Presented at the Applied Superconductivity Conference, Palo Alto, CA, August 17 - 20, 1976

LBL-5461

A COMPUTER MODEL FOR NOISE IN THE DC SQUID

Claudia D. Tesche and John Clarke

August 1976

Prepared for the U. S. Energy Research and Development Administration under Contract W-7405-ENG-48

LBL-5461

Invited paper to be given at the Applied Superconductivity Conference at Stanford University, Palo Alto, California, August 17-20, 1976.

LBL-5461

UNIVERSITY OF CALIFORNIA

Lawrence Berkeley Laboratory
Berkeley, California

AEC Contract No. W-7405-eng-48

A COMPUTER MODEL FOR NOISE IN THE DC SQUID

Claudia D. Tesche and John Clarke

Physics Department
University of California
Berkeley, California

and

Materials and Molecular Research Division
Lawrence Berkeley Laboratory
University of California
Berkeley, California

"This work was done with support from the U.S. Energy Research and Development Administration."

0 0 0 0 4 6 0 0 7 6 0

A COMPUTER MODEL FOR NOISE IN THE DC SQUID

Claudia D. Tesche and John Clarke*

ABSTRACT

A computer model for the dc SQUID is described which predicts signal and noise as a function of various SQUID parameters. Differential equations for the voltage across the SQUID including the Johnson noise in the shunted junctions are integrated stepwise in time. Noise-rounded I-V characteristics are computed as a function of applied flux, Φ_a, and ring inductance, L. A measure of the SQUID response, $dV/d\Phi_a$, is calculated as a function of bias current. Low frequency voltage power spectral densities S_V^o computed for various Φ_a and L show considerable variation from the corresponding single junction values. The flux resolution $(S_V^o)^{1/2}/(dV/d\Phi_a)$ as a function of bias current is computed for several values of L and Φ_a. The results are in good agreement with experiment.

I. INTRODUCTION

Although dc SQUIDs[1] have been used as instruments for over a decade, no detailed calculation of their behavior or of their noise limitations has been published. We have, therefore, investigated the characteristics of the dc SQUID both with and without intrinsic noise. The main purpose of this study is to understand the optimization of the SQUID sensitivity. In the present paper, we summarize our results for the intrinsic noise limitation of a symmetric SQUID as a function of SQUID inductance and bias current. Our results are in good agreement with the noise measured by Clarke et al.[2] in cylindrical dc SQUIDs.

In due course, an expanded version of this work, which includes the effects of asymmetry in the SQUID, details of the SQUID response in the absence of noise, and a full discussion of the results, will be published elsewhere.

II. EQUATIONS AND NUMERICAL TECHNIQUES

The dc SQUID consists of two resistively shunted Josephson junctions each of critical current I_o and resistance R mounted symmetrically on a superconducting ring of inductance L (Fig. 1). The junction capacitances are assumed to be zero, so that the current-voltage characteristic is non-hysteretic. The SQUID is biased with a constant current I. $I_1(t)$ and $I_2(t)$ are the time-dependent currents flowing in the two arms, and $J(t) = [I_2(t) - I_1(t)]/2$ is the net circulating current. In this model, the currents $I_1(t)$ and $I_2(t)$ are related to the phase differences $\delta_1(t)$ and $\delta_2(t)$ across the junctions by

$$I_1 = I_o \sin \delta_1 + (V_1 + V_{N1})/R \ , \qquad (1)$$

and

$$I_2 = I_o \sin \delta_2 + (V_2 + V_{N2})/R \ . \qquad (2)$$

V_1 and V_2 are the voltages across the junctions, and V_{N1} and V_{N2} are the Johnson noise voltages associated with the shunt resistors. The phases differences δ_1 and δ_2 develop in time as

$$d\delta_1/dt = (2\,e/\hbar)V_1 \ , \qquad (3)$$

and

$$d\delta_2/dt = (2\,e/\hbar)V_2 \ . \qquad (4)$$

The net phase difference $(\delta_1 - \delta_2)$ is related to the externally applied quasistatic flux Φ_a and the circulating current J via

$$\delta_1 - \delta_2 = 2\pi(\Phi_a + LJ)/\Phi_o \ . \qquad (5)$$

The total voltage drop across the SQUID is given by

$$V = V_1 + L_1\, dI_1/dt + M\, dI_2/dt$$
$$= V_2 + L_2\, dI_2/dt + M\, dI_1/dt \ , \qquad (6)$$

where M is the mutual inductance between the two arms of the SQUID, and $L_1 = L_2$ for the symmetric SQUID. For the case of constant bias current, Eq. (6) can be simplified to

$$V = V_1 - (L/2)\, dJ/dt$$
$$= V_2 + (L/2)\, dJ/dt \ . \qquad (7)$$

Equation (7) includes the effect of the mutual inductance, even though M does not appear explicitly.

