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Formulations of General Relativity and their Applications to Quantum Mechanical Systems (with an emphasis on gravitational waves interacting with superconductors)

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Formulations of General Relativity and their Applications to Quantum Mechanical Systems

(with an emphasis on gravitational waves interacting with superconductors)

A dissertation submitted in partial satisfaction
of the requirements for the degree of

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in

Physics

by

Nader Inan

Committee in charge:

Professor Raymond Chiao, Advisor

Professor Jay Sharping, Chair

Professor Kevin Mitchell

Professor Gerardo Muñoz

2018

The Dissertation of Nader Inan is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Raymond Chiao (advisor)

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Gerardo Muñoz

University of California, Merced

2018

Dedicated to the two greatest joys and inspirations of my life:
Athalia Grace Inan and Josiah Samuel Inan

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Acknowledgements

I have personally found that writing acknowledgments is one of the most rewarding and yet daunting parts of preparing a dissertation. It is rewarding because it provides the opportunity to express my heartfelt appreciation to all of the people in my life who have contributed in various ways to make it possible for me to achieve this goal. However, it is daunting because words can never adequately express the depth of my gratitude and the amount of credit that is due to each of these people. To me, this task seems more difficult than working with the tensor calculus of General Relativity! Nevertheless, the following is my humble attempt.

First and foremost, I am forever grateful to my mother and father for everything they have done for me throughout my entire life − raising me throughout my childhood, and supporting and encouraging me throughout my adulthood. There is no way I could describe the amount of time, energy, love, and resources they have poured into my growth and wellbeing since my very conception! Any accomplishments that I have, including earning a doctorate, I truly owe to them. Thanks mom and dad!

I am also deeply grateful to my sister who has been an incredible role model, friend, and encouragement throughout the PhD program, as well as throughout life in general. Watching her overcome severe trials in her own life, and ultimately earn a doctorate in her own field of study, has been an incredible source of inspiration to me. Not to mention, for her to attend my dissertation defense and endure hours of physics jargon is truly commendable. Thanks sis!

As stated on the dedication page of this dissertation, I am exceedingly grateful for my two kids who are the greatest source of joy and inspiration in my life! Not only do I thank them for being such a blessing to me, but I also thank them for being patient with me while I spent much of my free time trying to complete the requirements of the PhD. They watched as dad did his best to juggle work, school, and raising kids. In a sense, they went through the PhD program with me! (In fact, I can proudly say that they are probably the only kids in their elementary school who can explain what $E = mc^2$ actually means!). I love you so much, Thaly and Jojy!

On a related note, I have noticed that most individuals who earn a PhD and also have children, typically earn the PhD early in life and have children later in life. However, I am thankful that my kids are old enough to actually witness my graduation. My hope is that they might be inspired to pursue their own goals and dreams concerning their education and beyond.

Next I would like to acknowledge and thank Linda Inan who was still my wife during the first half of the PhD program. Despite all that has happened, I would like to acknowledge her for being supportive about my decision to start the PhD in 2012. We have now gone our separate ways, but I am grateful that the choice we made together at the time has now culminated in a degree that will last a lifetime.

The next category of people I would like to acknowledge are those who have influenced me in the field of physics. My interest in physics began in high school due to the inspiration of my high school physics teacher, James Kerker. Although I do not know him any longer, and he may never know that eventually I earned a PhD, I would still like to acknowledge his important contribution to inspire me at a young age. I would also like to acknowledge Dr. Dean Ayers who was the first person to mentor me in physics and hire me to work in his lab during my bachelor's degree program at CSU Long Beach. Likewise, I would like to acknowledge Dr. Frederick Ringwald who served as my thesis advisor while completing my master's degree at CSU Fresno. I truly appreciate how each of these early influences throughout my life caused me to develop a love for physics and a passion for research.

It was also during my time at CSU Fresno that I had the privilege of meeting Dr. Gerardo Muñoz who would continue to be a tremendous source of encouragement and insight during my PhD program. Dr. Muñoz has profoundly shaped my appreciation for the beauty and elegance of theoretical physics. In fact, almost all of my training in advanced topics (Relativistic Electrodynamics, Relativistic Quantum Mechanics, and General Relativity) is due to taking courses with him. As a result, his approach to physics has greatly shaped my own. I am so grateful that he agreed to be a member of my PhD committee and that he graciously agreed to teach General Relativity as an elective course for graduate students at UC Merced. I am also grateful for him sacrificing his time and effort to come to Merced for my qualifying exam as well as my dissertation defense, and for all of his ongoing contributions to our work.

Likewise, I am extremely grateful to Dr. Douglas Singleton for his support throughout my entire PhD program. I have often considered him as an "unofficial member" of my PhD committee because of his active role throughout my entire PhD experience, including coming to UC Merced for my dissertation defense. In addition to being a co-author on several of our papers, Dr. Singleton encouraged my participation at several conferences, such as the APS Far West Meetings and the Pacific Coast Gravity Meetings. In fact, he kindly invited me to give a colloquium talk at CSU Fresno, not once but twice! These talks occurred near the beginning and near the end of my PhD program, and hence served essentially as bookends to my whole PhD experience.

I am also grateful for the theory group meetings at CSU Fresno where I had the repeated opportunity to get helpful feedback on our research. In fact, it was at those meetings where I had the opportunity to meet Johnathon Thompson who became such a valuable collaborator and a dear friend. I would like to thank Johnathon for all of the helpful conversations, calculations, and visits to my home where we labored for hours preparing our work for publication.

Going back to 2006, when I initially began working at UC Merced, my first points of contact were Professor Kevin Mitchell and Professor Jay Sharping. Over the years of knowing them, my respect for their work as physicists, and my general fondness of them as individuals, naturally led me to ask them to be on my PhD committee. I am so grateful for their willingness to do so, and the tremendously valuable support and guidance they provided throughout the program. In fact, I am deeply grateful to my entire PhD committee for being so patient with me as I went through some of the most difficult circumstances of my personal life while being in the program. Their compassion and support during that difficult time truly cannot be overstated.

Of course, my greatest gratitude by far goes to my advisor, Professor Raymond Chiao, for whom I truly have the highest regard. I have intentionally saved my acknowledgement of Ray's role in my life until the end because there is so much that needs to be said! I simply cannot exaggerate the level of appreciation I have for Ray as a physicist, as a mentor, and as a friend. As a physicist, Ray is absolutely brilliant. I am continually amazed at his physical intuition and his ability to reduce complex systems to their most basic principles. I will always remember his detailed explanations which typically started with the phrase, "Now the heart of the physics here is this..."

Throughout my research, there were numerous times that I hit a difficult roadblock, yet Ray never failed to suggest insightful ideas and methods for moving forward. His creativity, ingenuity, and downright tenacity have always fueled our research efforts to find new ways to reformulate the physics and continue making progress. His promptness at providing feedback truly amazes me, sometimes answering emails within minutes! Furthermore, his ability to communicate so clearly both with words and mathematics, and his proficiency in both theoretical and experimental physics, is truly remarkable.

On a more personal level, Ray has been such an encouragement to me with his way of highlighting successes and commending achievements. Any time an important new result was produced, Ray was quick to recognize it and bring it to the attention of the rest of the research group. I often noticed how much this creates a positive environment and makes it fun to work hard and share results. In fact, Ray's friendly personality made our long theory meetings very enjoyable, despite lasting for several hours or even half of

the day! Sometimes our light-hearted joking would lead to full-blown laughter during our meetings. I will always have fond memories of those moments.

As my PhD program now comes to a close, I can honestly say that Ray helped to make it an experience that I will cherish for the rest of my life. In fact, he generously sent me on one of the most memorable trips I have ever had – a week long visit to Prague, Czech Republic, where I had the honor of speaking on his behalf and sharing our research with some of the highest quality physicists around the world. This stands out as one of the most climatic parts of the entire PhD program and I will always be grateful for it. I am also aware that I am one of the very last students whom Ray has taken as a PhD student, and I consider it such an honor and privilege to have that opportunity.

Lastly, but most of all, I would like to thank and acknowledge God for creating this marvelous universe and all of the amazing laws of physics that govern it. I am thankful that He created me and gave me the mind, passion, and enjoyment of physics that made it possible for me to write this dissertation. I also thank Him for giving me the time, resources, and opportunity to do research. Finally, I thank Him for allowing me to have all of the people in my life mentioned above, as well as my church family and many other friends who have all been so loving and supportive throughout this journey. If it was possible, I would list each and every person who has been an encouragement to me along the way, but unfortunately it would not be practical. I hope that those who were not listed here by name would still know in their hearts how much I appreciate the support and encouragement that has been shown to me.

With the deepest gratitude,
Nader Inan

Curriculum Vitae

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- "A New Approach to Detecting Gravitational Waves via the Coupling of Gravity to the Zero-point Energy of the Phonon Modes of a Superconductor," IJMPD, Vol. 26, No. 12 (2017) 1743031, http://www.worldscientific.com/doi/abs/10.1142/S0218271817430313. (Essay awarded Honorable Mention in the 2017 Essay Competition of the Gravity Research Foundation.)

- "Interaction of Gravitational Waves with Superconductors," Fortschritte der Physik, Vol. 65, Issue 6-8, (2016), http://onlinelibrary.wiley.com/doi/10.1002/prop.201600066/abstract.

- "Neutrino induced decoherence and variation in nuclear decay rates," Physics Letters A, Volume 379, Issues 12–13, 5 June 2015, Pages 941–946, https://arxiv.org/abs/1501.07665.

- "Observability of the scalar Aharonov-Bohm effect inside a 3D Faraday cage with time-varying exterior charges and masses," (2014) http://arxiv.org/abs/1411.3627.

- "A Gravitational Aharonov-Bohm Effect, and its Connection to Parametric Oscillators and Gravitational Radiation" published in Quantum Theory: A Two-Time Success Story, Yakir Aharonov Festschrift (2014), http://link.springer.com/chapter/10.1007%2F978-88-470-5217-8_14.

- "Lurching Waves in DPGraph" published in The Journal of the Acoustical Society of America (2003) http://adsabs.harvard.edu/abs/2003ASAJ..114Q2310A.

- "Viewer-controlled computer images for some basic wave phenomena" prepared for The Journal of the Acoustical Society of America (2002), http://www.csulb.edu/~rdayers/ARLOAyersInan.pdf.

Invited Colloquia and Talks

- "Formulations of General Relativity and their Applications to Quantum Mechanical Systems" (California State University, Fullerton, Physics Colloquium, April 2018)

- "Generating and Detecting Gravitational Waves via Superconductors" (California State University, Fresno, Physics Colloquium, February 2018)

- "The Interaction of Gravitational Waves with Superconductors" (Frontiers of Quantum and Mesoscopic Thermodynamics in Prague, Czech Republic, July 2015)

- "Relating Aharonov-Bohm Effects and Gravitational Waves" (Fresno State University Physics Colloquium, April 2013)

Contributed Talks

- "A New Approach to Detecting Gravitational Waves via a Charge Separation Effect in Superconductors" (34th Pacific Coast Gravity Meeting, March 2018)

- "A New Approach to Detecting Gravitational Waves via the Coupling of Gravity to the Zero-point Energy of the Phonon Modes of a Superconductor" (APS Far West Meeting, November 2017)

- "A Charge Separation Effect Induced by Gravitational Waves Incident on a Superconductor" (33rd Pacific Coast Gravity Meeting, March 2017)

- "The Interaction of Gravitational Waves with Superconductors" (APS Far West Meeting, October 2016)

- "The Coupling of Gravitational Waves to the Cooper Pair Density of a Superconductor" (31st Pacific Coast Gravity Meeting, March 2015)

- "Detection and Transmission of Gravitational Waves via Superconductors" (University of California, Merced, Physics Seminar, December 2013)

- "Kinetic Displays of Wave Behavior Applied to Acoustics" (Long Beach State Student Research Symposium, September 2002)

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Abstract

The linearization of General Relativity leads to various formulations of gravity often referred to as gravito-electromagnetism due to its resemblance to electromagnetism. Three methods are compared: (i) the harmonic gauge approach; (ii) the Parameterized Post-Newtonian (PPN) approach; and (iii) the Helmholtz Decomposition (HD) approach. New relationships are developed that are not generally found in the literature. These include the use of the linearized Bianchi identity, the Landau-Lifshitz pseudotensor, the Isaacson power formula, the geodesic equation of motion, and the geodesic deviation equation. The formalism is applied to examples such as a mass-solenoid and a gravitational mutual inductance system.

The HD approach is shown to be the most favorable of the three methods due to being gauge-invariant (to linear order in the metric), and because it shows explicitly that the transverse-traceless part of the metric contains the only radiative degrees of freedom. This is similar to the transverse-traceless (TT) *gauge* except that the HD formulation is fully valid in matter. Therefore, unlike the TT gauge, the HD formulation can be used to describe how gravitational waves interact with various types of material. Traditionally, it is believed that all known materials are essentially transparent to gravitational waves. However, this conclusion relies on a *classical* treatment which describes how gravitational waves (originating from astrophysical sources) are *passively* detected with no affect on the wave itself. As an alternative, we consider how gravitational waves could be coupled to *quantum* systems which may be used for detection as well as reflection and even generation of gravitational waves.

To investigate this possibility, a classical Hamiltonians is developed which describes the kinematics of charged, relativistic, massive particles in curved space-time. The coupling of quantum matter to gravitational fields is then described by *quantizing* the Hamiltonian. This leads to various gravitational quantum effects such gravitational Aharonov-Bohm effects, gravitational Casimir effects, and various time-holonomies. Furthermore, developing a quantized stress tensor and taking the expectation value allows the Einstein field equation to predict how quantum matter can produce classical gravitational fields. This semi-classical approach is used to describe how superconductors interact with gravitational waves. A London-like constitutive equation describes the response of the superconductor in terms of a “gravitational shear modulus” analogous to the standard shear modulus of elastic mechanics. Also using a “gravitational permeativity” (analogous to the magnetic permeability) leads to a gravitational plasma frequency, index of refraction, penetration depth, and impedance. The same analysis also done for a *normal* conductor using a gravitational Ohm-like constitutive equation, however, it is shown that a superconductor exhibits a gravitational Meissner-like effect, while a normal conductor does not.

For the case of a superconductor, the Cooper pairs are described by the Ginzburg-Landau free energy density embedded in curved spacetime. This leads to a new gravito-London gauge condition and a predicted graviton mass within the superconductor. Next, the ionic lattice is modeled by an ensemble of quantum harmonic oscillators coupled to gravitational waves and characterized by quasi-energy eigenvalues for the phonon modes. This formulation predicts a gravitationally-induced dynamical Casimir effect within the ionic lattice since the zero-point energy of the phonon modes is modulated by the gravitational wave. Applying periodic thermodynamics and the Debye model in the low-temperature limit leads to a free energy density for the ionic lattice. From these results it is shown that the response to a gravitational wave is far less for the Cooper pair density than for the ionic lattice. This predicts a charge separation effect which can be used to detect the passage of a gravitational wave, and the possibility of reflection of gravitational waves by a superconductor. Lastly, a long-range communication system is proposed based on the coupling of gravitational and electromagnetic waves via ellipsoidal superconducting cavities.

Introduction

The work in this dissertation utilizes several different areas of physics including General Relativity (GR[1]), electrodynamics, fluid mechanics, quantum mechanics, and some basic aspects of statistical mechanics and superconductivity. It is assumed that the reader has a general familiarity with these subjects. Several references are given throughout the discussion to provide the reader with supplemental resources which supply background information. Also, much of the mathematical detail is formulated "step-by-step" to enable the reader to easily follow the derivations.

Numerous appendices are included to provide what might be considered standard text book derivations (ranging from topics such as linearized GR, the ideal fluid stress tensor, the geodesic equation of motion, Lagrangians for relativistic particles in curved space-time, derivation of the London equation and penetration depth, electrodynamics in a superconductor, and other miscellaneous topics that are related to the work done in the body of this dissertation). Although many of these topics can be found in text books or published papers, some of the appendices include calculations that are unique to this dissertation and therefore are *not* found in text books (such as the Bianchi identity applied to the linearized Riemann tensor and the linearized transverse-traceless Landau-Lifshitz pseudotensor). These calculations are relegated to appendices simply because placement within the main body of the dissertation would distract from the flow of thought leading to the primary points being developed.

The general goal of this dissertation is to examine how gravitation (as described by GR) couples to quantum mechanical systems in ways that may yield experimentally testable predictions. In particular, the primary focus is to describe how gravitational waves interact with superconductors. However, a wide range of other related topics will also be covered which could lead to other experimental investigations of gravitational effects on quantum systems. Some of these other topics include gravitational Aharonov-Bohm effects, gravitational Casimir effects, and various models for coupling ultra-cold quantum systems to gravity.

Concerning the interaction of gravitational waves with superconductors, some of the most important results of this dissertation were published in [1] as well as [2]. These papers contain a condensed version of much of the material found in this dissertation.[2] Further developments concerning this topic are also currently in progress in [3]. The content pertaining to the gravitational vector and scalar Aharonov-Bohm effects has been published in [4] and [5], respectively. Lastly, an examination of the parallels between *neutrino* induced quantum decoherence and *gravitational* induced quantum decoherence has been published in [6].

One major goal of this dissertation is to develop a comprehensive framework which describes the interaction of gravity with superconductors and determines if there is a viable mechanism for gravitational wave reflection and generation in the lab. Much of the content associated with this topic is motivated by [7], [8], and [9] which claim that superconductors can be mirrors for gravitational waves. The formulation in these references begins with gravitational Maxwell-like equations which can be obtained using linearized GR in the harmonic gauge for non-relativistic sources. For this reason, Chapter 1 of this dissertation begins with a formal derivation of these gravitational Maxwell-like equations. It is found that for non-relativistic sources, the equations are not identical to Maxwell's equations, contrary to what is found in many papers on this topic. In fact, the absence of a gravitational Faraday's law leads to the conclusion that gravitational waves do not occur in this restricted framework. It may be referred to as a gravitational *magneto-static* formulation.

[1]The acronym "GR" is commonly used to refer to "General Relativity." However, in some sections of this dissertation, "GR" will be used to represent "gravitation" just as "EM" is commonly used to represent "electromagnetism."

[2]Note that [2] is an essay that was awarded Honorable Mention in the 2017 Essay Competition of the Gravity Research Foundation before being pubslied in the International Journal of Modern Physics.

In contrast to this, using linearized GR in the harmonic gauge for *relativistic* sources leads to gravitational Maxwell-like equations that contain a gravitational analog to Faraday's law. Therefore, it is possible to arrive at a formulation of gravitational waves involving *vector* fields similar to the case in electromagnetism. However, it can be shown that these "waves" do not actually propagate out with a $1/r$ dependence since the sources are the conserved mass and the conserved linear momentum. As a result, the formulation predicts that gravitational waves are necessarily *tensor* fields. However, since this approach is gauge-dependent, it leads to gravitational effects which can be made to vanish by an appropriate coordinate transformation.

In Chapter 2, it is found that the Parametrized Post-Newtonian (PPN) approach also leads to gravitational Maxwell-like equations that contain a gravitational analog to Faraday's law. However, attempting to derive a wave equation leads to the erroneous result that vector gravitational waves propagate with a speed $c/2$ in vacuum. As a result of these problems, a new approach was sought to describe gravitational waves in a gauge-invariant formulation that correctly predicts the known properties of gravitational waves, namely, that they are *tensor* waves that propagate with speed c in vacuum.

In Chapter 3, the Helmholtz Decomposition (HD) approach is introduced which provides a way to describe gravitational waves as a tensor field which is gauge-invariant (to linear order in the metric). The only requirement of the formulation is that space-time is asymptotically flat at infinity (in other words, the metric perturbation vanishes infinitely far away from all gravitational sources). This approach leads to a new set of gravito-electromagnetic Maxwell-like equations which are used throughout the remainder of the dissertation to describe how gravitational waves interact with superconductors, normal conductors, and other quantum mechanical systems.

Chapters 4 and 5 introduce equations of motion in terms of the HD metric. These include the geodesic equation, the geodesic deviation equation, and the Lorentz force in curved space-time. The results are obtained for the purpose of using them later in the dissertation to describe how gravitational waves interact with matter in general. Furthermore, chapters 6 and 7 introduce Hamiltonians which can be used to describe relativistic, charged, massive particles in the presence of curved space-time. A comparison is made between a "four-velocity invariant" Hamiltonian and a "four-momentum invariant" Hamiltonian which are each found in the literature by other authors who have disputed the correct approach to this topic. In this dissertation, it is shown in that by developing particular relations involving the metric, these Hamiltonians can all be shown to be equivalent, and therefore the discrepancies emphasized by other authors is resolved. The investigation concerning these Hamiltonians is ultimately guided by the motivation to formulate a *classical* Hamiltonian that includes the coupling of gravitational waves to matter, so that the Hamiltonian can then be *quantized* in order to describe the interaction of gravitational fields with *quantum* matter.

Chapter 8 is the first formal introduction of quantum mechanics into the dissertation. The Hamiltonians developed in previous chapters are quantized and thereby used to describe how quantum matter responds to classical gravitational fields. A quantized stress tensor for an ideal fluid is also developed in anticipation of describing how quantum matter may also generate classical gravitational fields via Einstein's field equations of General Relativity. Some of the implications of this semi-classical approach are also discussed, such as the inherent difficulties of using a *classical* field theory (GR) in conjunction with *quantum* sources of gravitation.

Chapter 9 further explores some interesting Special Relativistic and General Relativistic phenomena such as a time-dilation holonomy (due to a relativistic rotating frame) and a gravitational time-holonomy (due to a rotating mass cylinder). These phenomena are studied in order to consider their connection to related quantum phenomena such as the Aharonov-Bohm (AB) effects introduced in Chapter 10. In fact, a quantum phase in curved space-time is developed in Chapter 10 in order to describe the possibility of AB effects involving scalar, vector, and tensor gravitational fields. In this context, the role of gauge-freedom is examined, and the possibility of a gauge-invariant phase in terms of the linearized Riemann tensor is considered.

Chapter 11 introduces the topic of gravito-electromagnetic fields interacting with superconductors. In this chapter, it is shown that contrary to the work by previous authors, there is no gravitational Meissner-like effect for the gravito-magnetic field. However, there is a small correction to the electromagnetic London penetration depth due to the presence of gravito-magnetic fields in a superconductor.

Chapter 12 is arguably one of the most important chapters of the dissertation. First, the electromagnetic London equation and London penetration depth are shown to be modified by the presence of gravitational waves. Then, a gravito-London constitutive equation is used to derive a gravitational plasma frequency, index of refraction, and penetration depth. The result is that a gravitational Meissner-like effect is predicted to occur in the DC limit. By contrast, in Chapter 13 a gravito-Ohm constitutive equation is found for gravitational waves interacting with a *normal* conductor. A corresponding gravitational plasma frequency, index of refraction, and skin depth is determined for normal conductors. However, it is found that there is an *absence* of a gravitational Meissner-like effect in the DC limit.

Chapters 14, 15, and 16 deal with the interaction of gravitational waves with the Cooper pair density, ionic lattice, and electromagnetic fields of a superconductor, respectively. In each chapter, there is a formal derivation of the gravito-London constitutive equation. Chapter 13 deals specifically with the Cooper pairs by developing the Ginzburg-Landau free energy density embedded in curved space-time. The formulation leads to a gravitational shear modulus for the Cooper density which characterizes the response of the Cooper pair density to gravitational waves. The formulation also predicts a gravito-London gauge condition in a superconductor, as well as an effective graviton mass within a superconductor.

Chapter 14 deals specifically with the ionic lattice by using the Debye model in the low-temperature limit in order to obtain a free energy density embedded in curved space-time. The formulation leads to a gravitational shear modulus for the ionic lattice which characterizes the response of the ionic lattice to gravitational waves. The formulation also predicts a gravitationally induced dynamical Casimir effect where the ground state energy of the phonon modes of the ionic lattice of a superconductor is modulated by the presence of a gravitational wave. The result of the Cooper pair density and the ionic lattice responding differently to the gravitational wave leads to a charge-separation effect in the superconductor. Consequently, Chapter 15 deals specifically with the resulting electromagnetic fields arising from the charge-separation effect, and describes how these electromagnetic fields also interact with the incident gravitational wave. The Maxwell stress tensor is developed in curved space-time to describe this interaction.

Chapter 17 continues to investigate the charge separation effect as a possible mechanism for superconductors to act as "mirrors" for gravitational waves as claimed in [7], [8], and [9]. In particular, the claim is examined that superluminal supercurrents are predicted to occur in the superconductor in response to a gravitational wave. Also, a possible reformulation of the approach used in these references is suggested.

In Chapter 18, the topic of reflection and expulsion of gravitational waves is treated in further detail. A gravitational shear modulus value is determined for the reflection of microwave frequency gravitational waves to be expelled from a superconductor. Also, a complete model is formulated for the stress tensor induced in a superconductor in response to a gravitational wave. This stress tensor is then used to predict the relationship between the gravitational and electromagnetic penetration depths. Furthermore, a gravitational reflectivity and transmissivity is found by following the approach commonly used for electromagnetism for the case of linear media and the case of a gravitational conductor. Lastly, a summary is provided which lists all of the quantities and relationships commonly found in electromagnetism, and their corresponding gravitational analogs which have been developed throughout the dissertation.

In Chapter 19, boundary conditions are given for a gravitational wave field at the surface of a gravitational conductor. Next, the ratio of output to input gravitational wave power (scattering cross-section) is found for time-varying mass-quadrupole moments. Similarly, the transduction of gravitational to electromagnetic power is found by calculating the ratio of output electromagnetic to input gravitational wave power for a modified Lorentz oscillator.

Chapter 20 takes a less quantitative and more heuristic approach to considering possibilities for coupling gravity to quantum mechanical systems. The Compton wavelength and the Schwarzschild radius are discussed in terms of their relevance to systems exhibiting both quantum mechanical and strong gravitational effects. This leads to considerations of various Planck quantities (such as the Planck mass and the Planck length) as a means to describe the physical scales at which Quantum Mechanics and General Relativity are both relevant. Since the Planck length is so miniscule, it is generally believed that such investigations are completely beyond what can be explored experimentally. However, it is shown that for ultra-cold quantum systems, it is possible for quantum effects and gravitational effects to be exhibited simultaneously without the need for remaining at Planck length and mass scales.

Chapters 21 and 22 investigate the possibility of constructing a gravitational wave transmitter-receiver system consisting of small charged superconducting spheres levitated at the foci of superconducting ellipsoidal cavities. Numerical results are calculated for the power that can be transmitted and detected by such a system. Also, gravitational-electromagnetic transduction efficiency factor are determined.

Finally, Chapter 23 investigates the possibility of electromagnetic and gravitational Casimir effects using a parallel-plate waveguide configuration. The electromagnetic energy and gravitational energy are quantized and the resulting force of attraction between the plates is compared to the standard Casimir force. The Isaacson power flux formula is also used to determine the gravitational wave strain field for the quantum ground state of a cavity with a high quality factor for gravitational wave reflection. This *quantum* result is compared to the corresponding *classical* velocity and number density of Cooper pairs that would be needed within the penetration of a superconductor in order to generate the same gravitational wave energy.

The conclusion section summarizes some of the key discoveries and important achievements made in this dissertation. There is also a discussion of further research that is yet to be pursued as a natural extension of this work. As with most research, the process of answering questions often leads to even more questions that have yet to be explored!

1 Gravito-electromagnetism via the harmonic gauge (for non-relativistic sources)

1.1 Overview of the harmonic gauge for non-relativistic sources

In the following sections, we derive the gravito-electromagnetic equations for the first-order post-Newtonian fields in the harmonic gauge. (A similar treatment of this approach can also be found in [10], pp. 39-45, 51-56, as well as other references on gravito-electromagnetism.) First we describe the Einstein field equations in the trace-reversed harmonic gauge and solve them via a Green's function. We then use the stress tensor for non-relativistic dust (a pressureless ideal fluid with the highest order relativistic terms neglected). By defining a gravitational four-potential (analogous to the electromagnetic four-potential) we find that the Einstein equations in the trace-reversed harmonic gauge take the same form as the Maxwell equations in the Lorenz gauge. This enables us to define a "gravito-strength tensor" analogous to the electromagnetic strength tensor and write the field equations in covariant form. From there, we produce Maxwell-like vector field equations for the associated gravito-electric and gravito-magnetic fields. We also discuss some important features and limitations of this formulation as well as the example of a uniformly rotating mass solenoid cylinder.

1.2 The linearized Einstein equation in the harmonic gauge

We begin with a first-order perturbation to the flat space-time metric written as[3]

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \text{where} \qquad \left|h_{\mu\nu}\right| << 1 \tag{1}$$

Next, we employ the usual process for evaluating the Einstein tensor as shown in Appendix A. This involves finding the linearized Christoffel symbols (2367), Riemann tensor (2370), Ricci tenor (2372), and Ricci scalar (2375) which are obtained, respectively, as

$$\Gamma^{\mu}_{\nu\gamma} = \tfrac{1}{2}\eta^{\mu\rho}\left(\partial_\gamma h_{\rho\nu} + \partial_\nu h_{\gamma\rho} - \partial_\rho h_{\nu\gamma}\right) \tag{2}$$

$$R^{\mu}{}_{\nu\gamma\delta} = \tfrac{1}{2}\eta^{\mu\rho}\left(\partial_\gamma\partial_\nu h_{\delta\rho} - \partial_\gamma\partial_\rho h_{\nu\delta} - \partial_\delta\partial_\nu h_{\gamma\rho} + \partial_\delta\partial_\rho h_{\nu\gamma}\right) \tag{3}$$

$$R_{\nu\delta} = \tfrac{1}{2}\left(\partial^\rho\partial_\nu h_{\delta\rho} - \Box h_{\nu\delta} - \partial_\delta\partial_\nu h + \partial_\delta\partial^\gamma h_{\nu\gamma}\right) \tag{4}$$

$$R = -\Box h + \partial^\nu\partial^\gamma h_{\nu\gamma} \tag{5}$$

where h is the trace of $h_{\mu\nu}$ and

$$\Box = \partial_\mu\partial^\mu = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{6}$$

The linearized Einstein tensor can then be constructed as the trace-reversed Ricci tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}R\eta_{\mu\nu} \tag{7}$$

The result is found in (2379) as

$$G_{\mu\nu} = \tfrac{1}{2}\left(\partial^\gamma\partial_\mu h_{\gamma\nu} + \partial^\gamma\partial_\nu h_{\gamma\mu} + \eta_{\mu\nu}\Box h - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial^\rho\partial^\gamma h_{\rho\gamma}\right) \tag{8}$$

We can also define the *trace-reversed* metric perturbation as

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}h \qquad \text{where} \qquad h = \eta^{\mu\nu}h_{\mu\nu} = h_\mu{}^\mu \tag{9}$$

Then the Einstein tensor becomes

$$G_{\mu\nu} = \tfrac{1}{2}(\partial_\mu\partial^\rho\bar{h}_{\rho\nu} + \partial_\nu\partial^\rho\bar{h}_{\rho\mu} - \eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} - \partial^\rho\partial_\rho\bar{h}_{\mu\nu}) \tag{10}$$

As shown in Appendix A, the gauge freedom in linearized GR is found in (2418) to be

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu \tag{11}$$

We also show that ξ_μ can be selected so as to yield the trace-reversed harmonic gauge given in (2444) as

$$\partial^\nu\bar{h}_{\mu\nu} = 0 \tag{12}$$

[3]This dissertation will follow the sign conventions and notation of MTW [11]. The signature of the Minkowski metric is diag$(-1,+1,+1,+1)$. Greek space-time indices $\alpha,\beta,\ldots$ run from 0 to 3. Latin spatial indices $i,j,\ldots$ run from 1 to 3. Repeated indices imply a summation according to Einstein notation.

This gauge involves four constraint equations (one for each value of the index μ). Therefore it reduces the 10 components of the metric to 6 independent degrees of freedom. This gauge also simplifies the Einstein tensor (10) so that it becomes

$$G_{\mu\nu} \quad = \quad -\tfrac{1}{2}\Box\bar{h}_{\mu\nu} \tag{13}$$

Then using the Einstein field equation, $G_{\mu\nu} = \kappa T_{\mu\nu}$, in the trace-reversed harmonic gauge gives

$$\Box\bar{h}_{\mu\nu} = -2\kappa T_{\mu\nu} \tag{14}$$

where $\kappa = 8\pi G/c^4$. So we can also write this as

$$\Box\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu} \tag{15}$$

The solution to this equation is given by the following Green's function

$$\bar{h}_{\mu\nu}(t,\vec{x}) \quad = \quad -\frac{1}{4\pi}\left(-\frac{16\pi G}{c^4}\right)\int\frac{T_{\mu\nu}(t_r,\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x' \tag{16}$$

Simplifying the prefactor leads to

$$\boxed{\bar{h}_{\mu\nu}(t,\vec{x}) = \frac{4G}{c^4}\int\frac{T_{\mu\nu}(t_r,\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x' \qquad \begin{array}{c}\textit{Green's function solution for the metric}\\ \textit{perturbation in the trace-reversed harmonic gauge}\end{array}} \tag{17}$$

Here $\vec{x}'$ is the spatial coordinate of each infinitesimal element of $T_{\mu\nu}$ occupying a differential volume element d^3x. Also, $T_{\mu\nu}(t_r,\vec{x}')$ is the stress-energy-momentum contribution at $\vec{x}'$ evaluated at a retarded time t_r and located at a distance $|\vec{x}-\vec{x}'|$ from the field point where $\bar{h}_{\mu\nu}$ is measured. We can therefore express the retarded time as $t_r = t - |\vec{x}-\vec{x}'|/c$. From this expression we find that each component of $T_{\mu\nu}$ is directly related to the corresponding component of $h_{\mu\nu}$.

1.3 The linearized field equation for non-relativistic dust

The stress-energy-momentum tensor for a perfect fluid is given by

$$T^{\mu\nu} = (\rho + P/c^2)u^\mu u^\nu + Pg^{\mu\nu} \tag{18}$$

where μ is the mass density (as measured by an observer at rest with respect to the fluid) and P is the pressure. To develop the GEM framework, we impose a non-relativistic (slow moving) approximation for the sources of gravitation. For slow moving particles, the energy density due to the rest mass is much greater than any pressures: $\rho c^2 >> P$. So the pressures can be neglected and we just have

$$T^{\mu\nu} \approx \rho u^\mu u^\nu \tag{19}$$

Since $u^\mu = \gamma\left(c, v^i\right)$, then we have

$$T^{00} \approx \rho c^2, \qquad T^{0i} \approx \rho c u^i, \qquad T^{ij} \approx \rho u^i u^j \tag{20}$$

Comparing these, we can see that $T^{00} >> T^{0i} >> T^{ij}$ which means T^{ij} is the smallest contribution. For slow moving sources we have $v/c << 1$ and $v^2/c^2 \approx 0$. This means that we can consider $T^{ij} \approx 0$ which only leaves

$$T^{0\mu} \approx \rho c\left(c, u^i\right) \tag{21}$$

Since $T^{ij} \approx 0$, then (15) gives $\Box\overline{h}^{ij} \approx 0$ over all space. Assuming that $\overline{h}^{ij} = \partial_t \overline{h}^{ij} = 0$ at infinity, then such boundary conditions require that $\overline{h}^{ij} = 0$ is the unique solution. This means that the only non-zero components of $\overline{h}^{\mu\nu}$ are $\overline{h}^{0\mu}$ and $\overline{h}^{\mu 0}$ (by symmetry). To lower the indices of $\overline{h}^{0\mu}$ we can apply the metric twice:

$$\overline{h}_{0\mu} \quad = \quad g_{0\rho} g_{\sigma\mu} \overline{h}^{\rho\sigma} \tag{22}$$

Since $g_{\mu\nu} \approx \eta_{\mu\nu}$ to first order, then $h_{0\mu} = \eta_{0\rho}\eta_{\sigma\mu}\overline{h}^{\rho\sigma}$. But $\eta_{\mu\nu}$ is diagonalized so we have

$$h_{0\mu} \approx \eta_{00}\eta_{\mu\mu}\overline{h}^{0\mu} \tag{23}$$

This gives

$$h_{00} \quad \approx \quad h^{00} \qquad \text{and} \qquad h_{0i} \approx -h^{0i} \tag{24}$$

Therefore (15) is only non-trivial for one index being zero. Setting $\nu = 0$ and writing the equation with upper indices gives

$$\boxed{\Box\overline{h}^{0\mu} = -\frac{16\pi G}{c^4} T^{0\mu} \qquad \textit{Linearized field equation for non-relativistic dust}} \tag{25}$$

1.4 The gravitational four-potential and four-current density

To obtain field equations similar to electromagnetism, we first define a gravitational four-potential as

$$A_G^{\mu} \equiv \left(\frac{\tilde{\varphi}_G}{c}, \vec{h}\right) \tag{26}$$

where the subscript "g" is to indicate that this is a *gravitational* four-potential in terms of $\tilde{\varphi}_G$, the gravito-scalar potential and $\vec{h}$, the gravito-vector potential. We can also define the mass four-current density as

$$J_m^{\mu} \equiv \left(J_0, \vec{J}_m\right) \tag{27}$$

where the subscript "m" is to indicate that this is a *mass* four-current. Using $\vec{J}_m = \rho u_i$ for the mass current density vector and associating J_0 with the mass density, $J_0 = c\rho$, means we can write (27) as

$$J_m^{\mu} \quad = \quad (c\rho, \rho u_i) \tag{28}$$

In order to make use of (25), we can express A_G^{μ} and J_m^{μ} from (26) and (28) in terms of $\bar{h}^{0\mu}$ and $T^{0\mu}$, respectively. Since we know that $T^{00} = \rho c^2$ and $T^{0i} = \rho c u^i$, then we can write

$$T^{0\mu} \approx \left(\rho c^2, \rho c u_i\right) \tag{29}$$

Relating J_m^{μ} from (28) to $T^{0\mu}$ in (29) means we have

$$J_m^{\mu} = T^{0\mu}/c \tag{30}$$

We also can choose appropriate prefactors when relating A_G^{μ} and $\bar{h}^{0\mu}$. We will find that to obtain Maxwell-like equations, we must choose to define[4]

$$A_G^{\mu} \equiv -\frac{c}{4}\bar{h}^{0\mu} \tag{31}$$

Matching this to (26), we can define the (gauge-dependent) gravito-scalar potential and gravito-vector potential each in terms of the trace-reversed metric perturbation as, respectively,

$$\tilde{\varphi}_G \quad \equiv \quad -\frac{c^2}{4}\bar{h}^{00} \qquad \text{and} \qquad \vec{h} \equiv \frac{c}{4}\left(\bar{h}_{01},\ \bar{h}_{02},\ \bar{h}_{03}\right) \tag{32}$$

Solving for $T^{0\mu}$ and $\bar{h}^{0\mu}$ from (30) and (31), respectively, gives

$$T^{0\mu} = cJ_m^{\mu} \qquad \text{and} \qquad \bar{h}^{0\mu} = -\frac{4}{c}A_G^{\mu} \tag{33}$$

We can now express the field equation of (25) in terms of the gravitational four-potential and the mass four-current density in a manner analogous to the usual treatment in electromagnetism.

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[4]Alternatively, it is also possible to define $A_g^{\mu} \equiv -c\bar{h}^{0\mu}$ and use $\varepsilon_G \equiv \dfrac{1}{16\pi G}$ and $\mu_G \equiv 16\pi G/c^2$ in (35). This will still lead to Maxwell-like equations, however, there will no longer be a parallel between $\varepsilon_0 = \dfrac{1}{4\pi K}$ and $\varepsilon_G = \dfrac{1}{4\pi G}$.

1.5 The gravito-electromagnetic potential field equations

Inserting the relations in (33) into the field equation of (25) and simplifying gives

$$\Box A_G^{\mu} = \frac{4\pi G}{c^2} J_m^{\mu} \tag{34}$$

We can define the gravito-electric permittivity and gravito-magnetic permeability of free space as, respectively,

$$\varepsilon_G \equiv \frac{1}{4\pi G} \qquad \text{and} \qquad \mu_G \equiv \frac{4\pi G}{c^2} \qquad \text{where} \qquad \varepsilon_G \mu_G = 1/c^2 \tag{35}$$

These are the gravitational analogs of the electric permittivity and magnetic permeability of free space. (These quantities are discussed in much greater detail in Section 23.) Therefore, we can write (34) as

$$\Box A_G^{\mu} = \mu_G J_m^{\mu} \tag{36}$$

Recall that to arrive at (34) we used the field equation in (15) which is valid only in the trace-reversed harmonic gauge given by (12) as $\partial_\nu \overline{h}^{\mu\nu} = 0$. For $\mu = 0$, we can use $A_G^{\mu} = -\frac{c}{4}\overline{h}^{0\mu}$ from (31). Then the trace-reversed harmonic gauge for $\mu = 0$ can be written as

$$\partial_\nu A_G^{\nu} = 0 \tag{37}$$

Summing over ν gives $\partial_0 A^0 + \partial_i A^i = 0$. Then using $A_G^{\mu} = \left(\frac{1}{c}\tilde{\varphi}_G, \vec{h}\right)$ from (26) gives

$$\frac{1}{c^2}\dot{\tilde{\varphi}}_G + \nabla \cdot \vec{h} = 0 \tag{38}$$

This has the exact same form as the Lorenz gauge in electromagnetism. Then the field equation (36) and the gauge condition (37), together make up the gravito-electromagnetic framework for non-relativistic sources.

$$\boxed{\Box A_G^{\mu} = \mu_G J_m^{\mu}, \qquad \partial_\nu A_G^{\nu} = 0 \qquad \textit{Gravito-electromagnetic four-potential field equations}} \tag{39}$$

This clearly has the same form as the covariant Maxwell's equations in electromagnetism when using the Lorenz gauge. This implies that we can define gravito-electromagnetic vector fields in terms of the gravito-potentials and then formulate vector field equations which are similar to the electromagnetic Maxwell equations. This process is carried out in the next section.

However, as a final remark, it is important to recognize that $\partial_\nu A_G^{\nu} = 0$ in (39) does not completely describe the gauge conditions required to arrive at the field equation $\Box A_G^{\mu} = \mu_G J_m^{\mu}$. The complete gauge condition is given by the trace-reversed harmonic gauge, $\partial_\nu \overline{h}^{\mu\nu} = 0$, in (12). This gives essentially *four* gauge equations, one for each value of μ. We obtained $\partial_\nu A_G^{\nu} = 0$ by setting $\mu = 0$. However, using $\mu = i$ gives the *additional* condition

$$\partial_0 \overline{h}^{i0} + \partial_j \overline{h}^{ij} = 0 \tag{40}$$

By using non-relativistic sources, we set $T^{ij} \approx 0$ and therefore from (17) we have $h^{ij} \approx 0$. Then (40) means $\partial_0 \overline{h}^{i0} \approx 0$ and therefore $\overline{h}^{i0}$ is static. Since $\vec{h}$ is defined in terms of $\overline{h}_{0i}$ in (32), then $\vec{h}$ must be a static vector field in this approximation. Therefore, if we write (39) with $\mu = 0$ and $\mu = i$ separately, we must recognize that the equation with $\mu = i$ should really be a Poisson equation, not a wave equation. Furthermore, using $\partial_\nu \overline{h}^{\mu\nu} = 0$ with $\mu = 0$ leads to

$$\partial_0 \overline{h}^{00} + \partial_j \overline{h}^{0j} = 0 \tag{41}$$

Taking ∂_0 of this expression gives

$$\partial_0^2 h^{00} + \partial_j \partial_0 \overline{h}^{0j} = 0 \tag{42}$$

Since $\partial_0 \overline{h}^{i0} \approx 0$ then $\partial_0^2 h^{00} \approx 0$ as well. This means that the scalar potential field equation also cannot be written with a box operator. Therefore, we have

$$\nabla^2 A_G^0 = \mu_G J_m^0 \qquad \text{and} \qquad \nabla^2 A_G^i = \mu_G J_m^i \tag{43}$$

We can use $A_G^\mu = \left(\frac{1}{c}\tilde{\varphi}_G, \vec{h}\right)$ from (26) as well as $J_m^\mu = \left(c\rho, \vec{J}_m\right)$ from (27) and (28). Also, since $\mu_G c^2 = 1/\varepsilon_G$ from (35) then we have

$$\boxed{\nabla^2 \tilde{\varphi}_G = \rho/\varepsilon_G, \qquad \nabla^2 \vec{h} = \mu_G \vec{J}_m \qquad \textit{Gravito-electromagnetic potential field equations}} \tag{44}$$

Note that in electromagnetism, we have

$$\Box\varphi = -\rho/\varepsilon_0, \qquad \Box\vec{A} = -\mu_0 \vec{J} \tag{45}$$

which is often mistakenly assumed to be true for gravitation (in the non-relativistic limit) as well. This error is easily made since it follows naturally from (39) if one does not carefully consider the additional gauge condition in (40) combined with the non-relativistic approximation $h^{ij} \approx 0$. This error is found to occur in equation 3.11 and table 3.1 of [10].

1.6 The gravito-electromagnetic Maxwell-like field equations

We can define $\tilde{\vec{E}}_G$ as the gravito-electric field (the standard Newtonian gravitational field) and $\tilde{\vec{B}}_G$ as the gravito-magnetic field (also known as the Lense-Thirring field).[5] Describing $\tilde{\vec{E}}_G$ and $\tilde{\vec{B}}_G$ in terms of $\tilde{\varphi}_G$ and $\vec{h}$ gives[6]

$$\tilde{\vec{E}}_G \equiv -\nabla\tilde{\varphi}_G \qquad \text{and} \qquad \tilde{\vec{B}}_G \equiv \nabla\times\vec{h} \tag{46}$$

We can also define a "gravito-electromagnetic strength" tensor as

$$F_G^{\mu\nu} \equiv \partial^\mu A_G^\nu - \partial^\nu A_G^\mu \tag{47}$$

where the superscript "g" again indicates that these are gravitational quantities. Since this is an anti-symmetric tensor, then $F_G^{\mu\nu} = -F_G^{\nu\mu}$. The components are given by

$$F_G^{0i} = \frac{1}{c}\tilde{E}_G^i, \qquad F_G^{ij} = \varepsilon^{ijk}\tilde{B}_G^k, \qquad F_G^{\mu\mu} = 0 \tag{48}$$

We can also define the dual of $F_G^{\mu\nu}$ as $G_G^{\mu\nu}$ obtained by the usual transformation: $\tilde{\vec{E}}_G/c \to \tilde{\vec{B}}_G$ and $\tilde{\vec{B}}_G \to -\tilde{\vec{E}}_G/c$. Therefore the components of $G_G^{\mu\nu}$ are

$$G_G^{0i} = \tilde{B}_G^i, \qquad G_G^{ij} = -\frac{1}{c}\varepsilon^{ijk}\tilde{E}_G^k, \qquad G_G^{\mu\mu} = 0 \tag{49}$$

Then we can write the field equations as

$$\boxed{\partial_\nu F_G^{\mu\nu} = -\mu_G J_m^\mu, \quad \partial_\nu G_G^{\mu\nu} = 0 \qquad \textit{Gravito-electromagnetic strength tensor field equations}} \tag{50}$$

Notice that unlike electromagnetism, the sign in the first field equation is negative in order to recover the correct gravito-Gauss law which leads to the correct Newtonian gravitational force. Similar to the usual process in electromagnetism, we can obtain "Maxwell-like" field equations by evaluating the field equations in (50) for spatial and temporal indices, and then substituting in terms of the gravito-electromagnetic vector fields in (46). For $\mu = 0$, we have

$$\begin{aligned} \partial_0 F_G^{00} + \partial_i F_G^{0i} &= -\mu_G J_m^0, \qquad & \partial_0 G_G^{00} + \partial_i G_G^{0i} = 0 \qquad (51)\\ \partial_i \tilde{E}_G^i/c &= -\mu_G \rho c, \qquad & \partial_i \tilde{B}_G^i = 0 \qquad (52)\\ \nabla\cdot\tilde{\vec{E}}_G &= -\rho/\varepsilon_G, \qquad & \nabla\cdot\tilde{\vec{B}}_G = 0 \qquad (53)\end{aligned}$$

Here we obtain Gauss's law for the gravito-electric field and gravito-magnetic fields.

[5]We use a tilda on the gauge-*dependent* gravito-electric and gravito-magnetic fields in order to distinguish them from the gauge-*invariant* gravito-electric and gravito-magnetic fields which are defined later in (352) of Part IV.

[6]Ordinarily we would define the gravito-electric field as $\tilde{\vec{E}}_g \equiv -\nabla\tilde{\varphi}_g - \partial_t\vec{h}$ in analogy with electromagnetism. However, as shown in the previous section, in this approximation we have $\partial_t\vec{h} \approx 0$. As a result, (47) should really be written as $F_g^{0i} = -\partial^i A^0$, $F_g^{ij} = \partial^i A^j - \partial^j A^i$. Otherwise, evaluating (47) for $\mu = 0$, $\nu = i$, and inserting $A_g^i \equiv \vec{h}$ and $F_g^{0i} = \frac{1}{c}\tilde{E}_g^i$ will give $\tilde{E}_g^i = -\partial_t h^i - \partial^i\phi$.

For $\mu = i$, (50) gives

$$\partial_0 F_G^{i0} + \partial_j F_G^{ij} = -\mu_G J_m^i, \qquad \partial_0 G_G^{i0} + \partial_j G_G^{ij} = 0 \tag{54}$$

$$\frac{1}{c}\partial_t\left(-\tilde{E}_G^i/c\right) + \partial_j\left(\varepsilon^{ijk}\tilde{B}_G^k\right) = -\mu_G J_m^i, \qquad \frac{1}{c}\partial_t\left(-\tilde{B}_G^i\right) + \partial_j\left(-\varepsilon^{ijk}\tilde{E}_G^k/c\right) = 0 \tag{55}$$

We can write $\varepsilon_{ijk}\partial_j$ as the curl of a vector to obtain the following vector equations.

$$-\frac{1}{c^2}\partial_t\tilde{\tilde{\vec{E}}}_G + \nabla\times\tilde{\tilde{\vec{B}}}_G = -\mu_G\vec{J}_m, \qquad \partial_t\tilde{\tilde{\vec{B}}}_G + \nabla\times\tilde{\tilde{\vec{E}}}_G = 0 \tag{56}$$

$$\nabla\times\tilde{\tilde{\vec{B}}}_G = -\mu_G\vec{J}_m + \frac{1}{c^2}\partial_t\tilde{\tilde{\vec{E}}}_G, \qquad \nabla\times\tilde{\tilde{\vec{E}}}_G = -\partial_t\tilde{\tilde{\vec{B}}}_G \tag{57}$$

Here we obtain a gravito-Ampere law and a gravito-Faraday law. However, in (46) we defined $\vec{E}_G \equiv -\nabla\tilde{\varphi}_G$, so taking the curl leads to $\nabla\times\vec{E}_G \approx 0$. Hence there is no gravito-Faraday law in this approximation. Then we find that (47) yields the following results.

$$\boxed{\begin{array}{ll} \nabla\cdot\tilde{\tilde{\vec{E}}}_G = -\rho/\varepsilon_G & \nabla\cdot\tilde{\tilde{\vec{B}}}_G = 0 \\ \nabla\times\tilde{\tilde{\vec{E}}}_G = 0 & \nabla\times\tilde{\tilde{\vec{B}}}_G = \mu_G\left(-\vec{J}_m + \varepsilon_G\partial_t\tilde{\tilde{\vec{E}}}_G\right) \end{array}} \tag{58}$$

Gravito-electromagnetic field equations
in the trace-reversed harmonic gauge (v/c sources)

These are the gravito-electromagnetic "Maxwell-like" equations in the trace-reversed harmonic gauge. They are only valid for what may be referred to as gravitational "magneto-statics." In other words, they are only valid for non-time varying gravito-magnetic fields and hence we see the absence of a gravito-Faraday law. As a result, we do not obtain field equations that have the exact same form as the Maxwell equations. This conclusion is also obtained by Clark and Tucker [12] where the curl of the gravito-electric field is zero in their in equation (4.44b), and the time rate of change of the gravitational vector potential is zero in equation (4.44b).[7] On the other hand, the error of including a gravito-Faraday law is found to occur in equation (40b) of [7], equation (14.2) in [4], and table 3.2 of [10], as well as several other papers in the literature where the "Maxwell-like" equations (for non-relativistic sources) show the gravito-Faraday law appearing.

In the case of [10], Thorsrud states[8] that "to formulate the equation of motion in terms of $\mathbf{E}_g$ and $\mathbf{B}_g$ one must assume that the magnetic field is stationary, ie. $\frac{\partial\hat{A}}{\partial t} = 0$. . . . I will show in a detailed manner that this is not due to our choice of definitions for $\hat{\phi}$, $\hat{\mathbf{A}}$, $\mathbf{E}_g$ and $\mathbf{B}_g$. It is not possible to define these variable such that both the field equations *and* the equation of motion can be formulated in terms of $\mathbf{E}_g$ and $\mathbf{B}_g$ without assuming a stationary $\hat{\mathbf{A}}$ field."

[7] Alternatively, Clark and Tucker show in [12], section 7, pp. 17-19 that there is a different gauge choice in equation (7.2) that will lead to field equations identical to Maxwell in equation (7.13). However, they point out that in the "gravito-electromagnetic limit," the gravito-Faraday law still vaishes in equation (7.15).

[8] Thorsrud uses $\hat{\phi}$, $\hat{\mathbf{A}}$, $\mathbf{E}_g$ and $\mathbf{B}_g$ instead of our corresponding $\tilde{\varphi}_g$, $\vec{h}$, $\tilde{\tilde{\vec{E}}}_g$, and $\tilde{\tilde{\vec{B}}}_g$.

In actuality, the inconsistency pointed out by Thorsrud is not due to a discrepancy between the field equations and the equation of motion. It is due to the additional gauge condition in (40) combined with the non-relativistic approximation $h^{ij} \approx 0$. This requires that the gravito-electric field would be defined as $\mathbf{E}_g \equiv -\nabla\hat{\phi}$ for *both* the equation of motion *and* the field equations. Thorsrud discovered the condition that $\partial_t \vec{h} = 0$ while deriving the equation of motion but *not* while deriving the field equations and therefore interpreted the result as a discrepancy between the equation of motion and the field equations.

Recognizing that $\mathbf{E}_g \equiv -\nabla\hat{\phi}$ for all the equations also eliminates the need for the entire analysis in section 3.2 of [10]. There Thorsrud shows that one cannot obtain consistent prefactors for the definition of $\hat{\phi}$, $\hat{\mathbf{A}}$, $\mathbf{E}_g$ and $\mathbf{B}_g$ in terms of the metric if one includes $\frac{\partial \hat{A}}{\partial t}$ in the definition of $\mathbf{E}_g$ from equation (3.24) as well as using the gravitational Lorentz gauge condition from equation (3.24), and field equations that include a gravito-Faraday law in equation (3.26). In particular, he begins by defining

$$\bar{h}_{00} = -l\frac{4\hat{\phi}}{c^2}, \qquad \bar{h}_{0i} = \frac{\kappa \hat{A}_i}{c^2}, \qquad \mathbf{E}_g = -\nabla\hat{\phi} - \lambda\frac{\partial \hat{\mathbf{A}}}{\partial t}, \qquad \mathbf{B}_g = \nabla \times \hat{\mathbf{A}} \tag{59}$$

where he sets $l = 1$ to ensure $\hat{\phi} = -GM/r$ outside a spherically symmetric mass distribution, and $\kappa = 4c$ to ensure the harmonic gauge condition corresponds to the Lorenz gauge in electromagnetism: $\frac{1}{c^2}\partial_t\hat{\phi} - \nabla\cdot\hat{A} = 0$. He also indicates that $\lambda = 1$ ensures that $\mathbf{E}_g$ corresponds to the same definition in electromagnetism. However, he finds that the field equations and gauge condition lead to $\lambda = \frac{\kappa}{4lc}$ in his equation (3.29), while the equation of motion (3.37) leads to $\lambda = \frac{\kappa}{lc}$. We would argue here that the appropriate value for λ in this context is simply $\lambda = 0$. This gives the correct definition of the field, $\mathbf{E}_g = -\nabla\hat{\phi}$, and completely removes the inconsistency described above.

Finally, we note that in order to have a gravito-Faraday law, we must allow sources with velocities of order $(v/c)^2$ so that $h_{ij} \neq 0$ and therefore $\partial\vec{h}/\partial t \neq 0$. However, this will result in a change in the gravitational Gauss's law and Ampere-Maxwell law. This can be seen from the other sets of gravito-electromagnetic field equations derived in other sections, such as the field equations for *relativistic* sources in the harmonic gauge, the PPN formalism, or the gauge-invariant Helmholtz Decomposition formulation.

1.7 Mass-current conservation and the gravito-displacement current

We also note that the equations in (58) are only valid for *non*-time varying (or "steady-state") mass currents. This can be observed by considering the linearized conservation of energy-momentum found in (2394) of Appendix A as $\partial_\nu T^{\mu\nu} = 0$. For $\mu = i$, we have

$$\partial_0 T^{i0} + \partial_j T^{ij} = 0 \tag{60}$$

Since we are working with non-relativistic sources, then $T^{ij} \approx 0$ and therefore we have $\partial_0 T^{i0} = 0$. Using (33) to express this in terms of the four-current gives $\dot{\vec{J}}_m = 0$. Therefore, we see that in this approximation, the current can not be time-varying. We also point out that writing the linearized conservation of energy-momentum, $\partial_\nu T^{\mu\nu} = 0$, with $\mu = 0$ gives

$$\partial_0 T^{00} + \partial_i T^{0i} = 0 \tag{61}$$

Using (28) and (33), we can express this in terms of the mass density and mass-current density as

$$-\frac{1}{c}\partial_t\left(\rho c^2\right) + \partial_i\left(-J_m^i c\right) = 0 \tag{62}$$

$$\dot{\rho} + \nabla\cdot\vec{J}_m = 0 \tag{63}$$

So we find that the linearized conservation of energy-momentum leads to a continuity equation for mass-current. This can also be written as conservation of the mass-four-current: $\partial_\nu J_m^\mu = 0$ where $J_m^\mu = \left(c\rho, \vec{J}_m\right)$. We can easily show that this requires that there is a gravitational displacement current as given in Ampere's law in (353). If we solve the gravito-Gauss law in (58) for the charge density and take the time-derivative, then we have $\dot{\rho} = -\varepsilon_G \nabla\cdot\partial_t \tilde{\tilde{\vec{E}}}_G$. Inserting this into the continuity equation (63) gives

$$\left(-\varepsilon_G \nabla\cdot\partial_t \tilde{\tilde{\vec{E}}}_G\right) + \nabla\cdot\vec{J}_m = 0 \tag{64}$$

$$\nabla\cdot\left(\vec{J}_m - \varepsilon_0 \partial_t \tilde{\tilde{\vec{E}}}_G\right) = 0 \tag{65}$$

We can therefore define the "gravito-displacement current density" as

$$\boxed{\vec{J}_{m\,(D)} = -\varepsilon_G \partial_t \tilde{\tilde{\vec{E}}}_G \qquad \textit{Gravito-displacement current density to order } v/c} \tag{66}$$

The *full* mass current density can be written as $\vec{J}_{m\,(full)} = \vec{J}_m + \vec{J}_{m\,(D)}$. Then the gravito-Ampere law gives

$$\nabla\times\tilde{\tilde{\vec{B}}}_G = -\mu_G \vec{J}_{m\,(full)} \tag{67}$$

$$\nabla\times\tilde{\tilde{\vec{B}}}_G = -\mu_G \vec{J}_m + \varepsilon_G \mu_G \partial_t \tilde{\tilde{\vec{E}}}_G \tag{68}$$

This result agrees with the gravito-Ampere law found in (58).

1.8 Gauge freedom in the gravito-electromagnetic fields

The gravito-electromagnetic fields, $\vec{E}_G = -\nabla\varphi_G$ and $\vec{B}_G = \nabla\times\vec{h}$ in (46) are essentially ad hoc definitions since these quantities cannot be constructed as gauge-invariant quantities in the context of linearized GR. This is contrast to $\vec{E}$ and $\vec{B}$ in electromagnetism. Recall that in electromagnetism,the gauge freedom is given by $A'^{\mu} = A^{\mu} + \partial^{\mu}\chi$ where χ is a scalar gauge function. Using $A^{\mu} = \left(\varphi/c, A^i\right)$ gives

$$\varphi' = \varphi + \dot{\chi} \qquad \text{and} \qquad A'^i = A^i + \partial^i\chi \tag{69}$$

The gauge freedom can be removed from the second equation by simply taking the curl so that $\nabla\times\vec{A}' = \nabla\times\vec{A}$. Then $\vec{B} = \nabla\times\vec{A}$ can be identified as a gauge-invariant quantity. A second gauge-invariant quantity can also be constructed by applying ∂^i to the first equation and applying ∂_t to the second equation, then adding them. Canceling terms involving $\partial^i\chi$ lead to $\partial^i\varphi' + \dot{A}'^i = \partial^i\varphi + \dot{A}^i$. Then $E^i = -\partial^i\varphi + \dot{A}^i$ can be identified as another gauge-invariant quantity (where the negative sign is introduced so that $\vec{E}$ points in the opposite direction of the gradient φ). Therefore, we find that $\vec{E}$ and $\vec{B}$ are not ad hoc definitions but rather arise naturally in electromagnetism as gauge-invariant quantities.

The gauge freedom of the gravito-electromagnetic fields can be shown by starting with the gauge freedom in linearized GR as shown in (2418) of Appendix A.

$$h'^{\mu\nu} = h^{\mu\nu} + \partial^{\mu}\xi^{\nu} + \partial^{\nu}\xi^{\mu} \tag{70}$$

Using $\partial^0 = -\frac{1}{c}\partial_t$ (to linear order in the metric) gives the following three relations.

$$h'^{00} = h^{00} - \tfrac{2}{c}\dot{\xi}^0 \qquad h'^{0i} = h^{0i} - \tfrac{1}{c}\dot{\xi}^i + \partial^i\xi^0 \qquad h'^{ij} = h^{ij} + \partial^i\xi^j + \partial^j\xi^i \tag{71}$$

In (2435), the metric perturbation is written in terms of the trace-reversed metric perturbation as $h^{\mu\nu} = \bar{h}^{\mu\nu} - \dfrac{1}{2}\eta^{\mu\nu}\bar{h}$,where the trace is $h = \eta^{\mu\nu}h_{\mu\nu} = -\bar{h}^{00} + \eta_{ij}\bar{h}^{ij}$. This leads to

$$h^{\mu\nu} = \bar{h}^{\mu\nu} + \frac{1}{2}\eta^{\mu\nu}\left(\bar{h}^{00} - \eta_{ij}\bar{h}^{ij}\right) \tag{72}$$

From (32), we also have $\bar{h}^{00} = -\frac{4}{c^2}\varphi_G$ and $h^{0i} = \frac{4}{c}h^i$. Also using $\bar{h}_{ij} = 0$ (for non-relativistic sources in the unprimed frame) leads to the following three relations.

$$h^{00} = -\frac{2}{c^2}\varphi_G \qquad h^{0i} = \frac{4}{c}h^i \qquad h^{ij} = -\frac{2}{c^2}\varphi_G\eta^{ij} \tag{73}$$

Using these in (71) leads to

$$\varphi'_G = \varphi_G - c\dot{\xi}^0, \qquad h'^i = h^i - \tfrac{1}{4}\dot{\xi}^i + \tfrac{c}{4}\partial^i\xi^0, \qquad \varphi'_G\eta^{ij} = \varphi_G\eta^{ij} + \tfrac{c^2}{2}\left(\partial^j\xi^i - \partial^i\xi^j\right) \tag{74}$$

Taking $\partial^i$of the first equation above, and ∂_t of the second equation above gives

$$\partial^i\varphi'_G = \partial^i\varphi_G - c\partial^i\dot{\xi}^0 \qquad \text{and} \qquad \dot{h}'^i = \dot{h}^i - \tfrac{1}{4}\ddot{\xi}^i + \partial^i\dot{\xi}^0 \tag{75}$$

Multiplying the second equation in (75) by 4 and adding it to the first equation in (75) gives

$$\partial^i\varphi'_G + 4\dot{h}'^i = \partial^i\varphi_G + 4\dot{h}^i - \ddot{\xi}^i \tag{76}$$

Defining the gravito-electric field as $\vec{E}_G = -\nabla\varphi_G - 4\partial_t\vec{h}$ leads to[9]

$$\boxed{\vec{E}'_G = \vec{E}_G + \partial_t^2\vec{\xi}^i \qquad \textit{Gauge freedom in the gravito-electric field}} \tag{77}$$

[9]As shown earlier, for non-relativistic sources, $\partial_t\vec{h} = 0$ which means that $\vec{E}_G = -\nabla\varphi_G$. Therefore, taking the gradient of the first equation in (75) would lead to $\vec{E}'_G = \vec{E}_G + c\nabla\dot{\xi}^0$. However, taking a time derivative of the second equation in (75) and using $\partial_t\vec{h} = 0$ would lead to $c\partial^i\dot{\xi}^0 = \ddot{\xi}^i$. Therefore, the gauge freedom for $\vec{E}_G$ would still remain the same.

The last term above demonstrates the gauge freedom in the gravito-electric field. The gauge freedom can be interpreted as a manifestation of the Equivalence Principle. If one were to boost into a frame with an acceleration $a^i_{boost} = -\ddot{\xi}^i$ then $\vec{E}'_G = 0$ in the primed frame.

Next, taking the curl of the second relation in (74) and using $\vec{B}_G = \nabla \times \vec{h}$ gives

$$\boxed{\vec{B}'_G = \vec{B}_G - \frac{1}{4}\nabla \times \dot{\vec{\xi}} \qquad \textit{Gauge freedom in the gravito-magnetic field}} \tag{78}$$

The last term above demonstrates the gauge freedom in the gravito-magnetic field. The gauge freedom can be interpreted as a type of rotational Equivalence Principle. If one were to boost into a frame with a velocity $v^i_{boost} = \dot{\xi}^i$ such that $\nabla \times \vec{v} = 4\vec{B}_G$, then $\vec{B}'_G = 0$ in the primed frame. This can be better understood by considering the equation of motion involving $\vec{B}_G$. The geodesic equation (to lowest order in the metric and in particle velocity) gives $m\vec{a} = m\vec{v} \times \vec{B}_G$. For circular motion, the acceleration is centripetal which gives $a_c = vB_G$. One could boost to a primed frame which introduces a new centripetal force that is equal and opposite to the gravito-magnetic force. Then the net force experienced would be zero. This corresponds to choosing $v^i_{boost} = \dot{\xi}^i$ such that $\nabla \times \vec{v} = 4\vec{B}_G$.

1.9 The geodesic equation of motion for relativistic test masses

To find an equation of motion, we will need $h^{\mu\nu}$ (the *non*-trace-reversed metric perturbation). We can use (2435) from Appendix B to express $h^{\mu\nu}$ in terms of $\overline{h}^{\mu\nu}$.

$$h^{\mu\nu} = \overline{h}^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu}\overline{h} \tag{79}$$

For $\overline{h}^{\mu\nu}$, we can use (32) rewritten as

$$\overline{h}^{00} = -\frac{4}{c^2}\tilde{\varphi}_G \qquad \text{and} \qquad \overline{h}^{0i} = \frac{4}{c}\left(\vec{h}\right)^i \tag{80}$$

For $\overline{h}$, we can take the trace of $\overline{h}^{\mu\nu}$. To first order in the metric, we have $\overline{h} = \eta_{\mu\nu}\overline{h}^{\mu\nu}$. Summing the indices gives $\overline{h} = \eta_{00}\overline{h}^{00} + \eta_{ii}\overline{h}^{ii}$. Since $\eta_{00} = -1$ and $\overline{h}^{ii} = 0$, then the trace is $\overline{h} = -\overline{h}^{00}$. Using (80), we have

$$\overline{h} = -\overline{h}^{00} = \frac{4}{c^2}\tilde{\varphi}_G \tag{81}$$

Inserting (80) and (81) into (79) gives

$$h^{\mu\nu} = \overline{h}^{\mu\nu} - \tfrac{2}{c^2}\tilde{\varphi}_G\eta^{\mu\nu} \tag{82}$$

Writing the equation above in matrix form gives

$$h^{\mu\nu} = \begin{pmatrix} -\frac{4}{c^2}\tilde{\varphi}_G & \frac{4}{c}\overline{h}_1 & \frac{4}{c}\overline{h}_2 & \frac{4}{c}\overline{h}_3 \\ \frac{4}{c}h_1 & 0 & 0 & 0 \\ \frac{4}{c}h_2 & 0 & 0 & 0 \\ \frac{4}{c}h_3 & 0 & 0 & 0 \end{pmatrix} - \frac{2}{c^2}\tilde{\varphi}_G \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{83}$$

$$h^{\mu\nu} = \begin{pmatrix} -\frac{2}{c^2}\tilde{\varphi}_G & \frac{4}{c}\overline{h}_1 & \frac{4}{c}\overline{h}_2 & \frac{4}{c}\overline{h}_3 \\ \frac{4}{c}h_1 & -\frac{2}{c^2}\tilde{\varphi}_G & 0 & 0 \\ \frac{4}{c}h_2 & 0 & -\frac{2}{c^2}\tilde{\varphi}_G & 0 \\ \frac{4}{c}h_3 & 0 & 0 & -\frac{2}{c^2}\tilde{\varphi}_G \end{pmatrix} \tag{84}$$

This result is consistent with the fact that in the Newtonian limit, the metric perturbation simply contains $-\frac{2}{c^2}\tilde{\varphi}_G$ for each term on the diagonal, where $\tilde{\varphi}_G$ is the Newtonian potential which satisfies the Newtonian Poisson equation. Now we can use the components of $h^{\mu\nu}$ to determine the equation of motion. The geodesic equation of motion is given by

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \tag{85}$$

It is shown in (2729) of Appendix J that by reparameterizing in terms of t instead of τ, we obtain

$$a^\mu = -\Gamma^\mu_{\rho\sigma}v^\rho v^\sigma - \gamma^{-1}\dot{\gamma}v^\mu \tag{86}$$

where $\gamma = dt/d\tau$, $v^\mu = (c,\dot{x}) = \left(c, v^i\right)$ and $a^\mu = \left(0, \dot{v}^i\right)$. It is also shown in (2738) of Appendix J that the "Lorentz factor" in terms of the metric perturbation is

$$\gamma = \left(1 - h_{00} - 2h_{0i}\frac{v^i}{c} - \frac{v^2}{c^2} - h_{ij}\frac{v^i v^j}{c^2}\right)^{-1/2} \tag{87}$$

Note that (86) requires the time derivative of γ. It also requires summing the Christoffel symbol over ρ and σ. Again, from Appendix J, we found in (2764) that to first order in $h_{\mu\nu}$ and second order in v_i/c, the geodesic equation of motion becomes[10]

$$\begin{aligned} a_i &= \tfrac{c^2}{2}\partial_i h_{00} - c\dot{h}_{0i} + cv^j\left(\partial_i h_{0j} - \partial_j h_{0i}\right) - \tfrac{1}{2}\dot{h}_{00}v_i - \frac{1}{c}\dot{h}_{0j}v^j v_i \\ &\quad -\dot{h}_{ij}v^j + v^j v^k\left(\tfrac{1}{2}\partial_i h_{jk} - \partial_k h_{ij}\right) \end{aligned} \tag{88}$$

In a previous section it was shown that $\partial_t \overline{h}_{0i} \approx 0$ due to the trace-reversed harmonic gauge and the condition that $\overline{h}_{ij} \approx 0$. We can also easily verify from (79) that $h_{0i} = \overline{h}_{0i}$ since $\eta_{0i} = 0$. Therefore $\dot{h}_{0i} \approx 0$ and we can immediately eliminate the two terms involving $\dot{h}_{0i}$ in (88). However, although $\overline{h}_{ij} \approx 0$, we found in (84) that $h_{ij} \neq 0$ for $i = j$, therefore we do not eliminate these terms. We can see in (88) that they involve summations of the velocity over the diagonal components of (84). First, we can simplify the term involving $v^j v^k \partial_i h_{jk}$ since (84) shows that $h_{jj} = -\frac{2}{c^2}\tilde{\varphi}_G$. We have

$$v^j v^k\left(\partial_i h_{jk}\right) = v_1 v_1\left(\partial_i h_{11}\right) + v_2 v_2\left(\partial_i h_{22}\right) + v_3 v_3\left(\partial_i h_{33}\right) = -\tfrac{2}{c^2}v^2\left(\partial_i \tilde{\varphi}_G\right) \tag{89}$$

This term can be combined with the first term in (88) since $h_{00} = -\frac{2}{c^2}\tilde{\varphi}_G$. We can use (84) to substitute for h_{00} and h_{0i} in the other terms of the first line as well. After some rearranging we have

$$a_i = -\partial_i \tilde{\varphi}_G\left(1 + \frac{v^2}{c^2}\right) + 4v^j\left(\partial_i h_j - \partial_j h_i\right) + \frac{v_i}{c^2}\dot{\tilde{\varphi}}_G - \dot{h}_{ij}v^j - v^j v^k \partial_k h_{ij} \tag{90}$$

We can also write this equation in vector form. In the first line we can simply make the following replacements

$$\partial \implies \nabla \qquad v_i \implies \vec{v} \qquad v^j\left(\partial_i h_j - \partial_j h_i\right) \implies \vec{v}\times\left(\nabla\times\vec{h}\right) \tag{91}$$

This last relation follows from writing the curl in terms of a Levi-Civita.[11] For the last two terms in (90), we can sum over the repeated indices to obtain

$$\dot{h}_{ij}v^j = \dot{h}_{i1}v_1 + \dot{h}_{i2}v_2 + \dot{h}_{i3}v_3 \tag{92}$$

$$v^j v^k \partial_k h_{ij} = v_1\left(\vec{v}\cdot\nabla\right)h_{i1} + v_2\left(\vec{v}\cdot\nabla\right)h_{i2} + v_3\left(\vec{v}\cdot\nabla\right)h_{i3} \tag{93}$$

We can evaluate (92) for $i = 1, 2, 3$, multiply each result by the appropriate unit vector and add them to build the vector. Note that $h_{ij} = 0$ for $i \neq j$, therefore each value of i will only contribute one term. Also, using

[10] Since we are working to first order in the metric, then spatial indices can be freely raised and lowered. Therefore, we choose to write the equation of motion with a lower index i.

[11] Since $\left(\nabla\times\vec{h}\right)_k = \varepsilon_{klm}\partial_l h_m$ then $\left(\vec{v}\times\nabla\times\vec{h}\right)_i = \varepsilon_{ijk}v_j\left(\varepsilon_{klm}\partial_l h_m\right) = \varepsilon_{ijk}\varepsilon_{klm}v_j\partial_l h_m$. We can permute the first Levi-Civita twice to make k the first index in order to use the relation $\varepsilon_{kij}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$. Then we have

$$\varepsilon_{kij}\varepsilon_{klm}v_j\partial_l h_m = \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)v_j\partial_l h_m = v_j\partial_i h_j - v_j\partial_j h_i = v_j\left(\partial_i h_j - \partial_j h_i\right)$$

Hence we have shown that $\left(\vec{v}\times\nabla\times\vec{h}\right)_i = v_j\left(\partial_i h_j - \partial_j h_i\right)$ which is what appears in (90).

$h_{jj} = -\frac{2}{c^2}\tilde{\varphi}_G$ gives

$$\dot{h}_{ij}v^j \quad \Longrightarrow \quad \dot{h}_{11}v_1\hat{x}_1 + \dot{h}_{22}v_2\hat{x}_2 + \dot{h}_{33}v_3\hat{x}_3 = -\tfrac{2}{c^2}\dot{\tilde{\varphi}}_G\vec{v} \tag{94}$$

Carrying out the same process for (93) gives

$$v^j v^k \partial_k h_{ij} \quad \Longrightarrow \quad v_1\left(\vec{v}\cdot\nabla\right)h_{11}\hat{x}_1 + v_2\left(\vec{v}\cdot\nabla\right)h_{22}\hat{x}_2 + v_3\left(\vec{v}\cdot\nabla\right)h_{33}\hat{x}_3 = -\tfrac{2}{c^2}\left(\vec{v}\cdot\nabla\varphi_G\right)\vec{v} \tag{95}$$

We can now use (91), (94), and (95) to write (90) as a vector equation of motion.

$$\vec{a} \quad = \quad -\nabla\tilde{\varphi}_G\left(1+\frac{v^2}{c^2}\right) + 4\vec{v}\times\nabla\times\vec{h} + \frac{\dot{\tilde{\varphi}}_G}{c^2}\vec{v} + \frac{2}{c^2}\dot{\tilde{\varphi}}_G\vec{v} + \frac{2}{c^2}\left(\vec{v}\cdot\nabla\varphi_G\right)\vec{v} \tag{96}$$

Combining similar terms and rearranging gives

$$\vec{a} \quad = \quad -\nabla\tilde{\varphi}_G\left(1+\frac{v^2}{c^2}\right) + 4\vec{v}\times\nabla\times\vec{h} + \left(\frac{3}{2}\dot{\tilde{\varphi}}_G + 2\vec{v}\cdot\nabla\tilde{\varphi}_G\right)\frac{\vec{v}}{c^2} \tag{97}$$

We can also express this in terms of the gravito-electric and gravito-magnetic fields. These were defined in (46) as $\tilde{\vec{E}}_G = -\nabla\tilde{\varphi}_G$ and $\tilde{\vec{B}}_G = \nabla\times\vec{h}$, respectively. This gives

$$\boxed{\vec{a} = \tilde{\vec{E}}_G\left(1+\frac{v^2}{c^2}\right) + 4\vec{v}\times\tilde{\vec{B}}_G + \left(\frac{3}{2}\frac{\dot{\tilde{\varphi}}_G}{c^2} - 2\frac{\vec{v}}{c^2}\cdot\tilde{\vec{E}}_G\right)\vec{v} \qquad \begin{array}{c}\textit{Geodesic equation}\\ \textit{in the harmonic gauge}\\ (v/c \textit{ sources, } v^2/c^2 \textit{ test masses})\end{array}} \tag{98}$$

This expression gives the acceleration of a test mass to first order in the metric perturbation $\left(h_{\mu\nu}\right)$ with sources to order v_s/c (which means $\bar{h}_{ij}\approx 0$) and test masses with velocity $\left(v^2/c^2\right)$. This equation of motion does *not* describe the motion of test particles in the presence of gravitational waves since the approximation $\bar{h}_{ij}\approx 0$ has effectively removed the radiation degrees of freedom from the fields. A similar expression is obtained in [10], p. 43, 54-55.

The term with $\vec{v}\times\tilde{\vec{B}}_G$ does not contain an explicit division by c^2 and therefore appears to be a lower order term when compared to the terms with v^2/c^2. However, from the field equations in (58), we find that the quasi-static gravitational Ampere law is

$$\nabla\times\tilde{\vec{B}}_G = \mu_G\vec{J}_m \tag{99}$$

where $\vec{J}_m = \rho v_s$ with v_s being the speed of the source and $\mu_G = 4\pi G/c^2$ according to (35). Therefore, we find that $\tilde{B}_G \sim v_s/c^2$. This means that

$$\vec{v}\times\tilde{\vec{B}}_G \sim v_t v_s/c^2 \tag{100}$$

where v_t is the speed of the test mass. From this we recognize that if *both* v_t and v_s are non-relativistic, then $\vec{v}\times\tilde{\vec{B}}_G$ is negligible.[12]

[12] For a more concrete example, consider the gravito-magnetic field found in (125) for the mass solenoid. We have $\tilde{B}_g = \frac{1}{2}\mu_g R^2\rho\omega$. Using $v_s = R\omega$ and $\mu_g = 4\pi G/c^2$ gives $\tilde{B}_g = 2\pi G R\rho v_s/c^2$. Then the equation of motion for a test mass with speed v_t will give $\vec{v}_t\times\tilde{\vec{B}}_g \sim (2\pi G R\rho)\,v_s v_t/c^2$.

In our case here, we must have that v_s is non-relativistic (so that $\bar{h}_{ij} \approx 0$), but v_t can be relativistic which would keep (98) from being totally negligible. However, it would likely be much smaller than the other terms in (98).

Notice that there are two terms in (98) which involve relativistic corrections to the acceleration associated with $\tilde{\tilde{\vec{E}}}_G$. The first terms has a relativistic correction of $\left(1+v^2/c^2\right)$ which enhances the effective acceleration experienced by the test mass. The velocity appears as a scalar which is independent of the direction of the acceleration. For example, for a test mass that is orbiting the earth[13] (and hence moving perpendicular to the direction of the gravito-electric field), the test mass will experience an *enhancement* in the gravitational acceleration *toward* the earth due to this factor. For this particular case, if we neglect $\vec{v}\times\tilde{\tilde{\vec{B}}}_G$ and consider $\tilde{\varphi}_G$ to be essentially static (so that $\dot{\tilde{\varphi}}_G \approx 0$) we find that (98) reduces to just[14]

$$\vec{a} \quad = \quad \tilde{\tilde{\vec{E}}}_G\left(1+\frac{v^2}{c^2}\right) \tag{101}$$

By contrast, the last term of (98) contains a relativistic correction involving $\tilde{\tilde{\vec{E}}}_G$, however, it appears as $\vec{v}\cdot\tilde{\tilde{\vec{E}}}_G$ which means it is only associated with motion of the test mass in the direction of $\tilde{\tilde{\vec{E}}}_G$. Therefore, this correction would *not* affect the acceleration of a test mass that is orbiting the earth. However, it *would* affect the acceleration of a test mass that has any component of its motion directed toward the center of the earth. Since this term appears with a *negative* sign, it acts to *retard* rather than enhance the net acceleration of the test mass.

For the particular case of a test mass falling toward the center of the earth, we would have $\vec{v}\cdot\tilde{\tilde{\vec{E}}}_G = v_t\tilde{\tilde{E}}_G$. Neglecting $\vec{v}\times\tilde{\tilde{\vec{B}}}_G$ and considering $\tilde{\varphi}_G$ to be essentially static (so that $\dot{\tilde{\varphi}}_G \approx 0$) we find that (98) reduces to just

$$\vec{a} \quad = \quad \tilde{\tilde{\vec{E}}}_G\left(1-\frac{v^2}{c^2}\right) \tag{102}$$

Therefore, we find that a particle that is moving toward the center of the earth would have an overall relativistic correction that *decreases* the acceleration by v^2/c^2 compared to the acceleration for non-relativistic velocities. This result is consistent with that obtained in [**?**].

Lastly, we point out that the factor of 4 in the gravito-magnetic force can be traced back to the definition of the gravito-vector potential in (31) where we chose the prefactor in order to make the gravitational Maxwell-like equations in (58) appear almost identical in form to the electromagnetic Maxwell equations. The factor of 4 in the gravito-magnetic force is obviously a distinguishing feature which makes the gravito-Lorentz force unlike the electromagnetic Lorentz force. We could certainly choose to redefine the prefactors used when defining the potentials and/or the fields to remove the factor of 4, but we would find that the field equations are correspondingly altered in the process which would make them less similar to the electromagnetic Maxwell equations. Regardless of what choice is made with these prefactors, the physics will fundamentally remain the same and it is evident that we cannot obtain a perfect match between gravitation and electromagnetism.

[13]Here we are treating the earth as perfectly spherical and considering circular orbits concentric with the equator.

[14]This does not seem to match the result obtained from the Schwarzschild metric in (2811) of Appendix K. The expression there can be written as $\vec{a} = \vec{E}_G\left(1+3\frac{v^2}{c^2}\right)\hat{r}$ where $\vec{E}_G = -\frac{GM}{r^2}\hat{r}$, and $v = l/r$ .where $l = L/m$.

1.10 The gravitational Lorentz-like force for non-relativistic test masses

Next we consider the limit as $(v/c)^2$ becomes negligible. Notice that taking this limit properly requires specifying the strength of $\dot{\tilde{\varphi}}_G$ compared to the strength of $\nabla\tilde{\varphi}_G$. For example, in principle there could be a time-varying scalar potential that varies extremely rapidly (such as at microwave frequencies) and the test mass could be moving through a region where the scalar potential has very small spatial variance. In that case, it is possible that

$$\dot{\tilde{\varphi}}_G \vec{v} \quad > \quad \nabla\varphi\vec{v}^2 \tag{103}$$

However, for the case of a scalar potential that is slowly varying in time (or static), and also has a significant spatial variation, then it may be possible that[15]

$$\dot{\tilde{\varphi}}_G \vec{v} \quad < \quad \nabla\varphi\vec{v}^2 \tag{104}$$

Therefore, it may not be immediately obvious what is the appropriate approach when taking the non-relativistic limit of the test mass since the velocity of the test mass is not the only relevant quantity. However, assuming that both $\dot{\tilde{\varphi}}_G\vec{v}$ and $\nabla\varphi\vec{v}^2$ are negligible compared to the other remaining terms, then (98) would reduce to

$$\boxed{\vec{a} = \tilde{\tilde{\vec{E}}}_G + 4\vec{v}\times\tilde{\tilde{\vec{B}}}_G \qquad \begin{array}{c}\textit{Geodesic equation of motion}\\ \textit{in trace-reversed harmonic gauge}\\ \textit{(sources and test masses with } v << c\textit{)}\end{array}} \tag{105}$$

This is the gravitational "Lorentz-like" force law which often appears in the literature. It is important to recognize from the discussion above that this force law involves several assumptions and approximations concerning both the source of gravitation as well as the test masses experiencing them. It was obtained from linearized GR in the trace-reversed harmonic gauge for slow moving sources $(v_s << c)$, slow moving test masses, $(v_t << c)$, and fields that do not vary greatly in space or time (low frequency, nearly uniform gravitational fields). If any of these assumptions are violated, then a more general equation of motion is given by (98). In fact, depending on the properties of the physical system, it may be necessary to go back to (88) which also includes the effects of gravitational waves. In the case that non-linear gravitational interactions become relevant, it would be necessary to return all the way to the geodesic equation of motion given in (85) where the Christoffel symbols are completely general (not evaluated to first order in the metric).

