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N. Whitaker
(Ph.D. Thesis)

April 1987

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NUMERICAL SOLUTION OF THE HELE-SHAW EQUATIONS¹

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Ph.D. Thesis

April 1987

¹Supported in part by the Applied Mathematical Sciences Subprogram of the Office of Energy Research, U.S. Department of Energy under contract DE-AC03-76SF00098.

Numerical Solution of the Hele-Shaw Equations

Nathaniel Whitaker

Abstract

An algorithm is presented for approximating the motion of the interface between two immiscible fluids in a Hele-Shaw cell. The interface is represented by a set of volume fractions. We use the Simple Line Interface Calculation method along with the method of fractional steps to transport the interface.

The equation of continuity leads to a Poisson equation for the pressure. The Poisson equation is discretized. Near the interface where the velocity field is discontinuous, the discretization is based on a weak formulation of the continuity equation. Interpolation is used on each side of the interface to increase the accuracy of the algorithm. The weak formulation as well as the interpolation are based on the computed volume fractions. This treatment of the interface is new. The discretized equations are solved by a modified conjugate gradient method. Surface tension is included and the curvature is computed through the use of osculating circles.

For perturbations of small amplitude, a surprisingly good agreement is found between the numerical results and linearized perturbation theory. Numerical results are presented for the finite amplitude growth of unstable fingers.

Acknowledgements

I would like to first thank my thesis advisor, Alexandre Chorin, for his encouragement and enthusiastic interest in my work. He has been a great source of inspiration for me. It has been a pleasure and a privilege to be his student.

I would like to thank Harry Morrison for his guidance and support.

I would like to thank Ronald DiPerna, Ole Hald, Andrew Majda, John Neu and Murray Protter for their valuable instruction and support.

I would like to thank Stanley Berger and Alberto Grunbaum for serving on my thesis committee.

I would like to thank Paul Concus for having an open door and being so willing to discuss my research.

I thank the Mathematics Department and the Computer Science Research Department at Lawrence Berkeley Laboratory for providing me financial support and computing resources. I especially thank Les Tabata for his assistance.

I thank IBM and the Graduate Minority Program for supporting me with fellowships while I was a graduate student at Berkeley. I thank the Mathematics Opportunity Committee and associates for their support.

I would like to thank Christoph Borgers, Dalia Fishelov and Pierre Haldenwang for many helpful discussions about my work while they were visiting Lawrence Berkeley Laboratory. I thank my fellow students Tom Buttke, Chien Cheng Chang and Gerry Puckett with whom I have had many enjoyable conversations.

I thank my parents and sisters for their moral support and I thank my wife Shirley for her love, patience and understanding and for just being there.

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CHAPTER 1

INTRODUCTION

In this paper, we present a numerical method for evolving an interface between two immiscible fluids in a Hele-Shaw cell (two closely placed parallel plates). A Hele-Shaw cell is illustrated in a rectangular coordinate system (x, y, z) in figure 1.1 with the plates located at $z = 0$ and $z = b$. These fluids are immiscible in the sense that there is a finite surface tension which stabilizes small-scale disturbances in the interface. We consider here only horizontal flow, i.e., the acceleration due to gravity is neglected. The equations describing the flow of a fluid in a Hele-Shaw cell are the equation of motion

$$\vec{u} = -\frac{b^2}{12\mu}\nabla p \quad (1.1)$$

and the equation of continuity

$$\nabla \cdot \vec{u} = 0 \quad (1.2)$$

where μ is the fluid viscosity, b is the spacing between the plates in the Hele-Shaw cell. The plates are taken as horizontal, the positive y -axis is in the direction of the flow, the x -axis is parallel to the plates and the z -axis is perpendicular to the plates. The velocity vector $\vec{u}$ has two components u and v which are functions of t (time), x and y . These velocities result from taking an average of the three-dimensional velocity field. The pressure p is also an averaged pressure. We use (1.1) and (1.2) to find the velocity field of two fluids separated by an interface in a Hele-Shaw cell.

The problem of tracking an interface in a Hele-Shaw cell is of interest for many reasons. When one fluid displaces another fluid in a porous medium, the interface between the two fluids can be stable or unstable. For an unstable interface, small disturbances in an initially horizontal interface grow into long 'fingers'. This instability is of interest, for example, in secondary oil

recovery where residual oil is forced out of the ground by the injection of another fluid, usually water. Other examples of fingering in a porous medium occur during the underground storage of gas (see Paterson (1981) and references within) and in water infiltration (see Wooding & Morel-Seytoux (1976)). Some numerical models to evolve interfaces between two fluids in a porous medium are presented by Concus & Proskurowski (1979), Glimm et al (1980, 1981), Lotstedt (1982), Chorin (1983) and Colella et al (1983).

Usually these interfaces are three dimensional and difficult to study both experimentally and numerically. Furthermore, the equations describing the interfaces are usually nonlinear and difficult to analyze.

The Hele-Shaw cell is often used as a device for studying two-dimensional flow in porous medium. This is due to the fact that the differential equations for the velocity field in a porous medium are the same equations for the velocity field in a Hele-Shaw cell, apart from scaling. The Hele-Shaw cell is a simple system which is easier to visualize than a porous medium. The application of viscous fingering in a Hele-Shaw cell to the flow in actual reservoir conditions is not unquestionable (see Wooding & Morel-Seytoux (1976)). Nevertheless, there is interest in the Hele-Shaw equations for their own sake. This problem is presented by Meyer (1981) as a non-trivial model problem for two dimensional transient free boundary problems. As pointed out by Howison (1986), it is a member of an important class of Stefan-type moving boundary problems receiving significant attention lately as models of processes such as crystal growth from a super-saturated solution, solidification of supercooled liquids, and freezing or melting of a dilute binary alloy (Langer (1980); Chorin (1985)). We have developed a code which appears to handle this type of problem in an accurate way. Thus this paper is of interest from a methodological point of view.

In a Hele-Shaw cell, for horizontal flow an interface becomes unstable when a less viscous fluid displaces a more viscous fluid. Unstable fingering in a Hele-Shaw cell has been studied experimentally by many authors (see Saffman & Taylor (1958); Chuoke et al (1959); Gupta et al

(1973); White et al (1976, 1977); Pitts (1980); Park et al (1984)). Saffman & Taylor (1958) performed a linearized stability analysis for the interface, neglected surface tension, and found an analytic solution for the shape of single fingers. Chuoke et al (1959) performed experiments in a Hele-Shaw cell and in packed bed models. They also performed a linearized stability analysis, included surface tension, and showed that surface tension prevents fingers from growing below a certain wavelength. Paterson (1980) recently performed experiments investigating radial fingering, i.e., an initially circular interface is perturbed and fingering occurs. Paterson argues that the radial model is more appropriate to practical situations.

In this paper we present a numerical method which demonstrates high accuracy in evolving interfaces represented by waves of small amplitude thereby suggesting that it gives reasonable results for large amplitude waves. Numerical methods simulating the evolution of an interface in a Hele-Shaw cell have been presented by Meng & Thomson (1978), Tryggvason & Aref (1983, 1985) and by Degregoria & Schwartz (1985, 1986). To our knowledge, only in Tryggvason & Aref (1985) are results compared with the results predicted by the linearized theory. In Tryggvason & Aref (1985) there is a modest agreement, for the case shown, between the numerical results and the predictions of the linearized theory. We compared our numerical method extensively with the predictions of the linearized theory and the results are surprisingly good. This gives us some indication of how well our numerical method can expect to handle unstable interfaces, which evolve into fingers. Some of these evolved interfaces are presented here and they compare well with those obtained experimentally and numerically by other authors.

In our numerical method, the interface we evolve is represented by volume fractions. We use the Simple Line Interface Calculation (SLIC) method along with the method of fractional steps (see Richtmyer & Morton (1967)) to transport our interface of volume fractions. This method (SLIC) is presented initially by Noh & Woodward (1976). Chorin (1980) modifies this method to increase accuracy and applies it to flame propagation. Lotstedt (1982) makes further modifications to increase accuracy in solving Burgers' equation and two phase porous flow.

Sethian (1984) and Ghonien et al (1982) use SLIC to attain highly successful results in combustion. SLIC relies on local reconstructions of the interface instead of global reconstructions and thereby becomes highly effective in handling interfaces with such complex geometry as fingers and cusps. We present a method for finding the velocity field by discretizing a weak formulation of the continuity equation, equation (1.2). Interpolation is used to increase accuracy. A modified conjugate gradient method is used to solve the system resulting from the discretization. Curvature is computed using a method presented by Chorin (1985). This curvature is then used to put surface tension in our model.

The remainder of this paper is organized in the following way. In chapter 2, we give a derivation of the Hele-Shaw equations. Similar derivations can be found in Lamb (1932), Bear (1972) and Thompson (1968). In chapter 3, we give the formulation of the problem. In chapter 4, the results of the linearized theory are presented (see Saffman & Taylor (1959); Chuoke et al (1958)). In chapter 5, we present our numerical method. In section 5.1, the algorithm (SLIC) for transporting the interface, represented by volume fractions, is presented. In section 5.2, our method of approximating the velocity field is presented. In section 5.3, our modified conjugate gradient method is presented. In section 5.4, we discuss how curvature is computed and surface tension is included. In chapter 6, we show how the rate of growth of a wave of small amplitude is approximated by our numerical method. The numerical results are then compared with the results as predicted by the linearized theory in various tables and graphs. In chapter 7, we show the evolution of unstable interfaces for long periods of time. Finally, chapter 8 is devoted to a general discussion of the results, our conclusions about the method, and some ideas for future work.

CHAPTER 2

DERIVATION OF THE HELE-SHAW EQUATIONS

In this chapter, we derive the equations describing the flow of a fluid in a Hele-Shaw cell. The two parallel plates are located in the rectangular coordinate system (x, y, z) at $z = 0$ and $z = b$, see figure 1.1. The equations to be derived consist of the equation of motion

$$\vec{u}(x, y) = -\frac{b^2}{12\mu} \nabla p(x, y) \quad (2.1)$$

and the equation of continuity

$$\nabla \cdot \vec{u}(x, y) = 0 \quad (2.2)$$

where $\vec{u}(x, y) = (u(x, y), v(x, y))$ is an averaged velocity field given by

$$\vec{u}(x, y) = \frac{1}{b} \int_0^b \vec{U}(x, y, z) dz, \quad (2.3)$$

$\vec{U}(x, y, z)$ is the three-dimensional velocity field, b is the spacing between the plates, μ is the dynamic viscosity of the fluid and $p(x, y)$ is a similarly averaged pressure. We derive (2.1) and (2.2) from the Navier-Stokes equations for an incompressible fluid in three dimensions. The Hele-Shaw equations are also derived in Lamb (1932) and Bear (1972). The Navier-Stokes equations for an incompressible fluid in three dimensions are

$$\frac{\partial \vec{U}}{\partial t} + (\vec{U} \cdot \nabla) \vec{U} = \frac{1}{\rho} \nabla p + \nu \Delta \vec{U} \quad (2.4)$$

$$\nabla \cdot \vec{U} = 0 \quad (2.5)$$

where $\vec{U} = (U, V, W)$ are the velocity components in the x, y and z directions respectively, p is the pressure, ρ is the density and ν is the kinematic viscosity. With a change of variables, we nondimensionalize equation (2.4). Using the nondimensionalized equation and the fact that the flow in a Hele-Shaw cell is a slow, creeping flow, we are able to neglect the inertia term $(\vec{U} \cdot \nabla) \vec{U}$ to get the Stokes equations (see Chorin (1979), Landau & Lifshitz (1959), and Milne &

Thomson (1968)). The remaining term on the left hand side of the Stokes equations is then neglected by assuming that the flow is nearly steady. The remaining equations are then integrated to find an expression for the averaged velocities. These integrations result into the Hele-Shaw equations. The details of this derivation are presented below.

We derive the Stokes equations by assuming a slow, creeping flow. Let us choose a constant speed $\bar{U}$ and a constant length $\bar{L}$ which are typical of the flow. The equations (2.4) are nondimensionalized by making the following change of variables

$$\bar{U} = \bar{U}U_1, \quad \bar{x} = \bar{L}x_1, \quad t = \frac{\bar{L}}{\nu}t_1, \quad p = \frac{\nu\bar{U}}{\bar{L}}p_1 \quad (2.6)$$

where $\bar{x} = (x, y, z)$, $\bar{x}_1 = (x_1, y_1, z_1)$ and $\bar{U}_1 = (U_1, V_1, W_1)$. Let us make this change of variables for the x component of (2.4), i.e., to

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\Delta U. \quad (2.7)$$

We then have, using (2.6) and (2.7),

$$\begin{aligned} \frac{\partial \bar{U}U_1}{\partial t_1} \frac{\partial t_1}{\partial t} + \bar{U}U_1 \frac{\partial \bar{U}U_1}{\partial x_1} \frac{\partial x_1}{\partial x} + \bar{U}V_1 \frac{\partial \bar{U}U_1}{\partial y_1} \frac{\partial y_1}{\partial y} + \bar{U}W_1 \frac{\partial \bar{U}U_1}{\partial z_1} \frac{\partial z_1}{\partial z} \\ = -\frac{1}{\rho} \frac{\nu\bar{U}}{\bar{L}} + \frac{\partial^2(\bar{U}U_1)}{\partial(\bar{L}x_1)^2} + \frac{\partial^2(\bar{U}U_1)}{\partial(\bar{L}y_1)^2} + \frac{\partial^2(\bar{U}U_1)}{\partial(\bar{L}z_1)^2} \end{aligned} \quad (2.8)$$

After a little algebra we have,

$$\begin{aligned} \frac{\partial U_1}{\partial t} + R \left(U_1 \frac{\partial U_1}{\partial x_1} + V_1 \frac{\partial U_1}{\partial y_1} + W_1 \frac{\partial U_1}{\partial z_1} \right) \\ = -\frac{1}{\rho} \frac{\partial p_1}{\partial x_1} + \Delta U_1 \end{aligned} \quad (2.9)$$

where R is known as the Reynolds number and is given by $\frac{\bar{U}\bar{L}}{\nu}$. When the flow is very slow, then our typical speed $\bar{U}$ is small implying R is small. The inertia term is then neglected and (2.7) becomes

$$\frac{\partial U}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \Delta U \quad (2.10)$$

We handle the y and z components in a similar manner and obtain the Stokes equations for an

incompressible fluid,

$$\frac{\partial \vec{U}}{\partial t} = -\frac{1}{\rho} \nabla p + \nu \Delta \vec{U}. \quad (2.11)$$

In addition, the flow is assumed to be almost steady, therefore we assume that

$$\frac{\partial \vec{U}}{\partial t} = 0, \quad (2.12)$$

giving us

$$\frac{1}{\mu} \nabla p = \Delta \vec{U}. \quad (2.13)$$

where μ is the dynamic viscosity given by $\rho\nu$. Writing (2.13) componentwise gives us

$$\frac{1}{\mu} \frac{\partial p}{\partial x} = \Delta U \quad (2.14)$$

$$\frac{1}{\mu} \frac{\partial p}{\partial y} = \Delta V \quad (2.15)$$

$$\frac{1}{\mu} \frac{\partial p}{\partial z} = \Delta W. \quad (2.16)$$

The fluid flows through a very narrow space of width b and it is assumed that, the velocity in the z direction, W , is 0. Since the fluid sticks to the plates ($\vec{U}(x, y, 0) = \vec{0}$ and $\vec{U}(x, y, b) = \vec{0}$), the velocity gradients in the z direction are much larger than those in the x and y directions. Therefore $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial y^2}$ may be neglected when compared with $\frac{\partial^2}{\partial z^2}$.

Equations (2.14), (2.15) and (2.16) then become

$$\frac{1}{\mu} \frac{\partial p}{\partial x} = \frac{\partial^2 U}{\partial z^2} \quad (2.17)$$

$$\frac{1}{\mu} \frac{\partial p}{\partial y} = \frac{\partial^2 V}{\partial z^2} \quad (2.18)$$

$$\frac{\partial p}{\partial z} = 0. \quad (2.19)$$

By (2.19), p is a function of x and y only, therefore (2.17) can be integrated in z by treating the left hand side of (2.17) as a function of x and y only. Performing this integration gives us

$$U = \frac{z^2}{2\mu} \frac{\partial p}{\partial x} + a_1 z + a_2 \quad (2.20)$$

where a_1 and a_2 are functions of x and y only. Applying the boundary conditions

$U = V = 0$ when $z = 0$ and $z = b$, we have

$$U = \left(\frac{z^2}{2\mu} - \frac{bz}{2\mu} \right) \frac{\partial p}{\partial x}. \quad (2.21)$$

The average value of $U(x, y, z)$ over a segment perpendicular to the plates between $z = 0$ and $z = b$ is given by (2.3). We use (2.3) and (2.21) and find that

$$u(x, y) = \frac{1}{b} \int_0^b \left(\frac{z^2}{2\mu} - \frac{bz}{2\mu} \right) \frac{\partial p}{\partial x} dz \quad (2.22)$$

and with a little algebra we have

$$u(x, y) = -\frac{b^2}{12\mu} \frac{\partial p(x, y)}{\partial x}. \quad (2.23)$$

In a similar manner, we obtain from (2.18)

$$v(x, y) = -\frac{b^2}{12\mu} \frac{\partial p(x, y)}{\partial y}. \quad (2.24)$$

Equations (2.23) and (2.24) are equivalent to Darcy's law in a porous medium with permeability $\frac{b^2}{12}$ (see Collins(1961)).

Since $W = 0$, equation (2.5) is written

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0. \quad (2.25)$$

Integrating both sides over z from 0 to b and multiplying by $\frac{1}{b}$, we have

$$\frac{1}{b} \int_0^b \frac{\partial U}{\partial x} dz + \frac{1}{b} \int_0^b \frac{\partial V}{\partial y} dz = 0 \quad (2.26)$$

and using (2.3), we have that the averaged velocities satisfy

$$\frac{\partial u(x, y)}{\partial x} + \frac{\partial v(x, y)}{\partial y} = 0. \quad (2.27)$$

Equations (2.23), (2.24) and (2.27) give us the equations describing the flow of a fluid in a Hele-Shaw cell.

CHAPTER 3

FORMULATION OF THE PROBLEM

In this chapter, we give the formulation of the problem which we solve numerically. We first interpret the three dimensional problem of evolving an interface in a Hele-Shaw cell in terms of a two dimensional problem. We then give a formulation of this problem in terms of the differential equations and conditions to be satisfied. The Hele-Shaw equations are solved for on each side of the interface to obtain a velocity field. These two solutions are connected by the following three conditions. First, the Rankine-Hugoniot condition is satisfied on the interface, i.e., the normal component of the velocity field is continuous across the interface. Second, a prescribed pressure jump at the interface related to a given surface tension is required. Third, a compatibility condition between the velocity behind the interface and the velocity in front of the interface is satisfied. After finding the velocity field, the fluids are advected in a way such that the interface is transported. Finally, we derive the compatibility condition.

Suppose that a Hele-Shaw cell is given in a rectangular coordinate system (x, y, z) where the two parallel plates lie in the planes $z = 0$ and $z = b$ and b is as before the spacing between the plates (see figure 1.1). The plates are bounded on the sides by the planes $x = 0$ and $x = a$. It is assumed that black fluid, lying behind the interface, is injected at some constant velocity $(0, V, 0)$ across the plane $y = 0$ in the direction of the positive y axis the white fluid, lying in front of the interface, is expelled at the constant velocity $(0, V, 0)$ across the plane $y = c$ in the direction of the positive y axis. We are interested in the interface between the two fluids. In deriving the equations governing the flow of a fluid in a Hele-Shaw cell, we find that $\frac{\partial p}{\partial z} = 0$, implying that p is a function of x and y only. The velocity field $\vec{u}$ is an averaged velocity field which is also a function of x and y only. The flow is considered as two dimensional in the x - y plane. In the x - y plane, we consider the flow to lie in a rectangular box R with corners lying at $(0,0)$, $(a,0)$, $(0,c)$, and (a,c) where c is a positive real number

(see figure 1.2). Black fluid flows into the box from behind the interface at a constant velocity $(0, V)$ across the line segment between the points $(0, 0)$ and $(a, 0)$. White fluid flows out of the box in front of the interface at a constant velocity $(0, V)$ across the line segment between the points $(0, c)$ and (a, c) . All functions defined on the box are periodic in the horizontal direction with period a . The region in which the black fluid lies is R_b and the region in which the white fluid lies is R_w such that the union of those regions is the region R . We are interested in the evolution of the interface between the white and the black fluids with viscosities μ_w and μ_b respectively. The problem to be solved with the appropriate conditions to be satisfied is as follows.

In R_w , find a solution to the equations

$$\vec{u}_w = -\frac{b^2}{12\mu_w} \nabla p_w \quad (3.1)$$

$$\nabla \cdot \vec{u}_w = 0 \quad (3.2)$$

such that all quantities are periodic in the x direction with period a and the velocity, $v_w(x, c)$ is prescribed.

In R_b , find a solution to the equations

$$\vec{u}_b = -\frac{b^2}{12\mu_b} \nabla p_b \quad (3.3)$$

$$\nabla \cdot \vec{u}_b = 0 \quad (3.4)$$

such that all quantities are periodic in the x direction with period a and the velocity, $v_b(x, 0)$, is prescribed.

These two problems are connected by the following conditions.

1) The Rankine-Hugoniot condition is satisfied, i.e.,

$$\vec{u}_w \cdot \vec{n} - \vec{u}_b \cdot \vec{n} = 0 \quad (3.5)$$

where $\vec{n}$ is the normal to the interface pointing towards the black fluid.

2) Surface tension is included in this model by specifying a jump in the pressure across the

interface given by

$$p_w(x_0, y_0) - p_b(x_0, y_0) = -\tau(c_1 + c_2) \quad (3.6)$$

where $p_b(x_0, y_0)$ is the limit of $p_b(x, y)$ as (x, y) approaches (x_0, y_0) with (x, y) a point in the black fluid and p_w similarly defined, τ is the surface tension and c_1, c_2 are the signed principal curvatures of the interface at the point (x_0, y_0) .

3) The compatibility condition

$$\int_0^1 v_b(x, 0) dx = \int_0^1 v_w(x, c) dx, \quad (3.7)$$

is satisfied, i.e., the inflow velocity and the outflow velocity cannot be specified arbitrarily.

After finding the velocity field above, we advect the black and white fluids by approximately solving the equation

$$\rho_t + \vec{u} \cdot \nabla \rho = 0 \quad (3.8)$$

in such a way that the interface is transported where ρ is the density and t is the time.

Let us now derive (3.7), the compatibility condition. The equation of motion and the continuity equation imply that in R_w

$$\Delta p_w = 0 \quad (3.9)$$

and in R_b

$$\Delta p_b = 0 \quad (3.10)$$

where p_w is the pressure in the white fluid and p_b is the pressure in the black fluid. For these elliptic problems, we specify boundary conditions. On the vertical sides of R_w and R_b , we assume periodic boundary conditions. On the side of R_b given by the line segment between the points $(0, 0)$ and $(0, a)$, we specify the value of the normal derivative where the normal is the outward normal. Specifying this normal derivative is equivalent to giving the value of

$$-\frac{\partial p_b}{\partial y}(x, 0) \quad (3.11)$$

on the line segment. Note that, from (3.1),

$$v(x,0) = \frac{-b^2}{12\mu} \frac{\partial p_b}{\partial y}(x,0) \quad (3.12)$$

and by specifying the boundary condition (3.11), one is giving the value of the velocity on the line segment. On the part of the boundary of R_w given by the line segment between the points $(c,0)$ and (c,a) , we specify the value of the normal derivative where the normal is the outward normal. Giving the normal derivative is equivalent to giving

$$\frac{\partial p_w}{\partial y}(x,c) \quad (3.13)$$

on the line segment. Note that, from (3.1),

$$v(x,c) = \frac{-b^2}{12\mu} \frac{\partial p_w}{\partial y}(x,c) \quad (3.14)$$

and by specifying (3.13), one is giving the value of the velocity at the line segment.

The inflow velocity and the outflow velocity are not given arbitrarily. When the continuity equation is satisfied on both sides of the interface the Rankine-Hugoniot condition must be satisfied across the interface, i.e., the normal component of the velocity is continuous across the interface. In solving (3.10), in R_b , for a Neumann problem, the compatibility condition

$$\int_{\partial R_b} \vec{u}_b \cdot \vec{n} ds = 0 \quad (3.15)$$

is to be satisfied and in R_w , for a Neumann problem, the condition

$$\int_{\partial R_w} \vec{u}_w \cdot \vec{n} ds = 0 \quad (3.16)$$

is to be satisfied where $\vec{n}$ is the outward normal above, and ∂R_b and ∂R_w denote the boundaries of R_b and R_w respectively. The parts of (3.15) along the vertical sides cancel each other out, line integral in (3.15) then reduces to

$$\int_0^1 -v(x,0) dx = \int_{\gamma} \vec{u}_b \cdot \vec{n}_{\gamma} ds \quad (3.17)$$

where γ is the curve representing the interface and $\vec{n}_{\gamma}$ is the normal to γ pointing into the black fluid. By the same argument for R_w , we have

$$\int_0^1 v(x, c) dx = - \int_{\partial \gamma} \vec{u}_w \cdot \vec{n}_\gamma ds. \quad (3.18)$$

By the Rankine-Hugoniot condition and by (3.17) and (3.18) we have the compatibility condition

$$\int_0^1 v(x, 0) dx = \int_0^1 v(x, c) dx. \quad (3.19)$$

The inflow velocity and the outflow velocity must satisfy (3.19).

CHAPTER 4

LINEARIZED STABILITY THEORY

In this chapter, the linearized stability theory will be presented, following Saffman & Taylor (1958), Chuoke et al (1959). As given in chapter 2, the equations describing the flow of a fluid in a Hele-Shaw cell are the equation of motion (Darcy's law)

$$\vec{u}(x, y, z) = -\frac{k}{\mu} \vec{\nabla} p(x, y, z) \quad (4.1)$$

and the equation of continuity

$$\vec{\nabla} \cdot \vec{u}(x, y, z) = 0 \quad (4.2)$$

where $\vec{u}(x, y, z) = (u, v, w)$ is the velocity vector, μ is the viscosity of the fluid, k is $\frac{b^2}{12}$ and p is the pressure. These equations are written in a three dimensional form so that the effects of curvature in the $x-z$ plane and the $y-z$ plane is included. We start with an initial flow consisting of a horizontal interface in a stationary state of uniform velocity $(0, 0, W)$. We suppose fluid 1 is below the interface and fluid 2 above. The subscripts 1 will denote variables in fluid 1 and subscript 2 denotes variables in fluid 2. It is supposed that the interface is perturbed by some wave of wavenumber n and that the amplitude of the wave, $\epsilon e^{\sigma t}$, is small where t is time, ϵ is a small parameter and σ is referred to as the rate of growth. The corresponding perturbed velocity field is substituted into (4.1) and (4.2) to get equations for the perturbations. By equating terms of first order in ϵ , and assuming continuity of normal velocities across the interface we get differential equations for the perturbations which can be easily solved. We use the solutions of the differential equations obtained on each side of the interface and our definition of the surface tension coefficient, τ , to get a formula for the rate of growth of the wave. The rate of growth, σ , is given by

$$\sigma = \frac{(\mu_2 - \mu_1)nW - \tau n^3 k}{\mu_2 + \mu_1}. \quad (4.3)$$

μ_1 and μ_2 are the viscosities from below and above respectively.