Equations (1)-(5) and (7) are used to eliminate V_1 and V_2. The final equations in dimensionless units are

$$j = (\delta_1 - \delta_2 - 2\pi\phi_a)/\pi\beta \ , \qquad (8)$$

$$2v = d\delta_1/d\theta + d\delta_2/d\theta \ , \qquad (9)$$

$$d\delta_1/d\theta = i/2 - j - \sin \delta_1 + v_{N1} \ , \qquad (10)$$

and

$$d\delta_2/d\theta = i/2 + j - \sin \delta_2 + v_{N2} \ . \qquad (11)$$

Here, $\beta = 2LI_o/\Phi_o$ is the dimensionless inductance in units of $\Phi_o/2I_o$, voltages are in units of I_oR, currents are in units of I_o, fluxes are in units of Φ_o, and times, θ, are in units of $\Phi_o/2\pi I_oR$. The Johnson noise voltages in the shunts, v_{N1} and v_{N2}, at each instant are approximated by a pair of uncorrelated voltage pulses each of duration ΔT. The amplitudes are Gaussian distributed about zero to yield a flat power spectrum for each train of pulses over the frequency range of interest. The voltage power spectral density for the noise in each shunt is $S_V^N = 4k_BTR$ in dimensioned units, and $S_v^N = 4\Gamma$ in dimensionless units. At a fixed temperature T, the value of $\Gamma = 2\pi k_BT/I_o\Phi_o$ depends solely on the value of I_o. For fixed Γ, β becomes a function of L only.

Equations (8)-(11) were integrated directly in time-steps DT on a CDC 7600 to determine $v(\theta)$ and $j(\theta)$ for the SQUID as functions of i, ϕ_a, β and Γ. We computed the time-averaged voltage, $\bar{v}$, as a function of i by averaging $v(\theta)$ over times long compared with the Josephson period. Two i-$\bar{v}$ characteristics for slightly different values of the applied flux ϕ_a were used to determine $d\bar{v}/d\phi_a$. The low frequency component, S_v^o, of the voltage power spectral density generated by $v(\theta)$

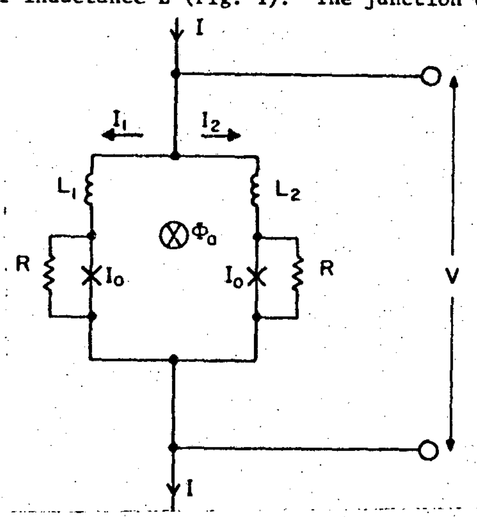

Fig. 1. Configuration of dc SQUID

Manuscript received August 17, 1976.

*Department of Physics, University of California, Berkeley, California 94720, and Materials and Molecular Research Division, Lawrence Berkeley Laboratory, Berkeley, California 94720.

was determined by averaging the values of S_v over frequencies well below the Josephson frequency.

The numerical techniques were checked in two limits. With $I_o = 0$ and $L = 0$, spectra appropriate to a resistance R/2 were obtained. For the case $L = 0$, $\phi_a = 0$, corresponding to a single resistively shunted junction, the noise rounded i-v characteristics agreed well with the calculations of Ambegaokar and Halperin[3], while the low frequency spectral densities were in good agreement with the calculations of Vystavkin *et al*[4].

III. RESULTS

The voltage $v(\theta)$ across the SQUID was computed as a function of the relevant parameters. Values for I_o, Γ, and L were selected to correspond to typical experimental values. At T = 4.2 K, a critical current of $I_o = 3.6$ μA determines a $\Gamma = 0.05$. With this value of I_o, $\beta = 1$ corresponds to L = 0.28 nH. Figure 2 shows the i-$\bar{v}$ characteristics for $\phi_a = 0$, 0.25 and 0.5. From curves of this type, we calculated the dependence of the SQUID voltage on ϕ_a for various bias currents, as shown in Fig. 3. The decrease in response as i is increased is apparent.