Lastly, we point out that formally speaking, the term with $\vec{v}\times\tilde{\tilde{\vec{B}}}_G$ should also be dropped in (105). This follows from (100) where we found $\vec{v}\times\tilde{\tilde{\vec{B}}}_G \sim v_t v_s/c^2$. Since the entire formulation here assumes that $v_s << c$ (so that $\overline{h}_{ij} \approx 0$) and we are now taking the non-relativistic limit for test masses, $v_t << c$, then $\vec{v}\times\tilde{\tilde{\vec{B}}}_G \sim v_t v_s/c^2$ is of the same order as the other terms with v^2/c^2 in (98) that we dropped. Therefore, we should simply have $\vec{a} = \tilde{\tilde{\vec{E}}}_G$. The only way to have other terms appear (while still having non-relativistic test masses), is to have relativistic sources or extremely dense mass sources. This would allow more terms in (88) to remain even for a slow-moving test mass. Therefore, we conclude that (105) is not a valid expression since it is obtained by an inconsistent use of approximations.

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[15] We are assuming that in either case, the field is still sufficiently weak so that the approximations of linearized GR are not violated. We still require $h^{\mu\nu} << \eta^{\mu\nu}$.

1.11 The gravito-electromagnetic fields of a “mass solenoid”

Here we consider a very long rotating cylindrical “mass solenoid” of length L and radius R (where $L >> R$) with the axis along the z-axis from $z = -L/2$ to $z = L/2$. This system provides an interesting application of the gravito-electromagnetic field equations derived from the trace-reversed harmonic gauge (58). We assume that the cylinder rotates at a constant angular velocity and hence has a *non*-time varying mass current as required for the equations in (58) to apply.[16]

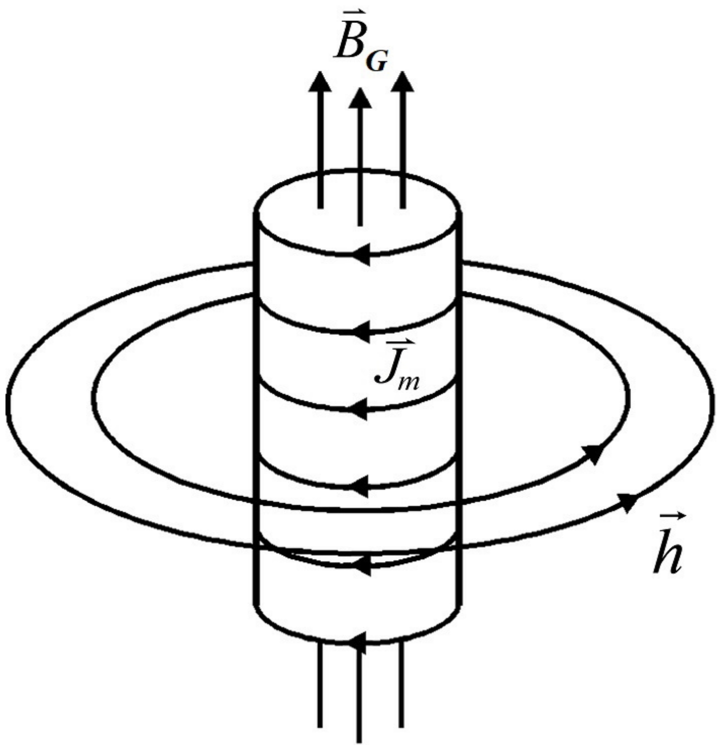

Figure 1: A “mass solenoid” with a mass current, $\vec{J}_m$, creating a gravito-vector potential, $\vec{h}$, and corresponding gravito-magnetic field, $\vec{B}_G$.

We can use the gravito-Gauss law from (58) to obtain the gravito-electric field, $\tilde{\vec{E}}_G$.

$$\nabla \cdot \vec{E}_G(r) = -\rho/\varepsilon_G \tag{106}$$

Taking the volume integral of both sides and applying the Divergence theorem gives

$$\iiint\limits_V \left(\nabla \cdot \vec{E}_G\right) dV \quad = \quad -\frac{1}{\varepsilon_G}\int\limits_V \rho(r)\, dV \tag{107}$$

$$\oint\limits_{\substack{Surface\\ of\ V}} \vec{E}_G \cdot d\vec{A} \quad = \quad -\frac{1}{\varepsilon_G}\int\limits_V \rho(r)\, dV \tag{108}$$

[16]Note that in the diagram, $\vec{h}$ points in the *opposite* direction of $\vec{J}_m$ as a result of the negative sign in $\nabla \times \left(\nabla \times \vec{h}\right) = -\mu_G \vec{J}_m$.

If we assume a uniform mass distribution[17], then $\rho(r) = \rho_0$. We can use a cylindrical Gaussian surface with radius r surrounding the mass solenoid concentrically. This gives

$$E_{G,r} 2\pi r L = -\rho_0 \left(\pi R^2 L\right)/\varepsilon_G \tag{109}$$

$$\boxed{\vec{E}_G = -\frac{R^2 \rho_0}{2\pi \varepsilon_G r}\hat{r}} \tag{110}$$

We can also use a line integral of $\vec{E}_G$ to find the change in the gravito-scalar potential, φ_G, when going from $r' = r_0$ to $r' = r$, where r_0 is an arbitrary distance away from the mass solenoid such that $\varphi_G(r_0) = 0$.

$$\Delta\varphi_G(r) = -\int_{r_0}^{r} \vec{E}_G \cdot d\vec{r} = -\int_{r_0}^{r} \left(-\frac{R^2\rho_0}{2\pi\varepsilon_G r'}\hat{r}\right)\cdot d\vec{r}' = \frac{R^2\rho_0}{2\pi\varepsilon_G}\int_{r_0}^{r}\frac{1}{r'}dr' = \frac{R^2\rho_0}{2\pi\varepsilon_G}\left[\ln\left(r'\right)\right]_{r_0}^{r} \tag{111}$$

$$\boxed{\varphi_G(r) = \frac{R^2\rho_0}{2\pi\varepsilon_G}\ln\left(\frac{r}{r_0}\right)} \tag{112}$$

Next we can calculate the gravito-vector potential, $\vec{h}$, outside the mass solenoid. Since the z-axis is the axis of symmetry as well as the axis of rotation, then $\vec{h} = h_\phi(\vec{r})\hat{\phi}$. To find an expression for $\vec{h}$, we take a line integral of $\vec{h}$ along a closed path around the solenoid, use $\vec{B}_G = \nabla\times\vec{h}$ from (46) and apply Stokes' theorem.

$$\oint\limits_{\substack{\text{Around}\\ \text{solenoid}}} \vec{h}\cdot d\vec{r} = \int\limits_{\substack{\text{Cross section}\\ \text{of solenoid}}} \left(\nabla\times\vec{h}\right)\cdot d\vec{S} = \int\limits_{\substack{\text{Cross section}\\ \text{of solenoid}}} \vec{B}_G\cdot d\vec{S} = \Phi_{\vec{B}_G} \tag{113}$$

where Φ_{gm} is the total gravito-magnetic flux of $\vec{B}_G$ through a cross-section of the solenoid. If we use a circular path along the $\hat{\phi}$ direction (with z in the upward direction), then we also have

$$\oint\limits_{\substack{\text{Around}\\ \text{solenoid}}} \vec{h}\cdot d\vec{r} = h_\phi 2\pi r \tag{114}$$

So equating (113) and (114) gives

$$\vec{h} = \frac{\Phi_{gm}}{2\pi r}\hat{\phi} \tag{115}$$

We can also develop an expression for Φ_{gm} (and hence for $\vec{h}$) by taking a surface integral of both sides of Ampere's law from (58) and applying Stokes' theorem to change the surface integral into a line integral.

$$\iint\left(\nabla\times\vec{B}_G\right)\cdot d\vec{S} = -\mu_G\iint\vec{J}_m\cdot d\vec{S} \tag{116}$$

[17] If the mass distrubution is *not* uniform then we can not explicitely evaluate the integral. However, if a function is known for the mass density, then because the integral is over the *entire* cylinder, we would simply use $M = \int\rho(\vec{r})\,dV$ where M is the total mass.

Using Stoke's theorem gives

$$\oint \vec{B}_G \cdot d\vec{l} \quad = \quad -\mu_G I_m \tag{117}$$

where I_m is the mass-current. We can use a line integral along a rectangular loop with one edge *inside* the solenoid parallel to the axis (where $\vec{B}_G \neq 0$) and the opposite edge *outside* the solenoid (where $\vec{B}_G = 0$). If the length of the edge is L, then we obtain

$$B_G L \quad = \quad -\mu_G I_m \tag{118}$$

The total current in a solenoid is $I_m = N i_m$ where i_m is the current in each loop. If the solenoid is a continuous mass shell, then it is effectively a "perfect" solenoid where the current is distributed continuously over the surface. Then we can use $J_m = \sigma\omega$ where σ is the effective surface mass density of the cylinder spinning with angular velocity ω. So the total current would be $I_m = J_m A_\perp$ where $A_\perp = RL$ is the area normal to the current and R is the radius of the solenoid. Then we have

$$B_G L \quad = \quad -\mu_G (\sigma\omega)(RL) \tag{119}$$

$$B_G \quad = \quad -\mu_G R\sigma\omega \tag{120}$$

We can now determine the magnitude of the gravito-magnetic flux Φ_{gm} through a cross-sectional area of the solenoid, $A_{cs} = \pi R^2$. When we determined the gravito-magnetic field, we already treated the cylinder as a "perfect" solenoid which means it is effectively one "loop" so $N = 1$. Then we have

$$\Phi_{gm} = N B_G A_{cs} = (\mu_G R\sigma\omega)\left(\pi R^2\right) = \mu_G \pi R^3 \sigma\omega \tag{121}$$

We can also express this in terms of a *volume* mass density since

$$\sigma A = \rho V \qquad \Longrightarrow \qquad \sigma(2\pi RL) = \rho\left(\pi R^2 L\right) \qquad \Longrightarrow \qquad \sigma = R\rho/2 \tag{122}$$

Then the gravito-magnetic flux in (121) becomes

$$\Phi_{gm} = \tfrac{1}{2}\mu_G \pi R^4 \rho\omega \tag{123}$$

We can substitute this back into (115) and (120) to express the gravito-vector potential and the gravito-magnetic field in terms of the physical parameters of the mass solenoid. This gives

$$\boxed{\vec{h} = \frac{\mu_G R^4 \rho\omega}{4r}\hat{\phi}} \tag{124}$$

and

$$\boxed{\vec{B}_G = -\tfrac{1}{2}\mu_G R^2 \rho\omega\hat{z}} \tag{125}$$

2 Gravito-electromagnetism via the Parametrized Post-Newtonian (PPN) approach

2.1 The PPN equations to second order with GR parameters

The Parameterized Post-Newtonian formalism does not begin with General Relativity but from a broader approach of simply assuming a metric theory of gravity and then relating derivatives of the metric to components of the energy-momentum-stress tensor. In the paper titled, “Laboratory experiments to test relativistic gravity” by Braginsky, Caves, and Thorne (BCT) [41], the following “Maxwell-like equations” (3.8a-3.8d) for the “electric-type” and “magnetic-type” gravitational fields are given.

$$\nabla\cdot\vec{g} = -4\pi G\rho_0\left(1+2\beta_1\frac{\vec{v}^2}{c^2}+\beta_3\frac{\Pi}{c^2}+3\beta_4\frac{P}{\rho_0 c^2}\right)+\left(\frac{7}{2}\Delta_1-\frac{1}{2}\Delta_2\right)\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi \tag{126}$$

$$\nabla\times\vec{g} = -\frac{1}{c}\frac{\partial\vec{H}}{\partial t} \tag{127}$$

$$\nabla\cdot\vec{H} = 0 \tag{128}$$

$$\nabla\times\vec{H} = \left(\frac{7}{2}\Delta_1+\frac{1}{2}\Delta_2\right)\left(-4\pi G\frac{\rho_0\vec{v}}{c}+\frac{1}{c}\frac{\partial}{\partial t}\vec{g}\right) \tag{129}$$

Throughout these equations $\beta_1,\ \beta_3,\ \beta_4,\ \Delta_1$, and Δ_2 are parameterized-post-Newtonian (PPN) parameters. Also, ρ_0 is the density of rest mass in the local frame of the matter, $\vec{v}$ is the ordinary (coordinate) velocity of the rest mass relative to the PPN coordinate frame, and Π is the specific internal energy. The scalar and vector potentials (3.3) are also defined, respectively, as

$$\Phi\equiv-(U+2\Psi) \qquad \text{and} \qquad \vec{A}\equiv-\frac{7}{2}\Delta_1\vec{V}-\frac{1}{2}\Delta_2\vec{W} \tag{130}$$

where $U,\Psi,\vec{V}$, and $\vec{W}$ are gravitational potentials as shown in Chapter 39 of MTW [11]. The “electric-type” gravitational field $\vec{g}$ and “magnetic-type” gravitational field $\vec{H}$ are then given, respectively, in (3.4) by

$$\vec{g}=-\nabla\Phi-\frac{1}{c}\frac{\partial\vec{A}}{\partial t} \qquad \text{and} \qquad \vec{H}=\nabla\times\vec{A} \tag{131}$$

In the Maxwell-type equations (126)-(129), we can consider the case when we have the PPN parameters of General Relativity which are simply $\beta_1=\beta_3=\beta_4=\Delta_1=\Delta_2=1$. Then the equations become

$$\nabla\cdot\vec{g} = -4\pi G\rho_0\left(1+2\frac{\vec{v}^2}{c^2}+\frac{\Pi}{c^2}+3\frac{P}{\rho_0 c^2}\right)+3\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi \tag{132}$$

$$\nabla\times\vec{g} = -\frac{1}{c}\frac{\partial\vec{H}}{\partial t} \tag{133}$$

$$\nabla\cdot\vec{H} = 0 \tag{134}$$

$$\nabla\times\vec{H} = 4\left(-4\pi G\frac{\rho_0\vec{v}}{c}+\frac{1}{c}\frac{\partial}{\partial t}\vec{g}\right) \tag{135}$$

These equations can be compared to the gravito-electromagnetic “Maxwell-like” equations in the trace-reversed harmonic gauge shown in (58). First, we can consider the case when the specific internal energy and pressure are negligible so that

$$\frac{\Pi}{c^2}\approx\frac{P}{\rho_0 c^2}\approx 0 \tag{136}$$

We can also change notation so that $\vec{g}=\tilde{\tilde{\vec{E}}}_G$ and $\vec{H}=\tilde{\tilde{\vec{B}}}_G$. Furthermore, we will use $\Phi=\tilde{\varphi}_G$ in the gravito-Gauss law and $\vec{J}_m=\rho\vec{v}$ in the gravito-Ampere law. Lastly, we recognize that the factor of $1/c^2$ that multiplies the displacement current in (58) is “split” in the BCT equations between a factor of $1/c$ in the gravito-Faraday law and a factor of $1/c$ in the time-dependent part of the gravito-Ampere law. This is simply a result of using cgs units so we can simply revert to SI units as used in (58). Then we can substitute ε_G and μ_G as defined in (35) and write (132)-(135) as

$$\boxed{\begin{array}{ll} \nabla\cdot\tilde{\tilde{\vec{E}}}_G = -\dfrac{\rho}{\varepsilon_G}\left(1+2\dfrac{\vec{v}^2}{c^2}\right)+3\dfrac{1}{c^2}\partial_t^2\tilde{\varphi}_G & \nabla\cdot\tilde{\tilde{\vec{B}}}_G = 0 \\ \nabla\times\tilde{\tilde{\vec{E}}}_G = -\partial_t\tilde{\tilde{\vec{B}}}_G & \nabla\times\tilde{\tilde{\vec{B}}}_G = \mu_G\left(-\vec{J}_m+\varepsilon_G\partial_t\tilde{\tilde{\vec{E}}}_G\right) \\ \multicolumn{2}{c}{\textit{Gravito-electromagnetic field equations from the PPN formalism}} \end{array}} \tag{137}$$

Comparing these equations to (58), we see that the gravito-Faraday law is preserved and the gravito-Ampere law takes the same form. However, since the sources are relativistic to order v^2/c^2 (as required for the gravito-Faraday law), then the gravito-Gauss law contains additional terms, namely, a term involving $\rho v^2/c^2$ and a term involving $\partial_t^2\varphi_G$.

We also note that unlike the gravito-electromagnetic equations in (58), the equations in (137) are valid for *time-varying* mass currents. To see this, we can consider the linearized conservation of energy-momentum, $\partial^\mu T_{0\mu} = 0$, with $\mu = i$ which gives

$$\partial^0 T_{0i} + \partial^j T_{ij} = 0 \tag{138}$$

Since $T_{ij} = \rho u_i u_j$ and we are keeping source terms to order $\rho v^2/c^2$ in (137), then T^{ij} can be non-zero in this approximation. Therefore, $\partial^0 T_{0i}$ can be non-zero and using (33) to express this in terms of the four-current density means $\dot{\vec{J}}_m = 0$ can be non-zero. Therefore, we see that in this approximation, the current can be time-varying.

2.2 The displacement current to second order in the velocity of sources

We can observe that the BCT equations in (132)-(135) are first-order in spatial derivatives of the vector fields. Therefore only a single curl or divergence of $\tilde{\tilde{\vec{E}}}_G$ or $\tilde{\tilde{\vec{B}}}_G$ appears to highest order. Also notice that the equations are second-order in time-derivatives of the potentials. Therefore, we find $\partial_t^2 \varphi_G$ in the gravito-Gauss law, we find $\partial_t \tilde{\tilde{\vec{B}}}_G = \nabla \times \partial_t \vec{h}$ in the gravito-Faraday law, and we find $\partial_t \tilde{\tilde{\vec{E}}}_G = -\partial_t^2 \vec{h}$ in the gravito-Ampere law.

In considering an expression for the displacement current, we must respect these approximation limits. Beginning from the BCT equations in (137), we can take a time-derivative to obtain

$$\frac{\dot{\rho}}{\varepsilon_G}\left(1+2\vec{v}^2/c^2\right)+\frac{\rho}{\varepsilon_G}\left(4\vec{v}\cdot\dot{\vec{v}}/c^2\right) = -\nabla\cdot\partial_t \tilde{\tilde{\vec{E}}}_G + 3\frac{1}{c^2}\partial_t^3\varphi_G \tag{139}$$

Since we are only keeping second-order time-derivatives of the potentials, then we must neglect $\partial_t^3 \varphi_G$. Also, because the sources in (137) are v^2/c^2 to highest order, we can consider $\vec{v}\cdot\dot{\vec{v}}/c^2$ to be negligible[18]. Solving for $\dot{\rho}$ gives

$$\dot{\rho} = \frac{-\varepsilon_G \nabla\cdot\partial_t \tilde{\tilde{\vec{E}}}_G}{(1+2\vec{v}^2/c^2)} \tag{140}$$

We can apply a binomial approximation and keep terms only to order v^2/c^2. Then we simply have

$$\dot{\rho} = -\varepsilon_G \nabla\cdot\partial_t \tilde{\tilde{\vec{E}}}_G \left(1-2\vec{v}^2/c^2\right)$$

Substituting this into the mass-current continuity equation in (63) and factoring out a common divergence gives

$$\nabla\cdot\left[-\varepsilon_G \partial_t \tilde{\tilde{\vec{E}}}_G \left(1-2\vec{v}^2/c^2\right)+\vec{J}_m\right]=0$$

Therefore, we can define a "mass displacement current" to order v^2/c^2 as

$$\boxed{\vec{J}_{m\,(D)} = -\varepsilon_G \partial_t \tilde{\tilde{\vec{E}}}_G \left(1-2\vec{v}^2/c^2\right) \qquad \textit{Mass displacement current to order } v^2/c^2} \tag{141}$$

Then the *full* mass current density is $\vec{J}_{m\,(full)} = \vec{J}_m + \vec{J}_{m\,(D)}$ and the gravito-Ampere law gives

$$\nabla\times\tilde{\tilde{\vec{B}}}_G = -\mu_G \vec{J}_{m\,(full)} \tag{142}$$

$$\nabla\times\tilde{\tilde{\vec{B}}}_G = -\mu_G \vec{J}_m + \varepsilon_G\mu_G \partial_t \tilde{\tilde{\vec{E}}}_G \left(1-2\vec{v}^2/c^2\right) \tag{143}$$

Matching this to the gravito-Ampere law found in (137) which comes from the equations by BCT, we find that in their approximation scheme they must have neglected terms of order $2\partial_t \tilde{\tilde{\vec{E}}}_G \vec{v}^2/c^2$. This could be justified by the fact that the sources are limited to second order in velocity $\left(v^2/c^2\right)$ and the time-derivatives of the potentials are limited to second order $\left(\partial_t \tilde{\tilde{\vec{E}}}_G = -\partial_t^2 \vec{h}\right)$ so the product of these two terms would be negligible.

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[18]We can write $\vec{v}\cdot\dot{\vec{v}}/c^2$ as $\omega v^2/c^2$ where ω represents the rate of change of $\vec{v}$. For low velocities, v^2/c^2 is extremely small which means that ω would not be able to compensate even with a high acceleration. On the other hand, for high velocities approaching c, the acceleration must be small since the velocity must approach c asymptotically. Therefore, we conclude that $\vec{v}\cdot\dot{\vec{v}}/c^2$ must be negligible compared to v^2/c^2.

2.3 The incompatibility of the PPN formalism with gravitational waves

From the BCT equations in (137), we consider a stationary scalar potential so that $\partial_t^2\varphi_G = 0$. To calculate the propagation speed of the field in vacuum, we also set $\rho = 0$ and $\vec{J}_m = 0$ for the mass density and mass current density. This makes the BCT equations in (137) become

$$\nabla\cdot\tilde{\tilde{\vec{E}}}_G = 0 \tag{144}$$

$$\nabla\times\tilde{\tilde{\vec{E}}}_G = -\partial_t\tilde{\tilde{\vec{B}}}_G \tag{145}$$

$$\nabla\cdot\tilde{\tilde{\vec{B}}}_G = 0 \tag{146}$$

$$\nabla\times\tilde{\tilde{\vec{B}}}_G = \frac{4}{c^2}\partial_t\tilde{\tilde{\vec{E}}}_G \tag{147}$$

Taking the curl of (145) and (147) gives

$$\nabla\times\left(\nabla\times\tilde{\tilde{\vec{E}}}_G\right) = -\partial_t\left(\nabla\times\tilde{\tilde{\vec{B}}}_G\right) \quad \text{and} \quad \nabla\times\left(\nabla\times\tilde{\tilde{\vec{B}}}_G\right) = \frac{4}{c^2}\partial_t\left(\nabla\times\tilde{\tilde{\vec{E}}}_G\right) \tag{148}$$

We can apply the vector calculus identity $\nabla\times\left(\nabla\times\vec{F}\right) = \nabla\left(\nabla\cdot\vec{F}\right) - \nabla^2\vec{F}$ and note that $\nabla\cdot\tilde{\tilde{\vec{E}}}_G = 0$ and $\nabla\cdot\tilde{\tilde{\vec{B}}}_G = 0$ from (144) and (146). So we have

$$-\nabla^2\tilde{\tilde{\vec{E}}}_G = -\frac{1}{c}\frac{\partial}{\partial t}\left(\nabla\times\tilde{\tilde{\vec{B}}}_G\right) \quad \text{and} \quad -\nabla^2\tilde{\tilde{\vec{B}}}_G = \frac{4}{c}\frac{\partial}{\partial t}\left(\nabla\times\tilde{\tilde{\vec{E}}}_G\right) \tag{149}$$

Substituting (145) and (147) into the expressions in (149) and returning to using gives

$$\nabla^2\tilde{\tilde{\vec{E}}}_G = \frac{4}{c^2}\frac{\partial^2}{\partial t^2}\tilde{\tilde{\vec{E}}}_G \quad \text{and} \quad \nabla^2\tilde{\tilde{\vec{B}}}_G = \frac{4}{c^2}\frac{\partial^2}{\partial t^2}\tilde{\tilde{\vec{B}}}_G \tag{150}$$

In general, a wave equation of the form $\nabla^2\vec{F} = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\vec{F}$ has a wave speed v. Therefore we find from (150) that the wave speed of the gravitational wave would be $v = c/2$. However, note that here we have obtained *vector* wave equations, not *tensor* wave equations. This would imply that GR permits vector wave equations similar to that of EM waves. However, that is not the case. In fact, in [11], p. 1075, the author state concerning the PPN formalism, "Changes with time of all quantities at fixed x_j are due primarily to the motion of the matter. As a result, time derivatives are small by $\mathscr{O}(\varepsilon)$ compared to space derivatives

$$\left|\frac{\partial A/\partial t}{\partial A/\partial x_j}\right| \sim \left|v_j\right| \lesssim \varepsilon \text{ for any quantity } A, \tag{151}$$

although *not* in the radiation zone, where outgoing gravitational waves flow . . . . Consequently, the radiation zone must be excluded from the analysis when one makes a post-Newtonian expansion. To treat it requires different techniques . . ." Therefore, we cannot consider the formalism above as relevant to gravitational waves.

Furthermore, there is an approximation error in the calculation above which can be traced back to taking the second derivative in (148). The equations derived by [41] in (132) – (135) are already approximations that have truncated second derivatives in the fields. Therefore, when taking second derivatives in (148), formally they would need to be set to zero to maintain consistency with (132) – (135). Otherwise, we are throwing out such terms to obtain (132) – (135) and then claiming that these equations permit higher order derivatives. This is obviously inconsistent.

Lastly, we point out that the wave equations shown in (150) implies there can be *vector* gravitational waves similar to EM waves. However, there are a number of properties of gravitation that are violated by having *vector* wave equations. This can be seen by comparing the following intrinsic properties of a tensor field theory, a vector field theory and a scalar field theory.

Tensor field theory properties (gravitation)

- Source is a rank-2 tensor (stress-energy-momentum tensor)
- Conservation of source obeys two fundamental laws (conservation of mass and momentum)
- "Field" is a symmetric rank-4 tensor (Riemann tensor)
- "Potential" is a symmetric rank-2 tensor (metric tensor)
- "Field" is related to "potential" by second-order, anti-symmetric derivatives
- Force is only attractive (corresponding to one sign for "charges")
- Polarization states are 45-degrees apart ("plus" and "cross" polarizations)
- Waves are transverse, traceless tensor fields (degrees of freedom isolated by TT gauge)
- Lowest order radiation is quadrupolar
- Force carrier is a spin-2 particle (graviton)

Vector field theory properties (electromagnetism)

- Source is a rank-1 tensor (four-current density)
- Conservation of source obeys one fundamental law (conservation of charge)
- Field is an anti-symmetric rank-2 tensor (EM field tensor)
- Potential is a rank-1 tensor (four-potential)
- Field is related to potential by a single order, anti-symmetric derivative
- Force can be attractive or repulsive (corresponding to two oppositely signed charges)
- Polarization states are 90-degrees apart (vertical and horizontal polarizations)
- Waves are transverse vector fields (degrees of freedom isolated by Coulomb gauge)
- Lowest order radiation is dipolar
- Force carrier is a spin-1 particle (photon)

Scalar field theory properties (sound)

- No "source" exists (no "sound charge")
- No conservation law of source since there is no source
- "Field" is a rank-0 tensor which is a scalar (pressure)
- No polarization states
- Waves are longitudinal scalar fields
- Lowest order radiation is monopolar
- Force carrier is a spin-0 particle (phonon)

2.4 Gravito-electromagnetic resistance and mutual inductance

The treatment of the mass solenoid in Section 9 can be extended to consider the characteristics of a gravito-electromagnetic "inductance" and "resistance." This can be done using the field equations derived from the PPN formalism in (137) since they permit time-varying mass currents. This is in contrast to the field equations derived in Part I from the trace-reversed metric perturbation.

Since electrons carry both mass and charge, we will compare the electromagnetic properties to the gravito-electromagnetic properties. We begin with a "particle current" which can then be easily related to a charge current as well as a mass current. The particle current is

$$I_n = \frac{dn}{dt} \tag{152}$$

where n represents the number of particles. Since the particles in an electrical current are electrons then we can write the charge and mass currents as, respectively,

$$I_q = eI_n \qquad \text{and} \qquad I_m = m_e I_n \tag{153}$$

If the current is sinusoidal, then the charge and mass currents in (153) become, respectively,

$$I_q = eI_o \sin(\omega t) \qquad \text{and} \qquad I_m = m_e I_o \sin(\omega t) \tag{154}$$

where I_o is the maximum current. If this charge/mass current exists in a solenoid, then the associated magnetic field and gravito-magnetic field would be, respectively,[19]

$$B = \mu_0 I_q n_{loop} \qquad \text{and} \qquad B_G = -\mu_G I_m n_{loop} \tag{155}$$

where $n_{loop} = N/L$ is the number of loops N per unit length L of the solenoid. We are using μ_0 and μ_G to represent the magnetic and gravito-magnetic permeability within the solenoid, respectively, assuming the inside of the solenoid is just vacuum. The magnetic flux and gravito-magnetic flux through the N loops of the solenoid would be, respectively,

$$\Phi_m = \frac{\mu_0 N^2 A I_q}{L} \qquad \text{and} \qquad \Phi_{gm} = -\frac{\mu_G N^2 A I_q}{L} \tag{156}$$

where A is the cross-sectional area of the solenoid. (The field is assumed to be relatively uniform over the cross-sectional area and along the length of the solenoid.) Substituting the currents shown in (154) into (156) gives

$$\Phi_m = \frac{N^2 A \mu_0 e}{L} I_o \sin(\omega t) \qquad \text{and} \qquad \Phi_{gm} = -\frac{N^2 A \mu_G m_e}{L} I_o \sin(\omega t) \tag{157}$$

Faraday's law for electromagnetism and gravito-electromagnetism is given by, respectively,

$$\Delta\varphi_e = -\partial_t \Phi_m \qquad \text{and} \qquad \Delta\varphi_G = -\partial_t \Phi_{gm} \tag{158}$$

where φ_e is the electric scalar potential and φ_G is the gravitational scalar potential. Substituting (157) into (158) and taking the time derivatives gives

$$\Delta\varphi_e = -\frac{N^2 A \mu_0 e}{L} I_o \omega \cos(\omega t) \qquad \text{and} \qquad \Delta\varphi_G = \frac{N^2 A \mu_G m_e}{L} I_o \omega \cos(\omega t) \tag{159}$$

[19] The negative sign appears for the gravito-magnetic field due to the fact that the gravito-Ampere law in (137) has a negative sign.

Now consider a ring that is placed coaxially above the solenoid as shown in the diagram below.

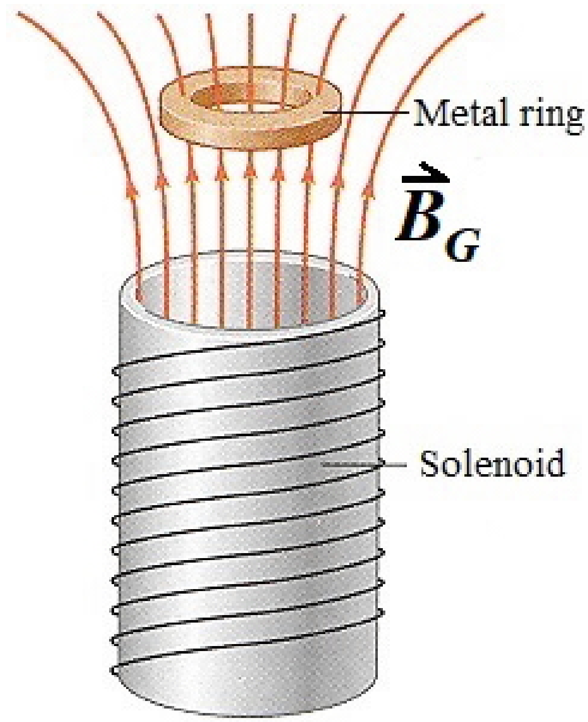

Figure 2: Metal ring placed concentircally above a cylindrical solenoid.

To determine the electric and gravito-electric current induced in the ring, we must consider how to compare electric resistance with "gravitational resistance." Electrical resistance can be written in terms of Ohm's law as

$$R_e = \frac{\varphi_e}{I_q} \tag{160}$$

Since electric potential is defined as energy per unit time, $\varphi_e \equiv U_e/q$, then volts are joules per coulomb. Therefore, the units of electrical resistance can be found accordingly.

$$[\Omega_e] = \frac{[J/C]}{[C]/[s]} = \frac{[J]\,[s]}{[C]^2} \tag{161}$$

Here we use Ω_e for electrical resistance to distinguish it from gravitational resistance. If we define an analogous gravitational "Ohm's law," we would have

$$R_G = \frac{\varphi_G}{I_m} \tag{162}$$

Since *gravitational* potential is defined as energy per unit *mass*, $\varphi_G \equiv U_G/m$, then "gravitational volts" are joules per kilogram. Therefore, the units of gravitational resistance can be found accordingly.[20]

$$[\Omega_G] \quad = \quad \frac{[J/kg]}{[kg]/[s]} = \frac{[J]\,[s]}{[kg]^2} \tag{163}$$

[20]Note that writing $[\Omega_g]$ in the basic MKS units gives $[\Omega_g] = \dfrac{[m]^2}{[kg]\,[s]}$. This is consistent with defining the gravito-electromagnetic impedance as $Z_g = \sqrt{\dfrac{\mu_g}{\varepsilon_g}} = \dfrac{4\pi G}{c}$ in (363) which also has units $[\Omega_g] = \dfrac{[m]^2}{[kg]\,[s]}$.

Hence we find on dimensional grounds that the unit of gravitational resistance is related to the unit of electrical resistance according to

$$[\Omega_G] \quad = \quad \left(\frac{[C]}{[kg]}\right)^2 [\Omega_e] \tag{164}$$

Formally, the total resistance that would be experienced by a current would therefore be $R_e + R_G$. However, it is known that in actuality the resistance upon a current of charged particles is generally due to electromagnetic interactions and not gravitational interactions. In other words, if a collection of charged particles (such as electrons) is accelerated by a *gravito*-electric field, the resistance that would determine the current would still be the *electrical* resistance, not the gravitational resistance. Therefore, we can simply use $R_G = \alpha R_G$ where $\alpha = 1C^2/kg^2$ is set to unity and is simply providing dimensional consistency.

Now the total current that would be induced in the metal ring above the solenoid will have a contribution due to *both* the electric and gravito-electric fields induced in the ring. So we have

$$I_{Total} = I_{q,induced} + I_{m,induced} \tag{165}$$

where

$$I_{q,induced} \quad = \quad \frac{\varphi_e}{R_e} \qquad \text{and} \qquad I_{m,induced} = \frac{\varphi_G}{R_G} \tag{166}$$

Using $R_G = \alpha R_e$ and substituting (159) into (166) gives

$$I_{q,induced} \quad = \quad -\frac{N^2 A I_o \omega \cos(\omega t)}{L R_e} \mu_0 e \qquad \text{and} \qquad I_{m,induced} = -\frac{N^2 A I_o \omega \cos(\omega t)}{L R_e \alpha} \mu_G m_e \tag{167}$$

We do not mean to imply that there are literally two currents. Rather, there is only a single current since the electrons are providing both the charge and mass. The two expressions are simply decomposing the contributions from the electric field and the gravitational field to produce a *single* current. If we take a ratio of the gravitational and electrical contributions, we obtain

$$\frac{I_{m,induced}}{I_{q,induced}} \quad = \quad \frac{\mu_G m_e}{\alpha \mu_0 e} \tag{168}$$

Since we know that $\mu_G = 4\pi G/c^2$ and $1/\mu_0 = \varepsilon_0 c^2$, then we can write the ratio above as[21]

$$\frac{I_{m,induced}}{I_{q,induced}} \quad = \quad (4\pi\varepsilon_0 G)\left(\frac{m_e}{\alpha e}\right) \tag{169}$$

Using $\alpha = 1$ and expressing the constants in SI units gives

$$\frac{I_{m,induced}}{I_{q,induced}} \quad \approx \quad \frac{4\pi\left(8.85\times10^{-12}\right)\left(6.67\times10^{-11}\right)\left(9.11\times10^{-31}\right)}{1.60\times10^{-19}} \approx 4.22\times10^{-32} \tag{170}$$

[21]The factor $4\pi\varepsilon_0 G$ is related to the “criticality condition” described in [91], pp. 3-7, where the Coulomb electrostatic force and Newtonian gravitational force are set equal in magnitude, $F_C = F_N$. This means that $\frac{q^2}{4\pi\varepsilon_0 r^2} = \frac{Gm^2}{r^2}$. Solving for the charge-to-mass ratio gives $(q/m)^2 = 4\pi\varepsilon_0 G$.

Hence we find that even if there is a way to completely shield the metal ring from the magnetic flux while allowing the gravito-magnetic flux to still penetrate, the resulting current would be 32 orders of magnitude weaker than the electric current had been. If we consider the case of a superconducting solenoid with a superconducting ring above it, then the only change to the treatment above would be with regard to R_e and R_G. Specifically, the relevant factor would be α where $\alpha = R_G/R_e$. If the value of α can be made extremely small, then this could help to offset the extremely small ratio in (170).

Let us consider some numeric values for the parameters appearing in the expression for $I_{m,induced}$ in (167) to determine if it could actually be detectable. We can let the number of loops in the solenoid be $N = 10^3$ while the cross-sectional area is $A = \pi (1cm)^2 = \pi \times 10^2 m$, and the length is $L = 50cm = .5m$. The frequency of oscillation can be microwaves, $\omega = 2\pi \times 10^9 Hz$. The current can be considered to be $I_q = 10A$ which means from (153) that for the case of electron pairs we have $I_o = I_q/2e = 5A/\left(1.602 \times 10^{-9} C\right)$. With these values, we have (in SI units)

$$I_{m,induced} = -\frac{2\left(10^3\right)^2\left(\pi \times 10^2\right)5\left(2\pi \times 10^9\right)\cos(\omega t)}{\left(1.602 \times 10^{-9}\right)(.5)R_G}\left(8.48 \times 10^{-57}\right) \approx \frac{2.1 \times 10^{-28}}{R_G} \tag{171}$$

Solving for R_G gives

$$R_G \approx \frac{2.1 \times 10^{-28}}{I_{m,induced}} \ (SI\ units) \tag{172}$$

If the smallest detectable current is on the order of picoAmps, then we have

$$R_G \approx 10^{-16} \ (SI\ units) \tag{173}$$

This is the maximum gravitational resistance that would still potentially allow for measurable effects of gravitational induction due to the oscillating current in the solenoid.

3 Gravito-electromagnetism via the Helmholtz Decomposition (HD) theorem

3.1 Overview of the HD theorem formulation

The use of cosmological perturbation theory for the purpose of decomposing the metric has been treated by a large assortment of authors including Lifshitz, Bardeen, Bertschinger, Carroll and others as shown in [13]-[27]. However, the treatment given here will closely follow that of Flanagan and Hughes [28] who essentially employ the Helmholtz Decomposition (HD) theorem in describing the metric perturbation tensor as well as the stress-energy-momentum tensor. Much of the notation and convention used by [28] will be retained although there will be a few important changes which are noted when appearing. Also, unlike [28] and most other treatments, we will retain all factors of G and c explicitly.[22]

First we develop the full framework of the gauge-invariant formulation of linearized GR via the Helmholtz Decomposition (HD) theorem. The HD theorem is never explicitly referenced in [28], however we recognize it as the premise upon which the entire formulation is built. We describe the HD metric, the associated HD boundary conditions (requiring the metric to be flat at infinity), and the relevance of the HD theorem. Specifically, this allows vectors to be uniquely defined in terms of a rotational component and an irrotational component.[23] As an extension of the HD theorem, a symmetric rank-2 tensor can also be uniquely defined in terms of longitudinal, rotational, and transverse degrees of freedom. This is discussed at some length by Bertschinger in [20], section 4.2 (pp. 40-43) as well as [21], sections 3 through 7 (pp. 4-14).

Next we apply the transformations given by the gauge freedom in linearized GR and construct four gauge-invariant quantities by enforcing the HD boundary conditions on the metric. These gauge-invariant quantities consist of two scalar potentials, a vector potential, and a tensor potential (Φ, Θ, Ξ_i, $h_{ij}^{\tau\tau}$). The scalar and vector potentials are all described by Poisson equations and are associated with *non*-radiative fields. The tensor potential satisfies a wave equation and gives the only metric degrees of freedom associated with gravitational radiation.

Next we formulate the linearized Einstein tensor components in terms of the gauge-invariant potentials. Then we consider a stress-energy-momentum tensor that is constructed in a manner similar to the HD metric and therefore also satisfies the HD theorem. We refer to this as the "HD stress tensor." Linearized conservation of stress-energy-momentum is used to formulate conservation laws relating the various stress tensor quantities to one another. The linearized Einstein tensor and the stress tensor are then used to write the Einstein field equations. Using the conservation laws leads to three Poisson equations, one for each of the non-radiative invariant potentials, and a wave equation for the radiative field. Up to this point, we have primarily just provided the mathematical details and elaborated on the conceptual framework that has already been summarized in the paper by Flanagan and Hughes [28].

After this we begin to expand on the results given in [28]. First we define gauge-*invariant* vector fields (the gravito-electric field and gravito-magnetic field) in terms of the gauge-invariant potentials. These vector fields are essentially a new form of gravito-electromagnetism (GEM). They are used to write vector field equations similar to the Maxwell equations of electromagnetism (EM). By appropriately defining the GEM vector fields in terms of the potentials, the four Poisson equations for the potentials can be turned into two divergence equations and two curl equations for the vector fields. These are referred to as gauge-invariant GEM field equations. To enhance the similarity with electromagnetism, we define a gravito-electric permittivity and

[22] Although this is often viewed as cumbersome, it will be shown that keeping factors of c proves highly instructive in revealing the relative strengths of fields and sources which would otherwise not be evident by omitting factors of c.

[23] Such a treatment is directly analogous to electromagnetism, although it is often not emphasized that the HD theorem is responsible for allowing the electric and magnetic fields to be uniquely defined in terms of just a scalar potential and a vector potential. For a discussion of this topic see Appendix B of Griffiths' *Introduction to Electrodynamics* [29].

gravito-magnetic permeability and discuss the properties of these quantities. It is also found that in this formulation there is no gravitational displacement current.

Next we relate the physical quantities of an ideal fluid tensor (mass density, velocity, and pressure) to the quantities of the HD stress tensor. This allows for the GEM field equations to be written in terms of physical sources such as mass density, mass current density, pressure and stress. Next we examine the Newtonian limit and first-order post-Newtonian limits to recover the familiar gravitational relations such as the Newtonian field and the traditional Lense-Thirring field. We find that in those limits, gauge-invariance is not be preserved. Lastly, we compare the gauge-invariant Lense-Thirring field with the traditional gauge-*dependent* Lense-Thirring field.

3.2 The HD metric perturbation

In the following sections, the Helmholtz Decomposition (HD) theorem is applied to the metric perturbation which will later be utilized in the formulation of linearized General Relativity (as developed in Appendix A). In the weak-field limit, the metric can be written in terms of a small perturbation about the flat Minkowski space-time metric to first order.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \text{where} \qquad \left|h_{\mu\nu}\right| << 1 \tag{174}$$

The HD theorem (which is commonly applied to the *vector* formulation of electromagnetism) can be extended to the *tensor* formulation of General Relativity. Applying the approach of Flanagan and Hughes [28], the components of the metric perturbation can be decomposed as follows.

$$h_{00} = -2\phi/c^2 \tag{175}$$
$$h_{0i} = (\beta_i + \partial_i\alpha)/c \tag{176}$$
$$h_{ij} = h_{ij}^{\tau\tau} + \tfrac{1}{3}\delta_{ij}H + \partial_{(i}\varepsilon_{j)} + \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\lambda \tag{177}$$

This formulation is identical to that found in [28] except for a few minor differences.[24] The factor of 2 in (175) will be shown later to simplify the transformation of h_{00} in (203) and consequently the definition of the gauge-invariant quantity Φ found in (246). Also, the notation $\partial_{(i}\varepsilon_{j)}$ found in (177) is a symmetric derivative given by[25]

$$\partial_{(i}\varepsilon_{j)} = \partial_i\varepsilon_j + \partial_j\varepsilon_i \tag{178}$$

Similar to [28], the following constraints are imposed on the components of the metric perturbation.

$$\partial_i\beta_i = 0 \tag{179}$$
$$\partial_i\varepsilon_i = 0 \tag{180}$$
$$\partial_i h_{ij}^{\tau\tau} = 0 \tag{181}$$
$$\delta^{ij}h_{ij}^{\tau\tau} = 0 \tag{182}$$

We also assume boundary conditions such that the metric (174) becomes the flat Minkowski space-time metric at infinity. That is

$$h_{\mu\nu} \rightarrow 0 \qquad \text{as} \qquad r \rightarrow \infty \tag{183}$$

Specifically, the following components of $h_{\mu n}$ are assumed to vanish independently.

$$\alpha \rightarrow 0, \qquad \varepsilon_i \rightarrow 0, \qquad \lambda \rightarrow 0, \qquad \nabla^2\lambda \rightarrow 0 \qquad \text{as} \qquad r \rightarrow \infty \tag{184}$$

The constraints in (179) – (182) and the boundary conditions in (183) and (184) insure that the Helmholtz decomposition of the perturbation metric given in (175) – (177) satisfy the conditions necessary for the vector h_{0i} to be separated into a rotational and irrotational component, and for the tensor h_{ij} to be separated

[24]Note that [28] define h_{00} with a positive ϕ versus a negative. We use a negative to maintain consistency with the Newtonian limit where $g_{00} \approx -1 + h_{00}$ with $h_{00} = -2\phi/c^2$ and ϕ playing the role of a gravitational potential. Also, [28] uses γ instead of α in (176). We reserve γ for the Lorentz factor in Special Relativity which will appear later in this treatment. Lastly, [28] use h_{ij}^{TT} instead of $h_{ij}^{\tau\tau}$. We reserve h_{ij}^{TT} for the transverse-traceless *gauge* which is not to be confused with $h_{ij}^{\tau\tau}$ which is the transverse-traceless *part* of the tensor in this gauge-invariant formulation.

[25]Note that [28] and others typically include a factor of $\frac{1}{2}$ in the symmetrization. Using this convention here would require introducing a factor of 2 in front of $\partial_{(i}\varepsilon_{j)}$ in (177) to compensate for the factor of $\frac{1}{2}$ in the symmertrization. Instead, to simplify the formulation, the symmertrization is redefined without the factor of $\frac{1}{2}$.

into longitudinal, rotational, and transverse components. Specifically, the constraint in (179) requires that β_i is a purely *rotational* vector component of h_{0i}. It is evident from (176) that $\partial_i \alpha$ is a purely *irrotational* vector component of h_{0i}.[26] The HD theorem states that the vector h_{0i} can be completely defined by a rotational component and irrotational component, provided the boundary conditions given in (183) and (184) are satisfied.

Likewise, the constraint in (180) requires that ε_i is a purely *rotational* vector so that $\partial\left({}_i\varepsilon_j\right)$ can be thought of as the rotational part of the h_{ij} tensor. Also, the constraint in (181) requires that $h_{ij}^{\tau\tau}$ is a *transverse* tensor while (182) requires that $h_{ij}^{\tau\tau}$ is *traceless*. Hence the superscript "$\tau\tau$" represents the "transverse-traceless" part of the entire h_{ij} tensor.[27]

It is shown in Section 16 that the metric in $(175)-(177)$ along with all the constraints given by $(179)-(184)$ lead to unique solutions for all of the scalar, vector and tensor quantities given by $\phi, \beta_i, \alpha, H, \varepsilon_i, H, \lambda$ and $h_{ij}^{\tau\tau}$. However, the solutions require knowledge of $h_{\mu\nu}$ over all space.

We can also note that the dimensions of the various metric quantities each differ. Since the metric is dimensionless, then from (175) we can see that ϕ has the dimensions of velocity squared. From (176) we see that β_i has the dimensions of velocity while α has the units of velocity/distance. From (177) we see that $h_{ij}^{\tau\tau}$ and H are dimensionless while ε_i has the dimensions of distance and λ has the dimensions of distance squared.

Lastly, $(175)-(177)$ can be used to write the metric in terms of the relativistic space-time invariant, $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. This gives

$$
\begin{aligned}
ds^2 &= g_{00}c^2dt^2 + 2g_{0i}cdtdx_i + g_{ij}dx_idx_j && (185)\\
&= \left(-1+h_{00}\right)c^2dt^2 + 2h_{0i}cdtdx_i + \left(1+h_{ij}\right)dx_idx_j && (186)\\
&= \left(-c^2-2\phi\right)dt^2 + 2\left(\beta_i+\partial_i\alpha\right)dtdx_i\\
&\quad + \left[1 + h_{ij}^{\tau\tau} + \tfrac{1}{3}\delta_{ij}H + \partial\left({}_i\varepsilon_j\right) + \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\lambda\right]dx_idx_j && (187)
\end{aligned}
$$

The metric perturbation takes the following form as an explicit matrix.

$$
h_{\mu\nu} = \begin{pmatrix}
-\frac{2}{c^2}\phi & \frac{1}{c}(\beta_1+\partial_1\alpha) & \frac{1}{c}(\beta_2+\partial_2\alpha) & \frac{1}{c}(\beta_3+\partial_3\alpha)\\
\frac{1}{c}(\beta_1+\partial_1\alpha) & h_{11}^{TT}+\frac{1}{3}H+2\partial_1\varepsilon_1+\left(\partial_1^2-\frac{1}{3}\nabla^2\right)\lambda & h_{12}^{TT}+\partial({}_1\varepsilon_2)+\partial_1\partial_2\lambda & h_{13}^{TT}+\partial({}_1\varepsilon_3)+\partial_1\partial_3\lambda\\
\frac{1}{c}(\beta_2+\partial_2\alpha) & h_{21}^{TT}+\partial({}_2\varepsilon_1)+\partial_2\partial_1\lambda & h_{22}^{TT}+\frac{1}{3}H+2\partial_2\varepsilon_2+\left(\partial_2^2-\frac{1}{3}\nabla^2\right)\lambda & h_{23}^{TT}+\partial({}_2\varepsilon_3)+\partial_2\partial_3\lambda\\
\frac{1}{c}(\beta_3+\partial_3\alpha) & h_{31}^{TT}+\partial({}_3\varepsilon_1)+\partial_3\partial_1\lambda & h_{32}^{TT}+\partial({}_3\varepsilon_2)+\partial_3\partial_2\lambda & h_{33}^{TT}+\frac{1}{3}H+2\partial_3\varepsilon_3+\left(\partial_3^2-\frac{1}{3}\nabla^2\right)\lambda
\end{pmatrix}
\tag{188}
$$

[26] This follows from the fact that the curl of a gradient is always zero. Taking the curl of $\vec{h} = (h_{01}, h_{02}, h_{03})$ gives $\nabla\times\vec{h} = \nabla\times\left(\vec{\beta}+\nabla\alpha\right) = \nabla\times\vec{\beta}$. This means that the contribution by α vanishes when taking the curl of $\vec{h}$ and it is therefore purely irrotational.

[27] Note that h_{ij}^{TT} (in the TT *gauge*) also satisfies the criteria of being transverse and traceless, however, it eliminates all other components of the metric as shown in Appendix F. Also, h_{ij}^{TT} can only be applied to vacuum solutions whereas $h_{ij}^{\tau\tau}$ will be shown to satisfy a field equations with sources (the transverse-traceless part of the stress tensor.)

3.3 Uniqueness of solutions for the Helmholtz decomposition metric

Here we show that given the constraints in $(179)-(182)$ and the boundary conditions in (184), then the scalars and vectors that make of the metric the components $(\phi,\ \alpha,\ \beta_i,\ H,\ \varepsilon_i,\ \lambda)$ as described in$(175)-(177)$ can be uniquely determined assuming one knows h_{00}, h_{0i}, and h_{ij} over all space. First, we take the divergence of (176) and note that $\partial_i\beta_i = 0$ according to (179).

$$\partial_i h_{0i} = \partial_i\left(\beta_i + \partial_i\alpha\right)/c = \tfrac{1}{c}\partial_i\partial_i\alpha \tag{189}$$

$$\nabla^2\alpha = c\partial_i h_{0i} \tag{190}$$

The solution to this equation will be in terms of h_{0i} and some added function $f(r,t)$ that satisfies

$$\nabla^2 f(r,t) = 0 \tag{191}$$

Since (183) requires that $\alpha \to 0$ as $r \to \infty$, then we must also have $f(r,t) \to 0$ as $r \to \infty$. So the unique solution to (191) is $f(r,t) = 0$ and therefore the unique solution to (190) is only in terms of h_{0i}. Provided we have a function describing h_{0i} over all space, then we can use (176) to also determine β_i.

Next, we consider the components that make up h_{ij} in (177). Note that if we have h_{ij} then we immediately have H as well since it is the spatial trace of h_{ij}. This can be easily seen by taking a spatial trace of (177) and noting that $\partial_i\varepsilon_i = 0$ and $\delta^{ij}h_{ij}^{\tau\tau} = 0$ according to (180) and (182), respectively.

$$\delta^{ij}h_{ij} = \delta^{ij}h_{ij}^{\tau\tau} + \tfrac{1}{3}H\delta^{ij}\delta_{ij} + \delta^{ij}\partial\left({}_i\varepsilon_j\right) + \left(\delta^{ij}\partial_i\partial_j - \tfrac{1}{3}\delta^{ij}\delta_{ij}\nabla^2\right)\lambda \tag{192}$$

$$= \tfrac{1}{3}H(3) + \delta^{ij}\partial_i\varepsilon_j + \delta^{ij}\partial_j\varepsilon_i + \left(\nabla^2 - \tfrac{1}{3}(3)\nabla^2\right)\lambda \tag{193}$$

$$= H \tag{194}$$

Next, we take a derivative of (177) and note that $\partial_i\varepsilon_i = 0$ and $\partial_i h_{ij}^{\tau\tau} = 0$ according to (180) and (181), respectively.

$$\partial_i h_{ij} = \partial_i h_{ij}^{\tau\tau} + \partial_i\partial\left({}_i\varepsilon_j\right) + \tfrac{1}{3}\partial_j H + \left(\nabla^2\partial_j - \tfrac{1}{3}\partial_j\nabla^2\right)\lambda \tag{195}$$

$$= \nabla^2\varepsilon_j + \tfrac{1}{3}\partial_j H + \tfrac{2}{3}\nabla^2\partial_j\lambda \tag{196}$$

$$3\nabla^2\varepsilon_j = 3\partial_i h_{ij} - \partial_j H - 2\nabla^2\partial_j\lambda \tag{197}$$

We can take another derivative of (195) and again note that $\partial_j\varepsilon_j = 0$.

$$\partial_i\partial_j h_{ij} = \tfrac{1}{3}\nabla^2 H + \tfrac{2}{3}\nabla^2\nabla^2\lambda \tag{198}$$

$$2\nabla^2\lambda = 3\partial_i\partial_j h_{ij} - \nabla^2 H \tag{199}$$

Since (184) requires that $\nabla^2\lambda \to 0$ as $r \to \infty$, then the differential equation above has a unique solution for $\nabla^2\lambda$ in terms of h_{ij} and H. This solution can be substituted into (197) to obtain a differential equation for ε_j in terms of h_{ij} and H. This differential equation will also have a unique solution for ε_j since (183) requires that $\varepsilon_j \to 0$ as $r \to \infty$. Hence if we have a function for h_{ij}, then we can determine H, ε_j, and λ from (194), (197) and (199), respectively. Then the solutions for ε_j, λ, and H can be used with h_{ij} in (177) to determine $h_{ij}^{\tau\tau}$. Therefore, we conclude that provided we have functions describing h_{00}, h_{0i} and h_{ij} over all space, we can uniquely determine all the scalars and vectors which make up the metric perturbation in $(175)-(177)$.

3.4 Gauge transformation of the HD metric perturbation components

In (2418) of Appendix A, we found that the coordinate transformation (gauge freedom) of the metric perturbation in linearized GR is given by

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{200}$$

where $\xi_\mu = (\xi_0, \xi_i)$ is an arbitrary 4-displacement vector which acts as a gauge function. If we decompose ξ_i into a purely rotational component and a purely irrotational component, then we can write

$$\xi_\mu = (cA,\ B_i + \partial_i C) \tag{201}$$

where A is a time-like four-vector component, B_i is a purely rotational vector ($\partial_i B_i = 0$), and $\partial_i C$ is a purely *irrotational* vector. Also, by requiring that $\xi_\mu \to 0$ as $r \to \infty$, then we preserve the condition that the transformed metric still goes to the flat Minkowski metric infinitely far away. Lastly, by requiring $C \to 0$ as $r \to \infty$, then by the HD theorem we know ξ_μ is completely described in terms of A, B, and C.

Applying the transformation in (200) to the h_{00} component gives[28] $h'_{00} = h_{00} + 2\partial_0 \xi_0$. Then using h_{00} from (175) and the gauge vector in (201) gives

$$-2\phi'/c^2 \quad = \quad -2\phi/c^2 + 2\left(\tfrac{1}{c}\partial_t\right) cA \tag{202}$$

$$\boxed{\phi' = \phi - c^2 \dot{A}} \tag{203}$$

Next, applying the transformation in (200) to the h_{0i} component gives $h'_{0i} = h_{0i} + \partial_0 \xi_i + \partial_i \xi_0$. Then using h_{0i} from (176) and the gauge vector in (201) gives

$$\left(\beta'_i + \partial_i \alpha'\right)/c \quad = \quad \left(\beta_i + \partial_i \alpha\right)/c + \tfrac{1}{c}\partial_t \left(B_i + \partial_i C\right) + \partial_i cA \tag{204}$$

$$\beta'_i + \partial_i \alpha' \quad = \quad \beta_i + \partial_i \alpha + \dot{B}_i + \partial_i \left(\dot{C} + c^2 A\right) \tag{205}$$

Taking a divergence of (205) and noting that $\partial_i \beta_i = 0$ and $\partial_i B_i = 0$ gives

$$\nabla^2 \alpha' = \nabla^2 \alpha + \nabla^2 \left(\dot{C} + c^2 A\right) \tag{206}$$

$$\nabla^2 \left(\alpha' - \alpha - \dot{C} - c^2 A\right) = 0 \tag{207}$$

Since α, α', A, and C all go to zero as $r \to \infty$, then the only unique solution[29] to the differential equation above is $\alpha' - \alpha - \dot{C} + c^2 A = 0$. Therefore we have

$$\boxed{\alpha' = \alpha + c^2 A + \dot{C}} \tag{208}$$

[28] We have shown in (2424) of Appendix A that to linear order, the four-derivative does not change under a coordinate transformation so $\partial_{\mu'} \approx \partial_\mu$. Therefore, we simply drop the prime notation on the derivative.

[29] For a discussion of the uniqueness of solutions with Dirichlet and Neumann boundary conditions, see-Jackson [40], Section 1.9, p. 37-38. In this context, we are applying Dirichlet boundary conditions since (183) and (184) require that all components of the metric vanish at the boundary $r \to \infty$.

If we write (205) as a vector equation and take the curl, then we obtain

$$\nabla\times\left(\vec{\beta}'+\nabla\alpha'\right) = \nabla\times\left(\vec{\beta}+\nabla\alpha+\dot{\vec{B}}+\nabla\left(\dot{C}+c^2A\right)\right) \tag{209}$$

$$\nabla\times\left(\vec{\beta}'-\vec{\beta}-\dot{\vec{B}}\right) = 0 \tag{210}$$

The solution to this differential equation is

$$\vec{\beta}'-\vec{\beta}-\dot{\vec{B}}+\nabla f(r,t)=0 \tag{211}$$

Since $\nabla\cdot\vec{\beta}=\nabla\cdot\vec{B}=0$, then taking the divergence of the equation above requires that $\nabla^2 f(r,t)=0$. Also, since $\vec{\beta}$ and $\vec{B}$ go to zero as $r\to\infty$, then we must also have that $\nabla f(r,t)\to 0$ as $r\to\infty$ which means that $f(r,t)\to$ *constant* as $r\to\infty$. Then the unique solution of $\nabla^2 f(r,t)=0$ is $f(r,t)=$ *constant* everywhere and therefore $\nabla f(r,t)=0$ everywhere.[30] Then (211) gives

$$\boxed{\beta'=\beta+\dot{B}_i} \tag{212}$$

Lastly, applying the transformation in (200) to the h_{ij} component gives $h'_{ij}=h_{ij}+\partial_i\xi_j+\partial_j\xi_i$. Then using h_{ij} from (177) and the gauge vector in (201) gives

$$\begin{aligned} & h'^{\tau\tau}_{ij}+\tfrac{1}{3}\delta_{ij}H'+\partial\left({}_i\varepsilon'_{j}\right)+\left(\partial_i\partial_j-\tfrac{1}{3}\delta_{ij}\nabla^2\right)\lambda' \\ & = h^{\tau\tau}_{ij}+\tfrac{1}{3}\delta_{ij}H+\partial\left({}_i\varepsilon_{j}\right)+\left(\partial_i\partial_j-\tfrac{1}{3}\delta_{ij}\nabla^2\right)\lambda+\partial_i\left(B_j+\partial_jC\right)+\partial_j\left(B_i+\partial_iC\right) \end{aligned} \tag{213}$$

We can take the spatial trace of (213) and note that $\delta^{ij}h^{\tau\tau}_{ij}=0$. Also, since $\delta^{ij}\delta_{ij}=3$, then upon taking the trace, the terms involving λ become

$$\left(\delta^{ij}\partial_i\partial_j-\tfrac{1}{3}\delta^{ij}\delta_{ij}\nabla^2\right)\lambda=\left(\nabla^2-\nabla^2\right)\lambda=0 \tag{214}$$

We also find that taking the trace of $\partial\left({}_i\varepsilon'_j\right)$ and using (178) with $\partial_i\varepsilon_i=0$ will give

$$\delta^{ij}\partial\left({}_i\varepsilon_j\right)=\delta^{ij}\partial_i\varepsilon_j+\delta^{ij}\partial_j\varepsilon_i=\partial_j\varepsilon_j+\partial_i\varepsilon_i=0 \tag{215}$$

So the trace of (213) will simply become

$$H' = H+\delta^{ij}\partial_i\left(B_j+\partial_jC\right)+\delta^{ij}\partial_j\left(B_i+\partial_iC\right) \tag{216}$$

$$H' = H+\partial_jB_j+2\nabla^2C+\partial_iB_i \tag{217}$$

Since $\partial_iB_i=0$, then we are left with

$$\boxed{H'=H+2\nabla^2C} \tag{218}$$

[30]We will use this argument repeatedly when solving a differential equation where the curl of a vector is zero and the vector is known to be purely rotational and vanishes at infinity. Hence we will not repeat the argument throughout the rest of this treatment.

Next we take two derivatives of (213) using $\partial_i\partial_j$ and note that $\partial_i\varepsilon_i = 0$ and $\partial_i h_{ij}^{\tau\tau} = 0$.

$$\begin{aligned}\tfrac{1}{3}\partial_i\partial_j\delta_{ij}H' + \partial_i\partial_j\left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\lambda' &= \tfrac{1}{3}\partial_i\partial_j\delta_{ij}H + \partial_i\partial_j\left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\lambda \\ &\quad + \partial_i\partial_j\partial_i\left(B_i + \partial_i C\right) + \partial_i\partial_j\partial_j\left(B_i + \partial_i C\right)\end{aligned} \tag{219}$$

Since $\partial_i B_i = 0$, then we have

$$\tfrac{1}{3}\nabla^2 H' + \tfrac{2}{3}\nabla^2\nabla^2\lambda' = \tfrac{1}{3}\nabla^2 H + \tfrac{2}{3}\nabla^2\nabla^2\lambda + 2\nabla^2\nabla^2 C \tag{220}$$

Substituting (218) gives

$$\tfrac{1}{3}\nabla^2\left(H + 2\nabla^2 C\right) + \tfrac{2}{3}\nabla^2\nabla^2\lambda' = \tfrac{1}{3}\nabla^2 H + \tfrac{2}{3}\nabla^2\nabla^2\lambda + 2\nabla^2\nabla^2 C \tag{221}$$

$$2\nabla^2\nabla^2 C + 2\nabla^2\nabla^2\lambda' = 2\nabla^2\nabla^2\lambda + 6\nabla^2\nabla^2 C \tag{222}$$

$$\nabla^2\left[\nabla^2\left(\lambda' - \lambda - 2C\right)\right] = 0 \tag{223}$$

We know that $\nabla^2\lambda$ and C both go to zero as $r \to \infty$ (and hence we can infer that $\nabla^2 C$ must also go to zero), so the unique solution to the outermost differential equation above is

$$\nabla^2\left(\lambda' - \lambda - 2C\right) = 0 \tag{224}$$

Since λ and C both go to zero as $r \to \infty$, then the solution to the differential equation above is $\lambda' - \lambda - 2C = 0$ and therefore we have

$$\boxed{\lambda' = \lambda + 2C} \tag{225}$$

Next we take a single derivative of (213) using ∂_j and note again that $\partial_i h_{ij}^{\tau\tau} = 0$.

$$\begin{aligned}&\tfrac{1}{3}\partial_j\delta_{ij}H' + \partial_j\partial\left({}_i\varepsilon'_j\right) + \partial_j\left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\lambda' \\ &= \tfrac{1}{3}\partial_j\delta_{ij}H + \partial\partial_j\left({}_i\varepsilon_j\right) + \partial_j\left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\lambda + \partial_j\partial_i\left(B_j + \partial_j C\right) + \partial_j\partial_j\left(B_i + \partial_i C\right)\end{aligned} \tag{226}$$

Using (178) and noting that $\partial_i\varepsilon_i = 0$ will cause the terms involving $\partial\left({}_i\varepsilon'_j\right)$ to become

$$\partial_j\partial\left({}_i\varepsilon_j\right) = \partial_j\left(\partial_i\varepsilon_j + \partial_j\varepsilon_i\right) = \nabla^2\varepsilon_i \tag{227}$$

So (226) gives

$$\begin{aligned}&\tfrac{1}{3}\partial_i H' + \nabla^2\varepsilon'_i + \left(\partial_i\nabla^2 - \tfrac{1}{3}\partial_i\nabla^2\right)\lambda' \\ &= \tfrac{1}{3}\partial_i H + \nabla^2\varepsilon_i + \left(\partial_i\nabla^2 - \tfrac{1}{3}\partial_i\nabla^2\right)\lambda + \partial_j\partial_i B_j + \nabla^2 B_i + 2\nabla^2\partial_i C\end{aligned} \tag{228}$$

Using $\partial_i B_i = 0$ and substituting in (218) and (225) gives

$$\tfrac{1}{3}\partial_i\left(H + 2\nabla^2 C\right) + \nabla^2\varepsilon'_i + \tfrac{2}{3}\partial_i\nabla^2\left(\lambda + 2C\right) = \tfrac{1}{3}\partial_i H + \nabla^2\varepsilon_i + \tfrac{2}{3}\partial_i\nabla^2\lambda + \nabla^2 B_i + 2\nabla^2\partial_i C \tag{229}$$

$$\tfrac{2}{3}\partial_i\nabla^2 C + \nabla^2\varepsilon'_i + \tfrac{4}{3}\partial_i\nabla^2 C = \nabla^2\varepsilon_i + \nabla^2 B_i + 2\nabla^2\partial_i C \tag{230}$$

$$\nabla^2\varepsilon'_i = \nabla^2\varepsilon_i + \nabla^2 B_i \tag{231}$$

$$\nabla^2\left(\varepsilon'_i - \varepsilon_i - B_i\right) = 0 \tag{232}$$

Since β_i and ε_i both go to zero as $r \to \infty$, then the unique solution to the differential equation above is $\varepsilon_i' - \varepsilon_i - B_i = 0$ and therefore we have

$$\boxed{\varepsilon_i' = \varepsilon_i + B_i} \tag{233}$$

Lastly, we can substitute (218), (225) and (233) into (213) to obtain

$$\begin{aligned} & h_{ij}'^{\tau\tau} + \frac{1}{3}\delta_{ij}\left(H + 2\nabla^2 C\right) + \partial\left(_i\left(\varepsilon_j + B_j\right)\right) + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)(\lambda + 2C) \\ & \quad = h_{ij}^{\tau\tau} + \frac{1}{3}\delta_{ij}H + \partial\left(_i\varepsilon_j\right) + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\lambda + \partial_i\left(B_j + \partial_j C\right) + \partial_j\left(B_i + \partial_i C\right) \end{aligned} \tag{234}$$

Canceling like terms gives

$$h_{ij}'^{\tau\tau} + \frac{2}{3}\delta_{ij}\nabla^2 C + \partial\left(_i B_j\right) + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)2C = h_{ij}^{\tau\tau} + \partial_i\left(B_j + \partial_j C\right) + \partial_j\left(B_i + \partial_i C\right) \tag{235}$$

Expanding $\partial\left(_i B_j\right)$ and canceling more terms gives

$$h_{ij}'^{\tau\tau} + \partial_i B_j + \partial_j B_i + \left(\partial_i\partial_j\right)2C \quad = \quad h_{ij}^{\tau\tau} + \partial_i B_j + \partial_i\partial_j C + \partial_j B_i + \partial_j\partial_i C \tag{236}$$

$$\boxed{h_{ij}'^{\tau\tau} = h_{ij}^{\tau\tau}} \tag{237}$$

Therefore we find that $h_{ij}^{\tau\tau}$ is gauge-invariant. We now list all the transformation results from this section below.

$$\phi' = \phi - c^2\dot{A} \tag{238}$$
$$\alpha' = \alpha + c^2 A + \dot{C} \tag{239}$$
$$\beta_i' = \beta_i + \dot{B}_i \tag{240}$$
$$H' = H + 2\nabla^2 C \tag{241}$$
$$\lambda' = \lambda + 2C \tag{242}$$
$$\varepsilon_i' = \varepsilon_i + B_i \tag{243}$$

3.5 Gauge invariant potentials

We now use the gauge transformations in $(238)-(243)$ to construct gauge-invariant quantities. Solving (242) for C gives $C=\left(\lambda'-\lambda\right)/2$. Substituting this into (241) and rearranging leads to $H'-\nabla^2\lambda'=H-\nabla^2\lambda$. Therefore, we can define the following gauge-invariant quantity.[31]

$$\boxed{\Theta \equiv \frac{1}{3}\left(H-\nabla^2\lambda\right)} \tag{244}$$

Next, taking a time-derivative of (243) and solving for $\dot{B}_i$ gives $\dot{B}_i=\dot{\varepsilon}'_i-\dot{\varepsilon}_i$. Substituting this into (240) and rearranging gives $\beta'_i-\dot{\varepsilon}'_i=\beta_i-\dot{\varepsilon}_i$. Therefore, we can also define the following gauge-invariant quantity.

$$\boxed{\Xi_i \equiv \beta_i-\dot{\varepsilon}_i} \tag{245}$$

Lastly, taking a time-derivative of (239) and two time-derivatives of (242) gives, respectively,

$$\dot{\alpha}'=\dot{\alpha}+c^2\dot{A}+\ddot{C} \qquad \text{and} \qquad \ddot{\lambda}'=\ddot{\lambda}+2\ddot{C}$$

Combining these two equations by eliminating $\ddot{C}$ gives $\dot{\alpha}'=\dot{\alpha}+c^2\dot{A}+\left(\ddot{\lambda}'-\ddot{\lambda}\right)/2$. Solving this for $c^2\dot{A}$ gives $c^2\dot{A}=\dot{\alpha}'-\dot{\alpha}-\left(\ddot{\lambda}'-\ddot{\lambda}\right)/2$. Substituting this into (238) and rearranging gives $\phi'+\dot{\alpha}'-\ddot{\lambda}'/2=\phi+\dot{\alpha}-\ddot{\lambda}/2$. Therefore, we can define another gauge-invariant quantity as

$$\boxed{\Phi \equiv \phi+\dot{\alpha}-\ddot{\lambda}/2} \tag{246}$$

Since we found in (237) that $h^{\tau\tau}_{ij}$ is also gauge-invariant, then we conclude that there are a total of four gauge-invariant quantities: Θ,Φ,Ξ, and $h^{\tau\tau}_{ij}$. The following are some observations concerning these gauge-invariant quantities.

- The metric component h_{ij} contains two gauge-invariant quantities: $h^{\tau\tau}_{ij}$ and $\Theta=\frac{1}{3}\left(H-\nabla\lambda\right)$. Therefore, we could simply write h_{ij} in (177) as $h_{ij}=h^{\tau\tau}_{ij}+\delta_{ij}\Theta+\partial_i\partial_j\lambda$. However, later when we express the Einstein tensor components in terms of the four gauge-invariant quantities, we will find that it is critical to have h_{ij} expressed using $\frac{1}{3}\left(H-\nabla\lambda\right)$ rather than Θ in order to be able to separate H and $\nabla^2\lambda$. This can be seen specifically in going from (262) to (266) where separating $\frac{1}{3}\left(H-\nabla\lambda\right)$ is necessary. This is due to the fact that λ appears in both Φ and Θ.

- Notice that $\Phi=\phi+\dot{\alpha}-\ddot{\lambda}/2$ consists of one scalar from each of h_{00}, h_{0i}, and h_{ij}. Also, the order of time derivatives of these scalars matches the order of the metric components they came from, that is, the scalar from h_{00} has no time-derivative, the scalar from h_{0i} has one time derivative and the scalar from h_{ij} has two time-derivatives.

- Notice that $\Xi_i=\beta_i+\varepsilon_i$ consists of one vector from each of h_{0i} and h_{ij}. Also, the order of time derivatives of these vectors matches the order of the metric components they came from, that is, the vector from h_{0i} has no time-derivative and the vector from h_{ij} has one time derivative.

- The dimensions of the invariant quantities in $(244)-(246)$ each differ. In Section 15 we identified the dimensions of each of the metric quantities. The quantity H is dimensionless while λ has units of distance squared. Therefore, Θ is dimensionless. We also know that β_i has the dimensions of velocity

[31] The factor of $\frac{1}{3}$ is inserted to simplify the Einstein tensor components found later.

and ε_i has the dimensions of distance. Therefore, Ξ_i has the dimensions of velocity. Lastly, we also know that ϕ has the units of velocity squared and α has the units of velocity/distance. Therefore, Φ has the units of velocity squared. The last invariant, $h_{ij}^{\tau\tau}$, is of course dimensionless since it is simply a metric component with no derivatives or prefactors involved.

- Since the rotational vector component of h_{0i} (given by β_i) is combined with the rotational vector in h_{ij} (given by ε_i) to produce $\Xi_i \equiv \beta_i - \dot{\varepsilon}_i$, then this gauge-invariant potential is also purely rotational: $\partial_i \Xi_i = 0$. This implies that if Ξ_i plays the role of a "gravito-vector potential" analogous to the magnetic vector potential, $\vec{A}$, in electromagnetism (EM), then the condition $\partial_i \Xi_i = 0$ means that our formulation is analogous to the Coulomb gauge in EM where $\nabla \cdot \vec{A} = 0$. However, the important difference is that here we *always* have $\partial_i \Xi_i = 0$. It is *not* the result of a gauge choice like it is in EM.

- Notice that the transformations of some of the metric quantities required substituting the transformation from other metric quantities. For example, the transformation for λ in (225) required using the transformation of H. Likewise, the transformation of ε_i in (233) required the transformation of H and λ. Lastly, the transformation of $h_{ij}^{\tau\tau}$ in (237) required the transformation of H, λ and ε_i. This implies that if any of these metric quantities $(H,\ \lambda$ and $\varepsilon_i)$ are absent (perhaps do to a particular space-time geometry), then the transformations of the quantities that depend on them will be affected. All of this interdependence is encapsulated in the single transformation expression shown in (226) which is the transformation for h_{ij}. The transformations for λ, ε_i and $h_{ij}^{\tau\tau}$ all follow from this expression.

  In fact, for the case of $h_{ij}^{\tau\tau}$, which is found to be a gauge-invariant quantity, we must recognize that its gauge-invariance is heavily dependent on the presence and behavior of H, λ and ε_i. If any of these quantities are missing, then the gauge-invariance of $h_{ij}^{\tau\tau}$ would fail. For example, consider a space-time geometry where $\varepsilon_i = 0$ in all frames. Then we could not substitute the transformation of ε_i' into the expression in (213). As a result, the gauge term $\partial_i B_j$ could not be canceled. This means that $h_{ij}^{\tau\tau}$ would have remaining gauge freedom and could not be classified among the gauge-invariant quantities. This is just a single example of the delicate connection between all the metric quantities and the gauge vector which conspire together to form gauge-invariant quantities.

- Lastly, as a general observation, we note an important difference between gauge invariance in electromagnetism (EM) and gauge invariance in linearized General Relativity (GR). In EM, it is not possible to have gauge-invariant potentials because the gauge freedom itself is *defined* in terms of the potentials. The fields are then constructed from the potentials so that the fields are gauge-invariant. However, here we find that in linearized GR it is indeed possible to have gauge-invariant potentials because the gauge freedom is *not* defined in terms of the potentials but rather it is defined in terms of the *coordinates*. Then using the gauge-invariant potentials, it is possible to construct vector fields that are also gauge-invariant by simply defining the vector fields in terms of the derivatives of the potentials. (Note that taking the Laplacian, curl, or time-derivative of the potentials involves using ∂_μ. However, it is shown in (2424) of Appendix A that a coordinate transformation satisfying the conditions for *linearized* GR leads to $\partial_\mu' \approx \partial_\mu$. Therefore, to first order, the derivatives of gauge-invariant potentials will give gauge-invariant vector fields.)

  We are careful here to refer to *linearized* GR where gauge invariant potentials can be constructed directly from the metric. From these we can then identify gauge invariant *vector* fields that satisfy vector field equations found later in (353). In that sense, we are able to partly reduce GR from a tensor theory to a vector theory. (It is only partly because the field equation given by (333) is still necessarily a tensor equation.) This vector formulation of linearized GR allows for a direct comparison between EM and gravity. However, a treatment that is second order or higher in the metric will obviously introduce *non*-linearities which would not allow for constructing gauge invariant potentials directly from the metric. Similarly, it would not be possible to identify gauge invariant *vector* fields. The only guaranteed gauge invariant quantity is the Einstein tensor since it is constructed from the gauge invariant Riemann tensor. In that case, we find that gravity is fundamentally a tensor theory which can not be even partly reduced to a vector theory.