For nonzero surface tension, (4.3) shows that the rate of growth is a linear function of wavenumber n . All waves grow when the viscosity above is greater than the viscosity below and they grow arbitrarily fast. When the viscosity above is smaller than below, then all waves decay. For nonzero surface tension ($\tau > 0$), σ is a cubic polynomial in the n . When the viscosity above is smaller than below, all waves decay. Alternatively, when $\mu_2 > \mu_1$, there exists a wavenumber, n_0 , at which σ has a positive maximum and a wavenumber, n_1 , such that for all waves with wavenumbers $n > n_1$, σ is negative, i.e., we have decay. These results are given in terms of the wavelength also. The details of these derivations are given below.

A rectangular coordinate system (x, y, z) is chosen with the positive z -axis directed from fluid 1 to fluid 2. Assume a steady state of uniform velocity $(0, 0, W)$ vertically upwards and a horizontal interface parallel to the plane $z = 0$. Let us perturb our velocity field $(0, 0, W)$ by $(\tilde{u}, \tilde{v}, \tilde{w})$. We substitute this into Darcy's law (4.1) and we find that the perturbed velocity field must satisfy

$$(\tilde{u}, \tilde{v}, \tilde{w} + W) = -\nabla \left[\left(\frac{k}{\mu} \right) p \right] \quad (4.4)$$

where μ is the appropriate viscosity and p is the pressure. W is subtracted from the z component of both sides of (4.4) and we find

$$(\tilde{u}, \tilde{v}, \tilde{w}) = -\nabla \left[\left(\frac{k}{\mu} \right) p + Wz \right] = \nabla \phi \quad (4.5)$$

where ϕ is the velocity potential. We substitute the velocity field $(\tilde{u}, \tilde{v}, \tilde{w} + W)$ into the continuity equation which gives us

$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z} = 0. \quad (4.6)$$

Equations (4.5) and (4.6) are the equations of motion and continuity respectively for the perturbation velocities in a coordinate system moving with velocity $(0, 0, W)$ with the unperturbed

horizontal interface at rest. The velocity potential for our new system is given in (4.6) by

$$\phi = -\left(\frac{k}{\mu}\right)p - Wz \quad (4.7)$$

and p is given by

$$p = -\left(\frac{\mu}{k}\right)\phi - \left(\frac{\mu}{k}\right)Wz. \quad (4.8)$$

We wish to study the evolution of an arbitrary interface between two fluids with different viscosities. Suppose we decompose an arbitrary perturbation of the horizontal interface into its Fourier modes and study each mode separately. The interface is evolving in time (t) and is of the form

$$z = \eta(x, y, t, \epsilon) = \epsilon e^{in_x x + in_y y + \sigma t} \quad (4.9)$$

where n_x and n_y are the wavenumbers of this perturbation in x and y , σ is the rate of growth, ϵ is the amplitude (assumed to be small) and $i = \sqrt{-1}$. Let $g(x, y, z, t) = z - \eta(x, y, t, \epsilon) = 0$. We differentiate g with respect to t , taking into account that x , y and z are functions of t , and find that

$$\begin{aligned} 0 &= \frac{\partial g}{\partial t} + \frac{\partial x}{\partial t} \frac{\partial g}{\partial x} + \frac{\partial y}{\partial t} \frac{\partial g}{\partial y} + \frac{\partial z}{\partial t} \frac{\partial g}{\partial z} \\ &= -\frac{\partial \eta}{\partial t} - \tilde{u} \frac{\partial \eta}{\partial x} - \tilde{v} \frac{\partial \eta}{\partial y} + \tilde{w} \end{aligned} \quad (4.10)$$

We then solve for $\frac{\partial \eta}{\partial t}$ and find that

$$\frac{\partial \eta}{\partial t} = -\tilde{u} \frac{\partial \eta}{\partial x} - \tilde{v} \frac{\partial \eta}{\partial y} + \tilde{w} = (\tilde{u}, \tilde{v}, \tilde{w}) \cdot \vec{n} \quad (4.11)$$

where $\vec{n} = (-\frac{\partial \eta}{\partial x}, -\frac{\partial \eta}{\partial y}, 1)$ is the outward normal from fluid 1. By the Rankine-Hugoniot condition discussed in Chapter 3, the normal components of the velocity are continuous across the interface. We use this fact and (4.11) which gives us

$$\frac{\partial \eta}{\partial t} = -\frac{\partial \eta}{\partial x} \frac{\partial \phi_1}{\partial x} - \frac{\partial \eta}{\partial y} \frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_1}{\partial z}$$

$$= -\frac{\partial \eta}{\partial x} \frac{\partial \phi_2}{\partial x} - \frac{\partial \eta}{\partial y} \frac{\partial \phi_2}{\partial y} + \frac{\partial \phi_2}{\partial z}. \quad (4.12)$$

For the perturbation velocities, (4.5) reduces to the following equations on the interface,

$$\begin{aligned} \tilde{u}(x, y, \eta) &= -\left(\frac{k}{\mu}\right) \left(\frac{\partial p}{\partial x}\right)(x, y, \eta) \\ \tilde{v}(x, y, \eta) &= -\left(\frac{k}{\mu}\right) \left(\frac{\partial p}{\partial y}\right)(x, y, \eta) \\ \tilde{w}(x, y, \eta) &= -\left(\frac{k}{\mu}\right) \left(\frac{\partial p}{\partial z}\right)(x, y, \eta) - W \end{aligned} \quad (4.13)$$

where μ is equal to μ_1 in fluid 1 and μ_2 in fluid 2. If we expand the right hand sides of equations (4.13) about $z=0$ we have

$$\begin{aligned} \tilde{u} &= -\left(\frac{k}{\mu}\right) \frac{\partial p}{\partial x}(x, y, 0) - \left(\frac{k}{\mu}\right) \frac{\partial^2 p}{\partial z \partial x}(x, y, \xi_x) \eta \\ \tilde{v} &= -\left(\frac{k}{\mu}\right) \frac{\partial p}{\partial y}(x, y, 0) - \left(\frac{k}{\mu}\right) \frac{\partial^2 p}{\partial z \partial y}(x, y, \xi_y) \eta \\ \tilde{w} &= -\left(\frac{k}{\mu}\right) \frac{\partial p}{\partial z}(x, y, 0) - \left(\frac{k}{\mu}\right) \frac{\partial^2 p}{\partial z^2}(x, y, \xi_z) \eta - W \end{aligned} \quad (4.14)$$

where ξ_x , ξ_y , and ξ_z are between 0 and $\eta(x, y, t, \epsilon)$. We observe that

$$\begin{aligned} -\left(\frac{k}{\mu}\right) \frac{\partial p}{\partial x}(x, y, 0) &= -\left(\frac{k}{\mu}\right) \lim_{\epsilon \rightarrow 0} \left(\frac{\partial p}{\partial x}\right)(x, y, \eta(x, y, t, \epsilon)) \\ -\left(\frac{k}{\mu}\right) \frac{\partial p}{\partial y}(x, y, 0) &= -\left(\frac{k}{\mu}\right) \lim_{\epsilon \rightarrow 0} \left(\frac{\partial p}{\partial y}\right)(x, y, \eta(x, y, t, \epsilon)) \\ -\left(\frac{k}{\mu}\right) \frac{\partial p}{\partial z}(x, y, 0) &= -\left(\frac{k}{\mu}\right) \lim_{\epsilon \rightarrow 0} \left(\frac{\partial p}{\partial z}\right)(x, y, \eta(x, y, t, \epsilon)). \end{aligned} \quad (4.15)$$

The terms on the right hand side are the original unperturbed velocity field which is $(0, 0, W)$.

(4.14) then reduces to

$$\begin{aligned} \tilde{u} &= -\left(\frac{k}{\mu}\right) \frac{\partial^2 p}{\partial z \partial x}(x, y, \xi_x) \eta = O(\epsilon) \\ \tilde{v} &= -\left(\frac{k}{\mu}\right) \frac{\partial^2 p}{\partial z \partial y}(x, y, \xi_y) \eta = O(\epsilon) \\ \tilde{w} &= -\left(\frac{k}{\mu}\right) \frac{\partial^2 p}{\partial z^2}(x, y, \xi_z) \eta = O(\epsilon) \end{aligned} \quad (4.16)$$

since η is $O(\epsilon)$. By (4.16), we then have that $\frac{\partial \phi_1}{\partial x}$, $\frac{\partial \phi_2}{\partial x}$, $\frac{\partial \phi_1}{\partial y}$, $\frac{\partial \phi_2}{\partial y}$, $\frac{\partial \phi_1}{\partial z}$ and $\frac{\partial \phi_2}{\partial z}$ are $O(\epsilon)$.

By (4.9), at the interface, $\frac{\partial \eta}{\partial x}$ and $\frac{\partial \eta}{\partial y}$ are both $O(\epsilon)$. We equate terms of first order in (4.12)

and we find

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi_1}{\partial z} = \frac{\partial \phi_2}{\partial z} \quad (4.17)$$

where both potentials ϕ_1 and ϕ_2 are evaluated at the interface $z = \eta(x, y, t, \epsilon)$. Substituting for $\frac{\partial \eta}{\partial t}$ we obtain the equations that our velocity potentials must satisfy:

$$\begin{aligned} \frac{\partial \phi_1}{\partial z} &= \epsilon \sigma e^{in_x x + in_y y + \sigma t} \\ \frac{\partial \phi_2}{\partial z} &= \epsilon \sigma e^{in_x x + in_y y + \sigma t} \end{aligned} \quad (4.18)$$

Since these potentials are perturbations of the initial flow they must vanish as $z \rightarrow +\infty$ for ϕ_2 and as $z \rightarrow -\infty$ for ϕ_1 . The appropriate solutions satisfying these properties are

$$\begin{aligned} \phi_1 &= \left(\frac{\epsilon \sigma}{n} \right) e^{nx + in_x x + in_y y + \sigma t} \\ \phi_2 &= - \left(\frac{\epsilon \sigma}{n} \right) e^{-nx + in_x x + in_y y + \sigma t} \end{aligned} \quad (4.19)$$

We expand $e^{\pm nx}$, where x is taken on the interface $z = \eta$, and find that $e^{\pm nx} \approx (1 + O(\epsilon))$.

Therefore

$$\begin{aligned} \phi_1 &= \left(\frac{\epsilon \sigma}{n} \right) (1 + O(\epsilon)) e^{in_x x + in_y y + \sigma t} \\ \phi_2 &= - \left(\frac{\epsilon \sigma}{n} \right) (1 + O(\epsilon)) e^{in_x x + in_y y + \sigma t} \end{aligned} \quad (4.20)$$

and therefore to the first order ϕ_1 and ϕ_2 satisfy (4.18).

Let (x_0, y_0, z_0) satisfy $z_0 = \eta(x_0, y_0, 0, \epsilon)$ for some fixed ϵ , i.e., (x_0, y_0, z_0) is a point on the interface. There is an assumed jump in the pressure defined by

$$p_2(x_0, y_0, z_0) - p_1(x_0, y_0, z_0) = -\tau(c_1 + c_2) \quad (4.21)$$

where $p_1(x_0, y_0, z_0)$ is the limit of $p_1(x, y, z)$ as (x, y, z) approaches (x_0, y_0, z_0) with (x, y, z) a point in fluid 1 and p_2 similarly defined. τ is the surface tension and c_1, c_2 are the signed principal curvatures of the interface at the point (x_0, y_0, z_0) . We combine equations (4.8) and (4.21) and at the point (x_0, y_0, z_0) we find

$$p_2 - p_1 = \frac{\mu_1}{k} \phi_1 + \frac{\mu_1}{k} Wz - \frac{\mu_2}{k} \phi_2 - \frac{\mu_2}{k} Wz = -\tau(c_1 + c_2). \quad (4.22)$$

We substitute ϕ_1 and ϕ_2 , given by (4.19), into (4.22) and find that

$$\begin{aligned} & \mu_1 \left(\frac{\epsilon \sigma}{n} \right) e^{n_x x + i n_z z + i n_y y + \sigma t} + \mu_2 \left(\frac{\epsilon \sigma}{n} \right) e^{-n_x x + i n_z z + i n_y y + \sigma t} \\ & + \mu_1 Wz - \mu_2 Wz = -\tau(c_1 + c_2)k \end{aligned} \quad (4.23)$$

The curvature on the interface is defined by

$$\begin{aligned} c_1 &= -\frac{\partial^2 \eta}{\partial x^2} = \epsilon n_x^2 e^{i n_z z + i n_y y + \sigma t} \\ c_2 &= -\frac{\partial^2 \eta}{\partial y^2} = \epsilon n_y^2 e^{i n_z z + i n_y y + \sigma t} \end{aligned} \quad (4.24)$$

We substitute back into (4.23) and we find that

$$\begin{aligned} & \mu_1 \left(\frac{\sigma}{n} \right) \left(1 + O(\epsilon) \right) \eta + \mu_2 \left(\frac{\sigma}{n} \right) \left(1 + O(\epsilon) \right) \eta \\ & - (\mu_2 - \mu_1) W \eta = -\tau(n_x^2 \eta + n_y^2 \eta)k \end{aligned} \quad (4.25)$$

We equate lower order terms, divide by η and find that

$$\mu_1 \left(\frac{\sigma}{n} \right) + \mu_2 \left(\frac{\sigma}{n} \right) - (\mu_2 - \mu_1) W = -\tau n^2 k. \quad (4.26)$$

(4.26) simplifies to

$$\sigma(\mu_2 + \mu_1) = (\mu_2 - \mu_1)nW - \tau n^3 k. \quad (4.27)$$

We solve for σ and find that

$$\sigma = \frac{(\mu_2 - \mu_1)nW - \tau n^3 k}{\mu_2 + \mu_1}. \quad (4.28)$$

By (4.9) we see that the growth of our perturbation depends on σ . If $\sigma > 0$ then we have exponential growth otherwise our perturbation decays exponentially. Equation (4.28) implies that $\sigma > 0$ if and only if

$$(\mu_2 - \mu_1)W > \tau n^2 k. \quad (4.29)$$

Let $n = \frac{2\pi}{l}$ for some wavelength l . There is a critical wavelength l_{crit} such that for all

wavelengths $l > l_{crit}$ we have stable modes and for $l < l_{crit}$ we have instability. By (4.28) the critical values of our parameters, i.e. the values such that σ is 0, occur when

$$(\mu_2 - \mu_1)W - \tau(n_{crit})^2k = 0. \quad (4.30)$$

Letting $n_{crit} = \frac{2\pi}{l_{crit}}$ in (4.30) we see that

$$l_{crit} = 2\pi \left(\frac{\tau k}{(\mu_2 - \mu_1)W} \right)^{\frac{1}{2}}. \quad (4.31)$$

To find the wavelength l_{max} associated with the maximum rate of growth we solve $\frac{\partial \sigma}{\partial n} = 0$

where σ is given by (4.28). We find

$$0 = \frac{\partial \sigma}{\partial n} = \frac{(\mu_2 - \mu_1)W - 3\tau n^2k}{\mu_2 - \mu_1}. \quad (4.32)$$

We then solve for n and find that the maximum value of σ occurs at the value

$$n_{max} = \left(\frac{(\mu_2 - \mu_1)W}{3\tau k} \right)^{\frac{1}{2}} \quad (4.33)$$

and since $n_{max} = \frac{2\pi}{l_{max}}$, we find that

$$l_{max} = 2\pi \left(\frac{3\tau k}{(\mu_2 - \mu_1)W} \right)^{\frac{1}{2}}. \quad (4.34)$$

We compare (4.31) with (4.34) and we find that

$$l_{max} = \sqrt{3} l_{crit}. \quad (4.35)$$

CHAPTER 5

THE NUMERICAL METHOD

In this chapter, a numerical method for evolving an interface between two fluids in a Hele-Shaw cell is presented. Our numerical method evolves the interface in a rectangular box in the $x-y$ plane. The four corners of the rectangular box lie at $(0,0)$, $(a,0)$, $(0,c)$, and (a,c) where a and c are positive real numbers (see figure 1.1). Black fluid flows into the box from behind the interface at a constant velocity $(0,V)$ across the line segment between the points $(0,0)$ and $(a,0)$. White fluid flows out of the box in front of the interface at a constant velocity $(0,V)$ across the line segment between the points $(0,c)$ and (a,c) . All functions defined on the box are periodic in the horizontal direction with period equal to a .

Our numerical method consists of two major parts;

- (1) Calculation of the velocity field.
- (2) Calculation of the motion of the interface.

In section 5.1, it is shown how the interface is transported with the Simple Line Interface Calculation (SLIC) method. A numerical grid is imposed over the region containing the interface and a number is assigned to each cell equal to the fraction of the cell located behind the interface. A local interface is drawn in each cell and the fractions are updated during each time step according to the appropriate differential equations. In section 5.2, the velocity field is approximated. An interface is reconstructed locally in each cell. The pressure is solved for at a point in each cell that depends on the type of interface in the cell. An equation involving the unknown pressure in each cell and its neighboring cells is obtained through the discretization of a weak (integral) form of the continuity equation. The velocity field is calculated from the weak form of the equations. After discretization of the continuity equation in each cell, a system of equations results and these equations are solved by a modified conjugate gradient method presented in section 5.3. In section 5.4, it is explained how surface tension is calculated and

how the curvature is approximated.

5.1. SLIC. A grid is imposed on a domain and a number is assigned to each cell corresponding to the fraction of that cell located behind the interface. The interface is reconstructed locally and the fractions are updated during each time step in accordance with the appropriate differential equations. This method relies on local reconstructions instead of global reconstructions and thereby becomes highly effective in handling interfaces with complex geometry such as fingers and cusps. Due to its simplicity and applicability there is a great interest in improving the accuracy of this method.

The original method proposed by Noh & Woodward (1976) uses horizontal and vertical lines for the local interfaces. Chorin (1980) applies SLIC to flame propagation and includes corners and thin randomly located fingers for accuracy. Lotstedt (1982) uses trapezoids to increase accuracy in solving Burgers' equation and two phase porous flow. Sethian (1984) and Ghoniem et al (1982) use SLIC to attain highly successful results in combustion.

Consider the two-fluid flow in the rectangular box described at the beginning of this chapter. Consider a numerical grid of uniform mesh size h imposed over this box. The centers of the cells are located at the points (ih, jh) where i, j are integers and h is a small parameter. Assign a number $f_{i,j}$ to each cell equal to the fraction of black fluid in that cell. A cell (i, j) has volume fraction $f_{i,j} = 1$ when it lies entirely behind the interface, volume fraction $f_{i,j} = 0$ when it lies entirely in front of the interface and $0 < f_{i,j} < 1$ when the interface passes through the cell. The differential equation that describes the advection of the fluid is

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} = 0 \quad (5.1.1)$$

where ρ is the density and u, v are the x and y components of the given velocity field. The solution to equation (5.1.1) is approximated using the method of fractional steps (Richtmyer & Morton (1967)). The fluid is first advected in the x -direction by the approximate equation

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = 0 \quad (5.1.2)$$

over a half time step and then in the y -direction by the approximate equation

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial y} = 0 \quad (5.1.3)$$

over a half time step. Equations (5.1.2) and (5.1.3) can be approximated by moving a fluid in the x and y directions a distance $u \cdot \frac{k}{2}$ and $v \cdot \frac{k}{2}$ respectively, where k is the time step. For each of these sweeps, a local interface is drawn in each cell where $0 < f_{i,j} < 1$. The black fluid is then advected in the x (respectively y) direction with velocity u (respectively v) and the volume fractions, $f_{i,j}$'s are updated.

The geometry of the interface is adapted to the particular sweep. The types of interfaces that we use are given below. They include horizontal interfaces, vertical interfaces, corners, and thin fingers as used by Chorin (1980). The local interface in cell (i,j) is determined by examining the volume fractions in cell (i,j) and its neighbors. For the horizontal sweep we give below the possible interfaces considered along with the volume fractions which determine them.

A) Vertical Interface: The interface in cell (i,j) is a vertical line and is located at $x = (i - .5)h + f_{i,j}h$. This case occurs when one of the following situations hold.

- (1) $0 < f_{i,j} < 1, 0 < f_{i-1,j}, f_{i+1,j} = 0$, and either $(f_{i,j+1} = 0, f_{i,j-1} = 0)$ or $(0 < f_{i,j+1}, 0 < f_{i,j-1})$ (see figure 5.1)
- (2) $0 < f_{i,j} < 1, f_{i-1,j} = 0, 0 < f_{i+1,j}$, and either $(f_{i,j+1} = 0, f_{i,j-1} = 0)$ or $(0 < f_{i,j+1}, 0 < f_{i,j-1})$
- (3) $0 < f_{i,j} < 1, f_{i-1,j} < 1, f_{i+1,j} = 1$, and either $(f_{i,j+1} = 1, f_{i,j-1} = 1)$ or $(f_{i,j+1} < 1, f_{i,j-1} < 1)$
- (4) $0 < f_{i,j} < 1, f_{i-1,j} = 1, f_{i+1,j} < 1$, and either $(f_{i,j+1} = 1, f_{i,j-1} = 1)$ or $(f_{i,j+1} < 1, f_{i,j-1} < 1)$

B) Horizontal Interface: $0 < f_{i,j} < 1, 0 < f_{i-1,j} < 1, 0 < f_{i+1,j} < 1$. The interface is a horizontal line in the (i,j) cell and is located at $y = (j - .5)h + f_{i,j}h$ (see figure 5.2).

C) Corner: $0 < f_{i,j} < 1$, $0 < f_{i-1,j} < 1$, $f_{i+1,j} = 0$, $f_{i,j+1} = 0$, $0 < f_{i,j-1}$ (see figure 5.3).

The black fluid lies in a rectangle in the lower left hand corner of the (i,j) cell. The rectangle has height b and width a . These parameters a and b are found by solving the equations

$$\frac{b}{a} = \frac{f_{i,j-1}}{f_{i-1,j}} \quad (5.1.4)$$

and

$$ab = f_{i,j} h^2 \quad (5.1.5)$$

If $b > h$ then let $b = h$ and $a = f_{i,j} h$. Also if $a > h$ then let $a = h$ and $b = f_{i,j} h$.

There are seven related cases. In three of these cases, the black fluid lies in a rectangle in the other three corners of the cell. In the remaining four cases, the role of the black fluid is interchanged with the white fluid and we have a rectangle of white fluid lying in the four corners.

D) Finger: $0 < f_{i,j} < 1$, $f_{i-1,j} = 0$, $f_{i+1,j} = 0$ (see figure 5.4). There is a thin finger of black fluid surrounded on the left and right by cells containing only white fluid. The location of the finger is determined randomly. Suppose the finger occupies the region $a \leq x \leq b$. The parameters a and b are given by $a = (i - .5)h + \theta(1 - f_{i,j})h$ and $b = a + f_{i,j}h$ where θ is a member of an equidistributed sequence of random numbers in $[0,1]$. We use the van der Corput sequence as used by Colella(1982). The same θ is used in all cells in which this case occurs for a fixed time. A new θ is chosen at each time step. Noh and Woodward (1976) originally proposed a finger placed in the middle of the cell. It is pointed out by Fenimore (1979) that the movement of the finger depends on $\frac{u \cdot k}{h}$ rather than u . This can be easily seen since the finger does not move at all unless $\frac{u \cdot k}{h}$ is large enough. Chorin (1980) proposed using the random finger above. This assures that the average motion of the finger is computed correctly. By interchanging the roles of black and white we get the case of a white finger.

E) No Interface: $f_{i,j} = 0$ or $f_{i,j} = 1$. These cases occur the most frequently and are the easiest. Nothing is usually done unless an interface is in an adjacent cell.

A velocity field is given on the associated staggered grid (Harlow and Welch (1965)). The horizontal velocities are given at the centers of the vertical sides of the cells. The vertical velocities are given at the centers of the horizontal sides of the cells. For cell (i, j) , there are two vertical velocities $v_{i,j-\frac{1}{2}}$ and $v_{i,j+\frac{1}{2}}$ given at the centers $((ih, (j - .5)h))$ and $((ih, (j + .5)h))$ respectively of the two horizontal sides. There are two horizontal velocities $u_{i-\frac{1}{2},j}$ and $u_{i+\frac{1}{2},j}$ given at the centers $((i - .5)h, jh)$ and $((i + .5)h, jh)$ respectively of the two vertical sides of cell (i, j) (see figure 5.5).

Below we will give several examples of the motion of an interface in a cell according to SLIC. The sweep will be assumed to be in the x -direction. In the first example, the moving of a horizontal interface will be illustrated. In the second, a vertical interface shall be studied. The vertical interface shall be used to illustrate how cells which are full or empty are handled. In example 2, it shall be assumed that a velocity u is given in the cell (i, j) for the black fluid.

Example 1. Suppose that $0 < f_{i-1,j} < 1$, $0 < f_{i,j} < 1$ and $0 < f_{i+1,j} < 1$, as in case B above, then we have locally in cell (i, j) a horizontal interface. The black fluid in cell (i, j) is advected horizontally as a rectangular unit with vertical dimension $f_{i,j} \cdot h$ and horizontal dimension h (see figure 5.2). Suppose that $u_{i-\frac{1}{2},j} < 0$ and $u_{i+\frac{1}{2},j} > 0$. The black fluid on the left hand side of the cell is transported a distance $(-u_{i-\frac{1}{2},j})k$ to the left. The fluid on the right side of the cell (i, j) is transported a distance $(u_{i+\frac{1}{2},j})k$ to the right. Cell $(i-1, j)$ gains $(-u_{i-\frac{1}{2},j})k \cdot f_{i,j} h$ of black fluid (in area). Cell $(i+1, j)$ gains $(u_{i+\frac{1}{2},j})k \cdot f_{i,j} h$ of black fluid in area. Cell (i, j) losses the amount gained by cells $(i-1, j)$ and $(i+1, j)$. The volume fractions are updated as follows:

$$f_{i,j}^{new} = f_{i,j}^{old} + (u_{i-\frac{1}{2},j}) \cdot k \cdot \frac{f_{i,j} h}{h^2} - (u_{i+\frac{1}{2},j}) \cdot k \cdot \frac{f_{i,j} h}{h^2}$$

$$f_{i+1,j}^{new} = f_{i+1,j}^{old} + (u_{i+\frac{1}{2},j}) \cdot k \cdot \frac{f_{i,j} h}{h^2}$$

$$f_{i-1,j}^{new} = f_{i-1,j}^{old} - (u_{i-\frac{1}{2},j}) \cdot k \cdot \frac{f_{i,j} h}{h^2}.$$

The updates for the horizontal sweep for cell (i, j) is complete but cells $(i-1, j)$ and $(i+1, j)$

must also interact with cells $(i-2, j)$ and $(i+2, j)$ respectively. If $u_{i-\frac{1}{2}, j} > 0$ then the horizontal sweep for cell (i, j) could only be completed by reconstructing the interface in cell $(i-1, j)$ and similarly for $u_{i+\frac{1}{2}, j}$.

