Practices of Questioning and Explaining in Learning to Model

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Abstract

Conceptual learning in mathematics involves more than getting to the right answer. Recent efforts in math education reform have focused on having students work in groups on open-ended projects that are based in realistic contexts. We extend previous analyses with hypotheses about conceptual and interactional aspects of understanding and solving problems by groups. The conceptual hypothesis focuses on integration of information in the group's *situation model* and its *problem model.* The interactional hypotheses involve patterns of interaction that make it easy or hard for the group to open up a discussion o f assumptions in its reasoning, and that make the group accountable to a wider audience for explaining relations between the situations and mathematical operations involved in their solutions. Regarding educational practice, these findings highlight a way in which student groups must coordinate their conceptual and interactional work to arrive at satisfactory solutions to the problems posed. The present study suggests the importance of students in these environments not only connecting the contextual situation and the attending mathematics, but also reconsidering the situation in light of their new mathematical understanding (bringing the mathematics back into play in their understanding of the situation). Interactional patterns in a group make this relatively easier or harder, and this must be accounted for in implementing new curricula and conducting teacher education.

Introduction

When a group works together on a problem involving mathematics, how does that work get done? How does the group arrive at its understanding of the problem on which they are working? How do they go about conducting the work? What happens when someone questions what another member of the group is doing or proposing to do? The person questioned may offer an explanation that justifies the claim or action. Alternatively, the group might collectively take up the question, and construct a new understanding.   Or the question could be ignored, deflected, or dismissed.

When students are involved in solving open-ended mathematics problems without one correct answer, it's not immediately obvious when an error has been made. So when a mistake *is* recognized by a member of a group, how does that happen? This paper reports results from an analysis of a student group engaged in a mathematical modeling unit. We present two episodes in which the group's problem solving involved mistaken assumptions. In one episode the group identified and corrected the mistake, but in the other they did not. In this paper we illustrate how a mistake was recognized and resolved, and how that differed from more common instances in which mistakes are not corrected. Implications for cognitive theory and for the design of learning environments are discussed.

An Interactional Aspect of Reasoning

The interactional aspect of activity that we focus on involves explanatory practices. We develop our analysis using the schema of conversational contributions provided by Clark and Schaefer (1989) and adapted for analyzing discourse in problem solving by Greeno and Engle (1995). In this schema, each contribution to the process of understanding and solving a problem includes a presentation of information or action and an acceptance, resulting in grounding the contribution in the participants' mutual understanding.

In much ordinary discourse, explanations occur mainly when someone questions or disagrees with something that someone else says or does (McLaughlin, Cocy & Reed, 1992). In Clark and Schaefer's scheme, occasions for explanation often arise when one participant presents some information and another other participant responds with a question, a challenge, or an alternative. When this happens, the group can take up the question, challenge, or the suggested alternatives. This "taking up" involves a kind of negotiation in which the question or challenge may be resolved, one of the alternatives may be chosen, or the group may leave the issue with the understanding that their uncertainty or disagreement remains. In such a negotiation, explanations may occur frequently between group members.

The discourse patterns of different groups or of groups in different situations vary in how open they are to presenting and considering questions, challenges, or alternatives. A presentation provides a possible continuation of a trajectory in the activity. A question, challenge, or alternative proposal presents a potential diversion from that trajectory, and the resolution may bring about a change in the group's direction. It is reasonable—perhaps necessary—for groups to maintain some level of inertia in their interactions in order to enable them to function productively. The amount of that inertia can vary depending on which participant has the floor. It is easier to challenge and

question some participants than others, and some participants are more likely than others to discuss an issue rather than to forge ahead, ignoring the intrusion, or to simply give a justificatory explanation.

Mathematics educators such as Lampert (e.g., 1990) and Cobb (e.g., Cobb Yackel & Wood, 1992) have emphasized the importance of students' developing practices of explanatory discourse that support valid reasoning and understanding in mathematics. Cobb and Yackel (1996) distinguished between *social norms, sociomathematical norms,* and *mathematical practices* in the activity patterns of mathematics classrooms. Social norms, or *participation structures* (Erickson, 1986; Lampert, 1990), include the extent to which participants expect each other to provide explanations and conduct their conversations so that it is easy to present questions, challenges, or alternative proposals and have them taken up. Sociomathematical norms include what counts as acceptable and valued mathematical explanation. Mathematical practices include methods that are established as acceptable without need for explanation.

