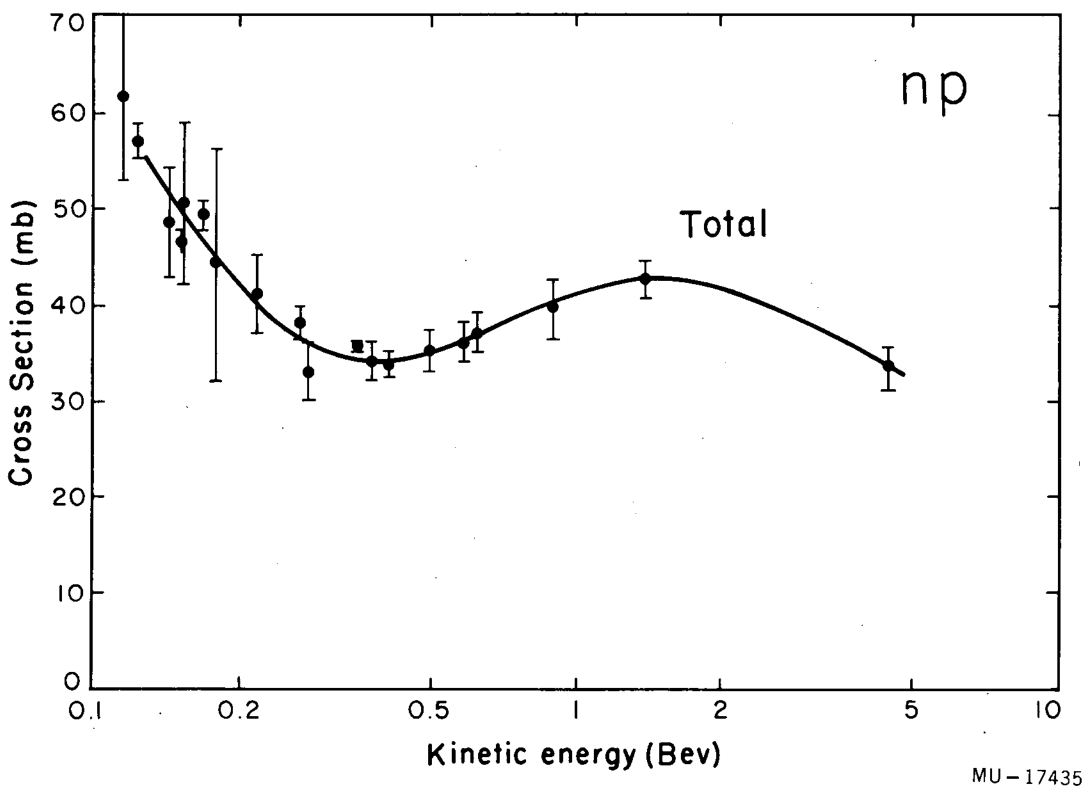

np
Total
Cross Section (mb)
70
60
50
40
30
20
10
0
0.1
0.2
0.5
1
2
5
10
Kinetic energy (Bev)
MU-17435

Fig. 9

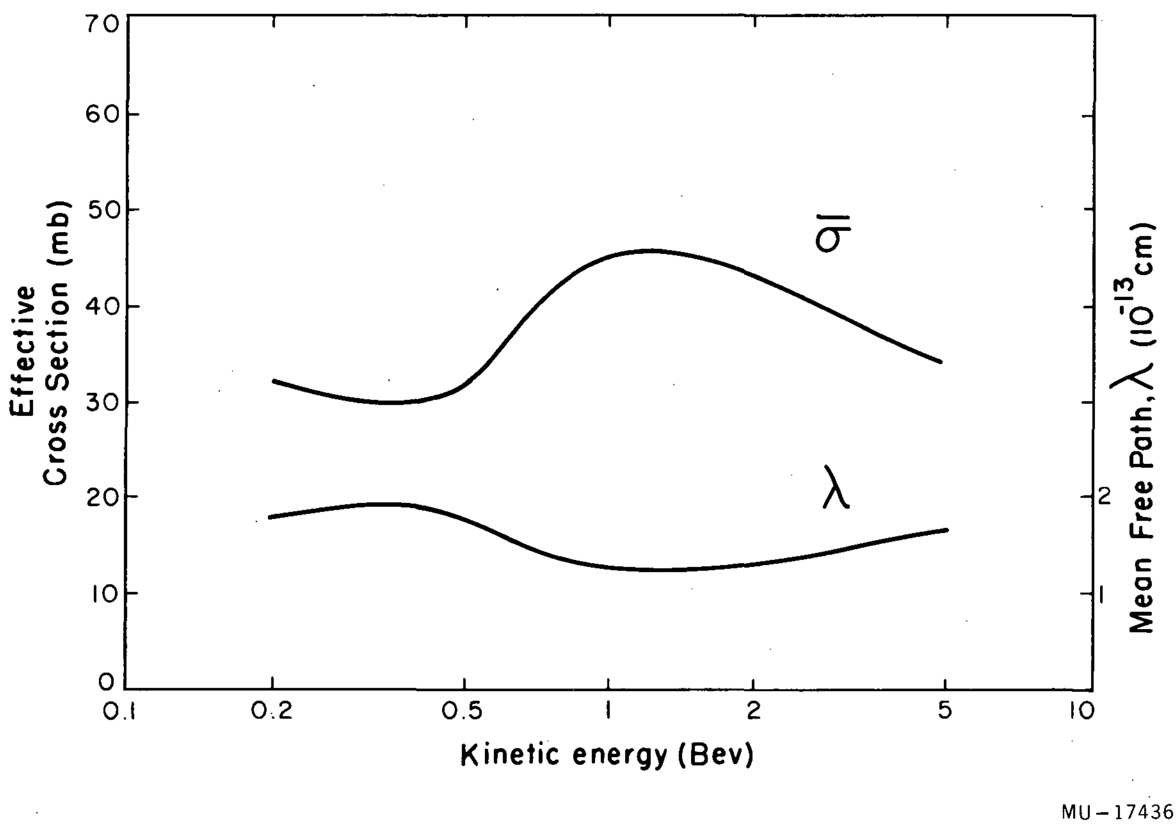

Effective
Cross Section (mb)
70
60
50
40
30
20
10
0
0.1
0.2
0.5
1
2
5
10
Kinetic energy (Bev)
$\bar{\sigma}$
λ
2
1
Mean Free Path, λ (10^{-13}cm)
MU-17436

Fig. 10