Example 2. Suppose we are performing the horizontal sweep in cell (i, j) with the volumes fractions satisfying the conditions of case A(1) above (see figure 5.1), i.e., we have locally in cell (i, j) a vertical interface. The black fluid lies in a rectangle with horizontal and vertical dimensions $f_{i,j} \cdot h$ and h respectively. The rectangle of black fluid lies against the left side of cell (i, j) . When $f_{i-1,j} < 1$, the rectangle of black fluid is advected as a rectangular unit in the same manner as example 1 and the volumes fractions are updated appropriately. When $f_{i,j} = 1$, the fluid is transported by advecting the interface that lies inside cell (i, j) and updating the volumes fractions depending on the new location of the interface. Suppose $f_{i-1,j} = 1$, then the interface advected a distance $u \cdot k$ where u is the velocity of the interface and k is the time step. If the interface remains in cell (i, j) then the only update is

$$f_{i,j}^{new} = f_{i,j}^{old} + \frac{u \cdot k \cdot h}{h^2}$$

If the interface moves into cell $(i+1, j)$, then cell (i, j) is updated to being full and the volume fraction of cell $(i+1, j)$ is updated based on the new location of the interface, i.e.,

$$f_{i,j}^{new} = 1$$

$$f_{i+1,j}^{new} = \frac{u \cdot k \cdot h}{h^2} + f_{i,j}^{old} - 1.$$

If the interface moves into cell $(i-1, j)$, then cell (i, j) is updated to being empty and the volume fraction of cell $(i-1, j)$ is updated based on the new location of the interface, i.e.,

$$f_{i,j}^{new} = 0$$

$$f_{i-1,j}^{new} = \frac{u \cdot k \cdot h}{h^2} + f_{i,j}^{old} + 1.$$

5.2. Approximation of the velocity field. Our problem is to solve

$$\vec{\nabla} \cdot \vec{u} = 0 \quad (5.2.1)$$

on both sides of the interface in the rectangular box. This equation may not be meaningful at the interface since $\vec{u}$ is not necessarily differentiable there. We therefore consider the weak formulation of equation (5.2.1) given by

$$\int_{\partial c_{i,j}} \vec{u} \cdot \vec{n} ds = 0, \quad (5.2.2)$$

for every cell (i, j) , where $\partial c_{i,j}$ is the boundary of the cell (i, j) , $\vec{u}$ is the velocity field, and $\vec{n}$ is the outward normal. (5.2.2) is derived using the divergence theorem:

$$0 = \int_{c_{i,j}} \vec{\nabla} \cdot \vec{u} dx dy = \int_{\partial c_{i,j}} \vec{u} \cdot \vec{n} ds \quad (5.2.3)$$

The main idea is as follows: The interface is reconstructed locally in each cell using the ideas in SLIC. Given these reconstructions we define an auxillary grid with each cell (i, j) in our original grid containing an auxillary grid point $\vec{z}_{i,j}$. For the case when cell (i, j) contains an interface the auxillary grid point $\vec{z}_{i,j}$ is located on the interface at a point that depends on what type of interface is present. For example, with horizontal and vertical interfaces, $\vec{z}_{i,j}$ is located at the center of the interface. Our possible interfaces also include fingers, fingertips, diagonals and corners. For diagonals and fingertips $\vec{z}_{i,j}$ lies at the center of the interface. For corners $\vec{z}_{i,j}$ lies at the corner of the rectangle which lies in the cell. For fingers $\vec{z}_{i,j}$ lies on a line segment in the middle of the finger. The location of $\vec{z}_{i,j}$ for each type of interface is given in appendix 1. For the case when cell (i, j) contains no interface $\vec{z}_{i,j}$ is located at the center of the cell. In each cell we seek the value of a grid function $p_{i,j}$ at the grid point $\vec{z}_{i,j}$ where $p_{i,j}$ represents an approximation to the pressure at $\vec{z}_{i,j}$. An equation for each $p_{i,j}$ is obtained through the discretization of (5.2.2). The line integral in (5.2.2) is divided into four parts, where each part pertains to a side of cell (i, j) . When the interface crosses a cell, the line integral along each side is divided into a part pertaining to the white fluid and a part pertaining to the black fluid. Each of these line integrals is approximated by the length of the line times $\vec{u} \cdot \vec{n}$ approximated at a convenient point on the line of integration. Approximating $\vec{u} \cdot \vec{n}$ along the sides of the cell is equivalent to approximating $\frac{\partial p}{\partial x}$ along the vertical sides and $\frac{\partial p}{\partial y}$ along

the horizontal sides of the cell. Suppose that $\frac{\partial p}{\partial y}$ is to be discretized along the top side of cell (i, j) . It is easiest to use, as an approximation to $\frac{\partial p}{\partial y}$, a difference quotient, equaling the value of the grid function $p_{i, j+1}$ in cell $(i, j+1)$ minus the value of the grid function $p_{i, j}$ in cell (i, j) divided by the distance between the locations of the auxillary grid points i.e., the distance between $\bar{x}_{i, j}$ and $\bar{x}_{i, j+1}$. However when an interface passes through either cell the auxillary grid points in cells (i, j) and $(i, j+1)$ in general do not have the same x -coordinate. Linear interpolation is used to approximate the value of a grid function in cell (i, j) or $(i, j+1)$ at a point which has the same x -coordinate as the auxillary grid point in cell $(i, j+1)$ or (i, j) respectively. This approximation of the grid function is then used in the difference quotient instead of the auxillary grid function in the cell. The choice of the cell in which interpolation is done and of which auxillary grid points to use in the interpolation depends on the locations of the grid points and the interface's structure. $\frac{\partial p}{\partial x}$ is approximated in the same manner.

Equation (5.2.2) can be rewritten as

$$\int_{RS} \vec{u} \cdot \vec{n} ds + \int_{TS} \vec{u} \cdot \vec{n} ds + \int_{LS} \vec{u} \cdot \vec{n} ds + \int_{BS} \vec{u} \cdot \vec{n} ds = 0, \quad (5.2.4)$$

where RS is the line segment on the right side of cell (i, j) , TS is the line segment on the top side of cell (i, j) , LS is the line segment on the left side of cell (i, j) , and BS is the line segment on the bottom side of cell (i, j) . For the discretization of the integral above one needs approximations to the locations of the local interfaces in each cell, an estimation of where the interface crosses the sides of the cells and the coordinates of where the pressure is to be solved for in that cell. This information is given in appendix 1 along with the discretization of equation (5.2.2) for all interfaces.

The equations for the cells along the boundary of the rectangular box are modified slightly due to the boundary conditions. The grid is periodic in the x -direction with period equal to the horizontal dimension of the rectangle. The velocity is prescribed on the horizontal sides of the rectangle. For cells on the bottom row in the rectangle, the first term in (5.2.4) is known. For

cells on the top row in the rectangle, the third term in (5.2.4) is known. Those terms are placed on the right hand side of equation (5.2.4) for each cell on the bottom row or the top row, and the remaining terms in (5.2.4) are discretized.

The discretization of (5.2.4) can be thought of as a weak discrete divergence. Suppose that an interface crosses the right side of cell (i, j) . The line integral on the right side is divided into two parts, one pertaining to the white fluid and one pertaining to the black fluid, and is given by

$$\int_{RS} \vec{u} \cdot \vec{n} ds = \int_{RS(white)} \vec{u}^w \cdot \vec{n} ds + \int_{RS(black)} \vec{u}^b \cdot \vec{n} ds \quad (5.2.5)$$

where we use the superscripts w and b to refer to the velocities in the white and black fluids respectively. We simplify (5.2.5) and get

$$\int_{RS} \vec{u} \cdot \vec{n} ds = \int_{RS(white)} u^w ds + \int_{RS(black)} u^b ds \quad (5.2.6)$$

where $\vec{u}^w = (u^w, v^w)$ and $\vec{u}^b = (u^b, v^b)$. Suppose that the length of the line of integration on which the black fluid lies is α_r . Let us discretize the integrals on the right hand side of (5.2.6) for cell (i, j) ; we have

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx (h - \alpha_r) u^w_{i+\frac{1}{2},j} + \alpha_r u^b_{i+\frac{1}{2},j} \quad (5.2.7)$$

where $u^w_{i+\frac{1}{2},j}$ and $u^b_{i+\frac{1}{2},j}$ are the velocity normals on the right side in the white and black fluids. Suppose that on the top, left and bottom sides, the lengths of the lines of integration for the black fluid are α_t , α_l and α_b , then we have the following approximations derived as above,

$$\begin{aligned} \int_{TS} \vec{u} \cdot \vec{n} ds &\approx (h - \alpha_t) v^w_{i,j+\frac{1}{2}} + \alpha_t v^b_{i,j+\frac{1}{2}} \\ \int_{LS} \vec{u} \cdot \vec{n} ds &\approx -((h - \alpha_l) u^w_{i-\frac{1}{2},j} + \alpha_l u^b_{i-\frac{1}{2},j}) \\ \int_{BS} \vec{u} \cdot \vec{n} ds &\approx -((h - \alpha_b) v^w_{i,j-\frac{1}{2}} + \alpha_b v^b_{i,j-\frac{1}{2}}) \end{aligned} \quad (5.2.8)$$

where $v^w_{i,j+\frac{1}{2}}$ and $v^b_{i,j+\frac{1}{2}}$ denote the normal velocities on the top side, $u^w_{i+\frac{1}{2},j}$ and $u^b_{i+\frac{1}{2},j}$ denote the normal velocities on the left side and $v^w_{i,j-\frac{1}{2}}$ and $v^b_{i,j-\frac{1}{2}}$ denote the normal velocities on the bottom side. Summing the terms above gives us a weak discrete diver-

gence, $D_{i,j}$, given by

$$\begin{aligned}
 D_{i,j} = & (h - \alpha_r) u^w_{i+\frac{1}{2},j} + \alpha_r u^b_{i+\frac{1}{2},j} \\
 & + (h - \alpha_t) v^w_{i,j+\frac{1}{2}} + \alpha_t v^b_{i,j+\frac{1}{2}} \\
 & - ((h - \alpha_l) u^w_{i-\frac{1}{2},j} + \alpha_l u^b_{i-\frac{1}{2},j}) \\
 & - ((h - \alpha_b) v^w_{i,j-\frac{1}{2}} + \alpha_b v^b_{i,j-\frac{1}{2}})
 \end{aligned} \tag{5.2.9}$$

In figure 5.6, these velocities are shown for a horizontal interface.

The following example illustrates the discretization of each term in (5.2.4) for a cell (i, j) with a typical interface passing through it. This example uses the following set of volume fractions: $0 < f_{i-1,j} < 1$, $0 < f_{i,j} < 1$, $0 < f_{i+1,j} < 1$, $f_{i-1,j-1} = 1$, $f_{i,j-1} = 1$, $f_{i+1,j-1} = 1$, $f_{i-1,j+1} = 0$, $f_{i,j+1} = 0$, $f_{i+1,j+1} = 0$. It is assumed that $f_{i-1,j} \leq f_{i,j} \leq f_{i+1,j}$ (see figure 5.7). By appendix 1, cells $(i-1, j)$, (i, j) and $(i+1, j)$ each contain a horizontal interface while the remaining cells given contain no interface. The auxiliary grid points in cells $(i-1, j-1)$, $(i, j-1)$, $(i+1, j-1)$, $(i-1, j+1)$, $(i, j+1)$ and $(i+1, j+1)$ are located in the center of the cell since they contain no interface. The auxiliary grid points in cells $(i-1, j)$, (i, j) and $(i+1, j)$ are located at the centers of the horizontal interfaces. Let us first evaluate the line integral on the right side of cell (i, j) . The interface crosses the right side of cell (i, j) at the average of the two local interfaces, i.e., at $y = (j - .5)h + .5h(f_{i,j} + f_{i+1,j})$. Since an interface crosses the right side of cell (i, j) , the line integral along the right side is divided into two parts, one pertaining to the white fluid and one pertaining to the black fluid which is given by

$$\int_{RS} \vec{u} \cdot \vec{n} ds = \int_{RS(white)} \vec{u} \cdot \vec{n} ds + \int_{RS(black)} \vec{u} \cdot \vec{n} ds. \tag{5.2.10}$$

On the right hand side of (5.2.10), $\vec{u} \cdot \vec{n}$ simplifies into $\kappa_w \frac{\partial p((i+.5)h, y)}{\partial x}$ when $((i+.5)h, y)$

lies in the white fluid and into $\kappa_b \frac{\partial p((i+.5)h, y)}{\partial x}$ when $((i+.5)h, y)$ lies in the black fluid

where $\kappa_b = -\frac{b^2}{12\mu_b}$ and $\kappa_w = -\frac{b^2}{12\mu_w}$. The line integrals on the right side of equation

(5.2.10) are approximated by $\vec{u} \cdot \vec{n}$ at some convenient point on the line times the length of the line. Using this and equations (5.2.10), we have the following approximations,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx .5h (f_{i,j} + f_{i+1,j}) \kappa_b \frac{\partial p((i+.5)h, \xi_1)}{\partial x} + (h - .5h (f_{i,j} + f_{i+1,j})) \kappa_w \frac{\partial p((i+.5)h, \xi_2)}{\partial x} \quad (5.2.11)$$

where ξ_1 and ξ_2 are the y -coordinates of convenient points on the right hand side of cell (i, j) in the black and white fluids respectively. The term $\frac{\partial p((i+.5)h, \xi_1)}{\partial x}$ in equation (5.2.11) is approximated by the difference quotient

$$\frac{\bar{p}_{i+1,j} - \bar{p}_{i,j}}{d(\bar{z}_{i+1,j}, \bar{z}_{i,j})} \quad (5.2.12)$$

where $\bar{z}_{i+1,j}$ and $\bar{z}_{i,j}$ are points located in the black fluid in the cells (i, j) and $(i+1, j)$ respectively with the same y -coordinate ξ_1 , and $d(x, y)$ is the Euclidean distance between the points x and y . $\bar{p}_{i+1,j}$ and $\bar{p}_{i,j}$ are approximations to the grid functions values at $\bar{z}_{i+1,j}$ and $\bar{z}_{i,j}$ respectively. Linear interpolation is used to approximate $\bar{p}_{i,j}$ and $\bar{p}_{i+1,j}$. $\bar{z}_{i,j}$ and $\bar{z}_{i+1,j}$ are chosen in a way to minimize the amount of interpolation done. The simplest thing to do is to use $\bar{z}_{i,j}$ for $\bar{z}_{i,j}$ in (5.2.12), which requires no interpolation in cell (i, j) . In cell $(i+1, j)$, the point $\bar{z}_{i+1,j}$ is chosen as the point on the line between $\bar{z}_{i+1,j}$ and $\bar{z}_{i+1,j-1}$ with the same y -coordinate as $\bar{z}_{i,j}$. We then interpolate linearly between $p_{i+1,j}$ located at $\bar{z}_{i+1,j}$ and $p_{i+1,j-1}$ located at $\bar{z}_{i+1,j-1}$ to approximate the grid functions value $\bar{p}_{i+1,j}$ at $\bar{z}_{i+1,j}$. $\bar{p}_{i+1,j}$ is given by the relation

$$\frac{p_{i+1,j} - \bar{p}_{i+1,j}}{d(\bar{z}_{i+1,j}, \bar{z}_{i+1,j})} = \frac{p_{i+1,j} - p_{i+1,j-1}}{d(\bar{z}_{i+1,j}, \bar{z}_{i+1,j-1})} \quad (5.2.13)$$

Therefore we have, for the first term on the right hand side of (5.2.11), the following approximation,

$$\begin{aligned} & .5h (f_{i,j} + f_{i+1,j}) \kappa_b \frac{\partial p((i+.5)h, \xi_1)}{\partial x} \\ & = .5h (f_{i,j} + f_{i+1,j}) \kappa_b \frac{\bar{p}_{i+1,j} - \bar{p}_{i,j}}{d(\bar{z}_{i+1,j}, \bar{z}_{i,j})} \end{aligned} \quad (5.2.14)$$

where $\bar{p}_{i+1,j}$ is given by (5.2.13).

The approximation of $\frac{\partial p((i+.5)h, \xi_2)}{\partial x}$ in (5.2.11) is similar to the approximation done above.

The term $\frac{\partial p((i+.5)h, \xi_2)}{\partial x}$ in equation is approximated by the difference quotient

$$\frac{\hat{p}_{i+1,j} - \hat{p}_{i,j}}{d(\hat{z}_{i+1,j}, \hat{z}_{i,j})} \quad (5.2.15)$$

where $\hat{z}_{i+1,j}$ and $\hat{z}_{i,j}$ are points located in the white fluid in the cells (i, j) and $(i+1, j)$ respectively with the same y -coordinate ξ_2 . $\hat{p}_{i+1,j}$ and $\hat{p}_{i,j}$ are approximations to the grid functions values at $\hat{z}_{i+1,j}$ and $\hat{z}_{i,j}$ respectively. Linear interpolation is used to approximate $\hat{p}_{i,j}$ and $\hat{p}_{i+1,j}$. $\hat{z}_{i,j}$ and $\hat{z}_{i+1,j}$ are chosen in a way to minimize the amount of interpolation done. The simplest thing to do is to use $\bar{z}_{i+1,j}$ for $\hat{z}_{i+1,j}$ in (5.2.15), which requires no interpolation in cell $(i+1, j)$. In cell (i, j) , the point $\hat{z}_{i,j}$ is chosen as the point on the line between $\bar{z}_{i,j}$ and $\bar{z}_{i,j+1}$ with the same y -coordinate as $\bar{z}_{i+1,j}$. We then interpolate linearly between $p_{i,j}$ located at $\bar{z}_{i,j}$ and $p_{i,j+1}$ located at $\bar{z}_{i,j+1}$ to approximate the grid functions value $\hat{p}_{i,j}$ at $\hat{z}_{i,j}$. $\hat{p}_{i,j}$ is given by the relation

$$\frac{\hat{p}_{i,j} - p_{i,j}}{d(\bar{z}_{i,j}, \bar{z}_{i,j})} = \frac{p_{i,j+1} - p_{i,j}}{d(\bar{z}_{i,j+1}, \bar{z}_{i,j})}. \quad (5.2.16)$$

Therefore we have, for the second term on the right hand side of (5.2.11), the following approximation,

$$\begin{aligned} & .5h(2 - f_{i,j} - f_{i+1,j})\kappa_w \frac{\partial p((i+.5)h, \xi_2)}{\partial x} \\ &= .5h(2 - f_{i,j} - f_{i+1,j})\kappa_w \frac{p_{i+1,j} - \hat{p}_{i,j}}{d(\bar{z}_{i+1,j}, \hat{z}_{i,j})} \end{aligned} \quad (5.2.17)$$

where $\hat{p}_{i,j}$ is given by (5.2.16). After summing the two terms (5.2.14) and (5.2.17), we have the approximation,

$$\begin{aligned} \int_{RS} \vec{u} \cdot \vec{n} ds &\approx .5h(f_{i,j} + f_{i+1,j})\kappa_b \frac{\bar{p}_{i+1,j} - p_{i,j}}{d(\bar{z}_{i+1,j}, \bar{z}_{i,j})} \\ &+ .5h(2 - f_{i,j} - f_{i+1,j})\kappa_w \frac{p_{i+1,j} - \hat{p}_{i,j}}{d(\bar{z}_{i+1,j}, \hat{z}_{i,j})}, \end{aligned} \quad (5.2.18)$$

where $\tilde{p}_{i+1,j}$ and $\hat{p}_{i,j}$ are given by (5.2.13) and (5.2.16) respectively.

There is no interface crossing the top side of cell (i,j) , only white fluid. $\vec{u} \cdot \vec{n}$ simplifies on the top side to $\kappa_w \frac{\partial p((x, (j+.5)h))}{\partial y}$. The line integral on the top side is approximated by $\vec{u} \cdot \vec{n}$ at some convenient point on the top side times the length of the side. We then have the approximation,

$$\int_{TS} \vec{u} \cdot \vec{n} ds \approx h \kappa_w \frac{\partial p(\xi, (j+.5)h)}{\partial y} \quad (5.2.19)$$

where ξ is the x -coordinate of some convenient point on the top side of cell (i,j) . The point is convenient in the sense that it minimizes the amount of interpolation required. Since $\bar{z}_{i,j}$ and $\bar{z}_{i,j+1}$ have the same x -coordinate, no interpolation is done. The term $\frac{\partial p(\xi, (j+.5)h)}{\partial y}$ in equation (5.2.11) is approximated by the difference quotient

$$\frac{p_{i,j+1} - p_{i,j}}{d(\bar{z}_{i,j+1}, \bar{z}_{i,j})} \quad (5.2.20)$$

and for our line integral on the top side we have the approximation,

$$\int_{TS} \vec{u} \cdot \vec{n} ds \approx h \kappa_w \frac{p_{i,j+1} - p_{i,j}}{d(\bar{z}_{i,j+1}, \bar{z}_{i,j})} \quad (5.2.21)$$

The line integral on the left side of cell (i,j) is discretized in a similar manner as the line integral on the right side. The line integral on the bottom side is done in a similar way to the line integral over the top side. These discretizations are substituted into equation (5.2.4) to give an equation for the grid function values around cell (i,j) .

The case that occurs most frequently is the one where (i,j) , $(i-1,j)$, $(i+1,j)$, $(i,j-1)$ and $(i,j+1)$ contain no interfaces and contain the same fluid. Without loss of generality, we assume the fluid in these cells to be black. The pressures are all cell centered and (5.2.4) can be approximated as follows,

$$h \kappa_b \frac{\partial p((i+.5)h, jh)}{\partial x} + h \kappa_b \frac{\partial p(ih, (j+.5)h)}{\partial y}$$

$$-h \kappa_b \frac{\partial p((i-.5)h, jh)}{\partial x} - h \kappa_b \frac{\partial p(ih, (j-.5)h)}{\partial y} = 0 \quad (5.2.22)$$

Using the centered difference approximation, (5.2.22) can be approximated by

$$\begin{aligned} h \kappa_b \frac{p_{i+1,j} - p_{i,j}}{h} + h \kappa_b \frac{p_{i,j+1} - p_{i,j}}{h} \\ - h \kappa_b \frac{p_{i,j} - p_{i-1,j}}{h} - h \kappa_b \frac{p_{i,j} - p_{i,j-1}}{h} = 0 \end{aligned} \quad (5.2.23)$$

which simplifies into the standard five-point difference formula for the Laplacian,

$$p_{i+1,j} + p_{i,j+1} + p_{i-1,j} + p_{i,j-1} - 4p_{i,j} = 0. \quad (5.2.24)$$

5.3. Modified Conjugate Gradient Method. To solve our system of equations for the unknown pressures obtained in section 5.2, two versions of the conjugate gradient method are used. The first algorithm, presented by Concus et al (1976), applies a preconditioned conjugate gradient method to the system

$$A\vec{x} = b. \quad (5.3.1)$$

Our second algorithm finds a least squares solution to (5.3.1) and is given by Hestenes (1980). Our matrix A does not satisfy all the conditions in the analysis of convergence of our first conjugate gradient algorithm and the norm of our residual

$$|A\vec{x}_k - b| \quad (5.3.2)$$

sometimes does not become small enough where $\vec{x}_k$ is the k^{th} approximate to $\vec{x}$ after iterating the conjugate gradient method k times. This is resolved by using algorithm 2 to find a least squares solution. Algorithm 2 requires many more multiplications than algorithm 1. Algorithm 1 works so often that it is much cheaper to use algorithm 1 with algorithm 2 as a backup than using algorithm 2 alone.

For algorithm 1, let us consider the system of equations

$$A\vec{x} = \vec{b} \quad (5.3.3)$$

where A is a $n \times n$ positive definite and symmetric matrix and $\vec{b}$ is a given vector. It is assumed that A can be written as

$$M - N \quad (5.3.4)$$

where M is positive-definite and symmetric and N is symmetric. It is also assumed that the system

$$M\vec{z} = \vec{c} \quad (5.3.5)$$

can be easily solved. The form of the conjugate gradient method derived by Concus et al (1976) is given by

$$\vec{x}_{k+1} = \vec{x}_{k-1} + \omega_{k+1}(\alpha_k \vec{z}_k + \vec{x}_k - \vec{x}_{k-1}) \quad (5.3.6)$$

where

$$M\vec{z}_k = \vec{b}^* - A\vec{x}_k. \quad (5.3.7)$$

The symmetric and positive definite matrix M is called the preconditioner. The convergence of the algorithm depends on

$$\kappa = \frac{\lambda_{\max}}{\lambda_{\min}} \quad (5.3.8)$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and the minimum eigenvalues of the matrix $I - M^{-1}N$ respectively. The closer κ is to unity, the better the convergence of (5.3.6). The eigenvalue problem

$$(I - M^{-1}N)x = \lambda x \quad (5.3.9)$$

can be rewritten as

$$A\vec{x} = M\lambda\vec{x}. \quad (5.3.10)$$

The closer the matrix M is to A in (5.3.10), the closer that κ is to 1. For our purposes, M is chosen to be the discrete 5-point Laplacian matrix which is easily invertible. The solver for the matrix M was obtained from the FISHPAK library (1980). Due to interpolation, our matrix A is not symmetric. Neumann boundary conditions are given on the top and bottom of our rectangle and periodic boundary conditions are given on the sides, making our matrix A singular. Our preconditioner M is the 5-point Laplacian matrix with boundary conditions the same as A , implying that it is also singular. The derivations of ω_{k+1} and α_k can be found in Concus et al (1976).

Algorithm 1 is given in a convenient form for computational purposes below.

Let $\vec{x}_0$ and $\vec{p}_{-1}$ be given. For $k = 0, 1, \dots$

1) Solve

$$M\vec{z}_k = \vec{b} - A\vec{x}_k. \quad (5.3.11)$$

2) Compute

$$\vec{p}_k = \vec{z}_k + b_k \vec{p}_{k-1}, \quad (5.3.12)$$

where

$$b_k = \frac{\vec{z}_k^t M \vec{z}_k}{\vec{z}_{k-1}^t M \vec{z}_{k-1}}, \quad \text{for } k \geq 1 \quad (5.3.13)$$

and

$$b_0 = 0, \quad \text{for } k=0. \quad (5.3.15)$$

3) Compute

$$\vec{x}_{k+1} = \vec{x}_k + a_k \vec{p}_k, \quad (5.3.16)$$

where

$$a_k = \frac{\vec{z}_k^t M \vec{z}_k}{\vec{p}_k^t A \vec{p}_k}. \quad (5.3.17)$$

In the computation of a_k and b_k , one need not compute $M \vec{z}_k$ since it can be saved from step 1.

Our second algorithm finds a least squares solution to

$$A\vec{x} = \vec{b}. \quad (5.3.18)$$

Our matrix meets the criteria for the convergence of this algorithm. This algorithm, obtained from Hestenes (1980), is given below.

Let $\vec{x}_1$ be given. $\vec{r}_1 = \vec{b} - A\vec{x}_1$, $\vec{g}_1 = A^t M^{-2} \vec{r}_1$, $\vec{p}_1 = \vec{g}_1$. For $k = 1, 2, \dots$

1) Compute

$$c_k = \bar{g}_k^t \bar{g}_k, \quad \bar{s}_k = A \bar{p}_k, \quad d_k = \bar{s}_k M^{-2} \bar{s}_k, \quad a_k = \frac{c_k}{d_k}. \quad (5.3.19)$$

2) Compute

$$\bar{x}_{k+1} = \bar{x}_k + a_k \bar{p}_k, \quad \bar{r}_{k+1} = \bar{r}_k - a_k \bar{s}_k, \quad \bar{g}_{k+1} = A^t M^{-2} \bar{r}_{k+1}. \quad (5.3.20)$$

3) Compute

$$\bar{p}_{k+1} = \bar{g}_{k+1} + b_k \bar{p}_k, \quad b_k = \frac{\bar{g}_{k+1}^t \bar{g}_{k+1}}{c_k}. \quad (5.3.21)$$

This algorithm terminates when the residual $\bar{r}_k$ is small enough.