A Conceptual Aspect of Reasoning

The conceptual activity that we focus on involves the coordination of cognitive resources from different conceptual domains in activities of reasoning. Like heterogeneous reasoning (e.g., Stenning & Sommerfeld, this volume), the cognitive process that we consider involves reasoning that is informed by different kinds of information sources. In heterogeneous reasoning, the sources are different representations (e.g., a diagram and a set of logical formulas). In the reasoning that we observed, the information sources were from different conceptual domains—one primarily involving ecology and the other primarily involving mathematics.

In professional practices that use mathematics, such as architecture or scientific research (e.g., Hall & Stevens, 1996), the integration of information drawn from mathematics and another domain is often seamless; often one cannot be understood without the other. This ubiquitous, implicit coordination of mathematics with another conceptual domain informs professionals' evaluations of their work, including identification of mistakes.

In school problem solving, the coordination of mathematics with other domains is often more problematic. In a study of primary-grade students solving word arithmetic problems, Kintsch and Greeno (1985) hypothesized two forms of understanding that they called *situation models* (following Kintsch & van Dijk, 1978), and *problem models.* According to this hypothesis, solving a mathematical problem includes understanding properties and relations of objects and events in the problem (the situation model), and using that information to construct an understanding in mathematical terms (the problem model). The problem model is often supported by material representations such as equations, which aid students in carrying out appropriate mathematical procedures. Nathan, Kintsch, & Young (1992)

hypothesized that difficulty in forming a model of a situation and coordinating that situation with a problem model is a significant impediment to students' success in learning algebra. They designed an interactive computer system that supports students' construction of algebra problem situations and makes relations needed for the problem model salient. Use of this program facilitated students' problem solving and learning.

Learning Environment

The curriculum materials in this study came from the Middle school Mathematics through Applications Project (MMAP), which was organized as a design experiment (Brown, 1992; Collins, 1992). The design team, housed at the Institute for Research on Learning at Stanford, included teachers and curriculum developers as well as cognitive science researchers. The team developed a middle-school mathematics curriculum in which students work in groups to solve extensive design problems using mathematics (Goldman, Moschkovich, & MMAP, 1995). Students work in interactive learning environments that are middle-school aged versions of design work in architecture, population biology, cryptography, or cartography. The purpose of the curriculum is to have students use math to address problems situated in non-mathematical contexts, often with the assistance of computer applications.

The data we analyzed come from an 8th grade MMAP classroom in the San Francisco Bay Area. They were collected by Rogers Hall and his colleagues (Hall, 1999; Hall, this volume). In the approximately 30-day unit we discuss, called Guppies, students created mathematical models of biological population growth. For their study, Hall and colleagues had collaborated with the teacher in designing a revision of the unit that had been taught and observed earlier. This revision included further emphasis on how to construct mathematical models of population growth and about the exponential functions that underlie them. In addition, they included more explicit attention to the relation between assumptions about guppies' reproduction and parameters of the mathematical model. Our analysis in this paper focuses on one group of students (Manuel, Lisa, Kera & Ned) whose improvement on pre/post assessments placed them about midway in learning of the focus groups videotaped by Hall.

Analysis

We examine two episodes from a videotape record of one student group. These episodes were chosen because each of them included a proposal for a move in the problem space that was incorrect. However, in one case the group identified the error and corrected it, while in the other the group did not identify the error, but instead proceeded using a flawed piece of information. We explain this difference between the successful episode and the unsuccessful one with two hypotheses about collaborative understanding and problem solving in interaction.

First, we hypothesize that the interaction of the group included a kind of threshold for taking up questions, challenges, and alternative proposals that could change the course of activity. Specifically, we hypothesize that detection of a misalignment between the group's situation model and its problem model could occur by a participant's questioning of an operation that was proposed or performed. Both episodes began with an operation initiated by one participant. In the first episode, another participant presented a new interpretation of the situation (which we hypothesize was based on her understanding of the situation model), which illustrated that the current operation was incorrect. In the second episode this participant questioned the initiated operation and expressed a lack of understanding of it. The first of these episodes resulted in the group changing their mathematical approach to accurately reflect their new understanding. In the second episode, the group did not change the operation it was carrying out. We hypothesize, then, that at least in these two cases, presenting a persuasive interpretation based on a situation model was sufficient to bring about a change, while merely expressing uncertainty and lack of understanding of the operation was not.