3.6 The Einstein tensor components in terms of gauge-invariant potentials

Here we evaluate the Einstein tensor components in terms of the gauge-invariant potentials. To determine G_{00} we can use (2385) from Appendix A.

$$G_{00} = \tfrac{1}{2}\left(\partial_i\partial_j h_{ij} - \nabla^2 H\right) \tag{247}$$

Inserting the metric components from (177) gives

$$G_{00} = \tfrac{1}{2}\partial_i\partial_j\left[h_{ij}^{\tau\tau} + \tfrac{1}{3}\delta_{ij}H + \partial\left(_i\varepsilon_j\right) + \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\lambda\right] - \tfrac{1}{2}\nabla^2 H \tag{248}$$

Since $\partial_i h_{ij}^{\tau\tau} = \partial_i\varepsilon_i = 0$, then we have

$$G_{00} = \tfrac{1}{6}\nabla^2 H + \tfrac{1}{2}\left(\nabla^2\nabla^2 - \tfrac{1}{3}\nabla^2\nabla^2\right)\lambda - \tfrac{1}{2}\nabla^2 H \tag{249}$$

$$= -\tfrac{1}{3}\nabla^2 H + \tfrac{1}{3}\nabla^2\nabla^2\lambda \tag{250}$$

Using $\Theta = \frac{1}{3}\left(H - \nabla^2\lambda\right)$ gives

$$\boxed{G_{00} = -\nabla^2\Theta} \tag{251}$$

Next, to determine G_{0i} we can use (2387) from Appendix A.

$$G_{0i} = \tfrac{1}{2}\left(\partial_k\dot{h}_{ki}/c - \partial_i\dot{H}/c + \partial_i\partial_k h_{k0} - \nabla^2 h_{0i}\right) \tag{252}$$

Substituting in the metric components from (176) and (177) gives

$$\begin{aligned} G_{0i} = {} & \tfrac{1}{2}\Big\{\partial_k\left[\dot{h}_{ki}^{TT} + \tfrac{1}{3}\delta_{ki}\dot{H} + \partial\left(_k\dot{\varepsilon}_i\right) + \left(\partial_k\partial_i - \tfrac{1}{3}\delta_{ki}\nabla^2\right)\dot{\lambda}\right]/c \\ & -\partial_i\dot{H}/c + \partial_i\partial_k\left(\beta_k + \partial_k\alpha\right)/c - \nabla^2\left(\beta_i + \partial_i\alpha\right)/c\Big\} \end{aligned} \tag{253}$$

Since $\partial_k h_{ki}^{TT} = \partial_k\beta_k = 0$, then we have

$$\begin{aligned} G_{0i} = {} & \tfrac{1}{2}\Big\{\left[\tfrac{1}{3}\partial_i\dot{H} + \nabla^2\dot{\varepsilon}_i + \left(\nabla^2\partial_i - \tfrac{1}{3}\partial_i\nabla^2\right)\dot{\lambda}\right]/c \\ & -\partial_i\dot{H}/c + \partial_i\nabla^2\alpha/c - \nabla^2\beta_i/c - \nabla^2\partial_i\alpha/c\Big\} \end{aligned} \tag{254}$$

$$= \tfrac{1}{2c}\left(-\tfrac{2}{3}\partial_i\dot{H} + \nabla^2\dot{\varepsilon}_i + \tfrac{2}{3}\nabla^2\partial_i\dot{\lambda} - \nabla^2\beta_i\right) \tag{255}$$

$$= -\tfrac{1}{c}\left(\tfrac{1}{3}\partial_i\dot{H} - \tfrac{1}{2}\nabla^2\dot{\varepsilon}_i - \tfrac{1}{3}\nabla^2\partial_i\dot{\lambda} + \tfrac{1}{2}\nabla^2\beta_i\right) \tag{256}$$

Using $\Theta = \frac{1}{3}\left(H - \nabla^2\lambda\right)$ and $\Xi_i = \beta_i - \dot{\varepsilon}_i$ gives

$$\boxed{G_{0i} = -\frac{1}{c}\left(\partial_i\dot{\Theta} + \tfrac{1}{2}\nabla^2\Xi_i\right)} \tag{257}$$

Lastly, to determine G_{ij}, we can use (2389) from Appendix A.

$$G_{ij} = \tfrac{1}{2}\left[-\Box h_{ij} - \left(\partial_i \dot{h}_{0j} + \partial_j \dot{h}_{0i}\right)/c + \partial_k \partial_i h_{kj} + \partial_k \partial_j h_{ki} + \partial_i \partial_j \left(h_{00} - H\right)\right.$$
$$\left. + 2\delta_{ij}\partial_k \dot{h}_{0k}/c - \delta_{ij}\partial_k\partial_l h_{kl} - \delta_{ij}\nabla^2 h_{00} + \delta_{ij}\nabla^2 H - \delta_{ij}\ddot{H}/c^2\right] \tag{258}$$

Substituting in the metric components from $(175)-(177)$ gives

$$\begin{aligned} G_{ij} = & -\tfrac{1}{2}\Box\left[h^{\tau\tau}_{ij} + \tfrac{1}{3}\delta_{ij}H + \partial\left({}_i\varepsilon_j\right) + \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\lambda\right] \\ & -\tfrac{1}{2}\left[\partial_i\left(\dot{\beta}_j + \partial_j\dot{\alpha}\right) + \partial_j\left(\dot{\beta}_i + \partial_i\dot{\alpha}\right)\right]/c^2 \\ & +\tfrac{1}{2}\partial_k\partial_i\left[h^{TT}_{kj} + \tfrac{1}{3}\delta_{kj}H + \partial\left({}_k\varepsilon_j\right) + \left(\partial_k\partial_j - \tfrac{1}{3}\delta_{kj}\nabla^2\right)\lambda\right] \\ & +\tfrac{1}{2}\partial_k\partial_j\left[h^{TT}_{ki} + \tfrac{1}{3}\delta_{ki}H + \partial\left({}_k\varepsilon_i\right) + \left(\partial_k\partial_i - \tfrac{1}{3}\delta_{ki}\nabla^2\right)\lambda\right] \\ & +\tfrac{1}{2}\partial_i\partial_j\left(-2\phi/c^2 - H\right) + \eta_{ij}\partial_k\left(\dot{\beta}_k + \partial_k\dot{\alpha}\right)/c^2 \\ & -\tfrac{1}{2}\delta_{ij}\partial_k\partial_l\left[h^{TT}_{kl} + \tfrac{1}{3}\delta_{kl}H + \partial\left({}_k\varepsilon_l\right) + \left(\partial_k\partial_l - \tfrac{1}{3}\delta_{kl}\nabla^2\right)\lambda\right] \\ & +\delta_{ij}\nabla^2\phi/c^2 + \tfrac{1}{2}\delta_{ij}\nabla^2 H - \tfrac{1}{2}\delta_{ij}\ddot{H}/c^2 \end{aligned} \tag{259}$$

Since $\partial_i h^{\tau\tau}_{ij} = \partial_i\beta_i = \partial_i\varepsilon_i = 0$, then we have

$$\begin{aligned} G_{ij} = & -\tfrac{1}{2}\Box h^{\tau\tau}_{ij} - \tfrac{1}{6}\Box\delta_{ij}H - \tfrac{1}{2}\Box\partial_i\varepsilon_j - \tfrac{1}{2}\Box\partial_j\varepsilon_i - \tfrac{1}{2}\Box\partial_i\partial_j\lambda + \tfrac{1}{6}\Box\delta_{ij}\nabla^2\lambda \\ & -\tfrac{1}{2}\left(\partial_i\dot{\beta}_j + 2\partial_i\partial_j\dot{\alpha} + \partial_j\dot{\beta}_i\right)/c^2 \\ & +\tfrac{1}{6}\partial_i\partial_j H + \tfrac{1}{2}\partial_i\nabla^2\varepsilon_j + \tfrac{1}{2}\left(\nabla^2\partial_i\partial_j - \tfrac{1}{3}\partial_i\partial_j\nabla^2\right)\lambda \\ & +\tfrac{1}{6}\partial_i\partial_j H + \tfrac{1}{2}\partial_j\nabla^2\varepsilon_i + \tfrac{1}{2}\left(\nabla^2\partial_i\partial_j - \tfrac{1}{3}\partial_i\partial_j\nabla^2\right)\lambda \\ & -\partial_i\partial_j\phi/c^2 - \tfrac{1}{2}\partial_i\partial_j H + \delta_{ij}\nabla^2\dot{\alpha}/c^2 \\ & -\tfrac{1}{6}\delta_{ij}\nabla^2 H - \tfrac{1}{2}\delta_{ij}\left(\nabla^2\nabla^2 - \tfrac{1}{3}\nabla^2\nabla^2\right)\lambda \\ & +\delta_{ij}\nabla^2\phi/c^2 + \tfrac{1}{2}\delta_{ij}\nabla^2 H - \tfrac{1}{2}\delta_{ij}\ddot{H}/c^2 \end{aligned} \tag{260}$$

We can combine three common terms involving $\partial_i\partial_j H$, two terms involving λ, and two terms involving $\delta_{ij}\nabla^2 H$.

$$\begin{aligned} G_{ij} \quad = \quad & -\tfrac{1}{2}\Box h^{\tau\tau}_{ij} - \tfrac{1}{6}\Box\delta_{ij}H - \tfrac{1}{2}\Box\partial_i\varepsilon_j - \tfrac{1}{2}\Box\partial_j\varepsilon_i - \tfrac{1}{2}\Box\partial_i\partial_j\lambda + \tfrac{1}{6}\Box\delta_{ij}\nabla^2\lambda \\ & -\tfrac{1}{2}\partial_i\dot{\beta}_j/c^2 - \partial_i\partial_j\dot{\alpha}/c^2 - \tfrac{1}{2}\partial_j\dot{\beta}_i/c^2 \\ & -\tfrac{1}{6}\partial_i\partial_j H + \tfrac{1}{2}\partial_i\nabla^2\varepsilon_j + \tfrac{1}{2}\partial_j\nabla^2\varepsilon_i + \tfrac{2}{3}\nabla^2\partial_i\partial_j\lambda \\ & -\partial_i\partial_j\phi/c^2 + \delta_{ij}\nabla^2\dot{\alpha}/c^2 + \tfrac{1}{3}\delta_{ij}\nabla^2 H \\ & -\tfrac{1}{3}\delta_{ij}\nabla^2\nabla^2\lambda + \delta_{ij}\nabla^2\phi/c^2 - \tfrac{1}{2}\delta_{ij}\ddot{H}/c^2 \end{aligned} \tag{261}$$

Next we expand the box operator as $\Box = \nabla^2 - \partial_t^2/c^2$.

$$\begin{aligned} G_{ij} \quad = \quad & -\tfrac{1}{2}\Box h^{\tau\tau}_{ij} - \tfrac{1}{6}\nabla^2\delta_{ij}H + \tfrac{1}{6}\delta_{ij}\ddot{H}/c^2 - \tfrac{1}{2}\nabla^2\partial_i\varepsilon_j + \tfrac{1}{2c^2}\partial_i\ddot{\varepsilon}_j - \tfrac{1}{2}\nabla^2\partial_j\varepsilon_i + \tfrac{1}{2c^2}\partial_j\ddot{\varepsilon}_i \\ & -\tfrac{1}{2}\nabla^2\partial_i\partial_j\lambda + \tfrac{1}{2c^2}\partial_i\partial_j\ddot{\lambda} + \tfrac{1}{6}\delta_{ij}\nabla^2\nabla^2\lambda - \tfrac{1}{6}\delta_{ij}\nabla^2\ddot{\lambda}/c^2 \\ & -\tfrac{1}{2}\partial_i\dot{\beta}_j/c^2 - \partial_i\partial_j\dot{\alpha}/c^2 - \tfrac{1}{2}\partial_j\dot{\beta}_i/c^2 \\ & -\tfrac{1}{6}\partial_i\partial_j H + \tfrac{1}{2}\partial_i\nabla^2\varepsilon_j + \tfrac{1}{2}\partial_j\nabla^2\varepsilon_i + \tfrac{2}{3}\nabla^2\partial_i\partial_j\lambda \\ & -\partial_i\partial_j\phi/c^2 + \delta_{ij}\nabla^2\dot{\alpha}/c^2 + \tfrac{1}{3}\delta_{ij}\nabla^2 H \\ & -\tfrac{1}{3}\delta_{ij}\nabla^2\nabla^2\lambda + \delta_{ij}\nabla^2\phi/c^2 - \tfrac{1}{2}\delta_{ij}\ddot{H}/c^2 \end{aligned} \tag{262}$$

We can combine two common terms involving $\nabla^2\delta_{ij}H$, two terms involving $\delta_{ij}\ddot{H}$, two terms involving $\nabla^2\partial_i\partial_j\lambda$, and two terms involving $\delta_{ij}\nabla^2\nabla^2\lambda$. We also cancel two terms involving $\nabla^2\partial_j\varepsilon_i$ and two terms involving $\nabla^2\partial_i\varepsilon_j$.

$$\begin{aligned} G_{ij} \quad = \quad & -\tfrac{1}{2}\Box h^{\tau\tau}_{ij} + \tfrac{1}{6}\nabla^2\delta_{ij}H - \tfrac{1}{3}\delta_{ij}\ddot{H}/c^2 + \tfrac{1}{2c^2}\partial_i\ddot{\varepsilon}_j + \tfrac{1}{2c^2}\partial_j\ddot{\varepsilon}_i \\ & +\tfrac{1}{6}\nabla^2\partial_i\partial_j\lambda + \tfrac{1}{2c^2}\partial_i\partial_j\ddot{\lambda} - \tfrac{1}{6}\delta_{ij}\nabla^2\nabla^2\lambda - \tfrac{1}{6}\delta_{ij}\nabla^2\ddot{\lambda}/c^2 \\ & -\tfrac{1}{2}\partial_i\dot{\beta}_j/c^2 - \partial_i\partial_j\dot{\alpha}/c^2 - \tfrac{1}{2}\partial_j\dot{\beta}_i/c^2 \\ & -\tfrac{1}{6}\partial_i\partial_j H - \partial_i\partial_j\phi/c^2 + \delta_{ij}\nabla^2\dot{\alpha}/c^2 + \delta_{ij}\nabla^2\phi/c^2 \end{aligned} \tag{263}$$

Next we write the coefficient of $\frac{1}{6}\delta_{ij}\nabla^2\ddot{\lambda}$ as $\left(\frac{1}{2}-\frac{1}{3}\right)\delta_{ij}\nabla^2\ddot{\lambda}$. We also group together terms with $\dot{\beta}_i$ and $\ddot{\varepsilon}_i$ to produce the invariant $\dot{\Xi}_i = \dot{\beta}_i - \ddot{\varepsilon}_i$.

$$\begin{aligned} G_{ij} &= -\tfrac{1}{2}\Box h_{ij}^{\tau\tau} + \tfrac{1}{6}\nabla^2\delta_{ij}H - \tfrac{1}{3}\delta_{ij}\ddot{H}/c^2 + \tfrac{1}{6}\nabla^2\partial_i\partial_j\lambda \\ &\quad + \tfrac{1}{2c^2}\partial_i\partial_j\ddot{\lambda} - \tfrac{1}{6}\delta_{ij}\nabla^2\nabla^2\lambda - \left(\tfrac{1}{2}-\tfrac{1}{3}\right)\delta_{ij}\nabla^2\ddot{\lambda}/c^2 \qquad (264) \\ &\quad -\tfrac{1}{2c^2}\partial_i\left(\dot{\beta}_j - \ddot{\varepsilon}_j\right) - \tfrac{1}{2c^2}\partial_j\left(\dot{\beta}_i - \ddot{\varepsilon}_i\right) - \partial_i\partial_j\dot{\alpha}/c^2 - \tfrac{1}{6}\partial_i\partial_j H \\ &\quad -\partial_i\partial_j\phi/c^2 + \delta_{ij}\nabla^2\dot{\alpha}/c^2 + \delta_{ij}\nabla^2\phi/c^2 \qquad (265) \end{aligned}$$

Now we substitute $\partial_i\left(\dot{\beta}_j - \ddot{\varepsilon}_j\right) + \partial_j\left(\dot{\beta}_i - \ddot{\varepsilon}_i\right) = \partial_i\left(\dot{\Xi}_j\right)$. We also rearrange terms in preparation to substitute the invariants $\Phi = \phi + \dot{\alpha} - \ddot{\lambda}/2$ and $\Theta = \frac{1}{3}\left(H - \nabla^2\lambda\right)$.

$$\begin{aligned} G_{ij} &= -\tfrac{1}{2}\Box h_{ij}^{\tau\tau} + \tfrac{1}{6}\nabla^2\delta_{ij}\left(H - \nabla^2\lambda\right) - \tfrac{1}{3c^2}\delta_{ij}\left(\ddot{H} - \nabla^2\ddot{\lambda}\right) \\ &\quad -\tfrac{1}{2c^2}\partial_i\left(\dot{\Xi}_j\right) - \tfrac{1}{6}\partial_i\partial_j\left(H - \nabla^2\lambda\right) \\ &\quad -\partial_i\partial_j\left(\phi + \dot{\alpha} - \tfrac{1}{2}\ddot{\lambda}\right)/c^2 + \delta_{ij}\nabla^2\left(\phi + \dot{\alpha} - \tfrac{1}{2}\ddot{\lambda}\right)/c^2 \qquad (266) \end{aligned}$$

Lastly we substitute in the invariant quantities and group similar terms.

$$\begin{aligned} G_{ij} &= -\tfrac{1}{2}\Box h_{ij}^{\tau\tau} + \tfrac{1}{2}\delta_{ij}\nabla^2\Theta - \tfrac{1}{c^2}\delta_{ij}\ddot{\Theta} \\ &\quad -\tfrac{1}{2c^2}\partial_i\left(\dot{\Xi}_j\right) - \tfrac{1}{2}\partial_i\partial_j\Theta - \partial_i\partial_j\Phi/c^2 + \delta_{ij}\nabla^2\Phi/c^2 \qquad (267) \end{aligned}$$

$$\boxed{G_{ij} = -\frac{1}{2}\Box h_{ij}^{\tau\tau} - \frac{1}{c^2}\delta_{ij}\ddot{\Theta} - \frac{1}{2c^2}\partial_i\left(\dot{\Xi}_j\right) + \delta_{ij}\nabla^2\left(\tfrac{1}{2}\Theta + \Phi/c^2\right) - \partial_i\partial_j\left(\tfrac{1}{2}\Theta + \Phi/c^2\right)} \qquad (268)$$

The following are some observations concerning the Einstein tensor components.

- The gauge-invariant quantity obey an interesting pattern We find that $\Theta, \dot{\Theta}, \ddot{\Theta}$ appear in G_{00}, G_{0i} and G_{ij}, respectively, while Ξ_i and $\ddot{\Xi}_i$ appear in G_{0i} and G_{ij}, respectively, and lastly, Φ appears only in G_{ij}.
- G_{ij} has a very similar form to h_{ij} in (177). In fact, if we define a quantity $\Lambda = -\frac{1}{c^2}\left(\Phi + \frac{c^2}{2}\Theta\right)$, then we can write $G_{ij} = -\frac{1}{2}\Box h_{ij}^{\tau\tau} - \frac{1}{c^2}\delta_{ij}\ddot{\Theta} - \frac{1}{2c^2}\partial_i\left(\dot{\Xi}_j\right) + \frac{1}{c^2}\left(\partial_i\partial_j - \delta_{ij}\nabla^2\right)\Lambda$ which further resembles h_{ij} in (177). This indicates that Λ contains all the longitudinal degrees of freedom of G_{ij}. It is therefore clearly related to the generalized gravito-scalar potential defined in (345) as $\varphi_G \equiv \frac{1}{2}\left(\Phi - \frac{c^2}{2}\Theta\right)$.
- The transverse-traceless part of G_{ij} is $G_{ij}^{\tau\tau} = -\frac{1}{2}\Box h_{ij}^{\tau\tau}$. This matches the linearized Einstein tensor in the trace-reversed harmonic gauge[32] which was found in (2454) of Appendix B as $G_{\mu\nu}^{(HG)} = -\frac{1}{2}\Box\bar{h}_{\mu\nu}$.

[32] The notation $G_{\mu\nu}^{(HG)}$ and $G_{\mu\nu}^{(HD)}$ is used here to distinguish between the Einstein tensor in the trace-reversed harmonic gauge and the Einstein tensor in terms of the Helmholtz Decomposition metric in (175) – (177).

Note that $G_{ij}^{(HG)}$ is *not* transverse-traceless, therefore it gives the impression that gravitational waves could have longitudinal as well as transverse components. Furthermore, $G_{\mu\nu}^{(HG)} = -\frac{1}{2}\Box\bar{h}_{\mu\nu}$ gives the impression that *all* the components of $\bar{h}_{\mu\nu}$ could be propagating degrees of freedom since they all satisfy wave equations. In actuality, the Helmholtz Decomposition approaches shows that the only (gauge-invariant) propagating degrees of freedom which satisfy a wave equation are $h_{ij}^{\tau\tau}$. In fact, $G_{00}^{(HD)}$ does not contain any time derivatives, $G_{0i}^{(HD)}$ only contains first-order time derivatives, and $G_{ij}^{(HD)}$ contains second-order time derivatives but only a wave operator on $h_{ij}^{\tau\tau}$.

3.7 The HD stress tensor components and conservation laws

Next we define a Helmholtz decomposition of the stress-energy-momentum tensor as follows.

$$
\begin{aligned}
T_{00} &= \rho c^2 && (269)\\
T_{0i} &= c\,(R_i + \partial_i I) && (270)\\
T_{ij} &= T_{ij}^{\tau\tau} + \mathbb{P}\delta_{ij} + \partial\,({}_i r_j) + \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right) L && (271)
\end{aligned}
$$

This formulation of the stress tensor follows [28] except for some notational differences.[33] Similar to [28], we also impose the following constraints

$$
\begin{aligned}
\partial_i R_i &= 0 && (272)\\
\partial_i r_i &= 0 && (273)\\
\partial_i T_{ij}^{\tau\tau} &= 0 && (274)\\
\delta^{ij} T_{ij}^{\tau\tau} &= 0 && (275)
\end{aligned}
$$

These constraints obviously match the constraints we had on the metric in $(179)-(182)$. We also assume boundary conditions such that the stress tensor vanishes at infinity. That is

$$T_{\mu\nu} \to 0 \qquad \text{as} \qquad r \to \infty \tag{276}$$

Specifically, the following components of $T_{\mu n}$ are assumed to vanish independently.

$$I \to 0, \qquad r_i \to 0, \qquad L \to 0, \qquad \nabla^2 L \to 0 \qquad \text{as} \qquad r \to \infty \tag{277}$$

The constraints in $(272)-(275)$ and the boundary conditions in (276) and (277) insure that the Helmholtz decomposition of the stress tensor given in $(269)-(271)$ satisfy the conditions necessary for the vector T_{0i} to be separated into a rotational and irrotational component, and for the tensor T_{ij} to be separated into longitudinal, rotational, and transverse components.

Specifically, the constraint in (272) requires that R_i is a purely *rotational* vector component of T_{0i}. It is evident from (270) that $\partial_i I$ is a purely *irrotational* vector component of T_{0i}. The Helmholtz decomposition theorem states that T_{0i} can be completely defined by a rotational component and irrotational component, provided the boundary conditions given in (276) and (277) are satisfied.

Likewise, the constraint in (273) requires that r_i is a purely *rotational* vector so that $\partial\,({}_i r_j)$ can be thought of as the rotational part of the T_{ij} tensor. Also, the constraint in (274) requires that $T_{ij}^{\tau\tau}$ is a transverse tensor while (275) requires that $T_{ij}^{\tau\tau}$ is traceless. Hence the superscript TT represents the "transverse-traceless" stress given by $T_{ij}^{\tau\tau}$. The stress tensor takes the following form as an explicit matrix.

---

[33]The notation in [28] uses S_i and S instead of R_i and I, respectively, in (270). They also use σ_{ij}, σ_i and σ instead of $T_{ij}^{\tau\tau}$, r_i and L, respectively, in (271). We use a different symbol for each quantity in (271) to avoid confusion that may come from using the same Greek letter for multiple quantities.

We also use $\mathbb{P}$ instead of P which we reserve to represent the pressure of an ideal fluid as found in (2681). The quantity $\mathbb{P}$ is a "pressure-like" quantity since it is a scalar which appears only on the diagonal of the stress tensor. However, it is not truly the pressure since we are describing $T_{\mu\nu}$, not $T^{\mu\nu}$. An importatnt distinction between $\mathbb{P}$ and P is that for an ideal fluid, we find that the pressure, P, appears in T_{0i} as well as in the off-diagonal elements of T_{ij} as seen in (376) and (377). This is *not* consistent with $T_{\mu\nu}$ described here in $(269)-(271)$ where $\mathbb{P}$ does *not* appear in T_{0i} or in the off-diagonal elements of T_{ij}. Later we find in (395) that the expression relating $\mathbb{P}$ and P actually involves ρ and v_i for an ideal fluid.

$$T_{\mu\nu} = \begin{pmatrix} \rho c^2 & c(R_1 + \partial_1 I) & c(R_2 + \partial_2 I) & c(R_3 + \partial_3 I) \\ c(R_1 + \partial_1 I) & T_{11}^{\tau\tau} + \mathbb{P} + 2\partial_1 r_1 + \left(\partial_1^2 - \frac{1}{3}\nabla^2\right)L & T_{12}^{\tau\tau} + \partial({}_1 r_2) + \partial_1\partial_2 L & T_{13}^{\tau\tau} + \partial({}_1 r_3) + \partial_1\partial_3 L \\ c(R_2 + \partial_2 I) & T_{21}^{\tau\tau} + \partial({}_2 r_1) + \partial_2\partial_1 L & T_{22}^{\tau\tau} + \mathbb{P} + 2\partial_2 r_2 + \left(\partial_2^2 - \frac{1}{3}\nabla^2\right)L & T_{23}^{\tau\tau} + \partial({}_2 r_3) + \partial_2\partial_3 L \\ c(R_3 + \partial_3 I) & T_{31}^{\tau\tau} + \partial({}_3 r_1) + \partial_3\partial_1 L & T_{32}^{\tau\tau} + \partial({}_3 r_2) + \partial_3\partial_2 L & T_{33}^{\tau\tau} + \mathbb{P} + 2\partial_3 r_3 + \left(\partial_3^2 - \frac{1}{3}\nabla^2\right)L \end{pmatrix} \tag{278}$$

In (2394) of Appendix A, it is shown that the linearized Einstein equation also leads to a linearized conservation law for the stress-energy-momentum tensor given by

$$\partial^\nu T_{\mu\nu} = 0 \tag{279}$$

Summing over ν gives

$$\partial^0 T_{\mu 0} + \partial^i T_{\mu i} = 0 \tag{280}$$

For $\mu = 0$ we have the following mass-momentum continuity equation.

$$\partial^0 T_{00} + \partial^i T_{0i} = 0 \tag{281}$$

Inserting (269) and (270) gives

$$-\tfrac{1}{c}\partial_t\left(\rho c^2\right) + \partial_i\left(R_i + \partial_i I\right)c = 0 \tag{282}$$

Since $\partial_i R_i = 0$, then the mass-momentum continuity equation becomes

$$\boxed{\dot{\rho} = \nabla^2 I} \tag{283}$$

If we let $\mu = i$ in (280), then we have the following momentum-stress conservation equation.

$$\partial^0 T_{i0} + \partial^j T_{ij} = 0 \tag{284}$$

Inserting (270) and (271) gives

$$-\tfrac{1}{c}\partial_t\left(R_i + \partial_i I\right)c + \partial_i\left[T_{ij}^{\tau\tau} + \delta_{ij}\mathbb{P} + \partial\left({}_i r_j\right) + \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)L\right] = 0 \tag{285}$$

Since $\partial_i T_{ij}^{\tau\tau} = \partial_i r_i = 0$, then we have

$$-\dot{R}_i - \partial_i \dot{I} + \partial_i \mathbb{P} + \nabla^2 r_i + \tfrac{2}{3}\partial_i \nabla^2 L = 0 \tag{286}$$

We can obtain the *irrotational* components of this conservation law by taking a divergence using ∂_i.

$$-\partial_i \dot{R}_i - \nabla^2 \dot{I} + \nabla^2 \mathbb{P} + \partial_i \nabla^2 r_i + \tfrac{2}{3}\nabla^2\nabla^2 L = 0 \tag{287}$$

Since $\partial_i R_i = \partial_i r_i = 0$, then we have

$$\nabla^2\left(-\dot{I} + \mathbb{P} + \tfrac{2}{3}\nabla^2 L\right) \quad = \quad 0 \tag{288}$$

Since I, $\mathbb{P}$, and $\nabla^2 L$ go to zero as $r \to \infty$, then $\dot{I} + \mathbb{P} + \frac{2}{3}\nabla^2 L = 0$ is the unique solution to the differential equation above. Therefore we have the following *irrotational* momentum-stress continuity equation.

$$\boxed{\tfrac{2}{3}\nabla^2 L = \dot{I} - \mathbb{P}} \tag{289}$$

We can also obtain the *rotational* components of the conservation law in (286) by expressing the equation as a vector equation taking the curl.

$$\nabla\times\left(-\dot{\vec{R}}-\nabla\dot{I}+\nabla\mathbb{P}+\nabla^2\vec{r}+\tfrac{2}{3}\nabla\nabla^2 L\right) \quad = \quad 0 \tag{290}$$

$$\nabla\times\left(\nabla^2\vec{r}-\dot{\vec{R}}\right) \quad = \quad 0 \tag{291}$$

Since $\nabla\cdot\vec{r}=\nabla\cdot\vec{R}=0$ and $\vec{r}$ and $\vec{R}$ both go to zero as $r\rightarrow\infty$, then the unique solution to the differential equation above is $\nabla^2\vec{r}-\dot{\vec{R}}=0$. So we have the following *rotational* momentum-stress continuity equation

$$\boxed{\nabla^2 r_i=\dot{R}_i} \tag{292}$$

The following are some observations concerning the conservation of stress-energy-momentum.

- We find that $T_{ij}^{\tau\tau}$ does not participate in any conservation law. This is associated with the fact that it contains the *only* degrees of freedom of the mass-energy source that produce gravitational radiation as shown later in $(330)-(333)$. In a sense, what we find here is that *all* the other degrees of freedom of the stress tensor have a conservation law and therefore do not need to radiate. Any change in one stress tensor quantity produces a change in another quantity so that the over all energy-momentum of the system is conserved without the system radiating. However, because the $T_{ij}^{\tau\tau}$ degrees of freedom do not have a conservation law that links them to other stress tensor quantities, then they have no other *internal* channel to transfer energy-momentum to within the mass-energy distribution. Consequently, they must radiate energy and momentum instead.

- The fact that $T_{ij}^{\tau\tau}$ does not participate in any conservation law implies that mathematically it is possible to have a situation where there is no other source in the stress tensor besides $T_{ij}^{\tau\tau}$ and therefore it is possible to have a mass-energy distribution that only produces an $h_{ij}^{\tau\tau}$ field and nothing else. This is disturbing since it implies that a physical system could have $\rho=\mathbb{P}=L=0$ and $R_i=\partial_i I=r_i=0$ and yet still produce radiation as long as $T_{ij}^{\tau\tau}\neq 0$. However, it is clearly nonphysical to have a system which produces gravitational waves and yet has $T_{00}=0$ which means no rest mass-energy, electromagnetic fields, or any other static source of Newtonian gravity.

3.8 Gauge invariant field equations

We now construct the Einstein field equations using the Einstein tensor components developed in Section 19 and the stress tensor components defined in Section 20. The Einstein field equations are given by $G_{\mu\nu} = \kappa T_{\mu\nu}$ where $\kappa = 8\pi G/c^4$. Using G_{00} from (251) and T_{00} from (269), we immediately identify the following field equation.

$$\boxed{\nabla^2\Theta = -\frac{8\pi G}{c^2}\rho} \tag{293}$$

Next, we use G_{0i} from (257) and T_{0i} from (270) to write $G_{0i} = \kappa T_{0i}$ as

$$-\tfrac{1}{c}\left(\partial_i\dot{\Theta} + \tfrac{1}{2}\nabla^2\Xi_i\right) = \kappa c\left(R_i + \partial_i I\right) \tag{294}$$

We can obtain an *irrotational* field equation by taking a divergence using ∂_i.

$$\nabla^2\dot{\Theta} + \tfrac{1}{2}\nabla^2\partial_i\Xi_i = -\kappa c^2\left(\partial_i R_i + \nabla^2 I\right) \tag{295}$$

Since $\partial_i\Xi_i = \partial_i R_i = 0$, then we have

$$\nabla^2\dot{\Theta} = -c^2\kappa\nabla^2 I \tag{296}$$

From (283) we know $\dot{\rho} = \nabla^2 I$ so we can write the field equation above as

$$\nabla^2\dot{\Theta} = -\frac{8\pi G}{c^2}\dot{\rho} \tag{297}$$

We can integrate with respect to time and recognize from (293) that any integration constant must be zero. Therefore we simply recover the same field equation as (293) again. We can also obtain a *rotational* field equation from (294) by writing the equation as a vector equation and taking a curl.

$$\nabla\times\left(\nabla\dot{\Theta} + \tfrac{1}{2}\nabla^2\vec{\Xi}\right) = -\kappa c^2\nabla\times\left(\vec{R} + \nabla I\right) \tag{298}$$

$$\nabla\times\left(\tfrac{1}{2}\nabla^2\vec{\Xi} + \kappa c^2\vec{R}\right) = 0 \tag{299}$$

The solution to this differential equation is

$$\tfrac{1}{2}\nabla^2\vec{\Xi} + \kappa c^2\vec{R} + \nabla f(r,t) = 0 \tag{300}$$

Since $\nabla\cdot\vec{\Xi} = \nabla\cdot\vec{R} = 0$ and both $\vec{\Xi}$ and $\vec{R}$ go to zero as $r\to\infty$, then (300) gives the following field equation

$$\boxed{\nabla^2\Xi_i = -\frac{16\pi G}{c^2}R_i} \tag{301}$$

Next, we use G_{ij} from (268) and T_{ij} from (271) to write $G_{ij} = \kappa T_{ij}$ as

$$\begin{aligned}&-\tfrac{1}{2}\Box h_{ij}^{\tau\tau} - \tfrac{1}{c^2}\delta_{ij}\ddot{\Theta} - \tfrac{1}{2c^2}\partial_i\left(\dot{\Xi}_j\right) + \delta_{ij}\nabla^2\left(\tfrac{1}{2}\Theta + \Phi/c^2\right) - \partial_i\partial_j\left(\tfrac{1}{2}\Theta + \Phi/c^2\right)\\ &= \kappa\left[T_{ij}^{\tau\tau} + \delta_{ij}\mathbb{P} + \partial\left({}_i r_j\right) + \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)L\right]\end{aligned} \tag{302}$$

Taking two derivatives using $\partial_i\partial_j$ and noting that $\partial_j h^{\tau\tau}_{ij} = \partial_j T^{\tau\tau}_{ij} = 0$ gives

$$-\tfrac{1}{c^2}\nabla^2\ddot{\Theta} + \nabla^2\nabla^2\left(\tfrac{1}{2}\Theta + \Phi/c^2\right) - \nabla^2\nabla^2\left(\tfrac{1}{2}\Theta + \Phi/c^2\right) = \kappa\left(\nabla^2\mathbb{P} + \tfrac{2}{3}\nabla^2\nabla^2 L\right) \tag{303}$$

$$-\tfrac{1}{c^2}\nabla^2\ddot{\Theta} = \kappa\left(\nabla^2\mathbb{P} + \tfrac{2}{3}\nabla^2\nabla^2 L\right) \tag{304}$$

From (289) we know $\frac{2}{3}\nabla^2 L = \dot{I} - P$ so we can write the field equation above as

$$-\tfrac{1}{c^2}\nabla^2\ddot{\Theta} = \kappa\left[\nabla^2\mathbb{P} + \nabla^2\left(\dot{I} - \mathbb{P}\right)\right] \tag{305}$$

$$\nabla^2\ddot{\Theta} = -c^2\kappa\nabla^2\dot{I} \tag{306}$$

From (289) we know $\ddot{\rho} = \nabla^2\dot{I}$ so we can write the field equation above as

$$\nabla^2\ddot{\Theta} = -\frac{8\pi G}{c^2}\ddot{\rho} \tag{307}$$

We can integrate with respect to time twice and recognize from (293) that any integration constant must be zero. Therefore we simply recover the same field equation as (293) again. Next we take a single derivative of (302) using ∂_j and note that $\partial_j h^{\tau\tau}_{ij} = \partial_j T^{\tau\tau}_{ij} = \partial_i\dot{\Xi}_i = 0$.

$$-\tfrac{1}{c^2}\partial_i\ddot{\Theta} - \tfrac{1}{2c^2}\nabla^2\dot{\Xi}_i + \partial_i\nabla^2\left(\tfrac{1}{2}\Theta + \Phi/c^2\right) - \partial_i\nabla^2\left(\tfrac{1}{2}\Theta + \Phi/c^2\right) = \kappa\left(\partial_i\mathbb{P} + \nabla^2 r_i + \tfrac{2}{3}\partial_i\nabla^2 L\right) \tag{308}$$

$$-\tfrac{1}{c^2}\partial_i\ddot{\Theta} - \tfrac{1}{2c^2}\nabla^2\dot{\Xi}_i = \kappa\left(\partial_i\mathbb{P} + \nabla^2 r_i + \tfrac{2}{3}\partial_i\nabla^2 L\right) \tag{309}$$

From (289) we know $\frac{2}{3}\nabla^2 L = \dot{I} - \mathbb{P}$ so we can write the field equation above as

$$-\tfrac{1}{c^2}\partial_i\ddot{\Theta} - \tfrac{1}{2c^2}\nabla^2\dot{\Xi}_i = \kappa\left(\partial_i\mathbb{P} + \nabla^2\vec{r} + \tfrac{2}{3}\partial_i\left(\dot{I} - \mathbb{P}\right)\right) \tag{310}$$

Gathering terms with common derivatives on each side gives

$$\nabla^2\left(-\tfrac{1}{2c^2}\dot{\Xi}_i - \kappa\dot{\vec{r}}\right) = \nabla\left(\kappa\mathbb{P} + \tfrac{2}{3}\kappa\left(\dot{I} - \mathbb{P}\right) + \tfrac{1}{c^2}\ddot{\Theta}\right) \tag{311}$$

Writing the equation as a vector equation and taking the curl gives

$$\nabla\times\left[\nabla^2\left(-\tfrac{1}{2c^2}\dot{\vec{\Xi}} - \kappa\vec{r}\right)\right] = \nabla\times\nabla\left(\kappa\mathbb{P} + \tfrac{2}{3}\kappa\left(\dot{I} - \mathbb{P}\right) + \tfrac{1}{c^2}\ddot{\Theta}\right) \tag{312}$$

$$\nabla\times\left[\nabla^2\left(\dot{\vec{\Xi}} + 2c^2\kappa\vec{r}\right)\right] = 0 \tag{313}$$

Since $\nabla\cdot\vec{\Xi} = \nabla\cdot\vec{r} = 0$ and both $\vec{\Xi}$ and $\vec{R}$ go to zero as $r \to \infty$, then we have the following field equation

$$\nabla^2\dot{\Xi}_i = -\frac{16\pi G}{c^2}\nabla^2 r_i \tag{314}$$

From (292) we know $\nabla^2 r_i = \dot{R}_i$ so we obtain

$$\nabla^2\dot{\Xi}_i = -\frac{16\pi G}{c^2}\dot{R}_i \tag{315}$$

We can integrate with respect to time and recognize from (293) that any integration constant must be zero. Therefore we simply recover the same field equation as (301) again. Next we can take the trace of (302) and note that $\delta^{ij}h^{\tau\tau}_{ij} = \delta^{ij}\partial_i\left(\dot{\Xi}_j\right) = \delta^{ij}T^{\tau\tau}_{ij} = \delta^{ij}\partial\left({}_ir_j\right) = 0$. This gives

$$-\tfrac{1}{c^2}3\ddot{\Theta} + 3\nabla^2\left(\tfrac{1}{2}\Theta + \Phi/c^2\right) - \nabla^2\left(\tfrac{1}{2}\Theta + \Phi/c^2\right) = 3\kappa\mathbb{P} \tag{316}$$

$$-\tfrac{1}{c^2}3\ddot{\Theta} + \nabla^2\Theta + 2\nabla^2\Phi/c^2 = 3\kappa\mathbb{P} \tag{317}$$

Substituting $\nabla^2\Theta = -c^2\kappa\rho$ from (293) gives

$$-\tfrac{1}{c^2}3\ddot{\Theta} - c^2\kappa\rho + 2\nabla^2\Phi/c^2 = 3\kappa\mathbb{P} \tag{318}$$

Applying ∇^2 to each term gives

$$-\tfrac{1}{c^2}3\nabla^2\ddot{\Theta} - c^2\kappa\nabla^2\rho + 2\nabla^2\nabla^2\Phi/c^2 = 3\kappa\nabla^2\mathbb{P} \tag{319}$$

Combining $\nabla^2\ddot{\Theta} = -c^2\kappa\ddot{\rho}$ from (293) with $\ddot{\rho} = \nabla^2\dot{I}$ from (289) gives $\nabla^2\ddot{\Theta} = -c^2\kappa\nabla^2\dot{I}$. Substituting this into (319) gives

$$3\kappa\nabla^2\dot{I} - c^2\kappa\nabla^2\rho + 2\nabla^2\nabla^2\Phi/c^2 = 3\kappa\nabla^2\mathbb{P} \tag{320}$$

$$\nabla^2\left(3\kappa\dot{I} - c^2\kappa\rho + 2\nabla^2\Phi/c^2 - 3\kappa\mathbb{P}\right) = 0 \tag{321}$$

Since I, ρ, $\mathbb{P}$, and Φ all go to zero as $r \to \infty$, then the only unique solution to the differential equation above is $3\kappa\dot{I} - c^2\kappa\rho + 2\nabla^2\Phi/c^2 - 3\mathbb{P} = 0$. Therefore we have

$$\nabla^2\Phi = \tfrac{c^2}{2}\left(c^2\kappa\rho + 3\kappa\mathbb{P} - 3\kappa\dot{I}\right) \tag{322}$$

$$\boxed{\nabla^2\Phi = 4\pi G\left(\rho + \frac{3}{c^2}\left(\mathbb{P} - \dot{I}\right)\right)} \tag{323}$$

Lastly, we again use (302) and take ∇^2 of each term.

$$\begin{aligned}
&-\tfrac{1}{2}\nabla^2\Box h^{\tau\tau}_{ij} - \tfrac{1}{c^2}\delta_{ij}\nabla^2\ddot{\Theta} - \tfrac{1}{2c^2}\partial_i\left(\nabla^2\dot{\Xi}_j\right)\\
&+\delta_{ij}\nabla^2\left(\tfrac{1}{2}\nabla^2\Theta + \nabla^2\Phi/c^2\right) - \partial_i\partial_j\left(\tfrac{1}{2}\nabla^2\Theta + \nabla^2\Phi/c^2\right)\\
&= \kappa\left[\nabla^2 T^{\tau\tau}_{ij} + \nabla^2\delta_{ij}\mathbb{P} + \partial\left({}_i\nabla^2 r_j\right) + \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\nabla^2 L\right]
\end{aligned} \tag{324}$$

Next we substitute $\nabla^2\Theta = -c^2\kappa\rho$ from (293) and $\nabla^2\Phi = \frac{c^4\kappa}{2}\left(\rho + \frac{3}{c^2}\left(\mathbb{P} - \dot{I}\right)\right)$ from (323). We can also use (301) to substitute $\partial\left({}_i\nabla^2\Xi_j\right) = -2\kappa\partial\left({}_iR_j\right)$. Lastly, we use the conservation laws $\nabla^2 L = \frac{3}{2}\left(\dot{I} - \mathbb{P}\right)$ and $\nabla^2 r_i = \dot{R}_i$ from (289) and (292), respectively.

$$\begin{aligned}
&-\tfrac{1}{2}\nabla^2\Box h^{\tau\tau}_{ij} + \kappa\delta_{ij}\ddot{\rho} + \tfrac{1}{c^2}\kappa\partial\left({}_iR_j\right)\\
&+\delta_{ij}\nabla^2\left[-\tfrac{1}{2}c^2\kappa\rho + \frac{c^2\kappa}{2}\left(\rho + \frac{3}{c^2}\left(\mathbb{P} - \dot{I}\right)\right)\right]\\
&-\partial_i\partial_j\left[-\tfrac{1}{2}c^2\kappa\rho + \frac{c^2\kappa}{2}\left(\rho + \frac{3}{c^2}\left(\mathbb{P} - \dot{I}\right)\right)\right]\\
&= \kappa\left[\nabla^2 T^{\tau\tau}_{ij} + \nabla^2\delta_{ij}P + \partial\left({}_i\dot{R}_j\right) + \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\tfrac{3}{2}\dot{I} - \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\tfrac{3}{2}\mathbb{P}\right]
\end{aligned} \tag{325}$$

Now we substitute $\ddot{\rho} = \nabla^2 \acute{I}$ from (280) and also cancel terms involving $\partial \left({}_i R_j \right)$ and ρ. We can consolidate the middle two lines and distribute to obtain

$$\begin{aligned} & -\tfrac{1}{2}\nabla^2 \Box h_{ij}^{\tau\tau} + \kappa \delta_{ij} \nabla^2 \dot{I} \\ & + \frac{3\kappa}{2}\delta_{ij}\nabla^2 \mathbb{P} - \frac{3\kappa}{2}\delta_{ij}\nabla^2 \dot{I} - \frac{3\kappa}{2}\partial_i\partial_j \mathbb{P} + \frac{3\kappa}{2}\partial_i\partial_j \dot{I} \\ & \qquad = \kappa \left[\nabla^2 T_{ij}^{\tau\tau} + \nabla^2 \delta_{ij}\mathbb{P} + \tfrac{3}{2}\partial_i\partial_j \dot{I} - \tfrac{1}{2}\delta_{ij}\nabla^2 \dot{I} - \tfrac{3}{2}\partial_i\partial_j \mathbb{P} + \tfrac{1}{2}\delta_{ij}\nabla^2 \mathbb{P} \right] \end{aligned} \tag{326}$$

We can cancel three terms involving $\delta_{ij}\nabla^2\mathbb{P}$, three terms involving $\delta_{ij}\nabla^2 \dot{I}$, two terms involving $\partial_i\partial_j \dot{I}$, and two terms involving $\partial_i\partial_j \mathbb{P}$.

$$-\tfrac{1}{2}\nabla^2 \Box h_{ij}^{\tau\tau} \quad = \quad \kappa \nabla^2 T_{ij}^{\tau\tau} \tag{327}$$

$$\nabla^2 \left(\Box h_{ij}^{\tau\tau} + 2\kappa T_{ij}^{\tau\tau} \right) \quad = \quad 0 \tag{328}$$

Since $h_{ij}^{\tau\tau}$ and $T_{ij}^{\tau\tau}$ go to zero as $r \to \infty$ (which implies that $\Box h_{ij}^{\tau\tau}$ and $T_{ij}^{\tau\tau}$ also go to zero as $r \to \infty$), then the unique solution to the differential equation above is $\Box h_{ij}^{\tau\tau} + 2\kappa T_{ij}^{\tau\tau} = 0$. Thus we have

$$\boxed{\Box h_{ij}^{\tau\tau} = -\frac{16\pi G}{c^4} T_{ij}^{\tau\tau}} \tag{329}$$

We now list the field equations resulting from (293), (301), (323), and (329).

$$\nabla^2 \Phi \quad = \quad 4\pi G \left(\rho + \frac{3}{c^2} \left(\mathbb{P} - \dot{I} \right) \right) \tag{330}$$

$$\nabla^2 \Theta \quad = \quad -\frac{8\pi G}{c^2} \rho \tag{331}$$

$$\nabla^2 \Xi_i \quad = \quad -\frac{16\pi G}{c^2} R_i \tag{332}$$

$$\Box h_{ij}^{\tau\tau} \quad = \quad -\frac{16\pi G}{c^4} T_{ij}^{\tau\tau} \tag{333}$$

The following are some observations concerning the gauge-invariant field equations.

- There are redundant field equations found in the Einstein equations.

  We find that $G_{00} = \kappa T_{00}$ produces the field equation for Θ. We also find that $G_{0i} = \kappa T_{0i}$ produces the field equation for Ξ_i as well as a time-derivative of the field equation for Θ as seen in (297). Lastly, we find that $G_{ij} = \kappa T_{ij}$ contains *all* the field equations above. Specifically, it contains a second time-derivative of the field equation for Θ as seen in (307) and also the first time-derivative of the field equation for Ξ_i as seen in (315). Then (330) and (333) are obtained exclusively from $G_{ij} = \kappa T_{ij}$. In this sense, it appears that all the field equations can simply be obtained from $G_{ij} = \kappa T_{ij}$, although the other equations uniquely determine that the time-independent integration constants are zero for the equations obtained for Θ and Ξ_i.

  This redundancy found in the Einstein equations is discussed by Bertschinger in [20] (pp. 50-51, 60). He states that it is due to the twice-contracted Bianchi identity written as $\nabla_\mu G^\mu{}_\nu = 0$, which causes the Einstein equations to enforce energy-momentum , $\nabla_\mu T^\mu_\nu = 0$. This can also be described in terms of Noether's theorem which relates gauge symmetries to conservation laws. In this case, it is coordinate invariance (a continuous symmetry) that leads to a conservation of energy-momentum (a conservation law). The role of the redundant scalar and vector equations is to enforce these conservation laws.

- Not all the stress tensor quantities appear in the field equations.

  The HD stress tensor quantities that appear in the Poisson equations are ρ, I, R_i and $\mathbb{P}$, while r_i and L are absent. Recall from (271) that r_i is a rotational vector $(\partial_i r_i = 0)$ such that the antisymmetric derivative, $\partial\left({}_i r_j\right)$, is found in T_{ij}. Also recall that L is a scalar which forms the traceless part of T_{ij} given by $\left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)L$. Therefore, the absence of these quantities in the field equations imply that they do no contribute to the fields. However, this does not mean that these quantities can simply be omitted from the HD stress tensor in (271). On the contrary, these quantities are required in order to obtain the conservation relations in (286), (289), and (292) which were critical for deriving the gauge-invariant field equations in $(330) - (333)$. This is directly analogous to the requirement that all of the quantities in the metric given in $(175) - (177)$ are non-zero in order to construct gauge-invariant fields as shown in Section 18.

- Some field equations apply only for highly relativistic sources

  Matching (330) and (331) shows that the invariant potentials Φ and Θ are related dimensionally by a factor of c^2. This is consistent with the observation in Section 18 (where the gauge-invariant potentials are derived) that Θ is dimensionless while Φ has dimensions of velocity squared. In fact, we observe that (331) is only relevant in the case of extremely high mass densities such that ρ/c^2 is not negligible.

- The Poisson equations appear to violate causality.

  The fact that Φ, Θ, and Ξ_i satisfy Poisson equations in (293), (301), and (323) implies that there is no retardation in the signal for these fields at a given field point when the sources of those fields (at an arbitrary distance away) changes with time. In other words, there is an instantaneous action at a distance which would seem to violate causality. Flanagan and Hughes [28] argue that this is a consequence of the fact that the metric must be known over all space to construct the invariant potentials and therefore the invariant potentials are essentially non-local quantities. However, it should be noted that although the metric must be determined over all space, it does *not* need to be known over all *time*. Knowing the metric over all space is necessary because the differential equations have all been *spatial* and the boundary conditions required specifying the metric all the way to $r \to \infty$. However, since the metric does not need to be specified for all *times*, then it does not seem that the violation of causality is a consequence of "non-local" fields which are defined by a metric known over all space.

Furthermore, it will be shown in a later section that the linearized Bianchi identities predict a time-dependent relationship between all four fields (Φ, Θ, Ξ_i, and $h_{ij}^{\tau\tau}$). Since $h_{ij}^{\tau\tau}$ satisfies a wave equation, and hence is a propagating field, it can be understood as essentially carrying the signal that relates changes in the source distribution to changes in the fields at a given point away from the distribution. This is analogous to the case in electromagnetism where φ also satisfies a Poisson equation but changes in the source distribution leads to changes in φ at an arbitrary distance away from the sources because the signal is carried by $\vec{E}$ which propagates as a wave into the far-field region.

This issue of causality being seemingly violated is similar to the case of the Coulomb gauge in electromagnetism where the fully time-dependent scalar potential satisfies a Poisson equation (rather than a wave equation) in terms of the charge density. This implies that changes in the charge density *instantly* determine changes in the scalar potential even at space-like distances. However, in electromagnetism, this issue is less troubling since it has been shown in [30] that the gauge-invariant quantity (the electric field) is still causal. In fact, because $\vec{E} = -\nabla\varphi - \partial_t\vec{A}$, where $\vec{A}$ still satisfies a wave equation even in the Coulomb gauge, then the causality of $\vec{E}$ can be thought of as being built into the vector potential despite causality being absent from the scalar potential. For the case we have here, we find that the time-dependent quantities Φ, Θ, and Ξ_i which satisfy Poisson equations, are already gauge-invariant. The causality that is preserved in the wave equation for $h_{ij}^{\tau\tau}$ does not "rescue" the violation of causality for Φ, Θ, and Ξ_i.

Bertschinger also discusses similar issues of instantaneous Poisson equations for gravitation in [20] section 4.7 (pp. 55-60). There he describes the "Poisson gauge" for the metric which is analogous to the Coulomb gauge in electromagnetism since both produce field equations where the fields are determined by the instantaneous source with no time integration required. Bertschinger also discusses this topic in [21], section 4 (pp. 7-8) where once again he draws a connection to the case in electromagnetism.

- <u>Conservation of energy-momentum removes the time-dependence of the field equations.</u>

  As a further related observation, it is evident that many of the field equations *do* in fact have a time dependence as seen in (307) and (315). However, in each case that a time-dependence appears, it is removed by the use of a conservation law. This results in a time-dependence of the same order on both sides of the equation (the field side and the source side) and therefore integrating allows for all time-dependence to vanish. Therefore, although time-dependence *does* naturally come out of the Einstein field equations, we find that conservation of stress-energy-momentum effectively removes it. We also do not find that any of the scalar or vector potential equations involve a wave operator, and therefore, they do not lend themselves to a Green's function solution and thereby to a retarded time in the solution.

  We also point out that the stress-energy-momentum conservation laws were derived using $\partial^\nu T_{\mu\nu} = 0$ for linearized GR. It is well known that linearized GR produces various inconsistency problems, including the linearized conservation law predicting that particles move on straight lines.[34] This may also be related to the apparent violation of causality due to the Poisson equations. It could be an artifact of linearized theory and *not* the choice of using a Helmholtz Decomposition method which requires the metric to be known over all space. In fact, it is an interesting contrast with the harmonic gauge approach which predicts that *all* the metric components are propagating degrees of freedom (satisfying wave equations) which preserves causality explicitly, but contradicts the fact that gravitational waves must be arise only from quadrupole sources (T_{ij}), not monopole or dipole sources $(T_{00}$ and $T_{0i})$ due to conservation of mass and conservation of momentum.

[34]For example, see the discussion in Carroll's text [22] (p. 307) or in Thorsrud's thesis [10] (pp. 49-50).

- Only the components of $h_{ij}^{\tau\tau}$ propagate to the far-field zone.

  Since (333) is the only wave equation among the field equations in (330) – (333), then it is expected that only $h_{ij}^{\tau\tau}$ is a *radiative* field while the other gauge-invariant fields (Φ, Θ, Ξ_i) are *non-radiative* fields. To show this explicitly, it is possible to expand all the gauge-invariant fields in powers of $1/r$. At sufficiently large distances, the leading order $\mathscr{O}(1/r)$ will dominate. However, for Φ, Θ, and Ξ_i, it can be shown that the coefficients of $1/r$ are the conserved mass or the conserved linear momentum. Therefore, by conservation of mass and conservation of linear momentum, the time-derivative of these quantities must vanish and therefore $\Phi,\Theta,\Xi_i \approx 0$ in the far field. On the other hand, a time-varying $T_{ij}^{\tau\tau}$ need not vanish and therefore $h_{ij}^{\tau\tau}$ gives the only remaining degrees of freedom in the far-field zone. Hence, one can conclude that

$$\Phi,\ \Theta,\ \Xi_i \approx 0 \qquad \text{in the far-field zone} \tag{334}$$

  Recall that the definitions of Φ, Θ, and Ξ_i are given in (244) – (246) in terms of the components of the HD metric perturbation as

$$\Theta \equiv \tfrac{1}{3}\left(H - \nabla^2\lambda\right), \qquad \Xi_i \equiv \beta_i - \dot{\varepsilon}_i, \qquad \Phi \equiv \phi + \dot{\alpha} - \ddot{\lambda}/2 \tag{335}$$

  Therefore, (334) is satisfied if

$$\phi, \beta_i, \alpha, H, \varepsilon_i, \lambda \quad \approx \quad 0 \tag{336}$$

  The HD metric perturbation is defined in (175) – (177) as

$$\begin{aligned} h_{00} &= -2\phi/c^2 && (337)\\ h_{0i} &= (\beta_i + \partial_i\alpha)/c && (338)\\ h_{ij} &= h_{ij}^{\tau\tau} + \tfrac{1}{3}\delta_{ij}H + \partial_{(i}\varepsilon_{j)} + \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\lambda && (339) \end{aligned}$$

  Then (336) is satisfied if[35]

$$h_{00},\ h_{0i} \approx 0 \qquad \text{and} \qquad h_{ij} \approx h_{ij}^{\tau\tau} \qquad \text{in the far-field zone} \tag{340}$$

[35] Note that the gauge-invariant potentials in (335) can also be made zero by choosing

$$H = \nabla^2\lambda, \qquad \beta_i = \dot{\varepsilon}_i, \qquad \phi = \ddot{\lambda}/2 - \dot{\alpha}$$

In that case, we do not have $h_{00},\ h_{0i} \approx 0$. Rather, the metric perturbation becomes

$$h_{00} = -2\left(\ddot{\lambda}/2 - \dot{\alpha}\right)/c^2, \qquad h_{0i} = (\dot{\varepsilon}_i + \partial_i\alpha)/c, \qquad h_{ij} = h_{ij}^{\tau\tau} + \partial_{(i}\varepsilon_{j)} + \partial_i\partial_j\lambda$$

However, α, ε_i, and λ are gauge-dependent quantities with gauge freedom shown in (208), (225) and (233) as

$$\alpha' = \alpha + c^2A + \dot{C}, \qquad \lambda' = \lambda + 2C, \qquad \varepsilon_i' = \varepsilon_i + B_i$$

Therefore, a gauge vector, $\xi_\mu = (cA,\ B_i + \partial_i C)$, can be chosen so as to make these quantities vanish. This leads back to $h_{00},\ h_{0i} \approx 0$.

- There are six components but only two radiative degrees of freedom for waves.

  It is shown in Appendix F that if we specialize to the TT gauge, then $h^{TT}_{\mu\nu}$ has only four components with *two* independent degrees of freedom: $h_{\oplus}$ for *plus*-polarization and $h_{\otimes}$ for *cross*-polarization.

  Here we are working with gauge-invariant quantities, therefore we cannot use the TT gauge in order to identify *plus*-polarization and *cross*-polarization fields. Rather, there are *six* components in $h^{\tau\tau}_{ij}$ which are *all* associated with gravitational radiation. However, within the six components there are really only two physical degrees of freedom due to the four constraint equations given by $\partial_i h^{\tau\tau}_{ij} = 0$ (for transversality) and $\delta^{ij} h^{\tau\tau}_{ij} = 0$ (for tracelessness).

  For example, consider a plane-fronted gravitational wave propagating in the z-direction given by $h^{\tau\tau}_{ij} = A_{ij}\cos(kz - \omega t)$ where A_{ij} is a constant amplitude and $\vec{k} = (0,0,k)$. Since the wave is transverse, then $\partial^i h^{\tau\tau}_{ij} = k^i A^{\tau\tau}_{ij} = 0$. This requires $A_{3j} = 0$. Since $h^{\tau\tau}_{ij}$ is traceless, then $\delta^{ij} h^{\tau\tau}_{ij} = 0$. This requires $A_{11} = -A_{22}$. We can use the notation $A_{11} = h_{\oplus}$ and $A_{12} = h_{\otimes}$ to write $h^{\tau\tau}_{ij}$ in a form that completely matches the transverse-traceless gauge.

$$h^{\tau\tau}_{ij} = \begin{pmatrix} h_{\oplus} & h_{\otimes} & 0 \\ h_{\otimes} & -h_{\oplus} & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos(kz - \omega t) \tag{341}$$

  Therefore we find that choosing an axis of wave propagation and applying the transverse and traceless conditions on $h^{\tau\tau}_{ij}$ reduces the six components to just the two polarization states of a gravitational wave.

3.9 Gravito-electromagnetic Maxwell-like field equations

We begin by observing that the field equation in (330) reduces to Newtonian gravity in the static, non-relativistic limit. In Section 26 we show that in the Newtonian limit, $T^{0i} = T^{ij} = 0$ which means $\frac{3}{c^2}\left(\mathbb{P} - \dot{I}\right)$ can be neglected so that the only source remaining is ρ. We also show that $\bar{h}_{0i} = \bar{h}_{ij} = 0$ which requires that the potential given by $\Phi = \phi + \dot{\alpha} - \ddot{\lambda}$ reduces to just $\Phi \approx \phi$. Therefore (330) becomes

$$\nabla^2 \phi = 4\pi G \rho \tag{342}$$

The Newtonian gravitational field can be defined as $\vec{g} = -\nabla\phi$ so that (342) can be written as $\nabla \cdot \vec{g} = -4\pi G\rho$, consistent with Newton's law of gravitation. We may extend the definition of the vector field to include relativistic gravitational effects as well. The natural choice would be to define the gravito-vector field as $\vec{E}_G = -\nabla\Phi$ so that the divergence of $\vec{E}_G$ satisfies (330)

$$\nabla \cdot \vec{E}_G = -4\pi G \left(\rho + \frac{3}{c^2}\left(\mathbb{P} - \dot{I}\right)\right) \tag{343}$$

However, we observe that (293) also contains ρ as a source and therefore would be expected to contribute to the gravito-electric field as well. The prefactor in (293) shows that the field is suppressed by a factor of c^2 and hence it is clearly a field that is relevant only for cases with extremely large ρ. Nevertheless, in order to define a completely general gravito-electric field, we can write (293) as $-\frac{c^2}{2}\nabla^2\Theta = 4\pi G\rho$ and then add it to (330) which gives[36]

$$\nabla^2 \left(\Phi - \frac{c^2}{2}\Theta\right) = 4\pi G \left(2\rho + \frac{3}{c^2}\left(\mathbb{P} - \dot{I}\right)\right) \tag{344}$$

In the non-relativistic limit, to keep the form of (342) for Newtonian gravity, we can divide both sides of (344) by 2 and define a new scalar potential as[37]

$$\boxed{\varphi_G \equiv \frac{1}{2}\left(\Phi - \frac{c^2}{2}\Theta\right) \qquad \textit{Generalized gravito-scalar potential}} \tag{345}$$

so that we have

$$\nabla^2 \varphi_G \quad = \quad 4\pi G \left(\rho + \frac{3}{2c^2}\left(\mathbb{P} - \dot{I}\right)\right) \tag{346}$$

Therefore, we define the *static* gravito-electric field as $\vec{E}_{g,\ static} \equiv -\nabla\varphi_G$ so that the fully relativistic gravito-Gauss law becomes

$$\nabla \cdot \vec{E}_G \quad = \quad 4\pi G \left(\rho + \frac{3}{2c^2}\left(\mathbb{P} - \dot{I}\right)\right) \tag{347}$$

Hence, from the equation above we find that ρ is the source of Newtonian gravity as expected while $\frac{3}{2c^2}\left(\mathbb{P} - \dot{I}\right)$ is the relativistic contribution to the post-Newtonian gravito-electric field.

Next we consider the field equation given by (332). Since this field equation involves a vector potential that is purely rotational ($\nabla \cdot \Xi_i = 0$) and because the source of this field equation comes from the T_{0i} component of the stress tensor (which is related to a "mass current density" in the case of an ideal fluid), then it is

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[36]Note that because Φ has units of velocity squared and Θ is dimensionless, then we cannot simply add them together. We *must* multiply Θ by c^2 before adding it to Φ to maintain dimensional consistency.

[37]We use a tilda on the gauge-*invariant* gravito-scalar potential in order to distinguish it from the gauge-*dependent* gravito-scalar potential defined in (26) of Part I.

natural to consider this equation to be similar to Ampere's law in electromagnetism. Therefore, we may consider a gravito-magnetic field defined as $\vec{B}_G \equiv \nabla \times \vec{\Xi}$ which satisfies the condition $\nabla \cdot \vec{B}_G = \nabla \cdot \left(\nabla \times \vec{\Xi}\right) = 0$ similar to EM. Also, because $\nabla \cdot \vec{\Xi} = 0$, then the vector calculus identity $\nabla \times \nabla \times \vec{\Xi} = \nabla \left(\nabla \cdot \vec{\Xi}\right) - \nabla^2 \vec{\Xi}$ becomes

$$\nabla \times \nabla \times \vec{\Xi} = -\nabla^2 \vec{\Xi} \tag{348}$$

Therefore we could write (332) as

$$\nabla \times \nabla \times \vec{\Xi} = \frac{16\pi G}{c^2} \vec{R} \tag{349}$$

Using $\vec{B}_G = \nabla \times \vec{\Xi}$ means that we have the following Ampere-like equation[38]

$$\nabla \times \vec{B}_G = \frac{16\pi G}{c^2} \vec{R} \tag{350}$$

Next we consider the relationship between $\vec{E}_G$ and $\vec{B}_G$. Similar to EM, we can define the *dynamic* gravito-electric field as $\vec{E}_G \equiv -\nabla \varphi_G - \dot{\vec{\Xi}}$ so that $\nabla \times \vec{E}_G = \nabla \times \left(-\nabla \varphi_G + \dot{\vec{\Xi}}\right) = \nabla \times \dot{\vec{\Xi}} = -\dot{\vec{B}}_G$. Therefore, we have a gravito-Faraday law given by

$$\nabla \times \vec{E}_G = -\partial_t \vec{B}_G \tag{351}$$

To summarize, we define the gravito-electromagnetic fields in terms of the invariant potentials as

$$\boxed{\vec{E}_G \equiv -\frac{1}{2} \nabla \left(\Phi + \frac{c^2}{2} \Theta\right) - \dot{\vec{\Xi}} \qquad \text{and} \qquad \vec{B}_G \equiv \nabla \times \vec{\Xi}} \tag{352}$$

Then the gravito-electromagnetic field equations (including the wave equation) are

$$\boxed{\begin{array}{ll} \nabla \cdot \vec{E}_G = -4\pi G \left(\rho + \dfrac{3}{2c^2} \left(\mathbb{P} - \dot{I}\right)\right) & \nabla \cdot \vec{B}_G = 0 \\ \\ \nabla \times \vec{E}_G = -\partial_t \vec{B}_G & \nabla \times \vec{B}_G = \dfrac{16\pi G}{c^2} \vec{R} \\ \\ \Box h_{ij}^{\tau\tau} = -\dfrac{16\pi G}{c^4} T_{ij}^{\tau\tau} & \end{array}} \tag{353}$$

The tensor wave equation for $h_{ij}^{\tau\tau}$ in (333) can be written in a form that also resembles Ampere's law. Recall that in electromagnetism, the propagating degrees of freedom are $\vec{E} = -\partial_t \vec{A}$ and $\vec{B} = \nabla \times \vec{A}$ which are the temporal and spatial derivatives of the vector potential. By analogy, the propagating degrees of freedom for gravitational waves can be written in terms of an electric-like tensor field and a propagating magnetic-like tensor defined as, respectively,

$$\mathscr{E}_{ij} \quad = \quad -\partial_t h_{ij}^{\tau\tau} \qquad \text{and} \qquad \mathscr{B}_{ijk} = \partial_k h_{ij}^{\tau\tau} \tag{354}$$

Using the wave equation in, $\Box h_{ij}^{\tau\tau} = -2\kappa T_{ij}^{\tau\tau}$, and expanding the box operator gives

$$-\frac{1}{c^2} \partial_t \left(\partial_t h_{ij}^{\tau\tau}\right) + \partial_k \left(\partial_k h_{ij}^{\tau\tau}\right) \quad = \quad -2\kappa T_{ij}^{\tau\tau} \tag{355}$$

[38]The gravito-Ampere law that appears in gravito-electromagnetism ordinarily has a negative source as seen in (58) or (137). It will be shown later that when a particular stress tensor is chosen, such as a perfect fluid in (429) or relativistic dust in (454), then the source term given by $T_{0i} = R_i$ will in fact be negative so as to be consistent with the gravito-Ampere law shown here.

Using (354) and rearranging gives

$$\partial_k \mathscr{B}_{ijk} \quad = \quad -\left(2\kappa T_{ij}^{\tau\tau} + \frac{1}{c^2}\partial_t \mathscr{E}_{ij}\right) \tag{356}$$

This is essentially an Ampere-like law in the sense that a spatial derivative of a magnetic-like field is proportional to a source term plus a time-derivative of the electric-like field. In vacuum, $T_{ij}^{\tau\tau} = 0$, which simply gives

$$\partial_k \mathscr{B}_{ijk} \quad = \quad -\frac{1}{c^2}\partial_t \mathscr{E}_{ij} \tag{357}$$

In contrast to electromagnetism where $\vec{E}$ and $\vec{B}$ are related by complimentary field equations (Faraday's law and Ampere's law), it is found here that $\mathscr{E}_{ij}$ and $\mathscr{B}_{ijk}$ are only related by an Ampere-like law but not a Faraday-like law. In other words, it is not found that the spatial derivative of the electric-like tensor field can be related to the time-derivative of the magnetic-like tensor.[39] In the next section, Faraday-like relationships involving $\mathscr{E}_{ij}$ and $\mathscr{B}_{ijk}$ will be found, but they will be more complicated then simply the time-derivatives of $\mathscr{B}_{ijk}$ related to a spatial derivative of $\mathscr{E}_{ij}$.

The following are some observations concerning the gravito-electromagnetic field equations in (353).

- The strength of the source terms varies by powers of c in the denominator.

  It is immediately evident that in the fully non-relativistic limit, the only source term which remains is ρ which produces a gravito-electric field, $\vec{E}_G$. The next higher order sources are $\left(\mathbb{P} - \dot{I}\right)$ and R_i which are all divided by c^2. These introduce corrections to the gravito-electric field and also provide a gravito-magnetic field. Lastly, the highest order sources are $T_{ij}^{\tau\tau}$ since they are divided by c^4. These sources are responsible for gravitational waves.

- The gravito-electric field contains the Newtonian field plus post-Newtonian corrections.

  The definition of the static gravito-electric field can be written using (345) as

$$\vec{E}_{g,\,static} \equiv -\nabla\varphi_G \equiv -\frac{1}{2}\nabla\left(\Phi - \frac{c^2}{2}\Theta\right) \tag{358}$$

  In Section 26 we show that in the Newtonian limit, $\Phi = \Phi_N$ and $\Theta_N = -\frac{2}{c^2}\Phi_N$ where Φ_N is the Newtonian potential. This means that the term in parentheses becomes $2\Phi_N$ and $\vec{E}_{g,\,static}$ reduces to $\vec{E}_N = -\nabla\Phi_N$ as we expect. Therefore, we cannot interpret either Θ or Φ alone as responsible for the Newtonian field or as responsible for post-Newtonian corrections. They both play a role in the Newtonian field and post-Newtonian corrections. In fact, in the Newtonian limit, we find that the field equations (331) and (330) given for Φ and for Θ, respectively, *both* reduce to the Newtonian field equation. Although, the contribution by Θ is multiplied by c^2 in the expression for $\vec{E}_{g,\,static}$ in (358),

[39] Recall that in electromagnetism, the 4-potential plays an analogous role to $h_{ij}^{\tau\tau}$ in that it also satisfies a wave equation, $\Box A^\mu = -2\mu J^\mu$. In that case, Ampere's law is obtained by using $\mu = i$ and the magnetic field, $\vec{B} = \nabla\times\vec{A}$. However, Faraday's law is not obtained from the wave equation. It is obtained from the Lorenz gauge condition, $\partial_\nu A^\nu = 0$, with the electric field, $\vec{E} = -\nabla\varphi - \partial_t\vec{A}$ and the magnetic field, $\vec{B} = \nabla\times\vec{A}$. Since there is no analogous gauge condition on $h_{ij}^{\tau\tau}$, then there is no corresponding Faraday-like law involving spatial and temporal derivatives of $h_{ij}^{\tau\tau}$. (Note that the transverality condition, $\partial_i h_{ij}^{\tau\tau} = 0$, is not a gauge condition and does not involve any time-derivatives of $h_{ij}^{\tau\tau}$, therefore, it still does not lead to a Faraday-like law.)

we must also recognize that the source of Θ is suppressed by c^2 in the field equation for Θ in (331). This means that it is impossible for the effect of Θ to exceed the effect of Φ and consequently reverse the sign of the Newtonian field. (In other words, this does not predict any kind of "anti-gravity.") Furthermore, because Θ and Φ have the *same* source, ρ, then there is no source that could ever make the strength of Θ comparable to the strength of Φ.

- The gravito-electric fields are only valid to first-order derivatives

  We cannot take derivatives of the field equations in (353) while maintaining consistency with the linearized GR approximation. In Appendix A, where the linearized Riemann tensor was developed, we showed that only terms that are third order in the metric perturbation (that is, terms involving *second* derivatives, $\partial_\sigma \partial_\rho h_{\mu\nu}$) were kept while all higher order terms were neglected. The field equations in (353) already involve the second derivative of the metric perturbation. Therefore, no further derivatives can be legitimately taken. For example, the curl of the Ampere law or Faraday law must be approximated to zero.

$$\nabla \times \nabla \times \vec{B}_G = 2\mu_G \nabla \times \vec{R} \approx 0 \qquad \text{and} \qquad \nabla \times \nabla \times \vec{E}_G \neq -\partial_t \nabla \times \vec{B}_G \approx 0 \tag{359}$$

  If we wish to take such derivatives, then we must return to the Riemann tensor and keep terms involving the third derivative of the metric and all other terms of similar order.

- The prefactor of 2 implies a spin-2 graviton

  Many authors claim that the factor of 2 in the gravito-Ampere law, $\nabla \times \vec{B}_G = 2\mu_G \vec{R}$, is indicative of gravity being a spin-2 tensor field. However, it may be a more valid argument to refer to the factor of 2 in the *wave equation*, $\Box h_{ij}^{\tau\tau} = -2\kappa T_{ij}^{\tau\tau}$, since this equation pertains to gravitational waves and hence to a spin-2 graviton in quantum theory (analogous to electromagnetic waves pertaining to a spin-1 photon).

3.10 The gravito-electric permittivity and gravito-magnetic permeability

To enhance the similarity with electromagnetism, we may also define a gravito-electric permittivity as well as a gravito-magnetic permeability given, respectively, as

$$\varepsilon_G \equiv \frac{1}{4\pi G} \qquad \text{and} \qquad \mu_G \equiv \frac{4\pi G}{c^2} \tag{360}$$

where $\varepsilon_G \mu_G = 1/c^2$. These are the gravitational analogs of the electric permittivity and magnetic permeability in electromagnetism. Then the field equations in (353) may be written as

$$\boxed{\begin{array}{ll} \nabla \cdot \vec{E}_G = -\dfrac{1}{\varepsilon_G}\left(\rho + \dfrac{3}{2c^2}\left(\mathbb{P} - \dot{I}\right)\right) & \nabla \cdot \vec{B}_G = 0 \\ \nabla \times \vec{E}_G = -\partial_t \vec{B}_G & \nabla \times \vec{B}_G = 2\mu_G \vec{R} \\ \Box h_{ij}^{\tau\tau} = -2\kappa T_{ij}^{\tau\tau} & \end{array}} \tag{361}$$

The following are some observations concerning the gravito-electric permittivity and gravito-magnetic permeability.

- The gravito-electric permittivity and gravito-magnetic permeability each depend on G.

  In the case of electromagnetism (EM), ε_0 and μ_0 cannot be *independently* expressed in terms of another fundamental constant of nature. They can only be expressed *together* in terms of c. However, ε_G and μ_G are each expressed *independently* in terms of another fundamental constant of nature which is Newton's gravitational constant, G. This means that if ε_G has a different value in a material than it does in vacuum, then the *effective* value of G in that medium would be different than in vacuum. For instance, if ε_G has a larger value in a given material, then this would imply a decrease of G . In other words, $G_{eff} < G$ in that material. It would also imply a decrease of μ_G by the same factor.

  Note that if a material has a G_{eff} which differs from G, then the expressions in (360) would imply that the gravito-electric polarizability is linked to the gravito-magnetic polarizability. This is rather strange since we do not find in EM that the electric polarizability and the magnetic polarizability of a material must always be related by the same factor. On the other hand, if we assume that G is the *same* in vacuum as well as in materials, then ε_G and μ_G are effectively just constants of nature as well and do not change depending on the material where the fields are present.

- The relation $\varepsilon_G \mu_G = 1/c^2$ is satisfied in all mediums.

  In the case of EM, we know that the relationship between ε and μ is derived by combining EM field equations to produce EM *wave* equations. The result is that the speed of EM waves is given by $v^2 = 1/\varepsilon\mu$. For vacuum, we find that ε_0 and μ_0 lead to a speed of $v = c$ and in other mediums we find that ε and μ lead to other speeds that we can express as $v = n/c$ with n being the index of refraction.

  However, in the case of gravitation, the origin of ε_G and μ_G is very different. We do *not* find the relationship between ε_G and μ_G by combining gravitational field equations to produce gravitational *wave* equations. Therefore, we do not have a general relationship given by $v^2 = 1/\varepsilon_G\mu_G$ where v is the speed of the gravitational wave. Rather, ε_G and μ_G are each intentionally *defined* to satisfy the relation $c^2 = 1/\varepsilon_G\mu_G$ without any regard to a medium for the fields. In other words, $c^2 = 1/\varepsilon_G\mu_G$ is established as an *absolute relationship*, not the special case of $v = c$ in vacuum.

  In fact, obtaining a relation other than $c^2 = 1/\varepsilon_G\mu_G$ would require not only an effective value for G in a medium, but the effective value of G would have to be *different* for the permittivity and the permeability. In other words, we would need G_{eff} for ε_G to be different from G_{eff} for μ_G. Otherwise, if they are the same then we still have $\varepsilon_G\mu_G = 1/c^2$ and there is still no dispersion relation involving ε_G and μ_G. As a result, this would imply that gravitational fields are not impaired at all by matter.

- The relation $\varepsilon_G\mu_G = 1/c^2$ may apply to gravito-electromagnetic oscillations but not waves.[40]

  It is important to recognize that the field equations in (361) describe gravito-electromagnetic (GEM) fields, $\vec{E}_G$ and $\vec{B}_G$, independently of gravitational *wave* fields, $h_{ij}^{\tau\tau}$. The GEM fields were derived from potentials that satisfy Poisson equations as shown in (331) – (330). Therefore, we know that the GEM fields are *bound* (or *non*-radiative) fields which fall off as $1/r^2$. On the other hand, $h_{ij}^{\tau\tau}$ are the only *radiative* fields which satisfy a wave equation and fall off as $1/r$ in the far-field. This is an important distinction from EM where $\vec{E}$ and $\vec{B}$ are the *only* fields and they *both* contribute to bound fields as well as radiative fields. In fact, the distinction between waves and oscillations is not necessary in EM because "waves" and "electromagnetic oscillations" are effectively the same.

  Now from the field equations in (361) we find that ε_G appears in the gravito-Gauss law (for the *bound* field $\vec{E}_G$) while μ_G appears in the gravito-Ampere law (for the *bound* field $\vec{B}_G$). However, we do *not* find that the *combination* of ε_G and μ_G appearing in the wave equation. Rather, the prefactor can only be expressed in terms of *either* ε_G or μ_G. For example, it may be expressed as either $\kappa = 2/\varepsilon_G c^4$ or $\kappa = 2\mu_G/c^2$. This indicates that the product $\varepsilon_G\mu_G = 1/c^2$ can only be describing the behavior of *bound* fields, *not* gravitational waves. If we interpret $\varepsilon_G\mu_G = 1/c^2$ as describing the speed of propagation of fields at the speed c, then it must be the propagation of $\vec{E}_G$ and/or $\vec{B}_G$.

  However, the field equations given in (361) do not predict a mutual inductance between $\vec{E}_G$ and $\vec{B}_G$. There is a gravito-Faraday law but there is no gravito-displacement current which is also required to have mutual inductance. There are other post-Newtonian gravitational field equations[41] which do contain a gravito-displacement current and therefore predict a mutual inductance between $\vec{E}_G$ and $\vec{B}_G$. However, in those cases we still cannot formally derive the relation $\varepsilon_G\mu_G = 1/c^2$ in order to justify the interpretation of c being the speed of propagation. Even if we insist that $\varepsilon_G\mu_G = 1/c^2$ predicts the speed of propagation is c, we still must recognize that it is speed of propagation for *bound* fields, *not* gravitational waves.

- An impedance expression for gravitational waves cannot be obtained from the fields.

  It is not clear what would constitute a gravitational "impedance" for waves such as the impedance that exists in EM. The EM wave impedance is given by $Z = \sqrt{\mu/\varepsilon}$ which in vacuum becomes $Z_0 = \sqrt{\mu_0/\varepsilon_0}$. However, this quantity is generally obtained by taking a ratio of $\vec{E}$ and $\vec{H}$, where $\vec{H} = \vec{B}/\mu$ and $\vec{B} = \vec{E}/c$ in vacuum. Therefore, the ratio of $\vec{E}$ and $\vec{H}$ gives

$$\frac{\vec{E}}{\vec{H}} = \frac{\vec{E}}{\vec{B}/\mu} = \frac{\vec{E}}{\vec{E}/c\mu} = c\mu = \sqrt{\frac{1}{\varepsilon\mu}}\mu = \sqrt{\frac{\mu}{\varepsilon}} = Z \tag{362}$$

  In the case of gravitation, we can *not* find a gravitational wave "impedance" by using an analogous procedure of taking the ratio of $\vec{E}_G$ and $\vec{H}_G$, where $\vec{H}_G = \vec{B}_G/\mu_G$ and $\vec{B}_G = \vec{E}_G/c$. First, we do not have

[40] We are careful to distinguish between "gravito-electromagnetic oscillations" which are oscillations of $\vec{E}_g$ and $\vec{B}_g$ and "gravitational waves" which are described by $h_{ij}^{\tau\tau}$ in the far-field.

[41] For example, a gravito-displacement current appears in the field equations (58) which are derived in linearized GR by applying the harmonic gauge to the trace-reversed metric perturbation $\left(\partial_\nu \bar{h}^{\mu\nu} = 0\right)$ and assuming non-relativistic sources. A gravito-displacement current also found in the field equations using the PPN formalism as shown in (137).

the relation $\vec{B}_G = \vec{E}_G/c$ for gravitation.[42] The analogous relationship in electromagnetism, $\vec{B} = \vec{E}/c$, occurs as a result of considering plane *wave* solutions to the wave equations involving $\vec{E}$ and $\vec{B}$ which do not exist for gravitation for $\vec{E}_G$ and $\vec{B}_G$.

Second, the ratio of $\vec{E}_G$ and $\vec{H}_G$ would not give a gravitational *wave* impedance because $\vec{E}_G$ and $\vec{H}_G$ are not associated with gravitational waves. Instead, the appropriate field is $h_{ij}^{\tau\tau}$ as shown in the wave equation in (361). For example, a gravitational wave with plus-polarization (in the far-field) only has strain fields given by $h_{xx}^{\tau\tau}$ and $h_{yy}^{\tau\tau}$. Taking the ratio of the fields does not produce any constant since the two fields are identical. (In fact, each of them is dimensionless.) This is in contrast to the case in EM where $\vec{E}$ and $\vec{H}$ have different dimensions and therefore the ratio leads to the physical quantity of impedance. If we disregard this fact and simply use the last equality in (362) expressed in terms of μ_G and ε_G, then we have

$$Z_G = \sqrt{\frac{\mu_G}{\varepsilon_G}} = \frac{4\pi G}{c} \tag{363}$$

This result has units that match the expected result for "gravitational resistance" as discussed in Section 13. Unfortunately it is not clear how to interpret the physical meaning and relevance of this quantity since it was not derived from field equations such as (361).

[42] Although we can not *derive* the relation $\vec{B}_g = \vec{E}_g/c$, it is interesting to point out that it is certainly possible *dimensionally*. This can be seen most easily from the gravito-Lorentz force obtained from the geodesic equation in (508). There we see that $\vec{E}_g$ and $\vec{v} \times \vec{B}_g$ are both accelerations. Therefore $\vec{E}_g$ and $\vec{B}_g$ could be related by a velocity.

3.11 The absence of a gravitational displacement current

In the gravito-electromagnetic field equations shown in (361), we find that there is no displacement current as there is in electromagnetism (EM). Recall that in EM, the displacement current is required to maintain consistency between Ampere's law, Gauss's law and the continuity equation. Specifically, if we solve Gauss's law for the charge density, $\rho = \varepsilon_0 \nabla \cdot \vec{E}$, and insert this into the continuity equation, $\dot{\rho} + \nabla \cdot \vec{J} = 0$, then we obtain

$$\left(\varepsilon_0 \nabla \cdot \dot{\vec{E}}\right) + \nabla \cdot \vec{J} = 0 \tag{364}$$

$$\nabla \cdot \left(\vec{J} + \varepsilon_0 \dot{\vec{E}}\right) = 0 \tag{365}$$

We can therefore define the *displacement* current density as $\vec{J}_D = \varepsilon_0 \dot{\vec{E}}$ and the *charge* current density as $\vec{J}$ so that the full current is $\vec{J}_{full} = \vec{J} + \vec{J}_D$. Then Ampere's law can be written as

$$\nabla \times \vec{B} = \mu_0 \vec{J}_{full} = \mu_0 \left(\vec{J} + \varepsilon_0 \dot{\vec{E}}\right) \tag{366}$$

Taking the divergence of Ampere's law in (366) gives zero on both sides. The left side is zero since the divergence of a curl is always zero. The right side is also zero by (365). Therefore, as stated above, we find that the displacement current is necessary to maintain consistency between Ampere's law, Gauss's law and the continuity equation.

Now we consider an analogous calculation for the case of the gravito-electromagnetic equations in (361). If we solve the gravito-Gauss law for the mass density, we obtain

$$\rho = -\left(\varepsilon_G \nabla \cdot \vec{E}_G + \frac{3}{2c^2}\left(\mathbb{P} - \dot{I}\right)\right) \tag{367}$$

Inserting this into the mass-momentum continuity equation from (283), $\dot{\rho} - \nabla^2 I = 0$, gives

$$\varepsilon_G \nabla \cdot \dot{\vec{E}}_G + \frac{3}{2c^2}\left(\dot{\mathbb{P}} - \ddot{I}\right) + \nabla^2 I = 0 \tag{368}$$

We can immediately observe that this expression is completely independent of the gravito-Ampere law given in (361) as

$$\nabla \times \vec{B}_G = 2\mu_G \vec{R} \tag{369}$$

Taking the divergence of this equation gives zero on both sides. The left side is zero since the divergence of a curl is always zero. The right side is zero due to the fact that R_i is a purely rotational vector and therefore $\partial_i R_i = 0$ as stated in (272). Therefore, we find that there is no need for a displacement current to maintain consistency between the gravito-Ampere law, gravito-Gauss law and the mass-momentum continuity equation. Ampere's law is completely independent because it does not involve any of the source terms that appear (368). In fact, if we wish, we could choose to define a gravito-displacement current density as $\vec{J}_D = \varepsilon_G \dot{\vec{E}}_G$ and from (368) this would have to satisfy

$$\nabla \cdot \vec{J}_D = -\frac{3}{2c^2}\left(\dot{\mathbb{P}} - \ddot{I}\right) - \nabla^2 I \tag{370}$$

However, this gravito-displacement current density would not play any role in the field equations shown in (361). The absence of a displacement current is also found by Bertschinger in [20] (equ 4.62 on p. 61). It also found in [21] (equ. 33 and 39) and discussed on p. 14. There are two reasons for the absence of a gravitational displacement current:

1. It is not needed to enforce energy-momentum conservation, $\partial_\nu T^{\mu\nu} = 0$. This is because mass conservation is enforced by the scalar potential, momentum conservation is enforced by the vector potential, and therefore the only remaining radiating degrees of freedom is the tensor potential.

2. The displacement current would lead to wave equations for the gravito-electromagnetic *vector* fields, $\vec{E}_G$ and $\vec{B}_G$. However, these are not propagating wave fields like they are in electromagnetism. Gravitational waves are purely *tensor* waves.

3.12 Gravito-electromagnetic field equations for an ideal fluid

The gravito-electromagnetic field equations in (361) are expressed in terms of quantities found in the Helmholtz Decomposition (HD) stress tensor, $T^{(HD)}_{\mu\nu}$, as given by $(269)-(271)$. To express the field equations in terms of an ideal fluid, $T^{(ideal\ fluid)}_{\mu\nu}$, we will need to relate the quantities found in the Helmholtz Decomposition (HD) stress tensor[43] ($\tilde{\rho}$, R_i, I, $T^{\tau\tau}_{ij}$, r_i, $\mathbb{P}$ and L) to the quantities found in an ideal fluid (ρ, P and v_i). The linearized stress tensor for an ideal fluid is given in (2685) of Appendix I as

$$T^{(ideal\ fluid)}_{\mu\nu} = \left(\rho + P/c^2\right)\left(\eta_{\mu\sigma}\eta_{\nu\rho}v^{\sigma}v^{\rho} + \eta_{\mu\sigma}h_{\nu\rho}v^{\sigma}v^{\rho} + h_{\mu\sigma}\eta_{\nu\rho}v^{\sigma}v^{\rho}\right)\gamma^2 + P\left(\eta_{\mu\nu} + h_{\mu\nu}\right) \tag{371}$$

Formally, this expression for the stress tensor does not violate the approximations used in this treatment since we only required two conditions so far:

1. Terms that are second order in $h_{\mu\nu}$ are neglected so that Christoffel symbols take the form of (2367).
2. Terms involving the product of Christoffel symbols and the stress tensor (such as $\Gamma^{\nu}_{\nu\sigma}T^{\sigma\mu}$) are neglected. This was required so that conservation of the stress-energy-momentum tensor

$$\nabla_{\nu}T^{\mu\nu} = \partial_{\nu}T^{\mu\nu} + \Gamma^{\nu}_{\nu\sigma}T^{\sigma\mu} + \Gamma^{\mu}_{\nu\sigma}T^{\nu\sigma} = 0 \tag{372}$$

   becomes simply

$$\partial_{\nu}T^{\mu\nu} = 0 \tag{373}$$

   From (2367), we know that to first order, the Christoffel symbols involve terms with $\partial_{\gamma}h_{\mu\nu}$. Therefore, using $\partial^{\nu}T_{\mu\nu} = 0$ as the *linear* conservation of stress-energy-momentum only requires neglecting terms of order $\left(\partial_{\gamma}h_{\mu\nu}\right)T^{\rho\sigma}$. This does *not* require neglecting terms of order $h_{\mu\nu}T^{\rho\sigma}$.