5.4. Surface Tension and Curvature. Surface tension is added to our model by imposing a jump in the pressure on the interface. Suppose that (x_0, y_0) is a point on the interface. This jump is given by

$$p_b(x_0, y_0) - p_t(x_0, y_0) = \frac{\tau}{R_{x_0, y_0}} \quad (5.4.1)$$

where $p_b(x_0, y_0)$ and $p_t(x_0, y_0)$ are the pressures associated with the fluid on the top and bottom respectively, τ is the surface tension (given) and R_{x_0, y_0} is the radius of curvature at the point (x_0, y_0) . Our method is designed so that if an interface passes through a cell then we solve for the pressure at some prescribed point on this interface. The equation involving this pressure is obtained by taking the line integral of $\vec{u} \cdot \vec{n}$ around the boundary of the cell discretely, where $\vec{u} = (u, v)$ is the velocity field and $\vec{n}$ is the outward normal. For each cell containing an interface the unknown pressure in that cell is involved in computing the velocities of the top fluid and the bottom fluid. Whenever we are computing the velocity of the bottom fluid, instead of using $p_{i,j}$ as the pressure on the interface for the bottom fluid we use $p_{i,j} + \frac{\tau}{R_{i,j}}$, where $R_{i,j}$ is the estimated radius curvature on the interface in cell (i, j) . In computing the velocity for the fluid on the top we use $p_{i,j}$. This appears to treat the black and white fluids differently. We now show that this is false and the fluids are treated symmetrically. We use two grid functions $\bar{p}$ and $\bar{q}$. $\bar{p}$ represents the grid function we solve for when treating the jump in the way above. $\bar{q}$ represents the grid function we solve for when considering $q_{i,j}$

on the interface as the pressure for the black fluid and $q_{i,j} - \frac{\tau}{R_{i,j}}$ as the pressure on the interface for the white fluid. We show that the problem resulting from each of these formulations of the jump can be expressed as a linear equation where the side of the equation containing the unknown pressures are decomposed into two linear expressions, one pertaining to the black pressure and one pertaining to the white pressure. We then compare the solutions of the two equations and show that the pressure used for the black fluid in each of the formulations of the jump are equal and similarly for the white fluid. We proceed as follows.

For each cell (i, j) , we discretize

$$\int_{\partial c_{i,j}} \vec{u} \cdot \vec{n} ds = 0 \quad (5.4.2)$$

to obtain an equation relating the pressure in cell (i, j) to the pressure in the neighboring cells. The term on the left hand side of (5.4.2),

$$\int_{\partial c_{i,j}} \vec{u} \cdot \vec{n} ds \quad (5.4.3)$$

is decomposed into a part pertaining to the white fluid and a part pertaining to the black fluid,

$$\int_{\partial c_{i,j}} \vec{u} \cdot \vec{n} ds = \int_{\partial c_{i,j}^{white}} \vec{u} \cdot \vec{n} ds + \int_{\partial c_{i,j}^{black}} \vec{u} \cdot \vec{n} ds. \quad (5.4.4)$$

For our first formulation of the jump, when we discretize

$$\int_{\partial c_{i,j}^{white}} \vec{u} \cdot \vec{n} ds \quad (5.4.5)$$

in each cell (i, j) , we obtain a linear expression $A_w \vec{p}$; similarly, when we discretize

$$\int_{\partial c_{i,j}^{black}} \vec{u} \cdot \vec{n} ds \quad (5.4.6)$$

in each cell (i, j) we obtain $A_b (\vec{p} + \vec{\tau})$, where $\vec{\tau}$ consists of components $\tau_{i,j}$ given by

$$\tau_{i,j} = \begin{cases} \tau/R_{i,j} & \text{if } (i, j) \text{ contains an interface} \\ 0 & \text{otherwise} \end{cases} \quad (5.4.7)$$

Equation (5.4.2) is then written in the form

$$A_w \vec{p} + A_b (\vec{p} + \vec{\tau}) = \vec{b} \quad (5.4.8)$$

where $\vec{b}$ comes from boundary terms. $\vec{p}$ and $\vec{p} + \vec{\tau}$ give the pressure for the white and black

fluids respectively.

We can similarly consider the jump in the direction of the white fluid. In the same way as above we have

$$A_w(\bar{q} - \bar{\tau}) + A_b \bar{q} = \bar{b}. \quad (5.4.9)$$

$\bar{q} + \bar{\tau}$ and $\bar{q}$ give the pressures for the white and black fluids respectively. We will show that the pressures, $\bar{p}$, used for the white fluid in (5.4.8), is equal to the pressures, $\bar{q} - \bar{\tau}$, used for the white fluid in (5.4.9). A similar relationship is shown for the black fluids. (5.4.8) can be rewritten as

$$A_w((\bar{p} + \bar{\tau}) - \bar{\tau}) + A_b(\bar{p} + \bar{\tau}) = \bar{b} \quad (5.4.10)$$

If we compare (5.4.9) and (5.4.10), we find

$$\bar{q} = \bar{p} + \bar{\tau} \quad (5.4.11)$$

Therefore the pressures used in the white fluid for the first formulation, $\bar{p}$, are equal to the pressures used in the white fluid for the second formulation, $\bar{q} - \bar{\tau}$. For the black fluid $\bar{p} + \bar{\tau}$ in the first formulation is obviously equal to $\bar{q}$ in the second formulation of the jump.

The curvature is computed using a method due to Chorin (1985). This method is designed for interfaces represented by volume fractions. Consider a cell (i, j) with a volume fraction $0 < f_{i,j} < 1$ and suppose we desire to estimate its curvature. The method considers the volume fractions of cell (i, j) and some of its neighbors. The main idea is to find a circle whose intersection with these cells cuts out the same volume fractions as those of cell (i, j) and its neighbors. This circle is identified with the osculating circle and its radius is taken to be the radius of curvature of the interface in cell (i, j) . The curvature is the inverse of the radius of curvature.

CHAPTER 6

NUMERICAL RESULTS FOR THE LINEARIZED THEORY

In this chapter, numerical results are compared with the predictions of the linearized theory. In the linearized theory presented in chapter 4, it is shown that an interface represented by a wave with wavenumber n and small amplitude ϵ , in a Hele-Shaw cell, evolves after a short time Δt into a wave with the same wavenumber but with an amplitude given by $\epsilon e^{\sigma \Delta t}$. A formula for σ , the rate of growth, is explicitly given. In section 6.1, a method for approximating the σ associated with the numerical method is presented. A cosine wave of small amplitude ϵ , representing an interface in the middle of the numerical grid, is converted to its corresponding volume fraction representation. The volume fractions are interpreted as a piecewise constant function on the grid for which we can compute the Fourier coefficients. The Fourier coefficient of the term with the same wavenumber as our original interface is computed. This represents an approximation to the amplitude ϵ of the original interface. The volume fractions representing the interface are then updated using the numerical method. The new volume fractions correspond to a piecewise constant function for which we can also compute the Fourier coefficients. Again, the Fourier coefficient of the term with the same wavenumber as our original interface is computed. This represents an approximation to the amplitude $\epsilon e^{\sigma \Delta t}$ of the updated interface. By comparing the Fourier coefficients associated with the original set of volume fractions and the new set of volume fractions, the numerical σ is estimated. In sections 6.2 and 6.3, the numerical results are presented and compared with the results predicted by the linearized theory for zero and non-zero surface tension respectively. We present comparisons of the numerical σ versus the σ predicted by the linearized theory. Our numerical σ is shown to be in excellent agreement with the σ predicted by the linearized theory.

6.1. Numerical Rate of Growth. In chapter 4, it is shown that when the Hele-Shaw

equations are satisfied on both sides of an interface of the form

$$z = \epsilon e^{in_x x + in_y y} \quad (6.1.1)$$

it then evolves over a short time Δt into an interface of the form

$$z = \epsilon e^{\sigma \Delta t} e^{in_x x + in_y y}, \quad (6.1.2)$$

assuming that $\epsilon e^{\sigma \Delta t}$ is small enough. The rate of growth σ is given by

$$\sigma = \frac{(\mu_w - \mu_b) W n - \tau k n^3}{\mu_w + \mu_b}. \quad (6.1.3)$$

n_x, n_y are the wave numbers in the $z-x, z-y$ planes respectively, ϵ is a small parameter, μ_w and μ_b are the viscosities of the white and black respectively, τ is the surface tension coefficient, $(0,0,W)$ is the velocity far ahead and far behind the interface, $k = \frac{b^2}{12}$ and $n = (n_x^2 + n_y^2)^{1/2}$. It is not totally resolved what the value of n_y should be (see McLean & Saffman (1981); Park et al (1984)). For convenience, we choose n_y to be zero. Under the assumptions of (6.1.1) and (6.1.2) above, an interface of the form

$$z = \epsilon \cos(nx) \quad (6.1.4)$$

then it evolves into an interface of the form

$$z = \epsilon e^{\sigma \Delta t} \cos(nx) \quad (6.1.5)$$

where σ is again given by (6.1.3).

Our interface is evolved in a rectangular box in the $x-z$ plane. The four corners of the rectangular box lie at $(0,0)$, $(1,0)$, $(0,c)$, and $(1,c)$ where c is positive real number. Black fluid flows into the box from behind the interface at a constant velocity $(0,W)$ across the line segment between the points $(0,0)$ and $(1,0)$. White fluid flows out of the box in front of the interface at a constant velocity $(0,W)$ across the line segment between the points $(0,c)$ and $(1,c)$. All functions defined on the box are periodic in the horizontal direction with period equal to 1 (see figure 1.1). The interface (6.1.4) lying in the middle of our rectangle is given by

$$z = \epsilon \cos(nx) + \frac{c}{2}. \quad (6.1.6)$$

For convenience we choose n to be of the form $n_0 2\pi$ for some integer n_0 . A numerical grid is

superposed over this rectangle with cells of dimension $h \times h$. There are m cells across horizontally and l cells vertically such that $m \cdot h = 1$ and $\frac{l}{m} = c$ where m and l are integers. l is assumed to be an odd for convenience. The interface given by (6.1.1) lies in the center of the grid in the j_0 row of cells where $j_0 = \frac{l}{2} + 1$. This is accomplished by choosing ϵ small enough. To transform the interface represented by a curve into one represented by volume fractions, the following is done. A cell (i, j) has volume fraction $f_{i,j} = 1$ when it lies behind the interface (i.e., when $j < j_0$), volume fraction $f_{i,j} = 0$ when it lies in front of the interface (i.e., when $j > j_0$), and volume fraction $0 < f_{i,j} < 1$ when the interface passes through the cell (i.e., when $j = j_0$). For cells (i, j_0) where $i = 1, \dots, m$, the amount of black fluid in area lying in each cell is

$$a_{i,j_0} = \int_{(i-1)h}^{ih} (\epsilon \cos(\pi x) + \frac{\epsilon}{2} - (j_0 - 1)h) dx \quad (6.1.7)$$

The fraction of black fluid in each cell, f_{ij_0} , $i = 1, \dots, m$ is given by

$$f_{ij_0} = \frac{a_{ij_0}}{h^2} \quad (6.1.8)$$

Each cell in the j_0 row is then assumed to contain a horizontal interface with height $f_{ij_0}h$. The interface is viewed globally as a piecewise constant function given by

$$f(x) = f_{ij_0}h + (j_0 - 1)h \quad \text{for } (i-1)h \leq x \leq ih. \quad (6.1.9)$$

Our numerical method is applied to the interface represented by the volume fractions above. This gives a new interface represented by a new set of volume fractions. The time step Δt is small enough so that the new interface remains inside the j_0 row of cells. The new set of volume fractions, $\bar{f}_{i,j}$, correspond also to a piecewise constant function, $\bar{f}(x)$, given by

$$\bar{f}(x) = \bar{f}_{ij_0}h + (j_0 - 1)h \quad \text{for } (i-1)h \leq x \leq ih. \quad (6.1.10)$$

The Fourier series representation of a function g is given by the formula

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos(2m\pi x) + b_m \sin(2m\pi x)) \quad (6.1.11)$$

where

$$a_m = 2 \int_0^1 g(x) \cos(2m\pi x) dx$$

$$b_m = 2 \int_0^1 g(x) \sin(2m\pi x) dx. \quad (6.1.12)$$

We approximate the rate of growth of the cosine wave with wavenumber n by the rate of growth of the corresponding Fourier component of $f(x)$ with wavenumber n . The corresponding Fourier component of $f(x)$ with wavelength n is given by

$$a_{n_0} \cos(n_0 2\pi x) \quad (6.1.13)$$

where $n_0 = \frac{n}{2\pi}$. The Fourier component of $\bar{f}(x)$ with wavelength n is given by

$$\bar{a}_{n_0} \cos(n_0 2\pi x) \quad (6.1.14)$$

which represents the result of applying the numerical method to (6.1.13). The numerical rate of growth $\bar{\sigma}$ satisfies

$$\bar{a}_{n_0} = a_{n_0} e^{\bar{\sigma} \Delta t} \quad (6.1.15)$$

where

$$\bar{\sigma} = \lim_{\Delta t \rightarrow 0} \frac{a_{n_0} e^{\bar{\sigma}(t+\Delta t)} \cos(nx) - a_{n_0} e^{\bar{\sigma}t} \cos(nx)}{(\Delta t) a_{n_0} e^{\bar{\sigma}t} \cos(nx)}. \quad (6.1.16)$$

or

$$\bar{\sigma} \approx \frac{a_{n_0} e^{\bar{\sigma}(\Delta t)} - a_{n_0}}{(\Delta t) a_{n_0}}, \quad (6.1.17)$$

for Δt small. From (6.1.15) a_{n_0} equals $\bar{a}_{n_0}$. Our numerical rate of growth for a cosine wave with wavenumber n is then approximated by

$$\frac{\bar{a}_{n_0} - a_{n_0}}{(\Delta t) a_{n_0}}. \quad (6.1.18)$$

6.2. Numerical Results for Zero Surface Tension. In this section, the surface tension coefficient is taken to be zero. The interface on our numerical grid is represented by a wave given by the equation

$$z = \epsilon \cos(nx) + \frac{c}{2}. \quad (6.2.1)$$

The wave is evolved for a short time $\Delta t = .00001$ on our grid and the numerical rate of growth is approximated by (6.1.20). Results are shown for wavenumbers ($n = 2\pi, 4\pi, \dots, 18\pi$). The amplitude, ϵ , of the wave given in (6.2.1) is .00001. Results are given for grid sizes 20x20, 40x40, 60x60, and 80x80. These results are presented in table 6.2.1 for the case when $\mu_b = 1$ and $\mu_w = 4$. They are presented in table 6.2.2 for the case when $\mu_b = 1$ and $\mu_w = 16$. In the two cases above, the linearized theory predicts growth in the amplitude of the wave. Table 6.2.3 presents results for the case when $\mu_b = 4$ and $\mu_w = 1$ and table 6.2.4 presents results for the case when $\mu_b = 16$ and $\mu_b = 1$. In these two cases the linearized theory predicts decay in the amplitude of the wave. In all four tables mentioned above there is excellent agreement between the numerical results and the linearized theory. In figure 6.2.1, we show a plot of σ , the rate of growth, versus $\frac{\mu_w}{\mu_b}$ where $\mu_b = 1$ and $\mu_w \geq 1$. Our wavenumber n is 2π and the computations are done on a 40x40 grid. In figure 6.2.2, we show a plot of σ versus $\frac{\mu_b}{\mu_w}$ where $\mu_w = 1$ and $\mu_b \geq 1$. Our wavenumber is 2π and the computations are done on a 40x40 grid. In figures 6.2.1 and 6.2.2, the σ from the linearized theory is represented by the solid curve and the numerical approximations for σ is represented by the circles. Again, there is excellent agreement between the numerical results and the linearized theory.

In solving our system of equations

$$A\vec{x} = \vec{b}, \quad (6.2.2)$$

we require that

$$\|\vec{b} - A\vec{x}\|_{\infty} < .0001 \quad (6.2.3)$$

where $\|\cdot\|_{\infty}$ denotes the maximum norm and $\vec{x}$ is the vector of grid function values for the pressure. Only the first version of the modified conjugate gradient algorithm is ever needed. About 13 iterations are required to satisfy (6.2.3) when $\mu_w = 4$ and $\mu_b = 1$. The number of iterations increase as the viscosity ratio increases.

6.3. Numerical Results For Nonzero Surface Tension. In this section, we include the effects of surface tension. Again, the interface on our grid is represented by the wave given by the equation (6.2.1). The wave is evolved for a short time $\Delta t = .0001$ and the numerical rate of growth is approximated. Table 6.3.1 presents a comparison of the numerical σ with that predicted by the linearized theory for a surface tension coefficient of 10. The viscosity in the white fluid is 4 and the viscosity in the black fluid is 1. Results are given for grids of size 20x20, 40x40, 60x60 and 80x80. The amplitude of the wave is .005. Using the same parameters, results are given in table 6.3.2 for a surface tension coefficient of 3. In figure 6.3.1, we show a plot of σ versus $\frac{\mu_w}{\mu_b}$ where $\mu_b = 1$ and $\mu_w \geq 1$. The surface tension coefficient is 3. Our wavenumber, n , is 2π and the computations are done on a 40x40 grid. The amplitude ϵ is taken to be .005. In figure 6.3.2, we show a plot of σ versus surface tension for a 2π -wave. $\mu_w = 1.3$ and $\mu_b = 1.0$. The time step Δt is .0001, the amplitude ϵ is taken to be .005 and computations are done on a 40x40 grid.

In solving the system

$$A\vec{x} = \vec{b} \quad (6.3.1)$$

again we require that

$$\|\vec{b} - A\vec{x}\|_{\infty} < .0001 \quad (6.3.2)$$

where $\|\cdot\|_{\infty}$ denotes the maximum norm. Only the first part of the modified conjugate gradient method is needed and on the average about 23 iterations are required to satisfy (6.3.2).

In this section, a larger amplitude is chosen than in section 6.2 due to the curvature algorithm. The curvature algorithm attempts to find the radius r of the osculating circle. The algorithm only considers circles with radii, in absolute value, less than some prescribed radius $r_0 > 0$. For cells with radius of curvature, in absolute value, greater than r_0 , the value of zero is given for the curvature. Our r_0 is chosen to be 1000. Our numerical method cannot expect to do well for waves with curvature less than $\frac{1}{1000}$. By choosing a larger amplitude, the region

where the curvature is less than $\frac{1}{1000}$ becomes smaller.

CHAPTER 7

NUMERICAL RESULTS FOR FINITE AMPLITUDE PERTURBATIONS

In this chapter, we apply the numerical method to interfaces of finite amplitude. Our interface is initially represented by a horizontal line with a small extrusion in the middle third. This interface is converted to its corresponding volume fraction representation. The numerical method is applied to the interface and an updated set of volume fractions is obtained. The interface is evolved by successively applying the numerical method to the updated set of volume fractions. Results are presented for several values of the surface tension coefficient.

Our interface is evolved in a rectangular box in the x - z plane. The four corners of the rectangular box lie at $(0,0)$, $(1,0)$, $(0,c)$, and $(1,c)$ where c is positive real number. Black fluid flows into the box from behind the interface at a constant velocity $(0,W)$ across the line segment between the points $(0,0)$ and $(1,0)$. White fluid flows out of the box in front of the interface at a constant velocity $(0,W)$ across the line segment between the points $(0,c)$ and $(1,c)$. All functions defined on the box are periodic in the horizontal direction with period 1.

A numerical grid is superposed over this rectangle with cells of dimension $h \times h$. There are m cells across horizontally and l cells vertically such that $m \cdot h = 1$ and $\frac{l}{m} = c$ where m and l are integers. Our interface is initially located in the j_0 cell where $j_0 = \frac{l}{2} + 1$ and is given by the formula

$$z = f(x) = \begin{cases} (j_0 - 1)h + (.001)h & 0 \leq x < 1/3 \\ (j_0 - 1)h + (.002)h & 1/3 \leq x < 2/3 \\ (j_0 - 1)h + (.001)h & 2/3 \leq x \leq 1 \end{cases} \quad (7.1)$$

(7.1) represents a horizontal line with an extrusion in the middle third. To transform the inter-

face represented by (7.1) into its corresponding volume fraction representation, the following is done. A cell (i, j) has volume fraction $f_{i,j}=1$ when it lies behind the interface (i.e., when $j < j_0$), volume fraction $f_{i,j}=0$ when it lies in front of the interface (i.e., when $j > j_0$), and volume fraction $0 < f_{i,j} < 1$ when the interface passes through the cell (i.e., when $j = j_0$). For cells (i, j_0) where $i = 1, \dots, m$, the amount of black fluid in area lying in each cell is

$$a_{i,j_0} = \int_{(i-1)h}^{ih} (f(x) - (j_0 - 1)h) dx. \quad (7.2)$$

The fraction of black fluid in each cell, f_{ij_0} , $i = 1, \dots, m$ is given by

$$f_{ij_0} = \frac{a_{ij_0}}{h^2}. \quad (7.3)$$

The interface given by the volume fractions above is evolved in figure 7.1 for the case of a surface tension coefficient of 3 with $\mu_w = 4$ and $\mu_b = 1$. The computations are done on a 30×60 grid where $m = 30$ and $l = 60$. The interface is evolved over a period of time $t = 2.75$ seconds (169 time steps). In figure 7.2, we evolve the interface with the same viscosities and grid size as above but with a surface tension of 7. The interface is evolved over a period of time $t = 20.06$ seconds (820 time steps). The figures in 7.1 and 7.2 are obtained by shading in each cell (i, j) where $f_{i,j} = 1$. To reduce the effects of the horizontal boundaries, after each time step the interface is translated back to the middle of the grid.

Our numerical method for advecting the flow satisfies a Courant-Friedrichs-Lewy condition that

$$\frac{\Delta t}{h} \leq \frac{1}{|U|} \quad (7.4)$$

where $|U|$ is the maximum speed of the flow and Δt is the time step. We choose

$$\Delta t = \frac{h}{2|U|} \quad (7.5)$$

as our time step.

In solving our system of equations

$$A\vec{x} = \vec{b}, \quad (7.6)$$

we require that

$$\|\vec{b} - A\vec{x}\|_{\infty} < .0001 \quad (7.7)$$

where $\|\cdot\|_{\infty}$ denotes the maximum norm and $\vec{x}$ is the vector of grid function values for the pressure. In the first conjugate gradient method, (7.7) sometimes does not become small enough. After 100 iterations of the first algorithm, we use the second algorithm.

In discretizing the weak formulation of the continuity equation to find the velocity field, a local reconstruction of the interface in each cell is required. This requires examining the volume fractions in the cell and its neighbors and determining the interface for that situation. Our numerical experiments seem to indicate that as the surface tension becomes smaller, the interface becomes less smooth and the chances of encountering a situation not covered increases. This can be resolved by using a finer grid. In the evolution of the interface above for a surface tension of 7, all cases encountered are covered. In the evolution of the interface above with a surface tension of 3, there is one situation where the interface in a cell is not able to be reconstructed from our possible cases. We handle this in the following way. When this happens we look for cells (i,j) that are partially full with none of its neighbors full, i.e., the eight cells sharing a side or a corner with cell (i,j) . We assign the amount of black fluid in cell (i,j) to the neighboring cell whose volume fraction is the largest. When the amount added to the neighboring cell makes neighboring cell's volume fraction greater than one, then the volume fraction of the neighbor is forced to be one and cell (i,j) gets the residual.

Our results are similar to experimental results observed in the laboratory (see Saffman & Taylor (1958); Chuoke et al (1958)) and to other numerical results (see Meng & Thomson (1978); Tryggvason & Aref (1983, 1985); Degregoria & Schwartz (1985, 1986)). The fingers which we observe are rounded at the ends with thin necks. Surface tension tends to fatten the fingers. At low surface tensions, the interface is also less smooth. We also noticed finger splitting at low surface tensions which is observed by Tryggvason & Aref and Degregoria & Schwartz. This will be presented in our future work.

CHAPTER 8

DISCUSSION AND CONCLUSIONS

We have presented a numerical method for evolving an interface between two immiscible fluids in a Hele-Shaw cell.

We represent our interface as a cosine wave of small amplitude for various wavenumbers. Our numerical method is applied to these waves for a wide range of viscosities and surface tensions. Excellent agreement is found between our numerical rates of growth and the rates of growth as predicted by the linearized theory. We are not aware of such agreement with the linearized theory by any other method. The linearized theory acts as a test problem for the numerical method. By solving this problem with some accuracy, one might expect acceptable results for finite amplitude waves.

For long runs, our numerical method is applied iteratively. The initial interface that we apply our method to is a horizontal interface with an extrusion in the middle third. Viscosities and surface tension are chosen such that there are unstable modes predicted by the linearized theory. The interfaces grow into long fingers similar to those observed in physical and numerical experiments. The finger for the higher surface tension is wider than the finger for the lower surface tension as expected.

In discretizing the weak formulation of the continuity equation to find the velocity field, a local reconstruction of the interface in each cell is required. This is done by examining the volume fractions in the cell and its neighbors and determining the interface for that configuration. Our numerical experiments seem to indicate that as the surface tension decreases, the interface becomes less smooth and the chances of encountering a configuration not covered increases. This can be resolved by using finer grid.

Corners, fingertips, horizontal and vertical interfaces are our most commonly used interfaces in discretizing the weak formulation of the continuity equation. They probably provide more accuracy in comparison with the other interfaces. The need for some of the other interfaces could probably be eliminated by using a fine enough grid.

The linearized theory only tested our horizontal interfaces. We expect similar results with vertical interfaces. The measuring of the growth rate is much more difficult when corners are involved.

In this paper we are concerned mostly about obtaining an accurate model. We implemented surface tension by choosing essentially a *ad hoc* pressure jump condition. There is no consensus as to what the correct jump condition should be (see McLean & Saffman (1981); Park et al (1984); Saffman (1986)).

Since we have a working model, there are many possibilities for future work. We might solve the Hele-Shaw equations with the gravity term included. In some of our long runs with small surface tension we noticed finger splitting (not shown here). This is also observed by Trygvason & Aref (1983, 1985) and Degregoria & Schwartz (1985, 1986). Our model can be used to investigate steady state finger shapes proposed by McLean & Saffman (1981). We might also investigate convergence of our algorithm. Another interesting question is whether the interface might be a fractal in the absence of surface tension.

APPENDIX 1

In this appendix, we show how the weak formulation of the continuity equation, given by (4.2.4), is discretized in cell (i, j) for each configuration of neighboring volume fractions. In section A, we give the location of the local interface in cell (i, j) and the coordinates of the point where the pressure is to be evaluated in the cell for each set of volume fractions. In section B, we give the point where the interface crosses the sides of cell (i, j) and in section C, we give the discretization.

A. Location of the grid function for the pressure. The pressure in our fluids will be solved for at certain locations determined by the orientation of the local interface. We reconstruct a local interface for each cell by volume fractions and the pressure is solved for at an appropriate position on the interface. For the case of no interface in a cell, we solve for the pressure in the center of the cell. For our possible interfaces we include those which are horizontal, vertical, diagonals, fingers, fingertips and corners. For horizontal, vertical, diagonal and fingertip interfaces we solve for the pressure at the middle of the interface. For corners, the pressure is solved for at the corner of the rectangle lying in the cell. For fingers, the pressure is solved for on a line segment in the middle of the finger and is assumed constant on that segment.

The following is a list of our considered cases of volume fractions, the local interfaces they determine and the location of the point on our grid where we seek the value of the pressure.