Second, we hypothesize that in both episodes the initial error involved an inadequate alignment between what the participants understood about the world of guppies (their situation model) and the mathematics (their problem model). That is, the students either did not attend to all the relevant details in the text of the problem when formulating their situation model, or they did not attend to all of the details of the situation model when formulating the problem model. This is consistent with the findings of Nathan, Kintsch & Young (1992), who suggested that when numbers are abstracted from their context, it is possible for students to perform operations that aren't faithful to the situations they are meant to represent (cf Hall et al., 1989). However, such mistakes can be recognized when the context the math is supposed to represent is considered, often using a simulation of some type. In the first episode such a simulation occurred, which led to a reconceptualization of both the problem and situation models. In the second episode however, the problem and situation models were not integrated, and the mistake was not recognized. It is important to note that we do not believe that situation models and problem models are static states, but that they ideally develop in coordination with each other in a recursive process. We suggest that although mathematics problems can be completed successfully without coordination of problem and situation models, integration of the two can highlight when mistakes have been made, leading to more successful problem solving.

Episode 1 - Pretest

Our first episode comes from the pretest in which the students were trying to answer the question: "Given an initial population of twenty mice who reproduce every season, how large will the population be at the end of two years?" The students had decided that each mouse couple would have four babies during each breeding season, and that the mice would reproduce eight times in four years. They were asked to show their solutions, which the students did by drawing a graph that depicted the size of the mouse population after each breeding season. Manuel had proposed that the vertical axis would need to extend to 340 mice. When questioned by Lisa and Kera, he explained this conclusion by repeating the mathematical procedures he used: there would be 40 mice born each season, resulting in 320 births, which would be added to the initial 20. When they began to construct the graph, the following exchange occurred:

180 **Manuel**: SO there's sixty… so let's see the first season is over here [*making a mark on the graph*]
185 **Lisa**: Wait a minute
186 **M**: and then sixty plus… is going to be a hundred
189 **L**: Wait a minute, it's forty, and then it's like [*put pencil down and placed fingertips together*] OK. It's forty, right?… And then you have to pair those up [*pressed palms of hands together*] and then they have kids [*spread the palms of her hands apart*]
195 **Kera**: pair the-
196 **M**: oh yeah [*laughing*]
202 **K**: …my gosh, that's a lot of nasty mice.

Manuel's participation at the beginning of this episode was consistent with the group's usual pattern, in which Manuel initiated actions and responded to questions by explaining why his proposals were satisfactory. Lisa indicated a question ("Wait a minute") then, when Manuel proposed adding 40 to the first data point to infer the next data point, Lisa took the floor, capturing attention with a gesture along with her speech, presenting a reasoned explanation for a different operation that would take account of the 40 mice that were in the population after the first season when they calculated the number born in the second season.

As the students began to graph their results, Lisa realized that the ending population had been miscalculated, and she interrupted the trajectory of the group with a suggestion that was recognized, acknowledged and finally implemented. Lisa's suggestion (line 189) recalled the context of the problem—how mice reproduce—which enabled the students to evaluate the mathematical model they were creating. The linear model that they had previously created had made sense to all three of the members until Lisa simulated a model of the situation they were supposed to be addressing. Thinking about the population growth in those terms enabled her to recognize the error of adding the same number of newborn mice every season. This served to relate the problem model back to the situation model, as the students were forced to think incrementally about the growth of their population.

We interpret this conceptually as follows. First, Manuel's proposal that there would be 320 births involved applying a familiar schema of mathematical practice. A process that increases a quantity may do so by producing a constant quantity during each of several intervals. Inferences about

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this kind of process can be made using a schema with three variables: an amount per unit, a number of units, and a total amount. We hypothesize that Manuel applied this schema, using the number of mice the group had calculated for the first year as the amount per unit (births per season) and the number of seasons as the number of units. When Manuel specified and represented 60 as the number mice after the first season, Lisa related this to a situation model in which the number of births in each season depends on the number of mice in the population that season. In that model, the number of births during the second season had to be calculated by considering how many mouse couples there were during the second season. This contradicted Manuel's method and required a change in the problem model, one in which there was a separate calculation of the number of births for each season, rather than a single calculation of the total number of births during eight seasons

Episode 2 - Birthrate worksheet

Our second example is more typical of the type of error recognition that we observed in this group. This episode comes from the middle of the unit. Students were trying to create their own model of guppy population growth. They were told that a population of ten guppies would be coming from Venezuela, and the students' task was to determine how large a tank they would need to hold the guppy population at the end of two years. In order to solve the problem the students needed to determine the specific composition of their original ten guppies (gender and age) and tabulate their birth rate- a complex equation that used multiple variables (see also Stenning & Sommerfeld, this volume).