However, using (371) to describe the sources in the field equations of (361) will lead to a non-linear self-coupling of the field to the sources due to the appearance of $h_{\mu\nu}$ in the source terms. Therefore, as a further approximation, we can choose to neglect the self-coupling of gravity (that is, the notion of the gravitational field acting back on the sources and hence altering the gravitational field through an iterative process). In that case, we are choosing a stricter approximation which neglects all terms of order $h_{\mu\nu}T^{\rho\sigma}$. This means that (371) becomes

$$T^{(ideal\ fluid)}_{\mu\nu} = \left(\rho + P/c^2\right)\gamma^2 v_{\mu}v_{\nu} + P\eta_{\mu\nu} \tag{374}$$

As shown in $(2692)-(2694)$ from Appendix I, the stress tensor components become

$$T^{(ideal\ fluid)}_{00} = \left(\rho c^2 + P\right)\gamma^2 - P \tag{375}$$

$$T^{(ideal\ fluid)}_{0i} = -\left(\rho c + P/c\right)\gamma^2 v_i \tag{376}$$

$$T^{(ideal\ fluid)}_{ij} = \left(\rho + P/c^2\right)\gamma^2 v_i v_j + P\eta_{ij} \tag{377}$$

[43]Here we use the notation $\tilde{\rho}$ to distinguish the scalar component of the HD stress tensor, $T^{(HD)}_{00} = \tilde{\rho}c^2$, from the rest mass energy density, ρc^2, of an ideal fluid. Nowhere in this treatment was it required that $T^{(HD)}_{00}$ would be the rest mass energy density. In fact, it was misleading to use $T^{(HD)}_{00} = \rho c^2$ for the HD stress tensor since this scalar quantity is *not* the rest mass density. This notation was simply used for consistency with the treatment in [28].

Relating the HD stress tensor quantities to the ideal fluid stress tensor quantities

We now relate the ideal fluid stress tensor components in $(375) - (377)$ to the HD stress tensor components given in $(269) - (271)$. Equating (375) and (269) gives

$$T_{00}^{(HD)} = T_{00}^{(ideal\ \ fluid)} \tag{378}$$

$$\boxed{\tilde{\rho}c^2 = \left(\rho c^2 + P\right)\gamma^2 - P} \tag{379}$$

Next, equating (270) and (376) gives

$$T_{0i}^{(HD)} \quad = \quad T_{0i}^{(ideal\ \ fluid)} \tag{380}$$

$$c\left(R_i + \partial_i I\right) \quad = \quad -\left(\rho c + P/c\right)\gamma^2 v_i \tag{381}$$

Taking the divergence of both sides[44] and noting that $\partial_i R_i = 0$ gives

$$\nabla^2 I \quad = \quad -\nabla\left[\left(\rho + P/c^2\right)\gamma^2\right]\cdot\vec{v} - \left(\rho + P/c^2\right)\gamma^2\left(\nabla\cdot v_i\right) \tag{382}$$

For a fluid with incompressible flow, we have $\nabla\cdot\vec{v} = 0$. Also, if the mass density remains uniform and there are no pressure gradients in the material, then $\nabla\left(\rho + P/c^2\right) = 0$ and we have

$$\nabla^2 I \quad = \quad -\left(\rho + P/c^2\right)\left(\nabla\gamma^2\right)\cdot\vec{v} \tag{383}$$

Here we need to evaluate $\nabla\gamma^2$ where $\gamma^2 = \left(1 - v^2/c^2\right)^{-1}$. This gives

$$\nabla\gamma^2 \quad = \quad -\left(1 - v^2/c^2\right)^{-2}\nabla\left(1 - v^2/c^2\right) \tag{384}$$

$$= \quad -\left(1 - v^2/c^2\right)^{-2}\left(-2v/c^2\right)\nabla v \tag{385}$$

For a fluid with incompressible flow, $\partial_i v_i = 0$. Also, if there is no spatial variation in the particle velocities (in other words, each streamline has the same velocity over a cross-sectional area of the flow) then $\partial_i v_{j\ (i\neq j)} = 0$. Therefore $\nabla v = 0$ and we have

$$\nabla\gamma^2 = 0 \tag{386}$$

Then (383) becomes

$$\nabla^2 I = 0 \tag{387}$$

Since I goes to zero as $r \to \infty$, then the only unique solution is

$$\boxed{I = 0} \tag{388}$$

[44] The divergence of a scalar S times a vector $\vec{V}$ is $\nabla\cdot\left(S\vec{V}\right) = S\nabla\cdot\vec{V} + \vec{V}\cdot\nabla S$.

Returning to (381), we can write the equation as a vector equation and take the curl.[45]

$$\nabla\times\vec{R} \quad = \quad -\left(\rho+P/c^2\right)\gamma^2\left(\nabla\times\vec{v}\right)-\nabla\left[\left(\rho+P/c^2\right)\gamma^2\right]\times v_i \tag{389}$$

Again, we assume $\left(\rho+P/c^2\right)$ is uniform so that $\nabla\left(\rho+P/c^2\right)=0$. We also use (386) to obtain[46]

$$\nabla\times\left[\vec{R}+\left(\rho+P/c^2\right)\gamma^2\vec{v}_\perp\right] \quad = \quad 0 \tag{390}$$

Since $\vec{R}$ and $\left(\rho+P/c^2\right)\vec{v}_\perp$ must go to zero as $r\to\infty$, then the only unique solution to the differential equation above is $\vec{R}+\left(\rho+P/c^2\right)\gamma^2\vec{v}_\perp=0$. Therefore we have

$$\boxed{\vec{R}=-\left(\rho+P/c^2\right)\gamma^2\vec{v}_\perp} \tag{391}$$

Next, equating (271) and (377) gives

$$T_{ij}^{(HD)} \quad = \quad T_{ij}^{(ideal\ \ fluid)} \tag{392}$$

$$T_{ij}^{\tau\tau}+\delta_{ij}\mathbb{P}+\partial\left(_i r_j\right)+\left(\partial_i\partial_j-\tfrac{1}{3}\delta_{ij}\nabla^2\right)L \quad = \quad \left(\rho+P/c^2\right)\gamma^2 v_i v_j+P\eta_{ij} \tag{393}$$

Taking the spatial trace of both sides using δ^{ij} gives

$$3\mathbb{P} \quad = \quad \left(\rho+P/c^2\right)\gamma^2 v^2+3P \tag{394}$$

$$\boxed{\mathbb{P}=\tfrac{1}{3}\left(\rho+P/c^2\right)\gamma^2 v^2+P} \tag{395}$$

Now we take two derivatives of (393) using $\partial_i\partial_j$ and note that $\partial_i T_{ij}^{\tau\tau}=0$ and $\partial_i r_i=0$.

$$\partial_i\partial_j\delta_{ij}\mathbb{P}+\partial_i\partial_j\left(\partial_i\partial_j-\tfrac{1}{3}\delta_{ij}\nabla^2\right)L \quad = \quad \partial_i\partial_j\left[\left(\rho+P/c^2\right)\gamma^2 v_i v_j\right]+\partial_i\partial_j\delta_{ij}P \tag{396}$$

Inserting (395) gives

$$\partial_i\partial_j\delta_{ij}\left[\tfrac{1}{3}\left(\rho+P/c^2\right)\gamma^2 v^2+P\right]+\tfrac{2}{3}\nabla^2\nabla^2 L \quad = \quad \partial_i\partial_j\left[\left(\rho+P/c^2\right)\gamma^2 v_i v_j\right]+\partial_i\partial_j\delta_{ij}P \tag{397}$$

$$\nabla^2\left[\tfrac{1}{3}\left(\rho+P/c^2\right)\gamma^2 v^2\right]+\tfrac{2}{3}\nabla^2\nabla^2 L \quad = \quad \partial_i\partial_j\left[\left(\rho+P/c^2\right)\gamma^2 v_i v_j\right] \tag{398}$$

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[45]The curl of a scalar S times a vector $\vec{V}$ is $\nabla\times\left(S\vec{V}\right)=S\nabla\times\vec{V}+\nabla S\times\vec{V}$.

[46]We do *not* assume that the fluid has no vorticies which would imply $\nabla\times\vec{v}=0$. Instead, we choose to permit the possibility of vortices so that later we can relate these results to a superfluid which has quantized vortices. We also use the notation $\vec{v}_\perp$ for the “transverse” component of $\vec{v}$ as a reminder that we have required $\nabla\cdot\vec{v}$ for an incompressible fluid flow.

Applying a tensor product rule[47] to the right side gives

$$\nabla^2\left[\tfrac{1}{3}\left(\rho+P/c^2\right)\gamma^2 v^2\right]+\tfrac{2}{3}\nabla^2\nabla^2 L \quad = \quad \left[\partial_i\partial_j\left(\rho+P/c^2\right)\gamma^2\right]v_iv_j+\left(\rho+P/c^2\right)\gamma^2\partial_i\partial_j\left(v_iv_j\right) \tag{399}$$

Note that $\partial_i\partial_j\left(v_iv_j\right)$ becomes

$$\partial_i\partial_j\left(v_iv_j\right) \quad = \quad \partial_i\left[\left(\partial_j v_i\right)v_j+\left(\partial_j v_j\right)v_i\right] \tag{400}$$

$$= \quad \left(\partial_i\partial_j v_i\right)v_j+\left(\partial_j v_i\right)\left(\partial_i v_j\right)+\left(\partial_i\partial_j v_j\right)v_i+\left(\partial_j v_j\right)\left(\partial_i v_i\right) \tag{401}$$

Once again, we let $\partial_i v_j=0$ and $\partial_i\left[\left(\rho+P/c^2\right)\gamma^2\right]=0.$ Then (399) reduces to $\nabla^2\nabla^2 L=0.$ Since $\nabla^2 L$ goes to zero as $r\to\infty$, then the only unique solution is $\nabla^2 L=0$. If L goes to zero as $r\to\infty$, then we have

$$\boxed{L=0} \tag{402}$$

Next, taking a single derivative, ∂_i, of both sides of (393) gives

$$\partial_i\delta_{ij}\mathbb{P}+\partial_i\partial\left({}_i r_j\right)+\partial_i\left(\partial_i\partial_j-\tfrac{1}{3}\delta_{ij}\nabla^2\right)L=\partial_i\left[\left(\rho+P/c^2\right)\gamma^2 v_iv_j\right]+\partial_i\delta_{ij}P \tag{403}$$

$$\partial_j\mathbb{P}+\nabla^2 r_j+\tfrac{2}{3}\partial_j\nabla^2 L \quad = \quad \partial_i\left[\left(\rho+P/c^2\right)\gamma^2\right]v_iv_j+\left(\rho+P/c^2\right)\gamma^2\partial_i\left(v_iv_j\right)+\partial_j P \tag{404}$$

Inserting (395) and (402) gives

$$\partial_j\left[\tfrac{1}{3}\left(\rho+P/c^2\right)\gamma^2 v^2+P\right]+\nabla^2 r_j$$

$$= \quad \partial_i\left[\left(\rho+P/c^2\right)\gamma^2\right]v_iv_j+\left(\rho+P/c^2\right)\gamma^2\left[\left(\partial_i v_i\right)v_j+v_i\left(\partial_i v_j\right)\right]+\partial_j P \tag{405}$$

Canceling common terms gives

$$\tfrac{1}{3}\partial_j\left[\left(\rho+P/c^2\right)\gamma^2 v^2\right]+\nabla^2 r_j \quad = \quad \partial_j\left[\left(\rho+P/c^2\right)\gamma^2\right]v_iv_j+\left(\rho+P/c^2\right)\gamma^2\left[\left(\partial_i v_i\right)v_j+\left(\partial_i v_j\right)v_i\right] \tag{406}$$

Again we can use $\partial_i v_i=0$ and $\nabla\left[\left(\rho+P/c^2\right)\gamma^2\right]=0.$ Then the equation above simply becomes $\nabla^2 r_j=0.$ Since r_j goes to zero as $r\to\infty$, then the only unique solution is

$$\boxed{r_i=0} \tag{407}$$

Lastly, we can substitute (395), (402) and (407) into (393) to obtain

$$T_{ij}^{\tau\tau}+\delta_{ij}\left[\tfrac{1}{3}\left(\rho+P/c^2\right)\gamma^2 v^2+P\right] \quad = \quad \left(\rho+P/c^2\right)\gamma^2 v_iv_j+P\delta_{ij} \tag{408}$$

[47] Just as $\nabla\cdot\left(S\vec{V}\right)=S\nabla\cdot\vec{V}+\vec{V}\cdot\nabla S$, so also $\partial_i\partial_j\left(Sv_iv_j\right)=S\partial_i\partial_j v_iv_j+v_iv_j\partial_i\partial_j S$.

$$\boxed{T_{ij}^{\tau\tau} = \gamma^2 \left(\rho + P/c^2\right)\left(v_i v_j - \tfrac{1}{3}\delta_{ij} v^2\right)} \tag{409}$$

As expected, we find that $T_{ij}^{\tau\tau}$ is transverse and traceless. It is transverse since $\partial_i T_{ij}^{\tau\tau} = 0$ due to the requirements that $\partial_i v_i = 0$ and $\partial_i \left[\left(\rho + P/c^2\right)\gamma^2\right] = 0$. It is traceless since taking the spatial trace of (409) will cause the two terms in the last parentheses to cancel.

In summary, we find that an ideal fluid with an incompressible flow and a uniform mass density and pressure requires that $L = I = r_i = 0$. This is consistent with the absence of these quantities in the field equations found in (361). Note that although we find that $L = I = r_i = 0$, this does not mean that these quantities can simply be omitted from the HD stress tensor in (269) – (271). These quantities are required in order to obtain the conservation relations in (286), (289), and (292) which were critical for deriving the gauge-invariant field equations in (330) – (333). These ultimately lead to the field equations in (361). This is directly analogous to the requirement that all of the quantities in the metric given in (175) – (177) are non-zero in order to construct gauge-invariant fields.

Now inserting the expressions for $\tilde{\rho}$, $\vec{R}$, $\mathbb{P}$ and $T_{ij}^{\tau\tau}$ from (379), (391), (395) and (409), respectively, into (269) – (271) allows us to express $T_{\mu\nu}^{(HD)}$ completely in terms of ρ, P and v_i. In doing so, we recover the correct results for $T_{\mu\nu}^{(ideal\ fluid)}$ as given by (375) – (377).

$$T_{00}^{(HD)} = \tilde{\rho} c^2 = \gamma^2 \left(\rho c^2 + P\right) - P \tag{410}$$

$$T_{0i}^{(HD)} = R_i = -\gamma^2 \left(\rho + P/c^2\right) v_{i,\perp} \tag{411}$$

$$T_{ij}^{(HD)} = T_{ij}^{\tau\tau} + \delta_{ij}\mathbb{P} = \gamma^2 \left(\rho + P/c^2\right) v_i v_j + P\delta_{ij} \tag{412}$$

Note that the trace is $T = T^{\mu}{}_{\mu} = T_{00} + T^{i}{}_{i}$ which gives

$$T = \gamma^2 \left(\rho c^2 + P\right) - P + \gamma^2 \left(\rho + P/c^2\right) v^2 + 3P \tag{413}$$

$$= \gamma^2 \left(\rho c^2 + P\right)\left(1 + v^2/c^2\right) + 2P \tag{414}$$

In the non-relativistic limit $\left(v^2/c^2 \approx 0 \text{ and } \gamma \approx 1\right)$, this gives $T = \rho c^2 + 3P$.

Conservation relations for the ideal fluid stress tensor quantities

Next, we can consider how the conservation laws in (283), (289) and (292) apply to these results. Starting with (283), we have $\dot{\tilde{\rho}} = \nabla^2 I$. Using $\tilde{\rho} c^2 = \left(\rho c^2 + P\right)\gamma^2 - P$ from (379) and $I = 0$ from (388) gives

$$\partial_t \left[\left(\rho c^2 + P\right)\gamma^2 - P\right] = 0 \tag{415}$$

$$\gamma^2 \partial_t \left(\rho c^2 + P\right) + \left(\rho c^2 + P\right)\partial_t \gamma^2 - \partial_t P = 0 \tag{416}$$

$$\gamma^2 c^2 \dot{\rho} + \left(\gamma^2 - 1\right)\dot{P} + \left(\rho c^2 + P\right)\partial_t \gamma^2 = 0 \tag{417}$$

Evaluating $\partial_t \gamma^2$ where $\gamma^2 = \left(1 - v^2/c^2\right)^{-1}$ gives

$$\begin{aligned}
\partial_t \gamma^2 &= -\left(1 - v^2/c^2\right)^{-2} \partial_t \left(1 - v^2/c^2\right) && (418) \\
&= -\left(1 - v^2/c^2\right)^{-2} \left(-2v/c^2\right) \partial_t v && (419) \\
&= \frac{2va}{c^2 \left(1 - v^2/c^2\right)^2} && (420) \\
&= 2va\gamma^4/c^2 && (421)
\end{aligned}$$

Then (417) becomes

$$\gamma^2 c^4 \dot{\rho} + \left(\gamma^2 - 1\right) c^2 \dot{P} + 2\gamma^4 \left(\rho c^2 + P\right) va = 0 \tag{422}$$

If the mass density and pressure of the ideal fluid do not change with time, then $\dot{\rho} = \dot{P} = 0$ and we simply have

$$\left(\rho c^2 + P\right) va = 0 \tag{423}$$

This expression is effectively an equation of motion for a volume element of the fluid. Using[48] (288) we also have $\nabla^2 \left(\frac{2}{3}\nabla^2 L - \dot{I} + \mathbb{P}\right) = 0$. Since (388) and (402) show that $I = 0$ and $\nabla^2 L = 0$, respectively, then we have

$$\nabla^2 \mathbb{P} = 0 \tag{424}$$

This means that $\partial_i \mathbb{P}$ is a constant. This is consistent with (395) where $\mathbb{P} = \frac{1}{3}\left(\rho + P/c^2\right)\gamma^2 v^2 + P$. Since we required a uniform mass density, no pressure gradients and a fluid with incompressible flow, then the derivative of $\mathbb{P}$ must vanish.

Lastly, from (292) we have $\nabla^2 r_i = \dot{R}_i$. Since $r_i = 0$ according to (407), then $\dot{R}_i = 0$. From (391) we have $\partial_t \left[\left(\rho + P/c^2\right)\gamma^2 v_{i,\perp}\right] = 0$. We can apply the product rule to obtain

$$\gamma^2 v_{i,\perp} \partial_t \left(\rho + P/c^2\right) + \left(\rho + P/c^2\right) v_{i,\perp} \partial_t \gamma^2 + \gamma^2 \left(\rho + P/c^2\right) \partial_t v_{i,\perp} = 0 \tag{425}$$

We can use the result of (421) to write

$$\gamma^2 v_{i,\perp} \left(\dot{\rho} + \dot{P}/c^2\right) + \left(\rho + P/c^2\right) v_{i,\perp} 2va\gamma^4/c^2 + \gamma^2 \left(\rho + P/c^2\right) a_{i,\perp} = 0 \tag{426}$$

Multiplying by c^2/γ^2 and grouping terms gives

$$\left[\left(\dot{\rho}c^2 + \dot{P}\right) + 2\left(\rho c^2 + P\right)\gamma^2 va/c^2\right] v_{i,\perp} + \left(\rho c^2 + P\right) a_{i,\perp} = 0 \tag{427}$$

[48]Note that for the conservation law we do *not* use (289) which gives $\frac{2}{3}\nabla^2 L = \dot{I} - \mathbb{P}$. This would lead to the erroneous result $\mathbb{P} = 0$ which would contradict (395) where $\mathbb{P}$ is clearly *not* zero. Rather, it is $\nabla^2 \mathbb{P}$ that is zero which is satisfied by the fact that $\mathbb{P} = \frac{1}{3}\left(\rho + P/c^2\right) v^2 + P$ has a vanishing derivative for uniform mass density, no pressure gradients and a fluid with incompressible flow.

If the mass density and pressure of the ideal fluid do not change with time, then $\dot{\rho} = \dot{P} = 0$ and we have

$$\left(\rho c^2 + P\right)\left(\frac{2\gamma^2 va}{c^2} v_{i,\,\perp} + a_{i,\,\perp}\right) \quad = \quad 0 \tag{428}$$

This expression is effectively an equation of motion for the transverse motion of a volume element of the fluid.

Gravito-electromagnetic field equations for an ideal fluid

We are now able to express the gravito-electromagnetic (GEM) field equations in (361) in terms of an ideal fluid. Inserting $L = I = r_i = 0$ and the expressions for $\tilde{\rho}$, $\vec{R}$, $\mathbb{P}$ and $T_{ij}^{\tau\tau}$ from (379), (391), (395) and (409), respectively, gives

$$\begin{array}{ll}
\nabla\cdot\vec{E}_G = -\dfrac{1}{\varepsilon_G}\left[\left(\rho + \dfrac{P}{c^2}\right)\gamma^2\left(1 + \dfrac{v^2}{2c^2}\right) + \dfrac{P}{2c^2}\right] & \nabla\cdot\vec{B}_G = 0 \\
\nabla\times\vec{E}_G = -\partial_t\vec{B}_G & \nabla\times\vec{B}_G = -2\mu_G\left(\rho + \dfrac{P}{c^2}\right)\gamma^2\vec{v}_\perp \\
\Box h_{ij}^{\tau\tau} = -2\kappa\left(\rho + \dfrac{P}{c^2}\right)\gamma^2\left(v_i v_j - \frac{1}{3}\delta_{ij}v^2\right) &
\end{array} \tag{429}$$

Gauge-invariant gravito-electromagnetic field equations for an ideal fluid

If we keep velocity terms only to order v^2/c^2, then $\gamma^2 = \left(1 - v^2/c^2\right)^{-1} \approx 1 + v^2/c^2$. Inserting this into (429) and keeping only terms to order v^2/c^2 gives[49]

$$\begin{array}{ll}
\nabla\cdot\vec{E}_G = -\dfrac{1}{\varepsilon_G}\left[\left(\rho + \dfrac{P}{c^2}\right)\left(1 + \dfrac{3v^2}{2c^2}\right) + \dfrac{P}{2c^2}\right] & \nabla\cdot\vec{B}_G = 0 \\
\nabla\times\vec{E}_G = -\partial_t\vec{B}_G & \nabla\times\vec{B}_G = -2\mu_G\left(\rho + \dfrac{P}{c^2}\right)\vec{v}_\perp \\
\Box h_{ij}^{\tau\tau} = -2\kappa\left(\rho + \dfrac{P}{c^2}\right)\left(v_i v_j - \frac{1}{3}\delta_{ij}v^2\right) &
\end{array} \tag{430}$$

Gauge-invariant gravito-electromagnetic field equations
for an ideal fluid (to order v^2/c^2)

For the case of relativistic dust, we can neglect terms with pressure so the field equations (still to order v^2/c^2)

[49]Since $\mu_g = 4\pi G/c^2$, then $\mu_g\vec{v}$ is of the order v/c^2 and therefore $\mu_g\left(1 + v^2/c^2\right)\vec{v}$ reduces to just $\mu_g\vec{v}$. Likewise, since $\kappa = 16\pi G/c^4$, then κv^2 is of the order v^2/c^4 and therefore $\kappa\left(1 + v^2/c^2\right)v^2$ reduces to just κv^2.

become[50]

$$\boxed{\begin{array}{ll} \nabla\cdot\vec{E}_G = -\dfrac{1}{\varepsilon_G}\rho\left(1+\dfrac{3v^2}{2c^2}\right) & \nabla\cdot\vec{B}_G = 0 \\[2ex] \nabla\times\vec{E}_G = -\partial_t\vec{B}_G & \nabla\times\vec{B}_G = -2\mu_G\vec{J}_{m_\perp} \\[2ex] \Box h_{ij}^{\tau\tau} = -2\kappa\rho\left(v_iv_j - \frac{1}{3}\delta_{ij}v^2\right) & \\[2ex] \multicolumn{2}{c}{\textit{Gauge-invariant gravito-electromagnetic field equations}} \\ \multicolumn{2}{c}{\textit{for relativistic dust (to order } v^2/c^2\textit{)}} \end{array}} \tag{431}$$

The following are some observations concerning the ideal fluid GEM field equations.

- The source in the gravito-Ampere law is $\vec{J}_{m_\perp}$ which is only the *transverse* component of the mass current density. This is similar to the results found by Bertschinger in [20] (equ 4.62 on p. 61). It also found in [21] (equ. 33 and 39). It is a consequence of $\vec{B}_G$ being a purely transverse field, $\vec{B}_G = \nabla\times\vec{\Xi}$ with $\nabla\cdot\vec{\Xi} = 0$. This is similar to the Coulomb gauge in electromagnetism which leads to $\Box\vec{A} = -\mu_0\vec{J}_\perp$. However, here we have $\nabla^2\vec{\Xi} = -c^2\kappa R_i$ where R_i is also a purely transverse current source.

- It is evident from the wave equation for gravitational waves why gravitational waves are so weak compared to other gravitational fields. First, the coupling constant in the wave equation is $2\kappa = 8\pi G/c^4 \approx 2\times 10^{-43}$ (SI units). (This is far weaker than the Lense-Thirring field which has a prefactor $\mu_G = 8\pi G/c^2 \approx 2\times 10^{-26}$ appearing in the gravito-Ampere law.) Second, gravitational waves are generated by sources of order v^2 rather than order v. This means that for non-relativistic sources, the gravitational wave field will be significantly weaker than the Lense-Thirring field.

- From (429) we find that the quantities in the ideal fluid stress tensor (ρ, $\vec{v}$ and P) appear in the wave equation for gravitational waves as well as the other field equations. This means that gravitational waves, $h_{ij}^{\tau\tau}$, are generated from the same sources as $\vec{E}_G$ and $\vec{B}_G$. This provides a clarification concerning the observation described in Section 20 where it was noted that $T_{ij}^{\tau\tau}$ does not participate in any conservation law and therefore does not seem to be linked to any other sources. This gave the impression that it is possible to have all the other quantities in the stress tensor be zero (including the rest mass energy) and yet have $T_{ij}^{\tau\tau}$ be *non*-zero. Clearly this doesn't match any physical system we observe in nature. Therefore, although the HD stress tensor and the conservation laws give the impression that this is possible, we find here that using an actual *physical* stress tensor (such as an ideal fluid or dust) shows it does *not* occur that $T_{ij}^{\tau\tau}$ can be non-zero while all the other quantities are zero.

[50] We define a "relativistic *transverse* mass current density" as $\vec{J}_{m_\perp} = \rho\vec{u}_\perp = \rho\gamma\vec{v}_\perp$. (However, in the approximation $\gamma\approx 1$, then we just have $\vec{J}_{m_\perp} = \rho\vec{v}_\perp$.) This is similar to $\vec{J}_m$ introduced in Section 3 except that here we have $\nabla\cdot\vec{J}_{m_\perp} = 0$ due to requiring the ideal fluid to have an incompressible flow. This is analogous to the *charge* current density in electromagnetism which is given by $\vec{J} = \rho_c\vec{u}$, with ρ_c being the *charge* density. Notice also that $\vec{J}_{m_\perp}$ is essentially just the transverse component of the relativistic momentum, $\vec{p} = \vec{u} = \gamma m\vec{v}$.

3.13 Newtonian and first-order post-Newtonian limits

We begin by defining the Newtonian and first-order post-Newtonian limit in terms of the velocity of gravitational sources and the corresponding components of the metric. In (2468) of Appendix B, it is shown that in the harmonic gauge, the linearized Einstein equation expressed in terms of the trace-reversed metric perturbation is

$$\Box\bar{h}_{\mu\nu} = -2\kappa T_{\mu\nu} \tag{432}$$

where $\bar{h}_{\mu\nu}$ is the trace-reversed metric perturbation given by

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}h \tag{433}$$

The Green's function solution to (432) is then given by

$$\bar{h}_{\mu\nu}(t,\vec{x}) = \frac{4G}{c^4}\int \frac{T_{\mu\nu}(t_r,\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x' \tag{434}$$

For simplicity, we can consider an ideal fluid with no pressure (relativistic dust). Then the stress tensor is $T_{\mu\nu} = \rho u_\mu u_\nu = \rho g_{\mu\rho}g_{\nu\sigma}u^\rho u^\sigma$. If we neglect self-coupling of the stress tensor to the field, then $g_{\mu\nu} \approx \eta_{\mu\nu}$ and we have

$$T_{00} = \rho\gamma^2c^2, \qquad T_{0i} = -\rho\gamma^2cv_s^i, \qquad T_{ij} = \rho\gamma^2v_s^iv_s^j \tag{435}$$

where v_s^i is the velocity of the gravitational sources. It is evident that $T_{00} << T_{0i} << T_{ij}$. Therefore, from (434) it also follows that $\bar{h}_{00} << \bar{h}_{0i} << \bar{h}_{ij}$. In particular, the components of the stress tensor are related by

$$T_{0i} \sim T_{00}\left(\frac{v_s}{c}\right) \qquad \text{and} \qquad T_{ij} \sim T_{00}\left(\frac{v_s}{c}\right)^2 \tag{436}$$

Therefore it follows from (434) that the components of the trace-reversed metric perturbation are related by[51]

$$\bar{h}_{0i} \sim \bar{h}_{00}\left(\frac{v_s}{c}\right) \qquad \text{and} \qquad \bar{h}_{ij} \sim \bar{h}_{00}\left(\frac{v_s}{c}\right)^2 \tag{437}$$

These relations are helpful for an order of magnitude approximation when considering Newtonian and first-order post-Newtonian limits. Therefore, in this context, we will consider "Newtonian order" to only involve non-moving matter $\left(T^{00}\right)$ and therefore only involves the Newtonian potential $\bar{h}_{00}$. We will consider "first-order post-Newtonian" to include mass currents $\left(T^{0i}\right)$ and therefore also includes $\bar{h}_{0i}$, the gravito-vector potential responsible for the Lense-Thirring field. Lastly, we will consider "second-order post-Newtonian" to include the stress $\left(T^{ij}\right)$ and therefore also includes $\bar{h}_{ij}$, the remaining part of the metric perturbation which includes GR waves and other strains. Conversely, we could summarize by saying "first-order post-Newtonian" simply sets $\bar{h}_{ij} \approx 0$ while "Newtonian order" simply sets $\bar{h}_{ij} \approx \bar{h}_{0i} \approx 0$.

Next we consider the potentials, the vector fields, and the field equations that result in each of these limits. In (2431) from Appendix B, we have the trace-reversed metric perturbation components given as

$$\bar{h}_{00} \;=\; \tfrac{1}{2}(h_{00}+H), \qquad \bar{h}_{0i} = h_{0i}, \qquad \bar{h}_{ij} = h_{ij} + \tfrac{1}{2}\delta_{ij}(h_{00}-H) \tag{438}$$

[51] These relations were motivated based on the field equation for linearized GR in the trace-reversed harmonic gauge, $\partial^\nu\bar{h}_{\mu\nu} = 0$. However, the same relations can be motivated for the *non*-trace-reversed metric perturbation and without the use of GR. We simply use the harmonic gauge, $\partial^\nu h_{\mu\nu} = 0$, and consider a metric perturbation given by $h_{\mu\nu} = A_{\mu\nu}e^{\left(\vec{k}\cdot\vec{x}-\omega t\right)}$. The details are shown in Section 40.

Inserting the HD metric components from $(175)-(177)$ gives

$$\bar{h}_{00} = -\phi/c^2 + \tfrac{1}{2}H \tag{439}$$

$$\bar{h}_{0i} = (\beta_i + \partial_i \alpha)/c \tag{440}$$

$$\bar{h}_{ij} = h^{\tau\tau}_{ij} - \tfrac{1}{6}\delta_{ij}H + \partial_{(i}\varepsilon_{j)} + \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\lambda - \delta_{ij}\phi/c^2 \tag{441}$$

The off-diagonal elements of $\bar{h}_{ij}$ are

$$\underset{(i\neq j)}{\bar{h}_{ij}} = h^{\tau\tau}_{ij} + \partial_{(i}\varepsilon_{j)} + \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\lambda \tag{442}$$

Since we set $\bar{h}_{ij} \approx 0$ in the first-order post-Newtonian limit, then we must have the off-diagonal elements above vanish. This means $h^{\tau\tau}_{ij} \approx \varepsilon_i \approx \lambda \approx 0$. In the case, (441) reduces to $0 = -\frac{1}{6}\delta_{ij}H - \delta_{ij}\phi/c^2$ which gives $H = -6\phi/c^2$. This condition holds in the Newtonian limit as well since we still require $\bar{h}_{ij} \approx 0$. However, for Newtonian order we also require $\bar{h}_{0i} \approx 0$ which means we have the added condition that $\beta_i \approx \alpha \approx 0$. We can summarize with the following set of conditions.

Newtonian order:	$\beta_i \approx \alpha \approx h^{\tau\tau}_{ij} \approx \varepsilon_i \approx \lambda \approx 0$ and $H = -6\phi/c^2$	
First post-Newtonian order:	$h^{\tau\tau}_{ij} \approx \varepsilon_i \approx \lambda \approx 0$ and $H = -6\phi/c^2$	(443)
Second post-Newtonian order:	no components are necessarily zero	

Defining Newtonian and first-order post-Newtonian potentials and vector fields

The gauge-invariant quantities given in $(244)-(246)$ are

$$\Phi = \phi + \dot{\alpha} - \ddot{\lambda}/2, \qquad \Theta = \tfrac{1}{3}\left(H - \nabla^2\lambda\right), \qquad \Xi_i = \beta_i - \dot{\varepsilon}_i \tag{444}$$

In the first-order post-Newtonian limit, these are reduced to

$$\boxed{\begin{array}{c} \Phi_{PN} \equiv \phi + \dot{\alpha}, \qquad \Theta_{PN} \equiv \tfrac{1}{3}H, \qquad \Xi_{i,\,PN} \equiv \beta_i \\ \textit{First-order post-Newtonian potentials} \end{array}} \tag{445}$$

where the subscript "*PN*" is used to denote "post-Newtonian." These are the post-Newtonian scalar and *vector* potentials. Because λ and ε_i were eliminated in the first post-Newtonian limit, the potentials in (445) are no longer gauge-invariant. This can be seen by considering the transformations given in (238), (239), (240) and (241). We can no longer carry out the process of Section 18 to construct gauge-invariant quantities. Specifically, without λ we can no longer eliminate the gauge freedom C from the metric perturbation quantity H in (241). Also, without ε_i we can no longer eliminate the gauge freedom $\dot{B}_i$ from β_i in (240). Lastly, without λ we can no longer eliminate the gauge freedom $\dot{C}$ from α in (239) and the gauge freedom $\dot{A}$ from the ϕ in (238). As a result, the gauge-invariance of Φ, Θ, and Ξ_i is broken.

Since the gauge-*invariant* GEM fields are defined according to (352) as

$$\vec{E}_G \equiv -\frac{1}{2}\nabla\left(\Phi - \frac{c^2}{2}\Theta\right) - \dot{\vec{\Xi}} \qquad \text{and} \qquad \vec{B}_G \equiv \nabla\times\vec{\Xi} \tag{446}$$

then we can define the first-order post-Newtonian GEM fields as

$$\boxed{\begin{array}{c} \vec{E}_{g(PN)} \equiv -\frac{1}{2}\nabla\left(\Phi_{PN} - \frac{c^2}{2}\Theta_{PN}\right) - \dot{\vec{\Xi}}_{PN} \qquad \text{and} \qquad \vec{B}_{g(PN)} \equiv \nabla\times\vec{\Xi}_{PN} \\ \textit{First-order post-Newtonian GEM fields} \end{array}} \tag{447}$$

These fields are gauge-*dependent* since they are defined in terms of the gauge-dependent first-order post-Newtonian potentials. In the Newtonian limit, the potentials in (445) are reduced further to

$$\boxed{\Phi_N \equiv \phi, \qquad \Theta_N \equiv \tfrac{1}{3}H, \qquad \Xi_{i,\,N} = 0 \qquad \textit{Newtonian potentials}} \tag{448}$$

where the subscript "N" is used to denote "Newtonian." Since (443) also shows that $H = -6\phi/c^2$, then dividing both sides by 3 and substituting Θ_N and Φ_N from (448) gives

$$\Theta_N = -\frac{2}{c^2}\Phi_N \qquad \text{in the Newtonian limit} \tag{449}$$

Inserting this into (447) shows that in the Newtonian limit, the GEM fields are simply

$$\boxed{\vec{E}_{g(N)} \equiv -\nabla\Phi_N \qquad \text{and} \qquad \vec{B}_{g(N)} = 0 \qquad \textit{Newtonian GEM fields}} \tag{450}$$

As expected, in the Newtonian limit, the gravito-magnetic (Lense-Thirring) field vanishes and the only remaining field is the Newtonian acceleration field expressed in terms of the gradient of the gravito-scalar potential. From these results, we can now decompose the gravito-electric field into a Newtonian field plus a post-Newtonian correction. Taking just the *static* part of the gravito-electric field in (446) we have

$$\vec{E}_{g,\,static} = -\frac{1}{2}\nabla\left(\Phi - \frac{c^2}{2}\Theta\right) \tag{451}$$

In the Newtonian limit, we found that inserting $\Theta_N = -\frac{2}{c^2}\Phi_N$ made the quantity in parentheses become $2\Phi_N$ and therefore $\vec{E}_{g,\,static}$ reduced to the Newtonian field in (450). Therefore, we could say that in the Newtonian limit, Φ and Θ combine *together* so as to produce the Newtonian field. Neither of them can be identified alone as responsible for the Newtonian gravitational field.

To illustrate this point further, we can express $\vec{E}_{g,\,static}$ in terms of metric quantities using (444).

$$\vec{E}_{g,\,static} = -\frac{1}{2}\nabla\left[\phi + \dot{\alpha} - \ddot{\lambda}/2 - \tfrac{c^2}{6}\left(H - \nabla^2\lambda\right)\right] \tag{452}$$

Rearranging gives

$$\vec{E}_{g,\,static} = -\frac{1}{2}\nabla\left(\phi - \tfrac{c^2}{6}H\right) + \nabla\left(\dot{\alpha}/2 - \ddot{\lambda}/4 + \tfrac{c^2}{6}\nabla^2\lambda\right) \tag{453}$$

Since $H = -6\phi/c^2$ in the Newtonian limit, then the first term above is clearly the Newtonian field. This means that the remaining quantities in the second term are responsible for all post-Newtonian corrections to the gravito-electric field (in the linearized GR approximation). These post-Newtonian corrections cannot be expressed purely in terms of the gauge-invariant potentials in (444). Rather, it is only when the Newtonian field in the first term is decomposed into ϕ and H (with appropriate prefactors) that gauge-invariant quantities can be formulated.

Furthermore, it is interesting to observe that the *first*-order post-Newtonian correction comes from $\dot{\alpha}$ which is a *first*-order time derivative of the longitudinal part of h_{0i}. Similarly, the *second*-order post-Newtonian correction comes from a *second*-order time derivative and *second*-order spatial derivative of λ which is the longitudinal part of h_{ij}. Therefore, as expected, all contributions post-Newtonian contributions to $\vec{E}_{g,\ static}$ come from longitudinal parts of the metric perturbation, with the order of derivatives (first and second) matching the order of the of the metric perturbation (h_{0i} and h_{ij}). In a similar manner, we find that the rotational part of h_{0i} (given by β_i) is the *first*-order contribution to Ξ_i while the time derivative of the rotational part of h_{ij} (given by $\dot{\varepsilon}_i$) is the second-order contribution to Ξ_i.

Newtonian and first-order post-Newtonian field equations

In (429) and (431) we found field equations for an ideal fluid and for relativistic dust, respectively. The associated stress tensors involve fully relativistic gravitational sources of order v^2/c^2 and therefore, the field equations are properly categorized as *second*-order post-Newtonian equations. If we consider the case of *non*-relativistic dust, we can neglect terms with v^2/c^2. Then the field equations in (431) (written with the appropriate notation for the fields) become[52]

$$\boxed{\begin{array}{ll} \nabla\cdot\vec{E}_{g(PN)} = -\rho/\varepsilon_G & \nabla\cdot\vec{B}_{g(PN)} = 0 \\[1em] \nabla\times\vec{E}_{g(PN)} = -\partial_t\vec{B}_{g(PN)} & \nabla\times\vec{B}_{g(PN)} = -2\mu_G\vec{J}_{m_\perp} \\ \text{(for low frequency } \vec{B}_{g(PN)}\text{)} & \\[1em] \Box h^{TT}_{ij} = 0 & \\[1em] \multicolumn{2}{c}{\textit{Gauge-dependent gravito-electromagnetic}} \\ \multicolumn{2}{c}{\textit{field equations for non-relativistic dust}} \end{array}} \qquad (454)$$

These field equations can be considered *first*-order post-Newtonian equations. They are *not* gauge-invariant since the fields are not gauge-invariant. The following are some additional observations concerning these field equations.

- The gravito-Faraday law is only valid for a low frequency $\vec{B}_{g(PN)}$ in order to maintain consistency with neglecting sources of order v^2/c^2. This can be observed by taking a time derivative of the gravito-

[52]Since $\mu_g = 4\pi G/c^2$, then the wave equation in (431) becomes $\Box h^{TT}_{ij} = -2\dfrac{\mu_g}{c^2}\left(\rho v_i v_j - \frac{1}{3}\delta_{ij}\rho v^2\right)$ which contains sources that are all of order v^2/c^2. Therefore the equation reduces to $\Box h^{TT}_{ij} = 0$ which indicates that only waves in vacuum are permitted. (Non-relativistic sources cannot generate gravitational waves.) However, the gravito-Ampere law is given by $\nabla\times\vec{B}_g = -2\mu_g\vec{J}_m$ and contains sources that are v/c^2. Therefore it is *not* neglected.

Ampere law which involves $\vec{B}_{g(PN)}$.[53] Since $\vec{J}_{m_\perp} = \rho \vec{v}_\perp$, then we have

$$\nabla \times \dot{\vec{B}}_G = -2\mu_G \left(\dot{\rho} \vec{v}_\perp + \rho \vec{a}_\perp\right) \tag{455}$$

where $\vec{a}_\perp$ is the transverse acceleration. For an oscillating source where $a_{\max} = v_{\max}\omega$, then $\mu_G a_{\max} \sim v_{\max}\omega/c^2$. Therefore if ω/c is comparable to $v_{\max}/c$, then we would have a source on the order of v^2/c^2 which violates the approximation required for those equations to hold.[54]

Recall that in Part I, we derived field equations (454) for relativistic dust in the trace-reversed harmonic gauge. In that case, we found that the gravito-Faraday law was *completely* absent. This was due to the fact that for non-relativistic dust $\left(T^{ij} \approx 0\right)$, we had $\bar{h}^{ij} \approx 0$ and as a result, the gauge condition explicitly required that $\partial_t \vec{h} = 0$. This eliminated the gravito-Faraday law. However, here we find that the gravito-Faraday law is only *limited* to low frequencies (with respect to v/c) in the non-relativistic approximation.

- There is no displacement current. This is *unlike* the harmonic gauge formulation in (454). The reason is because $\nabla \cdot \vec{J}_{m_\perp} = 0$ since $\vec{J}_{m_\perp} = \rho \vec{v}_\perp$ is transverse for an incompressible fluid.[55] Therefore ρ and $\vec{J}_{m_\perp}$ cannot be related by a continuity equation which would be required to lead to a displacement current.

- In the Newtonian limit, we find that Newtonian gravitation is encoded in *both* the field equations for Θ or Φ as given by (331) and (330), respectively. This can be seen by inserting $H = -6\phi/c^2$ from (443) into $\Theta_N = \frac{1}{3}H$ from (450) which gives $\Theta_N = -2\phi/c^2$. Then inserting this into $\nabla^2 \Theta_N = -\dfrac{8\pi G}{c^2}\rho$ from (331) gives

$$-\nabla^2 2\phi/c^2 \quad = \quad -\frac{8\pi G}{c^2}\rho \tag{456}$$

$$\nabla^2 \phi \quad = \quad 4\pi G \rho \tag{457}$$

  This is the form expected for Newtonian gravitation as pointed out in (342). Likewise, in the Newtonian limit, we can neglect $3\left(\mathbb{P} - \dot{I}\right)/c^2$ in (330) so that the field equation becomes $\nabla^2 \Phi = 4\pi G \rho$. In the Newtonian limit, we have $\Phi_N = \phi$ and therefore we again recover Newtonian gravitation. Hence we concluded that the field equations for *both* Θ or Φ reduce to the appropriate expression for Newtonian gravity in the Newtonian limit.

Gauge freedom in Newtonian and first-order post-Newtonian limits

In (2406) of Appendix A, we show that the full gauge freedom of *non*-linearized GR given as

$$g'^{\mu\nu} = \left(\delta_\sigma^{\ \mu} - \partial_\sigma \xi^\mu\right)\left(\delta_\rho^{\ \nu} - \partial_\rho \xi^\nu\right)\left(\eta^{\sigma\rho} - h^{\sigma\rho}\right) \tag{458}$$

[53] We still consider that $\dot{\rho} = 0$ from (417) so that it does not contribute to the time derivative.

[54] Likewise, we could argue that for an oscillating source where $\dot{\rho}\vec{v}_\perp = \rho_{\max}\omega\vec{v}_\perp$, then $\mu_g \dot{\rho}\vec{v}_\perp \sim \rho_{\max}\omega\vec{v}_\perp/c^2$. Therefore if ω/c is comparable to $v_{\max}/c$ then we would have a source on the order of v^2/c^2 which violates the approximation required for those equations to hold.

[55] This was an added restriction placed on the stress tensor while deriving the field equations in Section 25, however, without this added condition, the second-order, post-Newtonian equations in (429) and (454) would have been much more complicated.

It is evident that the gauge freedom is described completely in terms of the quantity $\partial^\mu \xi^\nu$ which introduces four different derivatives (one for each value of μ) of four different components of ξ^ν (one for each value of ν). Although we use a linearized version of this gauge freedom given by

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{459}$$

we find that the *degrees* of gauge freedom still remain the same since we still have all four derivatives of all four components of ξ^ν. The linearization process simply removes terms that are *small*, such as $\left(\partial^\rho \xi^\mu\right)\left(\partial_\rho \xi^\nu\right)$ and $\left(\partial_\sigma \xi^\mu\right)\left(h^{\sigma\nu}\right)$ but this does not remove any of the gauge degrees of freedom. However, we can show that because the first-order post-Newtonian limit sets $\bar{h}_{ij} = 0$, it effectively *removes* degrees of freedom from the gauge freedom in (459). Specifically, we showed in (442) that the off-diagonal elements of $\bar{h}_{ij}$ vanish. We can write (459) in terms of matching spatial indices and differing spatial indices.

$$h'_{ii} = h_{ii} + 2\partial_i \xi_i \qquad \text{and} \qquad \underset{(i\neq j)}{h'_{ij}} = \underset{(i\neq j)}{h_{ij}} + \partial_i \xi_j + \partial_j \xi_i \tag{460}$$

If the off-diagonal elements of h_{ij} vanish in all frame, then the second equation above is really

$$0 = \partial_i \xi_j + \partial_j \xi_i \tag{461}$$

$$\partial_i \xi_j = -\partial_j \xi_i \tag{462}$$

Since we know that $\partial_\mu \xi_\nu$ must be symmetrical (to keep the metric perturbation symmetric), then the anti-symmetric condition shown in (462) requires that $\partial_i \xi_j = 0$ for $i \neq j$. This means that 3 degrees of freedom are removed. (They are the off-diagonal elements of the spatial part of the tensor matrix, $\partial_\mu \xi_\nu$.) In that case, the only gauge freedom remaining is given by

$$h'_{0i} = h_{0i} + \partial_0 \xi_i + \partial_i \xi_0 \tag{463}$$

As a result, any formulation applying such an approximation cannot be gauge-invariant since we have effectively chosen the gauge given by $\partial_i \xi_j = 0$ for $i \neq 0$. Gauge-invariance is only preserved in the *second*-order post-Newtonian formulation involving the entire metric.

In terms of the HD metric, we can recognize that in going to the first-order post-Newtonian limit we have set $\lambda = \varepsilon_i = h^{\tau\tau}_{ij} = 0$ which means that the transformations given by (242) and (243) are eliminated. This will prevent constructing all four of the invariant quantities, Φ, Θ, Ξ_i, $h^{\tau\tau}_{ij}$.

In the Newtonian limit, we also set $\bar{h}_{0i} = 0$. Since $h_{0i} = \bar{h}_{0i}$, then writing (459) with a spatial index and temporal index gives

$$0 = \partial_0 \xi_i + \partial_i \xi_0 \tag{464}$$

$$\partial_0 \xi_i = -\partial_i \xi_0 \tag{465}$$

Once again, we know that $\partial_\mu \xi_\nu$ must be symmetrical therefore this anti-symmetric condition requires that $\partial_0 \xi_i = 0$. This means that 3 more degrees of gauge freedom are removed. (They are the time-space elements in the top row of the tensor matrix, $\partial_\mu \xi_\nu$.)

3.14 The gauge-dependent versus gauge-invariant Lense-Thirring fields

It is common in the literature to find that authors define the gauge-*dependent* gravito-magnetic field (or Lense-Thirring field)[56] as

$$\tilde{\vec{B}}_G \equiv \nabla \times \vec{h} \qquad \text{where} \qquad \vec{h} = c\,(h_{01}, h_{02}, h_{03}) \tag{466}$$

This formulation is adopted since it is directly analogous to the magnetic field, $\vec{B} = \nabla \times \vec{A}$. In fact, based on the expression found later in (854), it is predicted that the magnetic field, $\vec{B}$, and the gravito-magnetic field, $\tilde{\vec{B}}_G$, are *both* expelled from a superconductor.[57] This is famously claimed by by DeWitt in [42]. However, such a formulation of the gravito-magnetic field is *not* gauge-invariant since it is possible to choose a gauge which makes h_{0i} vanish. Namely, one could choose ξ_μ in (459) so that $h_{0i} = -\partial_0 \xi_i - \partial_i \xi_0$ which means $h'_{0i} = h_{0i} + \partial_0 \xi_i + \partial_i \xi_0$ gives $h'_{0i} = 0$.

On the other hand, in developing the gauge-invariant field equations in (361) we find that the gauge-*invariant* gravito-magnetic field is given by

$$\vec{B}_G = \nabla \times \vec{\Xi} \qquad \text{where} \qquad \vec{\Xi} = \vec{\beta} - \dot{\vec{\varepsilon}} \tag{467}$$

This gravito-magnetic field must be gauge-invariant (in terms of the gauge freedom given in (459) for linearized GR) simply because $\vec{B}_G$ is defined only in terms of Ξ_i which is a gauge-invariant potential. Notice that the gauge freedom in linearized GR given by $h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ in (200) contains four degrees of freedom while the gauge freedom in EM given by $A'_\mu = A_\mu + \partial_\mu \chi$, where χ is a scalar gauge function, contains only 1 degree of freedom. Therefore, it is evident that we must make the gauge choice $\xi_0 = \chi$ and $\xi_i = 0$ in order for the gauge freedom in linearized GR to be reduced to match the gauge freedom in EM.

Also notice that the gauge-*invariant* gravito-magnetic field is expressed in terms of *all* the "rotational" degrees of freedom from the metric, namely, $\vec{\beta}$ from h_{0i} as well as $\dot{\vec{\varepsilon}}$ from h_{ij}. In contrast, the gauge-*dependent* gravito-magnetic field is expressed only in terms of $\vec{\beta}$ and $\nabla\alpha$ (which is actually an *irrotational* component).

In addition, the gauge-*invariant* gravito-magnetic field, $\vec{B}_G$, is valid for fully relativistic sources and dynamic fields, whereas $\tilde{\vec{B}}_G = \nabla \times \vec{h}$ is not. This is because $\tilde{\vec{B}}_G$ is typically derived by applying the harmonic gauge to the trace-reversed perturbation in linearized GR $\left(\partial_\nu \bar{h}^{\mu\nu} = 0\right)$ with non-relativistic sources so that $T^{ij} \approx 0$ and consequently $\bar{h}^{ij} \approx 0$. In that context, the harmonic gauge leads to $\partial_t \bar{h}_{0i} = 0$ and therefore the formulation is only valid for steady-state (or gravitational "magneto-static") cases, not for fully dynamic cases where $\vec{B}_G$ is a time-varying field.

Lastly, we find that it is $\vec{B}_G$, not $\tilde{\vec{B}}_G$, that appears in an Ampere-like law in the field equations given in (361). Likewise, it is $\vec{B}_G$, not $\tilde{\vec{B}}_G$, that appears in a cross product with the velocity in the geodesic equation of motion found later in (501). Therefore, the field equations and the equation of motion clearly imply that $\vec{B}_G$ is the appropriate gravito-magnetic field.

However, it should be noted in the first-order post-Newtonian limit, we find that $\tilde{\vec{B}}_G = \vec{B}_{g(PN)}$. This is because $h_{0i} = \beta_i + \partial_i \alpha$ so taking the curl of h_{0i} gives

$$\tilde{\vec{B}}_G = \nabla \times \vec{h} = \nabla \times \left(\vec{\beta} + \nabla\alpha\right) = \nabla \times \vec{\beta} \tag{468}$$

[56] We use a tilda on the gauge-*dependent* gravito-magnetic field, $\tilde{\vec{B}}_g \equiv \nabla \times \vec{h}$, in order to distinguish it from the gauge-*invariant* gravito-magnetic field, $B_g = \nabla \times \vec{\Xi}$.

[57] However, it is shown in (1201) that because the gravito-Ampere law has a negative sign, the resulting differential equation for $B_{g(PN)}$ does not permit exponential decay solutions and therefore there is no expulsion of the gravito-magnetic field.

The term with $\nabla\alpha$ vanishes since the curl of a gradient is zero which means that we could just as well define the gauge-*dependent* gravito-magnetic field as $\tilde{\vec{B}}_G = \nabla\times\vec{\beta}$. We can compare this to the gauge-*invariant* gravito-magnetic field, $\vec{B}_G = \nabla\times\vec{\Xi}$ where $\vec{\Xi} = \vec{\beta} - \vec{\varepsilon}$. In the first-order post-Newtonian limit where $\vec{\varepsilon} \approx 0$, then $\vec{\Xi}_{PN} \approx \vec{\beta}$ and therefore

$$\vec{B}_{g(PN)} = \nabla\times\vec{\Xi}_{PN} \approx \nabla\times\vec{\beta} \tag{469}$$

Therefore, in the first-order post-Newtonian limit, we find $\vec{B}_{g(PN)} \approx \tilde{\vec{B}}_G$ which means that the gauge-*invariant* gravito-magnetic field effectively reduces to the gauge-*dependent* gravito-magnetic field.

4 Equations of motion in terms of the Helmholtz Decomposition (HD) metric

4.1 Overview of the equations of motion in terms of HD metric

Here we formulate the associated linearized geodesic equation of motion We find that the equation of motion contains gauge-*invariant* contributions identified as a “gravito-Lorentz” force in terms of GEM fields, as well as forces produced by gravitational waves. We also identify the remaining gauge-dependent terms that demonstrate the gauge freedom in the equations of motion which is expected for consistency with the Equivalence Principle. We develop the Newtonian and first-order post-Newtonian limits for the equation of motion and show that gauge freedom persists in those limits as well.

4.2 The linearized Christoffel symbols in terms of the HD metric

In preparation for evaluating the geodesic equation of motion, we first begin by evaluating the linearized Christoffel symbols in terms of the HD metric. The linearized Christoffel symbols were found in (2762) of Appendix J as

$$
\begin{aligned}
&\Gamma^0_{00} = -\tfrac{1}{2c}\dot{h}_{00} && \Gamma^i_{00} = \tfrac{1}{c}\dot{h}_{0i} - \tfrac{1}{2}\partial_i h_{00} \\
&\Gamma^0_{0i} = \Gamma^0_{i0} = -\tfrac{1}{2}\partial_i h_{00} && \Gamma^i_{0j} = \Gamma^i_{j0} = \tfrac{1}{2}\left(\partial_j h_{0i} + \tfrac{1}{c}\dot{h}_{ij} - \partial_i h_{0j}\right) \\
&\Gamma^0_{ij} = -\tfrac{1}{2}\left(\partial_j h_{0i} + \partial_i h_{j0} - \tfrac{1}{c}\dot{h}_{ij}\right) && \Gamma^i_{jk} = \tfrac{1}{2}\left(\partial_k h_{ij} + \partial_j h_{ki} - \partial_i h_{jk}\right)
\end{aligned}
\tag{470}
$$

Next we insert the metric components from (175) – (177) into each of the Christoffel symbols above. For Γ^0_{00} we have

$$\Gamma^0_{00} = -\frac{1}{2c}\dot{h}_{00} \tag{471}$$

$$= -\frac{1}{2c}\left(-\frac{2\dot{\phi}}{c^2}\right) \tag{472}$$

$$\boxed{\Gamma^0_{00} = \dot{\phi}/c^3} \tag{473}$$

For $\Gamma^0_{0i} = \Gamma^0_{i0}$ we have

$$\Gamma^0_{0i} = -\frac{1}{2}\partial_i h_{00} \tag{474}$$

$$= -\frac{1}{2}\partial_i\left(-2\phi/c^2\right) \tag{475}$$

$$\boxed{\Gamma^0_{0i} = \Gamma^0_{i0} = \nabla\phi/c^2} \tag{476}$$

For Γ^0_{ij} we have

$$
\begin{aligned}
\Gamma^0_{ij} &= -\frac{1}{2}\left(\partial_j h_{0i} + \partial_i h_{j0} - \frac{1}{c}\dot{h}_{ij}\right) \\
&= -\frac{1}{2c}\Big[\partial_j\left(\beta_i + \partial_i\alpha\right) + \partial_i\left(\beta_j + \partial_j\alpha\right) \\
&\quad -\dot{h}^{\tau\tau}_{ij} - \tfrac{1}{3}\delta_{ij}\dot{H} - \partial_{(i}\dot{\varepsilon}_{j)} - \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\dot{\lambda}\Big]
\end{aligned}
\tag{477}
$$

$$\boxed{\Gamma^0_{ij} = -\frac{1}{2c}\left[\partial_j\beta_i + \partial_i\beta_j + 2\partial_i\partial_j\alpha - \dot{h}^{\tau\tau}_{ij} - \tfrac{1}{3}\delta_{ij}\dot{H} - \partial_{(i}\dot{\varepsilon}_{j)} - \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\dot{\lambda}\right]} \tag{478}$$

For Γ^i_{00} we have

$$\Gamma^i_{00} = \frac{1}{c}\dot{h}_{0i} - \frac{1}{2}\partial_i h_{00} \tag{479}$$

$$= \frac{1}{c^2}\left(\dot{\beta}_i + \partial_i\dot{\alpha}\right) - \frac{1}{2}\partial_i\left(-2\phi/c^2\right) \tag{480}$$

$$\boxed{\Gamma^i_{00} = \frac{1}{c^2}\left(\dot{\beta}_i + \partial_i\dot{\alpha} + \partial_i\phi\right)} \tag{481}$$

For Γ^i_{0j} we have

$$\Gamma^i_{0j} = \Gamma^i_{j0} = \frac{1}{2}\left(\partial_j h_{0i} + \frac{1}{c}\dot{h}_{ij} - \partial_i h_{0j}\right) \tag{482}$$

$$\begin{aligned} &= \frac{1}{2c}\left[\partial_j\left(\beta_i + \partial_i\alpha\right) + \dot{h}^{\tau\tau}_{ij} + \tfrac{1}{3}\delta_{ij}\dot{H} + \partial\left({}_i\dot{\varepsilon}_j\right)\right. \\ &\quad \left. + \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\dot{\lambda} - \partial_i\left(\beta_j + \partial_j\alpha\right)\right] \end{aligned} \tag{483}$$

$$\boxed{\Gamma^i_{0j} = \frac{1}{2c}\left[\partial_j\beta_i - \partial_i\beta_j + \dot{h}^{\tau\tau}_{ij} + \tfrac{1}{3}\delta_{ij}\dot{H} + \partial\left({}_i\dot{\varepsilon}_j\right) + \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\dot{\lambda}\right]} \tag{484}$$

For Γ^i_{jk} we have

$$\Gamma^i_{jk} = \tfrac{1}{2}\left(\partial_k h_{ij} + \partial_j h_{ki} - \partial_i h_{jk}\right) \tag{485}$$

$$\begin{aligned} &= \tfrac{1}{2}\partial_k\left[h^{\tau\tau}_{ij} + \tfrac{1}{3}\delta_{ij}H + \partial\left({}_i\varepsilon_j\right) + \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\lambda\right] \\ &\quad + \tfrac{1}{2}\partial_j\left[h^{\tau\tau}_{ki} + \tfrac{1}{3}\delta_{ki}H + \partial\left({}_k\varepsilon_i\right) + \left(\partial_k\partial_i - \tfrac{1}{3}\delta_{ki}\nabla^2\right)\lambda\right] \\ &\quad - \tfrac{1}{2}\partial_i\left[h^{\tau\tau}_{jk} + \tfrac{1}{3}\delta_{jk}H + \partial\left({}_j\varepsilon_k\right) + \left(\partial_j\partial_k - \tfrac{1}{3}\delta_{jk}\nabla^2\right)\lambda\right] \end{aligned} \tag{486}$$

Grouping terms involving $h^{\tau\tau}_{ij}$, H, ε_i and λ separately gives

$$\begin{aligned} \Gamma^i_{jk} &= \tfrac{1}{2}\left(\partial_k h^{\tau\tau}_{ij} + \partial_j h^{\tau\tau}_{ki} - \partial_i h^{\tau\tau}_{jk}\right) + \tfrac{1}{2}\left(\tfrac{1}{3}\partial_k\delta_{ij}H + \tfrac{1}{3}\partial_j\delta_{ki}H - \tfrac{1}{3}\partial_i\delta_{jk}H\right) \\ &\quad + \tfrac{1}{2}\left[\partial_k\partial_i\varepsilon_j + \partial_k\partial_j\varepsilon_i + \partial_j\partial_k\varepsilon_i + \partial_j\partial_i\varepsilon_k - \partial_i\partial_j\varepsilon_k - \partial_i\partial_k\varepsilon_j\right] \\ &\quad + \tfrac{1}{2}\partial_i\partial_j\partial_k\lambda + \tfrac{1}{2}\left(-\tfrac{1}{3}\delta_{ij}\partial_k\nabla^2 - \tfrac{1}{3}\delta_{ki}\partial_j\nabla^2 + \tfrac{1}{3}\delta_{jk}\partial_i\nabla^2\right)\lambda \end{aligned} \tag{487}$$

Next we cancel four terms and combine two terms involving ε_i, as well as group together terms involving $H-\nabla^2\lambda$.

$$\begin{aligned}\Gamma^i_{jk} &= \tfrac{1}{2}\left(\partial_k h^{\tau\tau}_{ij}+\partial_j h^{\tau\tau}_{ki}-\partial_i h^{\tau\tau}_{jk}\right)+\partial_j\partial_k\varepsilon_i+\tfrac{1}{2}\partial_i\partial_j\partial_k\lambda \\ &\quad +\tfrac{1}{2}\left[\tfrac{1}{3}\partial_k\delta_{ij}\left(H-\nabla^2\lambda\right)+\tfrac{1}{3}\partial_j\delta_{ki}\left(H-\nabla^2\lambda\right)-\tfrac{1}{3}\partial_i\delta_{jk}\left(H+\nabla^2\lambda\right)\right]\end{aligned} \tag{488}$$

Lastly, substituting $\Theta=\frac{1}{3}\left(H-\nabla^2\lambda\right)$ gives

$$\boxed{\Gamma^i_{jk}=\tfrac{1}{2}\left(\partial_k h^{\tau\tau}_{ij}+\partial_j h^{\tau\tau}_{ki}-\partial_i h^{\tau\tau}_{jk}\right)+\partial_j\partial_k\varepsilon_i+\tfrac{1}{2}\partial_i\partial_j\partial_k\lambda+\tfrac{1}{2}\left(\partial_k\delta_{ij}+\partial_j\delta_{ki}-\partial_i\delta_{jk}\right)\Theta} \tag{489}$$

4.3 The linearized geodesic equation in terms of the HD metric

The geodesic equation of motion is given by

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} \quad = \quad 0 \tag{490}$$

It is shown in (2744) of Appendix J that by reparameterizing in terms of t instead of τ, we obtain

$$a^\mu+\Gamma^\mu_{\rho\sigma}v^\rho v^\sigma-\dot{\gamma}v^\mu=0 \tag{491}$$

where $\gamma=dt/d\tau$, $v^\mu=(c,\dot{x})=\left(c,v^i\right)$ and $a^\mu=\left(0,\dot{v}^i\right)$. It is also shown in (2738) of Appendix J that the "Lorentz factor" in terms of the metric perturbation is

$$\gamma \quad = \quad \left(1-h_{00}-2h_{0i}\frac{v^i}{c}-\frac{v^2}{c^2}-h_{ij}\frac{v^iv^j}{c^2}\right)^{-1/2} \tag{492}$$

We can insert the metric components from $(175)-(177)$ to obtain

$$\begin{aligned}\gamma \quad = \quad & \left[1+2\phi/c^2-2\left(\beta_i+\partial_i\alpha\right)\frac{v^i}{c^2}-\frac{v^2}{c^2}\right.\\ & \left.-\left(h^{\tau\tau}_{ij}+\tfrac{1}{3}\delta_{ij}H+\partial_{(i}\varepsilon_{j)}+\partial_i\partial_j\lambda-\tfrac{1}{3}\delta_{ij}\nabla^2\lambda\right)\frac{v^iv^j}{c^2}\right]^{-1/2}\end{aligned} \tag{493}$$

Note that (491) requires the time derivative of γ. It also requires that we expand the term involving the Christoffel symbol by summing over ρ and σ. Again, from Appendix J, we found in (2764) that to first order in $h_{\mu\nu}$ and second order in v^i/c, the geodesic equation of motion becomes

$$\begin{aligned}a^i \quad = \quad & \frac{c^2}{2}\partial_ih_{00}-c\dot{h}_{0i}+cv^j\left(\partial_ih_{0j}-\partial_jh_{0i}\right)-\dot{h}_{ij}v^j\\ & +\tfrac{1}{2}\left(\partial_ih_{jk}-\partial_kh_{ij}-\partial_jh_{ki}\right)v^jv^k-\frac{1}{2}\dot{h}_{00}v^i-\frac{1}{c}\dot{h}_{0j}v^jv^i\end{aligned} \tag{494}$$

Inserting the HD metric from $(175)-(177)$ gives[58]

$$\begin{aligned}a^i \quad = \quad & -\partial_i\phi-\left(\dot{\beta}_i+\partial_i\dot{\alpha}\right)+v^j\left[\partial_i\left(\beta_j+\partial_j\alpha\right)-\partial_j\left(\beta_i+\partial_i\alpha\right)\right]\\ & -\left[\dot{h}^{\tau\tau}_{ij}+\tfrac{1}{3}\delta_{ij}\dot{H}+\partial_i\dot{\varepsilon}_j+\partial_j\dot{\varepsilon}_i+\left(\partial_i\partial_j-\tfrac{1}{3}\delta_{ij}\nabla^2\right)\dot{\lambda}\right]v^j\\ & +\tfrac{1}{2}\left(\partial_ih_{jk}-\partial_kh_{ij}-\partial_jh_{ki}\right)v^jv^k+\dot{\phi}v^i/c^2-\left(\dot{\beta}_j+\partial_j\dot{\alpha}\right)v^jv^i/c^2\end{aligned} \tag{495}$$

[58] For brevity, we only insert the expression from (177) for $\dot{h}_{ij}$ but not for other appearances of h_{ij}.

In the first line, we cancel terms involving $\partial_i\partial_j\alpha$ and insert canceling terms involving $\ddot{\lambda}$ in order to construct $\Phi = \phi + \dot{\alpha} - \ddot{\lambda}/2$. In the second line we group terms in preparation to substitute $\dot{\Theta} = \frac{1}{3}\left(\dot{H} - \nabla^2\dot{\lambda}\right)$. We also move terms involving $\dot{\varepsilon}_i$ to the first line and insert canceling terms involving $\dot{\varepsilon}_i$ to construct $\Xi_i = (\beta_i - \dot{\varepsilon}_i)$.

$$
\begin{aligned}
a^i \quad = \quad & -\partial_i\left(\phi + \dot{\alpha} - \tfrac{1}{2}\ddot{\lambda}\right) - \tfrac{1}{2}\partial_i\ddot{\lambda} - \dot{\beta}_i + v^j\left(\partial_i\beta_j - \partial_i\dot{\varepsilon}_j - \partial_j\beta_i + \partial_j\dot{\varepsilon}_i\right) - 2v^j\partial_j\dot{\varepsilon}_i \\
& -v^j\dot{h}^{\tau\tau}_{ij} - v^j\partial_i\partial_j\dot{\lambda} - \tfrac{1}{3}v^i\left(\dot{H} - \tfrac{1}{3}\nabla^2\dot{\lambda}\right) \\
& +\tfrac{1}{2}\left(\partial_i h_{jk} - \partial_k h_{ij} - \partial_j h_{ki}\right)v^j v^k + \left(\dot{\phi} - v^j\dot{\beta}_j - v^j\partial_j\dot{\alpha}\right)v^i/c^2
\end{aligned} \tag{496}
$$

Substituting Φ, Ξ_i, and Θ gives

$$
\begin{aligned}
a^i \quad = \quad & -\partial_i\Phi - \tfrac{1}{2}\partial_i\ddot{\lambda} - \dot{\beta}_i + v^j\left(\partial_i\Xi_j - \partial_j\Xi_i\right) - 2v^j\partial_j\dot{\varepsilon}_i \\
& -v^j\dot{h}^{\tau\tau}_{ij} - v^j\partial_i\partial_j\dot{\lambda} - v^i\dot{\Theta} \\
& +\tfrac{1}{2}\left(\partial_i h_{jk} - \partial_k h_{ij} - \partial_j h_{ki}\right)v^j v^k + \left(\dot{\phi} - v^j\dot{\beta}_j - v^j\partial_j\dot{\alpha}\right)v^i/c^2
\end{aligned} \tag{497}
$$

We can write $v_j\left(\partial_i\Xi_j - \partial_j\Xi_i\right)$ as the i-component of $\vec{v}\times\nabla\times\vec{\Xi}$. We can also insert canceling terms involving $\ddot{\varepsilon}_i$ in order to construct $\dot{\Xi}_i$ in the first line. We then move all terms involving gauge-invariant quantities to the first line.

$$
\begin{aligned}
a^i \quad = \quad & -\partial_i\Phi - \left(\dot{\beta}_i - \ddot{\varepsilon}_i\right) - \ddot{\varepsilon}_i + \left(\vec{v}\times\nabla\times\vec{\Xi}\right)_i - v^i\dot{\Theta} - v^j\dot{h}^{\tau\tau}_{ij} \\
& -\tfrac{1}{2}\partial_i\ddot{\lambda} - v^j\left(\partial_i\partial_j\dot{\lambda} + 2\partial_j\dot{\varepsilon}_i\right) \\
& +\tfrac{1}{2}\left(\partial_i h_{jk} - \partial_k h_{ij} - \partial_j h_{ki}\right)v^j v^k + \left(\dot{\phi} - v^j\dot{\beta}_j - v^j\partial_j\dot{\alpha}\right)v^i/c^2
\end{aligned} \tag{498}
$$

We now substitute $\dot{\Xi}$ and move the remaining $\ddot{\varepsilon}$ to the second line. In the last line, we use the calculation shown in going from (485) to (489) to substitute for $\left(\partial_i h_{jk} - \partial_k h_{ij} - \partial_j h_{ki}\right)$.

$$
\begin{aligned}
a^i \quad = \quad & -\partial_i\Phi - \dot{\Xi}_i + \left(\vec{v}\times\nabla\times\vec{\Xi}\right)_i - v^i\dot{\Theta} - v^j\dot{h}^{\tau\tau}_{ij} \\
& -\left(\tfrac{1}{2}\partial_i\ddot{\lambda} + \ddot{\varepsilon}_i\right) - v^j\left(\partial_i\partial_j\dot{\lambda} + 2\partial_j\dot{\varepsilon}_i\right) \\
& -\left[\tfrac{1}{2}\left(\partial_k h^{\tau\tau}_{ij} + \partial_j h^{\tau\tau}_{ki} - \partial_i h^{\tau\tau}_{jk}\right) + \partial_j\partial_k\varepsilon_i + \tfrac{1}{2}\partial_i\partial_j\partial_k\lambda + \tfrac{1}{2}\left(\partial_k\delta_{ij} + \partial_j\delta_{ki} - \partial_i\delta_{jk}\right)\Theta\right]v^j v^k \\
& + \left(\dot{\phi} - v^j\dot{\beta}_j - v^j\partial_j\dot{\alpha}\right)v^i/c^2
\end{aligned} \tag{499}
$$

Next we distribute in the last line and move the terms involving Θ to the first line. We also move all the terms involving $h_{ij}^{\tau\tau}$ to the second line so that the first two lines contain only gauge-invariant quantities.

$$\begin{aligned} a^i \;=\;\; & -\partial_i\Phi - \dot{\Xi}_i + \left(\vec{v}\times\nabla\times\vec{\Xi}\right)_i - v^i\dot{\Theta} - \tfrac{1}{2}\left(2v_iv^j\partial_j\Theta - v^2\partial_i\Theta\right) \\ & -v^j\dot{h}_{ij}^{\tau\tau} - \tfrac{1}{2}v^jv^k\left(\partial_k h_{ij}^{\tau\tau} + \partial_j h_{ki}^{\tau\tau} - \partial_i h_{jk}^{\tau\tau}\right) \\ & -\left(\tfrac{1}{2}\partial_i\ddot{\lambda} + \ddot{\varepsilon}_i\right) - v^j\left(\partial_i\partial_j\dot{\lambda} + 2\partial_j\dot{\varepsilon}_i\right) - v^jv^k\partial_j\partial_k\varepsilon_i - \tfrac{1}{2}v^jv^k\partial_i\partial_j\partial_k\lambda \\ & +\left(\dot{\phi} - v^j\dot{\beta}_j - v^j\partial_j\dot{\alpha}\right)v^i/c^2 \end{aligned} \tag{500}$$

We can substitute $\vec{B}_G = \nabla\times\vec{\Xi}$ in the first line. We also observe that the combination $\frac{1}{2}\partial_i\lambda + \varepsilon_i$ appears repeatedly. Therefore we can define a "gauge" vector[59] given by $\chi_i = \frac{1}{2}\partial_i\lambda + \varepsilon_i$ so that we have

$$\boxed{\begin{aligned} & a^i = -\partial_i\Phi - \dot{\Xi}_i + \left(\vec{v}\times\vec{B}_G\right)_i + \left(\tfrac{1}{2}v^2\partial_i - v_i\partial_t - v_iv^j\partial_j\right)\Theta \\ & \qquad -v^j\dot{h}_{ij}^{\tau\tau} - \tfrac{1}{2}v^jv^k\left(\partial_k h_{ij}^{\tau\tau} + \partial_j h_{ki}^{\tau\tau} - \partial_i h_{jk}^{\tau\tau}\right) \\ & \qquad -\left(v^jv^k\partial_j\partial_k - 2v^j\partial_j\partial_t - \partial_t^2\right)\chi_i + \left(\dot{\phi} - v^j\dot{\beta}_j - v^j\partial_j\dot{\alpha}\right)v^i/c^2 \end{aligned}} \tag{501}$$

It is evident that we can no longer combine terms to formulate gauge-invariant quantities in the expression above. The first two terms comprise a gauge-invariant "gravito-Lorentz force" while the remaining terms in the first line are also gauge-invariant but do not have an analog in electromagnetism. The second line is also gauge-invariant and gives the force that would be caused by a gravitational wave. The last line contains quantities that are all gauge-dependent. It is not surprising that the geodesic equation cannot be written purely in terms of gauge-invariant quantities since Christoffel symbols are necessarily gauge-dependent.

We also find that (501) can not be easily expressed in terms of the gravito-electric field defined in (352) as $\vec{E}_G = -\frac{1}{2}\nabla\left(\Phi - \frac{c^2}{2}\Theta\right) - \dot{\vec{\Xi}}$. The reason is that there is no factor of $\frac{1}{2}$ in front of $-\nabla\Phi$ and there is no term with $\frac{c^2}{4}\nabla\Theta$. Therefore ,we conclude that although defining $\vec{E}_G = -\frac{1}{2}\nabla\left(\Phi - \frac{c^2}{2}\Theta\right) - \dot{\vec{\Xi}}$ offers the advantage of combining the two field equations involving Φ and Θ into a single equation involving $\vec{E}_G$, the disadvantage is that $\vec{E}_G$ does not appear in the equation of motion.

[59] It is interesting to note that χ has the form of a "Helmholtz vector" in the sence that it contains a purely rotational part, ε_i, and a purely irrotational part, $\partial_i\lambda$.

4.4 Low velocity and weak-field limits of the linearized geodesic equation

For slow-moving test particles, we neglect terms involving v^2 and v/c^2 in (501) which gives

$$\boxed{\underset{(low\text{-}velocity)}{a_i} = -\partial_i\Phi - \dot{\Xi}_i + \left(\vec{v}\times\vec{B}_G\right)_i - v_i\dot{\Theta} - v^j\dot{h}^{\tau\tau}_{ij} + 2v^j\partial_j\dot{\chi}_i + \ddot{\chi}_i} \tag{502}$$

Although the expression in (501) is greatly simplified, we can still observe that it contains the same features. There is a gauge-invariant gravito-Lorentz force, another gauge-invariant contribution with no analog in electromagnetism, a gauge-invariant force caused by gravitational waves, and gauge freedom remaining in the last term. In the case of stationary test masses, we have

$$\boxed{\underset{(static)}{a_{i(PN)}} = -\partial_i\Phi - \dot{\Xi}_i + \ddot{\chi}_i} \tag{503}$$

This is analogous to the electric force in electromagnetism expressed as $\vec{F} = q\left(-\nabla\varphi - \partial_t\vec{A}\right)$. Notice that in both (502) and (503) above, there is still gauge freedom due to the presence of $\chi_i = \frac{1}{2}\partial_i\lambda + \varepsilon_i$ which was defined as a "gauge" vector. Of course, we could choose to set $\chi_i = 0$ (by setting $\frac{1}{2}\partial_i\lambda = -\varepsilon_i$) however, this would effectively be a gauge choice. In fact, it removes three degrees of freedom from the metric (one for each index value in $\frac{1}{2}\partial_i\lambda = -\varepsilon_i$). In that case, it is questionable whether it is still valid to construct gauge-invariant quantities such as Φ, Θ, and Ξ_i if we have already chosen a gauge. This means that the results in (502) and (503) would still become gauge-*dependent* because Φ, Θ, and Ξ_i would no longer be gauge-invariant.

Returning to (501), we can consider the first-order post-Newtonian limit. In Section 26, it is shown that in the first-order post-Newtonian limit, the off-diagonal components of h_{ij} are neglected so that $\lambda = \varepsilon_i = h^{\tau\tau}_{ij} = 0$ which also means $\chi_i = 0$. The potentials become

$$\Phi_{PN} \approx \phi + \dot{\alpha}, \qquad \Theta_{PN} = \tfrac{1}{3}H, \qquad \vec{\Xi}_{PN} = \vec{\beta} \tag{504}$$

while the vector fields become

$$\vec{E}_{g(PN)} \equiv -\frac{1}{2}\nabla\left(\Phi_{PN} - \frac{c^2}{2}\Theta_{PN}\right) - \dot{\vec{\Xi}}_{PN} \qquad \text{and} \qquad \vec{B}_{g(PN)} \equiv \nabla\times\vec{\Xi}_{PN} \tag{505}$$

The equation of motion in (501) then becomes

$$\boxed{\begin{aligned} a_{i(PN)} = {} & -\partial_i\Phi_{PN} - \dot{\Xi}_{i(PN)} + \left(\vec{v}\times\vec{B}_{g(PN)}\right)_i + \left(\tfrac{1}{2}v^2\partial_i - v_i\partial_t - v_iv^j\partial_j\right)\Theta_{PN} \\ & + \left(\dot{\phi} - v^j\dot{\beta}_j - v^j\partial_j\dot{\alpha}\right)v^i/c^2 \end{aligned}} \tag{506}$$

Once again, we can consider slow-moving test particles and neglect terms involving v^2 and v/c^2 which gives

$$\boxed{\underset{(low\text{-}velocity)}{a_{i(PN)}} = -\partial_i\Phi_{PN} - \dot{\Xi}_{i(PN)} + \left(\vec{v}\times\vec{B}_{g(PN)}\right)_i - v_i\dot{\Theta}_{PN}} \tag{507}$$

The equation of motion in (507) is often described as the gravitational "Lorentz force." It is obviously very similar to the electromagnetic Lorentz force. However, the extra term involving $\dot{\Theta}_{PN}$ usually does not appear.

The reason it appears here is due to the fact that we decomposed the metric component h_{0i} into a rotational component and an irrotational component, $h_{0i} = \beta_i + \partial_i \alpha$. If we require it to be purely rotational, then $\alpha = 0$. In that case, $\Phi_{PN} \approx \phi + \dot{\alpha}$ becomes $\Phi_N \approx \phi$ and in Section 26 we found that this leads to $\Theta_N = -\frac{2}{c^2}\Phi_N$. Therefore, the last term in (507) becomes $2v_i\dot{\Phi}_N/c^2$ which must be neglected since we have neglected all terms involving v/c^2. Also, since $\Xi_i = h_i$ when $\alpha = \varepsilon = 0$, then we can replace Ξ_i with h_i.Therefore, in vector notation (507) becomes[60]

$$\boxed{\underset{(low\text{-}velocity)}{\vec{a}_{(PN)}} = -\nabla\Phi_N - \dot{\vec{h}} + \left(\vec{v} \times \vec{B}_{g(PN)}\right), \qquad \text{for } \nabla \cdot \vec{h} = 0} \tag{508}$$

This result is now extremely similar to the Lorentz force. Note that it differs slightly from the expression commonly obtained using the harmonic gauge applied to linearized GR for the case of weak-fields and slow-moving test particles. In that formulation, there is commonly a factor of 4 that appears in front of the gravito-magnetic force. However, the factor of 4 appears because the harmonic gauge is typically applied to the *trace-reversed* metric perturbation. When returning to the *non*-trace-reversed metric perturbation, the factor of 4 appears.

Lastly, we return again to (501) and consider the Newtonian limit. In Section 26, it is shown that in the Newtonian limit, we have $\beta_i = \alpha = \lambda = \varepsilon_i = h_{ij}^{\tau\tau} \approx 0$ so that only ϕ and H remain. The potentials become

$$\Phi_N \approx \phi, \qquad \Theta_N = \tfrac{1}{3}H, \qquad \vec{\Xi}_N = 0 \tag{509}$$

We also found $\Theta_N = -\frac{2}{c^2}\Phi_N$ so that vector fields become

$$\vec{E}_{g(N)} \equiv -\nabla\Phi_N \qquad \text{and} \qquad \vec{B}_{g(N)} \equiv 0 \tag{510}$$

The equation of motion in (501) then becomes

$$a_{i(N)} = -\partial_i\Phi_N - \tfrac{2}{c^2}\left(\tfrac{1}{2}v^2\partial_i - v_i\partial_t - v_iv^j\partial_j\right)\Phi_N + \dot{\Phi}_N v^i/c^2 \tag{511}$$

$$\boxed{a_{i(N)} = -\partial_i\Phi_N - \tfrac{v^2}{c^2}\partial_i\Phi_N - \tfrac{2}{c^2}v_iv^j\partial_j\Phi_N - \dot{\Phi}_N v^i/c^2} \tag{512}$$

For slow-moving test particles, we neglect terms involving v^2 and v/c^2 which gives

$$\boxed{\underset{(low\text{-}velocity)}{a_{i(N)}} = -\partial_i\Phi_N} \tag{513}$$

This is effectively just Newton's second law equated with Newton's universal law of gravitation. Notice that it is technically valid even for $v \neq 0$ since the gravito-magnetic force (and any other velocity dependent forces) are removed by the Newtonian limit. Therefore, the slow-moving limit and the static limit are the same for Newtonian gravity.