I. (no interface). $f_{i,j}=1$ or $f_{i,j}=0$. The pressure $p_{i,j}$ is solved for in the center of the cell, at the point (ih, jh) (figure A 1).

II. (horizontal interface). This occurs whenever $0 < f_{i,j} < 1$, $f_{i,j-1}=1$, $f_{i,j+1}=0$ with neither $f_{i-1,j}=1, f_{i+1,j}=0$ or $f_{i-1,j}=0, f_{i+1,j}=1$. The interface is located at $y=(j-.5)h + f_{i,j}h$ and the pressure $p_{i,j}$ is solved for at the point $(ih, (j-.5)h + f_{i,j}h)$ at the center of the interface (figure A2). The following cases cover this situation:

a. $0 < f_{i,j} < 1$, $f_{i,j-1}=1$, $f_{i,j+1}=0$, $0 < f_{i-1,j} < 1$ or $0 < f_{i+1,j} < 1$.

$$b. \quad 0 < f_{i,j} < 1, f_{i,j-1} = 1, f_{i,j+1} = 0, f_{i-1,j} = 0, f_{i+1,j} < 1.$$

$$c. \quad 0 < f_{i,j} < 1, f_{i,j-1} = 1, f_{i,j+1} = 0, f_{i-1,j} < 1, f_{i+1,j} = 0.$$

By interchanging the roles of white and black fluids, one gets three additional cases with the interface located at $y = (j+0.5)h - f_{i,j}h$ and the pressure solved for at the point $(ih, (j+0.5)h - f_{i,j}h)$ at the center of interface. These remaining cases are as follows:

$$d. \quad 0 < f_{i,j} < 1, f_{i,j+1} = 1, f_{i,j-1} = 0, 0 < f_{i-1,j} < 1 \text{ or } 0 < f_{i+1,j} < 1.$$

$$e. \quad 0 < f_{i,j} < 1, f_{i,j+1} = 1, f_{i,j-1} = 0, f_{i-1,j} = 0, f_{i+1,j} < 1.$$

$$f. \quad 0 < f_{i,j} < 1, f_{i,j+1} = 1, f_{i,j-1} = 0, f_{i-1,j} < 1, f_{i+1,j} = 0.$$

III. (vertical interface). This occurs whenever $0 < f_{i,j} < 1, f_{i-1,j} = 1, f_{i+1,j} = 0$ with neither $f_{i,j-1} = 1, f_{i,j+1} = 0$ or $f_{i,j-1} = 0, f_{i,j+1} = 1$. The interface is located at $x = (i-0.5)h + f_{i,j}h$ and the pressure $p_{i,j}$ solved for at the point $((i-0.5)h + f_{i,j}h, jh)$ at the center of the interface (figure A3). The following cases cover this situation:

$$a. \quad 0 < f_{i,j} < 1, f_{i-1,j} = 1, f_{i+1,j} = 0, 0 < f_{i,j-1} < 1 \text{ or } 0 < f_{i,j+1} < 1.$$

$$b. \quad 0 < f_{i,j} < 1, f_{i-1,j} = 1, f_{i+1,j} = 0, f_{i,j-1} = 0, f_{i,j+1} < 1.$$

$$c. \quad 0 < f_{i,j} < 1, f_{i-1,j} = 1, f_{i+1,j} = 0, f_{i,j-1} < 1, f_{i,j+1} = 0.$$

By interchanging the roles of white and black fluids, one gets three additional cases with the interface located at $x = (i+0.5)h - f_{i,j}h$ and the pressure is solved for at the point $((i+0.5)h - f_{i,j}h, jh)$ at the center of the interface. These remaining cases are as follows:

$$d. \quad 0 < f_{i,j} < 1, f_{i+1,j} = 1, f_{i-1,j} = 0, 0 < f_{i,j-1} < 1 \text{ or } 0 < f_{i,j+1} < 1.$$

$$e. \quad 0 < f_{i,j} < 1, f_{i+1,j} = 1, f_{i-1,j} = 0, f_{i,j-1} = 0, f_{i,j+1} < 1.$$

$$f. \quad 0 < f_{i,j} < 1, f_{i+1,j} = 1, f_{i-1,j} = 0, f_{i,j-1} < 1, f_{i,j+1} = 0.$$

IV. (corner). There is a rectangle composed of black fluid located in a corner of the (i, j) cell with white fluid in the remainder of the cell. Without loss of generality, let us study a black rectangle in the lower left hand corner of the (i, j) cell (see figure A4). The cells $(i-1, j)$, $(i, j-1)$, and (i, j) are part full and $(i, j+1)$, $(i+1, j)$ are both empty. The sides of this rectangle have horizontal and vertical dimensions a and b respectively. The parameters a and b are found by solving the following equations,

$$ab = f_{i,j} h^2$$

and

$$\frac{a}{b} = \frac{f_{i,j-1}}{f_{i-1,j}}$$

If this equation leads to $b > h$ then we set $b = h$ and $a = f_{i,j} h$. Alternatively, if $a > h$ then $a = h$ and $b = f_{i,j} h$. The following cases refer to corners of black fluid:

- a. $0 < f_{i,j} < 1, 0 < f_{i-1,j} < 1, 0 < f_{i,j-1} < 1, f_{i,j+1} = 0, f_{i+1,j} = 0$. There is a rectangle of black fluid located in the lower left hand corner of the (i, j) cell and $p_{i,j}$ is solved for at the corner of the rectangle lying inside the cell, i.e., at $((i-0.5)h + a, (j-0.5)h + b)$ (figure A 4).
- b. $0 < f_{i,j} < 1, 0 < f_{i-1,j} < 1, 0 < f_{i,j+1} < 1, f_{i+1,j} = 0, f_{i,j-1} = 0$. There is a rectangle of black fluid located in the upper left hand corner of the (i, j) cell and $p_{i,j}$ is solved for at the corner of the rectangle lying inside the cell, i.e., at $((i-0.5)h + a, (j+0.5)h - b)$.
- c. $0 < f_{i,j} < 1, 0 < f_{i+1,j} < 1, 0 < f_{i,j+1} < 1, f_{i-1,j} = 0, f_{i,j-1} = 0$. There is a rectangle of black fluid located in the upper right hand corner of the (i, j) cell and $p_{i,j}$ is solved for at the corner of the rectangle lying inside the cell, i.e., at $((i+0.5)h - a, (j+0.5)h - b)$.
- d. $0 < f_{i,j} < 1, 0 < f_{i+1,j} < 1, 0 < f_{i,j-1} < 1, f_{i-1,j} = 0, f_{i,j+1} = 0$. There is a rectan-

gle of black fluid located in the lower right hand corner of the (i, j) cell and $p_{i,j}$ is located at the corner of the rectangle lying inside the cell, i.e., at $((i+.5)h-a, (j-.5)h+b)$.

There are four corresponding cases for corners of white fluid which can be obtained by interchanging the role of black and white:

- e. $0 < f_{i,j} < 1, 0 < f_{i-1,j} < 1, 0 < f_{i,j-1} < 1, f_{i,j+1}=1, f_{i+1,j}=1$. There is a rectangle of white fluid located in the lower left hand corner of the (i, j) cell and $p_{i,j}$ is solved for at the corner of the rectangle lying inside the cell, i.e., at $((i-.5)h+a, (j-.5)h+b)$.
- f. $0 < f_{i,j} < 1, 0 < f_{i-1,j} < 1, 0 < f_{i,j+1} < 1, f_{i+1,j}=1, f_{i,j-1}=1$. There is a rectangle of white fluid located in the upper left hand corner of the (i, j) cell and $p_{i,j}$ is solved for at the corner of the rectangle lying inside the cell, i.e., at $((i-.5)h+a, (j+.5)h-b)$.
- g. $0 < f_{i,j} < 1, 0 < f_{i+1,j} < 1, 0 < f_{i,j+1} < 1, f_{i-1,j}=1, f_{i,j-1}=1$. There is a rectangle of white fluid located in the upper right hand corner of the (i, j) cell and $p_{i,j}$ is solved for at the corner of the rectangle lying inside the cell, i.e., at $((i+.5)h-a, (j+.5)h-b)$.
- h. $0 < f_{i,j} < 1, 0 < f_{i+1,j} < 1, 0 < f_{i,j-1} < 1, f_{i-1,j}=1, f_{i,j+1}=1$. There is a rectangle of white fluid located in the lower right hand corner of the (i, j) cell and $p_{i,j}$ is solved for at the corner of the rectangle lying inside the cell, i.e., at $((i+.5)h-a, (j-.5)h+b)$.

V. (diagonal). The next four cases are called diagonal interfaces. The interface is composed of two line segments. The two line segments imminiate from two diagonal corners of the cell and meet at a point on the diagonal through the other two corners which would

appropriately cut out the correct area $f_{i,j}$. There are four related cases as follows:

- a. $0 < f_{i,j} < 1$, $f_{i-1,j}=1$, $f_{i,j-1}=1$, $f_{i+1,j}=0$, $f_{i,j+1}=0$. The diagonal interface is located in the lower left hand corner of the (i,j) cell. Two line segments form our interface, imminating from the corners $((i+.5)h, (j-.5)h)$ and $((i-.5)h, (j+.5)h)$. They intersect on the diagonal through the other two corners, $((i-.5)h, (j-.5)h)$ and $((i+.5)h, (j+.5)h)$. This point is also where our pressure is solved, at $((i-.5)h + f_{i,j}h, (j-.5)h + f_{i,j}h)$ (figure A 5).
- b. $0 < f_{i,j} < 1$, $f_{i-1,j}=1$, $f_{i,j+1}=1$, $f_{i+1,j}=0$, $f_{i,j-1}=0$. The diagonal interface is located in the upper left hand corner of the (i,j) cell. Two line segments form our interface, imminating from the corners $((i-.5)h, (j-.5)h)$ and $((i+.5)h, (j+.5)h)$. They intersect on the diagonal through the other two corners, $((i-.5)h, (j+.5)h)$ and $((i+.5)h, (j-.5)h)$. This point is also where our pressure is solved, at $((i-.5)h + f_{i,j}h, (j+.5)h - f_{i,j}h)$.
- c. $0 < f_{i,j} < 1$, $f_{i+1,j}=1$, $f_{i,j+1}=1$, $f_{i-1,j}=0$, $f_{i,j-1}=0$. The diagonal interface is located in the upper right hand corner of the (i,j) cell. Two line segments form our interface, imminating from the corners $((i-.5)h, (j+.5)h)$ and $((i+.5)h, (j-.5)h)$. They intersect on the diagonal through the other two corners, $((i-.5)h, (j-.5)h)$ and $((i+.5)h, (j+.5)h)$. This point is also where our pressure is solved, at $((i+.5)h - f_{i,j}h, (j+.5)h - f_{i,j}h)$.
- d. $0 < f_{i,j} < 1$, $f_{i+1,j}=1$, $f_{i,j-1}=1$, $f_{i-1,j}=0$, $f_{i,j+1}=0$. The diagonal interface is located in the lower right hand corner of the (i,j) cell. Two line segments form our interface, imminating from the corners $((i-.5)h, (j-.5)h)$ and $((i+.5)h, (j+.5)h)$. They intersect on the diagonal through the other two corners, $((i-.5)h, (j+.5)h)$ and $((i+.5)h, (j-.5)h)$. This point is also where our pressure is solved, at $((i+.5)h - f_{i,j}h, (j-.5)h + f_{i,j}h)$.

VI. (finger). For the black vertical finger, the black fluid in cell (i, j) is assumed to be a thin finger lying in the center of the cell between two interfaces at $x = ih - f_{i,j} h/2$ and $x = ih + f_{i,j} h/2$. We assume that the pressure is constant along the line segment connected by the points $(ih - f_{i,j} h/2, jh)$ and $(ih + f_{i,j} h/2, jh)$. The pressures will be solved at the end-points $(ih - f_{i,j} h/2, jh)$ and $(ih + f_{i,j} h/2, jh)$ and at the center of the segment, (ih, jh) . The point chosen is the point which minimizes interpolation. There is a related case for the black vertical finger which can be obtained by interchanging the roles of black and white above. The two respective cases are as follows:

- a. $0 < f_{i,j} < 1, f_{i-1,j} = 1, f_{i+1,j} = 1, f_{i,j-1} < 1, f_{i,j+1} < 1$ (figure A 6).
- b. $0 < f_{i,j} < 1, f_{i-1,j} = 0, f_{i+1,j} = 0, f_{i,j-1} > 0, f_{i,j+1} > 0$. The pressures are solved for at $(ih - (1 - f_{i,j})h, jh)$, $(ih + (1 - f_{i,j})h, jh)$ and (ih, jh) .

Two cases are obtained by considering the horizontal fingers and are given as follows:

- c. $0 < f_{i,j} < 1, f_{i,j-1} = 1, f_{i,j+1} = 1, f_{i-1,j} < 1, f_{i+1,j} < 1$. The pressures are solved for at $(ih, jh - f_{i,j} h)$, $(ih, jh + f_{i,j} h)$ and (ih, jh) .
- d. $0 < f_{i,j} < 1, f_{i,j-1} = 0, f_{i,j+1} = 0, f_{i-1,j} > 0, f_{i+1,j} > 0$. The pressures are solved for at $(ih, jh - (1 - f_{i,j})h)$, $(ih, jh + (1 - f_{i,j})h)$ and (ih, jh) .

VII. (fingertip). This occurs in a cell which is partially full surrounded on three sides by black fluid and on the fourth side by a cell not totally black. The four related cases are as follows:

- a. $0 < f_{i,j} < 1, f_{i,j-1} = 1, f_{i,j+1} < 1, f_{i-1,j} = 1, f_{i+1,j} = 1$. The pressure is solved for at $(ih, (j - 0.5)h + f_{i,j} h)$ (figure A 7)
- b. $0 < f_{i,j} < 1, f_{i,j-1} = 1, f_{i,j+1} = 1, f_{i-1,j} < 1, f_{i+1,j} = 1$. The pressure is solved for at $((i + 0.5)h - f_{i,j} h, jh)$.

- c. $0 < f_{i,j} < 1, f_{i,j-1}=1, f_{i,j+1}=1, f_{i-1,j}=1, f_{i+1,j} < 1$. The pressure is solved for at $((i-.5)h + f_{i,j}h, jh)$.
- d. $0 < f_{i,j} < 1, f_{i,j-1} < 1, f_{i,j+1}=1, f_{i-1,j}=1, f_{i+1,j}=1$. The pressure is solved for at $(ih, (j+.5)h - f_{i,j}h)$.

There are four related cases involving white fingers which can be obtained by interchanging the roles of white and black:

- e. $0 < f_{i,j} < 1, f_{i,j-1}=0, f_{i,j+1}>0, f_{i-1,j}=0, f_{i+1,j}=0$. The pressure is solved for at $(ih, (j-.5)h + f_{i,j}h)$.
- f. $0 < f_{i,j} < 1, f_{i,j-1}=0, f_{i,j+1}=0, f_{i-1,j}>0, f_{i+1,j}=0$. The pressure is solved for at $((i+.5)h - f_{i,j}h, jh)$.
- g. $0 < f_{i,j} < 1, f_{i,j-1}=0, f_{i,j+1}=0, f_{i-1,j}=0, f_{i+1,j}>0$. The pressure is solved for at $((i-.5)h + f_{i,j}h, jh)$.
- h. $0 < f_{i,j} < 1, f_{i,j-1}>0, f_{i,j+1}=0, f_{i-1,j}=0, f_{i+1,j}=0$. The pressure is solved for at $(ih, (j+.5)h - f_{i,j}h)$.

VIII. (pseudo finger). The white pseudo finger is a white vertical finger which occurs if there is another white vertical finger below or above it. The pressures have the same location as the fingers in V. There are two similar cases:

- a. $0 < f_{i,j} < 1, 0 < f_{i,j-1} < 1, f_{i+1,j-1}=1, f_{i-1,j-1}=1, f_{i,j+1} < 1$ (figure A 8)
- b. $0 < f_{i,j} < 1, 0 < f_{i,j+1} < 1, f_{i+1,j+1}=1, f_{i-1,j+1}=1, f_{i,j-1} < 1$.

There are two related cases obtained by interchanging the roles of white and black:

- c. $0 < f_{i,j} < 1, 0 < f_{i,j-1} < 1, f_{i+1,j-1}=0, f_{i-1,j-1}=0, f_{i,j+1} > 0$.

$$d. \quad 0 < f_{i,j} < 1, 0 < f_{i,j+1} < 1, f_{i+1,j+1} = 0, f_{i-1,j+1} = 0, f_{i,j-1} > 0.$$

There are four additional cases obtained by considering the horizontal versions of the cases above:

$$e. \quad 0 < f_{i,j} < 1, 0 < f_{i-1,j} < 1, f_{i-1,j+1} = 1, f_{i+1,j+1} = 1, f_{i+1,j} < 0.$$

$$f. \quad 0 < f_{i,j} < 1, 0 < f_{i+1,j} < 1, f_{i-1,j-1} = 1, f_{i+1,j-1} = 1, f_{i-1,j} < 0.$$

$$g. \quad 0 < f_{i,j} < 1, 0 < f_{i-1,j} < 1, f_{i-1,j+1} = 0, f_{i+1,j+1} = 0, f_{i+1,j} > 0.$$

$$h. \quad 0 < f_{i,j} < 1, 0 < f_{i+1,j} < 1, f_{i-1,j-1} = 0, f_{i+1,j-1} = 0, f_{i-1,j} > 0.$$

B. Location of the interface across the sides of the cell. To evaluate the line integral of the velocity around a cell, the point where the interface crosses the sides must be known. In the previous section, for cell (i, j) , the location of the interface was approximated. We shall use the approximations from above to estimate the location of the interface across a side common to two adjacent cells. Without loss of generality we consider the side shared by cells (i, j) and $(i+1, j)$. In most cases to find where the interface crosses the side we take the average value of the two local interfaces. Below we shall give cases highlighting the black fluid. For each case given below there is a case obtained by interchanging the roles of white and black. This is done by interchanging the roles of $f_{i,j}$ and $f_{i+1,j}$ with $1-f_{i,j}$ and $1-f_{i+1,j}$ respectively. The possible situations are given below:

I. Cell (i, j) or $(i+1, j)$ is full ($f_{i,j} = 1$ or $f_{i+1,j} = 1$). There is no interface crossing the side.

II. Cell (i, j) or $(i+1, j)$ contain a black vertical finger or a black pseudo finger. There is no interface crossing the side, only white fluid crosses.

III. Cells (i, j) and $(i+1, j)$ both contain black horizontal fingers (may be pseudo). There is no interface crossing the side, only black fluid.

IV. Cells (i, j) and $(i+1, j)$ both contain horizontal interfaces. There are two cases.

- a. Cells $(i, j-1)$ and $(i+1, j-1)$ are full. The location of the interface on the shared side is given by the average of the two local interfaces in (i, j) and $(i+1, j)$, i.e., $((i+.5)h, (j-.5)h + (f_{i,j} + f_{i+1,j})h/2)$.
- b. Cells $(i, j+1)$ and $(i+1, j+1)$ are full. The location of the interface on the shared side is given by the average of the two local interfaces in (i, j) and $(i+1, j)$, i.e., $((i+.5)h, (j+.5)h - (f_{i,j} + f_{i+1,j})h/2)$.

V. Cell (i, j) contains a horizontal interface and cell $(i+1, j)$ is a black corner. There are two related cases. Case b comes from case a with the roles of top and bottom interchanged.

- a. The black rectangle is located in the lower left corner of cell $(i+1, j)$ with vertical and horizontal dimensions b and a respectively and $(i, j-1)$ full. The location of the interface on the shared side is given by the average of the two local interfaces in (i, j) and $(i+1, j)$, i.e., $((i+.5)h, (j-.5)h + (f_{i,j}h + b)/2)$.
- b. The black rectangle is located in the upper left corner of cell $(i+1, j)$ with vertical and horizontal dimensions b and a respectively and $(i, j+1)$ full. The location of the interface on the shared side is given by the average of the two local interfaces in (i, j) and $(i+1, j)$, i.e., $((i+.5)h, (j+.5)h - (f_{i,j}h + b)/2)$.

VI. Cell $(i+1, j)$ is a horizontal interface and cell (i, j) is a black corner. There are two cases which can be obtained by interchanging the roles of left and right in cases a and b from VI above.

- a. The black rectangle is located in the lower left corner of cell (i, j) with vertical and horizontal dimensions b and a respectively and $(i+1, j-1)$ full. The location of the interface on the shared side is given by the average of the two local interfaces in (i, j) and $(i+1, j)$, i.e., $((i+.5)h, (j-.5)h + (f_{i+1,j}h + b)/2)$.
- b. The black rectangle is located in the upper left corner of cell (i, j) with vertical and horizontal dimensions b and a respectively and $(i+1, j+1)$ full. The location

of the interface on the shared side is given by the average of the two local interfaces in (i, j) and $(i+1, j)$, i.e., $((i+0.5)h, (j+0.5)h - (f_{i+1,j}h + b)/2)$.

VII. Cell $(i+1, j)$ is a white corner and cell (i, j) is a black corner. There are two cases.

Case b can be obtained by interchanging the roles of top and bottom in case a.

a. The black rectangle is located in the lower left corner of cell (i, j) with vertical and horizontal dimensions b and a respectively and a white rectangle is located in the upper right corner of cell $(i+1, j)$ with dimensions b_1 and a_1 respectively. The location of the interface on the shared side is given by the average of the two local interfaces in (i, j) and $(i+1, j)$, i.e., $((i+0.5)h, (j-0.5)h + (h - b_1 + b)/2)$.

b. The black rectangle is located in the upper left corner of cell (i, j) with vertical and horizontal dimensions b and a respectively and a white rectangle is located in the lower right corner of cell $(i+1, j)$ with dimensions b_1 and a_1 respectively. The location of the interface on the shared side is given by the average of the two local interfaces in (i, j) and $(i+1, j)$, i.e., $((i+0.5)h, (j+0.5)h - (h - b_1 + b)/2)$.

C. Evaluating the line integral of cell (i, j) . In this section, we will show how the equation

$$\int_{\partial c_{i,j}} \vec{u} \cdot \vec{n} ds = 0, \quad (C1)$$

is discretized, where $\partial c_{i,j}$ is the boundary of the (i, j) cell and $\vec{n}$ is the outward normal. Equation (C1) can be rewritten as,

$$\int_{RS} \vec{u} \cdot \vec{n} ds + \int_{TS} \vec{u} \cdot \vec{n} ds + \int_{LS} \vec{u} \cdot \vec{n} ds + \int_{BS} \vec{u} \cdot \vec{n} ds = 0, \quad (C2)$$

where RS is the line segment on the right hand side on cell (i, j) , TS is the line segment on the top side on cell (i, j) , LS is the line segment on the left hand side on cell (i, j) , and BS is the line segment on the bottom side on cell (i, j) . Without loss of generality, we will show how to discretize the first term in equation (C2), i.e., we will show how to discretize the line integral along the side shared by cells (i, j) and $(i+1, j)$.

The line integral over RS is divided into two parts, one pertaining to the black fluid and one pertaining to the white fluid, and is given by

$$\int_{RS} \vec{u} \cdot \vec{n} ds = \int_{RS(black)} \vec{u} \cdot \vec{n} ds + \int_{RS(white)} \vec{u} \cdot \vec{n} ds. \quad (C3)$$

On the right hand side of (C3), $\vec{u} \cdot \vec{n}$ simplifies into $\kappa_w \frac{\partial p((i+.5)h, y)}{\partial x}$ when $((i+.5)h, y)$ lies in the white fluid and into $\kappa_b \frac{\partial p((i+.5)h, y)}{\partial x}$ when $((i+.5)h, y)$ lies in the black fluid where $\kappa_w = -\frac{b^2}{12\mu_w}$ and $\kappa_b = -\frac{b^2}{12\mu_b}$. The line integrals on the right hand side of (C3) are approximated by $\vec{u} \cdot \vec{n}$ at some convenient point on the line times the length of the line. Interpolation is used in the cases when $p_{i,j}$ and $p_{i+1,j}$ do not have the same y -coordinates. Cases highlighting the black fluid are given below with the remaining cases obtained by interchanging the roles of white and black.