The interaction in this episode was different than the first episode: although in this group ideas and questions were almost always acknowledged and attended to, one member's suggestion did not always serve to stop or change the trajectory of the group. In this episode Manuel proposed a "shortcut" through the mathematics of a worksheet (line 444). Apparently realizing that this suggestion did not fit into the expectations of the worksheet, Lisa questioned Manuel many times and attempted to stop the group. In this case however, Lisa's interjections (lines 451, 497, 501 and 505) did not successfully redirect the group.

444 **M**: …If four percent of the frys survive, why don't we just forget about the fry survival and just put that amount for the, for how much are born.
447 **L**: 'cause the number born are not how much survived.
448 **M**: Yes. Yes, the ones who survive are the ones we count, not the ones who are dead, because we don't make room for the ones that are dead.
451 **L**: …I'm kinda confused
462 **M**: …why don't we just put four percent on the guppies' birth, because that's how many are going to survive.
497 **L**: but what's that four percent?
498 **K**: the ones that survive

501 **L**: yeah, I know, but how many… of the guppies are four percent?
503 **M**: we don't know, we'll let that mechanical thing work and tell us
505 **L**: wait, are you answering assumption-
506 **M**: let's just try it out.

The group was working on a worksheet for calculating a value of the birth rate to enter into the computer model. The worksheet included four steps. First, the students made assumptions about the gender and age distribution of an initial population of ten guppies. Second, they were to calculate the total number of guppy fry that would be born according to data provided about the number of fry per female of each age in the population. Third, they were to apply a percentage of infant mortality, due to the fact that about 95% of newborn guppies are eaten by their mothers. Fourth, they were to calculate an effective birth rate by dividing the number of surviving fry by 10, the size of the initial population, and converting this to a percentage.

Manuel proposed that the survival rate (which he incorrectly remembered as 4% instead of 5%) could be entered in the computer model as the birth rate. Lisa questioned this, ("how many … of the guppies are four per cent?) but Manuel did not take up the question.

We interpret this episode using a hypothesis about a problem model that was based on an incomplete use of a situation model. The computer program and the worksheet required an entry labeled "birth rate," intended to be expressed as a percentage of the population in the previous season. The group understood that the value of this parameter should reflect the loss of most of the guppies that had been born. The percentage of surviving guppy fry — 4% — fit these specifications, and Manuel proposed to use that as the birth rate. Lisa's questions about this operation were analogous to the challenge she presented in Episode 1. The correct value should have taken into account the number or percentage of guppy fry born in relation to the previous total population, and then take 5% of that (or 4%, on Manuel's misremembered figure). If Lisa's questions had specified the neglected quantity in this case, as she did in Episode 1, she might have succeeded in having her alternative taken up and considered.

In this interaction, Lisa's question was not sufficient to force the group to recognize the mathematical error they had made. In the earlier episode, Lisa stopped the group with a suggestion that simulated the situation model they were supposed to be working from, enabling the group to identify an error in their mathematical reasoning. In this case, Lisa attempted to stop the trajectory of the group without either making a specific suggestion about the relation of the problem model to the situation model, or proposing a new situation model. The other members of the group were not forced to think differently about what they are saying, and consequently, no change was made.

Comparison

We chose these two episodes because they present a useful contrast for thinking about group problem solving. In both episodes, the group was working along a mathematically incorrect path, and one student questioned that path; but in one case the group corrected itself, and in the other it did not. Specifically, both episodes began with a proposal by one student (Manuel), which involved a mathematical shortcut. These shortcuts appear to have made some sense to the other students in the group, but neither proposal would have led to a successful solution to the problem at hand. What factors may have been involved in the first episode becoming a successful problem solving effort, while the second did not?