We emphasize once again that *all* of the expressions found above for the equation of motion were gauge-*dependent*. In some cases the gauge-dependence is manifest due to an obvious gauge term in the equation, such as in (502) and (503). In other cases, the gauge-dependence was implicit in the fact that in Newtonian and first-order post-Newtonian limits, the previously gauge-invariant potentials are no longer gauge invariant.

---

[60] Here we are using $\vec{h} = c\,(h_{01}, h_{02}, h_{03})$ which omits the prefactor of $1/4$ found in the definition of $\vec{h}$ in (32) of Section 3. Also, note that writing the condition $\nabla \cdot \vec{h} = 0$ is equivalent to requiring $\alpha = 0$. This follows from the fact that $\partial_i h_{0i} = \nabla^2\alpha = 0$. Since (184) requires that $\alpha \to 0$ as $r \to \infty$, then $\nabla^2\alpha = 0$ can only be satisfied if $\alpha = 0$.

We also emphasize that the entire treatment concerning the geodesic equation of motion required that we go beyond the strict condition of linearized GR which requires $\partial_\nu T^{\mu\nu} = 0$. Since the full covariant conservation of energy-momentum is

$$\nabla_\nu T^{\mu\nu} = \partial_\nu T^{\mu\nu} + \Gamma^\nu_{\nu\sigma} T^{\sigma\mu} + \Gamma^\mu_{\nu\sigma} T^{\nu\sigma} = 0 \tag{514}$$

then linearized GR requires that we omit terms of order $\Gamma^\nu_{\nu\sigma} T^{\sigma\mu}$. This means that we must also omit $\Gamma^\mu_{\rho\sigma} u^\rho u^\sigma$ from the geodesic equation of motion in (490) which reduces it to $\ddot{x}^\mu = 0$. This implies that matter does not respond to the gravitational field at all. In order to avoid this trivial result, we have kept $\Gamma^\mu_{\rho\sigma} u^\rho u^\sigma$ in (490) which means we have essentially kept $\Gamma^\nu_{\nu\sigma} T^{\sigma\mu}$ in the conservation relation in (514). Therefore, we are no longer justified in using the linearized Einstein equation since we found in Appendix A that the linearized Einstein equation necessarily leads to $\partial_\nu T^{\mu\nu} = 0$. In other words, the key point being made is that none of the results for the geodesic equation of motion in this treatment can be put back into the *linearized* Einstein equation to calculate fields. Doing so would be a contradiction in approximations.

On a related note, we should also emphasize that the designations of “first-order” and “second-order” in these sections was specifically with regard to *post-Newtonian* order as defined in Section 26. In other words, it is concerning fields which arise from sources involving v/c or $(v/c)^2$ and hence produce fields involving h_{0i} and h_{ij}, respectively. These terms are all still completely within the framework of *linearized* GR. They are not to be confused with “second-order” in the *full metric* which describes the self-coupling of gravity. In that case, “second-order ” refers to terms such as $h_{\rho\sigma} h^{\rho\sigma} \eta_{\mu\nu}$, “third order” refers to terms such as $h^{\rho\alpha} h_{\rho\mu} h_{\sigma\nu}$, etc. These considerations of “first-order” and “second-order” will always be referred to as higher order “coupling” terms rather than higher order “Post-Newtonian” terms.

4.5 The geodesic deviation equation in terms of gauge-invariant potentials

In (2834) of Appendix L, the linearized geodesic deviation equation was given as

$$\ddot{L}^{\mu} = -R^{\mu}{}_{\rho\gamma\sigma} L^{\gamma} u^{\rho} u^{\sigma} \tag{515}$$

where $L^{\mu} = \tilde{x}^{\mu} - x^{\mu}$ is the coordinate separation between two particles which follow two neighboring world-lines given by x^{μ} and $\tilde{x}^{\mu}$. Then $\ddot{L}^{\mu}$ gives the relative acceleration between the two particles. In (2473) of Appendix C, we also found that the linearized Riemann tensor (with lowered indices) is given by

$$R_{\beta\rho\gamma\sigma} \quad = \quad \tfrac{1}{2}\left(\partial_{\gamma}\partial_{\rho}h_{\sigma\beta} - \partial_{\gamma}\partial_{\beta}h_{\rho\sigma} - \partial_{\sigma}\partial_{\rho}h_{\gamma\beta} + \partial_{\sigma}\partial_{\beta}h_{\rho\gamma}\right) \tag{516}$$

In the local proper Lorentz frame of x^{μ}, we have $u^{\mu} = (c, 0)$ which means that (515) will only be non-zero for $\rho = \sigma = 0$ which means $R^{\mu}{}_{\rho\gamma\sigma} = R^{\mu}{}_{0\gamma 0}$ and $u^{\rho}u^{\sigma} = u^0 u^0 = c^2$. Also, if L^{γ} is initially only a spatial separation given by $L^{\gamma} = \left(0, L^j\right)$, then (515) will only be non-zero for $\gamma = j$. Lastly, if we are interested in the spatial acceleration, then we are looking for $\ddot{L}^i$ and therefore we choose $\mu = i$. Then (515) becomes

$$\ddot{L}^i \quad = \quad -R^i{}_{0j0} L^j c^2 \tag{517}$$

This means that the Riemann tensor component of interest will be $R^{\mu}{}_{\rho\gamma\sigma} = R^i{}_{0j0}$. If we lower indices (to first order in the metric) we have

$$R^i{}_{0j0} = \eta^{\mu i} R_{\mu 0j0} = \eta^{0i} R_{00j0} + \eta^{ki} R_{k0j0} = \eta^{ki} R_{kj00} \tag{518}$$

This is non-zero only for $k = i$ which gives $R^i{}_{0j0} = R_{i0j0}$. Therefore, (517) can be written as

$$\ddot{L}^i = -R_{i0j0} L^j c^2 \tag{519}$$

Using (516) gives

$$R_{i0j0} \quad = \quad \tfrac{1}{2}\left(\partial_j\partial_0 h_{0i} - \partial_j\partial_i h_{00} - \partial_0\partial_0 h_{ji} + \partial_0\partial_i h_{0j}\right) \tag{520}$$

$$= \quad \frac{1}{2}\left(\frac{1}{c}\partial_j \dot{h}_{0i} - \partial_j\partial_i h_{00} - \frac{1}{c^2}\ddot{h}_{ji} + \frac{1}{c}\partial_i \dot{h}_{0j}\right) \tag{521}$$

Inserting the metric components from $(175) - (177)$ gives

$$\begin{aligned} R_{i0j0} \quad = \quad & \frac{1}{2}\left\{\frac{1}{c^2}\partial_j\left(\dot{\beta}_i + \partial_i\dot{\alpha}\right) + \frac{2}{c^2}\partial_j\partial_i\phi \right. \\ & \left. -\frac{1}{c^2}\left[\ddot{h}^{\tau\tau}_{ij} + \tfrac{1}{3}\delta_{ij}\ddot{H} + \partial\left({}_i\ddot{\varepsilon}_j\right) + \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\ddot{\lambda}\right] + \frac{1}{c^2}\partial_i\left(\dot{\beta}_j + \partial_j\dot{\alpha}\right)\right\} \end{aligned} \tag{522}$$

Factoring out $1/c^2$ and distributing gives

$$\begin{aligned} R_{i0j0} \quad = \quad & \frac{1}{2c^2}\left[\partial_j\dot{\beta}_i + \partial_j\partial_i\dot{\alpha} + 2\partial_j\partial_i\phi \right. \\ & \left. -\ddot{h}^{\tau\tau}_{ij} - \tfrac{1}{3}\delta_{ij}\ddot{H} - \partial\left({}_i\ddot{\varepsilon}_j\right) - \partial_i\partial_j\ddot{\lambda} + \tfrac{1}{3}\delta_{ij}\nabla^2\ddot{\lambda} + \partial_i\dot{\beta}_j + \partial_i\partial_j\dot{\alpha}\right] \end{aligned} \tag{523}$$

Next we regroup terms in preparation to insert the gauge-invariant quantities given in $(244)-(246)$ as $\Phi = \phi + \dot{\alpha} - \ddot{\lambda}/2$, $\Theta = \frac{1}{3}\left(H - \nabla^2\lambda\right)$ and $\Xi_i = \beta_i - \dot{\varepsilon}_i$. We can write (523) as

$$R_{i0j0} = \frac{1}{c^2}\left[\frac{1}{2}\left(\partial_i\dot{\beta}_j + \partial_j\dot{\beta}_i - \partial\left(_i\ddot{\varepsilon}_{j}\right)\right) + \partial_j\partial_i\left(\phi + \partial_j\partial_i\dot{\alpha} - \frac{1}{2}\ddot{\lambda}\right) - \frac{1}{2}\ddot{h}^{\tau\tau}_{ij} - \frac{1}{6}\delta_{ij}\left(\ddot{H} - \nabla^2\ddot{\lambda}\right)\right] \tag{524}$$

In the first parentheses we have $\partial_i\dot{\beta}_j + \partial_j\dot{\beta}_i - \partial\left(_i\ddot{\varepsilon}_j\right) = \partial\left[_j\left(\dot{\beta}_j - \ddot{\varepsilon}_j\right)\right] = \partial\left(_i\ddot{\Xi}_j\right)$. Then writing the expression above completely in terms of the gauge-invariant quantities Φ, Θ, Ξ_i and $h^{\tau\tau}_{ij}$ gives

$$\boxed{R_{i0j0} = \frac{1}{2c^2}\left[2\partial_j\partial_i\Phi - \delta_{ij}\ddot{\Theta} + \partial\left(_i\ddot{\Xi}_j\right) - \ddot{h}^{\tau\tau}_{ij}\right]} \tag{525}$$

Arriving at an expression that is completely in terms of gauge-invariant quantities is consistent with the fact that the Riemann tensor is *always* gauge-invariant. As pointed out in [28], we find that the Riemann tensor component associated with measuring gravitational waves (such as at LIGO) can be expressed completely in terms of the gauge-invariant potentials. Therefore, unlike the geodesic equation of motion which is gauge-*dependent*, the geodesic deviation equation is gauge-*invariant* and provides a way to predict the relative motion of particles which does not depend on the coordinate frame of the observer.

We can now substitute this result into the linearized geodesic deviation equation in (519).

$$\boxed{\ddot{L}_i = -\frac{1}{2}\left[2\partial_j\partial_i\Phi - \delta_{ij}\ddot{\Theta} + \partial\left(_i\ddot{\Xi}_j\right) - \ddot{h}^{\tau\tau}_{ij}\right]L^j} \tag{526}$$

This is the *linearized* geodesic deviation equation expressed using the local proper Lorentz frame of x^{μ} and evaluated in terms of the *spatial* separation, L^j, for two particles along neighboring worldlines, x^{μ} and x^{μ}, which are separated by $L^{\mu} = \tilde{x}^{\mu} - x^{\mu}$. It is shown in (334) that $\Phi, \Theta, \Xi_i \approx 0$ in the far-field zone. In that case, we can write the geodesic deviation equation in (526) as simply

$$\ddot{L}_i = \frac{1}{2}\ddot{h}^{\tau\tau}_{ij}L^j \tag{527}$$

5 Equations of motion for matter in the presence of gravitational (GR) waves

5.1 Overview of the equations of motion in the presence of GR waves

Here we examine the topic of gravitational waves incident on a planar boundary. We make use of the fact that the HD formulation of linearized GR allows us to isolate the radiative degrees of freedom given by $h_{ij}^{\tau\tau}$ and neglect all other components in the far-field. We also follow the usual approach of specializing to the transverse-traceless (TT) gauge in order to describe gravitational waves moving in the z-direction in terms of *plus*-polarization and *cross*-polarization waves. Then we return to the geodesic equation of motion to develop expressions for the acceleration and velocity of particles in the presence of gravitational waves. We also return to the geodesic deviation equation to develop expressions for the relative acceleration, velocity and position of particles in the presence of gravitational waves.

5.2 The Lorentz force in curved space-time for GR waves

To describe the net force on a lattice ion, we can use the Lorentz force in curved space-time written as

$$m\frac{d^2x^\mu}{d\tau^2}+m\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}=q\frac{dx_\nu}{d\tau}F^{\mu\nu} \tag{528}$$

where $F^{\mu\nu}$ is the electromagnetic field strength tensor with components given by

$$F^{0i}=\frac{1}{c}E^i, \qquad F^{ij}=\varepsilon^{ijk}B^k, \qquad F^{\mu\mu}=0 \tag{529}$$

For the left side of (528) we can use (2729) from Appendix J with $v^\mu=\left(c,v^i\right)$ and $a^\mu=\left(0,a^i\right)$. On the right side, we can raise the index by using the metric and use $u^\mu=\gamma v^\mu$. Also dividing through by m gives

$$\gamma^2 a^\mu+\gamma\dot{\gamma}v^\mu+\gamma^2\Gamma^\mu_{\alpha\beta}v^\alpha v^\beta \quad = \quad \frac{q}{m}g_{\rho\nu}\gamma v^\rho F^{\mu\nu} \tag{530}$$

Following the approach of Appendix J, we can evaluate (528) for $\mu=0$ and for $\mu=i$, and use $a^\mu=\left(0,\dot{v}^i\right)$ and $v^\mu=\left(c,v^i\right)$. This gives

$$c\gamma\dot{\gamma}+\gamma^2\Gamma^0_{\rho\sigma}v^\rho v^\sigma \quad = \quad \frac{q\gamma}{m}g_{\rho\nu}v^\rho F^{0\nu} \qquad \text{and} \qquad \gamma^2 a^i+\gamma\dot{\gamma}v^i+\gamma^2\Gamma^i_{\rho\sigma}v^\rho v^\sigma=\frac{q\gamma}{m}g_{\rho\nu}v^\rho F^{i\nu} \tag{531}$$

Multiplying the first equation by v^i/c, subtracting it from the second, and dividing through by γ^2 gives

$$a^i+\Gamma^i_{\rho\sigma}v^\rho v^\sigma-\Gamma^0_{\rho\sigma}v^\rho v^\sigma v^i \quad = \quad \frac{q}{m\gamma}g_{\rho\nu}v^\rho\left(F^{i\nu}-F^{0\nu}v^i\right) \tag{532}$$

We can write the metric as a perturbation to flat space-time, $g_{\rho\nu}=\eta_{\rho\nu}+h_{\rho\nu}$, and solve for a^i to obtain

$$a^i \quad = \quad -\Gamma^i_{\rho\sigma}v^\rho v^\sigma+\Gamma^0_{\rho\sigma}v^\rho v^\sigma v^i+\frac{q}{m\gamma}\left(\eta_{\rho\nu}+h_{\rho\nu}\right)v^\rho\left(F^{i\nu}-F^{0\nu}v^i\right) \tag{533}$$

The first two terms on the right side is the acceleration due to gravity. The last term is the acceleration due to the electromagnetic (EM) field, including the coupling of the EM field to gravity: $h_{\rho\nu}v^\rho\left(F^{i\nu}-F^{0\nu}v^i\right)$. If we consider a gravitational wave produced by a distant source, then $h_{00},\ h_{0i}\approx 0$ and $h_{ij}\approx h^{\tau\tau}_{ij}$. Then to linear order in the metric, we can use (2765) from Appendix J to substitute for the first two terms of (533). Also summing over ρ and ν, and using $\eta_{00}=-1$ and $\eta_{i0},F^{00}=0$, gives

$$a^i \quad = \quad -v^j\dot{h}^{\tau\tau}_{ij}+\left(\frac{1}{2}\partial_i h^{\tau\tau}_{jk}-\partial_j h^{\tau\tau}_{ik}\right)v^j v^k-\frac{1}{2c^2}\dot{h}^{\tau\tau}_{jk}v^j v^k v^i$$

$$-\frac{q}{m\gamma}\left[cF^{i0}+\left(\eta_{jk}+h^{\tau\tau}_{jk}\right)v^j\left(F^{ik}-F^{0k}v^i\right)\right] \tag{534}$$

Next we can use (2738) from Appendix J to write γ in terms of $h^{\tau\tau}_{ij}$ (with $h_{00}=h_{0i}=0$ and $h_{ij}=h^{\tau\tau}_{ij}$).

$$\gamma^{-1} \quad = \quad \left(1-\frac{v^2}{c^2}-\frac{v^j v^k}{c^2}h^{\tau\tau}_{jk}\right)^{1/2} \tag{535}$$

In the weak-field limit, we can consider that $1 - v^2/c^2 >> \left|\frac{v^j v^k}{c^2} h^{\tau\tau}_{jk}\right|$ and use a binomial approximation. This leads to

$$\gamma^{-1} \quad \approx \quad \left(1 - \frac{v^2}{c^2} - \frac{v^l v^m}{2c^2} h^{\tau\tau}_{lm}\right) \tag{536}$$

Inserting this into (534) and using (529) gives

$$\begin{aligned} a^i \quad &= \quad -v^j \dot{h}^{\tau\tau}_{ij} + \left(\frac{1}{2}\partial_i h^{\tau\tau}_{jk} - \partial_j h^{\tau\tau}_{ik}\right) v^j v^k - \frac{1}{2c^2} \dot{h}^{\tau\tau}_{jk} v^j v^k v^i \\ & \quad -\frac{q}{m}\left(1 - \frac{v^2}{c^2} - \frac{v^l v^m}{2c^2} h^{\tau\tau}_{lm}\right)\left[E^i + \left(\eta_{jk} + h^{\tau\tau}_{jk}\right) v^j \left(\varepsilon^{ikn} B^n - E^k v^i/c\right)\right] \end{aligned} \tag{537}$$

Approximating to second order in velocity and rearranging gives

$$\boxed{\begin{aligned} a^i = -v^j \dot{h}^{\tau\tau}_{ij} + \left(\frac{1}{2}\partial_i h^{\tau\tau}_{jk} - \partial_j h^{\tau\tau}_{ik}\right) v^j v^k + \frac{q}{m}\left(1 - \frac{v^2}{c^2}\right) E^i - \frac{q}{m}\varepsilon^{ijk} v^j B^k \\ -\frac{q}{2mc^2} h^{\tau\tau}_{jk} v^j v^k E^i - \frac{q}{m} h^{\tau\tau}_{jk} v^j \varepsilon^{ikl} B^l \end{aligned}} \tag{538}$$

Lorentz force in the presence of gravitational waves
(*first order in the metric and second order in particle velocity*)

Note that the first two terms on the right are the acceleration terms due purely to the gravitational wave. The second two terms make up a relativistic Lorentz force which describes the acceleration due purely to the electromagnetic fields. The last two terms describe the acceleration terms due to the associated with the electromagnetic fields coupled to the gravitational wave field. Approximating to first order in velocity and rearranging gives

$$ma^i \quad = \quad qE^i - q\varepsilon^{ijk} v^j B^k - v^j \dot{h}^{\tau\tau}_{ij} - qv^j \varepsilon^{ikl} h^{\tau\tau}_{jk} B^l \tag{539}$$

Now it is even more apparent that the total force on a charged test particle is simply the Lorentz force (the first two terms), the gravitational wave force (the third term), and an additional magnetic force that is also coupled to the gravitational wave (the fourth term).

5.3 The geodesic deviation of spatial distance due to a GR wave

For the geodesic *deviation* in the far-field, we again neglect Φ, Θ, and Ξ_i. Then (526) gives

$$\ddot{L}_i = \frac{1}{2}\ddot{h}_{ij}^{\tau\tau} L^j \tag{540}$$

We can sum over j to obtain

$$\ddot{L}_i = \frac{1}{2}\ddot{h}_{ix}^{\tau\tau} L_x + \frac{1}{2}\ddot{h}_{iy}^{\tau\tau} L_y + \frac{1}{2}\ddot{h}_{iz}^{\tau\tau} L_z \tag{541}$$

For $\ddot{L}_x$ and $\ddot{L}_y$ we have, respectively,

$$\ddot{L}_x = \frac{1}{2}\ddot{h}_{xx}^{\tau\tau} L_x + \frac{1}{2}\ddot{h}_{xy}^{\tau\tau} L_y + \frac{1}{2}\ddot{h}_{xz}^{\tau\tau} L_z \quad \text{and} \quad \ddot{L}_y = \frac{1}{2}\ddot{h}_{yx}^{\tau\tau} L_x + \frac{1}{2}\ddot{h}_{yy}^{\tau\tau} L_y + \frac{1}{2}\ddot{h}_{yz}^{\tau\tau} L_z \tag{542}$$

For waves propagating in the z-direction, (341) gives

$$h_{xx}^{\tau\tau} = -h_{yy}^{\tau\tau} = A_{\oplus} \cos(kz - \omega t) \quad \text{and} \quad h_{xy}^{\tau\tau} = h_{yx}^{\tau\tau} = A_{\otimes} \cos(kz - \omega t) \tag{543}$$

where $h_{ij}^{\tau\tau} = 0$ for all other components and $A_{\oplus}$ and $A_{\otimes}$ are dimensionless, constant amplitudes for "plus" and "cross" polarization, respectively. The time derivatives of $h_{ij}^{\tau\tau}$ are

$$\dot{h}_{xx}^{\tau\tau} = -\dot{h}_{yy}^{\tau\tau} = A_{\oplus}\omega \sin(kz - \omega t) \quad \text{and} \quad \dot{h}_{xy}^{\tau\tau} = \dot{h}_{yx}^{\tau\tau} = A_{\otimes}\omega \sin(kz - \omega t) \tag{544}$$

Taking second time derivatives gives

$$\ddot{h}_{xx}^{\tau\tau} = -\ddot{h}_{yy}^{\tau\tau} = -A_{\oplus}\omega^2 \cos(kz - \omega t) \quad \text{and} \quad \ddot{h}_{xy}^{\tau\tau} = \ddot{h}_{yx}^{\tau\tau} = -A_{\otimes}\omega^2 \cos(kz - \omega t) \tag{545}$$

for "plus" and "cross" polarization, respectively. We still consider a planar surface at a fixed value of z. Inserting the appropriate expressions from (545) into (542) gives

$$\ddot{L}_x = -\frac{1}{2}(A_{\oplus} L_x + A_{\otimes} L_y)\,\omega^2 \cos(kz - \omega t) \tag{546}$$

and

$$\ddot{L}_y = -\frac{1}{2}(A_{\otimes} L_x - A_{\oplus} L_y)\,\omega^2 \cos(kz - \omega t) \tag{547}$$

Combining $\ddot{L}_x$ and $\ddot{L}_y$ into a single vector gives

$$\boxed{\begin{array}{c}\ddot{\vec{L}} = -\frac{1}{2}\left[(A_{\oplus} L_x + A_{\otimes} L_y)\,\hat{x} + (A_{\otimes} L_x - A_{\oplus} L_y)\,\hat{y}\right]\omega^2 \cos(kz - \omega t) \\ \textit{Geodesic deviation acceleration due to gravitational waves}\end{array}} \tag{548}$$

We can also write the expressions for plus-polarization and cross-polarization separately as

$$\ddot{\vec{L}}_{\oplus} = -\frac{1}{2}(L_x\hat{x} - L_y\hat{y})\,A_{\oplus}\omega^2 \cos(kz - \omega t) \tag{549}$$

and

$$\ddot{\vec{L}}_{\otimes} = -\frac{1}{2}(L_y\hat{x} + L_x\hat{y})\,A_{\oplus}\omega^2 \cos(kz - \omega t) \tag{550}$$

To visualize the solutions to the differential equations in (549) and (550), we can plot the trajectories on an x-y plane. Below is a diagram of the tidal wave acceleration fields for *plus* polarization and *cross* polarization (when $\ddot{h} > 0$) as shown in [11].

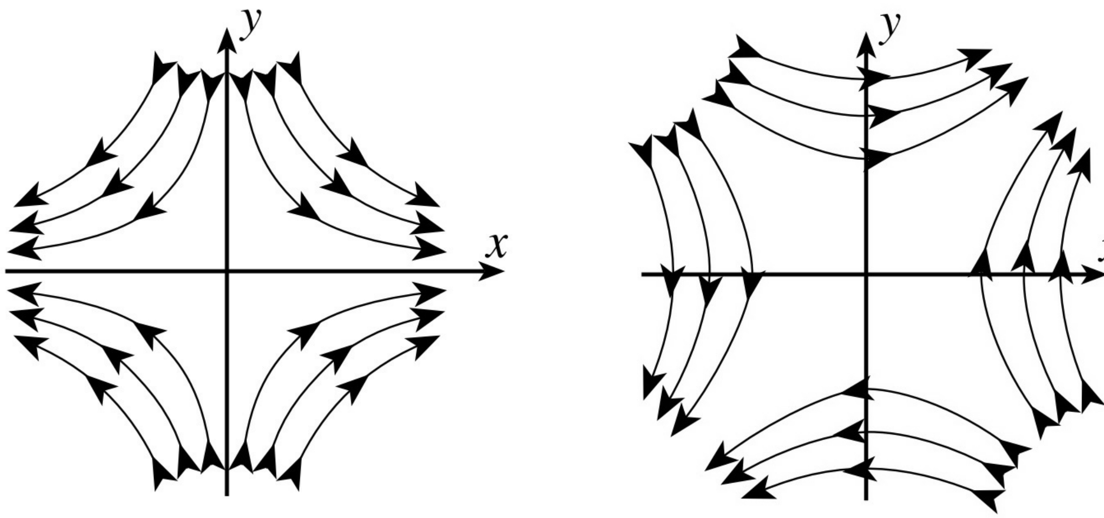

Figure 3: Plus-polarized and cross-polarized gravitational wave acceleration fields.

The solutions to the differential equations in (549) and (550) can be expressed in terms of Mathieu functions. For example, the solution for L_x for plus-polarization (549) has a solution given by

$$L_{x,\oplus}(t) = c_1 \text{MathieuC}\left[0, -A_\oplus, \tfrac{1}{2}(kz - \omega t)\right] + c_2 \text{MathieuC}\left[0, -A_\oplus, \tfrac{1}{2}(kz - \omega t)\right] \tag{551}$$

where c_1 and c_2 are constants determined by boundary conditions. For an *approximate* solution to (540), we can write L_i as a Taylor expansion to second order about $t = 0$.

$$L_i(t) \approx L_i(0) + \dot{L}_i(0)t + \tfrac{1}{2}\ddot{L}_i(0)t^2 \tag{552}$$

From (540), we can also use $\ddot{L}_i(0) = \frac{1}{2}\ddot{h}_{ij}^{\tau\tau}(0)L^j(0)$. Then the expression above becomes

$$L_i(t) \approx L_i(0) + \dot{L}_i(0)t + \tfrac{1}{4}\ddot{h}_{ij}^{\tau\tau}(0)L^j(0)t^2 \tag{553}$$

We can also Taylor expand $h_{ij}^{\tau\tau}(t)$ with respect to time to second order about $t = 0$.

$$h_{ij}^{\tau\tau}(t) \approx h_{ij}^{\tau\tau}(0) + \dot{h}_{ij}^{\tau\tau}(0)t + \tfrac{1}{2}\ddot{h}_{ij}^{\tau\tau}(0)t^2 \tag{554}$$

Solving this for $\frac{1}{2}\ddot{h}_{ij}^{\tau\tau}(0)t^2$ and inserting into (553) gives

$$\boxed{L_i(t) \approx L_i(0) + \dot{L}_i(0)t + \tfrac{1}{2}L^j(0)\left[h_{ij}^{\tau\tau}(t) - h_{ij}^{\tau\tau}(0) - \dot{h}_{ij}^{\tau\tau}(0)t\right]} \tag{555}$$

Lowest order solution to the geodesic deviation equation due to a gravitational wave

First we can confirm that this solution is a valid solution to (540). Since $L^j(0)$, $\dot{L}_i(0)$, $h_{ij}^{\tau\tau}(0)$, and $\dot{h}_{ij}^{\tau\tau}(0)$ are constants, then taking two time-derivatives of $L_i(t)$ gives

$$\ddot{L}_i(t) \approx \tfrac{1}{2} L^j(0)\, \ddot{h}_{ij}^{\tau\tau}(t) \tag{556}$$

Inserting this into (540) gives

$$\tfrac{1}{2}\ddot{h}_{ij}^{\tau\tau}(t)\, L^j(0) = \tfrac{1}{2}\ddot{h}_{ij}^{\tau\tau}(t)\, L^j(t) \tag{557}$$

This implies that $L^j(0) \approx L^j(t)$ which is consistent with solving $L_i(t)$ to lowest order. If we choose initial conditions such that neighboring particles are initially at rest at $t = 0$, then we can set $\dot{L}_i(0) = 0$ in (555). Evaluating L_x and L_y, and using the notation $L_{0,x} = L_x(0)$ and $L_{0,y} = L_y(0)$, gives

$$\begin{aligned} L_x(t) &\approx L_{0,x} + \tfrac{1}{2} L_{0,x} \left[h_{xx}^{\tau\tau}(t) - h_{xx}^{\tau\tau}(0) - \dot{h}_{xx}^{\tau\tau}(0)\, t \right] \\ &\quad + \tfrac{1}{2} L_{0,y} \left[h_{xy}^{\tau\tau}(t) - h_{xy}^{\tau\tau}(0) - \dot{h}_{xy}^{\tau\tau}(0)\, t \right] \end{aligned} \tag{558}$$

and

$$\begin{aligned} L_y(t) &\approx L_{0,y} + \tfrac{1}{2} L_{0,x} \left[h_{yx}^{\tau\tau}(t) - h_{yx}^{\tau\tau}(0) - \dot{h}_{yx}^{\tau\tau}(0)\, t \right] \\ &\quad + \tfrac{1}{2} L_{0,y} \left[h_{yy}^{\tau\tau}(t) - h_{yy}^{\tau\tau}(0) - \dot{h}_{yy}^{\tau\tau}(0)\, t \right] \end{aligned} \tag{559}$$

Then using (543) and (544) in the two expressions above gives

$$L_x(t) \approx L_{0,x} + \tfrac{1}{2}\left[L_{0,x} A_\oplus + L_{0,y} A_\otimes \right] \left[\cos(kz - \omega t) - \cos(kz) - \omega t \sin(kz) \right] \tag{560}$$

and

$$L_y(t) \approx L_{0,y} + \tfrac{1}{2}\left[L_{0,x} A_\otimes - L_{0,y} A_\oplus \right] \left[\cos(kz - \omega t) - \cos(kz) - \omega t \sin(kz) \right] \tag{561}$$

We can combine $L_x(t)$ and $L_y(t)$ into a single vector. We can also use the notation $\vec{L} = L_{0,x}\hat{x} + L_{0,y}\hat{y}$. This gives

$$\boxed{\begin{aligned} \vec{L}(z,t) &\approx \vec{L}_0 + \tfrac{1}{2}\left[\left(L_{0,x} A_\oplus + L_{0,y} A_\otimes \right)\hat{x} + \left(L_{0,x} A_\otimes - L_{0,y} A_\oplus \right)\hat{y} \right] \\ &\quad \cdot \left[\cos(kz - \omega t) - \cos(kz) - \omega t \sin(kz) \right] \end{aligned}} \tag{562}$$

Lowest order solution to the geodesic deviation equation for a gravitational wave propagating in the z-direction

We can also write the expressions for plus-polarization and cross-polarization separately as

$$\vec{L}(z,t) \approx \vec{L}_0 + \tfrac{1}{2}\left(L_{0,x}\hat{x} - L_{0,y}\hat{y} \right) A_\oplus \left[\cos(kz - \omega t) - \cos(kz) - \omega t \sin(kz) \right] \tag{563}$$

and

$$\vec{L}(z,t) \approx \vec{L}_0 + \tfrac{1}{2}\left(L_{0,y}\hat{x} + L_{0,x}\hat{y} \right) A_\otimes \left[\cos(kz - \omega t) - \cos(kz) - \omega t \sin(kz) \right] \tag{564}$$

Notice that for any value of z besides $kz = n\pi$, we find that the last term in both results above will increase indefinitely with time. This is a strange result since it implies that the distance between two neighboring particles would continue to increase linearly without bound, even though the gravitational wave is purely sinusoidal. If we choose to set $z=0$, then we simply have

$$\vec{L}(0,t) \approx \vec{L}_0 + \tfrac{1}{2}(L_{0,x}\hat{x} - L_{0,y}\hat{y})A_{\oplus}[\cos(\omega t) - 1] \tag{565}$$

and

$$\vec{L}(0,t) \approx \vec{L}_0 + \tfrac{1}{2}(L_{0,y}\hat{x} + L_{0,x}\hat{y})A_{\otimes}[\cos(\omega t) - 1] \tag{566}$$

We can check that our results are valid solutions to (540). To do this, we can take a double time-derivative of (560) and (561), and then insert the result into (546) and (547), respectively. The double time-derivative of (560) and (561) are

$$\ddot{L}_x(t) \approx -\frac{1}{2}[L_{0,x}A_{\oplus} + L_{0,y}A_{\otimes}]\,\omega^2\cos(kz - \omega t) \tag{567}$$

and

$$\ddot{L}_y(t) \approx -\tfrac{1}{2}[L_{0,x}A_{\otimes} - L_{0,y}A_{\oplus}]\,\omega^2\cos(kz - \omega t) \tag{568}$$

Inserting these into (546) and (547), respectively, and canceling common terms gives

$$L_{0,x}A_{\oplus} + L_{0,y}A_{\otimes} \approx L_x A_{\oplus} + L_y A_{\otimes} \tag{569}$$

and

$$L_{0,x}A_{\otimes} - L_{0,y}A_{\oplus} \approx L_x A_{\otimes} - L_y A_{\oplus} \tag{570}$$

Again we find that L_i is nearly constant to first order. We can also consider the case of *standing* waves in the z-direction. Then we have

$$h^{\tau\tau}_{xx} = -h^{\tau\tau}_{yy} = A_{\oplus}\cos(kz)\cos(\omega t) \quad \text{and} \quad h^{\tau\tau}_{xy} = h^{\tau\tau}_{yx} = A_{\otimes}\cos(kz)\cos(\omega t) \tag{571}$$

The time derivatives are

$$\dot{h}^{\tau\tau}_{xx} = -\dot{h}^{\tau\tau}_{yy} = -A_{\oplus}\omega\cos(kz)\sin(\omega t) \quad \text{and} \quad \dot{h}^{\tau\tau}_{xy} = \dot{h}^{\tau\tau}_{yx} = -A_{\otimes}\omega\cos(kz)\sin(\omega t) \tag{572}$$

Taking second time derivatives gives

$$\ddot{h}^{\tau\tau}_{xx} = -\ddot{h}^{\tau\tau}_{yy} = -A_{\oplus}\omega^2\cos(kz)\cos(\omega t) \quad \text{and} \quad \ddot{h}^{\tau\tau}_{xy} = \ddot{h}^{\tau\tau}_{yx} = -A_{\otimes}\omega^2\cos(kz)\cos(\omega t) \tag{573}$$

Now returning to (558) and (559), and making use of the expressions above gives

$$L_x(t) \approx L_{0,x} + \tfrac{1}{2}(L_{0,x}A_{\oplus} + L_{0,y}A_{\otimes})\cos(kz)[\cos(\omega t) - 1] \tag{574}$$

and

$$L_y(t) \approx L_{0,y} + \tfrac{1}{2}(L_{0,x}A_{\otimes} + L_{0,y}A_{\oplus})\cos(kz)[\cos(\omega t) - 1] \tag{575}$$

We can combine $L_x(t)$ and $L_y(t)$ into a single vector. We can also use the notation $\vec{L} = L_{0,x}\hat{x} + L_{0,y}\hat{y}$. This gives

$$\boxed{\begin{gathered}\vec{L}(z,t) \approx \vec{L}_0 + \tfrac{1}{2}\left[\left(L_{0,x}A_{\oplus} + L_{0,y}A_{\otimes}\right)\hat{x} + \left(L_{0,x}A_{\otimes} - L_{0,y}A_{\oplus}\right)\hat{y}\right] \\ \cdot\cos(kz)\left[\cos(\omega t) - 1\right] \\ \textit{Lowest order solution to the geodesic deviation equation} \\ \textit{for a gravitational standing wave in the z-direction}\end{gathered}} \tag{576}$$

We can also write the expressions for plus-polarization and cross-polarization separately as

$$\vec{L}(z,t) \approx \vec{L}_0 + \tfrac{1}{2}\left(L_{0,x}\hat{x} - L_{0,y}\hat{y}\right)A_{\oplus}\cos(kz)\left[\cos(\omega t) - 1\right] \tag{577}$$

and

$$\vec{L}(z,t) \quad \approx \quad \vec{L}_0 + \tfrac{1}{2}\left(L_{0,y}\hat{x} + L_{0,x}\hat{y}\right)A_{\otimes}\cos(kz)\left[\cos(\omega t) - 1\right] \tag{578}$$

Notice that we no longer have a term that increases indefinitely with time. This only seems to occur for the case of traveling waves.

5.4 The geodesic deviation of time due to a GR wave

In (2834) of Appendix L we found that the linearized geodesic deviation equation is given by

$$\ddot{L}^{\mu} = -R^{\mu}{}_{\rho\gamma\sigma}L^{\gamma}u^{\rho}u^{\sigma} \tag{579}$$

where $L^{\mu} = \tilde{x}^{\mu} - x^{\mu}$ is the coordinate separation between two particles which follow two neighboring world-lines given by x^{μ} and $\tilde{x}^{\mu}$. Then $\ddot{L}^{\mu}$ gives the relative acceleration between the two particles. In (2473) of Appendix C, we also found that the linearized Riemann tensor (with lowered indices) is given by

$$R_{\beta\rho\gamma\sigma} = \frac{1}{2}\left(\partial_{\gamma}\partial_{\rho}h_{\sigma\beta} - \partial_{\gamma}\partial_{\beta}h_{\rho\sigma} - \partial_{\sigma}\partial_{\rho}h_{\gamma\beta} + \partial_{\sigma}\partial_{\beta}h_{\rho\gamma}\right) \tag{580}$$

In the local proper Lorentz frame of x^{μ}, we have $u^{\mu} = \gamma(c,0)$. Then (579) will only be non-zero for $\rho = \sigma = 0$ which means $R^{\mu}{}_{\rho\gamma\sigma} = R^{\mu}{}_{0\gamma 0}$ and $u^{\rho}u^{\sigma} = u^{0}u^{0} = c^{2}$. Also, if L^{γ} is initially just a spatial separation given by $L^{\gamma} = \left(0, L^{j}\right)$, then (579) will only be non-zero for $\gamma = j$. Lastly, if we are interested in the effect on clocks, then we are looking for $\ddot{L}^{0}$ and therefore we choose $\mu = 0$. Then (579) becomes

$$\ddot{L}^{0} = -R^{0}{}_{0j0}L^{j}c^{2} \tag{581}$$

This means that the Riemann tensor component of interest will be $R^{\mu}{}_{\rho\gamma\sigma} = R^{0}{}_{0j0}$. If we lower indices (to first order in the metric) we have

$$R^{0}{}_{0j0} = \eta^{\mu 0}R_{\mu 0j0} = \eta^{00}R_{00j0} + \eta^{k0}R_{k0j0} = -R_{0j00} \tag{582}$$

Therefore, (581) can be written as

$$\ddot{L}^{0} = R_{00j0}L^{j}c^{2} \tag{583}$$

Using (580) gives

$$R_{00j0} = \frac{1}{2}\left(\partial_{j}\partial_{0}h_{00} - \partial_{j}\partial_{0}h_{00} - \partial_{0}\partial_{0}h_{j0} + \partial_{0}\partial_{0}h_{0j}\right) = 0 \tag{584}$$

This means that two observers separated only by a space-like vector (that is, two observers with synchronized clocks who are separated spatially by L^{j}) will not experience any deviation in their clocks (to first order in the metric).

Next we consider two observers initially separated by only a time-like vector (that is, two observers who are at the same position in space but at different times.) Again we can use the local proper Lorentz frame of x^{μ} so that $u^{\mu} = (c,0)$. As before, this means that (579) will only be non-zero for $\rho = \sigma = 0$ and therefore we have $R^{\mu}{}_{\rho\gamma\sigma} = R^{\mu}{}_{0\gamma 0}$ and $u^{\rho}u^{\sigma} = u^{0}u^{0} = c^{2}$. We will also let L^{γ} be a time-like separation so that $L^{\gamma} = \left(L^{0}, 0\right)$. Then (579) will only be non-zero for $\gamma = 0$. Lastly, if we are interested in the effect on clocks, then we are looking for $\ddot{L}^{0}$ and therefore we choose $\mu = 0$. Then (579) becomes

$$\ddot{L}^{0} = -R^{0}{}_{000}L^{0}c^{2} \tag{585}$$

If we lower indices (to first order in the metric) we have

$$R^{0}{}_{000} = \eta^{\mu 0}R_{\mu 000} = \eta^{00}R_{0000} + \eta^{k0}R_{k000} = -R_{0000} \tag{586}$$

Therefore, (585) can be written as

$$\ddot{L}^{0} = R_{0000}L^{0}c^{2} \tag{587}$$

Using (580) gives

$$R_{0000} = \frac{1}{2}\left(\partial_{0}\partial_{0}h_{00} - \partial_{0}\partial_{0}h_{00} - \partial_{0}\partial_{0}h_{00} + \partial_{0}\partial_{0}h_{00}\right) = 0 \tag{588}$$

This means that two observers separated only by a time-like vector will not experience any deviation in their clocks (to first order in the metric).

5.5 The geodesic deviation equation in an electromagnetic field

Here we apply the procedure used in Appendix L to derive a geodesic deviation equation that includes electromagnetic coupling as well gravitational coupling. Using the Lorentz force in curved space-time (528), we can consider a point particle with worldline $x^{\mu}(\tau)$ following an *effective* geodesic

$$\mathfrak{D}u^{\mu} = \frac{d}{d\tau}u^{\mu} + \Gamma^{\mu}_{\sigma\rho}(x)\,u^{\sigma}u^{\rho} - \overline{q}(x)\,u_{\nu}F^{\mu\nu}(x) = 0 \tag{589}$$

where $\overline{q}(x) = q/m$ is the charge-to-mass ratio of the particle.[61] Here $\mathfrak{D}$ can be thought of as a covariant derivative which incorporates the coupling to the electromagnetic field as well as the curvature of space-time. Notice that $\mathfrak{D}u^{\mu} = 0$ in (589) implies that this is an *effective* geodesic where a particle is "freely falling" in both gravitational and electromagnetic fields, with no other forces acting on it. In that sense, it follows a "curved path" where the "curvature" is determined by both gravitation and electromagnetism. We can also describe a second particle with worldline $\tilde{x}^{\mu}(t)$ following a neighboring *effective* geodesic

$$\mathfrak{D}\tilde{u}^{\mu} = \frac{d}{d\tau}\tilde{u}^{\mu} + \Gamma^{\mu}_{\sigma\rho}(\tilde{x})\,\tilde{u}^{\sigma}\tilde{u}^{\rho} - \overline{q}(\tilde{x})\,\tilde{u}_{\sigma}F^{\mu\sigma}(\tilde{x}) = 0 \tag{590}$$

where $\overline{q}(\tilde{x})$ is the charge-to-mass ratio of the particle on the worldline $\tilde{x}^{\mu}$. The two particles are separated by a coordinate distance $L^{\mu} = \tilde{x}^{\mu} - x^{\mu}$. From this relation we can find the proper acceleration[62] of L^{μ} which is the *relative* acceleration between x^{μ} and $\tilde{x}^{\mu}$. Inserting (589) and (590) into $L''^{\mu} = \tilde{u}'^{\mu} - u'^{\mu}$ gives

$$L''^{\mu} = -\Gamma^{\mu}_{\sigma\rho}(\tilde{x})\,\tilde{u}^{\sigma}\tilde{u}^{\rho} + \overline{q}(\tilde{x})\,\tilde{u}_{\sigma}F^{\mu\sigma}(\tilde{x}) + \Gamma^{\mu}_{\sigma\rho}(x)\,u^{\sigma}u^{\rho} - \overline{q}(x)\,u_{\sigma}F^{\mu\sigma}(x) \tag{591}$$

Since $L^{\mu} = \tilde{x}^{\mu} - x^{\mu}$, then $\tilde{u}^{\mu} = u^{\mu} + L'^{\mu}$. Inserting this above gives

$$\begin{aligned} L''^{\mu} &= -\Gamma^{\mu}_{\sigma\rho}(\tilde{x})\left(u^{\sigma} + L'^{\sigma}\right)\left(u^{\rho} + L'^{\rho}\right) + \Gamma^{\mu}_{\sigma\rho}(x)\,u^{\sigma}u^{\rho} \\ &\quad + \overline{q}(\tilde{x})\left(u_{\sigma} + L'_{\sigma}\right)F^{\mu\sigma}(\tilde{x}) - \overline{q}(x)\,u_{\sigma}F^{\mu\sigma}(x) \end{aligned} \tag{592}$$

Multiplying out terms and staying to first order in L'^{σ} gives

$$\begin{aligned} L''^{\mu} &= -\Gamma^{\mu}_{\sigma\rho}(\tilde{x})\left(u^{\sigma}u^{\rho} + L'^{\sigma}u^{\rho} + u^{\sigma}L'^{\rho}\right) + \Gamma^{\mu}_{\sigma\rho}(x)\,u^{\sigma}u^{\rho} \\ &\quad + \overline{q}(\tilde{x})\left(u_{\sigma} + L'_{\sigma}\right)F^{\mu\sigma}(\tilde{x}) - \overline{q}(x)\,u_{\sigma}F^{\mu\sigma}(x) \end{aligned} \tag{593}$$

Expanding $\Gamma^{\mu}_{\sigma\rho}(\tilde{x})$ to first order about x^{μ} gives

$$\Gamma^{\mu}_{\sigma\rho}(\tilde{x}) \approx \Gamma^{\mu}_{\sigma\rho}(x) + \left(\partial_{\gamma}\Gamma^{\mu}_{\sigma\rho}(x)\right)\left(\tilde{x}^{\gamma} - x^{\gamma}\right) \tag{594}$$

$$\approx \Gamma^{\mu}_{\sigma\rho}(x) + \left(\partial_{\gamma}\Gamma^{\mu}_{\sigma\rho}(x)\right)L^{\gamma} \tag{595}$$

---

[61] The worldline x^{μ} can be pictured as essentially a single space-time trajectory selected out of a continuum of space-time trajectories that make up a continuous medium. In that sense, rather than considering a charged massive particle on the worldline x^{μ}, we may think instead of an infinitesimal volume element of the continuous medium, where the volume element has a charge-to-mass ratio $\overline{q}(x)$. The entirety of the medium could have a non-uniform charge density mass density. However, we assume that $\overline{q}(x)$ is *uniform along* the x^{μ} worldline, and $\overline{q}(\tilde{x})$ is *uniform along* the $\tilde{x}^{\mu}$ worldline. Any non-uniformity only exists when comparing two worldlines, $\overline{q}(x)$ and $\overline{q}(\tilde{x})$, in the bulk of the continuous medium.

[62] In this section we use the notation $x'^{\mu} = dx^{\mu}/d\tau$ as the derivative with respect to *proper* time in order distinguish from $\dot{x}^{\mu} = dx^{\mu}/dt$ which is the derivative with respect to *coordinate* time.

Likewise, expanding $F^{\mu\sigma}(\tilde{x})$ to first order about x^{μ} gives

$$F^{\mu\sigma}(\tilde{x}) \approx F^{\mu\sigma}(x) + \left(\partial_{\gamma} F^{\mu\sigma}(x)\right)(\tilde{x}^{\gamma} - x^{\gamma}) \tag{596}$$

$$\approx F^{\mu\sigma}(x) + \left(\partial_{\gamma} F^{\mu\sigma}(x)\right) L^{\gamma} \tag{597}$$

Lastly, expanding the charge-to-mass distribution $\bar{q}(\tilde{x})$ to first order about x^{μ} gives

$$\bar{q}(\tilde{x}) \approx \bar{q}(x) + \left(\partial_{\gamma}\bar{q}(x)\right)(\tilde{x}^{\gamma} - x^{\gamma}) \tag{598}$$

$$\approx \bar{q}(x) + \left(\partial_{\gamma}\bar{q}(x)\right) L^{\gamma} \tag{599}$$

We can insert (595), (597), and (599) into (593). We also drop the function notation since all quantities are now functions of x.

$$\begin{aligned} L''^{\mu} = & -\left[\Gamma^{\mu}_{\sigma\rho} + \left(\partial_{\gamma}\Gamma^{\mu}_{\sigma\rho}\right) L^{\gamma}\right]\left(u^{\sigma}u^{\rho} + L'^{\sigma}u^{\rho} + u^{\sigma}L'^{\rho}\right) + \Gamma^{\mu}_{\sigma\rho}u^{\sigma}u^{\rho} \\ & + \left[\bar{q} + \left(\partial_{\rho}\bar{q}\right) L^{\rho}\right]\left(u_{\sigma} + L'_{\sigma}\right)\left[F^{\mu\sigma} + \left(\partial_{\gamma}F^{\mu\sigma}\right) L^{\gamma}\right] - \bar{q}u_{\sigma}F^{\mu\sigma} \end{aligned} \tag{600}$$

Canceling common terms and eliminating the higher order terms containing $L^{\gamma}L'^{\rho}$ gives

$$\begin{aligned} L''^{\mu} = & -\Gamma^{\mu}_{\sigma\rho}L'^{\sigma}u^{\rho} - \Gamma^{\mu}_{\sigma\rho}u^{\sigma}L'^{\rho} - L^{\gamma}\left(\partial_{\gamma}\Gamma^{\mu}_{\sigma\rho}\right)u^{\sigma}u^{\rho} \\ & + \bar{q}L'_{\sigma}F^{\mu\sigma} + \bar{q}u_{\sigma}L^{\gamma}\left(\partial_{\gamma}F^{\mu\sigma}\right) + L^{\rho}\,\partial_{\rho}\bar{q}u_{\sigma}F^{\mu\sigma} \end{aligned} \tag{601}$$

Since $\Gamma^{\mu}_{\sigma\rho}$ is symmetric in σ and ρ, then we can combine the first two terms to express the geodesic deviation equation in an electromagnetic field as

$$\boxed{\begin{aligned} & L''^{\mu} = -2\Gamma^{\mu}_{\sigma\rho}L'^{\sigma}u^{\rho} - \left(\partial_{\gamma}\Gamma^{\mu}_{\sigma\rho}\right)u^{\sigma}u^{\rho}L^{\gamma} \\ & \quad + \bar{q}\,L'_{\sigma}F^{\mu\sigma} + \bar{q}\,u_{\sigma}L^{\gamma}\left(\partial_{\gamma}F^{\mu\sigma}\right) + \left(\partial_{\rho}\bar{q}\,\right)L^{\rho}u_{\sigma}F^{\mu\sigma} \end{aligned} \quad \begin{array}{c}\textit{Coordinate dependent geodesic}\\ \textit{deviation equation with}\\ \textit{electromagnetic fields}\end{array}} \tag{602}$$

If we wish to express the geodesic deviation equation in terms of the Riemann tensor, we can start with the covariant derivative acting on L^{μ}.

$$\mathfrak{D}L^{\mu} = \frac{d}{d\tau}L^{\mu} + \Gamma^{\mu}_{\gamma\rho}L^{\gamma}u^{\rho} - \bar{q}\,L_{\rho}F^{\mu\rho} \tag{603}$$

$$= L'^{\mu} + \Gamma^{\mu}_{\gamma\rho}L^{\gamma}u^{\rho} - \bar{q}\,g_{\gamma\rho}L^{\gamma}F^{\mu\rho} \tag{604}$$

Then applying the covariant derivative to L^{μ} twice gives

$$\begin{aligned}\mathfrak{D}^2 L^{\mu} &= \frac{d}{d\tau}\left(L'^{\mu}+\Gamma^{\mu}_{\gamma\rho}L^{\gamma}u^{\rho}-\bar{q}\,L_{\rho}F^{\mu\rho}\right)\\ &\quad +\Gamma^{\mu}_{\alpha\beta}\left(L'^{\alpha}+\Gamma^{\alpha}_{\gamma\rho}L^{\gamma}u^{\rho}-\bar{q}\,L_{\rho}F^{\alpha\rho}\right)u^{\beta}\\ &\quad -\bar{q}\,g_{\alpha\sigma}\left(L'^{\alpha}+\Gamma^{\alpha}_{\gamma\rho}L^{\gamma}u^{\rho}-\bar{q}\,g_{\gamma\rho}L^{\gamma}F^{\alpha\rho}\right)F^{\mu\sigma}\end{aligned} \tag{605}$$

Using the product rule to evaluate the derivative and distributing gives

$$\begin{aligned}\mathfrak{D}^2 L^{\mu} &= L''^{\mu}+\Gamma'^{\mu}_{\gamma\rho}L^{\gamma}u^{\rho}+\Gamma^{\mu}_{\gamma\rho}L'^{\gamma}u^{\rho}+\Gamma^{\mu}_{\gamma\rho}L^{\gamma}u'^{\rho}-\bar{q}\,L'_{\rho}F^{\mu\rho}-\bar{q}\,L_{\rho}F'^{\mu\rho}-\bar{q}\,'\,L_{\rho}F^{\mu\rho}\\ &\quad +\Gamma^{\mu}_{\alpha\beta}L'^{\alpha}u^{\beta}+\Gamma^{\mu}_{\alpha\beta}\Gamma^{\alpha}_{\gamma\rho}L^{\gamma}u^{\rho}u^{\beta}-\bar{q}\,\Gamma^{\mu}_{\alpha\beta}L_{\rho}F^{\alpha\rho}u^{\beta}\\ &\quad -\bar{q}\,g_{\alpha\sigma}L'^{\alpha}F^{\mu\sigma}-\bar{q}\,g_{\alpha\sigma}\Gamma^{\alpha}_{\gamma\rho}L^{\gamma}u^{\rho}F^{\mu\sigma}+\bar{q}\,^{2}g_{\alpha\sigma}g_{\gamma\rho}L^{\gamma}F^{\alpha\rho}F^{\mu\sigma}\end{aligned} \tag{606}$$

Using the chain rule, we can write $\Gamma'^{\mu}_{\gamma\rho}\left(x^{\mu}\right)$ as

$$\Gamma'^{\mu}_{\gamma\rho}\left(x^{\mu}\right) = \frac{d}{d\tau}\Gamma^{\mu}_{\gamma\rho}\left(x^{\mu}\right)=\frac{dx^{\sigma}}{d\tau}\frac{d}{dx^{\sigma}}\Gamma^{\mu}_{\gamma\rho}=u^{\sigma}\left(\partial_{\sigma}\Gamma^{\mu}_{\gamma\rho}\right) \tag{607}$$

Likewise, using the chain rule, we can write $F'^{\mu\rho}\left(x^{\mu}\right)$ as

$$F'^{\mu\rho}\left(x^{\mu}\right) = \frac{d}{d\tau}F^{\mu\rho}\left(x^{\mu}\right)=\frac{dx^{\sigma}}{d\tau}\frac{d}{dx^{\sigma}}F^{\mu\rho}=u^{\sigma}\left(\partial_{\sigma}F^{\mu\rho}\right) \tag{608}$$

Finally, using the chain rule, we can write $\bar{q}\,'\left(x^{\mu}\right)$ as

$$\bar{q}\,'\left(x^{\mu}\right) = \frac{d}{d\tau}\bar{q}\left(x^{\mu}\right)=\frac{dx^{\sigma}}{d\tau}\frac{d}{dx^{\sigma}}\bar{q}=u^{\sigma}\left(\partial_{\sigma}\bar{q}\,\right) \tag{609}$$

We can now substitute (607), (608), and (609) into (606). We can also substitute (602) in the first term and (589) in the fourth term to obtain

$$\begin{aligned}\mathfrak{D}^2 L^{\mu} &= -2\Gamma^{\mu}_{\sigma\rho}L'^{\sigma}u^{\rho}-\left(\partial_{\gamma}\Gamma^{\mu}_{\sigma\rho}\right)u^{\sigma}u^{\rho}L^{\gamma}+\bar{q}\,L'_{\sigma}F^{\mu\sigma}+\bar{q}\,u_{\sigma}L^{\gamma}\left(\partial_{\gamma}F^{\mu\sigma}\right)+\left(\partial_{\rho}\bar{q}\,\right)L^{\rho}u_{\sigma}F^{\mu\sigma}\\ &\quad +u^{\sigma}\left(\partial_{\sigma}\Gamma^{\mu}_{\gamma\rho}\right)L^{\gamma}u^{\rho}+\Gamma^{\mu}_{\gamma\rho}L'^{\gamma}u^{\rho}+\Gamma^{\mu}_{\gamma\rho}L^{\gamma}\left(-\Gamma^{\rho}_{\sigma\nu}u^{\sigma}u^{\nu}+\bar{q}\,u_{\nu}F^{\rho\nu}\right)\\ &\quad -\bar{q}\,L'_{\rho}F^{\mu\rho}-\bar{q}\,L_{\rho}u^{\sigma}\left(\partial_{\sigma}F^{\mu\rho}\right)-u^{\sigma}\left(\partial_{\sigma}\bar{q}\,\right)L_{\rho}F^{\mu\rho}\\ &\quad +\Gamma^{\mu}_{\alpha\beta}L'^{\alpha}u^{\beta}+\Gamma^{\mu}_{\alpha\beta}\Gamma^{\alpha}_{\gamma\rho}L^{\gamma}u^{\rho}u^{\beta}-\bar{q}\,\Gamma^{\mu}_{\alpha\beta}L_{\rho}F^{\alpha\rho}u^{\beta}\\ &\quad -\bar{q}\,g_{\alpha\sigma}L'^{\alpha}F^{\mu\sigma}-\bar{q}\,g_{\alpha\sigma}\Gamma^{\alpha}_{\gamma\rho}L^{\gamma}u^{\rho}F^{\mu\sigma}+\bar{q}\,^{2}g_{\alpha\sigma}g_{\gamma\rho}L^{\gamma}F^{\alpha\rho}F^{\mu\sigma}\end{aligned} \tag{610}$$

The three terms involving $\Gamma^{\mu}_{\sigma\rho} L'^{\sigma} u^{\rho}$ cancel and two terms involving $\bar{q}\, L'_{\sigma} F^{\mu\sigma}$ cancel to give

$$\begin{aligned}
\mathfrak{D}^2 L^{\mu} \quad = \quad & -\left(\partial_{\gamma}\Gamma^{\mu}_{\sigma\rho}\right) u^{\sigma} u^{\rho} L^{\gamma} u^{\nu} + \bar{q}\, u_{\sigma}\left(\partial_{\gamma} F^{\mu\sigma}\right) L^{\gamma} + \left(\partial_{\rho}\bar{q}\right) L^{\rho} u_{\sigma} F^{\mu\sigma} + u^{\sigma}\left(\partial_{\sigma}\Gamma^{\mu}_{\gamma\rho}\right) L^{\gamma} u^{\rho} \\
& -\Gamma^{\mu}_{\gamma\rho}\Gamma^{\rho}_{\sigma\nu} L^{\gamma} u^{\sigma} L^{\gamma} + \bar{q}\,\Gamma^{\mu}_{\gamma\rho} L^{\gamma} u_{\nu} F^{\rho\nu} - \bar{q}\, L_{\rho} u^{\sigma}\left(\partial_{\sigma} F^{\mu\rho}\right) - u^{\sigma}\left(\partial_{\sigma}\bar{q}\right) L_{\rho} F^{\mu\rho} \\
& +\Gamma^{\mu}_{\alpha\beta}\Gamma^{\alpha}_{\gamma\rho} L^{\gamma} u^{\rho} u^{\beta} - \bar{q}\,\Gamma^{\mu}_{\alpha\beta} L_{\rho} F^{\alpha\rho} u^{\beta} \\
& -\bar{q}\, L'_{\sigma} F^{\mu\sigma} - \bar{q}\, g_{\alpha\sigma}\Gamma^{\alpha}_{\gamma\rho} L^{\gamma} u^{\rho} F^{\mu\sigma} + \bar{q}^{\,2} g_{\alpha\sigma} L_{\rho} F^{\alpha\rho} F^{\mu\sigma}
\end{aligned} \tag{611}$$

We can rearrange terms so that only terms with $\left(\partial_{\gamma}\Gamma^{\mu}_{\sigma\rho}\right)$ or $\Gamma^{\mu}_{\gamma\rho}\Gamma^{\rho}_{\sigma\nu}$ appear on the first line. Then we can change repeated indices to make the indices of u^{μ} match in each term of the first line. We can also make other repeated indices similar and collect common terms involving $F^{\mu\sigma}$.

$$\begin{aligned}
\mathfrak{D}^2 L^{\mu} \quad = \quad & -\left(\partial_{\gamma}\Gamma^{\mu}_{\sigma\rho}\right) u^{\sigma} u^{\rho} L^{\gamma} + \left(\partial_{\sigma}\Gamma^{\mu}_{\gamma\rho}\right) u^{\sigma} u^{\rho} L^{\gamma} - \Gamma^{\mu}_{\gamma\alpha}\Gamma^{\alpha}_{\sigma\rho} u^{\sigma} u^{\rho} L^{\gamma} + \Gamma^{\mu}_{\alpha\sigma}\Gamma^{\alpha}_{\gamma\rho} u^{\sigma} u^{\rho} L^{\gamma} \\
& +\bar{q}\,\Gamma^{\mu}_{\gamma\rho} L^{\gamma} u_{\sigma} F^{\rho\sigma} - \bar{q}\,\Gamma^{\mu}_{\sigma\rho} L_{\gamma} F^{\sigma\gamma} u^{\rho} - \bar{q}\, g_{\alpha\sigma}\Gamma^{\alpha}_{\gamma\rho} L^{\gamma} u^{\rho} F^{\mu\sigma} \\
& +\bar{q}\, L^{\gamma} u_{\sigma}\left(\partial_{\gamma} F^{\mu\sigma}\right) - \bar{q}\, L_{\gamma} u^{\sigma}\left(\partial_{\sigma} F^{\mu\gamma}\right) + \left(\partial_{\gamma}\bar{q}\right) L^{\gamma} u_{\sigma} F^{\mu\sigma} - \left(\partial_{\sigma}\bar{q}\right) u^{\sigma} L_{\gamma} F^{\mu\gamma} \\
& -\bar{q}\, L'_{\gamma} F^{\mu\gamma} + \bar{q}^{\,2} g_{\rho\sigma} L_{\gamma} F^{\rho\gamma} F^{\mu\sigma}
\end{aligned} \tag{612}$$

We can factor out $L^{\gamma} u^{\rho} u^{\sigma}$ from the top line and use the Riemann tensor given in (2368) as

$$R^{\mu}{}_{\rho\gamma\sigma} \quad = \quad \partial_{\gamma}\Gamma^{\mu}_{\sigma\rho} - \partial_{\sigma}\Gamma^{\mu}_{\gamma\rho} + \Gamma^{\mu}_{\gamma\alpha}\Gamma^{\alpha}_{\rho\sigma} - \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\alpha}_{\rho\gamma} \tag{613}$$

We can also rearrange and group terms in (612) to obtain

$$\begin{aligned}
\mathfrak{D}^2 L^{\mu} \quad = \quad & -R^{\mu}{}_{\rho\gamma\sigma} L^{\gamma} u^{\rho} u^{\sigma} - \bar{q}\, L'_{\gamma} F^{\mu\gamma} + \bar{q}^{\,2} g_{\rho\sigma} L_{\gamma} F^{\rho\gamma} F^{\mu\sigma} \\
& +\bar{q}\,\Gamma^{\mu}_{\gamma\rho} L^{\gamma} u_{\sigma} F^{\rho\sigma} - \bar{q}\,\Gamma^{\mu}_{\sigma\rho} L_{\gamma} F^{\sigma\gamma} u^{\rho} - \bar{q}\, g_{\alpha\sigma}\Gamma^{\alpha}_{\gamma\rho} L^{\gamma} u^{\rho} F^{\mu\sigma} \\
& +\bar{q}\, L^{\gamma} u_{\sigma}\left(\partial_{\gamma} F^{\mu\sigma}\right) - \bar{q}\, L_{\gamma} u^{\sigma}\left(\partial_{\sigma} F^{\mu\gamma}\right) + \left(\partial_{\gamma}\bar{q}\right) L^{\gamma} u_{\sigma} F^{\mu\sigma} - \left(\partial_{\sigma}\bar{q}\right) u^{\sigma} L_{\gamma} F^{\mu\gamma}
\end{aligned} \tag{614}$$

All of the purely gravitational effects are now encoded in the Riemann tensor in the first term. The other two terms in the first line describe the first order coupling to the electromagnetic field. The second line describes the coupling to the electromagnetic field due to the curved space-time manifold. The last line describes the higher order coupling to variations in the electromagnetic field and variations in the charge-to-mass distribution. We can also factor out L^γ by introducing the metric into various terms. This gives

$$
\begin{aligned}
\mathfrak{D}^2 L^\mu &= -R^\mu{}_{\rho\gamma\sigma} L^\gamma u^\rho u^\sigma - \bar{q}\, L'_\gamma F^{\mu\gamma} \\
&\quad + L^\gamma \Big[\bar{q}^{\,2} F_{\sigma\gamma} F^{\mu\sigma} + \bar{q}\, \Gamma^\mu_{\gamma\rho} u_\sigma F^{\rho\sigma} - \bar{q}\, \Gamma^\mu_{\sigma\rho} g^{\alpha\sigma} F_{\alpha\gamma} u^\rho - \bar{q}\, g_{\alpha\sigma} \Gamma^\alpha_{\gamma\rho} u^\rho F^{\mu\sigma} \\
&\quad + \bar{q}\, u_\sigma \left(\partial_\gamma F^{\mu\sigma}\right) - \bar{q}\, u^\sigma g^{\mu\alpha} \left(\partial_\sigma F_{\alpha\gamma}\right) + \left(\partial_\gamma \bar{q}\right) u_\sigma F^{\mu\sigma} - \left(\partial_\sigma \bar{q}\right) u^\sigma g^{\mu\alpha} F_{\alpha\gamma} \Big]
\end{aligned}
\tag{615}
$$

We can define the entire quantity in the bracket as an “electromagnetic curvature tensor.”

$$
\begin{aligned}
G^\mu_\gamma &\equiv \bar{q}^{\,2} F_{\sigma\gamma} F^{\mu\sigma} + \bar{q}\, \Gamma^\mu_{\gamma\rho} u_\sigma F^{\rho\sigma} - \bar{q}\, \Gamma^\mu_{\sigma\rho} g^{\alpha\sigma} F_{\alpha\gamma} u^\rho - \bar{q}\, g_{\alpha\sigma} \Gamma^\alpha_{\gamma\rho} u^\rho F^{\mu\sigma} \\
&\quad + \bar{q}\, u_\sigma \left(\partial_\gamma F^{\mu\sigma}\right) - \bar{q}\, u^\sigma g^{\mu\rho} \left(\partial_\sigma F_{\rho\gamma}\right) + \left(\partial_\gamma \bar{q}\right) u_\sigma F^{\mu\sigma} - \left(\partial_\sigma \bar{q}\right) u^\sigma g^{\mu\rho} F_{\rho\gamma}
\end{aligned}
\tag{616}
$$

For the special case of a uniform electromagnetic field and a uniform charge-to-mass distribution, we have $\partial_\gamma F^{\mu\sigma} = 0$ and $\partial_\gamma \bar{q} = 0$. Then the entire last line vanishes in the expression above and we have

$$
G^\mu{}_{\gamma\,(\text{uniform})} = \bar{q} \left(\bar{q}\, F_{\sigma\gamma} F^{\mu\sigma} + \Gamma^\mu_{\gamma\rho} u_\sigma F^{\rho\sigma} - \Gamma^\mu_{\sigma\rho} g^{\alpha\sigma} F_{\alpha\gamma} u^\rho - g_{\alpha\sigma} \Gamma^\alpha_{\gamma\rho} u^\rho F^{\mu\sigma} \right) \tag{617}
$$

For the general case of non-uniform fields and non-uniform charge-to-mass distributions, we can use (616) to write (615) as simply

$$
\boxed{\mathfrak{D}^2 L^\mu = -R^\mu{}_{\rho\gamma\sigma} L^\gamma u^\rho u^\sigma - \bar{q}\, L'_\gamma F^{\mu\gamma} + L^\gamma G^\mu_\gamma \qquad \textit{Coordinate-free geodesic deviation equation with electromagnetic fields}} \tag{618}
$$

We can choose to consider the local Lorentz frame of x^μ so that $\Gamma^\mu_{\sigma\rho}\left(x^\mu\right) = 0$. Then using (606) with $\Gamma^\mu_{\sigma\rho} = 0$ gives

$$
\mathfrak{D}^2 L^\mu = \left(L''^\mu - \bar{q}\, L'_\rho F^{\mu\rho} - \bar{q}\, L_\rho F'^{\mu\rho} \right) - \left(\bar{q}\, L'_\sigma F^{\mu\sigma} - \bar{q}^{\,2}\, L^\gamma F_{\sigma\gamma} F^{\mu\sigma} \right) \tag{619}
$$

We can combine terms involving $\bar{q}\, L'_\sigma F^{\mu\sigma}$. Also, if we are in the *proper* (rest) frame of x^μ, then $L''^\mu = \ddot{L}^\mu$. This gives

$$
\mathfrak{D}^2 L^\mu = \ddot{L}^\mu - \bar{q}\, L_\rho \dot{F}^{\mu\rho} - 2\bar{q}\, \dot{L}_\sigma F^{\mu\sigma} + \bar{q}^{\,2}\, L^\gamma F_{\sigma\gamma} F^{\mu\sigma} \tag{620}
$$

We can also observe that setting $\Gamma^\mu_{\sigma\rho}\left(x^\mu\right) = 0$ in (616) gives

$$
\begin{aligned}
G^\mu{}_{\gamma\,(\text{Lorentz frame})} &= \bar{q}^{\,2} F_{\sigma\gamma} F^{\mu\sigma} + \bar{q}\, u_\sigma \left(\partial_\gamma F^{\mu\sigma}\right) - \bar{q}\, u^\sigma g^{\mu\rho} \left(\partial_\sigma F_{\rho\gamma}\right) \\
&\quad + \left(\partial_\gamma \bar{q}\right) u_\sigma F^{\mu\sigma} - \left(\partial_\sigma \bar{q}\right) u^\sigma g^{\mu\rho} F_{\rho\gamma}
\end{aligned}
\tag{621}
$$

Then inserting (620) and (621) into (618) and solving for $\ddot{L}^\mu$ gives

$$
\ddot{L}^\mu = -R^\mu{}_{\rho\gamma\sigma} L^\gamma u^\rho u^\sigma + \bar{q}\, L_\rho \dot{F}^{\mu\rho} + 2\bar{q}\, \dot{L}_\sigma F^{\mu\sigma} - \bar{q}^{\,2}\, L^\gamma F_{\sigma\gamma} F^{\mu\sigma} + L^\gamma G^\mu{}_{\gamma\,(\text{Lorentz frame})} \tag{622}
$$

We can insert (621) and cancel terms involving $\bar{q}^{\,2}\, L^{\gamma} F_{\sigma\gamma} F^{\mu\sigma}$ to obtain

$$\begin{aligned} \ddot{L}^{\mu} &= -R^{\mu}{}_{\rho\gamma\sigma} L^{\gamma} u^{\rho} u^{\sigma} + \bar{q}\, L_{\rho} \dot{F}^{\mu\rho} + 2\bar{q}\, \dot{L}_{\sigma} F^{\mu\sigma} \\ &\quad + L^{\gamma} \left[\bar{q}\, u_{\sigma} \left(\partial_{\gamma} F^{\mu\sigma} \right) - \bar{q}\, u^{\sigma} g^{\mu\rho} \left(\partial_{\sigma} F_{\rho\gamma} \right) + \left(\partial_{\gamma} \bar{q} \right) u_{\sigma} F^{\mu\sigma} - \left(\partial_{\sigma} \bar{q} \right) u^{\sigma} g^{\mu\rho} F_{\rho\gamma} \right] \end{aligned} \tag{623}$$

We can also factor out u_{σ} from the bracket by raising the index of some of the derivatives.

$$\begin{aligned} \ddot{L}^{\mu} &= -R^{\mu}{}_{\rho\gamma\sigma} L^{\gamma} u^{\rho} u^{\sigma} + \bar{q}\, L_{\rho} \dot{F}^{\mu\rho} + 2\bar{q}\, \dot{L}_{\sigma} F^{\mu\sigma} \\ &\quad + L^{\gamma} u_{\sigma} \left[\bar{q} \left(\partial_{\gamma} F^{\mu\sigma} \right) - \bar{q}\, g^{\mu\rho} \left(\partial^{\sigma} F_{\rho\gamma} \right) + \left(\partial_{\gamma} \bar{q} \right) F^{\mu\sigma} - \left(\partial^{\sigma} \bar{q} \right) g^{\mu\rho} F_{\rho\gamma} \right] \end{aligned} \tag{624}$$

We can lower the index of L^{γ} and raise γ in the bracket. We can also define an "electromagnetic coupling tensor" in the Lorentz frame as

$$P^{\gamma\mu\sigma} \equiv \bar{q} \left(\partial^{\gamma} F^{\mu\sigma} \right) - \bar{q} \left(\partial^{\sigma} F^{\mu\gamma} \right) + \left(\partial^{\gamma} \bar{q} \right) F^{\mu\sigma} - \left(\partial^{\sigma} \bar{q} \right) F^{\mu\gamma} \tag{625}$$

Using this in (624) and changing some of the repeated indices gives

$$\boxed{\ddot{L}^{\mu} = -R^{\mu}{}_{\rho\gamma\sigma} L^{\gamma} u^{\rho} u^{\sigma} + \bar{q}\, L_{\gamma} \dot{F}^{\mu\gamma} + 2\bar{q}\, \dot{L}_{\gamma} F^{\mu\gamma} + L_{\gamma} u_{\sigma} P^{\gamma\mu\sigma}} \tag{626}$$

Geodesic deviation equation with electromagnetic fields in the proper Lorentz frame

6 The "four-velocity invariant" Hamiltonian for relativistic electron pairs

6.1 The vanishing covariant Hamiltonian

The Lagrangian for relativistic electron-pairs coupled to an electromagnetic field, A_μ, in curved space-time can be written as[63]

$$L_1 = -mc\sqrt{-g_{\mu\nu}u^\mu u^\nu} + eA_\mu u^\mu \tag{627}$$

It is effectively the same Lagrangian as used by DeWitt in equation (1) of [42]. It is derived in Appendix M and is referred to as the "four-velocity invariant Lagrangian" since it can be seen to be associated with the invariance of the four-velocity:

$$\sqrt{-g_{\mu\nu}u^\mu u^\nu} = c \tag{628}$$

The Hamiltonian that follows from the Lagrangian in (627) can be shown to vanish identically. To obtain the Hamiltonian, we use a Legendre transformation in terms of four-vectors written as $H_1 = p_\mu u^\mu - L_1$, where the canonical momentum is given by $p_\sigma = \dfrac{\partial L_1}{\partial u^\sigma}$. Using the Lagrangian from (627) we have

$$p_\sigma = \frac{\partial}{\partial u^\sigma}\left(mc\sqrt{-g_{\mu\nu}u^\mu u^\nu} + eA_\mu \dot{x}^\mu\right) \tag{629}$$

the metric and four-potential are not functions of velocity so the derivatives pass through them. Also, the components of the velocity are independent so $\dfrac{\partial u^\mu}{\partial u^\sigma} = \delta^\mu_\sigma$.

$$p_\sigma = \frac{mc}{2\sqrt{-g_{\mu\nu}u^\mu u^\nu}}\left(-g_{\mu\nu}\delta^\mu_\sigma u^\nu - g_{\mu\nu}u^\mu\delta^\nu_\sigma\right) + eA_\mu\delta^\mu_\sigma \tag{630}$$

Using (628) we can write this as

$$p_\sigma = \frac{m}{2}\left(-g_{\sigma\nu}u^\nu - g_{\mu\sigma}u^\mu\right) + eA_\sigma \tag{631}$$

$$= \frac{m}{2}\left(-u_\sigma - u_\sigma\right) + eA_\sigma \tag{632}$$

$$= -mu_\sigma + eA_\sigma \tag{633}$$

Applying an inverse metric $g^{\sigma\nu}$ to each term gives

$$g^{\sigma\nu}p_\sigma = -mg^{\sigma\nu}u_\sigma + eg^{\sigma\nu}A_\sigma \tag{634}$$

$$p^\nu = -mu^\nu + eA^\nu \tag{635}$$

Solving for u^ν gives

$$u^\nu = -\frac{1}{m}\left(p^\nu - eA^\nu\right) \tag{636}$$

[63] This Lagrangian is also found in Jackson [40] (eq. 12.31).

Substituting u^ν from (636) into the Lagrangian from (627) gives

$$L_1 = mc\sqrt{-g_{\mu\nu}\frac{1}{m^2}\left(p^\mu - eA^\mu\right)\left(p^\nu - eA^\nu\right)} + eA_\mu\left[-\frac{1}{m}\left(p^\mu - eA^\mu\right)\right] \tag{637}$$

$$= c\sqrt{-\left(p^\mu - eA^\mu\right)\left(p_\mu - eA_\mu\right)} - \frac{e}{m}A_\mu\left(p^\mu - eA^\mu\right) \tag{638}$$

Notice that this is just the Lagrangian in (627) but with $P_\mu \Longrightarrow P_\mu - eA_\mu$ which is standard for minimal coupling. The Hamiltonian can now be found from

$$H_1 = p_\mu u^\mu - L_1 \tag{639}$$

Putting the four-velocity (636) and Lagrangian (627) into the Hamiltonian (639) gives

$$H_1 = p_\mu\left[-\frac{1}{m}\left(p^\mu - eA^\mu\right)\right] - \left[c\sqrt{-\left(p^\mu - eA^\mu\right)\left(p_\mu - eA_\mu\right)} - \frac{e}{m}A_\mu\left(p^\mu - eA^\mu\right)\right] \tag{640}$$

$$= -\frac{1}{m}\left(p_\mu p^\mu - ep_\mu A^\mu\right) + \frac{1}{m}\left(eA_\mu p^\mu - e^2A_\mu A^\mu\right) - c\sqrt{-\left(p^\mu - eA^\mu\right)\left(p_\mu - eA_\mu\right)} \tag{641}$$

$$= -\frac{1}{m}\left(p_\mu p^\mu - ep_\mu A^\mu - eA_\mu p^\mu + e^2A_\mu A^\mu\right) - c\sqrt{-\left(p^\mu - eA^\mu\right)\left(p_\mu - eA_\mu\right)} \tag{642}$$

$$\boxed{H_1 = -\frac{1}{m}\left(p_\mu - eA_\mu\right)\left(p^\mu - eA^\mu\right) - c\sqrt{-\left(p^\mu - eA^\mu\right)\left(p_\mu - eA_\mu\right)}} \tag{643}$$

This result matches that of Jackson in equation (12.35) of [40] except for a negative in the front and inside the root, both owing to the fact that we are using a metric with signature $diag\left(-1,1,1,1\right)$ rather than Jackson's $diag\left(1,-1,-1,-1\right)$. We can show that his Hamiltonian vanishes identically by substituting the canonical momentum from (635) into the Hamiltonian in (643).

$$\begin{aligned} H_1 = & -\frac{1}{m}\left(-mu_\mu + eA_\mu - eA_\mu\right)\left(-mu^\mu + eA^\mu - eA^\mu\right) \\ & -c\sqrt{-\left(-mu^\mu + eA^\mu - eA^\mu\right)\left(-mu_\mu + eA_\mu - eA_\mu\right)} \end{aligned} \tag{644}$$

$$= -mu_\mu u^\mu - mc\sqrt{-u^\mu u_\mu} \tag{645}$$

From (628) we know that $u_\mu u^\mu = g_{\mu\nu}u^\mu u^\nu = -c^2$ so we have

$$H_1 = -m\left(-c^2\right) - mc\sqrt{c^2} \tag{646}$$

$$H_1 = 0 \tag{647}$$

Hence we find the Hamiltonian vanishes identically and cannot represent the energy of the system. It is therefore questionable if this Hamiltonian is the appropriate quantity to promote to an operator for describing a quantum mechanical system in curved space-time.

6.2 The space+time Hamiltonian

We begin with the Lagrangian given by (2864) as

$$L = -mc\sqrt{-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}} + eA_{\mu}\dot{x}^{\mu} \tag{648}$$

This is precisely[64] the Lagrangian used by DeWitt in equation (1) of [42]. The "dot derivative" is a derivative with respect to $t = x^0/c$ which matches the convention of DeWitt.[65] By expanding the summations in the Lagrangian we have

$$L = -mc\left(-g_{00}\dot{x}^0\dot{x}^0 - g_{0j}\dot{x}^0\dot{x}^j - g_{i0}\dot{x}^i\dot{x}^0 - g_{ij}\dot{x}^i\dot{x}^j\right)^{1/2} + eA_0\dot{x}^0 + eA_i\dot{x}^i \tag{649}$$

Using $\dot{x}^{\mu} = \left(c, v^i\right)$ and recognizing that the two middle terms inside the root are identical gives

$$L = -mc\left(-g_{00}c^2 - 2g_{0j}cv^j - g_{ij}v^iv^j\right)^{1/2} + ceA_0 + eA_iv^i \tag{650}$$

We can find the Hamiltonian using a Legendre transformation in terms of 3-vectors.

$$H = v^k p_k - L \tag{651}$$

where the canonical momentum can be found from $p_k = \dfrac{\partial \tilde{L}}{\partial v^k}$. Using the Lagrangian in (650) and recognizing that $\dfrac{\partial v^i}{\partial v^k} = \delta^i_k$ gives

$$\begin{aligned} p_k = & -\frac{mc}{2}\left(-c^2g_{00} - 2cg_{0j}v^j - g_{ij}v^iv^j\right)^{-1/2} \\ & \cdot\left(-2cg_{0j}\delta^j_k - g_{ij}\delta^i_kv^j - g_{ij}v^i\delta^j_k\right) + eA_i\delta^i_k \end{aligned} \tag{652}$$

$$p_k = \frac{mc}{2}\frac{2cg_{0k} + g_{kj}v^j + g_{ik}v^i}{\left(-c^2g_{00} - 2cg_{0j}v^j - g_{ij}v^iv^j\right)^{1/2}} + eA_k \tag{653}$$

$$p_k - eA_k = \frac{mc}{2}\frac{2cg_{0k} + g_{kj}v^j + g_{ik}v^i}{c\left(-g_{00} - 2g_{0j}v^j/c - g_{ij}v^iv^j/c^2\right)^{1/2}} \tag{654}$$

Once again we can observe that the last two terms in the numerator are essentially the same so we can combine them and factor out a 2. Then writing the kinetic momentum as $\pi_k = p_k - eA_k$ gives

$$\pi_k = m\left(-g_{00} - 2g_{0j}\frac{v^j}{c} - g_{ij}\frac{v^iv^j}{c^2}\right)^{-1/2}\left(cg_{0k} + g_{kj}v^j\right) \tag{655}$$

---

[64]There is only a difference in units since DeWitt sets $c = 1$, however, this calculaton will retain the constant c.

[65]Note that because $x^{\mu} = \left(ct, x^i\right)$ then $\dfrac{d}{dt}x^{\mu} = \dfrac{d}{dt}\left(ct, x^i\right) = \left(c, v^i\right)$. So we define $\dot{x}^{\mu} = \left(c, v^i\right)$. This is not to be confused with the four-velocity, $u^{\mu} = \dfrac{dx^{\mu}}{d\tau}$. By the chain rule, u^{μ} and $\dot{x}^{\mu}$ are related: $u^{\mu} = \dfrac{dx^{\mu}}{d\tau} = \dfrac{dt}{d\tau}\dfrac{dx^{\mu}}{dt} = \dfrac{dt}{d\tau}\dot{x}^{\mu}$.