I. The cell (i, j) is full ($f_{i,j}=1$). By section B, there is no interface crossing the side and only the first term on the right hand side of equation (C3) will be involved in the approximations, i.e., $h \kappa_b \frac{\partial p((i+.5)h, \xi)}{\partial x}$ where ξ is some convenient point on the right side of cell (i, j) .

a. The cell $(i+1, j)$ is full ($f_{i+1,j}=1$). The pressures in the two cells (i, j) and $(i+1, j)$ are both cell centered. These pressures have the same y -coordinates, therefore no interpolation is necessary and the approximation for our line integral is given by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_b \frac{p_{i+1,j} - p_{i,j}}{h}. \quad (C4)$$

b. $(i+1, j)$ contains a vertical interface with $(i+2, j)$ empty. $p_{i,j}$ and $p_{i+1,j}$ have the same y -coordinates, therefore no interpolation is necessary and we have the approximation

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_b \frac{p_{i+1,j} - p_{i,j}}{(.5 + f_{i+1,j})} \quad (C5)$$

c. $(i+1, j)$ contains a horizontal interface with $(i+1, j-1)$ full and $(i+1, j+1)$ empty. In this case, to get an estimate of $\frac{\partial p((i+.5)h, \xi)}{\partial x}$ on RS , the pressure in cell (i, j) should have the same y -coordinate as the pressure in cell $(i+1, j)$. We interpolate linearly through the

pressures $p_{i,j}$ and $p_{i+1,j-1}$ to obtain a pressure, $\tilde{p}_{i,j}$, in cell (i,j) with the same y -coordinate as $p_{i+1,j}$. The distance between the locations of the $p_{i,j}$ and $p_{i+1,j-1}$ is $\sqrt{2}h$. The distance between the locations of $\tilde{p}_{i,j}$ and $p_{i,j}$ is $\sqrt{2} |f_{i+1,j}h - .5h|$. The approximate grid function, $\tilde{p}_{i,j}$, is then given by

$$\tilde{p}_{i,j} = p_{i,j} + \sqrt{2}h \frac{(p_{i,j} - p_{i+1,j-1})}{\sqrt{2} |f_{i+1,j}h - .5h|} \quad (C6)$$

The line integral is approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_b \frac{p_{i+1,j} - \tilde{p}_{i,j}}{(.5 + f_{i+1,j})h} \quad (C7)$$

where $(.5 + f_{i+1,j})h$ is the distance between $p_{i+1,j}$ and $\tilde{p}_{i,j}$.

d. $(i+1,j)$ contains a horizontal interface with $(i+1,j+1)$ full and $(i+1,j-1)$ empty. This becomes equivalent to the previous case upon interchanging the roles of top and bottom. Interpolation takes place between $p_{i+1,j+1}$ and $p_{i,j}$. The new pressure, $\tilde{p}_{i,j}$, is given by

$$\tilde{p}_{i,j} = p_{i,j} - \sqrt{2}(f_{i+1,j}h - .5h) \frac{(p_{i+1,j+1} - p_{i,j})}{\sqrt{2}h}. \quad (C8)$$

The line integral is approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_b \frac{p_{i+1,j} - \tilde{p}_{i,j}}{(.5 + f_{i+1,j})h}. \quad (C9)$$

e. $(i+1,j)$ contains a white rectangle in the upper right corner of $(i+1,j)$. The white rectangle has vertical and horizontal dimensions b and a respectively. The pressure, $p_{i+1,j}$, is located at $((i+1.5)h - a, (j+.5)h - b)$. $p_{i,j}$ and $p_{i+1,j-1}$ are used to obtain the pressure $\tilde{p}_{i,j}$ with the same y -coordinate as $p_{i+1,j}$. $p_{i,j}$ and $p_{i+1,j-1}$ are located at (ih, jh) and $((i+1)h, (j-1)h)$. This new pressure, $\tilde{p}_{i,j}$ is located at $((i-.5)h + b, (j+.5)h - b)$. The distance between the points $p_{i,j}$ and $p_{i+1,j-1}$ is $\sqrt{2}h$. The distance between $p_{i,j}$ and $\tilde{p}_{i,j}$ is $\sqrt{2} |b - .5h|$. The new pressure, $\tilde{p}_{i,j}$, is given by,

$$\tilde{p}_{i,j} = p_{i,j} - \sqrt{2}(b - .5h) \frac{(p_{i,j} - p_{i+1,j-1})}{\sqrt{2}h}. \quad (C10)$$

The line integral is approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_b \frac{p_{i+1,j} - \tilde{p}_{i,j}}{(h - b + h - a)} \quad (C11)$$

where $(h - b + h - a)$ is the distance between $p_{i+1,j}$ and $\tilde{p}_{i,j}$.

f. $(i+1,j)$ contains a white rectangle in the lower right corner of $(i+1,j)$. This becomes equivalent to the case above upon interchanging the roles top and bottom. For this case $(i+1,j+1)$ is full and the interpolated pressure, $\tilde{p}_{i,j}$, is given by,

$$\tilde{p}_{i,j} = p_{i,j} + (b - .5h) \frac{(p_{i+1,j+1} - p_{i,j})}{h}. \quad (C12)$$

The line integral is approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_b \frac{p_{i+1,j} - \tilde{p}_{i,j}}{(h - b + h - a)}. \quad (C13)$$

g. $(i+1,j)$ contains a white vertical finger. The pressure $p_{i+1,j}$ is located at $((i + .5)h + .5f_{i+1,j}h, jh)$. Therefore no interpolation is required and the line integral can be approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_b \frac{p_{i+1,j} - p_{i,j}}{(.5h + .5f_{i+1,j}h)}. \quad (C14)$$

where $.5h + .5f_{i+1,j}h$ is the distance between the locations of $p_{i,j}$ and $p_{i+1,j}$.

h. $(i+1,j)$ contains a white horizontal fingertip. For a white horizontal fingertip, there is only one possible case since (i,j) is full. The pressure is located at $((i + .5)h + f_{i+1,j}h, jh)$. The line integral can then be approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_b \frac{p_{i+1,j} - p_{i,j}}{(.5h + f_{i+1,j}h)}. \quad (C15)$$

where $.5h + f_{i+1,j}h$ is the distance between the locations of $p_{i,j}$ and $p_{i+1,j}$.

i. $(i+1,j)$ contains a white vertical fingertip. For white vertical fingertips, there are two possibilities. One possibility has the cell $(i+1,j+1)$ full and the pressure $p_{i+1,j}$ located at $((i+1)h, (j + .5)h - f_{i+1,j}h)$. For this case, the pressures $p_{i,j}$ and $p_{i+1,j+1}$ are used to

estimate $\tilde{p}_{i,j}$, the pressure in (i,j) with the same y -coordinate as $p_{i+1,j}$. Our pressure approximated by interpolation is located on the line through $p_{i,j}$ and $p_{i+1,j+1}$ at the point $((i+.5)h - f_{i+1,j}h, (j+.5)h - f_{i+1,j}h)$. $\tilde{p}_{i,j}$ is given by,

$$\tilde{p}_{i,j} = p_{i,j} - (f_{i+1,j}h - .5h) \frac{(p_{i+1,j+1} - p_{i,j})}{h}. \quad (C16)$$

The line integral can then be approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_b \frac{p_{i+1,j} - \tilde{p}_{i,j}}{(.5h + f_{i+1,j}h)}. \quad (C17)$$

There is a similar vertical fingertip obtained by a horizontal flip of the finger in $(i+1,j)$. For this case $(i+1,j-1)$ is full and the pressure $p_{i+1,j}$ is located at $((i+1)h, (j-.5)h + f_{i+1,j}h)$.

The interpolated pressure $\tilde{p}_{i,j}$ is given by,

$$\tilde{p}_{i,j} = p_{i,j} + (f_{i+1,j}h - .5h) \frac{(p_{i,j} - p_{i+1,j-1})}{h}. \quad (C18)$$

The line integral can then be approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_b \frac{p_{i+1,j} - \tilde{p}_{i,j}}{(.5h + f_{i+1,j}h)}. \quad (C19)$$

j. $(i+1,j)$ contains a diagonal interface. There are two possibilities, the black fluid is located in the lower or upper left corners of the cell. For the diagonal located in the lower left corner, $p_{i+1,j}$ is located at $((i+.5)h + f_{i+1,j}h, (j-.5)h + f_{i+1,j}h)$. The pressures $p_{i,j}$ and $p_{i+1,j-1}$, located at (ih, jh) and $((i+1)h, (j-1)h)$ respectively, are used to obtain a new pressure, $\tilde{p}_{i,j}$, with the same y -coordinate as $p_{i+1,j}$. The new pressure, $\tilde{p}_{i,j}$, is given by

$$\tilde{p}_{i,j} = p_{i,j} + (f_{i+1,j}h - .5h) \frac{(p_{i,j} - p_{i+1,j-1})}{h}. \quad (C20)$$

The line integral can then be approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_b \frac{p_{i+1,j} - \tilde{p}_{i,j}}{2f_{i+1,j}h}. \quad (C21)$$

For the case of the black fluid located in the upper left corner, $p_{i+1,j}$ is located at $((i+.5)h + f_{i+1,j}h, (j+.5)h - f_{i+1,j}h)$. The pressures $p_{i,j}$ and $p_{i+1,j+1}$, located at (ih, jh) and $((i+1)h, (j+1)h)$ respectively, are used to obtain a new pressure, $\tilde{p}_{i,j}$, with the same y -

coordinate as $p_{i+1,j}$. The new pressure, $\tilde{p}_{i,j}$, is given by

$$\tilde{p}_{i,j} = p_{i,j} - (f_{i+1,j} h - .5h) \frac{(p_{i+1,j+1} - p_{i,j})}{h}. \quad (C23)$$

The line integral can then be approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_b \frac{p_{i+1,j} - \tilde{p}_{i,j}}{2f_{i+1,j} h}. \quad (C24)$$

II. The cell (i,j) is a horizontal interface with $(i,j-1)$ full and $(i,j+1)$ empty.

a. $(i+1,j)$ is full. This is similar to Ic. Interpolation takes place between $p_{i,j-1}$ and $p_{i+1,j}$. The new pressure, $\tilde{p}_{i+1,j}$, is given by

$$\tilde{p}_{i+1,j} = p_{i+1,j} + (f_{i,j} h - .5h) \frac{(p_{i+1,j} - p_{i,j-1})}{h}. \quad (C25)$$

The line integral is approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_b \frac{\tilde{p}_{i+1,j} - p_{i,j}}{(.5 + f_{i,j})h}. \quad (C26)$$

b. $(i+1,j)$ contains a horizontal interface with $(i+1,j-1)$ full, $(i+1,j+1)$ empty and $f_{i,j}$ less than $f_{i+1,j}$. For this case, both terms in (C4) must be approximated since our interface crosses the common side of cells (i,j) and $(i+1,j)$. The interface crosses the side between cells (i,j) and $(i+1,j)$ at the point $((i + .5)h, (j - .5)h + .5h(f_{i,j} + f_{i+1,j}))$, i.e., the average of the two local interfaces. Let us first estimate the part of the line integral pertaining to the black fluid. $p_{i+1,j}$ and $p_{i+1,j-1}$ are used to obtain $\tilde{p}_{i+1,j}$, the pressure located at a point with the same y -coordinate as $p_{i,j}$ in the black fluid. This point lies on the line between $p_{i+1,j}$ and $p_{i+1,j-1}$ at the point $((i+1)h, (j-.5)h + f_{i,j} h)$. The interpolated pressure, $\tilde{p}_{i+1,j}$, is given by,

$$\tilde{p}_{i+1,j} = p_{i+1,j} - (f_{i+1,j} h - f_{i,j} h) \frac{(p_{i+1,j} - p_{i+1,j-1})}{(f_{i+1,j} h + .5h)}. \quad (C27)$$

The line integral over the black fluid is given by,

$$\int_{RS(\text{black})} \vec{u} \cdot \vec{n} ds \approx .5h (f_{i,j} + f_{i+1,j}) \kappa_b \frac{\tilde{p}_{i+1,j} - p_{i,j}}{h} \quad (\text{C28})$$

For the part of the line integral pertaining to the white fluid, $p_{i,j}$ and $p_{i,j+1}$ are used to obtain $\tilde{p}_{i,j}$, the pressure located at a point with the same y -coordinate as $p_{i+1,j}$ in the white fluid. This pressure is located on the line between the pressures $p_{i,j}$ and $p_{i,j+1}$ at the point $(ih, (j-.5)h + f_{i+1,j}h)$. The interpolated pressure, $\hat{p}_{i,j}$, is given by,

$$\hat{p}_{i,j} = p_{i,j} + (f_{i+1,j}h - f_{i,j}h) \frac{(p_{i,j+1} - p_{i,j})}{(-f_{i+1,j}h + 1.5h)}. \quad (\text{C29})$$

The line integral over the white fluid can then be approximated by,

$$\int_{RS(\text{white})} \vec{u} \cdot \vec{n} ds \approx .5h (2 - f_{i,j} - f_{i+1,j}) \kappa_w \frac{p_{i+1,j} - \hat{p}_{i,j}}{h}. \quad (\text{C30})$$

The complete line integral over RS can then be approximated by summing the quantities in (C28) and (C30) and obtaining,

$$\begin{aligned} \int_{RS} \vec{u} \cdot \vec{n} ds &\approx .5h (f_{i,j} + f_{i+1,j}) \kappa_b \frac{\tilde{p}_{i+1,j} - p_{i,j}}{h} \\ &+ .5h (2 - f_{i,j} - f_{i+1,j}) \kappa_w \frac{p_{i+1,j} - \hat{p}_{i,j}}{h}. \end{aligned} \quad (\text{C31})$$

where $\tilde{p}_{i+1,j}$ and $\hat{p}_{i,j}$ are given by (C27) and (C29) respectively.

c. $(i+1,j)$ contains a horizontal interface with $(i+1,j-1)$ full, $(i+1,j+1)$ empty and $f_{i+1,j}$ less than $f_{i,j}$. Again, the line integral must be divided into two parts, one pertaining to the velocity in the black fluid and one pertaining to the velocity in the white fluid. The interface crosses the side between cells (i,j) and $(i+1,j)$ at the point $((i+.5)h, (j-.5)h + .5h(f_{i,j} + f_{i+1,j}))$. Let us first estimate the part of the line integral pertaining to the black fluid. $p_{i,j}$ and $p_{i,j-1}$ are used to obtain $\tilde{p}_{i,j}$, a pressure located at a point with the same y -coordinate as $p_{i+1,j}$ in the black fluid. The interpolated pressure, $\tilde{p}_{i,j}$, is given by,

$$\tilde{p}_{i,j} = p_{i,j} + (f_{i+1,j}h - f_{i,j}h) \frac{(p_{i,j} - p_{i,j-1})}{(f_{i,j}h + .5h)}. \quad (\text{C32})$$

The line integral over the black fluid can then be approximated by,

$$\int_{RS(black)} \vec{u} \cdot \vec{n} ds \approx .5h (f_{i,j} + f_{i+1,j}) \kappa_b \frac{p_{i+1,j} - \bar{p}_{i,j}}{h}. \quad (C33)$$

For the part of the line integral pertaining to the white fluid, $p_{i+1,j}$ and $p_{i+1,j+1}$ are used to obtain $\hat{p}_{i+1,j}$, the pressure located at a point with the same y -coordinate as $p_{i,j}$. The interpolated pressure, $\bar{p}_{i+1,j}$, is given by,

$$\hat{p}_{i+1,j} = p_{i+1,j} - (f_{i+1,j}h - f_{i,j}h) \frac{(p_{i+1,j+1} - p_{i+1,j})}{(-f_{i+1,j}h + 1.5h)}. \quad (C34)$$

The line integral over the white fluid can then be approximated by,

$$\int_{RS(white)} \vec{u} \cdot \vec{n} ds \approx .5h (2 - f_{i,j} - f_{i+1,j}) \kappa_w \frac{\hat{p}_{i+1,j} - p_{i,j}}{h} \quad (C35)$$

The complete line integral on the RS can then be approximated by summing the quantities in (C33) and (C35) and obtaining,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx .5h (f_{i,j} + f_{i+1,j}) \kappa_b \frac{p_{i+1,j} - \bar{p}_{i,j}}{h} \quad (C36)$$

$$+ .5h (2 - f_{i,j} - f_{i+1,j}) \kappa_w \frac{\hat{p}_{i+1,j} - p_{i,j}}{h}. \quad (C37)$$

d. $(i+1,j)$ is a contains a black rectangle in the lower left hand corner with b (vertical height of the rectangle) greater than $f_{i,j}h$. Again, the line integral must be divided into two parts, one pertaining to the velocity in the black fluid and one pertaining to the velocity in the white fluid. It is estimated that the interface crosses the side between cells (i,j) and $(i+1,j)$ at the point $((i + .5)h, (j - .5)h + .5(f_{i,j}h + b))$, i.e., the average of the two local interfaces. Let us first estimate the part of the line integral pertaining to the black fluid. $p_{i+1,j}$ and $p_{i,j-1}$ are used to obtain $\bar{p}_{i+1,j}$, the pressure located at a point with the same y -coordinate as $p_{i,j}$ in the black fluid. Our objective is to find the pressure $\bar{p}_{i+1,j}$ on the line through the points $(ih, (j-1)h)$ and $((i + .5)h + a, (j - .5)h + b)$ with vertical height $((j - .5)h + f_{i,j}h)$. These points are the coordinates associated with $p_{i,j-1}$ and $p_{i+1,j}$ respectively. The location of the $\bar{p}_{i+1,j}$ is given by

$$((f_{i,j}h - b) \frac{.5h + a}{.5h + b} + (i + .5)h + a, (j - .5)h + f_{i,j}h). \quad (C38)$$

The distance between the points associated with the pressures $p_{i+1,j}$ and $p_{i,j-1}$ is easily computed and will be denoted d_1 . The distance between the points associated with $p_{i+1,j}$ and $\tilde{p}_{i+1,j}$ will be denoted by d_2 . $p_{i,j}$ is located at $(ih, (j-.5)h + f_{i,j}h)$ and the distance between its location and the location of $\tilde{p}_{i+1,j}$ will be denoted by d_3 . The interpolated pressure, $\tilde{p}_{i+1,j}$, is given by,

$$\tilde{p}_{i+1,j} = p_{i+1,j} - d_1 \frac{(p_{i+1,j} - p_{i,j-1})}{d_2}. \quad (C39)$$

The line integral over the black fluid can then be approximated by,

$$\int_{RS(\text{black})} \vec{u} \cdot \vec{n} ds \approx .5(f_{i,j}h + b)\kappa_b \frac{\tilde{p}_{i+1,j} - p_{i,j}}{d_3} \quad (C40)$$

For the part of the line integral pertaining to the white fluid, $p_{i,j}$ and $p_{i,j+1}$ are used to obtain $\hat{p}_{i,j}$, the pressure located at a point with the same y -coordinate as $p_{i+1,j}$. The new pressure, $\hat{p}_{i+1,j}$, is given by,

$$\hat{p}_{i+1,j} = p_{i+1,j} + (b - f_{i,j}h) \frac{(p_{i,j+1} - p_{i,j})}{(-f_{i,j}h + 1.5h)}. \quad (C41)$$

The line integral over the white fluid can then be approximated by,

$$\int_{RS(\text{white})} \vec{u} \cdot \vec{n} ds \approx .5(h - f_{i,j}h - h - b)\kappa_w \frac{p_{i+1,j} - \hat{p}_{i,j}}{.5h + a} \quad (C42)$$

The complete line integral on the RS can then be approximated by summing the quantities in (C40) and (C42) and obtaining,

$$\begin{aligned} \int_{RS} \vec{u} \cdot \vec{n} ds &\approx .5(f_{i,j}h + b)\kappa_b \frac{\tilde{p}_{i+1,j} - p_{i,j}}{d_3} \\ &+ .5(h - f_{i,j}h - h - b)\kappa_w \frac{p_{i+1,j} - \hat{p}_{i,j}}{.5h + a}. \end{aligned} \quad (C43)$$

e. $(i+1,j)$ is a contains a black rectangle in the lower left hand corner with b (vertical height of the rectangle) less than $f_{i,j}h$. Again, the line integral must be divided into two parts, one pertaining to the velocity in the black fluid and one pertaining to the velocity in the white fluid. The interface crosses the side between cells (i,j) and $(i+1,j)$ at the point

$((i + .5)h, (j - .5)h + .5(f_{i,j}h + b))$. Let us first estimate the part of the line integral pertaining to the white fluid. $p_{i+1,j}$ and $p_{i+1,j+1}$ are used to obtain $\hat{p}_{i+1,j}$, the pressure located at a point with the same y -coordinate as $p_{i,j}$ in the white fluid. Our objective is to find the pressure $\hat{p}_{i+1,j}$ on the line through the points $((i + 1)h, (j + 1)h)$ and $((i + .5)h + a, (j - .5)h + b)$ with vertical height $((j - .5)h + f_{i,j}h)$. These points are the coordinates associated with $p_{i+1,j+1}$ and $p_{i+1,j}$ respectively. The location of the $\hat{p}_{i+1,j}$ is given by

$$((f_{i,j}h - 1.5h) \frac{-.5h + a}{-1.5h + b} + (i + 1)h, (j - .5)h + f_{i,j}h). \quad (C44)$$

The distance between the points associated with the pressures $p_{i+1,j}$ and $p_{i+1,j+1}$ is easily computed and is denoted d_1 . The distance between the points associated with $p_{i+1,j}$ and $\hat{p}_{i+1,j}$ will be denoted by d_2 . $p_{i,j}$ is located at $(ih, (j - .5)h + f_{i,j}h)$ and the distance between its location and the location of $\hat{p}_{i+1,j}$ will be denoted by d_3 . The interpolated pressure, $\hat{p}_{i+1,j}$, is given by,

$$\hat{p}_{i+1,j} = p_{i+1,j} + d_1 \frac{(p_{i+1,j+1} - p_{i+1,j})}{d_2}. \quad (C45)$$

The line integral over the white fluid can then be approximated by,

$$\int_{RS(white)} \vec{u} \cdot \vec{n} ds \approx .5(h - f_{i,j}h + h - b) \kappa_w \frac{\hat{p}_{i+1,j} - p_{i,j}}{d_3}. \quad (C46)$$

For the part of the line integral pertaining to the black fluid, $p_{i,j}$ and $p_{i,j-1}$ are used to obtain $\tilde{p}_{i,j}$, the pressure located at a point with the same y -coordinate as $p_{i+1,j}$. The new pressure, $\tilde{p}_{i,j}$, is given by,

$$\tilde{p}_{i,j} = p_{i,j} + (b - f_{i,j}h) \frac{(p_{i,j} - p_{i,j-1})}{(f_{i,j}h + .5h)}. \quad (C47)$$

The line integral over the black fluid can then be approximated by,

$$\int_{RS(black)} \vec{u} \cdot \vec{n} ds \approx .5(f_{i,j}h + b) \kappa_b \frac{p_{i+1,j} - \tilde{p}_{i,j}}{.5h + a}. \quad (C48)$$

The complete line integral on the RS can then be approximated by summing the quantities in (C46) and (C48) and obtaining,

$$\begin{aligned} \int_{RS} \vec{u} \cdot \vec{n} ds &\approx .5(h - f_{i,j}h + h - b)\kappa_w \frac{\hat{p}_{i+1,j} - p_{i,j}}{d_3} \\ &+ .5(f_{i,j}h + b)\kappa_b \frac{p_{i+1,j} - \tilde{p}_{i,j}}{.5h + a}. \end{aligned} \quad (C49)$$

f. $(i+1, j)$ is a pseudo white finger. There is no white fluid crossing the right hand side. The pressure in $(i+1, j)$ is located at $((i + .5)h + .5hf_{i+1,j}, jh)$. The interpolation to be done is determined by whether $f_{i,j} > .5$ or $f_{i,j} < .5$. For $f_{i,j} > .5$, interpolation between the points $p_{i,j}$ and $p_{i,j-1}$ is used to get an approximation for the pressure $\tilde{p}_{i,j}$ in the black fluid with the same y -coordinate as $p_{i+1,j}$. The revised pressure, $\tilde{p}_{i,j}$ is given by

$$\tilde{p}_{i,j} = p_{i,j} + (.5h - f_{i,j}h) \frac{(p_{i,j} - p_{i,j-1})}{(f_{i,j}h + .5h)}. \quad (C50)$$

The line integral over RS which only contains black fluid is approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h\kappa_b \frac{p_{i+1,j} - \tilde{p}_{i,j}}{.5h + .5hf_{i+1,j}}. \quad (C51)$$

Alternatively, if $f_{i,j} < .5$, interpolation for $\tilde{p}_{i+1,j}$ is performed using $p_{i+1,j}$ and $p_{i,j-1}$. This interpolation is performed such that $\tilde{p}_{i+1,j}$ has the same y -coordinate as $p_{i,j}$. The pressure $\tilde{p}_{i+1,j}$ will be found on the line through the points $(ih, (j-1)h)$ and $((i + .5)h + .5hf_{i+1,j}, jh)$ with vertical height $((j - .5)h + f_{i,j}h)$. These points are the coordinates associated with $p_{i,j-1}$ and $p_{i+1,j}$ respectively. The location of the $\tilde{p}_{i+1,j}$ is given by

$$((.5f_{i,j} + .5)(.5h + f_{i,j}h) + ih, (j - .5)h + f_{i,j}h). \quad (C52)$$

The distance between the points associated with the pressures $p_{i+1,j}$ and $p_{i,j-1}$ is easily computed and will be denoted d_1 . The distance between the points associated with $p_{i+1,j}$ and $\tilde{p}_{i+1,j}$ will be denoted by d_2 . $p_{i,j}$ is located at $(ih, (j-.5)h + f_{i,j}h)$ and the distance between its location and the location of $\tilde{p}_{i+1,j}$ will be denoted by d_3 . The interpolated pressure, $\tilde{p}_{i+1,j}$, is given by,

$$\tilde{p}_{i+1,j} = p_{i+1,j} - d_2 \frac{(p_{i+1,j} - p_{i,j-1})}{d_1}. \quad (C53)$$

The line integral over the fluid can then be approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_b \frac{\tilde{p}_{i+1,j} - p_{i,j}}{d_3}. \quad (C54)$$

III. The cell (i, j) is a horizontal interface with $(i, j+1)$ full and $(i, j-1)$ empty. The following cases are equivalent to the cases in II except for the interchanging of the roles of top and bottom.

a. $(i+1, j)$ is full. This case is equivalent to IIa with roles of top and bottom interchanged. Interpolation takes place between $p_{i,j+1}$ and $p_{i+1,j}$. The new point, $\tilde{p}_{i+1,j}$, is given by

$$\tilde{p}_{i+1,j} = p_{i+1,j} - (f_{i,j} h - .5h) \frac{(p_{i,j+1} - p_{i+1,j})}{h}. \quad (C55)$$

The line integral is approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_b \frac{\tilde{p}_{i+1,j} - p_{i,j}}{(.5 + f_{i,j})h}. \quad (C56)$$

b. $(i+1, j)$ contains a black rectangle in the upper left hand corner with b (vertical height of the rectangle) greater than $f_{i,j} h$. This case is equivalent to IIb with the roles of top and bottom interchanged. Again, the line integral must be divided into two parts, one pertaining to the velocity in the black fluid and one pertaining to the velocity in the white fluid. The interface crosses the side between cells (i, j) and $(i+1, j)$ at the point $((i + .5)h, (j + .5)h - .5(f_{i,j} h + b))$. Let us first estimate the part of the line integral pertaining to the black fluid. $p_{i+1,j}$ and $p_{i,j+1}$ are used to obtain $\tilde{p}_{i+1,j}$, the pressure located at a point with the same y -coordinate as $p_{i,j}$ in the black fluid. Our objective is to find the pressure $\tilde{p}_{i+1,j}$ on the line through the points $(ih, (j + 1)h)$ and $((i + .5)h + a, (j + .5)h - b)$ with vertical height $((j + .5)h - f_{i,j} h)$. These points are the coordinates associated with $p_{i,j+1}$ and $p_{i+1,j}$ respectively. The location of the $\tilde{p}_{i+1,j}$ is given by

$$((f_{i,j} h + .5h) \frac{.5h + a}{.5h + b} + ih, (j + .5)h - f_{i,j} h), \quad (C57)$$

The distance between the points associated with the pressures $p_{i+1,j}$ and $p_{i,j+1}$ is easily

computed and will be denoted d_2 . The distance between the points associated with $p_{i+1,j}$ and $\tilde{p}_{i+1,j}$ will be denoted by d_1 . $p_{i,j}$ is located at $(ih, (j+.5)h - f_{i,j}h)$ and the distance between its location and the location of $\tilde{p}_{i+1,j}$ will be denoted by d_3 . The interpolated pressure, $\tilde{p}_{i+1,j}$, is given by,

$$\tilde{p}_{i+1,j} = p_{i+1,j} + d_1 \frac{(p_{i,j+1} - p_{i+1,j})}{d_2}. \quad (C58)$$

The line integral over the black fluid can then be approximated by,

$$\int_{RS(black)} \vec{u} \cdot \vec{n} ds \approx .5(f_{i,j}h + b)\kappa_b \frac{\tilde{p}_{i+1,j} - p_{i,j}}{d_3}. \quad (C59)$$

For the part of the line integral pertaining to the white fluid, $p_{i,j}$ and $p_{i,j-1}$ are used to obtain $\hat{p}_{i,j}$, the pressure at a point with the same y -coordinate as $p_{i+1,j}$. The new pressure, $\tilde{p}_{i,j}$, is given by,

$$\hat{p}_{i,j} = p_{i,j} - (b - f_{i,j}h) \frac{(p_{i,j} - p_{i,j-1})}{(-f_{i,j}h + 1.5h)}. \quad (C60)$$

The line integral over the white fluid can then be approximated by,

$$\int_{RS(white)} \vec{u} \cdot \vec{n} ds \approx .5(h - f_{i,j}h + h - b)\kappa_w \frac{p_{i+1,j} - \hat{p}_{i,j}}{.5h + a}. \quad (C61)$$

The complete line integral on the RS can then be approximated by summing the quantities in (C59) and (C61) and obtaining,

$$\begin{aligned} \int_{RS} \vec{u} \cdot \vec{n} ds &\approx .5(f_{i,j}h + b)\kappa_b \frac{\tilde{p}_{i+1,j} - p_{i,j}}{d_3} \\ &+ .5(h - f_{i,j}h + h - b)\kappa_w \frac{p_{i+1,j} - \hat{p}_{i,j}}{.5h + a}. \end{aligned} \quad (C62)$$

c. $(i+1,j)$ contains a black rectangle in the upper left hand corner with b (vertical height of the rectangle) less than $f_{i,j}h$. This case is equivalent to IIc with the roles of top and bottom interchanged. Again, the line integral must be divided into two parts, one pertaining to the velocity in the black fluid and one pertaining to the velocity in the white fluid. The interface crosses the side between cells (i,j) and $(i+1,j)$ at the point