We hypothesize that the principal conceptual difference between these two interactions lay in the students relating the situation model and the problem model. The curriculum was designed so that students necessarily developed a situation model about guppies, and used that situation model to construct their mathematical problem model. When they constructed problem models that neglected significant aspects of the situations, incorrect assumptions and conclusions could be easily missed.

The patterns of interaction between the two episodes also differed. In this group, one student (Manuel) consistently took the function of directing the process, initiating and performing operations. Two other students (Lisa and Kera) frequently asked questions or expressed uncertainty. Generally Manuel responded to these by justifying the operation he had initiated or performed. In the first episode (the pretest problem), Lisa not only questioned Manuel's line of thinking, she presented a definite alternative to Manuel's operation. It appears that this met the threshold required for Manuel and Kera to attend to Lisa's idea and to accept it. In the second episode (the birthrate worksheet), Lisa questioned Manuel's shortcut and referred to a critical property of the situation. But, apparently, raising a question rather than proposing an alternative was insufficient to meet the threshold needed to open up a negotiation of how they should proceed.

Discussion

The hypotheses arising from analysis of these two episodes point to potential contributions both in fundamental cognitive science and the design of learning environments.

First, consistent with Nathan, Kintsch, and Young (1992), we see that students working on contextual problems get into conceptual difficulty when they do not adequately align their situation model with their domain-specific problem model. However, we extend that finding to suggest that for problem solving in real-world contexts it is also important to realign the problem model to the situation model, checking for sensibility in the integrated understanding of the context-mathematics relationship. In this way the problem and situation models develop in coordination with each other and are constantly changing in response to one another. The details of this mapping

between situation model and problem model and back again are subject to further study. From an educational standpoint, it seems that it is important to do more than provide students with a contextual situation from which they can extrapolate a problem model. Another important step is for students to connect the numbers back to the situation model, for it is all too easy to get lost in the abstract world of numbers and forget about their meanings. (See also, Stenning and Sommerfeld, this volume.)

Links between a situation model and a problem model could also be accomplished through the use of material representations. In this unit the students are given worksheets and a computer program in an effort to help them understand the components of making a model, and to guide their understanding of how math can work to create that model. In its current state, the MMAP technology presents a problem model in the form of a network of problem quantities. However, it does not have provisions to facilitate relating the math back to a situation model. One way that the technology might be changed is to present a mathematical representation (as it currently does) alongside a simulation of the components that are taking place. Simulations of that sort might serve to create more links from the problem model back to the situation model, forcing students to think situationally about the math they are producing, making it more likely that they will notice their own mistakes if the simulation doesn't work as they expected.

Still another way to increase students' attention to links between situation models and problem models would be to develop a socio-mathematical norm (Cobb & Yackel, 1996) in which students expect to be accountable for explanations that justify mathematical operations and representations in terms of properties and relations of quantities in situations.

Additionally, interactional patterns create thresholds for questioning, which affect how a suggestion is taken up or explained. Although in this group the students seemed to feel that a mutual understanding was important in order to proceed, oftentimes that understanding was not a consensus. One potential way that such a pattern may be altered is through the larger classroom learning practices, including aspects of reasoning and explaining for which students are accountable to each other and to the teacher. Accountability is provided through discourse activities at different levels, as Hall and Rubin (1998) discussed in their analysis of Magdalene Lampert's teaching. Lampert had an explicit policy that any member of a group could be called on to provide an explanation for any of the group's results. This policy made each group of students accountable for achieving mutual understanding so that the individual members could satisfy the expectation that they would be able to understand the group's results.

Introducing this order of accountability into classroom practices can serve the conceptual linking as well. It is necessary to include some sort of provision for making sure that students understand that finding the "correct" mathematical answer is only part of their responsibility. They also should be responsible for relating what they

found back to the model that they were trying to create and to share that information with their peers. In another example involving FCL classrooms (Brown and Campione, 1994), at the end of a unit students are made accountable for what they have learned by sharing their findings with the rest of their class. Therefore, assumptions that are made throughout the unit need to be accounted for and explained. In the course of doing the birthrate sheet, Manuel made many assumptions about the number of guppies, their survival rate, and how that affected the birth rate. If the group had felt some accountability to a "larger audience," that is, if they had to present their findings to the class and explain the elements of their model, they might have been less likely to take logical leaps without trying to understand how they related to the model being created.

Acknowledgements

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