In (2738) of Appendix J, we defined a "Lorentz factor" in GR as[66]

$$\gamma \equiv \left(-g_{00} - \frac{2}{c} g_{0j} v^j - \frac{1}{c^2} g_{ij} v^i v^j\right)^{-1/2} \tag{656}$$

Therefore we can write the kinetic momentum in (655) more compactly as

$$\pi_k = \gamma m \left(c g_{0k} + g_{ik} v^i\right) \tag{657}$$

In order to find the Hamiltonian in (651), we must find the velocity in terms of the canonical momentum. Note that g_{ik} *cannot* lower the index of v^i because the indices are only *spatial* indices, not *space-time* indices[67]. Also note that v^i is summed with g_{ik} in (657) therefore we cannot simply solve for v^i algebraically[68]. First we must construct the inverse of g_{ik} which we will refer to as the "spatial inverse metric." From (2875) in Appendix N, we have

$$\tilde{g}^{ik} = g^{ik} - g^{0i} g^{0k} / g^{00} \tag{658}$$

where $\tilde{g}^{jk} g_{ik} = \delta^j{}_i$. Applying $\tilde{g}^{jk}$ to both sides of (657) and solving for the velocity gives

$$\tilde{g}^{jk} \pi_k = \gamma m \left(c \tilde{g}^{jk} g_{0k} + \tilde{g}^{jk} g_{ik} v^i\right) \tag{659}$$

$$\frac{\tilde{g}^{jk} \pi_k}{\gamma m} = c \tilde{g}^{jk} g_{0k} + \delta^j{}_i v^i \tag{660}$$

$$v^j = \frac{\tilde{g}^{jk} \pi_k}{\gamma m} - c \tilde{g}^{jk} g_{0k} \tag{661}$$

We can also write the Lagrangian in (650) using γ as defined in (656)

$$L = -\frac{mc^2}{\gamma} + ceA_0 + eA_i v^i \tag{662}$$

Now we can find the Hamiltonian, $H = p_j v^j - L$, using the Lagrangian in (662).

$$H = p_j v^j + \frac{mc^2}{\gamma} - ceA_0 - eA_j v^j \tag{663}$$

$$= \left(p_j - eA_j\right) v^j + \frac{mc^2}{\gamma} - ceA_0 \tag{664}$$

---

[66] In the case of a flat Minkowski space-time, we have $\eta_{00} = -1$, $\eta_{0i} = 0$, and $\eta_{ij} = \delta_{ij}$. Then the Lorentz factor reduces to the familiar form in Special Relativity: $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$.

[67] To ellaborate on this point, note that $v_\nu = g_{\mu\nu} v^\mu = g_{\nu 0} v^0 + g_{\nu j} v^j$ where $v^0 = c$. If we choose $\nu = i$ then this gives $v_i = c g_{0i} + g_{ij} v^j$. Therefore, we see that $v_i \neq g_{ij} v^j$. Rather, there is an additional term, $c g_{0i}$.

[68] Because v^i is summed with g_{ik}, then what we have in the equation is $v^i g_{ik} = v^1 g_{1k} + v^2 g_{2k} + v^3 g_{3k}$. Therefore, it is obvious that we cannot solve for $\vec{v} = \left(v^1, v^2, v^3\right)$ while it is summed with g_{ik}.

Substituting π_j for $(p_j - eA_j)$ and using the velocity in (661) gives

$$H = \pi_j \left(\frac{\tilde{g}^{jk}\pi_k}{\gamma m} - c\tilde{g}^{jk}g_{0k} \right) + \frac{mc^2}{\gamma} - ceA_0 \tag{665}$$

$$= \frac{\tilde{g}^{jk}\pi_j\pi_k}{\gamma m} - c\pi_j\tilde{g}^{jk}g_{0k} + \frac{mc^2}{\gamma} - ceA_0 \tag{666}$$

Factoring $1/\gamma m$ from the first and third terms gives

$$H = \frac{1}{\gamma m}\left(m^2c^2 + \tilde{g}^{jk}\pi_j\pi_k \right) - c\pi_j\tilde{g}^{jk}g_{0k} - ceA_0 \tag{667}$$

We must now find an expression for γ in terms of the kinetic momentum, π^j. Squaring and taking the reciprocal of (656) and substituting (661) for the velocity gives

$$\gamma^{-2} = -g_{00} - \frac{2}{c}g_{0j}\left(\frac{\tilde{g}^{jk}\pi_k}{\gamma m} - c\tilde{g}^{jk}g_{0k} \right) - \frac{1}{c^2}g_{ij}\left(\frac{\tilde{g}^{ik}\pi_k}{\gamma m} - c\tilde{g}^{ik}g_{0k} \right)\left(\frac{\tilde{g}^{jl}\pi_l}{\gamma m} - c\tilde{g}^{jl}g_{0l} \right) \tag{668}$$

Multiplying out these terms gives

$$\begin{aligned}\gamma^{-2} = {} & -g_{00} - \frac{2g_{0j}\tilde{g}^{jk}\pi_k}{\gamma mc} + 2g_{0j}\tilde{g}^{jk}g_{0k} \\ & - \frac{1}{c^2}g_{ij}\left(\frac{\tilde{g}^{ik}\tilde{g}^{jl}\pi_l\pi_k}{\gamma^2m^2} - \frac{c\tilde{g}^{jl}g_{0l}\tilde{g}^{ik}\pi_k}{\gamma m} - \frac{c\tilde{g}^{ik}g_{0k}\tilde{g}^{jl}\pi_l}{\gamma m} + c^2\tilde{g}^{ik}g_{0k}\tilde{g}^{jl}g_{0l} \right)\end{aligned} \tag{669}$$

The middle two terms in the second parentheses can be combined. Also, by distributing g_{ij} through the parentheses, we can use $g_{ij}\tilde{g}^{il} = \delta^l{}_j$ to simplify each term. We will also multiply the entire expression by γ^2 to obtain

$$1 = -g_{00}\gamma^2 - \gamma\frac{2\tilde{g}^{jk}g_{0j}\pi_k}{mc} + 2\gamma^2\tilde{g}^{jk}g_{0j}g_{0k} - \frac{\tilde{g}^{ik}\pi_j\pi_k}{m^2c^2} + \gamma\frac{2g_{0i}\tilde{g}^{ik}\pi_k}{mc} - \gamma^2 g_{0j}\tilde{g}^{jl}g_{0l} \tag{670}$$

The second and fifth terms on the right side cancel. Also, the third and sixth terms can be combined.

$$1 = -g_{00}\gamma^2 + \gamma^2\tilde{g}^{jk}g_{0j}g_{0k} - \frac{\tilde{g}^{ik}\pi_j\pi_k}{m^2c^2} \tag{671}$$

Now we solve for γ^2.

$$1 + \frac{\tilde{g}^{ik}\pi_j\pi_k}{m^2c^2} = \gamma^2\left(\tilde{g}^{jk}g_{0j}g_{0k} - g_{00} \right) \tag{672}$$

$$\gamma^2 = \frac{m^2c^2 + \tilde{g}^{ik}\pi_j\pi_k}{m^2c^2\left(\tilde{g}^{jk}g_{0j}g_{0k} - g_{00} \right)} \tag{673}$$

$$\gamma = \frac{1}{mc}\sqrt{\frac{m^2c^2 + \tilde{g}^{ik}\pi_j\pi_k}{\tilde{g}^{jk}g_{0j}g_{0k} - g_{00}}} \tag{674}$$

Substituting this result into the Hamiltonian found in (667) gives

$$H = c\sqrt{\frac{\tilde{g}^{jk}g_{0j}g_{0k}-g_{00}}{m^2c^2+\tilde{g}^{ik}\pi_j\pi_k}}\left(m^2c^2+\tilde{g}^{jk}\pi_j\pi_k\right)-c\pi_j\tilde{g}^{jk}g_{0k}-ceA_0 \tag{675}$$

$$H = c\left(\tilde{g}^{jk}g_{0j}g_{0k}-g_{00}\right)^{1/2}\left(m^2c^2+\tilde{g}^{jk}\pi_j\pi_k\right)^{1/2}-c\tilde{g}^{jk}g_{0k}\pi_j-ceA_0 \tag{676}$$

Lastly, we can use $\pi_j = p_j - eA_j$ to express the Hamiltonian in terms of the canonical momentum and obtain

$$\boxed{\begin{aligned} H = c\left(\tilde{g}^{jk}g_{0j}g_{0k}-g_{00}\right)^{1/2}\left[m^2c^2+\tilde{g}^{jk}\left(p_j-eA_j\right)\left(p_k-eA_k\right)\right]^{1/2} \\ -c\tilde{g}^{jk}g_{0k}\left(p_j-eA_j\right)-ceA_0 \end{aligned}} \tag{677}$$

This matches DeWitt's Hamiltonian in equation (3) of [42], except that DeWitt has g^{jk} rather than $\tilde{g}^{jk}$. It is shown in (2877) of Appendix N, that $\tilde{g}^{jk} \approx g^{jk}$ is only true to first order in the metric. We also develop two relations in Appendix N given by (2880) and (2886) as

$$\tilde{g}^{ik}g_{0k} = -\frac{g^{0i}}{g^{00}} \qquad \text{and} \qquad \tilde{g}^{jk}g_{0j}g_{0k}-g_{00} = -\frac{1}{g^{00}} \tag{678}$$

Inserting these into (676) gives

$$\boxed{H = c\left[\frac{m^2c^2+\tilde{g}^{jk}\pi_j\pi_k}{-g^{00}}\right]^{1/2}+c\frac{g^{0i}}{g^{00}}\pi_j-ceA_0} \tag{679}$$

This result matches that of Cognola, et al. [45] and Bertschinger [48].

6.3 The weak field, low velocity Hamiltonian to second order

We now write the metric as a perturbation of flat space-time: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. In the weak field approximation we require $\left|h_{\mu\nu}\right| << 1$. The Hamiltonian in (676) can be expanded as

$$\begin{aligned} H &= c\left[\tilde{g}^{jk}\left(\eta_{0j}+h_{0j}\right)\left(\eta_{0k}+h_{0k}\right)-\left(\eta_{00}+h_{00}\right)\right]^{1/2}\left[m^2c^2+\tilde{g}^{jk}\pi_k\pi_j\right]^{1/2} \\ &\quad -c\tilde{g}^{jk}\left(\eta_{0k}+h_{0k}\right)\pi_j - ceA_0 \end{aligned} \tag{680}$$

Since $\eta^{jk} = \delta^k_j$, $\eta_{0j} = 0$ and $\eta_{00} = -1$, then we have

$$H = c\left[\tilde{g}^{jk}h_{0j}h_{0k}+1-h_{00}\right]^{1/2}\left[m^2c^2+\tilde{g}^{jk}\pi_k\pi_j\right]^{1/2} - c\tilde{g}^{jk}h_{0k}\pi_j - ceA_0 \tag{681}$$

To expand this Hamiltonian to second order in the metric, we will need to express $\tilde{g}^{jk}$ (the "spatial inverse metric") in terms of $h^{\mu\nu}$ to second order as

$$\tilde{g}^{jk} \approx \delta^j_k - h^{jk} + h^{ij}h^k_i \tag{682}$$

Substituting this into the Hamiltonian in (681) gives

$$\begin{aligned} H &= c\left[\left(\delta^j_k - h^{jk} + h^{ij}h^k_i\right)h_{0j}h_{0k}+1-h_{00}\right]^{1/2} \\ &\quad \cdot\left[m^2c^2+\left(\delta^j_k - h^{jk} + h^{ij}h^k_i\right)\pi_k\pi_j\right]^{1/2} \\ &\quad -c\left(\delta^j_k - h^{jk} + h^{ij}h^k_i\right)h_{0k}\pi_j - ceA_0 \end{aligned} \tag{683}$$

Distributing terms and omitting results that are third order in $h_{\mu\nu}$ gives

$$\begin{aligned} H &= c\left[(h_{0k})^2+1-h_{00}\right]^{1/2}\left[m^2c^2+\pi_k^2+\left(-h^{jk}+h^{ij}h^k_i\right)\pi_k\pi_j\right]^{1/2} \\ &\quad -ch_{0k}\pi^k + ch^{jk}h_{0k}\pi_j - ceA_0 \end{aligned} \tag{684}$$

In the first square root above, the weak field limit requires $\left|(h_{0k})^2 - h_{00}\right| << 1$. Since expanding a square root to second order gives $(1+x)^{1/2} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2$ when $|x| << 1$, then we can write

$$\left[1+(h_{0k})^2-h_{00}\right]^{1/2} \approx 1+\tfrac{1}{2}\left[(h_{0k})^2-h_{00}\right]-\frac{1}{8}\left[(h_{0k})^2-h_{00}\right]^2 \tag{685}$$

$$\approx 1+\tfrac{1}{2}(h_{0k})^2-\tfrac{1}{2}h_{00}-\frac{1}{8}(h_{00})^2+\mathscr{O}\left(\varepsilon^3\right) \tag{686}$$

Eliminating third order terms and using this expression to replace the first square root in the Hamiltonian gives

$$\begin{aligned} H &\approx c\left\{1+\tfrac{1}{2}\left(h_{0k}\right)^2-\tfrac{1}{2}h_{00}-\frac{1}{8}\left(h_{00}\right)^2\right\} \\ &\quad \cdot\left[m^2c^2+\pi_k^2+\left(h^{ij}h_i^k-h^{jk}\right)\pi_k\pi_j\right]^{1/2} \\ &\quad -ch_{0k}\pi^k+ch^{jk}h_{0k}\pi_j-ceA_0 \end{aligned} \tag{687}$$

Next, factoring out m^2c^2 from the second root makes it become

$$mc\left[1+\frac{\pi_k^2}{m^2c^2}+\frac{h^{ij}h_i^k-h^{jk}}{m^2c^2}\pi_k\pi_j\right]^{1/2} \tag{688}$$

In the low velocity limit, we have

$$\left|\frac{\pi_k^2}{m^2c^2}+\frac{h^{ij}h_i^k-h^{jk}}{m^2c^2}\pi_k\pi_j\right| \quad < \quad <1 \tag{689}$$

If we only keep terms out to $(v/c)^2$ for the speed of the electrons, then expanding the root to first order gives

$$mc\left(1+\frac{\pi_k^2}{2m^2c^2}+\frac{h^{ij}h_i^k-h^{jk}}{2m^2c^2}\pi_k\pi_j\right) \tag{690}$$

This result now replaces the second square root in the Hamiltonian so that the Hamiltonian becomes

$$\begin{aligned} \underset{\substack{(weak\ field, \\ low\ velocity)}}{H} &\approx c\left\{1+\tfrac{1}{2}\left(h_{0k}\right)^2-\tfrac{1}{2}h_{00}-\frac{1}{8}\left(h_{00}\right)^2\right\} \\ &\quad \cdot mc\left(1+\frac{\pi_k^2}{2m^2c^2}+\frac{h^{ij}h_i^k-h^{jk}}{2m^2c^2}\pi_k\pi_j\right) \\ &\quad -ch_{0k}\pi^k+ch^{jk}h_{0k}\pi_j-ceA_0 \end{aligned} \tag{691}$$

Multiplying out all terms and omitting any terms that are third order or higher in $h^{\mu\nu}$ gives

$$\begin{aligned} \underset{(2nd\text{-}order)}{H} &\approx mc^2+\frac{\pi_k^2}{2m}-ch_{0k}\pi^k+\tfrac{1}{2}mc^2\left(h_{0k}\right)^2-\tfrac{1}{2}mc^2h_{00}-ceA_0 \\ &\quad -\tfrac{1}{8}mc^2\left(h_{00}\right)^2-\frac{h_{00}\pi_k^2}{4m}-\frac{h^{jk}\pi_k\pi_j}{2m}-\frac{\left(h_{00}\right)^2\pi_k^2}{16m} \\ &\quad +ch^{jk}h_{0k}\pi_j+\frac{\left(h_{0j}\right)^2\pi_k^2}{4m}+\frac{h^{ij}h_i^k\pi_k\pi_j}{2m}+\frac{h_{00}h^{jk}\pi_k\pi_j}{4m} \end{aligned} \tag{692}$$

Writing the second, third, and fourth terms each as a perfect square gives

$$\boxed{\begin{aligned} \underset{(2nd\text{-}order)}{H} \approx mc^2 + \frac{1}{2m}\left(\pi_k - mch_{0k}\right)^2 - \tfrac{1}{2}mc^2 h_{00} - ceA_0 \\ -\tfrac{1}{8}mc^2\left(h_{00}\right)^2 - \frac{h_{00}\pi_k^2}{4m} - \frac{h^{jk}\pi_k\pi_j}{2m} - \frac{\left(h_{00}\right)^2\pi_k^2}{16m} + ch^{jk}h_{0k}\pi_j \\ + \frac{\left(h_{0j}\right)^2\pi_k^2}{4m} + \frac{h^{ij}h_{ik}\pi^k\pi_j}{2m} + \frac{h_{00}h^{jk}\pi_k\pi_j}{4m} \end{aligned}} \tag{693}$$

This is the weak field, low velocity Hamiltonian that is second order in $h^{\mu\nu}$, second order in (v/c), and also second order in *products* of $h^{\mu\nu}$ and (v/c). It can be seen that the top line is DeWitt's "first-order" Hamiltonian. In reality, we have shown here that one must go to *second* order to obtain the term with $\frac{1}{2}mc^2\left(h_{0k}\right)^2$ which is required to write the perfect square.

We may consider further approximations that would remove more terms while preserving the terms in the perfect square in the first line. We cannot employ a stricter low velocity approximation since this would mean only keeping terms that are *first* order in (v/c) and hence removing the kinetic energy term. This would break the perfect square in the first line. We also cannot employ a stricter weak field approximation since keeping terms that are only first order in $h^{\mu\nu}$ would mean removing $\left(h_{0k}\right)^2$. Again, this would break the perfect square in the first line. Therefore, the only other approximations we could make that would preserve the conditions for the perfect square would involve removing *products* of $(h^{\mu\nu})$ and (v/c).

However, since there is no physical relationship between the value of (v/c) (the speed of the electrons) and the value of $h^{\mu\nu}$ (the strength of the gravitational field), then we must impose an additional condition which relates these values in order to justify omitting more terms. For example, if we consider that $h^{\mu\nu}$ and (v/c) are of the same order, then terms like $(h^{\mu\nu})^2(v/c)$ or $h^{\mu\nu}(v/c)^2$ would be considered third order and could therefore be omitted.

To approach this process more formally, we know in general that for approximation purposes we can consider that

$$h_{0i} \sim \left(\frac{v_s}{c}\right)h_{00} \qquad \text{and} \qquad h_{ij} \sim \left(\frac{v_s}{c}\right)^2 h_{00} \tag{694}$$

where v_s is the speed of the gravitational sources (not the speed of the electrons which we are just calling v). These relations can be motivated by considering the harmonic gauge: $\partial^\nu h_{\mu\nu} = 0$. Summing over ν gives

$$\partial^0 h_{\mu 0} + \partial^i h_{\mu i} = 0 \tag{695}$$

For $\mu = 0$ we have

$$\frac{1}{c}\dot{h}_{00} \quad = \quad \partial^i h_{0i} \tag{696}$$

We can consider a metric perturbation given by $h_{\mu\nu} = A_{\mu\nu}e^{\left(\vec{k}\cdot\vec{x}-\omega t\right)}$ where $A_{\mu\nu}$ is a constant amplitude and $k = \omega/v_s$ with v_s being the speed of the sources. Then the expression above becomes

$$-\frac{\omega}{c}h_{00} = kh_{0i} \qquad \Longrightarrow \qquad h_{0i} \sim h_{00}\left(\frac{v_s}{c}\right) \tag{697}$$

This matches the first relation in (694). Next, using $\mu = i$ in (695) gives

$$\frac{1}{c}\dot{h}_{i0} = \partial^i h_{ij} \tag{698}$$

Again using $h_{\mu\nu} = A_{\mu\nu} e^{\left(\vec{k}\cdot\vec{x}-\omega t\right)}$ in the expression above gives

$$-\frac{\omega}{c} h_{i0} = k h_{ij} \qquad \Longrightarrow \qquad h_{ij} \sim h_{0i}\left(\frac{v_s}{c}\right) \sim h_{00}\left(\frac{v_s}{c}\right)^2 \tag{699}$$

This matches the second relation in (694). Therefore, we find that the relations in (694) can be obtained by simply examining the harmonic gauge with $h_{\mu\nu}$ expressed as a wave solution.[69] Since the harmonic gauge is perfectly applicable in non-linear GR, then we find that the relations in (437) are valid in the higher order Hamiltonian we have in (693).

Now using the relations in (694), we can observe that the squared parentheses in (693) implies that we are keeping terms of order

$$\frac{\pi^j \pi_j}{2m} \sim mv^2, \qquad c h_{0k} \pi^k \sim v_s h_{00} m v, \qquad \tfrac{1}{2} mc^2 \left(h_{0k}\right)^2 \sim m v_s^2 \left(h_{00}\right)^2 \tag{700}$$

This implies that the following quantities are comparable

$$mv^2 \sim m v_s v h_{00} \sim m v_s^2 h_{00}^2 \qquad \Longrightarrow \qquad v \sim v_s h_{00} \tag{701}$$

Clearly the first term in the second line of the Hamiltonian in (693) must remain since it is of order

$$mc^2 \left(h_{00}\right)^2 \gg m v_s^2 h_{00}^2 \tag{702}$$

However, substituting $v \sim v_s h_{00}$ into the second term in the second line of the Hamiltonian shows that it is of order

$$\frac{h_{00} \pi_k^2}{4m} \sim mv^2 h_{00} \qquad \Longrightarrow \qquad m v_s^2 h_{00}^3 \tag{703}$$

This is a higher order term that can be neglected. Likewise, continuing this process with the remaining terms in the Hamiltonian will show that all the rest can be neglected as well. Then the Hamiltonian would become

$$\boxed{\underset{(2nd\text{-}order)}{H} \approx mc^2 + \frac{1}{2m}\left(\pi_k + mch_{0k}\right)^2 - \tfrac{1}{2} mc^2 h_{00} - ceA_0 - \frac{1}{8} mc^2 \left(h_{00}\right)^2} \tag{704}$$

This is the Hamiltonian to second order in $h^{\mu\nu}$ and (v/c) but *first* order in any *products* of $h^{\mu\nu}$ and (v/c). Notice that this result still matches DeWitt's result except for an additional term involving $(h_{00})^2$ which DeWitt neglected. This term must be included for consistency as was shown above.

[69] This approach is in contrast to the one used in Section 26 based on the field equation for linearized GR in the trace-reversed harmonic gauge, $\partial^\nu \bar{h}_{\mu\nu} = -2\kappa T_{\mu\nu}$. Since we are not dealing with the field equations here, then this approach is not appropriate. Also, since we are working with a Hamiltonian that is in terms of $h_{\mu\nu}$, not $\bar{h}_{\mu\nu}$, and because it contains terms that are higher than first order in $h_{\mu\nu}$, then using the approach from Section 26 would not be suitable. Rather, the approach here with the harmonic gauge, $\partial^\nu h_{\mu\nu} = 0$, and a wave solution, $h_{\mu\nu} = A_{\mu\nu} e^{\left(\vec{k}\cdot\vec{x}-\omega t\right)}$, is ideal.

6.4 The weak field, low velocity Hamiltonian to first order

If we are only interested in a *minimal* (post-Newtonian) coupling between the electron-pair and the gravitational field, then we can see that we only need $ch_{0k}\pi^k$ which is the lowest order coupling term. However, to maintain consistency in our approximation we must continue to check for other comparable terms before eliminating them. Note that in general, $h_{0k} \sim (v_s/c)\,h_{00}$ where v_s is the velocity of the gravitational source. Then using $\tilde{\pi} = m\vec{v}_e - e\vec{A}$, we see that $ch_{0k}\pi^k$ is of order

$$ch_{0k}\pi^k \quad \sim \quad v_s h_{00} m v_e, \quad v_s h_{00} e A \tag{705}$$

where v_e is the velocity of the electron-pair. Since $h_{00} << 1$, then keeping this term naturally requires keeping the first and fourth terms of (693) as well. This is obviously expected since these terms are the rest mass energy and Newtonian gravitational coupling energy, respectively. The kinetic energy of an electron-pair in the absence of a gravitational field (but still coupled to an electromagnetic field) is given by the fifth term in (693). It is of order

$$\frac{\pi^j \pi_j}{2m} \quad \sim \quad m v_e^2, \quad e v_e A, \quad \frac{e^2 A^2}{m} \tag{706}$$

The first term of (706) is comparable to the first term of (705) when

$$v_e \sim v_s h_{00} \tag{707}$$

This means that the (post-Newtonian) gravitational coupling energy would be comparable to the kinetic energy of the electron-pair.[70] Typically, this will not be the case but rather $v_e >> v_s h_{00}$ since $h_{00} << 1$. In other words, even when $v_s = v_e$, then still the gravitational coupling energy will be weaker than the kinetic energy by a factor of $\sim h_{00}$.

The third term of (706) is comparable to the second term of (705) if $e^2A^2/m \sim v_s h_{00} e A$. This means we would have[71]

$$A \sim \frac{m v_s h_{00}}{e} \tag{708}$$

Since $h_{00} << 1$ and the charge-to-mass ratio of an electron-pair is very small, then the vector potential can be very weak and still have a coupling energy comparable to gravitation. The only other term in (693) that is of lowest order is the second term which is of order

$$\tfrac{1}{2} mc^2 (h_{0k})^2 \sim m v_s^2 (h_{00})^2 \tag{709}$$

For (709) to be comparable to the first term of (705), again we find $v_e \sim v_s h_{00}$. This means that the second-order gravitational coupling with the mass in (709) will be weaker by a factor of $\sim h_{00}$ compared to the first-order coupling with the momentum in (705). Likewise, for (709) to be comparable to the second term of (705), again we find $m v_s h_{00}/e$. Hence we see that the term in (709) is comparable to the other terms discussed above and must be preserved in the Hamiltonian. All other terms involving $h_{\mu\nu}$ and π_j in (693) are of higher order and therefore the two conditions given by (707) and (708) will not be adequate to preserve them. Therefore, we neglect the following terms:

$$\begin{gathered} \tfrac{1}{2} mc^2 h^{jk} h_{0j} h_{0k}, \qquad \frac{h_{jk}\pi^j\pi^k}{2m}, \qquad \frac{h_{k0}h_{0k}\pi^j\pi_j}{4m}, \qquad \frac{(h_{0k})^2 h_{jk}\pi^j\pi^k}{2m}, \\ \frac{h^{jk}h_{jk}h_{0j}h_{0k}\pi^j\pi^k}{4m}, \qquad \frac{h_{00}\pi^j\pi_j}{4m}, \qquad \frac{h_{00}h_{jk}\pi^j\pi^k}{4m}, \qquad ch_{jk}h^{0j}\pi^k \end{gathered} \tag{710}$$

[70]The same relation emerges from comparing the *second* term of (706) with the *second* term of (705).

[71]The same relation emerges from comparing the *second* term of (706) with the *first* term of (705).

So the Hamiltonian in (693) for first-order coupling becomes

$$\underset{(first\ order)}{\tilde{H}_1} \quad = \quad mc^2 + \tfrac{1}{2}mc^2\left(h_{0k}\right)^2 - \tfrac{1}{2}mc^2 h_{00} + \frac{\pi^j \pi_j}{2m} - ch_{0k}\pi^k - ceA_0 \tag{711}$$

Since k and j are both repeated indices, then we can just use a single index i. Grouping together the second, fourth and fifth terms gives

$$\underset{(first\ order)}{\tilde{H}_1} \quad = \quad mc^2 + \frac{1}{2m}\left(\pi^i\pi_i - 2mch_{0i}\pi^i + m^2c^2\left(h_{0i}\right)^2\right) - \tfrac{1}{2}mc^2 h_{00} - ceA_0 \tag{712}$$

Writing the terms in the parenthesis as a perfect square gives

$$\underset{(first\ order)}{\tilde{H}_1} \quad = \quad mc^2 + \frac{1}{2m}\left(\pi_i - mch_{0i}\right)^2 - \tfrac{1}{2}mc^2 h_{00} - ceA_0 \tag{713}$$

Substituting $\pi_i = p_i - eA_i$ gives

$$\boxed{\underset{(first\ order)}{\tilde{H}_1} = mc^2 + \frac{1}{2m}\left(p_i - eA_i - mch_{0i}\right)^2 - \tfrac{1}{2}mc^2 h_{00} - ceA_0} \tag{714}$$

Therefore, we find that the canonical momentum under *minimal* (post-Newtonian) coupling for a non-relativistic charged particle in electromagnetic and weak gravitational fields is

$$\vec{p}_{can} \rightarrow \vec{p}_{can} - q\vec{A} - m\vec{h} \tag{715}$$

where $\vec{h} = c\left(h_{01,}h_{02}, h_{03}\right)$ is the gravitational vector potential or "gravito-vector potential." Note that (714) is precisely the Hamiltonian that DeWitt obtained in his equation (3). However, he states "*after removal of the rest mass*" and writes the Hamiltonian (using $c = 1$) as

$$H = \frac{1}{2m}\left(\vec{p} - \vec{B}\right)^2 + V \tag{716}$$

where

$$V \quad = \quad -eA_0 - \tfrac{1}{2}mh_{00}, \qquad \vec{B} = e\vec{A} + m\vec{h}_0, \qquad \vec{h} = \left(h_{01,}h_{02,}h_{03}\right) \tag{717}$$

The coupling rule in (715) gives a prescription for expressing the canonical momentum, $\vec{p}$, in the presence of a magnetic vector potential, $\vec{A}$, and a *gravito*-vector potential, $\vec{h}$. It is used by DeWitt [42] to argue that there should be a *gravitational* Meissner effect directly analogous to the magnetic Meissner effect for superconductors. Hence the DeWitt minimal coupling rule would provide a means by which to consider the coupling of gravity to quantum matter. In the last two sections we have shown the full derivation beginning with the Lagrangian used by DeWitt in (648). However, we also show in Section 44 that this coupling rule can be derived via the gravito-electromagnetic framework which is valid as a weak field, low velocity approximation for gravity.

7 The “four-momentum invariant” Hamiltonian for relativistic electron pairs

7.1 The non-vanishing covariant Hamiltonian

It was previously shown that the "four-velocity invariant" Lagrangian, L_1, in (627) leads to a Hamiltonian, H_1, that vanishes identically and therefore does not properly represent energy. Here we will show that the "four-momentum invariant Hamiltonian," H_2, does *not* identically vanish. In (2857) of Appendix M, we show that the "four-*momentum* invariant Lagrangian" is

$$L_2 = \frac{1}{2m} g_{\mu\nu} p^\mu p^\nu + eA_\mu u^\mu \tag{718}$$

It is apparent that this Lagrangian is associated with the invariance of the four-momentum:

$$g_{\mu\nu} p^\mu p^\nu = -m^2 c^2 \tag{719}$$

Using $p^\mu = mu^\mu$ we can write this as

$$L_2 = \frac{m}{2} g_{\mu\nu} u^\mu u^\nu + eA_\mu u^\mu \tag{720}$$

To obtain a Hamiltonian, we use a Legendre transformation in terms of four-vectors given by $H_2 = p_\mu u^\mu - L_2$ where the canonical momentum is $p_\sigma = \dfrac{\partial L_2}{\partial u^\sigma}$. Using the Lagrangian from (720) gives

$$p_\sigma = \frac{\partial}{\partial u^\sigma}\left(\frac{m}{2} g_{\mu\nu} u^\mu u^\nu + eA_\mu u^\mu\right) \tag{721}$$

The metric and four-potential are not functions of velocity so the derivatives pass through them. Also, the components of the velocity are independent so $\dfrac{\partial u^\mu}{\partial u^\sigma} = \delta^\mu_\sigma$. Then we have

$$p_\sigma = \frac{m}{2}\left(g_{\mu\nu}\delta^\mu_\sigma u^\nu + g_{\mu\nu} u^\mu \delta^\nu_\sigma\right) + eA_\mu \delta^\mu_\sigma \tag{722}$$

$$= \frac{m}{2}\left(g_{\sigma\nu} u^\nu + g_{\mu\sigma} u^\mu\right) + eA_\sigma \tag{723}$$

$$= \frac{m}{2}\left(u_\sigma + u_\sigma\right) + eA_\sigma \tag{724}$$

$$= mu_\sigma + eA_\sigma \tag{725}$$

Applying an inverse metric $g^{\sigma\nu}$ to each side gives

$$g^{\sigma\nu} p_\sigma = mg^{\sigma\nu} u_\sigma + eg^{\sigma\nu} A_\sigma \tag{726}$$

$$p^\nu = mu^\nu + eA^\nu \tag{727}$$

Solving for u^ν gives

$$u^\nu = \frac{1}{m}\left(p^\nu - eA^\nu\right) \tag{728}$$

Substituting u^ν from (728) into the Lagrangian from (720) gives

$$L_2 \quad = \quad \frac{1}{2m} g_{\mu\nu}\left(p^\mu - eA^\mu\right)\left(p^\nu - eA^\nu\right) + eA_\mu\left(p^\mu - eA^\mu\right) \tag{729}$$

Notice that this is just the Lagrangian in (718) but with $p_\mu \Longrightarrow p_\mu - eA_\mu$ which is standard for minimal coupling. Thus introducing an electromagnetic four-potential can be summarized as

$$L_2 \quad \Longrightarrow \quad L_2 + eA_\mu u^\mu \qquad \text{and} \qquad p^\mu \Longrightarrow p^\mu - eA^\mu \tag{730}$$

The Hamiltonian can now be found from a Legendre transformation using

$$H_2 = p_\mu u^\mu - L_2 \tag{731}$$

Putting the four-velocity (728) and Lagrangian (720) into the Hamiltonian (731) gives

$$H_2 \quad = \quad p_\mu\left[\frac{1}{m}\left(p^\mu - eA^\mu\right)\right] - \left[\frac{1}{2m} g_{\mu\nu}\left(p^\mu - eA^\mu\right)\left(p^\nu - eA^\nu\right) + \frac{e}{m}A_\mu\left(p^\mu - eA^\mu\right)\right] \tag{732}$$

$$= \quad \frac{1}{m}p_\mu p^\mu - \frac{e}{m}p_\mu A^\mu - \left[\frac{1}{2m}\left(p_\nu - eA_\nu\right)\left(p^\nu - eA^\nu\right) + \frac{e}{m}A_\mu p^\mu - \frac{e^2}{m}A_\mu A^\mu\right] \tag{733}$$

$$\begin{aligned} = \quad & \frac{1}{m}p_\mu p^\mu - \frac{e}{m}p_\mu A^\mu - \frac{1}{2m}p_\nu p^\nu + \frac{e}{2m}A_\nu p^\nu \\ & + \frac{e}{2m}p_\nu A^\nu - \frac{e^2}{2m}A_\nu A^\nu - \frac{e}{m}A_\mu p^\mu + \frac{e^2}{m}A_\mu A^\mu \end{aligned} \tag{734}$$

$$= \quad \frac{1}{2m}p_\mu p^\mu - \frac{e}{m}A_\mu p^\mu + \frac{e^2}{2m}A_\mu A^\mu \tag{735}$$

Here we find a term involving the free particle energy $\left(p_\mu p^\mu/2m\right)$, followed by a coupling of the particle with the electromagnetic four-potential $\left(eA_\mu p^\mu/m\right)$, and finally a term showing an effective diamagnetism $\left(e^2 A_\mu A^\mu/2m\right)$. Since the terms in (735) make up a perfect square, then we can write the Hamiltonian as

$$H_2 \quad = \quad \frac{1}{2m}\left(p^\mu - eA^\mu\right)^2 \tag{736}$$

Hence we find that the Hamiltonian does *not* vanish but rather takes the same form as the *non-relativistic* Hamiltonian for a charged particle in an electromagnetic field. Notice that if $A^\mu = 0$, then the Lagrangian in (718) becomes

$$L_2 \quad = \quad \frac{1}{2m}g_{\mu\nu}p^\mu p^\nu = \frac{1}{2m}p^\mu p_\mu = \frac{1}{2m}\left(-m^2c^2\right) = -\frac{1}{2}mc^2 \tag{737}$$

Similarly, if $A^\mu = 0$, then the Hamiltonian in (736) becomes

$$H_2 \quad = \quad \frac{1}{2m}\left(p^\mu\right)^2 = \frac{1}{2m}p^\mu p_\mu = -\frac{1}{2}mc^2 \tag{738}$$

So we conclude that for a free particle, we have

$$L_2 \quad = \quad H_2 = \frac{1}{2m} g_{\mu\nu} p^\mu p^\nu = -\frac{1}{2} mc^2 \qquad \text{for a free particle } (A^\mu = 0) \tag{739}$$

We can now go back and see that finding H_2 for a particle in an electromagnetic field $(A^\mu \neq 0)$ can be done very easily. From the minimal coupling rule in (730), we know that introducing an electromagnetic field just causes $p^\mu \Longrightarrow p^\mu - eA^\mu$ in the Hamiltonian. Then the Hamiltonian for the free particle in (739) simply becomes

$$H_2 = \frac{1}{2m} g_{\mu\nu} \left(p^\mu - eA^\mu\right)\left(p^\nu - eA^\nu\right) \tag{740}$$

which is precisely the result we obtained in (736). We can write the kinetic four-momentum as $\pi^\mu = p^\mu - eA^\mu$ and summarize the results of (730), (736), and (739) as follows.

$$\boxed{\begin{array}{rl} \text{For a free particle } (A^\mu = 0): & L_2 = H_2 = \frac{1}{2m} g_{\mu\nu} p^\mu p^\nu = -\frac{1}{2} mc^2, \qquad \pi^\mu = p^\mu \\[2em] \text{For a particle in an EM field } (A^\mu \neq 0): & L_2 = \frac{1}{2m} g_{\mu\nu} p^\mu p^\nu + eA_\mu u^\mu, \qquad \pi^\mu = p^\mu - eA^\mu \\[1em] & H_2 = \frac{1}{2m}\left(p^\mu - eA^\mu\right)^2 \\[2em] \text{Invariance of kinetic four-momentum:} & g_{\mu\nu} \pi^\mu \pi^\nu = -m^2 c^2 \end{array}} \tag{741}$$

7.2 The "space + time" Hamiltonian

The Lagrangian in (718) is a function of four-vectors, (x^μ, u^μ), which are parameterized by τ. However, we can also write the Lagrangian as a function of 3-vectors, $\left(x^i, \dot{x}^i\right)$, which are parameterized by x^0. The process for reparameterizing the Lagrangian is described in Appendix M. It was found in (2865) that the Lagrangian becomes

$$\tilde{L}_2 = \dot{x}^\mu p_\mu + eA^\mu \dot{x}_\mu \quad (742)$$

$$= cp_0 + v^i p_i + eA^\mu \dot{x}_\mu \quad (743)$$

The "dot derivative" is a derivative with respect to $t = x^0/c$ so we can use $\dot{x}_\mu = v_\mu = (v_0, v_i)$. Applying a Legendre transformation in terms of 3-vectors, $\tilde{H}_2 = v^i p_i - \tilde{L}_2$, gives

$$\tilde{H}_2 = v^i p_i - \left(cp_0 + v^i p_i + eA^\mu v_\mu\right) \quad (744)$$

$$\tilde{H}_2 = -cp_0 - eA^\mu v_\mu \quad (745)$$

We will use this relation later in the calculation. For now we may simply notice that if there is no electromagnetic field, then $A_\mu = 0$ and we are left with $\tilde{H}_2 = -cp_0$ where p_0 is the time-like component of the four-momentum.[72] In local flat space-time coordinates we have $p_0 = -p^0 = -E/c$ so $\tilde{H}_2 = E$. Therefore we find that the Hamiltonian represents the energy of the particle in a given inertial reference frame. Although this Hamiltonian can be made to vanish in a particular frame (by an appropriate Lorentz boost), it does *not* vanish identically in all frames.

Returning to the Hamiltonian found in (735) and writing it in terms of the metric gives

$$H_2 = \frac{1}{2m} p_\mu p^\mu - \frac{e}{m} A_\mu p^\mu + \frac{e^2}{2m} A_\mu A^\mu \quad (746)$$

$$= \frac{1}{2m} g^{\mu\nu} p_\mu p_\nu - \frac{e}{m} g^{\mu\nu} p_\mu A_\nu + \frac{e^2}{2m} A_\mu A^\mu \quad (747)$$

We are choosing to use *covariant* components of the four-vectors since the Hamiltonian in (745) is also in terms of covariant components. By expanding the summations in terms of the metric components we have

$$\begin{aligned} H_2 = {} & \frac{1}{2m}\left(g^{00} p_0 p_0 + g^{i0} p_i p_0 + g^{0j} p_0 p_j + g^{ij} p_i p_j\right) \\ & - \frac{e}{m}\left(g^{00} p_0 A_0 + g^{i0} p_i A_0 + g^{0j} p_0 A_j + g^{ij} p_i A_j\right) + \frac{e^2}{2m} A_\mu A^\mu \end{aligned} \quad (748)$$

[72] Note that we use a standard four-velocity, $u^\mu = dx^\mu/d\tau$, which in flat space-time becomes $u^\mu = \gamma\left(c, v^i\right)$, and we also use a "space+time" velocity, $\dot{x}^\mu = v^\mu = \left(c, v^i\right)$. However, for the momentum, we are *only* using a standard four-momentum, $p^\mu = mu^\mu$, which in flat space-time becomes $p^\mu = m\gamma\left(c, v^i\right) = \left(\gamma mc, \gamma v^i\right) = \left(E/c, \gamma v^i\right)$. Notice we do *not* have a "space+time" momentum which could be written in terms of the "space+time" velocity as $mv^\mu = \left(mc, mv^i\right)$. Therefore, p_0 should not be misunderstood as mc but rather it is the time-like component of the standard four-momentum and therefore $p^0 = E/c$ where E is the full energy.

Notice from (745) that in order to determine $\tilde{H}_2$, we can simply solve the equation above for $\tilde{H}_2 = -cp_0 - eA_\mu \dot{x}^\mu$. However, we will need to substitute for H_2 on the left side. To do so, we can notice that substituting the canonical momentum from (727) back into the Hamiltonian in (736) gives

$$H_2 \quad = \quad \frac{1}{2m}\left(mu^\mu + eA^\mu - eA^\mu\right)^2 = \frac{1}{2m}\left(mu^\mu\right)^2 = \frac{m}{2}u^\mu u_\mu = -\frac{1}{2}mc^2 \tag{749}$$

We can substitute this into the left side of (748) and also combine the middle two terms inside the first parenthesis which are essentially identical.

$$\begin{aligned} -\frac{1}{2}mc^2 \quad &= \quad \frac{1}{2m}\left(g^{00}p_0p_0 + 2g^{0i}p_ip_0 + g^{ij}p_ip_j\right) \\ &\quad -\frac{e}{m}\left(g^{00}p_0A_0 + g^{0j}p_0A_j + g^{i0}p_iA_0 + g^{ij}p_iA_j\right) + \frac{e^2}{2m}A_\mu A^\mu \end{aligned} \tag{750}$$

What we have done is effectively use the constraint $\left(g_{\mu\nu}p^\mu p^\nu = -\frac{1}{2}mc^2\right)$ applied to H_2 (the covariant Hamiltonian), in order to solve for $\tilde{H}_2$ (the "space + time" Hamiltonian). Note that this was not possible in the case of H_1 because applying the corresponding constraint $\left(g_{\mu\nu}u^\mu u^\nu = -c^2\right)$ led to H_1 vanishing as was shown in Section 35.

Next we rearrange the terms in (750) to write a quadratic in p_0.

$$\begin{aligned} -m^2c^2 \quad &= \quad g^{00}\left(p_0\right)^2 + 2g^{0i}p_ip_0 + g^{ij}p_ip_j \\ &\quad -2eg^{00}p_0A_0 - 2eg^{0j}p_0A_j - 2eg^{i0}p_iA_0 - 2eg^{ij}p_iA_j + e^2A_\mu A^\mu \end{aligned} \tag{751}$$

$$\begin{aligned} 0 \quad &= \quad g^{00}\left(p_0\right)^2 + \left(2g^{0i}p_i - 2eg^{00}A_0 - 2eg^{0j}A_j\right)p_0 \\ &\quad +g^{ij}p_ip_j - 2eg^{i0}p_iA_0 - 2eg^{ij}p_iA_j + e^2A_\mu A^\mu + m^2c^2 \end{aligned} \tag{752}$$

In the first parenthesis, we can use i for the repeated index for the first and last term and factor out a common $2g^{i0}$. In the second parenthesis we can expand $A_\mu A^\mu$ in terms of the metric as $A_\mu A^\mu = g^{00}\left(A_0\right)^2 + 2g^{0i}A_0A_i + g^{ij}A_iA_j$. Then we have

$$\begin{aligned} 0 \quad &= \quad g^{00}\left(p_0\right)^2 + \left[2g^{0i}\left(p_i - eA_i\right) - 2eg^{00}A_0\right]p_0 \\ &\quad +g^{ij}p_ip_j - 2eg^{i0}p_iA_0 - 2eg^{ij}p_iA_j + e^2g^{00}\left(A_0\right)^2 + 2e^2g^{0i}A_0A_i + e^2g^{ij}A_iA_j + m^2c^2 \end{aligned} \tag{753}$$

In the second parenthesis we can group the second and fifth terms to factor out $2eg^{i0}A_0$ and also group the first, third, and sixth terms to factor out g^{ij}.

$$\begin{aligned} 0 \quad &= \quad g^{00}\left(p_0\right)^2 + \left[2g^{0i}\left(p_i - eA_i\right) - 2eg^{00}A_0\right]p_0 \\ &\quad +g^{ij}\left(p_ip_j - 2ep_iA_j + e^2A_iA_j\right) - 2eg^{i0}A_0\left(p_i - eA_i\right) + e^2g^{00}\left(A_0\right)^2 + m^2c^2 \end{aligned} \tag{754}$$

Notice in the second parenthesis that $\left(p_i p_j - 2ep_iA_j + e^2A_iA_j\right) = (p_i - eA_i)(p_j - eA_j)$. If we write the kinetic momentum as $\pi_i = p_i - eA_i$, then we have

$$0 = g^{00}(p_0)^2 + \left[2g^{0i}\pi_i - 2eg^{00}A_0\right]p_0 + \left[g^{ij}\pi_i\pi_j - 2eg^{i0}A_0\pi_i + e^2g^{00}(A_0)^2 + m^2c^2\right] \tag{755}$$

Using the quadratic formula to solve for p_0 gives

$$\begin{aligned} p_0 = & \frac{-2g^{0i}\pi_i + 2eg^{00}A_0}{2g^{00}} \\ & \pm\frac{1}{2g^{00}}\sqrt{(2g^{i0}\pi_i - 2eg^{00}A_0)^2 - 4g^{00}\left(g^{ij}\pi_i\pi_j - 2eg^{i0}A_0\pi_i + e^2g^{00}(A_0)^2 + m^2c^2\right)} \end{aligned} \tag{756}$$

Simplifying the fraction in the front of the expression, pulling a 4 out of the root, and multiplying out the squared parenthesis in the root gives

$$\begin{aligned} p_0 = & \frac{-g^{0i}\pi_i}{g^{00}} + eA_0 \\ & \pm\frac{1}{g^{00}}\sqrt{(g^{i0}\pi_i)^2 - 2eg^{00}A_0g^{i0}\pi_i + (eg^{00}A_0)^2 - g^{00}\left(g^{ij}\pi_i\pi_j - 2eg^{i0}A_0\pi_i + e^2g^{00}(A_0)^2 + m^2c^2\right)} \end{aligned} \tag{757}$$

Canceling common terms inside the root and bringing $1/g^{00}$ inside the root gives

$$p_0 = -\frac{g^{0i}\pi_i}{g^{00}} + eA_0 \pm\sqrt{\left(\frac{g^{i0}\pi_i}{g^{00}}\right)^2 - \frac{g^{ij}\pi_i\pi_j + m^2c^2}{g^{00}}} \tag{758}$$

Now we use $\tilde{H}_2 = -cp_0 - eA^\mu v_\mu$ from (745) to obtain $\tilde{H}_2$. Solving for p_0 gives $p_0 = \left(-eA^\mu v_\mu - \tilde{H}_2\right)/c$. Substituting this into the equation above gives

$$\left(-eA^\mu v_\mu - \tilde{H}_2\right)/c = -\frac{g^{0i}\pi_i}{g^{00}} + eA_0 \pm\sqrt{\left(\frac{g^{i0}\pi_i}{g^{00}}\right)^2 - \frac{g^{ij}\pi_i\pi_j + m^2c^2}{g^{00}}} \tag{759}$$

Solving for $\tilde{H}_2$ and using the positive root[73] gives

$$\tilde{H}_2 = \frac{cg^{0i}\pi_i}{g^{00}} + c\sqrt{\frac{m^2c^2 + g^{ij}\pi_i\pi_j}{-g^{00}} + \left(\frac{g^{i0}\pi_i}{g^{00}}\right)^2} - eA^\mu v_\mu - ceA_0 \tag{760}$$

Finally, we use $\pi_i = p_i - eA_i$ to expression the Hamiltonian in terms of the canonical momentum.

$$\boxed{\tilde{H}_2 = \frac{cg^{0i}(p_i - eA_i)}{g^{00}} + c\sqrt{\frac{m^2c^2 + g^{ij}(p_i - eA_i)(p_j - eA_j)}{-g^{00}} + \left(\frac{g^{i0}p_i - eg^{i0}A_i}{g^{00}}\right)^2} - eA^\mu v_\mu - ceA_0} \tag{761}$$

[73]We have taken the positive root so that for zero momentum, we recover a positive rest energy, mc^2.

A similar Hamiltonian has been found by Cognola, et al. [45] and by Bertschinger [48]. In the case of Bertschinger, it appears that he uses the same method as used here expect he does not include an electromagnetic field. His equation (14) has

$$H_{Bertschinger} = \frac{g^{0i}p_i}{g^{00}} + \left[\frac{\left(g^{ij}p_i p_j + m^2\right)}{-g^{00}} + \left(\frac{g^{i0}p_i}{g^{00}}\right)^2\right]^{1/2} \tag{762}$$

with $c = 1$. This matches (761) for the case of a free particle where $A^\mu = 0$ (no electromagnetic field) and $\pi_i = p_i$ (since the kinetic and canonical momenta become identical). A similar Hamiltonian is also obtained by Cognola, et al. in equation (2.5) of [45] which they give as

$$H_{CVZ} = \left(\frac{m^2 - \tilde{g}^{ij}\pi_i\pi_j}{g^{00}}\right)^{1/2} - \frac{g^{0i}\pi_i}{g^{00}} + eA_0 \tag{763}$$

where $\pi_i = p_i - eA_i$ and $\tilde{g}^{ij} = g^{ij} - g^{0i}g^{0j}/g^{00}$ is the inverse of g_{ij}. Substituting $\tilde{g}^{ij}$ into (763) and rearranging gives

$$H_{Cognola} = -\frac{g^{0i}\pi_i}{g^{00}} + \left[\frac{m^2 - g^{ij}\pi_i\pi_j}{g^{00}} + \frac{g^{0i}g^{0j}\pi_i\pi_j}{\left(g^{00}\right)^2}\right]^{1/2} + eA_0 \tag{764}$$

Now we can compare $H_{Cognola}$ in (764) with $\tilde{H}_2$ in (761). It is evident that there are several signs that are opposite but this is simply because Cognola, et al. use a metric with $diag\,(1,-1,-1,-1)$ rather than our $diag\,(-1,1,1,1)$. However, it is not a trivial discrepancy that $H_{Cognola}$ does not have the additional term $-eA^\mu v_\mu$ which is in $\tilde{H}_2$. Notice that this term was introduced in going from (758) to (759) where we used $\tilde{H}_2 = -cp_0 - eA^\mu v_\mu$ from (745) to substitute for p_0. Obtaining the expression for $H_{Cognola}$, requires using $\tilde{H}_2 = -cp_0$ which would only be valid if $A_\mu = 0$. However, this is obviously not the case if $\pi_\mu = p_\mu - eA_\mu$. Therefore this appears to be an inconsistency in $H_{Cognola}$.

It is important to note that Cognola, et al. started from the same Lagrangian that DeWitt started from. First they write a Lagrangian in equation (2.1) of [45] as

$$L_0 = -m\left(-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right)^{1/2} - eA_\mu\dot{x}^\mu \tag{765}$$

which they acknowledge is singular. However, they simply separate the Lagrangian into space+time components to write the Lagrangian in equation (2.2) of [45] as

$$L = -m\left(g_{00} + 2g_{oi}v^i + g_{ij}v^iv^j\right)^{1/2} - eA_iv^i - eA_0 \tag{766}$$

This Lagrangian is essentially identical to DeWitt's Lagrangian in (650) except for a different sign being used inside the root. They also obtain a canonical momentum that is identical to (655) when (656) is inserted. However, the velocity they obtain in their equation (2.4) is

$$v^i = \frac{\tilde{g}^{ij}\pi_j}{\left[g^{00}\left(m^2 - \tilde{g}^{rs}\pi_r\pi_s\right)\right]^{1/2}} + g^{0i}/g^{00} \tag{767}$$

where $\tilde{g}^{ij} = g^{ij} - g^{0i}g^{0j}/g^{00}$ just as we had for the "spatial inverse metric" in (658). The velocity obtained here in (769) is

$$v^j = \frac{\tilde{g}^{jk}\pi_k}{\gamma m} - c\tilde{g}^{jk}g_{0k} \tag{768}$$

If we substitute (656) into (661), then we obtain

$$v^j = c\sqrt{\frac{\tilde{g}^{jk}g_{0j}g_{0k} - g_{00}}{m^2c^2 + \tilde{g}^{ik}\pi_j\pi_k}}\tilde{g}^{jk}\pi_k - c\tilde{g}^{jk}g_{0k} \tag{769}$$

From (2886) and (2872) in Appendix N, we also have

$$\tilde{g}^{jk}g_{0j}g_{0k} - g_{00} = -\frac{1}{g^{00}} \quad \text{and} \quad \tilde{g}^{jk}g_{0k} = -\frac{g^{0i}}{g^{00}} \tag{770}$$

Substituting the first expression above into the first term of (769) and the second expression above into the second term of (769) gives

$$v^j = \frac{c\tilde{g}^{jk}\pi_k}{\sqrt{-g^{00}\left(m^2c^2 + \tilde{g}^{ik}\pi_j\pi_k\right)}} + c\frac{g^{0i}}{g^{00}} \tag{771}$$

This result matches that of Cognola, et al. in equation (2.4) of [45]. Similarly, substituting (770) into the Hamiltonian in (676) gives

$$H = c\left(\frac{m^2c^2 + \tilde{g}^{jk}\pi_j\pi_k}{-g^{00}}\right)^{1/2} + c\frac{g^{0i}}{g^{00}}\pi_j - ceA_0 \tag{772}$$

This also matches $H_{Cognola}$ as shown in (763) except for some sign differences which are due to a difference in metric signature. As pointed out by Cognola, et al. in [45], this result only matches DeWitt's result in (677) only when g_{0i} is zero. Otherwise, DeWitt's result is in error.

7.3 The weak field, low velocity Hamiltonian

We now express the metric as a perturbation of flat Minkowski space-time: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Then the Hamiltonian in (760) can be expanded as

$$\begin{aligned}\tilde{H}_2 &= \frac{c\left(\eta^{0j}+h^{0j}\right)\pi_j}{\eta^{00}+h^{00}} + c\sqrt{\frac{m^2c^2+\left(\eta^{ij}+h^{ij}\right)\pi_i\pi_j}{-\left(\eta^{00}+h^{00}\right)} + \left[\frac{\left(\eta^{0j}+h^{0j}\right)\pi_j}{\eta^{00}+h^{00}}\right]^2} \\ &\quad -eA^{\mu}v_{\mu} - ceA_0\end{aligned} \tag{773}$$

Since $\eta^{jk} = \delta_j^k$, $\eta^{0j} = 0$ and $\eta^{00} = -1$, then we have

$$\begin{aligned}\tilde{H}_2 &= \frac{ch^{0j}\pi_j}{-1+h^{00}} + c\sqrt{\frac{m^2c^2+\pi^j\pi_j+h^{ij}\pi_i\pi_j}{-\left(-1+h^{00}\right)} + \left(\frac{h^{0j}\pi_j}{-1+h^{00}}\right)^2} \\ &\quad -eA^{\mu}v_{\mu} - ceA_0\end{aligned} \tag{774}$$

We will need to isolate m^2c^2 inside the square root since it will be the dominant term in a weak field approximation.

$$\begin{aligned}\tilde{H}_2 &= \frac{ch^{0j}\pi_j}{-1+h^{00}} + c\sqrt{\frac{m^2c^2}{-\left(-1+h^{00}\right)} + \frac{\left(\pi^j\pi_j+h^{ij}\pi_i\pi_j\right)\left(-1+h^{00}\right)+\left(h^{0j}\pi_j\right)^2}{-\left(-1+h^{00}\right)^2}} \\ &\quad -eA^{\mu}v_{\mu} - ceA_0\end{aligned} \tag{775}$$

Factoring m^2c^2 out of the square root gives

$$\begin{aligned}\tilde{H}_2 &= \frac{ch^{0j}\pi_j}{-1+h^{00}} + mc^2\sqrt{\frac{1}{-\left(-1+h^{00}\right)} + \frac{\left(\pi^j\pi_j+h^{ij}\pi_i\pi_j\right)\left(-1+h^{00}\right)+\left(h^{0j}\pi_j\right)^2}{-\left(-1+h^{00}\right)^2 m^2c^2}} \\ &\quad -eA^{\mu}v_{\mu} - ceA_0\end{aligned} \tag{776}$$

For weak fields and low velocities, the contents of the square root will satisfy

$$\frac{\left(\pi^j\pi_j+h^{ij}\pi_i\pi_j\right)\left(-1+h^{00}\right)+\left(h^{0j}\pi_j\right)^2}{-\left(-1+h^{00}\right)^2 m^2c^2} \quad << \frac{1}{-\left(-1+h^{00}\right)} \tag{777}$$

or simplifying,

$$\frac{\left(\pi^j\pi_j+h^{ij}\pi_i\pi_j\right)\left(-1+h^{00}\right)+\left(h^{0j}\pi_j\right)^2}{\left(-1+h^{00}\right)m^2c^2} \;<<\; 1 \tag{778}$$

A binomial approximation to first order can be applied since any second order terms will involve π^4 which is beyond the non-relativistic limit considered here. The binomial approximation[74] leads to the Hamiltonian becoming

$$\begin{aligned}\underset{\substack{(weak\ field,\\ low\ velocity)}}{\tilde{H}_2} &= \frac{ch^{0j}\pi_j}{-1+h^{00}} - eA^{\mu}v_{\mu} - ceA_0 \\ &+mc^2\left[\frac{1}{-\left(-1+h^{00}\right)} + \frac{1}{2}\left(\frac{\left(\pi^j\pi_j + h^{ij}\pi_i\pi_j\right)\left(-1-h^{00}\right) + \left(h^{0j}\pi_j\right)^2}{-\left(-1+h^{00}\right)^2 m^2c^2}\right)\right] \qquad (779)\end{aligned}$$

Multiplying out terms gives

$$\begin{aligned}\underset{\substack{(weak\ field,\\ low\ velocity)}}{\tilde{H}_2} &= \frac{ch^{0j}\pi_j}{-1+h^{00}} - \frac{mc^2}{-1+h^{00}} + \frac{\pi^j\pi_j + h^{ij}\pi_i\pi_j + \pi^j\pi_j h^{00} + h^{ij}\pi_i\pi_j h^{00} + \left(h^{0j}\pi_j\right)^2}{2m\left(-1+h^{00}\right)^2} \\ &-eA^{\mu}v_{\mu} - ceA_0 \qquad (780)\end{aligned}$$

We can use a series expansion to eliminate the denominators involving $-1+h^{00}$. Note that in using the binomial approximation in (778) we only kept terms up to order $\left(h^{\mu\nu}\right)^2$ hence we can only keep terms of order $\left(h^{00}\right)^2$ in the series expansion.

$$\frac{1}{-1-h^{00}} \approx -1 + h^{00} + \left(h^{00}\right)^2 \qquad (781)$$

We also need $1/\left(-1+h^{00}\right)^2$ which becomes

$$\begin{aligned}\frac{1}{\left(-1-h^{00}\right)^2} &\approx \left[-1+h^{00}+\left(h^{00}\right)^2\right]^2 \\ &\approx 1 - h^{00} - \left(h^{00}\right)^2 - h^{00} + \left(h^{00}\right)^2 + \left(h^{00}\right)^3 - \left(h^{00}\right)^2 + \left(h^{00}\right)^3 + \left(h^{00}\right)^4 \\ &\approx 1 - 2h^{00} - \left(h^{00}\right)^2 + \mathscr{O}\left(\left(h^{00}\right)^3\right) \qquad (782)\end{aligned}$$

Using (781) and (782) makes the Hamiltonian in (780) become

$$\begin{aligned}\underset{\substack{(weak\ field,\\ low\ velocity)}}{\tilde{H}_2} &= ch^{0j}\pi_j\left[-1+h^{00}+\left(h^{00}\right)^2\right] - mc^2\left[-1+h^{00}+\left(h^{00}\right)^2\right] \\ &+\frac{1}{2m}\left[\pi^j\pi_j + h^{ij}\pi_i\pi_j + \pi^j\pi_j h^{00} + h^{ij}\pi_i\pi_j h^{00} + \left(h^{0j}\pi_j\right)^2\right]\left[1 - h^{00} - \left(h^{00}\right)^2\right] \\ &-eA^{\mu}v_{\mu} - ceA_0 \qquad (783)\end{aligned}$$

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[74] Here we are using $(a+b)^{1/2} \approx a + \frac{1}{2}b$ for $b/a << 1$ where according to (777) we have $a = \dfrac{1}{-\left(-1+h^{00}\right)}$ and $b = \left[\dfrac{\left(\pi^j\pi_j + h^{ij}\pi_i\pi_j\right)\left(-1+h^{00}\right) + \left(h^{0j}\pi_j\right)^2}{-\left(-1+h^{00}\right)^2 m^2c^2}\right]$.

Again, we may only keep terms of order $(h^{\mu\nu})^2$ for consistency. Then distributing, canceling terms and removing terms of order $(h^{\mu\nu})^3$ and above gives

$$\begin{aligned}\underset{(second\text{-}order)}{\tilde{H}_2} &= -ch^{0j}\pi_j + ch^{00}h^{0j}\pi_j + mc^2 - mc^2h^{00} - mc^2\left(h^{00}\right)^2 \\ &+\frac{1}{2m}\left[\pi^j\pi_j + h^{ij}\pi_i\pi_j + \left(h^{0j}\pi_j\right)^2 - 2\left(h^{00}\right)^2\pi^j\pi_j\right] \\ &-eA^\mu v_\mu - ceA_0\end{aligned}$$

To lower the metric perturbation indices, we can apply the metric. Since $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, then staying to first order in the metric means we can use $g_{\mu\nu} \approx \eta_{\mu\nu}$ to lower the indices. This leads to

$$h^{00} = h_{00}, \qquad h^{0i} = -h_{0i}, \qquad h^{ij} = h_{ij} \tag{784}$$

Then the Hamiltonian becomes

$$\begin{aligned}\underset{(second\text{-}order)}{\tilde{H}_2} &= ch_{0i}\pi^i - ch_{00}h_{0i}\pi^i + mc^2 - mc^2h_{00} - mc^2\left(h_{00}\right)^2 - eA^\mu v_\mu - ceA_0 \\ &+\frac{1}{2m}\left[\pi^j\pi_j + h_{ij}\pi^i\pi^j + \left(h_{0i}\pi^i\right)^2 - 2\left(h^{00}\right)^2\pi^j\pi_j\right]\end{aligned} \tag{785}$$

Rearranging terms with rest-energy and kinetic energy first, and higher order terms last gives

$$\boxed{\begin{aligned}\underset{(second\text{-}order)}{\tilde{H}_2} = mc^2 + \frac{\pi^j\pi_j}{2m} + ch_{0i}\pi^i - mc^2h_{00} - mc^2\left(h_{00}\right)^2 - ch_{00}h_{0i}\pi^i - eA^\mu v_\mu - ceA_0 \\ -\frac{\left(h^{00}\right)^2\pi^j\pi_j}{m} + \frac{\left(h_{0i}\pi^i\right)^2}{2m} + \frac{h_{ij}\pi^i\pi^j}{2m}\end{aligned}}$$

Higher order "four-momentum invariant" Hamiltonian for a relativistic electron-pair coupled to electromagnetic and gravitational fields

(786)

We can compare $\tilde{H}_2$ in (786) with $H_{(2nd\text{-}order)}$ from (693) which was found to be

$$\begin{aligned}\underset{(2nd\text{-}order)}{H} &\approx mc^2 + \frac{1}{2m}\left(\pi_k - mch_{0k}\right)^2 - \tfrac{1}{2}mc^2h_{00} - ceA_0 \\ &-\tfrac{1}{8}mc^2\left(h_{00}\right)^2 - \frac{h_{00}\pi_k^2}{4m} - \frac{h^{jk}\pi_k\pi_j}{2m} - \frac{\left(h_{00}\right)^2\pi_k^2}{16m} + ch^{jk}h_{0k}\pi_j \\ &+\frac{\left(h_{0j}\right)^2\pi_k^2}{4m} + \frac{h^{ij}h_{ik}\pi^k\pi_j}{2m} - \frac{x^\sigma\partial_\sigma h^{jk}\pi_k\pi_j}{2m} + \frac{h_{00}h^{jk}\pi_k\pi_j}{4m}\end{aligned} \tag{787}$$

We may immediately observe that $\tilde{H}_2$ has a factor of 2 in front of $-mc^2h_{00}$ while $H_{(2nd\text{-}order)}$ has a factor of 1/2. Therefore, $\tilde{H}_2$ does not correctly recover the Newtonian gravitational potential energy. For example, a spherical mass M has $h_{00} \approx -2U/c^2$ where $U = -GM/r$ in the Newtonian limit. This gives $h_{00} \approx -\dfrac{2GM}{c^2r}$. Therefore, the Newtonian potential energy term in the Hamiltonian must be $-\frac{1}{2}mc^2h_{00}$ in order to obtain

$$-\tfrac{1}{2}mc^2h_{00} = -\tfrac{1}{2}mc^2\left(-\frac{2GM}{c^2r}\right) = \frac{GmM}{r} \tag{788}$$

as we expect. So we find that the factor of 2 rather than 1/2 in (786) violates the Newtonian limit for the Hamiltonian.

7.4 The Hamiltonian for particles in the presence of gravitational waves

For the purpose of describing the coupling of relativistic particles to gravitational waves, we may consider the far-field where $h_{00} = h_{0i} = 0$, and $h_{ij} = h_{ij}^{\tau\tau}$. We can see from (693) that the Hamiltonian (to first order in h_{ij}) becomes

$$H^{\tau\tau} = mc^2 + \frac{\pi^j \pi_j}{2m} - \frac{h_{ij}^{\tau\tau} \pi^i \pi^j}{2m} - ceA_0 \tag{789}$$

Since $\pi_j = p_j - eA_j$, then neglecting the electromagnetic field means $A_\mu = 0$ and $\pi_j = p_j$. So the Hamiltonian becomes

$$\boxed{H^{\tau\tau} = mc^2 + \frac{p^j p_j}{2m} - \frac{h_{ij}^{\tau\tau} p^i p^j}{2m}} \tag{790}$$

If we separate the free particle Hamiltonian and the interaction Hamiltonian, then $H = H_{free} + H_{int}^{TT}$ where

$$H_{int}^{\tau\tau} = -\frac{h_{ij}^{\tau\tau} p^i p^j}{2m} \tag{791}$$

This result is similar to the interaction Hamiltonian obtained by Rothman and Boughn (RB) in equation (5.6) of [54] and on page 8 of [55], which is a related paper by the same authors. However, there is a crucial difference between the result in (790) and that of (RB). Here we obtained a *negative* sign for the interaction Hamiltonian while RB obtained a *positive* sign. Below we will demonstrate that the positive sign is an error due to the use of an approximation that was applied prematurely in the calculation. First, we provide the detailed process that was used by RB to obtain their result. We start with the Lagrangian in (650) given as

$$L = -mc\left(-g_{00}c^2 - 2g_{0j}cv^j - g_{ij}v^i v^j\right)^{1/2} + ceA_0 + eA_i v^i \tag{792}$$

Eliminating the electromagnetic field and substituting in $g_{00} = -1 + h_{00}$ and $g_{0i} = h_{0i}$ gives

$$L = -mc\left[-\left(-1 + h_{00}\right)c^2 - 2h_{0i}cv^j - g_{ij}v^i v^j\right]^{1/2} \tag{793}$$

Again we consider the far-field where $h_{00} = h_{0i} = 0$ and $h_{ij} = h_{ij}^{\tau\tau}$ so that we have[75]

$$L^{\tau\tau} = -mc\left(c^2 - g_{ij}v^i v^j\right)^{1/2} \tag{794}$$

$$= -mc^2\left(1 - g_{ij}v^i v^j/c^2\right)^{1/2} \tag{795}$$

Using a first-order binomial approximation gives

$$\left(1 - g_{ij}v^i v^j/c^2\right)^{1/2} \approx 1 - g_{ij}v^i v^j/2c^2 \tag{796}$$

Then the Lagrangian becomes

$$L^{\tau\tau} = -mc^2\left(1 - g_{ij}v^i v^j/2c^2\right) \tag{797}$$

We can find the Hamiltonian using a Legendre transformation in terms of 3-vectors.

$$H^{\tau\tau} = p_k v^k - L^{\tau\tau} \tag{798}$$

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[75] RB refer to using the TT gauge in order to set $h_{00} = h_{0i} = 0$ and $h_{ij} = h_{ij}^{TT}$, however, it is effectively the same exact thing that we have done here. In our case, we do not need to refer to the TT gauge since we find that in the HD formulation, $h_{ij}^{\tau\tau}$ is the only radiative field. Therefore, instead of resorting to a gauge choice, we can simply make reference to the far-field.

where the canonical momentum can be found from $p_k = \dfrac{\partial L^{\tau\tau}}{\partial v^k}$. Making us of the Lagrangian in (797) and recognizing that $\dfrac{\partial v^i}{\partial v^k} = \delta^i_k$ gives

$$p_k = -mc^2\left(-g_{ij}\delta^i_k v^i/2c^2 - g_{ij}v^i\delta^j_k/2c^2\right) \quad (799)$$

$$= \frac{m}{2}\left(g_{kj}v^j + g_{ik}v^i\right) \quad (800)$$

$$= mv^i g_{ik} \quad (801)$$

Now we substitute the canonical momentum from (801) and the Lagrangian from (797) into the Hamiltonian in (798) to obtain

$$H^{\tau\tau} = mv^i v^k g_{ik} + mc^2\left(1 - g_{ij}v^i v^j/2c^2\right) \quad (802)$$

$$= mv^i v^k g_{ik} + mc^2 - mg_{ij}v^i v^j/2 \quad (803)$$

$$= mc^2 + mg_{ij}v^i v^j/2 \quad (804)$$

Then substituting $g_{ij} = \eta_{ij} + h^{\tau\tau}_{ij}$ gives

$$H^{TT} = mc^2 + m\left(\eta_{ij} + h^{\tau\tau}_{ij}\right)v^i v^j/2 \quad (805)$$

$$= mc^2 + \frac{m}{2}\eta_{ij}v^i v^j + \frac{m}{2}h^{\tau\tau}_{ij}v^i v^j \quad (806)$$

Since $\eta_{ij} = 0$ for $i \neq j$ and $\eta_{ij} = 1$ for $i = j$, then the second term becomes $mv^2/2$. So we have

$$H^{\tau\tau} = mc^2 + \tfrac{1}{2}mv^2 + \tfrac{1}{2}mh^{\tau\tau}_{ij}v^i v^j \quad (807)$$

Since the kinetic momentum and canonical momentum are the same when A_i and h_{0i} are absent, then the canonical momentum is just $p = mv$ and we can write the result above as

$$H^{\tau\tau} = mc^2 + \frac{p^2}{2m} + \frac{h^{\tau\tau}_{ij}p^i p^j}{2m} \qquad \text{(invalid result)} \quad (808)$$

This is the invalid result obtained in [54] and [55].

As stated earlier, if we compare the result in (808) with the expression in (790), we find that there is a different sign for the coupling term. To identify the cause of this sign difference, we can return to the derivation of the Hamiltonian provided in Section 36. There we began with the same Lagrangian given by (650) however, we did *not* use the first-order binomial approximation given in (796). This approximation effectively removes the square root in the Lagrangian and therefore changes the results obtained when taking the derivative of the Lagrangian to find the canonical momentum.

Rather than applying such an approximation, we use the *exact* expression for the Lagrangian to obtain the canonical momentum. The result obtained in (657) is $\pi_k = \gamma m\left(cg_{0k} + g_{ik}v^i\right)$. In the absence of electromagnetic fields, $\pi_j = p_j - eA_j$ simply becomes $\pi_j = p_j$. Also, for the far-field limit (or equivalently, in the TT gauge), we have $g_{0k} = 0$, so we may write the canonical momentum as

$$p_k = \gamma m g_{ik}v^i \quad (809)$$

This expression is in contrast to the result obtained using the *approximated* approach which was found in (801) to be

$$p_k = mv^i g_{ik} \quad (810)$$

The obvious difference is the presence of γ which is defined in (656) as

$$\gamma \equiv \left(-g_{00} - \frac{2}{c} g_{0j} v^j - \frac{1}{c^2} g_{ij} v^i v^j\right)^{-1/2} \tag{811}$$

This factor is a result of keeping the square root in the Lagrangian before taking the derivative to obtain the canonical momentum. Again applying the TT gauge, we have $g_{00} = -1$, $g_{0i} = 0$ and $g_{ij} = \eta_{\iota j} + h_{ij}$. Then the expression for γ becomes

$$\gamma = \left[1 - \frac{1}{c^2}\left(\eta_{ij} + h_{ij}\right) v^i v^j\right]^{-1/2} = \left(1 - \frac{v^2}{c^2} - \frac{h_{ij} v^i v^j}{c^2}\right)^{-1/2} \tag{812}$$

Applying a first-order binomial approximation to γ^{-1} gives

$$\gamma^{-1} \approx 1 - \frac{v^2}{2c^2} - \frac{h_{ij} v^i v^j}{2c^2} \tag{813}$$

Using the canonical momentum in (809), the Hamiltonian was found in (667) to be

$$H = \frac{1}{\gamma m}\left(m^2 c^2 + \tilde{g}^{jk} p_j p_k\right) - c p_j \tilde{g}^{jk} g_{0k} - ceA_0 \tag{814}$$

where $\tilde{g}^{ik} = g^{ik} - g^{0i} g^{0k} / g^{00}$ is the “spatial inverse metric.” Again applying the TT gauge (where $g_{0k} = 0$) and omitting electromagnetic fields makes the Hamiltonian become

$$H^{\tau\tau} = \frac{1}{\gamma m}\left(m^2 c^2 + \tilde{g}^{jk} p_j p_k\right) \tag{815}$$

In the TT gauge, we simply have $\tilde{g}^{jk} = g^{jk}$ where $g^{jk} = \eta^{jk} - h^{jk}$ to first order in the metric (as found in (2415) of Appendix A). So the second term in the Hamiltonian above will become

$$\tilde{g}^{jk} p_j p_k \quad = \quad \left(\eta^{jk} - h^{jk}\right) p_j p_k = p^2 - h^{jk} p_j p_k \tag{816}$$

Substituting (813) and (816) into the Hamiltonian in (815) gives

$$H^{\tau\tau} = \frac{1}{m}\left(1 - \frac{v^2}{2c^2} - \frac{h_{ij} v^i v^j}{2c^2}\right)\left(m^2 c^2 + p^2 - h^{jk} p_j p_k\right) \tag{817}$$

Multiplying terms and only keeping p^2 or v^2 to highest order gives

$$H^{\tau\tau} \quad = \quad \frac{1}{m}\left(m^2 c^2 + p^2 - h^{jk} p_j p_k - \frac{m^2 v^2}{2} - \frac{m^2 h_{ij} v^i v^j}{2}\right) \tag{818}$$

$$= \quad mc^2 + \frac{p^2}{m} - \frac{h^{jk} p_j p_k}{m} - \frac{mv^2}{2} - \frac{m h_{ij} v^i v^j}{2} \tag{819}$$

Lastly, substituting $p = mv$ and combining terms gives

$$\boxed{H^{\tau\tau} = mc^2 + \frac{p^2}{2m} - \frac{h_{ij} p^i p^j}{2m}} \tag{820}$$

This obviously matches the result stated in (790). We emphasize that this result was obtained starting from the general Hamiltonian found in (639) which was obtained *immediately* after applying a Legendre transformation. It was obtained well before the elaborate calculations and approximations of Section 37 which were required to obtain the second-order Hamiltonian in (693). In fact, (639) was obtained even *prior* to the calculation required to determine an expression for γ in (674) or the process required to reach the general Hamiltonian in (677).

We have provided the detailed analysis here to show explicitly the point at which the calculation by BR is in error. Specifically, we find that using the approximation in (796) to simplify the Lagrangian *before* applying the Legendre transformation leads to a sign error in the coupling term of the Hamiltonian. In general, one should only introduce approximations *after* applying the Legendre transformation, particularly because the derivative of the Lagrangian could be altered by an approximation applied to the Lagrangian beforehand.

As a final observation, we note that there is a much faster way of showing how to arrive at the correct result for the Hamiltonian in the TT gauge (or far field). We start with the general Hamiltonian obtained in (676). Omitting electromagnetic fields and using $\pi_k = p_k$ gives

$$H = c\left(\tilde{g}^{jk} g_{0j} g_{0k} - g_{00}\right)^{1/2}\left(m^2c^2 + \tilde{g}^{jk} p_j p_k\right)^{1/2} - c\tilde{g}^{jk} g_{0k} p_j \tag{821}$$

In the TT gauge, we have $g_{00} = -1$ and $g_{0j} = 0$ so the first square root simply becomes 1 and the last term vanishes. Applying a first-order binomial approximation to the remaining square root gives

$$\left(m^2c^2 + \tilde{g}^{jk} p_j p_k\right)^{1/2} = mc\left(1 + \frac{\tilde{g}^{jk} p_j p_k}{m^2c^2}\right)^{1/2} \approx mc\left(1 + \frac{\tilde{g}^{jk} p_j p_k}{2m^2c^2}\right) = mc + \frac{\tilde{g}^{jk} p_j p_k}{2mc} \tag{822}$$

We can substitute this into the Hamiltonian and recall that in the TT gauge, $\tilde{g}^{jk} = g^{jk} = \eta^{jk} - h^{jk}$.

$$H^{\tau\tau} = c\left[mc + \frac{\left(\eta^{jk} - h^{jk}\right) p_j p_k}{2mc}\right] = mc^2 + \frac{p^2}{2m} - \frac{h^{jk} p_j p_k}{2m} \tag{823}$$

Once again, we arrive at a result that is consistent with (790).

7.5 Summary of relativistic Hamiltonians and their quantization

The Lagrangian used by DeWitt [42] is referred to here as the “four-momentum invariant Lagrangian” L_1 which is given in (627) as

$$L_1 = -mc\sqrt{-g_{\mu\nu}u^{\mu}u^{\nu}} + eA_{\mu}u^{\mu} \tag{824}$$

In Section 35, it was shown that this Lagrangian leads to a Hamiltonian, H_1, that vanishes identically and therefore does not properly represent energy. This was pointed out by Jackson in [40] (p. 585) where he refers to Barut [44] for a discussion of some alternative formulations. This issue was also discussed by Cognola, et al. in [45] who recognize that the Lagrangian in (627) is singular and argue that the results of DeWitt [42] and Papini [46][47] are not valid. They begin with the same Lagrangian as DeWitt but they arrive at a very different Hamiltonian. Then they consider the same physical system considered by DeWitt (an axially symmetric, uniformly rotating superconductor) and show that their Hamiltonian makes different predictions.

There is also a treatment by Bertschinger [48] who begins with L_2, the “four-momentum invariant Lagrangian” found in (2857) of Appendix M. Bertschinger derives a Hamiltonian H_2 in terms of four-vectors as well as a Hamiltonian $\tilde{H}_2$ in terms of 3-vectors. This will be examined in detail in the following sections. However, it can be seen that Cognola, et al. also arrive at the same Hamiltonian as Bertschinger and therefore we may refer to $\tilde{H}_2$ as the “Cognola-Bertschinger Hamiltonian.” It is important to note that although Bertschinger begins with L_2 in order to obtain H_2 and $\tilde{H}_2$, it can be seen that Cognola, et al. begin with L_1 (the same Lagrangian as DeWitt) and yet they still obtain the *same* $\tilde{H}_2$ that Bertschinger obtains by use of a different method. This indicates that the critical issue is *not the Lagrangian* that one begins with, but rather *the method* one uses to obtain the Hamiltonian. This fact seems to be further supported by a paper by Cisneros-Parra [49] which discusses the broad issue of singular Lagrangians and shows in general how to still obtain a valid Hamiltonian using methods such as the Dirac method [50]-[52]. Specifically, in Section 5 of [49], which is titled, “*5. Relativistic Lagrangians*,” we find that Cisneros-Parra begins with the same Lagrangian as L_1 from (627), and then shows the reason why it is singular from the equations of motion as well as from the determinant of the Hessian matrix of the Lagrangian:

$$\det\left|\frac{\partial p_i}{\partial \dot{q}_i}\right| = \det\left|\frac{\partial^2 L}{\partial q^i \partial q^j}\right| = 0 \tag{825}$$

He then shows that it is ideal use L_2 which is *not* singular and therefore can lead to the Hamiltonian $\tilde{H}_2$ by the standard Legendre transform method.

Lastly, it is stated by Bertschinger [48] that *all* of the Lagrangians and Hamiltonians discussed here will to lead to the same equations of motion. This is shown in detail in Appendix M. Therefore, they are all adequate for describing the *classical* dynamics of relativistic electron-pairs in electromagnetic and gravitational fields. However, the reason we scrutinize the distinctions between the various Lagrangians, and more importantly the various Hamiltonians, is because we are interested in promoting the Hamiltonian and canonical momentum to quantum operators in order to describe the *quantum* dynamics of electron-pairs interacting with electromagnetic and gravitational fields.

In order to promote canonical quantities to quantum operators, we must insure that they possess the necessary characteristics that warrant this procedure. This is issue is discussed in another important paper by Castellani, et al. [53] which investigates the canonical quantization rule proposed by Dirac,

$$\{\ \}^*_{qp} \Longrightarrow \frac{1}{i\hbar}[\ ] \tag{826}$$

where the classical Poisson brackets of q, p (the canonical conjugate quantities) become quantum commutators. We therefore conclude that either Lagrangian L_1 or L_2 can be used to describe the physical system of a charged, spinless particle in the presence of an electromagnetic field in curved space-time. However, if L_1 is used, then it is critical to recognize that it is singular and hence great care must be taken when determining the Hamiltonian. Using the standard Legendre transformation leads to "DeWitt's general Hamiltonian," $\tilde{H}_1$, which has been found to be flawed. (Although, in the weak field, low velocity limit, it is found to be valid since it matches the correct result.)

However, it is clear that using L_2 is simpler choice since it is non-singular and leads to a non-vanishing, fully covariant Hamiltonian, H_2, which exhibits the invariance of the rest-mass energy, mc^2. It is also ideal to use L_2, which when reparameterized into a function of 3-vectors, $\tilde{L}_2$, leads to the correct expression for the Hamiltonian given by $\tilde{H}_2$. This Hamiltonian is then found to be the time-like component of the momentum four-vector (the frame-dependent energy) which is an appropriate physical interpretation for the Hamiltonian as a function of 3-vectors. The expression found for $\tilde{H}_2$ is in fact the correct Hamiltonian for the purpose of promoting the Hamiltonian and canonical momentum to quantum operators in order to describe the *quantum* dynamics of Cooper pairs interacting with electromagnetic and gravitational fields.

8 The quantized Hamiltonian, stress tensor, and coupling rules

8.1 Overview of the quantized Hamiltonian and stress tensor

In the sections that follow, the framework of gravito-electromagnetism is applied to various quantum mechanical systems. First is the development of a Lagrangian a non-relativistic charged particles in electromagnetic and gravito-electromagnetic fields (to first post-Newtonian order). The canonical momentum and associated coupling rule is found from the Lagrangian. The classical Hamiltonian is also developed and then quantized to obtain a modified Schrödinger equation.

Next we consider the Hamiltonian for a fully relativistic charged particle and develop a similar expression for the canonical momentum and the associated coupling rule. Extending the discussion of coupling rules, we then identify the lowest order scalar, vector, and tensor coupling rules and show that they are manifestations of the four-momentum invariance. We also consider the case of quantum particles in a zero-momentum state (where the canonical momentum is zero).

We also consider a quantized ideal fluid stress tensor developed by promoting all the momenta using the usual momentum operator while keeping the mass density and pressure as macroscopic quantities rather than eigen-valued operators. We briefly discuss an interpretation of the resulting stress tensor components and then develop quantum wave equations using the linearized conservation law for the stress-energy-momentum tensor. These quantum wave equations are found to be Schrödinger-like equations which have corresponding dispersion relations. We examine the limit of an ideal gas as well as a quantum system of completely non-interacting particles in a zero-momentum state. We find that a Cooper-pair superfluid can still exert a pressure while remaining in a superconductive state. Using these results, we develop field equations describing the response of a superconductor to external gravitational fields consisting of the gravito-electric field, gravito-magnetic field, and gravitational wave strain fields.