$((i + .5)h, (j + .5)h - .5(f_{i,j}h + b))$. Let us first estimate the part of the line integral pertaining to the white fluid. $p_{i+1,j}$ and $p_{i+1,j-1}$ are used to obtain $\hat{p}_{i+1,j}$, the pressure at a point with the same y -coordinate as $p_{i,j}$ in the white fluid. Our objective is to find the pressure $\hat{p}_{i+1,j}$ on the line through the points $((i + 1)h, (j - 1)h)$ and $((i + .5)h + a, (j + .5)h - b)$ with vertical height $((j + .5)h - f_{i,j}h)$. These points are the coordinates associated with $p_{i+1,j-1}$ and $p_{i+1,j}$ respectively. The location of the $\hat{p}_{i+1,j}$ is given by

$$((-f_{i,j}h + 1.5h) \frac{-.5h + a}{1.5h - b} + (i + 1)h, (j + .5)h - f_{i,j}h), \quad (C63)$$

The distance between the points associated with the pressures $p_{i+1,j}$ and $p_{i+1,j-1}$ is easily computed and will be denoted d_2 . The distance between the points associated with $p_{i+1,j}$ and $\hat{p}_{i+1,j}$ will be denoted by d_1 . $p_{i,j}$ is located at $(ih, (j + .5)h - f_{i,j}h)$ and the distance between its location and the location of $\hat{p}_{i+1,j}$ will be denoted by d_3 . The interpolated pressure, $\hat{p}_{i+1,j}$, is given by,

$$\hat{p}_{i+1,j} = p_{i+1,j} - d_1 \frac{(p_{i+1,j} - p_{i+1,j-1})}{d_2}. \quad (C64)$$

The line integral over the white fluid can then be approximated by,

$$\int_{RS(white)} \vec{u} \cdot \vec{n} ds \approx .5(h - f_{i,j}h + h - b) \kappa_w \frac{\hat{p}_{i+1,j} - p_{i,j}}{d_3} \quad (C65)$$

For the part of the line integral pertaining to the black fluid, $p_{i,j}$ and $p_{i,j+1}$ are used to obtain $\tilde{p}_{i,j}$, the pressure at a point with the same y -coordinate as $p_{i+1,j}$. The new pressure, $\tilde{p}_{i,j}$, is given by,

$$\tilde{p}_{i,j} = p_{i,j} - (b - f_{i,j}h) \frac{(p_{i,j+1} - p_{i,j})}{(f_{i,j}h + .5h)}. \quad (C66)$$

The line integral over the black fluid can then be approximated by,

$$\int_{RS(black)} \vec{u} \cdot \vec{n} ds \approx .5(f_{i,j}h + b) \kappa_b \frac{p_{i+1,j} - \tilde{p}_{i,j}}{.5h + a}. \quad (C67)$$

The complete line integral on the RS can then be approximated by summing the quantities in (C65) and (C67) and obtaining,

$$\begin{aligned} \int_{RS} \vec{u} \cdot \vec{n} ds \approx & .5(h - f_{i,j}h + h - b)\kappa_w \frac{\bar{p}_{i+1,j} - p_{i,j}}{d_3} \\ & + .5(f_{i,j}h + b)\kappa_b \frac{p_{i+1,j} - \bar{p}_{i,j}}{.5h + a}. \end{aligned} \quad (C68)$$

d. $(i+1, j)$ is a pseudo white finger. This case is equivalent to IId with the roles of top and bottom interchanged. The pressure in $(i+1, j)$ is located at $((i + .5)h + .5hf_{i+1,j}, jh)$. The interpolation to be done is determined by whether $f_{i,j} > .5$ or $f_{i,j} < .5$. For $f_{i,j} > .5$, interpolation between the pressures $p_{i,j}$ and $p_{i,j+1}$ is used to get an approximation for the pressure $\bar{p}_{i,j}$ in the black fluid on the same level as $p_{i+1,j}$. The revised pressure, $\bar{p}_{i,j}$ is given by

$$\bar{p}_{i,j} = p_{i,j} - (.5h - f_{i,j}h) \frac{(p_{i,j+1} - p_{i,j})}{(f_{i,j}h + .5h)}. \quad (C69)$$

The line integral over the RS which only contains black fluid is approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h\kappa_b \frac{p_{i+1,j} - \bar{p}_{i,j}}{.5h + .5hf_{i+1,j}}. \quad (C70)$$

Alternatively, if $f_{i,j} < .5$ then interpolation for $\bar{p}_{i+1,j}$ is performed using $p_{i+1,j}$ and $p_{i,j+1}$. This interpolation is performed such that $\bar{p}_{i+1,j}$ has the same y -coordinate as $p_{i,j}$. The pressure $\bar{p}_{i+1,j}$ then will be found on the line through the points $(ih, (j+1)h)$ and $((i + .5)h + .5hf_{i+1,j}, jh)$ with vertical height $(j + .5)h - f_{i,j}h$. These points are the coordinates associated with $p_{i,j+1}$ and $p_{i+1,j}$ respectively. The location of the $\bar{p}_{i+1,j}$ is given by

$$((.5f_{i,j} + .5)(.5h + f_{i,j}h) + ih, (j + .5)h - f_{i,j}h) \quad (C71)$$

The distance between the points associated with the pressures $p_{i+1,j}$ and $p_{i,j+1}$ is easily computed and will be denoted d_2 . The distance between the points associated with $p_{i+1,j}$ and $\bar{p}_{i+1,j}$ will be denoted by d_1 . $p_{i,j}$ is located at $(ih, (j + .5)h - f_{i,j}h)$ and the distance between its location and the location of $\bar{p}_{i+1,j}$ is denoted by d_3 . The interpolated pressure, $\bar{p}_{i+1,j}$ is given by,

$$\tilde{p}_{i+1,j} = p_{i+1,j} + d_1 \frac{(p_{i,j+1} - p_{i+1,j})}{d_2}. \quad (C72)$$

The line integral over the fluid can then be approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_b \frac{\tilde{p}_{i+1,j} - p_{i,j}}{d_3}. \quad (C73)$$

IV. The cell (i, j) contains a vertical interface with $(i+1, j)$ empty. No interpolation is necessary and the line integral is approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_w \frac{p_{i+1,j} - p_{i,j}}{1.5h - f_{i,j} h} \quad (C74)$$

V. The cell (i, j) contains a black rectangle in the lower right corner.

a. $(i+1, j)$ contains a horizontal interface with $(i+1, j-1)$ full and $(i+1, j+1)$ empty and b (vertical height of the rectangle) greater than $f_{i+1,j} h$. This case is equivalent IIId with the roles of left and right interchanged. Again, the line integral must be divided into two parts, one pertaining to the velocity in the black fluid and one pertaining to the velocity in the white fluid. The interface crosses the side between cells (i, j) and $(i+1, j)$ at the point $((i + .5)h, (j - .5)h + .5(f_{i,j} h + b))$. Let us first estimate the part of the line integral pertaining to the black fluid. $p_{i,j}$ and $p_{i+1,j-1}$ are used to obtain $\tilde{p}_{i,j}$, the pressure at a point with the same y -coordinate as $p_{i+1,j}$ in the black fluid. Our objective is to find the pressure $\tilde{p}_{i,j}$ on the line through the points $((i+1)h, (j-1)h)$ and $((i + .5)h - a, (j - .5)h + b)$ with vertical height $((j - .5)h + f_{i+1,j} h)$. These points are the coordinates associated with $p_{i+1,j-1}$ and $p_{i,j}$ respectively. The location of $\tilde{p}_{i,j}$ is given by,

$$((f_{i+1,j} h + .5h) \frac{-.5h - a}{.5h + b} + (i + 1)h, (j - .5)h + f_{i+1,j} h), \quad (C75)$$

The distance between the points associated with the pressures $p_{i,j}$ and $p_{i+1,j-1}$ is easily computed and will be denoted d_2 . The distance between the points associated with $p_{i,j}$ and $\tilde{p}_{i,j}$ will be denoted by d_1 . $p_{i+1,j}$ is located at $(ih, (j-.5)h + f_{i+1,j} h)$ and the distance between

its location and the location of $\tilde{p}_{i,j}$ will be denoted by d_3 . The interpolated pressure, $\tilde{p}_{i,j}$ is given by,

$$\tilde{p}_{i,j} = p_{i,j} - d_1 \frac{(p_{i,j} - p_{i+1,j-1})}{d_2}. \quad (C76)$$

The line integral over the black fluid then can be approximated by,

$$\int_{RS(black)} \vec{u} \cdot \vec{n} ds \approx .5(f_{i+1,j} h + b) \kappa_b \frac{p_{i+1,j} - \tilde{p}_{i,j}}{d_3} \quad (C77)$$

For the part of the line integral pertaining to the white fluid, $p_{i+1,j}$ and $p_{i+1,j+1}$ are used to obtain $\hat{p}_{i+1,j}$, a point with the same y -coordinate as $p_{i,j}$. The new pressure, $\hat{p}_{i+1,j}$, is given by,

$$\hat{p}_{i+1,j} = p_{i+1,j} + (b - f_{i,j} h) \frac{(p_{i+1,j+1} - p_{i+1,j})}{(-f_{i+1,j} h + 1.5h)}. \quad (C78)$$

The line integral over the white fluid can then be approximated by,

$$\int_{RS(white)} \vec{u} \cdot \vec{n} ds \approx .5(h - f_{i+1,j} h - h - b) \kappa_w \frac{\hat{p}_{i+1,j} - p_{i,j}}{.5h + a}. \quad (C79)$$

The complete line integral on the RS can then be approximated by summing the quantities in (C77) and (C79) and obtaining,

$$\begin{aligned} \int_{RS} \vec{u} \cdot \vec{n} ds &\approx .5(f_{i+1,j} h + b) \kappa_b \frac{p_{i+1,j} - \tilde{p}_{i,j}}{d_3} \\ &+ .5(h - f_{i+1,j} h - h - b) \kappa_w \frac{\hat{p}_{i+1,j} - p_{i,j}}{.5h + a}. \end{aligned} \quad (C80)$$

b. $(i+1,j)$ contains a horizontal interface with $(i+1,j-1)$ full and b (vertical height of the rectangle) less than $f_{i+1,j} h$. This case is equivalent to IIe with the roles of left and right interchanged. Again, the line integral must be divided into two parts, one pertaining to the velocity in the black fluid and one pertaining to the velocity in the white fluid. The interface crosses the side between cells (i,j) and $(i+1,j)$ at the point $((i + .5)h, (j - .5)h + .5(f_{i+1,j} h + b))$. Let us first estimate the part of the line integral pertaining to the white fluid. $p_{i,j}$ and $p_{i,j+1}$ are used to obtain $\hat{p}_{i,j}$, the pressure at a point with the same y -coordinate as $p_{i+1,j}$ in the white fluid. Our objective is to find the pressure

$\hat{p}_{i,j}$ on the line through the points $(ih, (j+1)h)$ and $((i+.5)h - a, (j-.5)h + b)$ with vertical height $((j-.5)h + f_{i+1,j}h)$. These points are the coordinates associated with $p_{i,j+1}$ and $p_{i,j}$ respectively. The location of the $\hat{p}_{i,j}$ is given by

$$((f_{i+1,j}h - 1.5h) \frac{.5h - a}{-1.5h + b} + ih, (j-.5)h + f_{i+1,j}h), \quad (C81)$$

The distance between the points associated with the pressures $p_{i,j}$ and $p_{i,j+1}$ is easily computed and will be denoted d_2 . The distance between the points associated with $p_{i,j}$ and $\hat{p}_{i,j}$ will be denoted by d_1 . $p_{i+1,j}$ is located at $(ih, (j-.5)h + f_{i+1,j}h)$ and the distance between its location and the location of $\hat{p}_{i,j}$ will be denoted by d_3 . The interpolated pressure, $\hat{p}_{i,j}$ is given by,

$$\hat{p}_{i,j} = p_{i,j} + d_1 \frac{(p_{i,j+1} - p_{i,j})}{d_2}. \quad (C82)$$

The line integral over the white fluid can then be approximated by,

$$\int_{RS(white)} \vec{u} \cdot \vec{n} ds \approx .5(h - f_{i,j}h + h - b) \kappa_w \frac{p_{i+1,j} - \hat{p}_{i,j}}{d_3} \quad (C83)$$

For the part of the line integral pertaining to the black fluid, $p_{i+1,j}$ and $p_{i+1,j-1}$ are used to obtain $\tilde{p}_{i+1,j}$, the pressure at a point with the same y -coordinate as $p_{i,j}$. The new pressure, $\tilde{p}_{i+1,j}$, is given by,

$$\tilde{p}_{i+1,j} = p_{i+1,j} + (b - f_{i+1,j}h) \frac{(p_{i+1,j} - p_{i+1,j-1})}{(f_{i+1,j}h + .5h)}. \quad (C84)$$

The line integral over the black fluid then be can approximated by,

$$\int_{RS(black)} \vec{u} \cdot \vec{n} ds \approx .5h(f_{i+1,j}h + b) \kappa_b \frac{\tilde{p}_{i+1,j} - p_{i,j}}{.5h + a}. \quad (C85)$$

The complete line integral on the RS can then be approximated by summing the quantities in (C83) and (C85) and obtaining,

$$\begin{aligned} \int_{RS} \vec{u} \cdot \vec{n} ds &\approx .5(h - f_{i+1,j}h + h - b) \kappa_w \frac{\hat{p}_{i+1,j} - p_{i,j}}{d_3} \\ &+ .5(f_{i+1,j}h - b) \kappa_b \frac{p_{i+1,j} - \tilde{p}_{i,j}}{.5h + a}. \end{aligned} \quad (C86)$$

c. $(i+1, j)$ contains a white rectangle in the upper left corner. The dimensions of the black rectangle in (i, j) are horizontally and vertically a_1 and b_1 respectively. The white rectangle in (i, j) has horizontal and vertical dimensions a_2 and b_2 respectively. The interface crosses the side RS at $((i + .5)h, (j - .5)h + .5(b_1 + h - b_2))$. For the case with $b_1 < h - b_2$ we proceed as follows. $p_{i+1, j}$ and $p_{i+1, j-1}$ are used to obtain $\bar{p}_{i+1, j}$, the pressure with the same y -coordinate as $p_{i, j}$. $p_{i, j}$, $p_{i+1, j}$ and $p_{i+1, j-1}$ are located at $((i + .5)h - a_1, (j - .5)h + b_1)$, $((i + .5)h + a_2, (j + .5)h - b_2)$ and $((i + 1)h, (j - 1)h)$, respectively. $\bar{p}_{i+1, j}$ is located therefore at

$$((b_1 + .5h) \frac{-.5h + a_2}{1.5h - b_2} + (i+1)h, (j - .5)h + b_1), \quad (C87)$$

The distance between the points associated with the pressures $p_{i+1, j}$ and $p_{i+1, j-1}$ is easily computed and is denoted d_1 . The distance between the points associated with $p_{i+1, j}$ and $\bar{p}_{i+1, j}$ will be denoted by d_2 . The interpolated pressure $\bar{p}_{i+1, j}$ is given by,

$$\bar{p}_{i+1, j} = p_{i+1, j} + d_2 \frac{(p_{i+1, j} - p_{i+1, j-1})}{d_1}. \quad (C88)$$

The distance between the locations of $\bar{p}_{i+1, j}$ and $p_{i, j}$ is denoted by d_3 . The line integral over the black fluid can then be approximated by,

$$\int_{RS(black)} \vec{u} \cdot \vec{n} ds \approx .5(b_1 + h - b_2)\kappa_b \frac{\bar{p}_{i+1, j} - p_{i, j}}{d_3}. \quad (C89)$$

For the part of the line integral pertaining to the white fluid, $p_{i, j}$ and $p_{i, j+1}$ are used to obtain $\hat{p}_{i, j}$, the pressure with the same y -coordinate as $p_{i+1, j}$. $p_{i, j+1}$ is located at the point $(ih, (j + 1)h)$. $\hat{p}_{i, j}$ is then located at

$$((-b_2 - .5h) \frac{.5h - a_1}{-1.5h + b_1} + ih, (j + .5)h - b_2), \quad (C90)$$

The distance between the points associated with $p_{i, j+1}$ and $p_{i, j}$ is denoted by d_1 and the distance between $p_{i, j}$ and $\hat{p}_{i, j}$ is denoted by d_2 . The pressure $\hat{p}_{i, j}$ is given by

$$\hat{p}_{i, j} = p_{i, j} + d_2 \frac{(p_{i, j+1} - p_{i, j})}{d_1}. \quad (C91)$$

The line integral over the white fluid then be approximated by,

$$\int_{RS(white)} \vec{u} \cdot \vec{n} ds \approx .5(b_2 + h - b_1)\kappa_w \frac{p_{i+1,j} - \hat{p}_{i,j}}{d_4}, \quad (C92)$$

where d_4 is the distance between the location of the pressures $p_{i+1,j}$ and $\tilde{p}_{i,j}$. The complete line integral on the RS can then be approximated by summing the quantities in (C89) and (C92), obtaining

$$\begin{aligned} \int_{RS} \vec{u} \cdot \vec{n} ds &\approx .5(b_2 + h - b_1)\kappa_w \frac{p_{i+1,j} - \hat{p}_{i,j}}{d_4} \\ &+ .5(b_1 + h - b_2)\kappa_b \frac{\tilde{p}_{i+1,j} - p_{i,j}}{d_3}. \end{aligned} \quad (C93)$$

For the case with $b_1 > h - b_2$ we proceed as follows. In the black fluid, $p_{i,j}$ and $p_{i+1,j-1}$ are used to obtain $\tilde{p}_{i,j}$, the pressure with the same y -coordinate as $p_{i+1,j}$. $p_{i,j}$, $p_{i+1,j}$ and $p_{i+1,j-1}$ are located at $((i + .5)h - a_1, (j - .5)h + b_1)$, $((i + .5)h + a_2, (j + .5)h - b_2)$ and $((i + 1)h, (j - 1)h)$, respectively. $\tilde{p}_{i+1,j}$ is located therefore at

$$((-b_2 + 1.5h) \frac{-.5h - a_1}{.5h + b_1} + (i+1)h, (j + .5)h - b_2). \quad (C94)$$

The distance between the points associated with the pressures $p_{i,j}$ and $p_{i+1,j-1}$ is easily computed and will be denoted d_1 . The distance between the points associated with $p_{i,j}$ and $\tilde{p}_{i,j}$ will be denoted by d_2 . The interpolated pressure, $\tilde{p}_{i,j}$, is given by,

$$\tilde{p}_{i,j} = p_{i,j} - d_2 \frac{(p_{i,j} - p_{i+1,j-1})}{d_1}. \quad (C95)$$

The distance between the locations of $\tilde{p}_{i,j}$ and $p_{i+1,j}$ is denoted by d_3 . It is estimated that the interface crosses the side RS at $((i + .5)h, (j - .5)h + .5(b_1 + h - b_2))$. The line integral over the black fluid can then be approximated by,

$$\int_{RS(black)} \vec{u} \cdot \vec{n} ds \approx .5(b_1 + h - b_2)\kappa_b \frac{p_{i+1,j} - \tilde{p}_{i,j}}{d_3} \quad (C96)$$

For the part of the line integral pertaining to the white fluid, $p_{i+1,j}$ and $p_{i,j+1}$ are used to obtain $\hat{p}_{i+1,j}$, the pressure with the same y -coordinate as $p_{i,j}$. $\hat{p}_{i+1,j}$ is then located at

$$((b_1 - 1.5h) \frac{.5h + a_2}{-.5h - b_2} + ih, (j - .5)h + b_1). \quad (C97)$$

The distance between the points associated with $p_{i,j+1}$ and $p_{i+1,j}$ is denoted by d_1 and the

distance between $p_{i+1,j}$ and $\hat{p}_{i+1,j}$ is denoted by d_2 . The pressure $\hat{p}_{i+1,j}$ is given by

$$\hat{p}_{i+1,j} = p_{i+1,j} + d_2 \frac{(p_{i,j+1} - p_{i+1,j})}{d_1}. \quad (C98)$$

The line integral over the white fluid then be can approximated by,

$$\int_{RS(white)} \vec{u} \cdot \vec{n} ds \approx .5(b_2 + h - b_1)\kappa_w \frac{\hat{p}_{i+1,j} - p_{i,j}}{d_4}, \quad (C99)$$

where d_4 is the distance between the location of the pressures $p_{i+1,j}$ and $\tilde{p}_{i,j}$. The complete line integral on the RS can then be approximated by summing the quantities in (C96) and (C99), obtaining

$$\begin{aligned} \int_{RS} \vec{u} \cdot \vec{n} ds &\approx .5(b_2 + h - b_1)\kappa_w \frac{\hat{p}_{i+1,j} - p_{i,j}}{d_4} \\ &+ .5(b_1 + h - b_2)\kappa_b \frac{p_{i+1,j} - \tilde{p}_{i,j}}{d_3}. \end{aligned} \quad (C100)$$

V. The cell (i, j) contains a black rectangle in the upper right corner.

a. $(i+1, j)$ contains a horizontal interface with $(i+1, j+1)$ full and $(i+1, j-1)$ empty and b (vertical height of the rectangle) greater than $f_{i+1,j}h$. Again, the line integral must be divided into two parts, one pertaining to the velocity in the black fluid and one pertaining to the velocity in the white fluid. The interface crosses the side between cells (i, j) and $(i+1, j)$ at the point $((i + .5)h, (j + .5)h - .5(f_{i+1,j}h + b))$. Let us first estimate the part of the line integral pertaining to the black fluid. $p_{i,j}$ and $p_{i+1,j+1}$ are used to obtain $\tilde{p}_{i,j}$, a point with the same y -coordinate as $p_{i+1,j}$ in the black fluid. Our objective is to find the pressure $\tilde{p}_{i,j}$ on the line through the points $((i + 1)h, (j + 1)h)$ and $((i + .5)h - a, (j + .5)h - b)$ with vertical height $((j + .5)h - f_{i+1,j}h)$. These points are the coordinates associated with $p_{i+1,j+1}$ and $p_{i,j}$ respectively. $\tilde{p}_{i,j}$ is located at the point

$$((-f_{i+1,j}h - .5h) \frac{.5h + a}{.5h + b} + (i + 1)h, (j + .5)h - f_{i+1,j}h). \quad (C101)$$

The distance between the points associated with the pressures $p_{i,j}$ and $p_{i+1,j+1}$ is easily computed and will be denoted d_2 . The distance between the point associated with $p_{i,j}$ and $\tilde{p}_{i,j}$

will be denoted by d_1 . $p_{i+1,j}$ is located at $((i+.5)h, (j+.5)h - f_{i+1,j}h)$ and the distance between its location and the location of $\tilde{p}_{i,j}$ will be denoted by d_3 . The interpolated pressure, $\tilde{p}_{i,j}$, is given by,

$$\tilde{p}_{i,j} = p_{i,j} + d_1 \frac{(p_{i+1,j+1} - p_{i,j})}{d_2}. \quad (C102)$$

The line integral over the black fluid can then be approximated by,

$$\int_{RS(black)} \vec{u} \cdot \vec{n} ds \approx .5(f_{i,j}h + b)\kappa_b \frac{p_{i+1,j} - \tilde{p}_{i,j}}{d_3} \quad (C103)$$

For the part of the line integral pertaining to the white fluid, $p_{i+1,j}$ and $p_{i+1,j-1}$ are used to obtain $\hat{p}_{i+1,j}$, the pressure with the same y -coordinate as $p_{i,j}$. The new pressure, $\hat{p}_{i+1,j}$, is given by,

$$\hat{p}_{i+1,j} = p_{i+1,j} - (b - f_{i+1,j}h) \frac{(p_{i+1,j} - p_{i+1,j-1})}{(-f_{i+1,j}h + 1.5h)}. \quad (C104)$$

The line integral over the white fluid can then be approximated by,

$$\int_{RS(white)} \vec{u} \cdot \vec{n} ds \approx .5(h - f_{i,j}h - h - b)\kappa_w \frac{\hat{p}_{i+1,j} - p_{i,j}}{.5h + a}. \quad (C105)$$

The complete line integral on the RS can then be approximated by summing the quantities in (C103) and (C105) and obtaining,

$$\begin{aligned} \int_{RS} \vec{u} \cdot \vec{n} ds &\approx .5(f_{i,j}h + b)\kappa_b \frac{p_{i+1,j} - \tilde{p}_{i,j}}{d_3} \\ &+ .5(h - f_{i,j}h - h - b)\kappa_w \frac{\hat{p}_{i+1,j} - p_{i,j}}{.5h + a}. \end{aligned} \quad (C106)$$

b. $(i+1,j)$ contains a horizontal interface with $(i+1,j+1)$ full and b (vertical height of the rectangle) less than $f_{i+1,j}h$. Again, the line integral must be divided into two parts, one pertaining to the velocity in the black fluid and one pertaining to the velocity in the white fluid. The interface crosses the side between cells (i,j) and $(i+1,j)$ at the point $((i+.5)h, (j+.5)h - .5(f_{i+1,j}h + b))$. Let us first estimate the part of the line integral pertaining to the white fluid. $p_{i,j}$ and $p_{i,j-1}$ are used to obtain $\hat{p}_{i,j}$, the pressure with the same y -coordinate as $p_{i+1,j}$ in the white fluid. Our objective is to find the pressure $\hat{p}_{i,j}$ on the

line through the points $(ih, (j-1)h)$ and $((i+.5)h-a, (j+.5)h-b)$ with vertical height $((j+.5)h - f_{i+1,j}h)$. These points are the coordinates associated with $p_{i,j-1}$ and $p_{i,j}$ respectively. $\hat{p}_{i,j}$ is located at the point

$$((-f_{i+1,j}h + 1.5h) \frac{.5h-a}{1.5h-b} + ih, (j+.5)h - f_{i+1,j}h). \quad (C107)$$

The distance between the points associated with the pressures $p_{i,j}$ and $p_{i,j-1}$ is easily computed and will be denoted d_2 . The distance between the points associated with $p_{i,j}$ and $\hat{p}_{i,j}$ will be denoted by d_1 . $p_{i+1,j}$ is located at $(ih, (j+.5)h - f_{i+1,j}h)$ and the distance between its location and the location of $\hat{p}_{i,j}$ will be denoted by d_3 . The interpolated pressure, $\hat{p}_{i,j}$, is given by,

$$\hat{p}_{i,j} = p_{i,j} - d_1 \frac{(p_{i,j} - p_{i,j-1})}{d_2}. \quad (C108)$$

The line integral over the white fluid can then be approximated by,

$$\int_{RS(white)} \vec{u} \cdot \vec{n} ds \approx .5(h - f_{i,j}h + h - b) \kappa_w \frac{p_{i+1,j} - \hat{p}_{i,j}}{d_3} \quad (C109)$$

For the part of the line integral pertaining to the black fluid, $p_{i+1,j}$ and $p_{i+1,j+1}$ are used to obtain $\tilde{p}_{i+1,j}$, the pressure with the same y -coordinate as $p_{i,j}$. The new pressure, $\tilde{p}_{i+1,j}$, is given by,

$$\tilde{p}_{i+1,j} = p_{i+1,j} - (b - f_{i+1,j}h) \frac{(\tilde{p}_{i+1,j+1} - p_{i+1,j})}{(f_{i+1,j}h + .5h)}. \quad (C110)$$