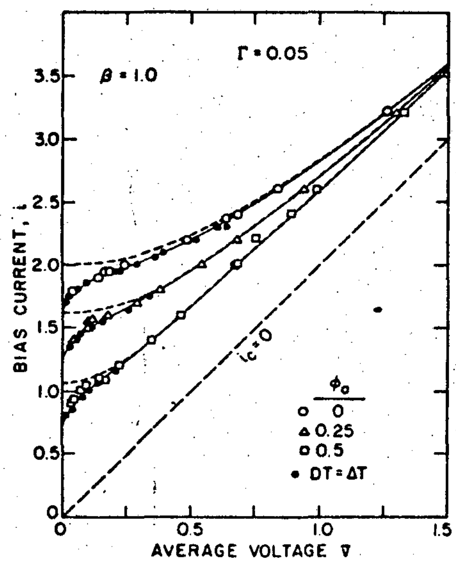

Fig. 2. i-$\bar{v}$ characteristics of dc SQUID with (—) and without (---) noise.

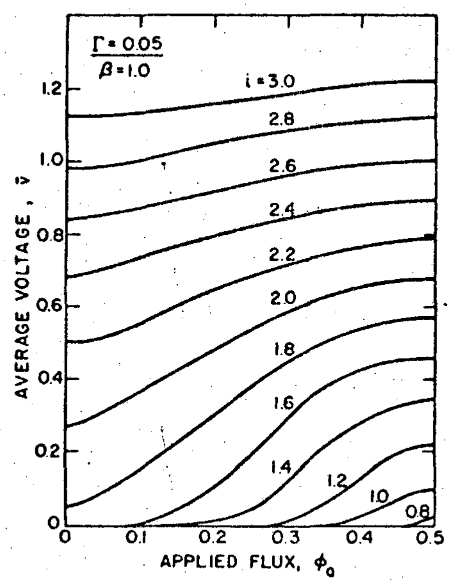

Fig. 3. $\bar{v}$ vs. ϕ_a for several values of i.

In Fig. 4 $d\bar{v}/d\phi_a$, a measure of the SQUID response, is plotted as a function of i. The peak values of $d\bar{v}/d\phi_a$ occur at bias currents near $i_c(\phi_a)$, the critical current in the absence of noise. We also computed noise-rounded i-$\bar{v}$ characteristics and values of $d\bar{v}/d\phi_a$ for several other values of β with $\Gamma = 0.05$. As is evident from Fig. 5, the peak value of $d\bar{v}/d\phi_a$ increases as β (i.e. L) is reduced.

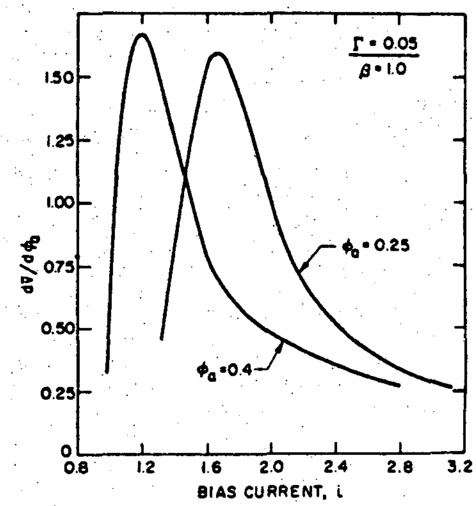

Fig. 4. $d\bar{v}/d\phi_a$ vs. i for ϕ_a = 0.25 and 0.4.

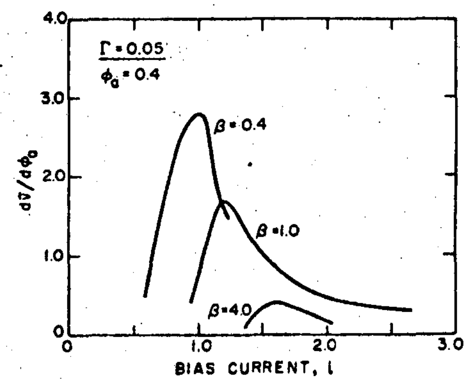

Fig. 5. $d\bar{v}/d\phi_a$ vs. i for ϕ_a = 0.4 and β = 0.4, 1.0, and 4.0.