8.2 The quantized Hamiltonian for non-relativistic charged particles

Here we develop the non-relativistic Lagrangian for a charged particle in an electromagnetic field as well as a *gravito*-electromagnetic field. The Euler-Lagrange equation of motion can be written as

$$\nabla L - \frac{d}{dt}\left(\nabla_{\vec{v}} L\right) = 0 \tag{827}$$

where $\nabla_{\vec{v}}$ is the gradient with respect to $\vec{v}$. To determine the Lagrangian, we must match this result to the equation of motion obtained from Newton's Second law applied to the sum of the electromagnetic Lorentz force and gravitational Lorentz force.[76]

$$q\left(\vec{E} + \vec{v}\times\vec{B}\right) + m(\vec{E}_{g(PN)} + \vec{v}\times\vec{B}_{g(PN)}) \quad = \quad m\vec{a} \tag{828}$$

We can write the electromagnetic fields in terms of potentials,

$$\vec{E} = -\nabla\varphi - \partial_t\vec{A} \qquad \text{and} \qquad \vec{B} = \nabla\times\vec{A} \tag{829}$$

as well as the *gravito*-electromagnetic fields in terms of their potentials,

$$\vec{E}_{g(PN)} \quad = \quad -\nabla\Phi_N - \partial_t\vec{h} \qquad \text{and} \qquad \vec{B}_{g(PN)} = \nabla\times\vec{h} \tag{830}$$

Substituting the fields from (829) and (830) into (828) gives

$$q\left[-\nabla\varphi - \partial_t\vec{A} + \vec{v}\times\left(\nabla\times\vec{A}\right)\right] + m\left[-\nabla\Phi_N - \partial_t\vec{h} + \vec{v}\times\left(\nabla\times\vec{h}\right)\right] \quad = \quad m\frac{d\vec{v}}{dt} \tag{831}$$

Applying the identity $\vec{a}\times\left(\vec{b}\times\vec{c}\right) = \vec{b}\left(\vec{a}\cdot\vec{c}\right) - \left(\vec{a}\cdot\vec{b}\right)\vec{c}$ in both brackets gives

$$\begin{aligned} &q\left[-\nabla\varphi - \partial_t\vec{A} + \nabla\left(\vec{v}\cdot\vec{A}\right) - (\vec{v}\cdot\nabla)\vec{A}\right] \\ &+m\left[-\nabla\Phi_N - \partial_t\vec{h} + \nabla\left(\vec{v}\cdot\vec{h}\right) - (\vec{v}\cdot\nabla)\vec{h}\right] \quad = \quad m\frac{d\vec{v}}{dt} \end{aligned} \tag{832}$$

Rearranging terms gives

$$\begin{aligned} &q\left\{-\nabla\varphi + \nabla\left(\vec{v}\cdot\vec{A}\right) - \left[(\vec{v}\cdot\nabla)\vec{A} + \partial_t\vec{A}\right]\right\} \\ &+m\left\{-\nabla\Phi_N + \nabla\left(\vec{v}\cdot\vec{h}\right) - \left[(\vec{v}\cdot\nabla)\vec{h} + \partial_t\vec{h}\right]\right\} \quad = \quad m\frac{d\vec{v}}{dt} \end{aligned} \tag{833}$$

It is helpful to recognize that the full time derivative of $\vec{A}$ can be written using the chain rule as

$$\frac{d\vec{A}}{dt} \quad = \quad \frac{\partial\vec{A}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial\vec{A}}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial\vec{A}}{\partial x}\frac{\partial z}{\partial t} + \frac{\partial\vec{A}}{\partial t} \tag{834}$$

$$\frac{d\vec{A}}{dt} \quad = \quad (\vec{v}\cdot\nabla)\vec{A} + \partial_t\vec{A} \tag{835}$$

[76]For the gravitational part of this force equation, we are essentially using the geodesic equation of motion given by (508) for the first-order post-Newtonian limit with a non-relativistic test mass. As discussed in Section 26, the equation of motion in this limit is gauge-*dependent* and requires that $\nabla\cdot\vec{h} = 0$ where $\vec{h} = c\left(h_{01}, h_{02}, h_{03}\right)$. We are also using $\vec{E}_{g(PN)} = -\nabla\Phi_N - \partial_t\vec{h}$ and $\vec{B}_{g(PN)} = \nabla\times\vec{h}$ as stated in (830).

Likewise, for the full time derivative of $\vec{h}$ we have

$$\frac{d\vec{h}}{dt} = (\vec{v}\cdot\nabla)\vec{h} + \partial_t\vec{h} \tag{836}$$

Substituting (835) into the first bracket of (833) and substituting (836) into the second bracket of (833) gives

$$q\left[-\nabla\varphi + \nabla\left(\vec{v}\cdot\vec{A}\right) - \frac{d\vec{A}}{dt}\right] + m\left[-\nabla\Phi_N + \nabla\left(\vec{v}\cdot\vec{h}\right) - \frac{d\vec{h}}{dt}\right] = m\frac{d\vec{v}}{dt} \tag{837}$$

Now we distribute the q and m, then collect terms that have ∇ and terms that have $\dfrac{d}{dt}$.

$$\nabla\left[-q\varphi + q\left(\vec{v}\cdot\vec{A}\right) - m\Phi_N + \left(\vec{v}\cdot\vec{h}\right)\right] - \frac{d}{dt}\left(q\vec{A} - m\vec{h} - m\vec{v}\right) = 0 \tag{838}$$

Lastly, we can make use of the following expressions.

$$m\vec{v} = \nabla_{\vec{v}}\left(\frac{1}{2}m\vec{v}^2\right), \qquad \vec{A} = \nabla_{\vec{v}}\left(\vec{v}\cdot\vec{A}\right), \qquad \vec{h} = \nabla_{\vec{v}}\left(\vec{v}\cdot\vec{h}\right) \tag{839}$$

Substituting these into (838) gives

$$\nabla\left[-q\varphi + q\left(\vec{v}\cdot\vec{A}\right) - m\Phi_N + \left(\vec{v}\cdot\vec{h}\right)\right] - \frac{d}{dt}\nabla_{\vec{v}}\left[q\left(\vec{v}\cdot\vec{A}\right) + m\left(\vec{v}\cdot\vec{h}\right) - \frac{1}{2}m\vec{v}^2\right] = 0 \tag{840}$$

Matching this equation of motion with the equation of motion in (827) requires that the Lagrangian is given by

$$\boxed{L = \tfrac{1}{2}m\vec{v}^2 - q\varphi + q\left(\vec{v}\cdot\vec{A}\right) - m\Phi_N + m\left(\vec{v}\cdot\vec{h}\right)} \tag{841}$$

This is the Lagrangian for a non-relativistic charged particle in electromagnetic and *gravito*-electromagnetic fields. In general, $L = T - V$ so we see here that the kinetic energy is $\frac{1}{2}mv^2$, the electromagnetic potential energy is $q\left[\varphi - \left(\vec{v}\cdot\vec{A}\right)\right]$, and the gravitational potential energy is $m\left[\Phi_N - \left(\vec{v}\cdot\vec{h}\right)\right]$.

Next, the canonical momentum can be found using $\vec{p}_{can} = \dfrac{\partial L}{\partial v}$ which gives

$$\vec{p}_{can} = m\vec{v} + q\vec{A} + m\vec{h} \tag{842}$$

This means that $m\vec{v} = \vec{p}_{can} - q\vec{A} + m\vec{h}$ so the potentials $\vec{A}$ and $\vec{h}$ cause the canonical momentum to transform as

$$\boxed{\vec{p}_{can} \implies \vec{p}_{can} - q\vec{A} - m\vec{h}} \tag{843}$$

This is the coupling rule for the momentum of a *non-relativistic* charged particle in a magnetic and gravito-magnetic field (to first post-Newtonian order). Note that because the coupling rule involves $\vec{h} = \vec{\beta}$, *not* $\vec{\Xi} = \vec{\beta} - \dot{\vec{\varepsilon}}$, then the coupling rule is gauge-*dependent*. We can apply a Legendre transformation to find the Hamiltonian, $H = \vec{v}\cdot\vec{p}_{can} - L$. Using (842) for $\vec{p}_{can}$ gives

$$H = \vec{v}\cdot\left(m\vec{v} + q\vec{A} + m\vec{h}\right) - \left[\frac{1}{2}m\vec{v}^2 - q\varphi + q\left(\vec{v}\cdot\vec{A}\right) - m\Phi_N + m\left(\vec{v}\cdot\vec{h}\right)\right] \tag{844}$$

$$= \frac{1}{2}m\vec{v}^2 + q\varphi + m\Phi_N \tag{845}$$

From (842) we also have $m\vec{v} = \vec{p}_{can} - q\vec{A} - m\vec{h}$. Substituting this into the Hamiltonian gives[77]

$$\boxed{H = \frac{1}{2m}\left(\vec{p}_{can} - q\vec{A} - m\vec{h}\right)^2 + q\varphi + m\Phi_N} \tag{846}$$

We can promote the Hamiltonian and canonical momentum to operators using $\hat{p}_{can} = -i\hbar\nabla$ and $\hat{H} = i\hbar\partial_t$. Then acting the Hamiltonian on a wavefunction, $\Psi(\vec{r},t)$, gives

$$\boxed{i\hbar\partial_t\Psi(\vec{r},t) = \left[\frac{1}{2m}\left(-i\hbar\nabla - q\vec{A} - m\vec{h}\right)^2 + q\varphi + m\Phi_N\right]\Psi(\vec{r},t)} \tag{847}$$

This is the Schrödinger equation for a non-relativistic charged particle in an electromagnetic and gravito-electromagnetic (first-order post-Newtonian) fields.

[77]We could also arrive at this result by assuming the Hamiltonian is the total energy, $H = T + V$ with $T = m\vec{v}^2/2$ and $V = q\varphi + m\Phi_{PN}$. Then substituting (842) would lead to the same Hamiltonian we have in (846).

8.3 The Hamiltonian for relativistic charged particles

In (2863) of Appendix M, it is shown that the Lagrangian for relativistic electron-pairs coupled to an electromagnetic field, A_μ, in curved space-time is[78]

$$L = -mc\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} + eA_\mu \dot{x}^\mu \tag{848}$$

where x^μ is parameterized in terms of the coordinate time t, rather than proper time τ. This leads to the Hamiltonian found in (677) as

$$H = c\left(\tilde{g}^{jk} g_{0j} g_{0k} - g_{00}\right)^{1/2}\left[m^2c^2 + \tilde{g}^{jk}(p_j - eA_j)(p_k - eA_k)\right]^{1/2} - c\tilde{g}^{jk} g_{0k}(p_j - eA_j) - ceA_0 \tag{849}$$

with the "spatial inverse metric" given by

$$\tilde{g}^{ik} = g^{ik} - \frac{g^{0i}g^{0k}}{g^{00}} \qquad \text{where} \qquad \tilde{g}^{ik} g_{jk} = \delta^i_j \qquad \text{and} \qquad \tilde{g}^{ik} g_{0k} = \delta^k_i g_{0k} = g_{0i} \tag{850}$$

The weak-field, low-velocity Hamiltonian that is second order in $h^{\mu\nu}$, in (v/c), and also in *products* of $h^{\mu\nu}$ and (v/c) is found in (194) to be

$$\begin{aligned} \underset{(2nd\text{-}order)}{H} \approx\ & mc^2 + \frac{1}{2m}(\pi_k - mch_{0k})^2 - \tfrac{1}{2}mc^2 h_{00} - ceA_0 - \tfrac{1}{8}mc^2(h_{00})^2 - \frac{h_{00}\pi_k^2}{4m} - \frac{h^{jk}\pi_k\pi_j}{2m} \\ & - \frac{(h_{00})^2\pi_k^2}{16m} + ch^{jk}h_{0k}\pi_j + \frac{(h_{0j})^2\pi_k^2}{4m} + \frac{h^{ij}h_{ik}\pi^k\pi_j}{2m} - \frac{x^\sigma\partial_\sigma h^{jk}\pi_k\pi_j}{2m} + \frac{h_{00}h^{jk}\pi_k\pi_j}{4m} \end{aligned} \tag{851}$$

We also show that a Hamiltonian that is second order in $h^{\mu\nu}$ and (v/c) but *first* order in any *products* of $h^{\mu\nu}$ and (v/c) gives

$$\underset{(2nd\text{-}order)}{H} \approx mc^2 + \frac{1}{2m}(\pi_k - mch_{0k})^2 - \frac{1}{2}mc^2 h_{00} - ceA_0 - \frac{1}{8}mc^2(h_{00})^2 \tag{852}$$

We can substitute the gauge-*dependent* vector potential, $\vec{h} = c(h_{01}, h_{02}, h_{03})$, the kinetic momentum in terms of the canonical momentum, $\vec{\pi} = \vec{p} - e\vec{A}$, and the metric component $h_{00} = -2\phi/c^2$ to obtain

$$\underset{(2nd\text{-}order)}{H} \approx mc^2 + \frac{1}{2m}\left(\vec{p} - e\vec{A} - m\vec{h}\right)^2 + m\phi - ceA_0 - \frac{1}{2c^2}m\phi^2 \tag{853}$$

From this we see that the potentials $\vec{A}$ and $\vec{h}$ cause the canonical momentum to transform as

$$\boxed{\vec{p} \implies \vec{p} - e\vec{A} - m\vec{h}} \tag{854}$$

This is the first-order coupling rule for a non-relativistic charged particle in an electromagnetic and *gravito*-electromagnetic field. As expected, this matches the result obtained in (843) which was derived using a completely non-relativistic approach.

[78] We are using e as a *positive* quantity in this expression.

8.4 Coupling rules and the "canonical velocity"

Here we consider the minimal coupling rules in terms of electromagnetic fields so that the kinetic momentum, $\pi_i = p_i - eA_i$ just becomes the canonical momentum, $\pi_i = p_i$. From (851), we see that to lowest order in h_{00}, we have

$$H \sim mc^2 - \frac{1}{2}mc^2 h_{00} \tag{855}$$

This implies that introducing the *scalar* potential, h_{00}, leads to the following *scalar* minimal coupling rule.

$$\boxed{mc^2 \implies mc^2 - \tfrac{1}{2}mc^2 h_{00}} \tag{856}$$

Again from (851), we find that to lowest order in h_{0i} and p_i, we have

$$H \sim \frac{1}{2m}\left(p_i - mch_{0i}\right)^2 \tag{857}$$

This implies that introducing the *vector* potential, h_{0i}, leads to the following *vector* minimal coupling rule.

$$\boxed{p_i \implies p_i - mch_{0i}} \tag{858}$$

Lastly, from (851), we find that to lowest order in h_{ij} and p_i we have[79]

$$H^{TT} \sim \frac{p^2}{2m} - \frac{h_{ij}p^i p^j}{2m} \tag{859}$$

This implies that introducing the *tensor* potential, h_{ij}, leads to the following *tensor* minimal coupling rule.

$$\boxed{p^2 \implies p^2 - p^i p^j h_{ij}} \tag{860}$$

In (801), we found that in the far-field[80], where $h_{00} = 0$, $h_{0i} = 0$, and $h_{ij} = h_{ij}^{\tau\tau}$, the canonical momentum is shown to be $p_j = mv^i g_{ij}$. If we substitute $g_{ij} = \eta_{ij} + h_{ij}^{\tau\tau}$, then we have

$$p_j = mv_j + mv^i h_{ij}^{\tau\tau} \tag{861}$$

This expression implies that we could have a tensor coupling given by $p_j \implies p_j - p^i h_{ij}$. However, notice that there is no term in the Hamiltonian in (851) that involves $p^i h_{ij}$. In fact, because the Hamiltonian is a scalar, then expressions like (861) could only appear squared in the Hamiltonian, such as we have for (858). However, we also do not find terms associated with the square of (861) in the Hamiltonian. Therefore, since we only find $p^i p^j h_{ij}$ in the Hamiltonian, then we conclude that the tensor coupling rule can only be in terms of p^2 and is *not* expressed as

$$p_j \quad \not\Rightarrow \quad p_j + p^i h_{ij} \tag{862}$$

[79] In Appendix G we point out that some papers such as [54] and [55] obtain a Hamiltonian that has a *positive* sign in front of the coupling term. However, in Section 39 we show in detail that this positive sign is an approximation error and the sign should in fact be *negative* as used here.

[80] As shown in (361), we find that Φ, Θ, and Ξ_i satisfy Poisson equations while h_{ij}^{TT} satisfies a wave equation. Therefore we know that in the far-field, all the metric components producing non-radiative fields will fall off as $1/r^2$ while h_{ij}^{TT}, which is a radiative field, will fall off as $1/r$.

In summary, we conclude that to lowest order in $h_{\mu\nu}$ and p^μ, we have the following coupling rules (or canonical transformations) for the scalar, vector, and tensor potentials.

$$\text{Scalar transformation} \qquad mc^2 \quad \Longrightarrow \quad mc^2 - \tfrac{1}{2}mc^2 h_{00} \tag{863}$$

$$\text{Vector transformation} \qquad p_i \quad \Longrightarrow \quad p_i - mch_{0i} \tag{864}$$

$$\text{Tensor transformation} \qquad p^2 \quad \Longrightarrow \quad p^2 - p^i p^j h_{ij} \tag{865}$$

Notice here that that h_{00} couples to the *mass* and transforms the rest energy, mc^2, while h_{0i} couples to the *mass* but transforms the *momentum*, p^i, and lastly h_{ij} couples to the *momentum* but transforms the *kinetic energy*, p^2. This result is consistent with the mass-energy-momentum invariant, $E^2 = m^2c^4 + p^2c^2$, where m is transformed by h_{00}, while $\vec{p}$ is transformed by h_{0i}, and lastly p^2 is transformed by h_{ij}. We can also argue these relationships by considering the four-momentum invariant quantity given by $p^\mu p_\mu = g_{\mu\nu} p^\mu p^\nu$. Expanding the metric gives

$$p^\mu p_\mu = g_{00}\left(p^0\right)^2 + 2g_{0i}p^0p^i + g_{ij}p^ip^j \tag{866}$$

Expressing the metric as a perturbation to flat Minkowski space-time gives

$$p^\mu p_\mu = \left(-1+h_{00}\right)E^2/c^2 + 2h_{0i}\frac{E}{c}p^i + \left(\eta_{ij}+h_{ij}\right)p^ip^j \tag{867}$$

On the right we substitute $E^2/c^2 = p^2 + m^2c^2$.

$$\begin{aligned}
p^\mu p_\mu &= \left(-1+h_{00}\right)\left(p^2+m^2c^2\right) + 2h_{0i}p^i\sqrt{p^2+m^2c^2} + \left(\eta_{ij}+h_{ij}\right)p^ip^j && (868)\\
&= -p^2 - m^2c^2 + h_{00}p^2 + h_{00}m^2c^2 + 2h_{0i}p^i\sqrt{p^2+m^2c^2} + p^2 + h_{ij}p^ip^j && (869)\\
&= -m^2c^2 + h_{00}p^2 + h_{00}m^2c^2 + 2h_{0i}p^i\sqrt{p^2+m^2c^2} + h_{ij}p^ip^j && (870)
\end{aligned}$$

Expanding the square root to first order gives

$$\sqrt{p^2+m^2c^2} = mc\sqrt{1+p^2/m^2c^2} \approx mc\left(1+\frac{1}{2}\frac{p^2}{m^2c^2}\right) = mc + \tfrac{1}{2}p^2/mc \tag{871}$$

Substituting this into (870) and dropping the rest energy term gives

$$p^\mu p_\mu = h_{00}p^2 + h_{00}m^2c^2 + 2h_{0i}p^i\left(mc + \tfrac{1}{2}p^2/mc\right) + h_{ij}p^ip^j \tag{872}$$

Keeping only the lowest order of p^μ for each metric component h_{00}, h_{0i}, h_{ij} and dividing through by $-2m$ gives

$$-\frac{p^\mu p_\mu}{2m} = \frac{1}{2}mc^2 - \tfrac{1}{2}h_{00}mc^2 - h_{0i}p^ic - \frac{h_{ij}p^ip^j}{2m} \tag{873}$$

The three interaction terms are the exact three couplings that we find in $(863)-(865)$. Specifically, we find that to lowest order in $h_{\mu\nu}$ and p^μ, we have the *scalar* potential coupling to the mass, the *vector* potential coupling to the momentum, and the *tensor* potential coupling to the kinetic energy.

The following are some observations concerning the coupling rules found here.

- We can define a "canonical velocity" using $\vec{p}_{can} = m\vec{v}_{can}$. Writing the *vector* coupling in terms of the "canonical velocity" gives $m\vec{v}_{can} = m\vec{v} - m\vec{h}$. Then the "canonical velocity" is simply

$$\vec{v}_{can} = \vec{v} - \vec{h} \tag{874}$$

For a quantum particle in a zero-momentum state (where the canonical momentum is zero), then $\vec{v}_{can} = 0$ and we simply have

$$\vec{v} = \vec{h} \qquad \text{for} \qquad p_{can} = 0 \tag{875}$$

In the case of the *tensor* coupling, using the "canonical velocity" we would write the coupling rule as $m^2 v_{can}^2 = m^2 v^2 - m^2 v^i v^j h_{ij}$. Then the "canonical velocity" is

$$v_{can}^2 = v^2 - v^i v^j h_{ij} \tag{876}$$

For a quantum particle in a zero-momentum state (where the canonical momentum is zero), then $\vec{v}_{can} = 0$ and we have

$$v^2 \quad = \quad v^i v^j h_{ij} \qquad \text{for} \qquad p_{can} = 0 \tag{877}$$

- We can also quantize the couplings found in (863) – (865) by using $\hat{E} = i\hbar\partial_t$ and $\hat{p} = -i\hbar\nabla$. Doing this gives

$$\text{Scalar transformation} \qquad \hat{E} \quad = \quad i\hbar\partial_t \implies i\hbar\partial_t - \tfrac{1}{2}mc^2 h_{00} \tag{878}$$

$$\text{Vector transformation} \qquad \hat{p} \quad = \quad -i\hbar\nabla \implies -i\hbar\nabla - mh_{0i} \tag{879}$$

$$\text{Tensor transformation} \qquad \hat{p}^2 \quad = \quad -\hbar^2\nabla^2 \implies -\hbar^2\nabla^2 - p^i p^j h_{ij} \tag{880}$$

Notice that there is an ambiguity when substituting these relations into $E^2 = m^2c^4 + p^2c^2$. We may wonder whether to use (879) or (880). If we use (879), then we obtain

$$\left(i\hbar\partial_t - mc^2 h_{00}\right)^2 = m^2c^4 + c^2\left(-i\hbar\nabla - mh_{0i}\right)^2 \tag{881}$$

On the other hand, if we use (880), then we obtain

$$\left(i\hbar\partial_t - mc^2 h_{00}\right)^2 = m^2c^4 - c^2\hbar^2\nabla^2 - c^2 p^i p^j h_{ij} \tag{882}$$

For this reason, it is clear that the best approach is to begin with the four-momentum invariant quantity given by $p^\mu p_\mu = g_{\mu\nu} p^\mu p^\nu$ and then expand the metric to obtain the result found in (866). In that result, we find that *both* h_{0i} and h_{ij} appear.

- The difference between (864) and (865) implies that there is an important difference between a Meissner effect for the *vector* potential and a Meissner effect for the *tensor* potential (associated with gravitational waves). In formulating a Meissner effect for the *tensor* potential, we can not use the standard *magnetic* Meissner effect formulation (where $p_i \implies p_i - eA_i$) as a model because the magnetic effect involves a *vector* transformation. Therefore, unlike DeWitt [42], we don't have the luxury of simply saying, "*All of the apparatus of the BCS theory may be applied to this Hamiltonian, with the result that the Meissner effect implies the vanishing of the vector*

$$\vec{G} = \left(e\nabla \times \vec{A}\right) + \left(m\nabla \times \vec{h}\right) \tag{883}$$

inside the superconductor." Notice that DeWitt does not go into the details of why we can simply treat the Lense-Thirring field like we do the magnetic field for a superconductor, although the logic is certainly reasonable. It relies on the strong similarity between the magnetic field and the Lense-Thirring field due to both of them being rotational vector fields. This similarity is especially compelling since the Cooper pairs carry *charge* and *mass* and therefore the Cooper-pair current must generate both a magnetic field and a *gravito*-magnetic (Lense-Thirring) field.[81]

Since the magnetic field and the gravito-magnetic field are *vector* fields, it is reasonable to compare them in an effort to determine the behavior of the gravito-magnetic field inside a superconductor. In fact, because the magnetic case has been studied and modeled extensively (by use of the London equations, the Ginzburg-Landau formulation, and ultimately the BCS theory), it is possible for us to have some intuition concerning the response of the superconductor to a *gravito*-magnetic field. At the very least, we know that there must be either a diamagnetic effect (which is taken to the extreme limit with the Meissner effect) or there must be a paramagnetic effect (which would be essentially the *opposite* of a Meissner-like effect).

However, for the case of gravitational waves which have a *tensor* nature, we do not have an electromagnetic analog that can be used as a starting place for understanding the response of the superconductor. As a result, we must attempt to formulate an analog to the London equations (as shown in Part XII. of this paper) or we must even directly employ the BCS theory to demonstrate that *tensor* fields are expelled from superconductors. Specifically, we must determine how the coupling rule given for h_{ij} in (865) would apply.

- We have derived coupling rules using a Hamiltonian formulation for massive, scalar particles in curved space-time. It is also possible to derive coupling rules using other formulations such as those shown by Speliotopoulos and Chiao in $[60]$ – $[62]$.

[81] It should be noted that there is a subtle but very important difference between the magnetic field and the gravito-magnetic (or Lense-Thirring). The gravitational Ampere law in (361) contains a negative sign that does *not* appear for the electromagnetic Ampere law. It is shown in a later section that this negative sign changes the associated gravitational London equations and consequently does *not* lead to a penetration depth or Meissner-like effect.

8.5 The quantized ideal fluid stress tensor

The ideal fluid stress tensor is given by

$$T^{\mu\nu}_{(ideal\ fluid)} = \left(\rho + P/c^2\right) u^\mu u^\nu + Pg^{\mu\nu} \tag{884}$$

where u^μ is the four-velocity and P is the pressure. Terms of order $h_{\mu\nu}T^{\sigma\rho}$ can be neglected so that there is no self-coupling of the source $(T^{\mu\nu})$ with its own gravitational field $(h^{\mu\nu})$. Then the components of the stress tensor can be written as[82]

$$T^{00} = \rho c^2 \tag{885}$$
$$T^{0i} = \left(\rho + P/c^2\right) cv_i \tag{886}$$
$$T^{ij} = \left(\rho + P/c^2\right) v_i v_j + P\eta_{ij} \tag{887}$$

For simplicity, the case of a uniform mass density can be considered so that $\rho = M/V$, where M is the total mass, and V is the total volume of the material described by the stress tensor. If the material is composed of N quantum particles, each of mass m_0, then $M = Nm_0$ and therefore

$$\rho = \frac{Nm_0}{V} \tag{888}$$

Substituting this mass density into the stress tensor components above gives

$$T^{00} = \frac{Nm_0c^2}{V} \tag{889}$$
$$T^{0i} = \left(\frac{N}{V} + \frac{P}{m_0c^2}\right) cm_0 v_i \tag{890}$$
$$T^{ij} = \left(\frac{N}{m_0V} + \frac{P}{m_0^2c^2}\right) m_0^2 v_i v_j + P\eta_{ij} \tag{891}$$

The usual quantization method can be applied in order to promote the stress tensor to a quantum mechanical operator. Since there is no minimal coupling involved here, then the kinetic momentum and canonical momentum are the same, $m_0\vec{v} = \vec{p}_{can}$. Then quantizing the canonical momentum gives the usual gradient operator, $\hat{p}_i = -i\hbar\partial_i$. In this context, the mass density and pressure can be simply considered numerical parameters. (This choice will be discussed further in a later section.) The *quantized* stress tensor components are therefore

$$\hat{T}^{00} = \frac{Nm_0c^2}{V} \tag{892}$$
$$\hat{T}^{0i} = -i\hbar c\left(\frac{N}{V} + \frac{P}{m_0c^2}\right)\partial_i \tag{893}$$
$$\hat{T}^{ij} = -\frac{\hbar^2}{m_0}\left(\frac{N}{V} + \frac{P}{m_0c^2}\right)\partial_i\partial_j + P\eta_{ij} \tag{894}$$

To simplify these expressions, the following constant can be defined

$$n \equiv \frac{N}{V} + \frac{P}{m_0c^2} \tag{895}$$

[82]To first order in the metric, upper and lower spatial indices are effectively the same.

Inserting (888) and (895) into the stress tensor components gives the following "semi-classical" stress tensor components.

$$\boxed{\hat{T}^{00} = \rho c^2, \qquad \hat{T}^{0i} = -icn\hbar\partial_i, \qquad \hat{T}^{ij} = -\frac{n\hbar^2}{m_0}\partial_i\partial_j + P\eta_{ij}} \tag{896}$$

Below are some observations concerning these quantized stress tensor components.

- Each component of the stress tensor has a different mathematical classification. $\hat{T}^{00}$ is simply a numerical parameter, $\hat{T}^{0i}$ is a *vector* quantum operator, and $\hat{T}^{0i}$ is a *tensor* quantum operator. The physical significance of these mathematical properties will be discussed in a later section.
- $\hat{T}^{00}$ is just a numerical parameter since it consists of only the mass density of the system which is considered here to be a classical value, not subject to the quantum state of the system. It is not considered an "observable" in the quantum mechanical sense because it does not have an operator and eigenvalues associated with it. As an alternative, one may choose to treat the mass density as well as the pressure as linear operators with corresponding eigenvalues such that

$$\hat{\rho}\Psi = \rho\Psi \qquad \text{and} \qquad \hat{P}\Psi = P\Psi \tag{897}$$

  This choice increases the complexity of the information that must be provided by the state of the system and opens the possibility of superposition states for the mass density, the pressure, and the momentum. However, in this treatment, the simpler choice was made to consider the pressure to be effectively a classical, macroscopic quantity (perhaps given by classical electromagnetic interactions of the particles). The mass density is also considered to be a classical, macroscopic quantity given simply in terms of the volume of the material, the number of particles making up the material, and the mass of each particle, as shown in (888). This means that the wave function only needs to give information concerning the *momentum*-state of the particles.

- The $\hat{T}^{0i}$ component plays a similar role to the standard momentum operator in standard quantum mechanics. However, the factor of n has the units of number volume density (the number of particles per unit volume). It is interesting to note from (895) that n is also directly affected by the pressure. This might be physically interpreted as the pressure being an effective increase (or decrease) in the density of quantum particles in the system. In other words, when the pressure is increased (or decreased) due to an external field or internal particle interactions, it is equivalent to having a number of quantum particles added (or removed) from the volume of the system. However, since P is divided by m_0c^2, then this pressure only becomes significant when it is on the order of the rest mass energy of a quantum particle.
- The $\hat{T}^{ij}$ component plays a similar role to a kinetic energy operator. However, its tensor nature is obviously unique. Specifically, the two derivatives with differing indices form a quantum operator not normally found in quantum mechanics. Note that along the diagonal, (where $i = j$), the stress becomes

$$\hat{T}^{ii} = -\frac{n\hbar^2}{m_0}\nabla^2 + P \tag{898}$$

  This expression bears a further resemblance to the kinetic energy operator. However, the factor of n in front and the additional term involving the pressure indicates that the operator is again affected by the particle number density as well as the pressure in the system. In this case we find that the pressure plays a more critical role since it doesn't appear only in n (where it suppressed by a factor of c^2) but it also appears as an additional scalar term. This result in contrast to the *off*-diagonal elements of $\hat{T}^{ij}$ which are given by

$$\hat{T}^{ij}_{(i\neq j)} = -\frac{n\hbar^2}{m_0}\partial_i\partial_j \tag{899}$$

  In this case, the additional scalar pressure term does not appear and the kinetic energy is due to *stresses* (the product of orthogonal momenta).This is an aspect of continuum mechanics which does not normally appear in a quantum mechanical context.

- To enforce symmetry of the stress tensor, the only allowable wavefunction solutions are those for which the operators ∂_i and ∂_j commute. If these operators do not commute (that is, switching the order of differentiation gives a different result), then the stress tensor will no longer be symmetric. Fortunately, the vast majority of functions satisfy this condition regardless.

- An expression similar to $\hat{T}^{ij}$ appears in equation (4.4) of [55] where they write an interaction Hamiltonian in terms of the metric perturbation and stress tensor as[83]

$$H_I = \frac{1}{2} h_{\mu\nu} T^{\mu\nu} \tag{900}$$

  They proceed to quantize the stress tensor and apply it to the case of gravitational waves where they obtain a result which is effectively a simple harmonic oscillator. Also, in equation (5.6) of [54], which is a similar paper by the same authors, the interaction Hamiltonian is written more explicitly as an operator with the following form

$$H_I = \frac{\hbar^2}{2m} h_{jk} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \tag{901}$$

  This serves as an example of another case where the classical stress can be formulated into a quantum mechanical tensor operator similar to $\hat{T}^{ij}$ given in (896).

- To describe the gravitational fields of the system, it will be necessary to apply a semi-classical equation for gravity such as

$$G_{\mu\nu} = \kappa \langle\Psi| \hat{T}_{\mu\nu} |\Psi\rangle \tag{902}$$

  This introduces difficulties since the Einstein tensor is a *non-linear* function of the metric while the wave function and all operators obey standard *linear* quantum mechanical commutation relations. If one uses a semi-classical version of *linearized* General Relativity, such as

$$\Box \bar{h}_{\mu\nu} = -2\kappa \langle\Psi| \hat{T}_{\mu\nu} |\Psi\rangle \tag{903}$$

  (which is in terms of the trace-reversed metric perturbation in the harmonic gauge), or a semi-classical equation for gravity in the Newtonian limit, such as

$$\nabla^2 \Phi_N = 4\pi G \langle\Psi| \hat{T}_{00} |\Psi\rangle \tag{904}$$

  there is still a problem with interpreting the meaning of a quantum measurement. In the common Copenhagen viewpoint, the effectively instantaneous "collapse" of the wave function upon making a quantum measurement will introduce a discontinuity in the gravitational field as it correspondingly "collapses" from a superposition state to a measured state.[84]

This indicates that the mathematical and conceptual roadblocks of formulating field equations using the quantized fluid stress tensor could make the quantized stress tensor unpractical. Nevertheless, for the purpose of describing a Cooper pair superfluid, it will be shown in the next section that the quantized stress tensor can be used to formulate quantum dispersion relations. These relations reveal what conditions must be imposed on the stress tensor to satisfy the fact that the Cooper pairs are in a zero-momentum eigenstate. It is found that this requires the pressure terms in the stress tensor to vanish. Also, in a later section it is shown that setting the expectation value of the *canonical* momentum to zero for a superconductor makes it possible to couple the gravitational field to the *kinetic* momentum and hence express the stress tensor as a *classical* stress tensor. Then it is possible to return to describing the superfluid using General Relativity and hence to describe the resulting gravitational fields.

[83] Note that for the case of non-interacting particles, the stress tensor is $T^{\mu\nu} = \rho u^\mu u^\nu$ and the Lagrangian is found to be identical to the Hamiltonian as given in (900). To show this, the Lagragian, $L = \frac{1}{2} h_{\mu\nu} \rho u^\mu u^\nu$, can be used to find the canonical momentum, $p^\mu_{can} = \partial L / \partial u^\mu = h_{\mu\nu} \rho u^\mu u^\nu$. Then the Hamiltonian is $H = u^\mu p_\mu - L = \frac{1}{2} h_{\mu\nu} \rho u^\mu u^\nu$ which is identical to the Lagrangian.

[84] For more discussion of these topics, see Wald [57], (pp. 382-383, 410-411).

8.6 Matter wave equations for the quantized ideal fluid stress tensor

The linearized conservation law for the stress-energy-momentum tensor is given by

$$\partial^{\nu} T_{\mu\nu} = 0 \tag{905}$$

Summing over ν gives

$$\partial^{0} T_{\mu 0} + \partial^{i} T_{\mu i} \quad = \quad 0 \tag{906}$$

For $\mu = 0$ we have the following mass-momentum continuity equation.

$$\partial^{0} T_{00} + \partial^{i} T_{0i} \quad = \quad 0 \tag{907}$$

Inserting T^{00} and T^{0i} from (896) gives[85]

$$-\tfrac{1}{c}\partial_t \left(\rho c^2\right) + \partial_i \left(icn\hbar\partial_i\right) \quad = \quad 0 \tag{908}$$

$$-\rho\partial_t + in\hbar\nabla^2 \quad = \quad 0 \tag{909}$$

Multiplying through by $-i\hbar/\rho$ and acting on a wavefunction, $\Psi(\vec{x},t)$, gives

$$\boxed{i\hbar\partial_t\Psi(\vec{x},t) = -\frac{n}{\rho}\hbar^2\nabla^2\Psi(\vec{x},t) \qquad \textit{First wave equation}} \tag{910}$$

This is effectively a time-dependent "Schrödinger-like equation" for a quantum fluid. Notice that the only difference between (910) and the Schrödinger equation is a factor of n/ρ replacing the usual factor of $1/2m$. Therefore, it is expected that the dispersion relation from the Schrödinger equation[86] $\left(\omega = \hbar k^2/2m\right)$ will likewise have $1/2m$ replaced with n/ρ. To show this explicitly, a plane wave solution can be used which is given by $\Psi(\vec{x},t) = Ae^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$. This leads to

$$\left(\omega\hbar - \frac{n}{\rho}\hbar^2 k^2\right)\Psi(\vec{x},t) = 0 \tag{911}$$

This equation is satisfied when $\Psi(\vec{x},t) = 0$ or when $\left(\omega\hbar - \dfrac{n}{\rho}\hbar^2 k^2\right) = 0$. Solving for ω leads to the following dispersion relation.

$$\boxed{\omega = \frac{n\hbar}{\rho}k^2 \qquad \textit{First dispersion equation}} \tag{912}$$

Returning to (906) and letting $\mu = i$ gives a momentum-stress conservation equation.

$$\partial^{0} T_{i0} + \partial^{j} T_{ij} \quad = \quad 0 \tag{913}$$

[85] Recall from Section 25 that if we neglect the self-coupling of the gravitational field back on the sources, then we neglect $h_{\mu\nu}T^{\rho\sigma}$ and therefore we raise and lower indices using $g_{\mu\nu} \approx \eta_{\mu\nu}$. This means that $T^{00} = T_{00}$, $T^{0i} = -T_{0i}$, and $T^{ij} = T_{ij}$.

[86] The dispersion relation for the Schrödinger equation is easily found by simply inserting $E = \hbar\omega$ and $p = \hbar k$ into $E = p^2/2m$ which gives $\omega = \hbar k^2/2m$.

Inserting T^{0i} and T^{ij} from (896) gives

$$-\frac{1}{c}\partial_t\left(icn\hbar\partial_i\right)+\partial_j\left(-\frac{n\hbar^2}{m_0}\partial_i\partial_j+P\eta_{ij}\right) = 0 \tag{914}$$

$$-in\hbar\partial_i\partial_t-\frac{n\hbar^2}{m_0}\nabla^2\partial_i+\left(\eta_{ij}\partial_jP+P\partial_i\right) = 0 \tag{915}$$

If the pressure within the material remains uniform, then $\partial_jP=0$ which gives

$$-in\hbar\partial_i\partial_t = \left(\frac{n\hbar^2}{m_0}\nabla^2-P\right)\partial_i \tag{916}$$

Allowing this expression to act on a wavefunction, $\Psi\left(\vec{x},t\right)$, gives

$$\boxed{-in\hbar\partial_i\partial_t\Psi\left(\vec{x},t\right)=\left(\frac{n\hbar^2}{m_0}\nabla^2-P\right)\partial_i\Psi\left(\vec{x},t\right) \qquad \textit{Second wave equation}} \tag{917}$$

This is an *additional* time-evolution wave equation which also describes the quantum ideal fluid. Note that for a given quantum state of the quantum ideal fluid, the wave function describing the state, $\Psi\left(\vec{x},t\right)$, must simultaneously satisfy the wave equation in (910) as well as the wave equation in (917). Once again, a plane wave solution given by $\Psi\left(\vec{x},t\right)=Ae^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$ leads to

$$-in\hbar\left(ik\right)\left(-i\omega\right)\Psi\left(\vec{x},t\right) = \left(\frac{n\hbar^2}{m_0}\left(ik\right)^2-P\right)\left(ik\right)\Psi\left(\vec{x},t\right) \tag{918}$$

$$-n\hbar k\omega\Psi\left(\vec{x},t\right) = \left(-\frac{n\hbar^2}{m_0}k^2-P\right)k\Psi\left(\vec{x},t\right) \tag{919}$$

$$\left(n\hbar k\omega-\frac{n\hbar^2}{m_0}k^3-Pk\right)\Psi\left(\vec{x},t\right) = 0 \tag{920}$$

This equation is satisfied when $\Psi\left(\vec{x},t\right)=0$ or when $\left(n\hbar k\omega-\frac{n\hbar^2}{m_0}k^3-Pk\right)=0$ which gives

$$\boxed{k\omega-\frac{\hbar}{m_0}k^3-\frac{P}{n\hbar}k=0 \qquad \textit{Second dispersion equation}} \tag{921}$$

Note that $k=0$ is a solution to (921). For $k\neq 0$, it is possible to divide through by k which gives

$$\boxed{\omega=\frac{\hbar}{m_0}k^2+\frac{P}{n\hbar} \qquad \text{for} \qquad k\neq 0} \tag{922}$$

It is also possible to substitute (912) into (921) to obtain

$$\frac{n\hbar}{\rho}k^3-\frac{\hbar}{m_0}k^3-\frac{P}{n\hbar}k = 0 \tag{923}$$

$$\left(\frac{m_0n-\rho}{\rho m_0}\right)n\hbar^2k^3-Pk = 0 \tag{924}$$

Again, $k = 0$ is a solution. For $k \neq 0$, it is possible to divide through by k to obtain

$$\boxed{k^2 = \frac{P\rho m_0}{n\hbar^2 (m_0 n - \rho)} \qquad \text{for} \qquad k \neq 0} \tag{925}$$

Now substituting this result into (922) gives

$$\omega = \frac{P\rho}{n\hbar (m_0 n - \rho)} + \frac{P}{n\hbar} \tag{926}$$

$$\omega = \frac{P\rho + P(m_0 n - \rho)}{n\hbar (m_0 n - \rho)} \tag{927}$$

$$\boxed{\omega = \frac{P m_0}{\hbar (m_0 n - \rho)} \qquad \text{for} \qquad k \neq 0} \tag{928}$$

From (925) and (928), it is found that k and ω can each be expressed independently in terms of the macroscopic bulk properties ρ, P, V and the microscopic quantity m_0. (Recall that n is given in (895) in terms of ρ, V, m_0 and P.) This means that the quantum wave dispersion relationship in (922) is really a relationship between the macroscopic bulk properties of the material. From (910) it was found that $\partial_t \Psi(\vec{x},t) = \frac{in\hbar}{\rho} \nabla^2 \Psi(\vec{x},t)$. Inserting this into (917) results in a single, combined time-evolution equation.

$$\frac{n^2\hbar^2}{\rho} \partial_i \nabla^2 \Psi(\vec{x},t) = \left(\frac{n\hbar^2}{m_0}\nabla^2 - P\right) \partial_i \Psi(\vec{x},t) \tag{929}$$

$$\left[n\hbar^2 \left(\frac{n}{\rho} - \frac{1}{m_0}\right) \nabla^2 + P\right] \partial_i \Psi(\vec{x},t) = 0 \tag{930}$$

$$\boxed{\left[n\hbar^2 \left(\frac{m_0 n - \rho}{\rho m_0}\right) \nabla^2 + P\right] \partial_i \Psi(\vec{x},t) = 0 \qquad \textit{Combined time-evolution equation}} \tag{931}$$

This single equation combines the results from both (910) and (917). Unlike the "Schrödinger-like equations" in (910) and (917), the equation in (931) is no longer a *wave* equation since it contains only *spatial* derivatives and no *time* derivatives. It also does not contain any imaginary terms and therefore is not a complex equation. Inserting a plane wave solution given by $\Psi(\vec{x},t) = Ae^{i(\vec{k}\cdot\vec{x} - \omega t)}$ into (931) gives

$$\left[n\hbar^2 \left(\frac{m_0 n - \rho}{\rho m_0}\right) \left(-ik^3\right) + P(ik)\right] \Psi(\vec{x},t) = 0 \tag{932}$$

This equation is satisfied when $\Psi(\vec{x},t) = 0$ or when

$$\left(\frac{m_0 n - \rho}{\rho m_0}\right) n\hbar^2 k^3 - Pk = 0 \tag{933}$$

This matches the result obtained in (924) and is therefore consistent with all the other results obtained for k and ω. However, it is interesting to note that because the single, combined time-evolution equation in (931) does not contain any *time* derivatives, it therefore cannot yield a dispersion equation involving ω and k. Such a relationship requires making use of a wave equation from either (910) or (917).

The ideal gas limit

Using (895), n can be written as

$$n = \frac{Nm_0c^2 + PV}{Vm_0c^2} \tag{934}$$

where N is the number of quantum particles in the material, m_0 is the mass of each particle, V is the total volume of the material, and P is the pressure in the material (assumed to be uniform). In the limit of an ideal gas (*non*-interacting particles), it is expected that $Nm_0c^2 >> PV$ since the total rest mass energy of the particles (Nm_0c^2) will be far greater than the thermal energy due to collisions (PV). In that case, the expression for n reduces to just

$$n_{ideal\ gas} \approx N/V \tag{935}$$

Then dividing by ρ and using $M = Nm_0$ gives

$$\frac{n_{ideal\ gas}}{\rho} \approx \frac{N/V}{M/V} = \frac{N}{Nm_0} = \frac{1}{m_0} \tag{936}$$

Therefore $n_{ideal\ gas} = \rho/m_0$. Substituting this into (933) eliminates the first term and leaves $Pk = 0$. This implies that either the pressure is zero or the particles have zero momentum.[87] Here the case of zero pressure is considered. (Later, the case of zero momentum is considered.) For zero pressure, (922) gives the following dispersion relation

$$\omega_{ideal\ gas} = \frac{\hbar k^2_{ideal\ gas}}{m_o} \tag{937}$$

which is essentially the Schrödinger dispersion relation (aside from a factor of 2 absent in the denominator).[88] Note that attempting to insert this result into (921) to obtain an expression for k will cause the equation to vanish. Likewise, it not possible to express ω independently of k. This indicates that in the ideal gas limit with vanishing pressure, k and ω can no longer be expressed independently in terms of the macroscopic bulk properties ρ, V. Rather, the wave dispersion relationship in (937) uniquely determines k and ω only with respect to one another.[89]

Using $P = 0$ and $n = \rho/m_0$ from (936) makes the quantum wave equations from (910) and (917) become, respectively,

$$i\hbar\partial_t\Psi(\vec{x},t) \quad = \quad -\frac{\hbar^2}{m_0}\nabla^2\Psi(\vec{x},t) \qquad \text{and} \qquad -i\hbar\partial_i\partial_t\Psi(\vec{x},t) = \frac{\hbar^2}{m_0}\nabla^2\partial_i\Psi(\vec{x},t) \tag{938}$$

[87]Note that it would be an error to substitute $n_{ideal\ gas} = \rho/m_0$ into (925) or into (928) since this would lead to the conclusion that k and ω diverge. It should be recognized that $(m_0n - \rho)$ was originally in the *numerator* of the left side of (933).

[88]Notice that multiplying both sides of (937) by $\hbar$ gives $\hbar\omega_{ideal\ gas} = \hbar^2k^2_{ideal\ gas}/m_0$. Using $E = \hbar\omega$ and $p = \hbar k$ makes this become $E = p^2/m_0$ which is similar to the kinetic energy of a free particle. In that sense, removing the pressure term from the fluid makes the fluid become a "free fluid" analogous to a free particle that has no coupling to any external force (pressure).

[89]This is analogous to the dispersion equation for light which reduces to just $\omega = kc$ in vacuum and therefore no longer has any dependence on the properties of the medium it propagates through (such as the index of refraction, $n = v/c$).

It can be observed immediately that these two equations are effectively identical except for a derivative, ∂_i, on each term in the second equation. This means that in the quantum coherent limit, the dynamics are completely governed by a single Schrödinger-like equation. This is expected since the Schrödinger equation describes the dynamics of a scalar quantum particle (in the non-relativistic limit). Although the Schrödinger-like equation found here is extremely similar to the Schrödinger equation in its mathematical form, there are important physical and conceptual differences that will be described in a later section.

The quantum coherent limit and the possibility of non-thermal pressure on a superfluid

In the limit of quantum coherence, such as when electrons form Cooper pairs in a superconductor, the particles fall into the same quantum state and therefore become completely *non*-interacting. They behave as effectively a single particle in what may be referred to as a "zero-momentum eigenstate." Observe from (921) that $k = 0$ is in fact a valid solution and hence the momentum, $p = \hbar k$, is zero. Likewise, inserting $k = 0$ into (912) requires that $\omega = 0$ and therefore the energy, $E = \hbar\omega$, is also zero. Provided the temperature of the superconductor is kept below the critical temperature (so that thermal energy does not exceed the BCS energy gap), then the Cooper pairs remain in this fermionic superfluid state.

Note that it was *not* necessary to set the pressure to zero in this limit. This can be observed from (933) where it is evident that setting $k = 0$ still allows $P \neq 0$ as an allowable condition. However, this pressure cannot be interpreted as an *internal thermal* pressure associated with particle collisions. Rather, the pressure can only be due to a response to an *external* force. For example, since the Cooper pairs are composed of electrons which are negatively charged, then it is possible that an *electromagnetic* pressure could be produced in the superfluid by introducing an appropriate electromagnetic field.

To further illustrate this possibility, consider the case of a superconductor in the shape of a hollow spherical shell. If a charge was placed at the center of the hollow shell, the response of the Cooper pairs would be repulsion from (or attraction to) the charge. As a result there would be a radial pressure on the superfluid (inward or outward depending on the sign of the charge placed at the center). Therefore, the superfluid would experience a net pressure, not due to any *internal thermal* degrees of freedom, but due to an *external electrostatic* Coulomb force. Similarly, because the Cooper pairs possess mass as well as charge, one could consider the same situation of a hollow spherical superconductor but now with a *mass* placed at the center. Again, the response of the Cooper pairs would be an attraction toward the center and hence there would be an inward directed pressure due to the Newtonian gravitational force.

Comparison of the Schrödinger equation and *Schrödinger-like* equation

In (910), a *Schrödinger-like* equation was obtained which is given by

$$i\hbar\partial_t\Psi(\vec{x},t) = -\frac{n}{\rho}\hbar^2\nabla^2\Psi(\vec{x},t) \tag{939}$$

This result obviously bears a striking resemblance to the Schrödinger equation. It was also found in (938) that in the limit with no pressure (relativistic dust), the *Schrödinger-like* equation became

$$i\hbar\partial_t\Psi(\vec{x},t) = -\frac{\hbar^2}{m_0}\nabla^2\Psi(\vec{x},t) \tag{940}$$

This result is now almost identical to the Schrödinger equation except for a factor of 2 missing in the denominator of the right side. However, it is important to recognize that although the *mathematical form* of the equation is extremely similar to the Schrödinger equation, the *physical interpretation* is significantly different. First recall that (910) was obtained using the mass-momentum continuity equation in (907) which is essentially

$$\partial_t\rho + \nabla\cdot\vec{J}_m = 0 \tag{941}$$

where $\vec{J}_m = \rho \vec{v} = \frac{M}{V} \vec{v}$ is the momentum density of an element of the fluid with mass M and volume V. If there is no minimal coupling, then the kinetic momentum and canonical momentum are the same, $M\vec{v} = \vec{p}_{can}$. Promoting the canonical momentum to a quantum operator gives $M\vec{v} = -i\hbar\nabla$. Therefore $\vec{J}_m = -\frac{1}{V} i\hbar\nabla$ and (941) becomes

$$\partial_t \rho - \frac{1}{V} i\hbar\nabla^2 = 0 \tag{942}$$

Multiplying through by $i\hbar/\rho$ and acting on a wavefunction, $\Psi(\vec{r},t)$ gives

$$i\hbar\partial_t \Psi(\vec{x},t) = -\frac{1}{\rho V} \hbar^2 \nabla^2 \Psi(\vec{x},t) \tag{943}$$

If the mass density of the element of fluid is uniform, then $\rho = M/V$ which leads to

$$i\hbar\partial_t \Psi(\vec{x},t) = -\frac{\hbar^2}{M} \nabla^2 \Psi(\vec{x},t) \tag{944}$$

Again, this is extremely similar to the Schrödinger equation except that the Schrödinger equation has a factor of 2 in the denominator on the right side. The factor of 2 is easily observed to be the result of quantizing the non-relativistic kinetic energy of a particle, $E = p^2/2m$, so that quantizing gives

$$i\hbar\partial_t \Psi(\vec{x},t) = -\frac{\hbar^2}{2m} \nabla^2 \Psi(\vec{x},t) \tag{945}$$

Therefore it is evident that the Schrödinger equation is a quantization of the *non-relativistic energy* of a *free particle* in the context of *discrete particle* mechanics. It is effectively the equation of motion of a test mass. On the other hand, the *Schrödinger-like* equation is a quantization of the *mass continuity*, $\dot{\rho} = -\nabla \cdot \vec{J}_m$, for a volume element of a quantum *fluid* in the context of *continuous fluid* mechanics. It is effectively a constraint on the time-evolution of quantum matter in order to preserve local mass conservation. In fact, once can observe that the mass m in the Schrödinger equation (945) is the *discrete*, point mass of a moving quantum particle while the mass M in the *Schrödinger-like* equation (944) is representing the entire mass of a *continuous* quantum fluid volume element where there are internal mass-current fluxes that obey (941).

Elaborating further on this distinction, it can also be pointed out that the *wavefunction* in the Schrödinger equation is the wavefunction of a single particle with mass m. However, in the case of the *Schrödinger-like* equation in (944), the wavefunction is describing the entire quantum superfluid as an extended quantum object consisting of an ensemble of particles that are effectively all in a single particle state. For the case of the *Schrödinger-like* equation in (940), the equation is expressed in terms of m_0 (for a single quantum particle), however, the wavefunction still necessarily involves the *entire* ensemble of particles that are together in a single coherent state. The only reason that the equation is expressed in terms of m_0 (a discrete parameter) is because of the special case of no pressure and uniform mass density which makes it possible to express n/ρ, a continuous parameter in (939), in terms of m_0, as shown in (936). However, this change in the parameter from n/ρ to m_0 does not change the physical representation of the wavefunction as describing the *entire* superfluid, not just a single quantum particle.

Furthermore, the Schrödinger equation is applicable only in the *non*-relativistic limit of slow-moving test particles so that the kinetic energy is $p^2/2m$. (In fact, this is obviously the reason for the factor of 2 in the Schrödinger equation which does not appear in the *Schrödinger-like* equation.) In contrast to this, the *Schrödinger-like* equation is *not* limited to any non-relativistic limit since the mass-momentum continuity equation does not involve such a limit. Rather, it comes from the stress tensor conservation law (when self-coupling of the stress tensor to the gravitational field is neglected). This means that the limit imposed is not $v^2/c^2 << 1$ but rather $\left|h_{\mu\nu}\right| << 1$ so that $\left(\partial_\gamma h_{\mu\nu}\right) T^{\rho\sigma}$ is negligible. In this approximation, the conservation law given by the full covariant derivative,

$$\nabla_\nu T^{\mu\nu} = \partial_\nu T^{\mu\nu} + \Gamma^\nu_{\nu\sigma} T^{\sigma\mu} + \Gamma^\mu_{\nu\sigma} T^{\nu\sigma} = 0 \tag{946}$$

becomes simply

$$\partial_\nu T^{\mu\nu} = 0 \tag{947}$$

Lastly, to further highlight this difference, one can consider the stark contrast between the Schrödinger equation and the *Schrödinger-like* equation when the particular approximations associated with each equation are removed. For the case of the Schrödinger equation, this means removing the non-relativistic limit and generalizing to a fully relativistic equation of motion which would be the Klein-Gordon equation. This can be derived by quantizing $E^2 = m^2c^4 + p^2c^2$ which leads to

$$-\hbar^2\partial_t^2\Psi(\vec{x},t) = \left(m^2c^4 - \hbar^2c^2\nabla^2\right)\Psi(\vec{x},t) \tag{948}$$

In contrast to this, for the case of the stress tensor conservation law, removing the weak-field approximation and generalizing to the full covariant derivative in (946) gives

$$\left(\partial_\nu \hat{T}^{\mu\nu} + \Gamma^\nu_{\nu\sigma}\hat{T}^{\sigma\mu} + \Gamma^\mu_{\nu\sigma}\hat{T}^{\nu\sigma}\right)\Psi(\vec{x},t) = 0 \tag{949}$$

It is obvious that quantizing the conservation law (in terms of the full covariant derivative) does not produce anything remotely close to the fully relativistic quantum wave equation given by the Klein-Gordon equation. It is only when comparing the *non*-relativistic limit of the Klein-Gordon equation (which gives the Schrödinger equation) and also quantizing the *weak-field limit* of the stress tensor conservation law, then it is found that the two results share a surprising similarity.

Summary

In summary, one can observe that *before* taking the ideal gas limit with vanishing pressure, there are *two* time-evolution equations of state that come out of the conservation of the quantized stress tensor, $\partial_\nu \hat{T}^{\mu\nu} = 0$. This was a consequence of the *semi-classical* approach of quantizing the momentum (and Hamiltonian) while preserving the pressure as a classical, bulk property. The reason for two equations, (910) and (917), was due to the fact that $\partial_\nu T^{\mu\nu} = 0$ is a *tensor* relationship which can be decomposed into *two* equations: a mass-momentum continuity equation and a momentum-stress continuity equation. However, it was found is that in the quantum coherent limit, the zero-momentum state of the Cooper pairs requires a vanishing of the pressure and therefore the *tensor* conservation law (which gave *two* time-evolution equations) reduces to a *single scalar* conservation law, namely, a Schrödinger-like equation. Nonetheless, although the Schrödinger equation and the Schrödinger-like equation have a very similar *mathematical* form, they actually describe completely different *physical systems*.

8.7 The Klein-Gordon equation in curved space-time

Lastly, we consider the Klein-Gordon equation in curved space-time as another example of the coupling of gravity to a scalar quantum field. The Klein-Gordon equation can be found from the relativistic energy

$$E^2 = p^2c^2 + m^2c^4 \tag{950}$$

by promoting the energy to a quantum mechanical Hamiltonian operator and promoting the momentum to a canonical momentum operator,

$$E \quad \Longrightarrow \quad \hat{H} = -i\hbar\partial_t \qquad\qquad \vec{p} \quad \Longrightarrow \quad \hat{p} = -i\hbar\nabla \tag{951}$$

Substituting these into (950) and acting the equation on a state φ gives

$$(-i\hbar\partial_t)^2\varphi \quad = \quad (-i\hbar\nabla)^2 c^2\varphi + m^2c^4\varphi \tag{952}$$

$$-\hbar^2\partial_t^2\varphi + \hbar^2c^2\nabla^2\varphi - m^2c^4\varphi \quad = \quad 0 \tag{953}$$

Dividing by $c^2\hbar^2$, using $k_c = mc/\hbar$ as the Compton wave vector, and combining the space and time derivatives into a 4-derivative, $\partial_\mu = \left(\frac{1}{c}\partial_t, \nabla\right)$, gives

$$\partial^\mu\partial_\mu\varphi - k_c^2\varphi \quad = \quad 0 \tag{954}$$

The Klein-Gordon equation can be embedded in curved space-time by promoting the partial derivative to covariant derivative. When acting on an arbitrary vector V_ν, we have

$$\partial_\mu V_\nu \qquad \Longrightarrow \qquad \nabla_\mu V_\nu = \partial_\mu V_\nu + \Gamma^\sigma_{\mu\nu}V_\sigma \tag{955}$$

Then (954) becomes $\nabla_\mu\nabla^\mu\varphi - k_c^2\varphi = 0$. We can pull out the metric explicitly by writing this as

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi - k_c^2\varphi \quad = \quad 0 \tag{956}$$

Since φ is a scalar, then the first covariant derivative of φ is just a partial derivative: $\nabla_\nu\varphi = \partial_\nu\varphi$. However, since $\partial_\nu\varphi$ is a vector, then acting the second covariant derivative brings in the Christoffel symbol.

$$\nabla_\mu\nabla_\nu\varphi \quad = \quad \nabla_\mu\left(\partial_\nu\varphi\right) \tag{957}$$

$$= \quad \partial_\mu\left(\partial_\nu\varphi\right) + \Gamma^\sigma_{\mu\nu}\left(\partial_\sigma\varphi\right) \tag{958}$$

Substituting this into (956) gives

$$g^{\mu\nu}\partial_\mu\partial_\nu\varphi + g^{\mu\nu}\Gamma^\sigma_{\mu\nu}\left(\partial_\sigma\varphi\right) - k_c^2\varphi \quad = \quad 0 \tag{959}$$

We can write the Christoffel symbols as $\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\left(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\nu\rho} - \partial_\rho g_{\mu\nu}\right)$ so that $g^{\mu\nu}\Gamma^\sigma_{\mu\nu}$ becomes

$$g^{\mu\nu}\Gamma^\sigma_{\mu\nu} \quad = \quad \frac{1}{2}g^{\mu\nu}g^{\sigma\rho}\left(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\nu\rho} - \partial_\rho g_{\mu\nu}\right) \tag{960}$$

Since the metric can be written as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, then the derivatives of the metric become $\partial_\gamma g_{\delta\beta} = \partial_\gamma \left(\eta_{\delta\beta} + h_{\delta\beta}\right) = \partial_\gamma h_{\delta\beta}$. Also, if we choose a lowest order approximation, we can use $g^{\mu\delta} \approx \eta^{\mu\delta}$ so that we only keep terms of order $\partial_\gamma h_{\delta\beta}$. Then (960) becomes

$$g^{\mu\nu}\Gamma^{\sigma}_{\mu\nu} \approx \tfrac{1}{2}\eta^{\mu\nu}\eta^{\sigma\rho}\left(\partial_\nu h_{\rho\mu} + \partial_\mu h_{\nu\rho} - \partial_\rho h_{\mu\nu}\right) \tag{961}$$

We can use $g^{\mu\nu} \approx \eta^{\mu\nu}$ to contract indices (to remain first order in the metric) so that we have

$$g^{\mu\nu}\Gamma^{\sigma}_{\mu\nu} \approx \tfrac{1}{2}\eta^{\sigma\rho}\left(\partial^\mu h_{\rho\mu} + \partial^\nu h_{\nu\rho} - \partial_\rho h\right) \tag{962}$$

where h is the trace of $h_{\mu\nu}$. We can simplify this expression further by immediately fixing the gauge. To maintain consistency with the formulation in Part I, we can choose the trace-reversed harmonic gauge, $\partial^\nu \bar{h}_{\mu\nu} = 0$. First, we must write (962) in terms of the trace-reversed harmonic gauge. From (2434) and (2435) in Appendix B, we have $h = -\bar{h}$ and $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}$. Using these in (962) gives

$$g^{\mu\nu}\Gamma^{\sigma}_{\mu\nu} \approx \tfrac{1}{2}\eta^{\sigma\rho}\left[\partial^\mu\left(\bar{h}_{\rho\mu} - \tfrac{1}{2}\eta_{\rho\mu}\bar{h}\right) + \partial^\nu\left(\bar{h}_{\nu\rho} - \tfrac{1}{2}\eta_{\nu\rho}\bar{h}\right) + \partial_\rho \bar{h}\right] \tag{963}$$

Distributing the derivatives, contracting indices and combining terms gives

$$g^{\mu\nu}\Gamma^{\sigma}_{\mu\nu} \approx \tfrac{1}{2}\eta^{\sigma\rho}\left(\partial^\mu \bar{h}_{\rho\mu} - \tfrac{1}{2}\partial_\rho \bar{h} + \partial^\nu \bar{h}_{\nu\rho} + \tfrac{1}{2}\partial_\rho \bar{h}\right) \tag{964}$$

$$\approx \tfrac{1}{2}\eta^{\sigma\rho}\left(\partial^\mu \bar{h}_{\rho\mu} + \partial^\nu \bar{h}_{\nu\rho}\right) \tag{965}$$

Now applying $\partial^\nu \bar{h}_{\mu\nu} = 0$ gives $g^{\mu\nu}\Gamma^{\sigma}_{\mu\nu} \approx 0$. Therefore, we find that in the trace-reversed harmonic gauge (to first order in the metric), the second term in (959) vanishes. Using $g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$ in the first term and distributing gives

$$\boxed{\begin{gathered}\partial_\mu \partial^\mu \varphi + h^{\mu\nu}\partial_\mu \partial_\nu \varphi - k_c^2 \varphi = 0 \\[1em] \textit{Klein-Gordon equation in curved space-time} \\ \textit{(linearized trace-reversed harmonic gauge)}\end{gathered}} \tag{966}$$

Summing over μ and ν in (966) gives

$$-\frac{1}{c^2}\ddot{\varphi} + \nabla^2\varphi + \frac{1}{c^2}h^{00}\ddot{\varphi} + \frac{2}{c}\vec{h}\cdot\nabla\dot{\varphi} + h^{ij}\partial_i\partial_j\varphi - k_c^2\varphi = 0 \tag{967}$$

9 Relativistic rotating frames and gravitational time holonomies

9.1 The metric and inverse metric for a rotating frame

In cylindrical space-time coordinates (ct, r, ϕ, z), the invariant for the rest frame is

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\phi^2 + dz^2 \tag{968}$$

so the metric is

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{969}$$

The transformed space-time coordinates of a rotating frame with respect to the rest frame are (ct', r', ϕ', z'). For an instant when a clock in the moving frame is synchronized with a clock in the rest frame, the transformed space-time coordinates must satisfy the following relations.

$$r' = r, \qquad \phi' = \phi + \omega t, \qquad z' = z, \qquad t' = t \tag{970}$$

where ω is the angular velocity of the rotating frame with the axis of rotation corresponding to the $z = z'$ axis. From (970) we also have

$$dr' = dr, \qquad d\phi' = d\phi + \omega dt, \qquad dz' = dz, \qquad dt' = dt \tag{971}$$

Similar to (968), the invariant of the rotating frame with respect to the rest frame is

$$ds^2 = -c^2dt'^2 + dr'^2 + r'^2d\phi'^2 + dz'^2 \tag{972}$$

Substituting (971) into (972) gives

$$\begin{aligned} ds^2 &= -c^2dt^2 + dr^2 + r^2(d\phi + \omega dt)^2 + dz^2 && (973) \\ &= -c^2dt^2 + dr^2 + r^2\left(d\phi^2 + \omega dt d\phi + \omega dt d\phi + \omega^2dt^2\right) + dz^2 && (974) \\ &= -c^2dt^2 + dr^2 + r^2d\phi^2 + \left(\omega r^2/c\right)cdtd\phi + \left(\omega r^2/c\right)d\phi cdt + r^2\omega^2dt^2 + dz^2 && (975) \\ &= \left(-1 + r^2\omega^2/c^2\right)c^2dt^2 + dr^2 + \left(\omega r^2/c\right)cdtd\phi + \left(\omega r^2/c\right)d\phi cdt + r^2d\phi^2 + dz^2 && (976) \end{aligned}$$

The transformed metric is therefore

$$g'_{\mu\nu} = \begin{pmatrix} -1 + r^2\omega^2/c^2 & 0 & \omega r^2/c & 0 \\ 0 & 1 & 0 & 0 \\ \omega r^2/c & 0 & r^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{977}$$

Dropping the prime and writing the out the metric components gives

$$g_{00} = -1 + r^2\omega^2/c^2, \qquad g_{11} = g_{33} = 1, \qquad g_{22} = r^2, \qquad g_{02} = g_{20} = \omega r^2/c \tag{978}$$

The inverse of the metric can be found by using Gauss-Jordan elimination which requires transforming the augmented matrix $[A|I]$ into $\left[I|A^{-1}\right]$. To do this we augment the metric with the identity matrix and perform row operations to diagonalize the left side. Using $\alpha = r\omega/c$ gives

$$\left[g_{\mu\nu}|I\right] = \left(\begin{array}{cccc|cccc} -1+\alpha^2 & 0 & r\alpha & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ r\alpha & 0 & r^2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \tag{979}$$

We can use α/r times the third row and subtract it from the first row.

$$\left[g_{\mu\nu}|I\right] = \left(\begin{array}{cccc|cccc} -1 & 0 & 0 & 0 & 1 & 0 & -\alpha/r & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ r\alpha & 0 & r^2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \tag{980}$$

Now we multiply the first row by -1. We can also multiply the first two by $r\alpha$ and subtract it from the third row.

$$\left[g_{\mu\nu}|I\right] = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 0 & \alpha/r & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 & r\alpha & 0 & 1+\alpha^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \tag{981}$$

Now we divide the third row by r^2.

$$\left[g_{\mu\nu}|I\right] = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 0 & \alpha/r & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \alpha/r & 0 & \left(1+\alpha^2\right)/r^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \tag{982}$$

The 4x4 matrix on the left has now become the identity matrix so we observe that the matrix on the right is the inverse metric since for any matrix A we have $[A|I] = \left[I|A^{-1}\right]$. Substituting $\alpha = r\omega/c$ back in gives

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & \omega/c & 0 \\ 0 & 1 & 0 & 0 \\ \omega/c & 0 & \dfrac{1+r^2\omega^2/c^2}{r^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{983}$$

This result can be compared to equation (10) of Cabrera, et al. [58]. We can write the out the inverse metric components as

$$g^{00} = -1, \qquad g^{11} = g^{33} = 1, \qquad g^{22} = \frac{c^2 + r^2\omega^2}{c^2 r^2}, \qquad g^{02} = g^{20} = \omega/c \tag{984}$$

9.2 The Hamiltonian for a rotating frame expressed as a gravitational field

The relativistic Hamiltonian for relativistic Cooper pairs in curved space-time was found in (677) as

$$H = c\left(\gamma^{jk}g_{0j}g_{0k} - g_{00}\right)^{1/2}\left(m^2c^2 + \gamma^{jk}\pi_j\pi_k\right)^{1/2} - c\gamma^{jk}g_{0k}\pi_j - ceA_0 \qquad (985)$$

where the "spatial inverse metric" is

$$\gamma^{jk} = g^{jk} - \frac{g^{0j}g^{0k}}{g^{00}} \qquad \text{so that} \qquad \gamma^{ij}g_{ik} = \delta^j_k \qquad (986)$$

From the metric components in (978), we see that $g_{00} = -1 + r^2\omega^2/c^2$ and $g_{0j} = 0$ so the Hamiltonian immediately simplifies to

$$H = c\left(1 - r^2\omega^2/c^2\right)^{1/2}\left(m^2c^2 + \gamma^{jk}\pi_j\pi_k\right)^{1/2} - ceA_0 \qquad (987)$$

From the inverse metric components in (984), we see that $g^{00} = -1$ so the "spatial inverse metric" in (986) becomes $\gamma^{jk} = g^{jk} + g^{0j}g^{0k}$. Substituting this into the Hamiltonian and distributing gives

$$H = c\left(1 - r^2\omega^2/c^2\right)^{1/2}\left[m^2c^2 + g^{jk}\pi_j\pi_k + g^{0j}g^{0k}\pi_j\pi_k\right]^{1/2} - ceA_0 \qquad (988)$$

We can expand the summations and recognize from (984) that $g^{jk} \neq 0$ only for $j = k$ and also $g^{0j} \neq 0$ only for $j = 2$.

$$\begin{aligned} H = {} & c\left(1 - r^2\omega^2/c^2\right)^{1/2} \\ & \left[m^2c^2 + g^{11}(\pi_1)^2 + g^{22}(\pi_2)^2 + g^{33}(\pi_3)^2 + g^{02}g^{0k}\pi_2\pi_k + g^{0j}g^{02}\pi_j\pi_2\right]^{1/2} - ceA_0 \end{aligned} \qquad (989)$$

Substituting the inverse metric components from (984) gives

$$\begin{aligned} H = {} & c\left(1 - r^2\omega^2/c^2\right)^{1/2} \\ & \left[m^2c^2 + (\pi_1)^2 + \frac{c^2 + r^2\omega^2}{c^2r^2}(\pi_2)^2 + (\pi_3)^2 + 2\left(\frac{\omega}{c}\right)(\pi_2)^2\right]^{1/2} - ceA_0 \qquad (990) \\ = {} & c\left(1 - r^2\omega^2/c^2\right)^{1/2}\left[m^2c^2 + \pi_r^2 + \pi_z^2 + \pi_\phi^2/r^2 + (\omega/c)^2\pi_\phi^2 + 2(\omega/c)\pi_\phi^2\right]^{1/2} - ceA_0 \qquad (991) \end{aligned}$$

$$\boxed{H = c\left(1 - r^2\omega^2/c^2\right)^{1/2}\left[m^2c^2 + \pi_r^2 + \pi_z^2 + \left(1/r^2 + \omega^2/c^2 + 2\omega/c\right)\pi_\phi^2\right]^{1/2} - ceA_0} \qquad (992)$$

This Hamiltonian is exact for the electron motion and the metric (expressed in terms of the angular velocity).

9.3 The Hamiltonian in the weak-field, low-velocity limit

In the first square root above, we can consider the low velocity limit (for the rotational speed) which is effectively a weak field limit since the rotating frame of reference can be considered equivalent to a solenoidal gravitational field (or Lense-Thirring field.) If we consider that $r^2\omega^2/c^2 << 1$, then we can approximate the square root as

$$\left(1-r^2\omega^2/c^2\right)^{1/2} \approx 1-\frac{r^2\omega^2}{2c^2} \tag{993}$$

So the Hamiltonian becomes

$$H = c\left(1-\frac{r^2\omega^2}{2c^2}\right)\left[m^2c^2+\pi_r^2+\pi_z^2+\left(1/r^2+\omega^2/c^2+2\omega/c\right)\pi_\phi^2\right]^{1/2}-ceA_0 \tag{994}$$

Factoring out m^2c^2 from the second root gives

$$mc\left[1+\frac{\pi_r^2+\pi_z^2+\left(1/r^2+\omega^2/c^2+2\omega/c\right)\pi_\phi^2}{m^2c^2}\right]^{1/2} \tag{995}$$

In the low velocity limit (for non-relativistic, slow moving electrons), we have

$$\left|\frac{\pi_r^2+\pi_z^2+\pi_\phi^2/r^2+\left(1/r^2+\omega^2/c^2+2\omega/c\right)\pi_\phi^2}{m^2c^2}\right| \quad < \quad < 1 \tag{996}$$

So we can approximate the square root as

$$\left[1+\frac{\pi_r^2+\pi_z^2+\left(1/r^2+\omega^2/c^2+2\omega/c\right)\pi_\phi^2}{m^2c^2}\right]^{1/2} \approx 1+\frac{\pi_r^2+\pi_z^2+\left(1/r^2+\omega^2/c^2+2\omega/c\right)\pi_\phi^2}{2m^2c^2} \tag{997}$$

Then the Hamiltonian becomes

$$\begin{aligned}
\underset{\substack{(weak\ field,\\ low\ velocity)}}{H} &= c\left(1-\frac{r^2\omega^2}{2c^2}\right)mc\left[1+\frac{\pi_r^2+\pi_z^2+\left(1/r^2+\omega^2/c^2+2\omega/c\right)\pi_\phi^2}{2m^2c^2}\right]-ceA_0 && (998)\\
&= \left(1-\frac{r^2\omega^2}{2c^2}\right)\left[mc^2+\frac{\pi_r^2+\pi_z^2+\left(1/r^2+\omega^2/c^2+2\omega/c\right)\pi_\phi^2}{2m}\right]-ceA_0 && (999)
\end{aligned}$$

Multiplying out terms and factoring out powers of ω/c leads to

$$\begin{aligned}
\underset{\substack{(weak\ field,\\ low\ velocity)}}{H} &= mc^2-\frac{1}{2}m^2r\omega^2+\frac{1}{2m}\left(\pi_r^2+\pi_z^2+\pi_\phi^2/r^2\right)+\frac{\pi_\phi^2}{m}\left(\frac{\omega}{c}\right)\\
&\quad+\frac{1}{4m}\left(3\pi_\phi^2-r^2\pi_r^2+r^2\pi_z^2\right)\left(\frac{\omega}{c}\right)^2+\frac{r^2\pi_\phi^2}{2m}\left(\frac{\omega}{c}\right)^3+\frac{r^2\pi_\phi^2}{4m}\left(\frac{\omega}{c}\right)^4-ceA_0 && (1000)
\end{aligned}$$

In (993) we neglected terms that are fourth order in ω/c therefore we must also do so here for consistency. If we also neglect third order terms and keep only second order in ω/c, then we have

$$\boxed{\begin{aligned} \underset{(2^{nd}\ order)}{H} &= mc^2 - \frac{1}{2}m^2 r\omega^2 + \frac{1}{2m}\left(\pi_r^2 + \pi_z^2 + \pi_\phi^2/r^2\right) + \frac{\pi_\phi^2}{m}\left(\frac{\omega}{c}\right) \\ &+ \frac{1}{4m}\left(3\pi_\phi^2 - r^2\pi_r^2 + r^2\pi_z^2\right)\left(\frac{\omega}{c}\right)^2 - ceA_0 \end{aligned}} \tag{1001}$$

9.4 A time-dilation holonomy due to a rotating frame

An interesting consequence of Relativity (General and Special) is the concept of a *time holonomy*.[90] In *Special Relativity*, the invariant interval ds between events is given by $ds^2 = -c^2dt^2 + dr^2$. In *General Relativity*, for any space-time $g_{\mu\nu}$, the invariant interval ds between events with an incremental coordinate separation dx^μ is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{1002}$$

If we write out the metric components from (1002) explicitly, we have

$$\begin{aligned} ds^2 &= c^2 g_{00}dt^2 + 2cg_{01}dtdx + 2cg_{02}dtdy + 2cg_{03}dtdz \\ &\quad + g_{11}dx^2 + g_{22}dy^2 + g_{33}dz^2 \\ &\quad + g_{12}dxdy + g_{23}dydz + g_{13}dxdz + g_{21}dydx + g_{32}dzdy + g_{31}dzdx \end{aligned} \tag{1003}$$

By defining $x^0 = ct$, $x^1 = x$, $x^2 = y$, and $x^3 = z$, then we can write this more concisely as

$$ds^2 = g_{00}\left(dx^0\right)^2 + 2g_{0i}dx^0dx^i + g_{ij}dx^idx^j \tag{1004}$$

If a light signal is sent between two events, say A and B, then the invariant interval is $ds^2 = 0$ for a light-like interval. Imposing this condition on (1004) gives a quadratic equation in dx^0. We can solve for dx^0 using the quadratic formula with

$$A = g_{00}, \qquad B = 2g_{0i}dx^i, \qquad C = g_{ij}dx^idx^j \tag{1005}$$

This gives

$$dx^0_{\pm} = \frac{-g_{0i}dx^i \pm \sqrt{\left(g_{0i}dx^i\right)^2 - g_{00}g_{ij}dx^idx^j}}{g_{00}} \tag{1006}$$

There are two solutions, dx^0_+ and dx^0_-, due to the $\pm$ root. The physical significance of these two solutions can be visualized in the following worldline diagram.Here it can be seen that if a signal is sent from A to B and

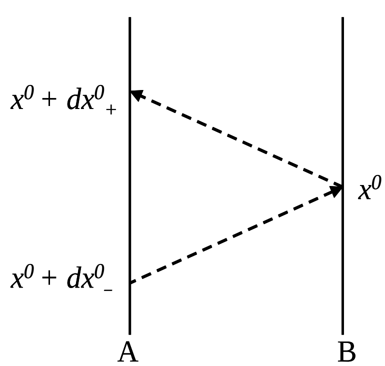

Figure 4: Worldline diagram of two points on rotating frame.

then back from B to A, then the overall time would be the sum of dx^0_+ and dx^0_-.

[90] We develop the details of the time holonomy following the treatment found in L. Landau and E. Lifshitz, *The Classical Theory of Fields*, [59], pp. 226-227, 233-234, 236, 248, 253-254.

Using (1006) gives

$$dx_+^0 + dx_-^0 = -\frac{2g_{0i}dx^i}{g_{00}} \tag{1007}$$

If we are attempting to synchronize events A and B, then it can be seen from the diagram that x^0 occurs half-way between $x^0 + dx_-^0$ and $x^0 + dx_+^0$ thus we need Δx_0 to be half of the sum of dx_+^0 and dx_-^0. This gives

$$\Delta x^0 \quad = \quad -\frac{g_{0i}dx^i}{g_{00}} \tag{1008}$$

If we add all of the time intervals associated with synchronizing clocks along a closed path we obtain a closed-loop integral. Using $\Delta x^0 = c\Delta t$, we have

$$\boxed{\Delta t = -\frac{1}{c}\oint \frac{g_{0i}}{g_{00}} dx^i} \tag{1009}$$

This is effectively a *time holonomy*.[91] We will find in the following treatment that there is a time holonomy arising from both *Special Relativity* and *General Relativity*.

We now consider a particle moving with angular velocity ω in a circular path of radius r surrounding a stationary cylindrical shell coaxially. Alternatively, we could also consider a *stationary* particle positioned on the same circular contour which surrounds a cylindrical shell spinning about its axis.[92] These two scenarios are similar in the sense that they both exhibit the same time holonomy.[93]

In cylindrical spacetime coordinates (ct, r, ϕ, z), the invariant for the rest frame is

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\phi^2 + dz^2 \tag{1010}$$

so the metric is

$$g_{\mu\nu} \quad = \quad \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1011}$$

[91]"In differential geometry, the holonomy of a connection on a smooth manifold is a general geometrical consequence of the curvature of the connection measuring the extent to which parallel transport around closed loops fails to preserve the geometrical data being transported."[63] In this case, the "geometrical data being transported" is essentially the measurement of *time*. The fact that moving around the contour can result in a larger or smaller time interval depending on the direction of motion, despite perfect contour symmetry of the two directions and identical speed in each direction, can be considered a *time holonomy.*

[92]The cylindrical shell does not need to possess any mass for this time holonomy to exist. It is simply used here as a way to describe a frame of reference separate from the circular contour surrounding the cylinder.

[93]This is not to imply that the two scenarios are completely equivalent since a rotating frame is not inertial and therefore cannot be transformed into an inertial rest frame by a Galilean coordinate transformation. In other words, an observer in either frame can experimentally determine whether the particle is rotating around the cylinder or whether the particle is stationary and the cylinder is rotating about its axis. (This could be related to the concept of Mach's principle which states that the rotational inertia of a rotating frame can be determined by the rotational motion of an otherwise "fixed background" such as the stars in the surrounding universe.) The only similarity between the two scenarios (the particle rotating or the cylinder rotating) is that the time holonomy calculation would yield the same result in either case.

The transformed spacetime coordinates of the rotating frame with respect to the rest frame are (ct', r', ϕ', z'). For an instant when a clock in the moving frame is synchronized with a clock in the rest frame, the transformed spacetime coordinates must satisfy the following relations.

$$r' = r, \qquad \phi' = \phi + \omega t, \qquad z' = z, \qquad t' = t \tag{1012}$$

where ω is the angular velocity of the rotating frame with the axis of rotation corresponding to the $z = z'$ axis. From (1012) we also have

$$dr' = dr, \qquad d\phi' = d\phi + \omega dt, \qquad dz' = dz, \qquad dt' = dt \tag{1013}$$

Similar to (1010), the invariant of the rotating frame with respect to the rest frame is

$$ds^2 = -c^2 dt'^2 + dr'^2 + r'^2 d\phi'^2 + dz'^2 \tag{1014}$$

Substituting (1013) into (1014) gives

$$\begin{aligned} ds^2 &= -c^2 dt^2 + dr^2 + r^2 (d\phi + \omega dt)^2 + dz^2 && (1015) \\ &= -c^2 dt^2 + dr^2 + r^2 \left(d\phi^2 + 2\omega d\phi dt + \omega^2 dt^2\right) + dz^2 && (1016) \\ &= \left(-c^2 + r^2\omega^2\right) dt^2 + 2\omega r^2 d\phi dt + dr^2 + r^2 d\phi^2 + dz^2 && (1017) \end{aligned}$$

Then the transformed metric $g'_{\mu\nu}$ is therefore

$$g'_{\mu\nu} = \begin{pmatrix} \left(-c^2 + \omega^2 r^2\right)/c^2 & 0 & \omega r^2/c & 0 \\ 0 & 1 & 0 & 0 \\ \omega r^2/c & 0 & r^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1018}$$

We can also confirm this is the correct metric by a direct transformation of the metric $g_{\mu\nu}$ in (1011) using

$$g'_{\mu\nu} = \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial x^\rho}{\partial x'^\nu} g_{\sigma\rho} \tag{1019}$$

We can consider the clocks in the two frames to be synchronized at a given point such as point A in the mass solenoid diagram on page 26. Then we can use the expression found in (1009),

$$\Delta t = -\frac{1}{c} \oint \frac{g_{0i}}{g_{00}} dx^i \tag{1020}$$

to find the time holonomy that results from traversing a closed path beginning at point A and returning back to point A again[94]. Using the transformed metric in (1018) above, we note that

$$g_{01} = g_{03} = 0, \qquad g_{02} = \omega r^2/c, \qquad g_{00} = \left(-c^2 + \omega^2 r^2\right)/c^2 \tag{1021}$$

[94] The diagram on page 26 could be considered as an infinitesimal segment of the circular path with worldline A and worldline B representing the contour and the cylinder wall, respectively.