The line integral over the black fluid can then be approximated by,

$$\int_{RS(black)} \vec{u} \cdot \vec{n} ds \approx .5(f_{i+1,j}h + b) \kappa_b \frac{\tilde{p}_{i+1,j} - p_{i,j}}{.5h + a} \quad (C111)$$

The complete line integral on the RS can then be approximated by summing the quantities in (109) and (111) and obtaining,

$$\begin{aligned} \int_{RS} \vec{u} \cdot \vec{n} ds &\approx .5(h - f_{i,j}h + h - b) \kappa_w \frac{\hat{p}_{i+1,j} - p_{i,j}}{d_3} \\ &+ .5(f_{i+1,j}h + b) \kappa_b \frac{\tilde{p}_{i+1,j} - p_{i,j}}{.5h + a}. \end{aligned} \quad (C112)$$

c. $(i+1, j)$ contains a white rectangle in the lower left corner. The dimensions of the black rectangle in (i, j) are horizontally and vertically a_1 and b_1 respectively. The white rectangle in (i, j) has horizontal and vertical dimensions a_2 and b_2 respectively. For the case with $b_1 < h - b_2$ we proceed as follows. In the black fluid, $p_{i+1, j}$ and $p_{i+1, j+1}$ are used to obtain $\tilde{p}_{i+1, j}$, the pressure with the same y -coordinate as $p_{i, j}$. $p_{i, j}$, $p_{i+1, j}$ and $p_{i+1, j+1}$ are located at $((i + .5)h - a_1, (j + .5)h - b_1)$, $((i + .5)h + a_2, (j - .5)h + b_2)$ and $((i + 1)h, (j + 1)h)$, respectively. $\tilde{p}_{i+1, j}$ is located therefore at

$$((-b_1 - .5h) \frac{-.5h + a_2}{-1.5h + b_2} + (i+1)h, (j + .5)h - b_1), \quad (C113)$$

The distance between the points associated with the pressures $p_{i+1, j}$ and $p_{i+1, j+1}$ is easily computed and will be denoted d_1 . The distance between the points associated with $p_{i+1, j}$ and $\tilde{p}_{i+1, j}$ will be denoted by d_2 . The interpolated pressure, $\tilde{p}_{i+1, j}$, is given by,

$$\tilde{p}_{i+1, j} = p_{i+1, j} + d_2 \frac{(p_{i+1, j+1} - p_{i+1, j})}{d_1}. \quad (C114)$$

The distance between the points associated with $\tilde{p}_{i+1, j}$ and $p_{i, j}$ is denoted by d_3 . The line integral over the black fluid can then be approximated by,

$$\int_{RS(\text{black})} \vec{u} \cdot \vec{n} ds \approx .5(b_1 + h - b_2)\kappa_b \frac{\tilde{p}_{i+1, j} - p_{i, j}}{d_3} \quad (C115)$$

For the part of the line integral pertaining to the white fluid, $p_{i, j}$ and $p_{i, j-1}$ are used to obtain $\hat{p}_{i, j}$, the pressure with the same y -coordinate as $p_{i+1, j}$. $\hat{p}_{i, j}$ is located at the point

$$((b_2 + .5h) \frac{.5h - a_1}{1.5h - b_1} + ih, (j - .5)h + b_2), \quad (C116)$$

The distance between the points associated with $p_{i, j-1}$ and $p_{i, j}$ is denoted by d_1 and the distance between $p_{i, j}$ and $\hat{p}_{i, j}$ is denoted by d_2 . The pressure $\hat{p}_{i, j}$ is given by

$$\hat{p}_{i, j} = p_{i, j} + d_2 \frac{(p_{i, j} - p_{i, j-1})}{d_1}. \quad (C117)$$

The line integral over the white fluid can then be approximated by,

$$\int_{RS(white)} \vec{u} \cdot \vec{n} ds \approx .5(b_2 + h - b_1)\kappa_w \frac{p_{i+1,j} - \tilde{p}_{i,j}}{d_4}, \quad (C118)$$

where d_4 is the distance between the location of the pressures $p_{i+1,j}$ and $\tilde{p}_{i,j}$. The complete line integral on the RS can then be approximated by summing the quantities in (C115) and (C118), obtaining

$$\begin{aligned} \int_{RS} \vec{u} \cdot \vec{n} ds &\approx .5(b_2 + h - b_1)\kappa_w \frac{p_{i+1,j} - \hat{p}_{i,j}}{d_4} \\ &+ .5(b_1 + h - b_2)\kappa_b \frac{\tilde{p}_{i+1,j} - p_{i,j}}{d_3}. \end{aligned} \quad (C119)$$

For the case with $b_1 > h - b_2$ we proceed as follows. In the black fluid, $p_{i,j}$ and $p_{i+1,j+1}$ are used to obtain $\tilde{p}_{i,j}$, the pressure with the same y -coordinate as $p_{i+1,j}$. $p_{i,j}$, $p_{i+1,j}$ and $p_{i+1,j+1}$ are located at $((i + .5)h - a_1, (j + .5)h - b_1)$, $((i + .5)h + a_2, (j - .5)h + b_2)$ and $((i + 1)h, (j + 1)h)$, respectively. $\tilde{p}_{i,j}$ is located therefore at

$$((b_2 + 1.5h) \frac{.5h + a_1}{.5h + b_1} + (i+1)h, (j - .5)h + b_2). \quad (C120)$$

The distance between the points associated with the pressures $p_{i,j}$ and $p_{i+1,j+1}$ is easily computed and will be denoted d_1 . The distance between the points associated with $p_{i,j}$ and $\tilde{p}_{i,j}$ will be denoted by d_2 . The interpolated pressure, $\tilde{p}_{i,j}$ is given by,

$$\tilde{p}_{i,j} = p_{i,j} + d_2 \frac{(p_{i+1,j+1} - p_{i,j})}{d_1}. \quad (C121)$$

The distance between the locations of $\tilde{p}_{i,j}$ and $p_{i+1,j}$ is denoted by d_3 . The interface crosses RS at $((i + .5)h, (j - .5)h + .5(b_1 + h - b_2))$. The line integral over the black fluid can then be approximated by,

$$\int_{RS(black)} \vec{u} \cdot \vec{n} ds \approx .5(b_1 + h - b_2)\kappa_b \frac{p_{i+1,j} - \tilde{p}_{i,j}}{d_3} \quad (C122)$$

For the part of the line integral pertaining to the white fluid, $p_{i+1,j}$ and $p_{i,j-1}$ are used to obtain $\hat{p}_{i+1,j}$, the pressure with the same y -coordinate as $p_{i,j}$. $\hat{p}_{i+1,j}$ is located at the point

$$((b_1 - 1.5h) \frac{.5h + a_2}{-.5h - b_2} + ih, (j + .5)h - b_1). \quad (C123)$$

The distance between the points associated with $p_{i,j-1}$ and $p_{i+1,j}$ is denoted by d_1 and the distance between $p_{i+1,j}$ and $\hat{p}_{i+1,j}$ is denoted by d_2 . The pressure, $\hat{p}_{i+1,j}$, is given by

$$\hat{p}_{i+1,j} = p_{i+1,j} + d_2 \frac{(p_{i,j+1} - p_{i+1,j})}{d_1}. \quad (C124)$$

The line integral over the white fluid can then be approximated by,

$$\int_{RS(white)} \vec{u} \cdot \vec{n} ds \approx .5(b_2 + h - b_1)\kappa_w \frac{\hat{p}_{i+1,j} - p_{i,j}}{d_4}, \quad (C125)$$

where d_4 is the distance between the location of the pressures $p_{i,j}$ and $\hat{p}_{i+1,j}$. The complete line integral on the RS can then be approximated by summing the quantities in (C122) and (C125), obtaining

$$\begin{aligned} \int_{RS} \vec{u} \cdot \vec{n} ds &\approx .5(b_2 + h - b_1)\kappa_w \frac{\hat{p}_{i+1,j} - p_{i,j}}{d_4} \\ &+ .5(b_1 + h - b_2)\kappa_b \frac{p_{i+1,j} - \tilde{p}_{i,j}}{d_3}. \end{aligned} \quad (C126)$$

VI. The cell (i,j) contains a black rectangle in the lower left corner. The rectangle has vertical and horizontal dimensions b and a respectively. No interface crosses the common side of (i,j) and $(i+1,j)$. The pressure $p_{i,j}$ is located at the point $((i-.5)h + a, (j-.5)h + b)$. The pressures $p_{i+1,j}$ and $p_{i,j+1}$ are used to obtain the pressure $\hat{p}_{i+1,j}$ with the same y -coordinate as $p_{i,j}$. $\hat{p}_{i+1,j}$ is given by

$$\hat{p}_{i+1,j} = p_{i+1,j} - (.5h - b) \frac{(p_{i,j+1} - p_{i+1,j})}{h}. \quad (C127)$$

The line integral over RS which only contains white fluid is approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_w \frac{\hat{p}_{i+1,j} - p_{i,j}}{(h - a + h - b)}. \quad (C128)$$

VII. The cell (i,j) contains a black rectangle in the upper left corner. The rectangle has vertical and horizontal dimensions b and a respectively. No interface crosses the side shared by (i,j) and $(i+1,j)$. The pressure $p_{i,j}$ is located at the point

$((i - .5)h + a, (j + .5)h - b)$. The pressures $p_{i+1,j}$ and $p_{i,j-1}$ are used to obtain the pressure $\hat{p}_{i+1,j}$ with the same y -coordinate as $p_{i,j}$. $\hat{p}_{i+1,j}$ is given by

$$\hat{p}_{i+1,j} = p_{i+1,j} + (.5h - b) \frac{(p_{i+1,j} - p_{i,j-1})}{h}. \quad (C129)$$

The line integral over the RS which only contains white fluid is approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_w \frac{\hat{p}_{i+1,j} - p_{i,j}}{(h - a + h - b)}. \quad (C130)$$

VIII. The cell (i, j) contains a white diagonal in the lower right corner. No black fluid crosses the common side between (i, j) and $(i+1, j)$. The pressure $p_{i,j}$ is located at the point $((i - .5)h + f_{i,j}h, (j + .5)h - f_{i,j}h)$. The pressures $p_{i+1,j}$ and $p_{i,j-1}$ are used to obtain the pressure $\hat{p}_{i+1,j}$ with the same y -coordinate as $p_{i,j}$. $\hat{p}_{i+1,j}$ is given by

$$\hat{p}_{i+1,j} = p_{i+1,j} + (.5h - f_{i,j}h) \frac{(p_{i+1,j} - p_{i,j-1})}{h}. \quad (C131)$$

The line integral over the RS which only contains white fluid is approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_w \frac{\hat{p}_{i+1,j} - p_{i,j}}{(h - f_{i,j}h + h - f_{i,j}h)}. \quad (C132)$$

IX. The cell (i, j) contains a white diagonal in the upper right corner. No black fluid crosses the common side between (i, j) and $(i+1, j)$. The pressure $p_{i,j}$ is located at the point $((i - .5)h + f_{i,j}h, (j - .5)h + f_{i,j}h)$. The pressures $p_{i+1,j}$ and $p_{i,j+1}$ are used to obtain the pressure $\hat{p}_{i+1,j}$ with the same y -coordinate as $p_{i,j}$. $\hat{p}_{i+1,j}$ is given by

$$\hat{p}_{i+1,j} = p_{i+1,j} - (.5h - f_{i,j}h) \frac{(p_{i,j+1} - p_{i+1,j})}{h}. \quad (C133)$$

The line integral over the RS which only contains white fluid is approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_w \frac{\hat{p}_{i+1,j} - p_{i,j}}{(h - f_{i,j}h + h - f_{i,j}h)}. \quad (C134)$$

X. Cell (i, j) contains a white vertical finger. Cell $(i+1, j)$ must be full. No interpolation is required and the line integral can be approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_b \frac{p_{i+1,j} - p_{i,j}}{(.5h + .5f_{i,j}h)}. \quad (C135)$$

XI. Cell (i, j) contains a pseudo white vertical finger. There is no white fluid crossing the right hand side.

a. Cell $(i+1, j)$ is a horizontal interface with $(i+1, j-1)$ full. The pressure in $(i+1, j)$ is located at $((i+1)h, (j-.5)h + f_{i+1,j}h)$ and in (i, j) at $((i+.5)h - f_{i,j}h/2, jh)$. The interpolation to be done is determined by whether $f_{i+1,j} > .5$ or $f_{i+1,j} < .5$. For $f_{i+1,j} > .5$, interpolation between the points associated with $p_{i+1,j}$ and $p_{i+1,j-1}$ is used to get an approximation for the pressure $\tilde{p}_{i+1,j}$ in the black fluid on the same level as $p_{i,j}$. The revised pressure, $\tilde{p}_{i+1,j}$, is given by

$$\tilde{p}_{i+1,j} = p_{i+1,j} - (.5h - f_{i+1,j}h) \frac{(p_{i+1,j} - p_{i+1,j-1})}{(f_{i+1,j}h + .5h)}. \quad (C136)$$

The line integral over the RS which only contains black fluid is approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_b \frac{\tilde{p}_{i+1,j} - p_{i,j}}{.5h + .5hf_{i,j}}. \quad (C137)$$

Alternatively, if $f_{i,j} < .5$ then interpolation for $\tilde{p}_{i,j}$ is performed using $p_{i,j}$ and $p_{i+1,j-1}$. This interpolation is performed so that $\tilde{p}_{i,j}$ has the same y -coordinate as $p_{i+1,j}$. The pressure, $\tilde{p}_{i,j}$, is found on the line through the points $((i+1)h, (j-1)h)$ and $((i+.5)h - .5hf_{i,j}, jh)$ with vertical height $((j-.5)h + f_{i+1,j}h)$. These points are the coordinates associated with $p_{i+1,j-1}$ and $p_{i,j}$ respectively. The location of the $\tilde{p}_{i,j}$ is given by

$$(-.5f_{i,j} + .5)(.5h + f_{i+1,j}h) + (i+1)h, (j-.5)h + f_{i+1,j}h \quad (C138)$$

The distance between the points associated with the pressures $p_{i,j}$ and $p_{i+1,j-1}$ is easily computed and will be denoted d_1 . The distance between the points associated with $\tilde{p}_{i,j}$ and $p_{i+1,j}$ will be denoted by d_2 . $p_{i+1,j}$ is located at $((i+1)h, (j-.5)h + f_{i+1,j}h)$ and the distance

between its location and the location of $\tilde{p}_{i,j}$ is denoted by d_3 . The interpolated pressure, $\tilde{p}_{i,j}$, is given by,

$$\tilde{p}_{i,j} = p_{i,j} - d_2 \frac{(p_{i,j} - p_{i+1,j-1})}{d_1}. \quad (C139)$$

The line integral over the fluid can then be approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_b \frac{p_{i+1,j} - \tilde{p}_{i,j}}{d_3}. \quad (C140)$$

b. Cell $(i+1, j)$ is a horizontal interface with $(i+1, j+1)$ full. The pressure in $(i+1, j)$ is located at $((i+1)h, (j+.5)h - f_{i+1,j}h)$ and in (i, j) at $((i+.5)h - f_{i,j}h/2, jh)$. The interpolation to be done is determined by whether $f_{i+1,j} > .5$ or $f_{i+1,j} < .5$. For $f_{i+1,j} > .5$, interpolation between the points $p_{i+1,j}$ and $p_{i+1,j+1}$ is used to get an approximation for the pressure $\tilde{p}_{i+1,j}$ in the black fluid with the same y -coordinate as $p_{i,j}$. The revised pressure, $\tilde{p}_{i+1,j}$, is given by

$$\tilde{p}_{i+1,j} = p_{i+1,j} - (.5h - f_{i+1,j}h) \frac{(p_{i+1,j+1} - p_{i+1,j})}{(f_{i+1,j}h + .5h)}. \quad (C141)$$

The line integral over the RS which only contains black fluid is approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_b \frac{\tilde{p}_{i+1,j} - p_{i,j}}{.5h + .5hf_{i,j}}. \quad (C142)$$

Alternatively, if $f_{i,j} < .5$ then interpolation for $\tilde{p}_{i,j}$ is performed using $p_{i,j}$ and $p_{i+1,j+1}$. This interpolation is performed so that $\tilde{p}_{i,j}$ has the same y -coordinate as $p_{i+1,j}$. The pressure, $\tilde{p}_{i,j}$, will be found on the line through the points $((i+1)h, (j+1)h)$ and $((i+.5)h - .5hf_{i,j}, jh)$ with vertical height $((j+.5)h - f_{i+1,j}h)$. These points are the coordinates associated with $p_{i+1,j+1}$ and $p_{i,j}$ respectively. The location of the $\tilde{p}_{i,j}$ is given by

$$((.5f_{i,j} + .5)(.5h + f_{i+1,j}h) + (i+1)h, (j+.5)h - f_{i+1,j}h) \quad (C143)$$

The distance between the points associated with the pressures $p_{i,j}$ and $p_{i+1,j+1}$ is easily computed and will be denoted d_1 . The distance between the points associated with $\tilde{p}_{i,j}$ and $p_{i,j}$ will be denoted by d_2 . $p_{i+1,j}$ is located at $((i+1)h, (j-.5)h + f_{i+1,j}h)$ and the distance between its location and the location of $\tilde{p}_{i,j}$ is denoted by d_3 . The interpolated pressure, $\tilde{p}_{i,j}$,

is given by,

$$\bar{p}_{i,j} = p_{i,j} - d_2 \frac{(p_{i,j} - p_{i+1,j+1})}{d_1}. \quad (C144)$$

The line integral over the fluid can then be approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_b \frac{p_{i+1,j} - \bar{p}_{i,j}}{d_3}. \quad (C145)$$

XII. (i,j) contains a white vertical fingertip. For vertical fingertips, there are two possibilities. One possibility has the cell $(i,j+1)$ full and the pressure $p_{i,j}$ located at $(ih, (j + .5)h - f_{i,j}h)$. The pressure $p_{i+1,j}$ and $p_{i,j+1}$ are used to estimate $\bar{p}_{i+1,j}$, the pressure in (i,j) with the same y -coordinate as $p_{i,j}$. This pressure, $\bar{p}_{i+1,j}$, is given by,

$$\bar{p}_{i+1,j} = p_{i+1,j} - (f_{i,j}h - .5h) \frac{(p_{i,j+1} - p_{i+1,j})}{h}. \quad (C146)$$

The line integral can then be approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_b \frac{\bar{p}_{i+1,j} - p_{i,j}}{(.5h + f_{i,j}h)}. \quad (C147)$$

There is a similar vertical fingertip obtained by interchanging the roles of top and bottom. For this case $(i,j-1)$ is full and the pressure $p_{i,j}$ is located at $(ih, (j - .5)h + f_{i,j}h)$. The interpolated pressure, $\bar{p}_{i+1,j}$, is given by,

$$\bar{p}_{i+1,j} = p_{i+1,j} + (f_{i,j}h - .5h) \frac{(p_{i+1,j} - p_{i,j-1})}{h}. \quad (C148)$$

The line integral can then be approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_b \frac{\bar{p}_{i+1,j} - p_{i,j}}{(.5h + f_{i,j}h)}. \quad (C149)$$

XIII. (i,j) contains a white horizontal fingertip with $(i+1,j)$ full. The pressure, $p_{i,j}$, is located at $((i + .5)h - f_{i,j}h, jh)$ and the line integral is approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_b \frac{p_{i+1,j} - p_{i,j}}{(.5h + f_{i,j}h)}. \quad (C150)$$

XIV. (i, j) contains a white horizontal fingertip with $(i-1, j)$ full. The pressure, $p_{i,j}$, is located at $((i-0.5)h + f_{i,j}h, jh)$ and there is no black fluid crossing the common side. For the case of $(i+1, j)$ empty the line integral is approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_w \frac{p_{i+1,j} - p_{i,j}}{(1.5h - f_{i,j}h)}. \quad (C151)$$

The only other option we have is the existence of a white horizontal finger in $(i+1, j)$. In this situation we assume that the $p_{i+1,j}$ is cell centered and the line integral is given by

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_w \frac{p_{i+1,j} - p_{i,j}}{(1.5h - f_{i,j}h)}. \quad (C152)$$

XV. (i, j) contains a white horizontal finger. The pressure is considered cell centered for this case. There is either another white horizontal finger in $(i+1, j)$, $(i+1, j)$ is empty or $(i+1, j)$ contains a white horizontal fingertip. In the first two cases the pressure is cell centered. The line integral is approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_w \frac{p_{i+1,j} - p_{i,j}}{h}. \quad (C153)$$

In the last case, our fingertip must be such that $(i+2, j)$ is full. For this case, the line integral is approximated by,

$$\int_{RS} \vec{u} \cdot \vec{n} ds \approx h \kappa_w \frac{p_{i+1,j} - p_{i,j}}{1.5h - f_{i+1,j}h}. \quad (C154)$$

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Table 6.2.1. Growth rates for interfaces of the
form $\epsilon \cos(m\pi x)$.

$\mu_w = 4, \mu_b = 1$, Surface Tension = 0, $\epsilon = .00001, \Delta t = .00001$.

m	Numerical Grid(20x20)	Numerical Grid(40x40)	Numerical Grid(60x60)	Numerical Grid(80x80)	Linearized Prediction
2	3.929	3.806	3.780	3.770	3.770
4	8.566	7.889	7.711	7.641	7.540
6	13.817	12.320	11.833	11.627	11.310
8	19.300	17.130	16.200	15.776	15.080
10	24.550	22.265	20.823	20.111	18.850
12	29.191	27.630	25.693	24.634	22.619
14	32.969	33.111	30.780	29.389	26.389
16	35.735	38.589	36.045	34.254	30.159
18	37.415	43.948	41.439	39.316	33.929

Table 6.2.2. Growth rates for interfaces of the
form $\epsilon \cos(m\pi x)$.

$\mu_w = 16, \mu_b = 1$, Surface Tension = 0, $\epsilon = .00001, \Delta t = .00001$.

m	Numerical Grid(20x20)	Numerical Grid(40x40)	Numerical Grid(60x60)	Numerical Grid(80x80)	Linearized Prediction
2	5.843	5.615	5.566	5.548	5.544
4	13.079	11.732	11.401	11.272	11.090
6	21.832	18.541	17.598	17.212	16.632
8	31.659	26.155	24.274	23.463	22.176
10	41.794	34.559	31.480	30.074	27.720
12	51.402	43.656	39.228	37.077	33.264
14	59.708	53.299	47.505	44.485	38.808
16	66.088	63.303	56.272	52.299	44.352
18	70.096	73.461	65.473	60.511	49.900

Table 6.2.3. Growth rates for interfaces of the
form $\epsilon \cos(m\pi x)$

$\mu_w = 1, \mu_b = 4$, Surface Tension = 0, $\epsilon = .00001$, $\Delta t = .00001$.

m	Numerical Grid(20x20)	Numerical Grid(40x40)	Numerical Grid(60x60)	Numerical Grid(80x80)	Linearized Prediction
2	-3.751	-3.757	-3.757	-3.756	-3.770
4	-7.406	-7.531	-7.541	-7.542	-7.540
6	-10.675	-11.234	-11.297	-11.310	-11.310
8	-13.403	-14.813	-15.015	-15.062	-15.080
10	-15.555	-18.202	-18.666	-18.787	-18.850
12	-17.175	-21.351	-22.220	-22.468	-22.619
14	-18.339	-24.226	-25.647	-26.087	-26.389
16	-19.116	-26.811	-28.923	-29.627	-30.159
18	-19.561	-29.105	-32.030	-33.072	-33.929

Table 6.2.4. Growth rates for interfaces of the
form $\epsilon \cos(m\pi x)$

$\mu_w = 1, \mu_b = 16$, Surface Tension = 0, $\epsilon = .00001$, $\Delta t = .00001$.

m	Numerical Grid(20x20)	Numerical Grid(40x40)	Numerical Grid(60x60)	Numerical Grid(80x80)	Linearized Prediction
2	-5.457	-5.508	-5.516	-5.519	-5.544
4	-10.555	-10.960	-11.032	-11.057	-11.090
6	-14.901	-16.184	-16.632	-16.524	-16.632
8	-18.388	-21.111	-21.707	-21.916	-22.176
10	-21.058	-25.667	-26.800	-27.206	-27.720
12	-23.025	-29.804	-31.667	-32.369	-33.264
14	-24.417	-33.509	-36.290	-37.382	-38.808
16	-25.338	-36.784	-40.642	-42.225	-44.352
18	-25.861	-39.647	-44.711	-46.881	-49.900

Table 6.3.1. Growth rates for interfaces of the
form $\epsilon \cos(m\pi x)$.

$\mu_w = 4, \mu_b = 1$, Surface Tension = 10, $\epsilon = .005$, $\Delta t = .00001$.

n	Numerical Grid(20x20)	Numerical Grid(40x40)	Numerical Grid(60x60)	Numerical Grid(80x80)	Linearized Prediction
2	3.525	3.432	3.176	2.944	3.435
4	5.377	4.945	4.861	4.809	4.861
6	3.338	2.524	2.341	2.318	2.268
8	-4.581	-6.139	-6.107	-6.288	-6.352
10	-20.117	-23.692	-23.218	-22.979	-23.009
12	-41.079	-52.254	-50.895	-49.879	-49.712
14	-64.794	-93.645	-91.279	-89.118	-88.470
16	-86.650	-148.693	-146.559	-142.690	-141.293
18	-101.916	-218.403	-218.141	-212.087	-210.189

Table 6.3.2. Growth rates for interfaces of the
form $\epsilon \cos(m\pi x)$.

$\mu_w = 4, \mu_b = 1$, Surface Tension = 3, $\epsilon = .005$, $\Delta t = .00001$.

m	Numerical Grid(20x20)	Numerical Grid(40x40)	Numerical Grid(60x60)	Numerical Grid(80x80)	Linearized Prediction
2	3.769	3.669	3.584	3.512	3.669
4	7.357	6.855	6.751	6.718	6.736
6	9.981	8.926	8.663	8.595	8.597
8	10.960	9.286	8.666	8.671	8.650
10	9.866	7.409	6.736	6.494	6.292
12	7.664	2.950	1.930	1.598	0.920
14	4.502	-5.013	-6.583	-6.976	-8.069
16	1.348	-17.373	-19.595	-19.873	-21.276
18	-1.316	-33.913	-37.330	-37.415	-39.306

Hele-Shaw cell

figure 1.1

figure 1.2

Figure 5.1 (Vertical Interface)

Figure 5.2 (Horizontal Interface)

Figure 5.3 (Corner)

Figure 5.4 (Black Finger)

Figure 5.5 (Discrete Divergence)

Figure 5.6 (Weak Discrete Divergence)

Figure 5.7

σ versus $\frac{\mu_w}{\mu_b}$

Figure 6.2.1

σ versus $\frac{\mu_b}{\mu_w}$

Figure 6.2.2

Figure 6.3.1

σ versus Surface Tension

Figure 6.3.2

time = 1.20

time = 2.33

time = 2.75

Figure 7.1: Surface Tension = 3, $\mu_w = 4$, $\mu_b = 1$, 30x60 Grid.

time = 1.47

time = 3.71

time = 6.83

time = 20.06

Figure 7.2: Surface Tension = 7, $\mu_w = 4$, $\mu_b = 1$, 30x60 Grid.

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