In Fig. 6, low frequency voltage power spectra, S_v^o, are plotted as functions of i for four values of ϕ_a. For $\phi_a = 0$, the curve is close to (but not exactly equal to) the single-junction curve. As ϕ_a is increased from 0 to 0.5, the maximum value of S_v^o decreases in magnitude, and moves to a lower value of i. The decrease in the values of the peaks with increasing ϕ_a reflects the effect of the circulating current induced in the SQUID. Figure 7 shows S_v^o vs. i for three values

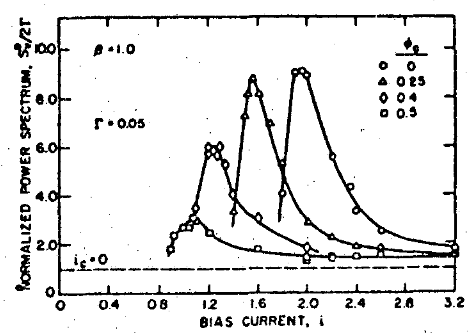

Fig. 6. $S_v^o/2\Gamma$ vs. i for ϕ_a = 0, 0.25. 0.4, 0.5.

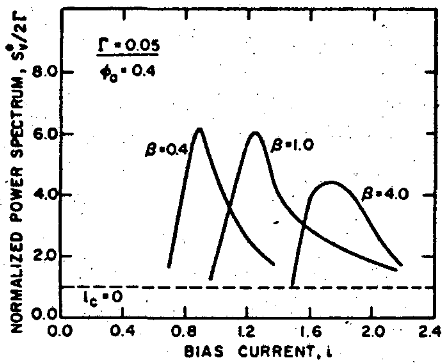

Fig. 7. $S_v^o/2\Gamma$ vs. i for ϕ_a = 0.4, and β = 0.4, 1.0, and 4.0.

of β with ϕ_a = 0.4 and Γ = 0.05.

We now use these results to estimate the flux resolution of a dc SQUID. In the simplest possible mode of operation, the SQUID is biased with a constant current i at a non-zero voltage. Changes in ϕ_a are measured by detecting the resulting changes in the voltage. No modulation flux is applied. In this case, a measure of the rms flux resolution is $S_\phi^{\frac{1}{2}} = (S_v^o/2\Gamma)^{\frac{1}{2}}/(d\bar{v}/d\phi_a)$. $S_\phi^{\frac{1}{2}}$ is plotted vs. i in Fig. 8 for β = 1 and Γ = 0.05. In Fig. 9, we show $S_\phi^{\frac{1}{2}}$ vs. i for Γ = 0.05, ϕ_a = 0.4, and β = 0.4, 1.0, and 4.0. The minimum value of S_ϕ scales approximately as β^2 (i.e. L^2). In most applications, flux is coupled into the SQUID by a superconducting coil. An appropriate figure of merit is then the energy resolution per Hz referred to the coil[2], $S_\phi/2\alpha^2(L)L$, where α(L) is the coupling coefficient between the coil and the SQUID. Since $S_\phi/2\alpha^2(L)L$ is proportional to L, we can improve the performance of the SQUID coil combination by reducing L, provided that α(L) is not also correspondingly reduced.

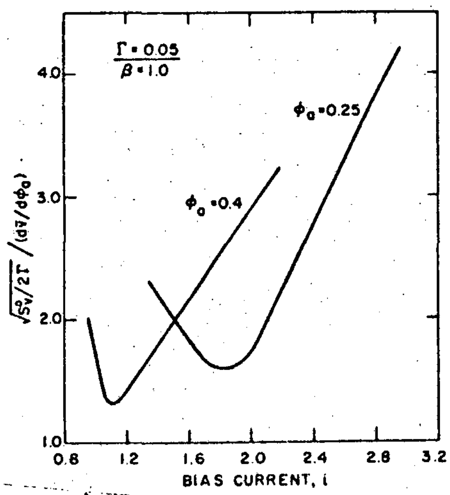

Fig. 8. $S_\phi^{\frac{1}{2}}$ vs. i for ϕ_a = 0.25 and 0.4.

For Γ = 0.05 and R = 1 Ω, $S_\phi^{\frac{1}{2}}$ is in units of approximately $3 \times 10^{-6}\ \phi_o\ Hz^{-\frac{1}{2}}$. Thus, Fig. 8 implies that, with β = 1, the optimum flux resolution is approximately $4 \times 10^{-6}\phi_o\ Hz^{-\frac{1}{2}}$. However, this resolution is not obtainable with SQUIDs operated in the conventional manner, in which an ac flux of peak-to-peak flux $\sim \phi_o/2$ is used to modulate the SQUID, and the ac voltage developed is subsequently lock-in detected. In this mode of operation, it is necessary that $\bar{v} > 0$ for all values of applied flux; consequently, i cannot be reduced to the value corresponding to the minimum value of $S_\phi^{\frac{1}{2}}$. In addition, the observed value of the flux noise is presumably an average over the modulation cycle. As an approximation we take the value of $S_\phi^{\frac{1}{2}}$ at ϕ_a = 0.25 as

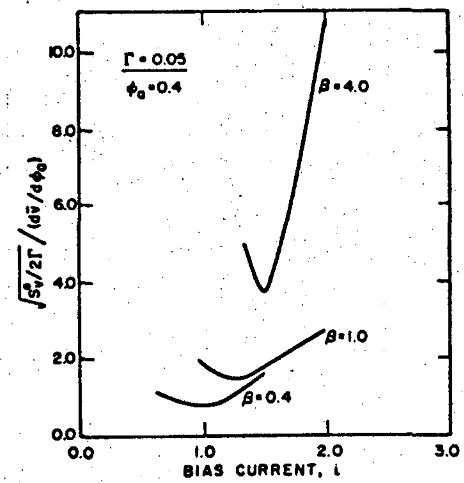

Fig. 9. $S_\phi^{\frac{1}{2}}$ vs. i for ϕ_a = 0.4 and β = 0.4, 1.0, and 4.0.

a reasonable estimate of this average.

Finally, we compare the predictions of the theory with the measured flux resolution of about $3.5 \times 10^{-5}\phi_o$ $Hz^{-\frac{1}{2}}$ obtained with the cylindrical dc SQUID of Clarke et al.[2]. The observed (noise-reduced) critical current was typically 4 μA, corresponding to $2\ I_o \approx 5$ μA, and the SQUID inductance, L, was about 1 nH. Thus, β ≈ 2.5, and Γ ≈ 0.072. The bias current was about 1.3 times the observed critical current, corresponding to i ≈ 2. For these values of β, Γ, and i, and with ϕ_a = 0.25, we compute a rms flux resolution of about $1.3 \times 10^{-5}\ \phi_o$ $Hz^{-\frac{1}{2}}$. Before comparing this result with experiment, we note that the modulation and detection scheme introduces an additional factor of $\sqrt{2}$ into the rms flux noise[2]. Therefore, the predicted resolution is about 1.8×10^{-5} $\phi_o\ Hz^{-\frac{1}{2}}$. In view of the uncertainty in the values of the experimental parameters and of the fact that the theory does not properly take into account the effects of the flux modulation, we conclude that the prediction of the flux resolution is in satisfactory agreement with the experimentally measured value.

ACKNOWLEDGEMENTS

We are grateful to Dr. W. M. Goubau and Dr. F. M. Tesche for helpful discussions. This work was performed under the auspices of U.S.E.R.D.A.

REFERENCES

1. R. C. Jaklevic, J. Lambe, A. H. Silver, and J. E. Mercereau, "Quantum Interference Effects in Josephson Tunneling", Phys. Rev. Lett., Vol. 12, pp. 159-160, 1964.

2. J. Clarke, W. Goubau, and M. Ketchen, "Thin-Film dc SQUID with Low Noise and Drift", Appl. Phys. Lett., Vol. 27, pp. 155-156, 1975; "Tunnel Junction dc SQUID: Fabrication, Operation, and Performance", J. Low Temp. Phys., Vol. 25, pp. 99-144, 1976.

3. V. Ambegaokar and B. I. Halperin, "Voltage due to Thermal Noise in the dc Josephson Effects", Phys. Rev. Lett., Vol. 22, pp. 1364-1368, 1969.

4. A. N. Vystavkin, V. N. Gubankov, L. S. Kuzmin, K. K. Likharev, V. V. Migulin, and V. K. Semenov, "S-c-S Junctions as Nonlinear Elements of Microwave Receiving Devices", Rev. Phys. Appl., Vol. 9, pp. 79-109, 1974.

TECHNICAL INFORMATION DIVISION
LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720