Then in (1020) we have $g_{0i}dx^i = g_{02}dx^2 = \left(\omega r^2/c\right)d\phi$ which means

$$\Delta t \quad = \quad -\frac{1}{c}\oint\frac{\omega r^2/c}{\left(-c^2+\omega^2 r^2\right)/c^2}d\phi \tag{1022}$$

$$\Delta t \quad = \quad -\frac{1}{c^2}\oint\frac{\omega r^2}{-1+\left(\frac{\omega r}{c}\right)^2}d\phi \tag{1023}$$

If the rotational velocity of the mass solenoid is much less than the speed of light (that is, $\omega r << c$), then (1023) becomes

$$\Delta t \quad \approx \quad \frac{\omega}{c^2}\oint r^2 d\phi \tag{1024}$$

Integrating around a circular path of radius r gives

$$\boxed{\Delta t = \pm\frac{2\pi\omega r^2}{c^2} \qquad \textit{Time-dilation holonomy}} \tag{1025}$$

The $+$ and $-$ signs are associated with integrating along a path that is in the *opposite* direction or in the *same* direction as the rotation. In the mass solenoid diagram (page 21), the "$-$" sign would be associated with taking Path 1 then Path 2, while the "$+$" sign would be associated with taking Path 2 then Path 1.

Therefore, in the context of *Special Relativity*, we obtain what might be described as a *time-dilation holonomy*. Each infinitesimal element along the contour path is a local inertial frame moving uniformly relative to the spinning frame. Therefore each infinitesimal element along the contour path demonstrates a time-dilation (or alternatively, non-synchronous clock difference) relative to the non-rotating frame. The integral is a sum of these infinitesimal time-dilation contributions around the contour path which results in a global time holonomy around the entire contour.[95]

[95]Notice that changing the sign of ω will switch the + and - signs as we expect since this would effectively switch the direction along the contour path that yields a shorter time and the direction along the contour path that yields a longer time.

9.5 A gravitational time holonomy due to a mass solenoid

We can now consider the *gravitational* time holonomy associated with an electron wave traveling around a mass solenoid as described in the gravitational AB effect.[4]. The mass solenoid could be described as essentially a cylindrical mass shell with mass current density $J_m = \sigma_m \omega$ where σ_m is the surface mass density of the cylinder spinning with constant angular velocity ω. (See the diagram in Section 9.)

In (32) of Section 4 we defined the (gauge-dependent) gravito-vector potential as[96]

$$\vec{h} = \frac{c}{4}\,(h_{01},\ h_{02},\ h_{03}) \tag{1026}$$

The gravitational field of an axially symmetric mass rotating uniformly about its axis (as in the case of the mass solenoid) has components h_{0i} that are nonzero and do not depend on x^0. To find the time holonomy associated with the path of the electron wave in the presence of this field, we again use the relation derived in (1009) as

$$\Delta t \quad = \quad -\frac{1}{c}\oint \frac{g_{0i}}{g_{00}} dx^i \tag{1027}$$

Since $h_{\mu\nu} << 1$ (for the weak-field approximation) and $\eta_{00} = -1$, then

$$g_{00} = \eta_{00} + h_{00} \approx -1 \tag{1028}$$

Also, since $\eta_{0i} = 0$, then

$$g_{0i} \quad = \quad \eta_{0i} + h_{0i} = h_{0i} \tag{1029}$$

Substituting (1028) and (1029) into (1027) gives

$$\Delta t \quad = \quad \frac{1}{c}\oint h_{0i} dx^i \tag{1030}$$

We already found in (113) that $\oint \vec{h}\cdot d\vec{r} = \tilde{\Phi}_{gm}$ so we can write the result here as

$$\boxed{\Delta t = \frac{\tilde{\Phi}_{gm}}{c^2} \qquad \textit{Gravitational time holonomy}} \tag{1031}$$

Note that $\vec{h}$ points in the *opposite* direction of $\vec{J}_m$ as shown in the diagram in Section 9. Therefore moving around the mass solenoid in the *same* direction as the rotation of the mass means moving *against* $\vec{h}$ which results in a *negative* value for Δt. Moving *against* the direction of the rotation of the mass means moving *with* $\vec{h}$ which results in a *positive* value for Δt. This is not reflected in final result, $\Delta t = \tilde{\Phi}_{gm}/c^2$, since the flux in this expression does not indicate whether one has integrated in the same direction or opposite direction to $\vec{h}$.

Therefore, we find that in the context of *General Relativity*, we can think of the result in (1031) as a *gravitational time holonomy*. It is a result of the metric which describes the curvature of spacetime around the contour. This curvature can be interpreted as a result of the gravito-vector potential due to a spinning cylindrical mass shell or "mass solenoid."

[96] We know from (2431) of Appendix B that $\bar{h}_{0i} = h_{0i}$ so we can drop the "bar" notation.

Note that this is *not* the same as a gravito-magnetic field (or "Lense-Thirring field") causing "frame dragging." For the mass solenoid, the Lense-Thirring field is only present *inside* but is *zero outside* where the particle is spinning (assuming a "perfect" mass solenoid which is infinitely long). However, in the case of frame-dragging, it is the gravito-magnetic *field*, not gravito-vector *potential*, that is acting on the non-rotating frame. The fact that it is purely the gravito-vector *potential* (not the field) that is affecting the particle makes this arrangement very similar to the AB effect (which involves the potential, not the field). However, the gravitational time holonomy is not necessarily quantum mechanical since the entire calculation was done classically. It applies to any wave which may propagate in the region where $\vec{h} \neq 0$, provided the wave couples to the potential field.

We can relate the *gravitational* time holonomy in(1031) to the *time-dilation* holonomy from (1025) to compare their relative values. This is done in the next section. We can also relate this gravitational time holonomy to the "phase holonomy," $\Delta\phi$, from the gravitational AB effect by multiplying Δt by the Compton frequency of the electron, $\omega_c = m_e c/\hbar$. This gives [39]

$$\Delta\phi = \omega_c \Delta t = \frac{m_e \Phi_{\vec{B}_G}}{\hbar} \tag{1032}$$

which is what we obtain later in (1138) for the gravito-vector AB effect. This implies that the *quantum mechanical phase shift* associated with the gravitational AB effect can also be regarded as a *classical time-holonomy* due to rotating reference frames.

9.6 Relating the time-dilation and gravitational time holonomies

In the previous two section on time holonomies, we derived two separate expression for a time holonomy. In (1025) the following expression for the *time-dilation holonomy* was obtained.

$$\Delta t_{td} = \pm \frac{2\pi \omega r^2}{c^2} \tag{1033}$$

This was interpreted as the result of adding all of the time-dilation contributions around a circular contour due to the motion of one rotating frame with respect to another frame positioned concentrically. Independently, in (1031) we also obtained the following expression for the *gravitational time holonomy*

$$\Delta t_G = \frac{\Phi_{gm}}{c^2} \tag{1034}$$

This was interpreted as the result of the curvature of spacetime around the contour. The curvature can be interpreted as a gravitational field, specifically, a uniform *gravito-magnetic field* pointing along the axis of the cylinder. This field is due to the spinning of a cylindrical mass shell and is described by the static gravito-Ampere law in (58) applied to a mass solenoid.

In order to relate the two time-holonomy results above, we can develop an expression for Φ_{gm} in (1034). Taking a surface integral of both sides of the static gravito-Ampere law (58) and applying Stoke's theorem gives

$$\iint \left(\nabla \times \vec{B}_G\right) \cdot d\vec{a} \quad = \quad -\mu_G \iint \vec{J}_m \cdot d\vec{a} \tag{1035}$$

$$\oint \vec{B}_G \cdot d\vec{l} \quad = \quad -\mu_G I_m \tag{1036}$$

The common treatment for a solenoid is to use a line integral along a rectangular loop with one edge *inside* the solenoid parallel to the axis (where $\vec{B}_G \neq 0$) and the opposite edge *outside* the solenoid (where $\vec{B}_G = 0$). If the length of the edge is L, then we obtain

$$B_G L = -\mu_G I_m \tag{1037}$$

The total current in a solenoid is $I = Ni$ where i is the current in each loop. However, since the mass solenoid is a continuous mass shell, then it is effectively a "perfect" solenoid where the current is distributed continuously over the surface. Then we can use $J_m = \sigma_m \omega$ where σ_m is the surface mass density of the cylinder spinning with angular velocity ω. Then the total current would be $I_m = J_m A_\perp$ where $A_\perp = rL$ is the area normal to the current. So we have

$$B_G L \quad = \quad -\mu_G \left(\sigma_m \omega\right)(rL) \tag{1038}$$

A uniform surface mass density is the total mass per total surface area of the cylindrical wall, $\sigma_m = \dfrac{m}{2\pi r L}$. We can also substitute $\mu_G = \frac{4\pi G}{c^2}$ to obtain

$$B_G L \quad = \quad -\left(\frac{4\pi G}{c^2}\right)\left(\frac{m}{2\pi r L}\right)\omega\,(rL) \tag{1039}$$

$$B_G \quad = \quad -\frac{2Gm\omega}{Lc^2} \tag{1040}$$

We can now determine the magnitude of the gravito-magnetic flux Φ_{gm} through a cross-sectional area of the solenoid, $A_{cs} = \pi r^2$. When we determined the magnetic field we already treated the cylinder as a "perfect" solenoid which means it is effectively one "loop" so $N = 1$. Then we have

$$\Phi_{gm} = NB_G A_{cs} = \frac{2Gm\omega}{c^2 L}\pi r^2 = \left(\frac{Gm}{c^2 L}\right) 2\pi\omega r^2 \tag{1041}$$

Substituting this into (1034) gives

$$\Delta t_G = \left(\frac{Gm}{Lc^2}\right)\frac{2\pi\omega r^2}{c^2} \tag{1042}$$

Relating this *gravitational* time holonomy to the *time-dilation* holonomy (1033) we have

$$\Delta t_G = \left(\frac{Gm}{c^2 L}\right)\Delta t_{td} \tag{1043}$$

This can be written in various forms depending on the mass density used for the cylinder (linear density λ_m, surface density σ_m or volume density ρ_m). Since

$$m = \lambda_m L = \sigma_m 2\pi r L = \rho_m \pi r^2 L \tag{1044}$$

then we can write (1043) as any of the following.

$$\frac{\Delta t_G}{\Delta t_{td}} = \left(\frac{G}{c^2}\right)\lambda_m = \left(\frac{G}{c^2}\right)2\pi r\sigma_m = \left(\frac{G}{c^2}\right)\pi r^2 \rho_m \tag{1045}$$

In any of these forms we see that the gravitational time-holonomy t_G is less than the time-dilation holonomy t_{td} by a factor of $G/c^2 \approx 10^{-27}$ (SI units). As is often the case, this demonstrates that the effects of General Relativity are much smaller than the effects of Special Relativity.

10 Gravitational Aharonov-Bohm (AB) effects and quantum phase interference

10.1 Overview of gravitational AB effects and phase interference

Here we relate the scalar, vector and tensor coupling rules previously developed to the quantum phase of a wavefunction. The phase is described in terms of the HD metric components. It is found that the only gauge *invariant* phase is the phase given in terms of $h_{ij}^{\tau\tau}$ in the far-field where $h_{ij}^{\tau\tau}$ is the gauge-*invariant*, transverse-traceless strain field of a gravitational wave. We also consider a comparison of the phase in electromagnetism versus the phase in gravitation. In the case of electromagnetism, the phase can be expressed in terms of the gauge-*dependent* four-potential, A_μ, or in terms of the gauge-*invariant* field strength tensor, $F_{\mu\nu}$. Likewise, we expect that in the case of gravitation, the phase should be expressible in terms of the gauge-*dependent* metric perturbation, $h_{\mu\nu}$, or in terms of a gauge-*invariant* quantity such as the Riemann tensor.

10.2 The quantum phase in curved space-time

In this section we consider the possibility of a quantum phase associated with quantum particles coupling to the scalar, vector or tensor gravitational potentials. We begin with the approach given by Stodolsky in [64]. In general, the phase factor of a quantum particle can be written in terms of the action, S, as $e^{i\tilde{\phi}} = e^{\frac{i}{\hbar}S}$ so that the phase is[97]

$$\tilde{\phi} = \frac{1}{\hbar}S \tag{1046}$$

From (2839) of Appendix M, we can parameterize the action for a relativistic particle in terms of proper time, $d\tau$, as

$$S = -\int_{\tau_1}^{\tau_2} mc^2 d\tau \tag{1047}$$

The relativistic invariant interval, ds, can be written in terms of the proper time as $ds = -cd\tau$. Then the action becomes

$$S = -mc\int ds \tag{1048}$$

The relativistic invariant interval can also be written in terms of the metric as $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. Then using $ds = -cd\tau$, we have

$$ds = \frac{g_{\mu\nu}dx^\mu dx^\nu}{ds} = \frac{g_{\mu\nu}dx^\mu dx^\nu}{-cd\tau} \tag{1049}$$

Since the four-momentum may be written as $p^\mu = m\dfrac{dx^\mu}{d\tau}$, then the expression above becomes

$$ds = -\frac{g_{\mu\nu}p^\mu dx^\nu}{mc} \tag{1050}$$

Substituting this into the action in (1048) gives

$$S = \int g_{\mu\nu}p^\mu dx^\nu \tag{1051}$$

Alternatively, we could begin with the "four-momentum invariant" action given in (2855) as

$$S = \int \frac{g_{\mu\nu}p^\mu p^\nu}{m}d\tau = \int \frac{g_{\mu\nu}p^\mu}{m}\left(m\frac{dx^\nu}{d\tau}\right)d\tau = \int g_{\mu\nu}p^\mu dx^\nu d\tau \tag{1052}$$

Then substituting for the phase in (1046) and writing the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ gives

$$\tilde{\phi} = \frac{1}{\hbar}\int \eta_{\mu\nu}p^\mu dx^\nu + \frac{1}{\hbar}\int h_{\mu\nu}p^\mu dx^\nu \tag{1053}$$

We may consider these two contributions to the phase as the "free particle" contribution and the "coupling to gravitation" contribution. Then $\tilde{\phi} = \tilde{\phi}_{free} + \tilde{\phi}_{coupling}$. If we consider the case of a particle moving through a closed space-time path, then we have

$$\tilde{\phi}_{coupling} = \frac{1}{\hbar}\oint h_{\mu\nu}p^\mu dx^\nu \tag{1054}$$

This is consistent with the phase as obtained by Stodolsky in [64]. Note that by using the action in (1052) we may recognize that this phase expression is essentially derived from four-momentum invariance which involves the entire metric, not just the metric perturbation. We will return to this point when discussion

[97] We will use the notation $\tilde{\phi}$ to distinguish the phase from the scalar potential found in $h_{00} = -2\phi/c^2$.

gauge-dependence. Summing over the indices in (1054), we can write the following phase expressions[98] for each of the metric components. We state them along with the associated coupling rules from (863) – (865).

$$\begin{aligned}
&\text{Scalar coupling:} \quad mc^2 \implies mc^2 - \tfrac{1}{2}mc^2 h_{00} \quad \text{corresponds to} \quad \tilde{\phi}_S = \frac{E}{\hbar}\oint h_{00}\,dt \\
&\text{Vector coupling:} \quad p_i \implies p_i - mh_{0i} \quad \text{corresponds to} \quad \tilde{\phi}_V = \frac{1}{\hbar}\oint \left(\frac{E}{c}h_{0i}dx^i + ch_{0i}p^i dt\right) \\
&\text{Tensor coupling:} \quad p^2 \implies p^2 - p^i p^j h^{\tau\tau}_{ij} \quad \text{corresponds to} \quad \tilde{\phi}_T = \frac{1}{\hbar}\oint h^{\tau\tau}_{ij} p^i dx^j
\end{aligned} \tag{1055}$$

Notice that for the *scalar* coupling, we can use the rest frame of the particle so that $E = mc^2$ and therefore $p^0 = E/c = mc^2/c = mc$. However, for the *vector* coupling, if we use the rest frame of the particle, then $p^0 = mc$ and $p^i = 0$ which means that $h_{0i}p^i dt$ does not contribute to the phase. Likewise for the tensor coupling, if we use the rest frame of the particle, then $p^i = 0$ and the contribution to the phase vanishes completely.

Lastly, we also point out that for the phase to be gauge-invariant, it is not sufficient that some component(s) of $h_{\mu\nu}$ be made gauge-invariant. Rather, it is necessary that the entire integrand be made gauge-invariant, which includes p^μ and dx^ν as well. Therefore, we must transform the entire integrand as a whole to examine the gauge freedom. This is done in the next section where we determine the irreducible gauge contribution to the coupling phase.

[98] The phase associated with the scalar, vector, and tensor couplings are each written separately as $\tilde{\phi}_S$, $\tilde{\phi}_V$, and $\tilde{\phi}_T$, respectively. However, the original expression given in (1054) is a *sum* over indices which means that the separate phases should really be added.

10.3 Gauge freedom in the coupling phase

In the case of electromagnetism, the phase can be expressed in terms of the four-potential, A_μ, or in terms of the field strength tensor, $F_{\mu\nu}$ where A_μ is gauge-*dependent* and $F_{\mu\nu}$ is gauge-*invariant*, and the two are related by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Specifically, we can use a higher dimensional version of Stokes' theorem to equate the four-dimensional closed-path integral of A_μ to a four-dimensional hyper-surface "flux" integral of $F_{\mu\nu}$. Therefore, the phase can be written as[99]

$$\tilde{\phi} \quad = \quad \frac{q}{\hbar}\oint_{\partial A} A_\mu dx^\mu = \frac{q}{\hbar}\iint_A F_{\mu\nu} dx^\mu \wedge dx^\nu \tag{1056}$$

Although A_μ is gauge-dependent, we find that the closed-path integral on the left side of (1056) is still gauge-invariant. This can be shown by inserting the gauge freedom given by $A'_\mu = A_\mu + \partial_\mu \chi$ into (1056).

$$\oint_{\partial A} A'_\mu dx^\mu \quad = \quad \oint_{\partial A} A_\mu dx^\mu + \oint_{\partial A} \left(\partial_\mu \chi\right) dx^\mu \tag{1057}$$

The differential of χ is $d\chi = \left(\partial_\mu \chi\right) dx^\mu$ which is exactly what we have in the last integral above. Therefore, by the fundamental theorem of Calculus, we have

$$\int_{x_A^\mu}^{x_B^\mu} \left(\partial_\mu \chi\right) dx^\mu \quad = \quad \int d\chi = \chi_{\left(x_B^\mu\right)} - \chi_{\left(x_A^\mu\right)} \tag{1058}$$

However, for a closed-loop integral, $\chi_B = \chi_A$ and the integral vanishes. Hence we find that the gauge freedom vanishes. Now for the case of the *gravitational* phase given by (1054), we can write the phase for the transformed metric perturbation as

$$\tilde{\phi}'_{coupling} \quad = \quad \frac{1}{\hbar}\oint h'_{\mu\nu} p'^\mu dx'^\nu \tag{1059}$$

Notice that for consistency, the potential, the four-momentum and the differential must all be transformed. The linearized gauge freedom for $h_{\mu\nu}$ found in (2418) of Appendix A is

$$h'_{\mu\nu} \quad = \quad h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{1060}$$

We must also transform p'^μ using $p'^\mu = \dfrac{\partial x^{\mu'}}{\partial x^\sigma} p^\sigma$. From (2402) we have $\dfrac{\partial x^{\mu'}}{\partial x^\sigma} = \delta_\sigma^{\ \mu} - \partial_\sigma \xi^\mu$ so the four-momentum transforms as

$$p'^\mu \quad = \quad \left(\delta_\sigma^{\ \mu} - \partial_\sigma \xi^\mu\right) p^\sigma = p^\mu - \left(\partial_\sigma \xi^\mu\right) p^\sigma \tag{1061}$$

Likewise, the differential dx^ν will transform as

$$dx'^\nu \quad = \quad dx^\nu - \left(\partial_\rho \xi^\nu\right) dx^\rho \tag{1062}$$

[99]Note that we use the exterior product (or wedge product), $\wedge$, between $dx^\mu dx^\nu$. This is necessary since $dx^\mu dx^\nu$ is a symmetric tensor while $F_{\mu\nu}$ is anti-symmetric which means that the product of the two tensors would vanish. In much of the literature this is neglected.

Then using (1060), (1061) and (1062), we can write the integrand of (1059) as

$$h'_{\mu\nu}p'^{\mu}dx'^{\nu} \quad = \quad \left(h_{\mu\nu}+\partial_{\mu}\xi_{\nu}+\partial_{\nu}\xi_{\mu}\right)\left[p^{\mu}-\left(\partial_{\sigma}\xi^{\mu}\right)p^{\sigma}\right]\left[dx^{\nu}-\left(\partial_{\rho}\xi^{\nu}\right)dx^{\rho}\right] \tag{1063}$$

In Appendix A, it is shown that to remain first order in $h_{\mu\nu}$, we must also remain first order in $\partial_{\mu}\xi_{\nu}$. This means that we also neglect any terms involving the square of $\partial_{\mu}\xi_{\nu}$ or the product of $\partial_{\mu}\xi_{\nu}$ and $h_{\mu\nu}$. Therefore, the additional terms due to transforming p^{μ} and dx^{ν} do not contribute to the transformation in the linearized approximation. So we are are left with

$$h'_{\mu\nu}p'^{\mu}dx'^{\nu} \quad = \quad h_{\mu\nu}p^{\mu}dx^{\nu}+\left(\partial_{\mu}\xi_{\nu}\right)p^{\mu}dx^{\nu}+\left(\partial_{\nu}\xi_{\mu}\right)p^{\mu}dx^{\nu} \tag{1064}$$

Inserting this into the phase expression in (1059) gives

$$\tilde{\phi}'_{coupling} \quad = \quad \frac{1}{\hbar}\oint h_{\mu\nu}p^{\mu}dx^{\nu}+\frac{1}{\hbar}\oint\left(\partial_{\mu}\xi_{\nu}+\partial_{\nu}\xi_{\mu}\right)p^{\mu}dx^{\nu} \tag{1065}$$

We can refer to the second integral as $\tilde{\phi}_{gauge}$ since it is the contribution to the phase coming purely from the gauge freedom. In the eikonal approximation[100], p^{μ} is a constant and can come out of the integral so we have

$$\tilde{\phi}'_{gauge} \quad = \quad \frac{p^{\mu}}{\hbar}\oint\left(\partial_{\mu}\xi_{\nu}+\partial_{\nu}\xi_{\mu}\right)dx^{\nu} \tag{1066}$$

In general, the differential of a four-vector ξ_{α} is

$$d\xi_{\alpha} \quad = \quad \left(\partial_{\beta}\xi_{\alpha}\right)dx^{\beta} \tag{1067}$$

For $\alpha=\nu$ and $\beta=\nu$, then we have $d\xi_{\mu}=\left(\partial_{\nu}\xi_{\mu}\right)dx^{\nu}$ which appears in the second term of the integral in (1066). Therefore the closed loop integral of that term will vanish and we are left with

$$\tilde{\phi}'_{gauge} \quad = \quad \frac{p^{\mu}}{\hbar}\oint\partial_{\mu}\xi_{\nu}dx^{\nu} \tag{1068}$$

We cannot use the same approach again to remove this integral. For example, using $\alpha=\mu$ and $\beta=\nu$ in (1067) gives $d\xi_{\nu}=\left(\partial_{\mu}\xi_{\nu}\right)dx^{\mu}$ however the integral above is in terms of dx^{ν}, not dx^{μ}. If we sum over indices in (1068), then we have

$$\tilde{\phi}'_{gauge} \quad = \quad \frac{p^{\tilde{0}}}{\hbar}\left(\oint\left(\partial_{\tilde{0}}\xi_{0}\right)dx^{0}+\oint\partial_{0}\xi_{i}dx^{i}\right)+\frac{p^{i}}{\hbar}\left(\oint\partial_{i}\xi_{0}dx^{0}+\oint\partial_{i}\xi_{j}dx^{j}\right) \tag{1069}$$

We can think of $\partial_0\xi_i$ as essentially just a vector a_i so that the closed path integral of $a_i dx^i$ must vanish (assuming a_i is a single-valued function of space). Likewise, we can think of $\partial_0\xi_i dx^i$ as essentially just a vector b_i so that the closed path integral of $b_i dx^0$ must vanish (assuming $\partial_i\xi_0$ is a single-valued function of time). The closed-path integral of $\partial_i\xi_j dx^j$ need not vanish since the values of $\partial_i\xi_j$ on a path parameterized by x^j from point A to B could be different than the values on a path from point B to A. (The reason would be

[100] In the eikonal approximation, we consider a wave expressed as $\psi=Ae^{ikx}$ to be highly localized along one dimension (essentially a ray) so that $\partial^2\psi=0$ and $\partial\psi=$ constant. Since $\partial\psi=k\psi$ then this means $k=$ constant. For a quantum particle, $p=\hbar k$ so $p=$ constant. Therefore, here we simply treat p^{μ} as a constant.

due to the derivative values of ξ_j with respect to ∂_i which are independent of the path parameterized by x^j.) Then the closed-path integral could be non-zero.

In (1068) we must be careful to recognize that the index of p^μ was contracted with the index of ∂_μ while the index of ξ_ν was contract with the index of dx^ν. This information is lost in the first integral of (1069) when 0 is simply substituted into both indices. Therefore we have used the labels $\mu = \tilde{0}$ and $\nu = 0$ to emphasize this distinction.

It is critical to pay attention to this issue since we can see that ∂_μ and dx^ν do not have matching indices and therefore, the closed-path integral does not necessarily vanish. The physical significance of $\mu = \tilde{0}$ is the selection of the time-like component of p^μ which is $p^{\tilde{0}} = E/c$ and to select the temporal derivative of ξ_ν with respect to a coordinate time $\tilde{t}$. However, the physical significance of $\nu = 0$ is to select the time-like component of ξ_ν which is ξ_0 and to integrate over a differential time element dt. Notice that there is no reason why the derivative of ξ_0 with respect to the coordinate time $\tilde{t}$ should be related to the integration with respect to the coordinate time dt. In other words, the rate at which some scalar ξ_0 is changing in time as measured by a clock with time $\tilde{t}$, need not be match the rate at which the integration occurs with respect to a clock measured by t. Therefore, even for a closed-path integral, we cannot expect that the integral of $(\partial_{\tilde{0}}\xi_0)\,dx^0$ necessarily vanishes. This only occurs in the special the case when $\mu = \nu = 0$ or in other words, $t = \tilde{t}$.

Therefore, (1069) can be reduced to

$$\boxed{\tilde{\phi}'_{gauge} = \frac{E}{\hbar c}\oint (\partial_{\tilde{t}}\xi_0)\,dt + \frac{p^i}{\hbar}\oint \partial_i \xi_j dx^j} \tag{1070}$$

This is the irreducible residual gauge freedom in the phase. It is very particular, with only two specific gauge functions persisting: $\partial_{\tilde{t}}\xi_0$ and $\partial_i\xi_j$. We can examine how this gauge freedom will affect each of the phase contributions for the scalar, vector, and tensor components of the metric perturbation. First we write the gauge freedom for each component separately as

$$h'_{00} = h_{00} + 2\partial_{\tilde{t}}\xi_0/c, \qquad h'_{0i} = h_{0i} + \dot{\xi}_i/c + \partial_i\xi_0, \qquad h'_{ij} = h^{\tau\tau}_{ij} + \partial_i\xi_j + \partial_j\xi_i \tag{1071}$$

The phase associated with the scalar, vector, and tensor components of the transformed metric perturbation in (1059) can also be written separately as

$$\tilde{\phi}'_S = \frac{E}{\hbar}\oint h'_{00}dt, \qquad \tilde{\phi}'_V = \frac{1}{\hbar}\oint\left(\frac{E}{c}h'_{0i}dx^i + ch'_{0i}p^i dt\right), \qquad \tilde{\phi}'_T = \frac{1}{\hbar}\oint h'_{ij}p^i dx^j \tag{1072}$$

Substituting each transformation from (1071) into the appropriate corresponding phase expression in (1072) and eliminating gauge terms not found in (1070) gives

$$\tilde{\phi}'_S = \frac{E}{\hbar}\oint (h_{00} + 2\partial_{\tilde{t}}\xi_0/c)\,dt \tag{1073}$$

$$\tilde{\phi}'_V = \frac{1}{\hbar}\oint\left(\frac{E}{c}h_{0i}dx^i + ch_{0i}p^i dt\right) \tag{1074}$$

$$\tilde{\phi}'_T = \frac{1}{\hbar}\oint\left(h^{\tau\tau}_{ij} + \partial_i\xi_j\right)p^i dx^j \tag{1075}$$

Hence we conclude that strangely only $\tilde{\phi}_V$ is found to be gauge-*invariant* while $\tilde{\phi}_S$ and $\tilde{\phi}_T$ are necessarily gauge-*dependent*. However, in the rest frame of the particle, then $p^i = 0$ and therefore $\tilde{\phi}_T$ vanishes as well as the term with $h_{0i}p^i$ in $\tilde{\phi}_V$. Also, in the rest frame we have $E = mc^2$. So adding up the phase expressions in $(1073)-(1075)$ gives

$$\boxed{\tilde{\phi}'_{coupling\ (rest\ \ frame)} = \frac{mc^2}{\hbar}\oint\left(h_{00}dt + \tfrac{1}{c}h_{0i}dx^i\right) + \frac{2mc}{\hbar}\oint\left(\partial_t\xi_0\right)dt} \tag{1076}$$

Note that the first integral contains the phase commonly found in the literature with regard to the scalar and vector gravitational AB effect. However, we find that even in the rest frame of the particle, the phase still contains an irreducible gauge contribution given by a single degree of freedom, ξ_0. This gauge freedom is commonly overlooked in the literature.

We might inquire if working in terms of this *full* gauge freedom would lead to the expression for the phase to be gauge-invariant. If we work with the full non-linear theory, then the gauge freedom of the metric was found in (2406) of Appendix A to be

$$g'^{\mu\nu} = g^{\mu\nu} - \partial^\nu\xi^\mu - \partial^\mu\xi^\nu + \left(\partial_\sigma\xi^\mu\right)\left(\partial^\sigma\xi^\nu\right) \tag{1077}$$

It follows that the gauge transformation for $h'^{\mu\nu}$ will be

$$h'^{\mu\nu} = h^{\mu\nu} + \partial^\nu\xi^\mu + \partial^\mu\xi^\nu - \left(\partial_\sigma\xi^\mu\right)\left(\partial^\sigma\xi^\nu\right) \tag{1078}$$

The contribution to the phase from the gauge freedom in (1066) will become

$$\tilde{\phi}'_{\substack{gauge \\ (non-linear)}} = \frac{p^\mu}{\hbar}\oint\left[\partial^\nu\xi^\mu + \partial^\mu\xi^\nu - \left(\partial_\sigma\xi^\mu\right)\left(\partial^\sigma\xi^\nu\right)\right]dx^\nu \tag{1079}$$

We have already demonstrated that the closed-loop integral of $\partial^\nu\xi^\mu dx_\nu$ must vanish. Then the remaining contribution to the phase due to the gauge freedom is

$$\tilde{\phi}_{gauge\ (full)} = -\frac{p_\mu}{\hbar}\oint\left[\partial^\mu\xi^\nu - \left(\partial_\sigma\xi^\mu\right)\left(\partial^\sigma\xi^\nu\right)\right]dx_\nu \tag{1080}$$

Since the integral is with respect to dx^ν, then $\left(\partial_\sigma\xi^\mu\right)$ in the second term can be treated as a constant. In that case, the second term has the same problem as the first term, namely, we cannot justify that the closed-loop integral of $\left(\partial^\sigma\xi^\nu\right)dx^\nu$ should necessarily be zero. Although we did not transform p_μ and dx^ν along with the metric, we showed earlier in (1063) that this will only introduce additional gauge terms which will not cancel. Therefore, we conclude that even in the fully non-linear theory, the phase expression in terms of the metric perturbation remains gauge-dependent.

10.4 Gauge-invariance of the full phase

Returning to the case of electromagnetism in (1056), we can reparameterize the integrals so they are expressed in terms of proper time, $d\tau$, by using

$$qdx^{\mu} = q\frac{dx^{\mu}}{d\tau}d\tau = qu^{\mu}d\tau = J^{\mu}d\tau \tag{1081}$$

where J^{μ} is the relativistic four-current density. Then (1056) becomes

$$\tilde{\phi} = \frac{1}{\hbar}\oint A_{\mu}J^{\mu}d\tau = \frac{1}{\hbar}\iint F_{\mu\nu}J^{\mu}\wedge dx^{\nu}d\tau \tag{1082}$$

This is could be considered a more natural way to express the phase for electromagnetism since it is written in terms of purely electromagnetic quantities. However, it is still important to note that the integral for A_{μ} is still a *closed*-path integral which is required to preserve gauge-invariance. In practice, this implies that one might integrate $d\tau$ along half of a closed path (such as along half of a circle that surrounds a cylindrical solenoid), and then continue the integration along the other half of the path with the interpretation that the integral is essentially summing *backwards* in time. In that sense, although the integration with respect to time, it is still essentially a *closed* path.

We may apply a similar process to reparameterize the gravitational phase in (1054) in terms of $d\tau$ by using

$$dx^{\mu} = \frac{dx^{\mu}}{d\tau}d\tau = u^{\mu}d\tau = \frac{1}{m}p^{\mu}d\tau \tag{1083}$$

Then the phase in (1054) becomes

$$\tilde{\phi} = \frac{1}{m\hbar}\oint h_{\mu\nu}p^{\mu}p^{\nu}d\tau \tag{1084}$$

We know that $p^{\mu}p_{\mu} = g_{\mu\nu}p^{\mu}p^{\nu}$ is a relativistic invariant quantity. Using $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ gives

$$p^{\mu}p_{\mu} = \left(\eta_{\mu\nu} + h_{\mu\nu}\right)p_{\mu}p_{\nu} = \eta^{\mu\nu}p^{\mu}p^{\nu} + h^{\mu\nu}p^{\mu}p^{\nu} \tag{1085}$$

Therefore we see that the phase expressed in (1084) is only *part* of the relativistic invariant quantity in (1085). In particular, it is the contribution to the phase which is due to coupling to gravitation. The *full* phase includes the free particle contribution as well as the contribution due to the gravitational coupling. In fact, we can write the full phase as

$$\tilde{\phi}_{full} = \frac{1}{m\hbar}\oint g_{\mu\nu}p^{\mu}p^{\nu}d\tau \tag{1086}$$

We can also return to writing the integral in terms of dx^{ν} (rather than $d\tau$), as well as use the inverse metric so that $p^{\mu}p_{\mu} = g^{\mu\nu}p_{\mu}p_{\nu}$. Then writing (1086) in terms of the transformed inverse metric, $g'^{\mu\nu}$, gives

$$\tilde{\phi}'_{full} = \frac{1}{\hbar}\oint g'^{\mu\nu}p_{\mu}dx_{\nu} \tag{1087}$$

If we write (1077) using $g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$, then we have

$$\tilde{\phi}'_{full} = \frac{1}{\hbar}\oint \left(\eta'^{\mu\nu} + h'^{\mu\nu}\right)p_{\mu}dx_{\nu} \tag{1088}$$

From (2417) we know that $h'^{\mu\nu}$ transforms as

$$h'^{\mu\nu} = h^{\mu\nu} + \partial^{\mu}\xi^{\nu} + \partial^{\nu}\xi^{\mu} \tag{1089}$$

Similarly, by replacing $g^{\mu\nu}$ with $\eta^{\mu\nu}$ in (2405), we can observe that $\eta'^{\mu\nu}$ transforms (to first order in $\partial^{\nu}\xi^{\mu}$) as

$$\eta'^{\mu\nu} = \eta^{\mu\nu} - \left(\partial_{\sigma}\xi^{\mu}\right)\eta^{\sigma\nu} - \left(\partial_{\rho}\xi^{\nu}\right)\eta^{\mu\rho} \tag{1090}$$

Substituting (1089) and (1090) into (1088) gives

$$\tilde{\phi}'_{full} = \frac{1}{\hbar}\oint\left[\eta^{\mu\nu} - \left(\partial_\sigma \xi^\mu\right)\eta^{\sigma\nu} - \left(\partial_\rho \xi^\nu\right)\eta^{\mu\rho} + h^{\mu\nu} + \partial^\mu \xi^\nu + \partial^\nu \xi^\mu\right] p_\mu dx_\nu$$

To lowest order in the metric, we can use $g^{\mu\nu} \approx \eta^{\mu\nu}$ to raise/lower the indices of $\partial_\sigma \xi^\mu$ in order to avoid terms involving the product of $\partial_\sigma \xi^\mu$ and $h^{\mu\nu}$ which are not consistent with a lowest order treatment. So we have

$$\tilde{\phi}'_{full} = \frac{1}{\hbar}\oint\left(\eta^{\mu\nu} - \partial^\nu \xi^\mu - \partial^\mu \xi^\nu + h^{\mu\nu} + \partial^\mu \xi^\nu + \partial^\nu \xi^\mu\right) p_\mu dx_\nu \tag{1091}$$

$$= \frac{1}{\hbar}\oint\left(\eta^{\mu\nu} + h^{\mu\nu}\right) p_\mu dx_\nu \tag{1092}$$

$$= \frac{1}{\hbar}\oint g^{\mu\nu} p_\mu dx_\nu \tag{1093}$$

$$= \tilde{\phi}_{full} \tag{1094}$$

Hence we find that the *full* phase (which involves the free particle motion as well as the coupling to gravitation) is in fact gauge-invariant. However, this gauge-invariance is not unique to the phase. It is simply due to the invariance of the four-momentum. The calculation above could have been done solely with the *integrand* without the closed path integral playing any role in the cancellation of the gauge. This is in stark contrast to the case in electromagnetism where it is specifically due to the closed path integration that the gauge freedom vanishes.

10.5 A generalized gauge-invariant phase

In the cause of electromagnetism, we find that the phase can be expressed in terms of the gauge-dependent quantity A_μ or the gauge-invariant quantity, $F_{\mu\nu}$. Since the Maxwell field equations can be written as $\partial^\nu F_{\mu\nu} = -\mu_0 J^\mu$, then we see that $F_{\mu\nu}$ is an appropriate quantity to relate to the phase.

A possible gauge-invariant phase in terms of the Einstein tensor

Likewise, for gravitation, the Einstein equation is $G_{\mu\nu} = \kappa T_{\mu\nu}$ therefore, we might expect that $G_{\mu\nu}$ is the appropriate quantity to express the phase in terms of.

As shown in (1056), the phase in electromagnetism is gauge-invariant and can be formulated in terms of the field strength tensor which is inherently a gauge-invariant quantity. Therefore we might expect that if there is a gauge-invariant phase for gravitation, then it might be formulated in terms of an inherently gauge-invariant quantity such as the Einstein tensor or the Riemann tensor. Therefore, the phase might be written as

$$\tilde{\phi} \stackrel{?}{=} \frac{A}{\hbar} \iint_A G_{\mu\nu} dx^\mu dx^\nu \tag{1095}$$

where A is an appropriate constant making $\tilde{\phi}$ dimensionless.[101] However, there are at least two problems we can identify with this formulation.

1. There is no coupling of the field, $G_{\mu\nu}$, to the particle, p^μ. Notice in (1056) that the physical quantity which couples the charged particle to the fields is q. This constant appears in both the expression for A_μ as well as the expression for $F_{\mu\nu}$. (Since it is a scalar, it simply comes out of the integral.) In the gravitational case, the quantity which couples to the field is p^μ. (In the case of $p^0 = mc$, where we have used $E = mc^2$, it is again a constant which comes out of the integral while in the case of p^i it remains in the integral.) The key point is that this coupling quantity does not appear on both sides of (1095). Attempting to include p^μ would require contracting another index. However, both indices of $G_{\mu\nu}$ have already been contracted with $dx^\mu dx^\nu$. We cannot omit one of these distance differentials otherwise there would no longer be a proper Stokes' theorem relationship that involves a line integral related to a surface integral.

2. The integral of $G_{\mu\nu}$ vanishes in vacuum even for *non*vanishing fields. In the case of *vacuum* solutions, the electromagnetic field equations involving sources, $\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$, become $\partial_\nu F^{\mu\nu} = 0$ which have non-trivial solutions for $F^{\mu\nu}$. This is obviously the case for electromagnetic waves in vacuum. Therefore, because the *derivative* of $F^{\mu\nu}$ is related to the sources, there is the possibility of setting the sources to zero and still having non-vanishing electromagnetic fields. Consequently, the phase given by (1056) can be non-zero even in vacuum.

However, in the case of gravity, the Einstein equation, $G_{\mu\nu} = \kappa T_{\mu\nu}$, involves a direct proportionality between $G_{\mu\nu}$ and the sources. As a result, vacuum solutions require $G_{\mu\nu} = 0$ and hence the phase in (1095) must vanish, even though there may be *non-zero* gravitational fields in the vacuum. For example, for gravitational waves in vacuum, (330) gives $\Box h_{ij}^{\tau\tau} = 0$. In fact, as (2468) shows, the linearized Einstein equation in the harmonic gauge for the case of vacuum $\left(T_{\mu\nu} = 0\right)$ gives $\Box \bar{h}_{\mu\nu} = 0$. This means that *all* the components of the metric perturbation (not just $h_{ij}^{\tau\tau}$) satisfy homogeneous wave equations which could have non-trivial solutions.

[101] Since $G_{\mu\nu}$ has dimensions of (distance)$^{-2}$ then integrating over $dx^\mu dx^\nu$ makes the integral dimensionless. However, $\hbar$ has dimensions of (momentum·distance) therefore A would need to have the same dimensions to keep $\tilde{\phi}$ dimensionless.

A possible gauge-invariant phase in terms of the Riemann tensor

Since the phase in terms of the metric perturbation was formulated in terms of a four-dimensional "closed-loop" integral of $h_{\mu\nu}p^{\mu}$, we might expect that applying a higher dimensional version of Stokes' theorem would allow us to formulate the phase in terms of a four-dimensional hyper-surface "flux" integral of $R_{\mu\alpha\beta\nu}$.

This leads us to consider an expression for the phase that would be even more general than (1095). The obvious remaining choice is to construct a phase expression in terms of the *Riemann* curvature tensor which is non-zero even for vacuum solutions of the metric as long as there is a truly non-vanishing curvature of space-time. The replacement of (1095) would be[102]

$$\tilde{\phi} = \frac{1}{\hbar} \oint_{\partial A} g_{\mu\nu} p^{\mu} dx^{\nu} \stackrel{?}{=} \frac{1}{\hbar} \iiint_{\tilde{A}} R_{\mu\alpha\beta\nu} p^{\mu} dx^{\alpha} dx^{\beta} \wedge dx^{\nu} \tag{1096}$$

Notice that we have expressed the left side in terms of $g_{\mu\nu}$, not $h_{\mu\nu}$, since we know that it is $g_{\mu\nu}p^{\mu}dx^{\nu}$ which is invariant, not $h_{\mu\nu}p^{\mu}dx^{\nu}$. On the right side, notice we have resolved the requirement to have a coupling of the field to the particle, p^{μ}, and to have a Stokes' theorem relationship between both sides. We also know that the Riemann tensor does not vanish if there are is non-zero curvature, even if the sources are zero for the case of vacuum solutions to the metric. Lastly, we find that the expression involving the integral of the Riemann tensor is dimensionless as we would expect for the phase. This follows from the fact that $R_{\mu\nu\gamma\delta}$ has dimensions of (distance)$^{-2}$ so integrating over $p^{\mu}dx^{\nu}dx^{\gamma}dx^{\delta}$ makes the integral have dimensions of angular momentum which cancels with the dimensions of $\hbar$.

To further motivate this expression, we can consider the linearized Riemann tensor (with lowered indices) in (2473) of Appendix C which we can write as[103]

$$R_{\mu\alpha\beta\nu} = \tfrac{1}{2}\left(\partial_{\nu}\partial_{\mu}g_{\beta\alpha} - \partial_{\nu}\partial_{\alpha}g_{\mu\beta} - \partial_{\beta}\partial_{\mu}g_{\nu\alpha} + \partial_{\beta}\partial_{\alpha}g_{\mu\nu}\right) \tag{1097}$$

Collecting common derivatives gives

$$R_{\mu\alpha\beta\nu} = \tfrac{1}{2}\left[\partial_{\beta}\left(\partial_{\alpha}g_{\mu\nu} - \partial_{\mu}g_{\nu\alpha}\right) - \partial_{\nu}\left(\partial_{\mu}g_{\beta\alpha} - \partial_{\alpha}g_{\mu\beta}\right)\right] \tag{1098}$$

We can define the following rank-3 anti-symmetric tensor

$$S_{\alpha\mu\nu} \equiv \partial_{\alpha}g_{\mu\nu} - \partial_{\mu}g_{\nu\alpha} \tag{1099}$$

Then the linearized Riemann tensor can be written as

$$R_{\mu\alpha\beta\nu} = \tfrac{1}{2}\left(\partial_{\beta}S_{\alpha\mu\nu} - \partial_{\nu}S_{\mu\beta\alpha}\right) \tag{1100}$$

Therefore the linearized Riemann tensor is an anti-symmetric derivative of $S_{\alpha\mu\nu}$, where $S_{\alpha\mu\nu}$ is also an anti-symmetric derivative of $g_{\mu\nu}$. In other words, the linearized Riemann tensor is essentially the second anti-symmetric derivative of $g_{\mu\nu}$. We can express these relationships using a higher dimensional Levi-Civita. Then (1099) would be written as

$$S_{\alpha\mu\nu} = \varepsilon_{\alpha\mu\nu\lambda\rho\sigma}\partial^{\lambda}h^{\rho\sigma} \tag{1101}$$

[102] Again we use the exterior product between $dx^{\beta}dx^{\nu}$ since $dx^{\mu}dx^{\nu}$ is a symmetric tensor while $R_{\mu\alpha\beta\nu}$ is anti-symmetric in $\beta\nu$.

[103] It is irrelevant whether we express the linearized Riemann tensor in terms of $g_{\mu\nu}$ or $h_{\mu\nu}$ since $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and the derivatives of $\eta_{\mu\nu}$ are zero anyhow. We simply use $g_{\mu\nu}$ to make the discussion concerning Stoke's theorem easier to follow.

and (1100) would be written as

$$R_{\mu\alpha\beta\nu} = \tfrac{1}{2}\varepsilon_{\mu\alpha\beta\nu\lambda\rho\sigma\delta}\partial^{\lambda}S^{\rho\sigma\delta} \tag{1102}$$

We can also use this notation to express the linearized Riemann tensor directly in terms of the metric perturbation. Since (1101) could be written as $S^{\rho\sigma\delta} = \varepsilon^{\rho\sigma\delta\gamma\kappa\tau}\partial_{\gamma}g_{\kappa\tau}$, then (1102) could be written as

$$R_{\mu\alpha\beta\nu} = \tfrac{1}{2}\varepsilon_{\mu\alpha\beta\nu\lambda\rho\sigma\delta}\partial^{\lambda}\left(\varepsilon^{\rho\sigma\delta\gamma\kappa\tau}\partial_{\gamma}g_{\kappa\tau}\right) \tag{1103}$$

Therefore, we would expect that a double application of Stokes' theorem should make it possible to write an integral relationship between $g_{\mu\nu}$ and $R_{\mu\alpha\beta\nu}$. In general, we know that Stokes' theorem can be used to change the closed boundary integral of a function, $\oint f_{\mu}dx^{\mu}$, into the surface integral of the anti-symmetric derivative of the function, $\int \left(\partial_{\mu}f_{\nu} - \partial_{\nu}f_{\mu}\right)dx^{\mu}dx^{\nu}$. Therefore, a repeated application of Stokes' theorem can be used to effectively keep increasing (or decreasing) the dimensions of integration while also decreasing (or increasing) the derivatives of the integrand. For example, we can begin with the closed boundary integral of a rank-2 tensor such as $g_{\mu\nu}$ and use Stokes' theorem to change the integral into a surface integral in a higher dimension.

$$\iint\limits_{\partial A} g_{\mu\nu}dx^{\mu}dx^{\nu} = \iiint\limits_{A}\left(\partial_{\alpha}g_{\mu\nu} - \partial_{\mu}g_{\nu\alpha}\right)dx^{\alpha}dx^{\mu}dx^{\nu} \tag{1104}$$

We can substitute (1099) for the integrand under the surface integral.

$$\iint\limits_{\partial A} g_{\mu\nu}dx^{\mu}dx^{\nu} = \iiint\limits_{A} S_{\alpha\mu\nu}dx^{\alpha}dx^{\mu}dx^{\nu} \tag{1105}$$

Stokes' theorem requires that we use a *closed* integral on ∂A which forms the boundary of the surface determined by $A = \int dx^{\alpha}dx^{\mu}dx^{\nu}$. Therefore, if we wish to extend the relationship in (1105) to a higher dimension, then we can require that A is now the closed *boundary* of a higher dimensional surface[104] given by $\tilde{A} = \int dx^{\beta}dx^{\alpha}dx^{\mu}dx^{\nu}$. In other words, we now consider that $A = \partial\tilde{A}$. Then applying Stokes' theorem again, we have

$$\iiint\limits_{\partial\tilde{A}} S_{\alpha\mu\nu}dx^{\alpha}dx^{\mu}dx^{\nu} = \iiiint\limits_{\tilde{A}}\left(\partial_{\beta}S_{\alpha\mu\nu} - \partial_{\nu}S_{\beta\alpha\mu}\right)dx^{\beta}dx^{\alpha}dx^{\mu}dx^{\nu} \tag{1106}$$

Because $S_{\beta\alpha\mu}$ is a completely anti-symmetric tensor, then permuting the indices twice leaves the sign unchanged. Then the integrand above becomes $\left(\partial_{\beta}S_{\alpha\mu\nu} - \partial_{\nu}S_{\mu\beta\alpha}\right)$ which matches the linearized Riemann tensor as shown in (1100). Therefore writing (1106) in terms of the Riemann tensor and equating it to (1105) gives

$$\iint\limits_{\partial A} g_{\mu\nu}dx^{\mu}dx^{\nu} = \iiiint\limits_{\tilde{A}} R_{\mu\alpha\beta\nu}dx^{\mu}dx^{\alpha}dx^{\beta}dx^{\nu} \tag{1107}$$

Multiplying both sides by $m/d\tau$ and using $p^{\mu} = m\left(dx^{\mu}/d\tau\right)$ gives

$$m\int\limits_{\partial A} g_{\mu\nu}p^{\mu}dx^{\nu} = m\iiint\limits_{\tilde{A}} R_{\mu\alpha\beta\nu}p^{\mu}dx^{\alpha}dx^{\beta}dx^{\nu} \tag{1108}$$

[104] It has been argued that Stokes' theorem cannot be applied repeatedly to reduce the dimension of the region of integration over and over, because the boundary of the boundary of a set is empty. (The consequence of this principle on fundamental theories of physics is discussed at length in [66].) This would mean that it is not possible for an open surface A (in n dimensions) to be turned into a boundary ∂A which encloses a higher dimensional open surface $\tilde{A}$ (in $n+1$ dimensions). For example, a solid sphere is 3-dimensional and has a unique boundary given by a spherical shell which is 2-dimensional. However, the spherical shell has no 1-dimensional enclosing "boundary." Similarly, a circle is a 1-dimensional boundary of a 2-dimensional disc. However, a disc cannot form a boundary that would enclose any 3-dimensional object.

Notice that the integrals have been effectively reduced by one dimension and there is now a coupling between the field (expressed as $g_{\mu\nu}$ or $R_{\beta\alpha\mu\nu}$) and a particle with four-momentum p^{μ}. Lastly, dividing both sides by $m\hbar$ allows us to write the phase as

$$\tilde{\phi} = \frac{1}{\hbar}\oint_{\partial A} g_{\mu\nu} p^{\mu} dx^{\nu} = \frac{1}{\hbar}\iiint_{\tilde{A}} R_{\mu\alpha\beta\nu} p^{\mu} dx^{\alpha} dx^{\beta} dx^{\nu} \tag{1109}$$

Note that this result applies only to the *linearized* Riemann tensor. We emphasize that we used the *linearized* Riemann tensor in (1097) to develop the phase expression in (1109). This led to the anti-symmetric properties that were exploited via Stokes' theorem in this treatment. It is unknown to the author whether the fully non-linear Riemann tensor could still lead to a relationship such as (1109). However, the primary objective was to develop an expression that is gauge-*invariant*. Since the linearized Riemann tensor is still gauge-invariant, then this purpose was fulfilled.

10.6 The quantum phase in terms of the Helmholtz Decomposition metric

Here we consider the quantum phase expressed in terms of the HD metric components. We can use the metric components from $(175)-(177)$ and the gauge vector given in (201) as

$$\xi_{\mu} = (cA,\ B_i + \partial_i C) \tag{1110}$$

Then the phase in $(1073)-(1075)$ for each metric component becomes

$$\text{Scalar coupling phase:} \qquad -\frac{2E}{\hbar}\oint\left(\phi/c^2 + \partial_{\bar{t}}A\right)dt \tag{1111}$$

$$\text{Vector coupling phase:} \qquad \tilde{\phi}_V = \frac{1}{\hbar}\oint\left[\frac{E}{c}\left(\beta_i + \partial_i\alpha\right)dx^i + c\left(\beta_i + \partial_i\alpha\right)p^i dt\right] \tag{1112}$$

$$\begin{aligned}\text{Tensor coupling phase :} \qquad \tilde{\phi}_T = \frac{1}{\hbar}\oint & \left[h_{ij}^{\tau\tau} + \tfrac{1}{3}\delta_{ij}H + \partial_i\varepsilon_j + \partial_j\varepsilon_i\right. \\ & \left. + \left(\partial_i\partial_j - \tfrac{1}{3}\delta_{ij}\nabla^2\right)\lambda + \partial_i B_j\right]p^i dx^j\end{aligned} \tag{1113}$$

Due to the closed-loop integrals, the following terms will vanish: $\partial_i\alpha dx^i$, $\partial_j\varepsilon_i dx^j$, and $\partial_j\left(\partial_i\lambda\right)dx^j$. We can also use $\Theta = \frac{1}{3}\left(H - \nabla^2\lambda\right)$. Then the phases reduce to

$$\tilde{\phi}_S = -\frac{2E}{\hbar}\oint\left(\phi/c^2 + \partial_{\bar{t}}A\right)dt \tag{1114}$$

$$\tilde{\phi}_V = \frac{1}{\hbar}\oint\left[\frac{E}{c}\beta_i dx^i + c\left(\beta_i + \partial_i\alpha\right)p^i dt\right] \tag{1115}$$

$$\tilde{\phi}_T = \frac{1}{\hbar}\oint\left[h_{ij}^{\tau\tau} + \Theta\delta_{ij} + \partial_i\varepsilon_j + \partial_i B_j\right]p^i dx^j \tag{1116}$$

Recall that $h_{ij}^{\tau\tau}$ is a gauge-*invariant* potential as well as

$$\Phi = \phi + \dot{\alpha} - \ddot{\lambda}/2, \qquad \Theta = \frac{1}{3}\left(H - \nabla^2\lambda\right), \qquad \Xi_i = \beta_i - \dot{\varepsilon}_i \tag{1117}$$

We can immediately identify the fact that the only gauge-invariant potential that can appear in the phase expressions above is $h_{ij}^{\tau\tau}$. Although, $\Theta = \frac{1}{3}\left(H - \nabla^2\lambda\right)$ also appears in the tensor coupling phase, the presence of $\partial_i\partial_j\lambda$ introduces gauge freedom that cannot be removed while still preserving Θ. Therefore, in the near-field, we find that all the quantum phase contributions must be gauge-*dependent* and therefore can be removed by transforming into an appropriate frame as given by the gauge transformations found in $(238)-(243)$. In the far-field, we know that $h_{ij}^{\tau\tau}$ is the only potential that remains since it is a radiative field. Therefore, in the far-field, the quantum phase is given by

$$\tilde{\phi}_T = \frac{1}{\hbar}\oint h_{ij}^{\tau\tau}p^i dx^j \tag{1118}$$

Notice that if we choose to integrate along a path that is always *parallel* to the particle velocity, then v^i and dx^i always have the same index values and therefore the only contribution to the phase comes from h_{ii}^{TT}. This means that only a *plus*-polarization wave will contribute to the phase since it involves h_{xx}^{TT} and h_{yy}^{TT}. On the other hand, if we choose to integrate along a path that is always *perpendicular* to the particle velocity, then v^i and dx^j always have a *different* index value and therefore the only contribution to the phase comes from $h_{ij\ (i\neq j)}^{\tau\tau}$. This means that only a *cross*-polarization wave will contribute to the phase since it involves $h_{xy}^{\tau\tau}$ and $h_{yx}^{\tau\tau}$. Lastly, if we choose to integrate along a path neither perpendicular nor perpendicular to the particle velocity, then v^i and dx^j can have both matching as well as differing index values and therefore both *plus*-polarization *and cross*-polarization waves will contribute to the phase.

We also note that the strain field given by $h_{ij}^{\tau\tau}$ does not necessarily have to be associated with gravitational waves. In a later section we will consider the case of a "mass solenoid" where it is possible to have a *steady-state* "mass current" so that the transverse-traceless stress, $T_{ij}^{\tau\tau}$, given in (409) as

$$T_{ij}^{\tau\tau} = v_i v_j \left(\rho + P/c^2\right) - \tfrac{1}{3}\delta_{ij} v^2 \left(\rho + P/c^2\right) \tag{1119}$$

does not vary with time. In that case, the wave equation given in (333) as $\Box h_{ij}^{\tau\tau} = -2\kappa T_{ij}^{\tau\tau}$ reduces to a Poisson equation, $\nabla^2 h_{ij}^{\tau\tau} = -2\kappa T_{ij}^{\tau\tau}$. Then $h_{ij}^{\tau\tau}$ is no longer a radiative field and does not drop off as $1/r$ in the far-field. Rather, it will drop off as $1/r^2$ like the other non-radiative fields. This means that even as we let $r \to \infty$, we do not have a region of space where all the quantities in h_{ij} become negligible except for $h_{ij}^{\tau\tau}$. Instead, all potentials are equally negligible (or non-negligible) in every region of space. As a result, the tensor coupling phase in (1113) cannot be reduced to (1118) and therefore there is no gauge-invariant phase expression. In fact, from this analysis we can simply state that there is no gauge-invariant phase for any non-radiative gravitational fields in the Helmholtz Decomposition (HD) formulation of linearized GR.[105]

[105] We are careful to state our conclusion here as being a result of using the HD formulation of linearized GR. This formulation required particular boundary conditions (such as $h_{\mu\nu} \to 0$ as $r \to \infty$). There may possibly be another formulation which would allow for a gauge-invariant phase expression for non-radiative gravitational fields. In fact, if a phase were to be expressed in terms of the Einstein tensor or Riemann tensor, as considered in a later section, then it would necessarily be gauge invariant.

10.7 An AB effect for the gravito–vector and gravito–scalar potentials

Next we describe the Aharonov-Bohm (AB) effect by considering the potentials and vector fields of a "mass solenoid." We show that the GEM field equations imply that there is a gauge-*invariant* AB effect for the gravito-*vector* and gravito-*scalar* potentials. However, because the geodesic equation of motion in terms of the GEM fields is necessarily gauge-*dependent,* the associated AB effect must also be gauge–*dependent*.

The original AB effect introduced a quantum mechanical manifestation of the electromagnetic scalar and vector potentials [67]. In a recent paper [4], we examined a gravitational vector AB effect in terms of the gravito-vector potential defined as $\vec{h} = c\,(h_{01}, h_{02}, h_{03})$. However, as we have shown in previous sections, the gravito-vector potential given by $\vec{h}$, and the associated gravito-magnetic field written as $\tilde{\vec{B}}_G = \nabla \times \vec{h}$, are gauge-*dependent* quantities. Therefore the associated AB effect is *not* gauge-invariant.

We might therefore speculate that there is a gauge-*invariant* AB effect given in terms of $\vec{\Xi}$ since this is a gauge-*invariant* potential. According to the usual procedure, we consider a configuration where the gravito-magnetic field vanishes $\left(\tilde{\vec{B}}_G = 0\right)$ and yet $\vec{\Xi} \neq 0$ so that only $\vec{\Xi}$ affects Ψ and $\tilde{\vec{B}}_G$ does not. We can consider a very long rotating cylindrical "mass solenoid".of length L and radius R with the axis along the z-axis from $z = -L/2$ to $z = L/2$.

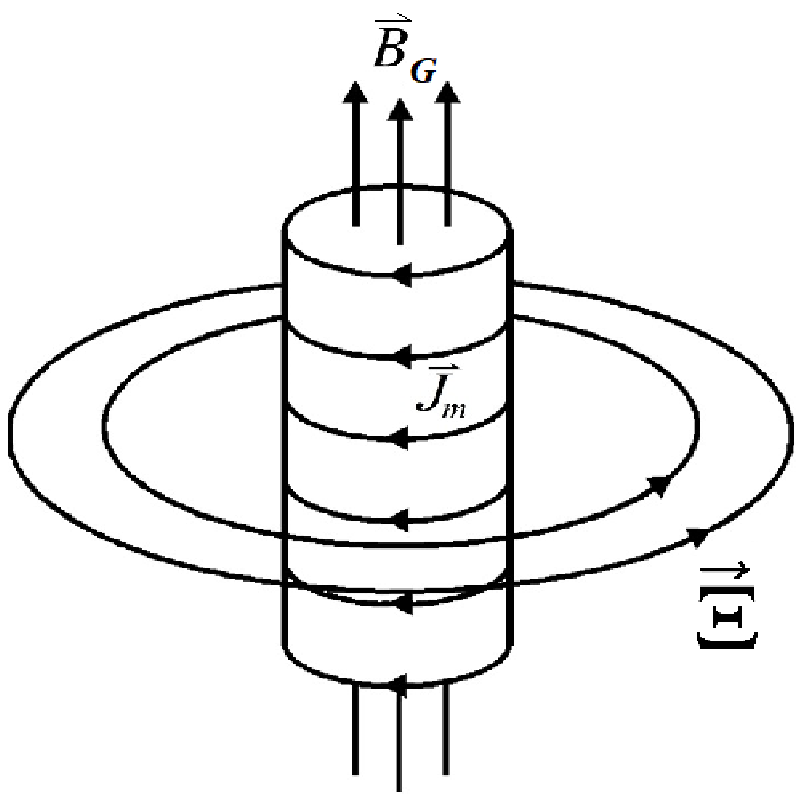

Figure 5: A mass solenoid producing a gauge-invariant gravito-magnetic field

At a point far from either end $(|z| << L)$ and close to the solenoid wall $(r \approx R$ but with $r > R)$, we effectively have $\tilde{\vec{B}}_G = 0$ while $\vec{\Xi} \neq 0$. We can apply the same methods commonly used for the magnetic AB effect. The details are identical to previous sections which were calculated in terms of the gauge-dependent potentials and fields, φ_G, $\vec{E}_G$, $\vec{h}$ and $\vec{B}_G$. These are simply replaced here with the gauge-invariant potentials and fields, $\tilde{\varphi}_G$, $\tilde{\vec{E}}_G$, $\vec{\Xi}_G$ and $\tilde{\vec{B}}_G$, respectively. In this model, we neglect terms involving pressure and use the field equations for relativistic dust given in (431). We can also define an effective mass density, $\rho_{eff} \approx \rho\left(1+\frac{v^2}{2c^2}\right)$. Then following the procedure in Section 9, we would find the gauge-invariant gravito-scalar potential, φ_G, to have the same form as (112) which gives

$$\varphi_G(r) \quad = \quad \frac{R^2\rho_{eff}}{2\pi\varepsilon_G}\ln\left(\frac{r}{r_0}\right) \tag{1120}$$

Likewise, the gauge-invariant gravito-vector potential, $\vec{\Xi}$, would have the same form as (115) which gives

$$\vec{\Xi} \quad = \quad \frac{\Phi_{\vec{B}_G}}{2\pi r}\hat{\phi} \tag{1121}$$

Also, like (124), this can be written in terms of the physical parameters of the mass solenoid as[106]

$$\vec{\Xi} \quad = \quad \frac{\mu_G R^4\rho\omega}{4r}\hat{\phi} \tag{1122}$$

We now consider the phase which would result from the gravito-scalar potential and the gravito-vector potential. From (1111) we know the gravito-scalar potential will introduce a phase generated by an integral in *time* rather than a *spatial* integral. For simplicity we will neglect this additional interaction and write the Schrödinger equation as

$$i\hbar\partial_t\Psi(\vec{r},t) \quad = \quad \frac{1}{2m}\left(i\hbar\nabla + m\vec{\Xi}\right)^2\Psi(\vec{r},t) \tag{1123}$$

If $\Psi(\vec{r},t)$ is a solution, then we can consider a wavefunction with a phase shift, $\tilde{\phi}$, so that

$$\Psi \quad = \quad e^{i\tilde{\phi}}\tilde{\Psi} \tag{1124}$$

For a quantum particle in the vicinity of the solenoid, we may expect the vector coupling phase[107] to be expressed in terms of $\vec{\Xi}$ as[108]

$$\tilde{\phi}_V \quad = \quad \frac{m}{\hbar}\int_0^r \vec{\Xi}\left(\vec{r}'\right)\cdot d\vec{r}' \tag{1125}$$

If we apply to (1123) the method shown in Appendix O, we arrive at

$$i\hbar\frac{\partial}{\partial t}\tilde{\Psi}(\vec{r},t) \quad = \quad -\frac{\hbar}{2m}\nabla^2\tilde{\Psi}(\vec{r},t) \tag{1126}$$

---

[106]Following the calculation in going from $(115)-(124)$ in Section 9, it can be seen that ρ, not ρ_{eff}, is what comes into the expression for $\vec{\Xi}$. It is only in the gravito-Gauss law that ρ_{eff} is relevant.

[107]A similar vector coupling phase is also derived by Chiao and Speliotopoulos in [65].

[108]The coupling rule leading to the phase in (1125) was developed in a previous section using a Hamiltonian for *non*-relativistic test particles. For the case of fully relativistic test particles, we must use the relativistic Hamiltonian given by (693). However, in this Hamiltonian, $\vec{\Xi}$ does not appear in the minimal coupling. Therefore, we would not have the phase given by (1125).

Hence we see that $\tilde{\Psi}(\vec{r},t)$ is also a solution of Schrödinger's equation but with the absence of $\vec{\Xi}$ which means that $\vec{\Xi}$ just produces a phase factor. We conclude the following:

$$\boxed{\text{Introducing } \vec{\Xi} \text{ causes} \qquad \Psi \implies e^{i\phi}\Psi \qquad \text{where} \qquad \tilde{\phi} = \frac{m}{\hbar}\int \vec{\Xi}\cdot d\vec{r}} \tag{1127}$$

Now we consider a beam of electrons directed toward a superconducting mass solenoid which contains circulating mass currents. The wavefunction of an electron splits at point A, takes paths 1 and 2 around the two sides of the solenoid, and recombines at point B on the other side. An interference pattern emerges.We can

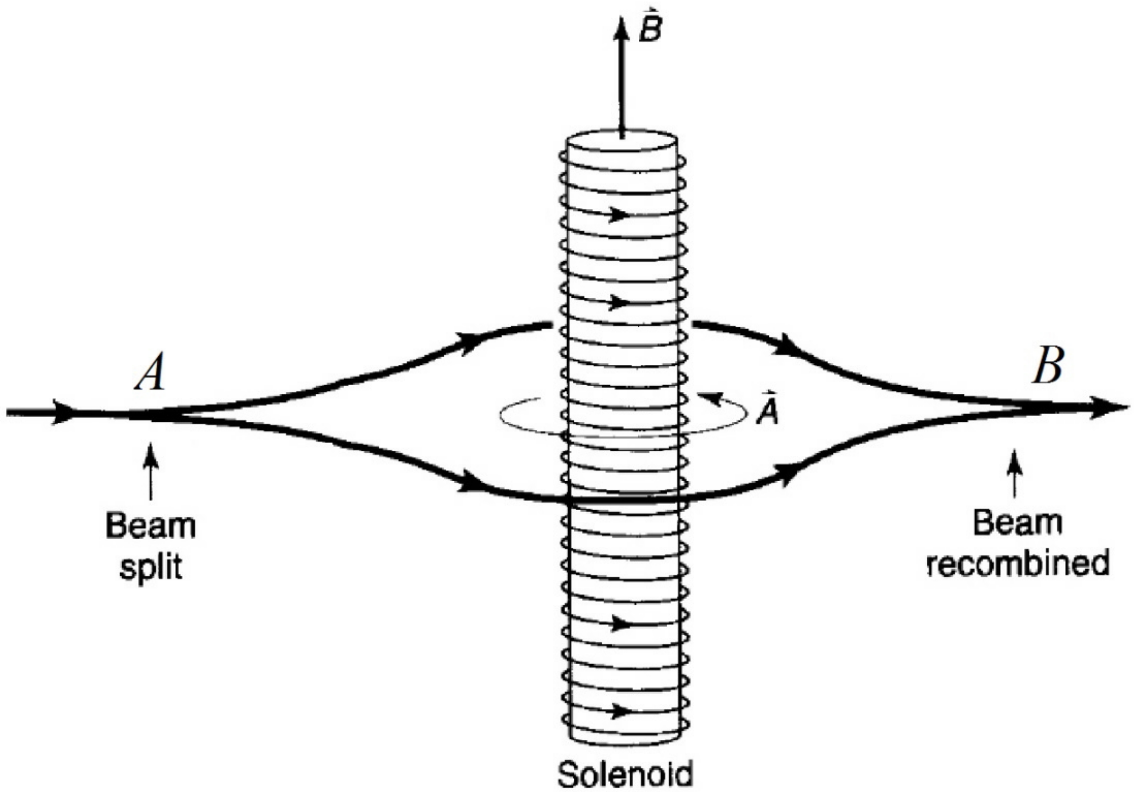

Figure 6: The Aharonov-Bohm effect occurs as an electron beam is split at point A and recombined at point B, on the other side of a solenoid.

consider the wave function of the electron to be a plane wave along paths 1 and 2.

$$\Psi_1 = Ae^{i\left(\vec{k}\cdot\vec{r}_1-\omega t\right)} \qquad \text{and} \qquad \Psi_2 = Ae^{i\left(\vec{k}\cdot\vec{r}_2-\omega t\right)} \qquad \text{where} \quad k = \frac{2\pi}{\lambda} = \frac{p}{\hbar} \tag{1128}$$

If points A and B are far from the solenoid, then we can use the double-slit model from classical optics to describe the difference in optical path lengths as shown in the following diagram.

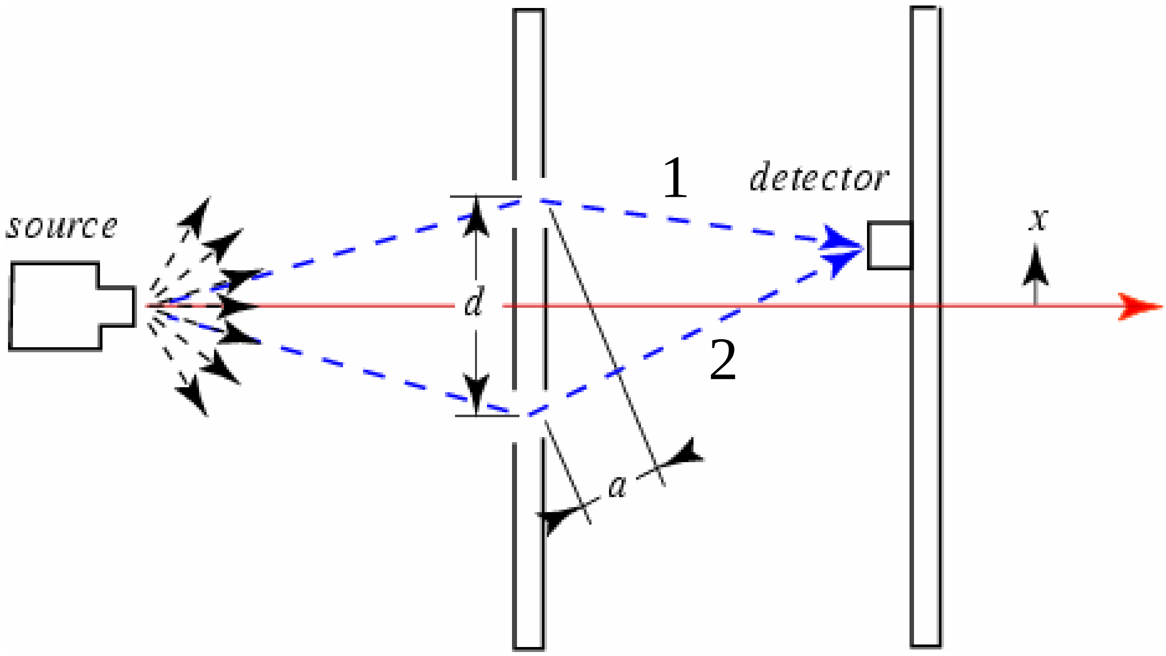

Figure 7: Double-slit experiment for observing the Aharonov-Bohm effect

If the cylindrical mass shell is not spinning, then we have

$$\vec{J}_m = 0, \qquad \tilde{\vec{B}}_G = 0, \qquad \Phi_{\vec{B}_G} = 0, \qquad \vec{\Xi} = 0 \tag{1129}$$

Then the interference pattern depends only on the difference between traveled paths so we have

$$\tilde{\phi} = \vec{k}\cdot\vec{r}_1 - \vec{k}\cdot\vec{r}_2 = k(r_1 - r_2) = \frac{2\pi}{\lambda}(r_1 - r_2) = \frac{2\pi}{\lambda}a \tag{1130}$$

If the interference pattern is detected far from the solenoid, then $x << L$ and $a \approx \left(\frac{x}{L}\right) d$. So we have

$$\tilde{\phi} = \frac{2\pi d x}{\lambda L} \tag{1131}$$

where $\tilde{\phi} = 2n\pi$ at maxima and $\tilde{\phi} = \left(n + \frac{1}{2}\right)\pi$ at minima. If the cylindrical mass shell is spinning, then we have

$$\vec{J}_m \neq 0, \qquad \vec{B}_G \neq 0, \qquad \Phi_{gm} \neq 0, \qquad \vec{\Xi} \neq 0 \tag{1132}$$

Since (1125) shows that $\tilde{\phi}_V = \frac{m_e}{\hbar}\int_0^r \vec{\Xi}(\vec{r}\,')\cdot d\vec{r}\,'$, then each wavefunction picks up a phase.

$$\Psi_1 = A e^{i\tilde{\phi}_{V,1}} e^{i\left(\vec{k}\cdot\vec{r}_1 - \omega t\right)} \quad \text{and} \quad \Psi_2 = A e^{i\tilde{\phi}_{V,2}} e^{i\left(\vec{k}\cdot\vec{r}_2 - \omega t\right)} \tag{1133}$$

where

$$\tilde{\phi}_{V,1} = \frac{m_e}{\hbar}\int_{Path\ 1} \vec{\Xi}(\vec{r})\cdot d\vec{r} \quad \text{and} \quad \tilde{\phi}_{V,2} = \frac{m_e}{\hbar}\int_{Path\ 2} \vec{\Xi}(\vec{r})\cdot d\vec{r} \tag{1134}$$

Note that Ψ_1 moves in a direction *opposite* of $\vec{\Xi}$ on path 1, while Ψ_2 moves in the *same* direction as $\vec{\Xi}$ on path 2. Therefore, the two waves pick up opposite phases on their paths. The phase difference is

$$\Delta\tilde{\phi}_V = \tilde{\phi}_{V,2} - \tilde{\phi}_{V,1} \tag{1135}$$

$$\Delta\tilde{\phi}_V = \frac{m_e}{\hbar}\left(\int\limits_{Path\ 2} \vec{\Xi}(\vec{r})\cdot d\vec{r} - \int\limits_{Path\ 1} \vec{\Xi}(\vec{r})\cdot d\vec{r}\right) \tag{1136}$$

Since both integrals have the same upper and lower bounds (from point A to point B in the earlier figure), then integrating along paths 1 and 2 forms a closed loop around the solenoid. Therefore we have

$$\Delta\tilde{\phi}_V = \frac{m_e}{\hbar}\oint\limits_{\substack{Around\\ solenoid}} \vec{\Xi}\cdot d\vec{r} \tag{1137}$$

By applying Stokes' theorem, as we did to obtain (1121), the integral becomes the gravito-magnetic flux.

$$\Delta\tilde{\phi}_V = \frac{m_e\Phi_{\vec{B}_G}}{\hbar} \tag{1138}$$

Since $\tilde{\phi} = \dfrac{2\pi dx}{\lambda L}$ from (1131), then

$$\Delta\tilde{\phi} = \frac{2\pi d\Delta x}{\lambda L} \tag{1139}$$

Equating (1138) and (1139) and solving for Δx gives

$$\Delta x = \frac{\lambda L m_e \Phi_{\vec{B}_G}}{hd} \tag{1140}$$

In (123) we found that the flux can also be expressed as $\Phi_{\vec{B}_G} = \mu_G \pi R^4 \rho\,\omega_{sol}/2$ where ρ is the uniform mass density of a solid mass solenoid, and ω_{sol} is the angular frequency of its rotation. Substituting this into (1140) gives a result completely in terms of the physical parameters of the system.

$$\boxed{\Delta x = \frac{\lambda L m_e \mu_G \pi R^4 \rho\,\omega_{sol}}{2hd}} \tag{1141}$$

From the result in (1140) we would expect that the effect of the gauge-*invariant* vector potential $\vec{\Xi}$ is to produce a *fringe shift* in the interference pattern. Since $\vec{\Xi}$ is gauge-*invariant*, then we would anticipate that this is a *gauge-invariant* gravitational Aharonov-Bohm effect.

In the electromagnetic case, the argument is often made that because the magnetic field is gauge-invariant and the flux of the magnetic field is non-zero, then the line integral of the magnetic vector potential cannot be removed by a choice of gauge. The argument is even stronger in the *gravitational* case considered here since the gravito-magnetic field, $\vec{B}_G$, is gauge-invariant, and the vector potential, $\vec{\Xi}$, is *also* gauge-invariant. This is further reason that the line integral in (1121) cannot be made to vanish by a gauge choice.[109]

[109] This is assuming the approximations for linearized GR used in Appendix A remain valid. Specifically, any linear coordinate transformation given by $x'^{\mu} = x^{\mu} - \xi^{\mu}$ must keep ξ^{μ} sufficiently small so that $\partial^{\mu}\xi^{\nu}$ is on the order of $h^{\mu\nu}$ which must satisfy $|h^{\mu\nu}| << 1$.

However, we also point out that $\vec{\Xi} = \beta_i - \dot{\vec{\varepsilon}}$ involves terms from both h_{0i} and h_{ij}. In the first-order post-Newtonian limit for slow moving gravitational sources, we have $h_{ij(i\neq j)} \approx 0$ and therefore $\vec{\varepsilon} = 0$. Then $\vec{\Xi} \approx \vec{\beta}$ which means $\vec{\Xi}$ is no longer gauge-invariant and consequently $\tilde{\vec{B}}_G$ is also no longer gauge-invariant. Therefore, the gauge-invariance of (1121) only holds in the case of fully relativistic gravitational sources for which we *cannot* approximate $h_{ij(i\neq j)} \approx 0$. This means that the field equations using an ideal fluid given by (429) or (431) would lead to (1121) being gauge-invariant. However, the field equations given by (454) for *non*-relativistic dust would *not* leave (1121) gauge-invariant.

The gauge-invariance described here was based on arguments concerning the potentials, fields, and field equations. Specifically, the argument was based on the gauge-invariance of the gravito-scalar and vector potentials, the gravito-electric and magnetic fields, and the gravito-Gauss and gravito-Ampere laws. However, we must also consider the *equation of motion*. We found in Section 17 that the geodesic equation of motion contains gauge-freedom regardless of whether one considers test particles that are non-relativistic or relativistic to order v/c or v^2/c^2. Additionally, the geodesic equation of motion contains gauge-freedom regardless of whether one considers the second post-Newtonian order, first post-Newtonian order, or the Newtonian limit. In other words, it is not possible to write the geodesic equation of motion purely in terms of Φ, Θ and $\vec{\Xi}$ which are the gauge-invariant, *near*-field potentials that would be relevant in close proximity to a mass solenoid.

In fact, it was also shown in the previous two sections that there is no coupling rule for the canonical momentum that involves $\vec{\Xi} = \vec{\beta} - \dot{\vec{\varepsilon}}$ which is the gauge-*invariant* gravito-vector potential. Rather, the coupling rule only involves $\vec{\Xi}_{PN} = \vec{\beta}$ which is a gauge-*dependent* gravito-vector potential in the first-order post-Newtonian limit. This result is ultimately due to the fact that the Hamiltonian in (851) for second-order post-Newtonian sources and fully relativistic test particles cannot be written in terms of the gauge-invariant potentials, Φ, Θ and $\vec{\Xi}$. This immediately indicates that there are no coupling rules in terms of these gauge-invariant quantities. In other words, we have

$$mc^2 \nRightarrow mc^2 - \tfrac{1}{2}mc^2\varphi_G \qquad \text{and} \qquad \vec{p}_{can} \nRightarrow \vec{p}_{can} - m\vec{\Xi} \tag{1142}$$

where φ_G was found in (345) as $\varphi_G \equiv \dfrac{1}{2}\left(\Phi + \dfrac{c^2}{2}\nabla^2\Theta\right)$. Accordingly, we have

$$\tilde{\phi}_S \neq \frac{m}{\hbar}\oint \varphi_G dt \qquad \text{and} \qquad \tilde{\phi}_V \neq \frac{m}{\hbar}\oint \vec{\Xi}\cdot d\vec{r} \tag{1143}$$

and therefore we never obtain a Schrödinger equation of the form

$$i\hbar\partial_t\Psi(\vec{r},t) \neq \left[\frac{1}{2m}\left(i\hbar\nabla + m\vec{\Xi}\right)^2 + m\varphi_G\right]\Psi(\vec{r},t) \tag{1144}$$

Consequently, although the procedure of demonstrating the local gauge invariance of the wavefunction (as shown in Appendix O) could be mathematically applied to $\vec{\Xi}$ and φ_G, the resulting phase expression and Schrödinger equation would not be valid since they would be consistent with the actual equation of motion of the quantum particle as derived from the classical relativistic Hamiltonian or the geodesic equation of motion. Therefore, we conclude that because the equation of motion of the quantum particle is necessarily gauge-*dependent*, then the associated AB effect (for the scalar and vector potentials) are also necessarily gauge-dependent.

However, as we will show in the following section, the AB effect associated with the *tensor* potential can be gauge-invariant. This follows from the fact that the only gauge-invariant equation of motion is in the far-field when there is a gravitational wave given by $h_{ij}^{\tau\tau}$ which is a gauge-*invariant* quantity. In the far-field, the geodesic equation of motion only contains $h_{ij}^{\tau\tau}$. Therefore, for the case of gravitational waves in the far-field, it is possible to have a gauge-*invariant* AB effect.

10.8 An AB effect for a spherically symmetric, time-varying scalar potential

A common approach to the gravitational scalar AB effect is to consider a quantum wave function which bifurcates into two paths which have different gravitational potential energy differences due to the gravitational potential of *the earth*. (An example is show in Appendix P.) However, here we consider a gravitational scalar AB effect via *hydrogen atom spectroscopy* inside a time-varying spherical mass shell. A sine-wave generator injects a charge $Q(t) = Q_0 \cos \omega t$ onto the surface of a spherical Faraday cage (blue). Inside a cavity (white) carved out of the cage is placed a hydrogen atom (red), which is at rest at the center of the cage. The lowest two unperturbed energy levels of the atom are indicated by E_1 and E_2. A photon γ is emitted upon a transition from E_2 to E_1Since the electrons deposited on the surface of the conductor carry both charge and

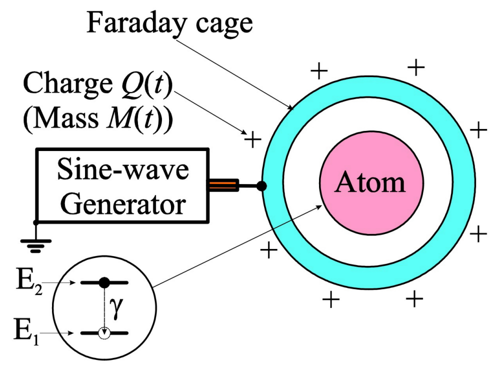

Figure 8: Two-level atom inside a spherical mass shell with a time-dependent mass, $M(t)$, that arises from a time-dependent charge, $Q(t)$.

mass, then there must be a time-varying mass on the surface of the conductor given by

$$m(t) = \frac{m_e}{e} Q(t) \tag{1145}$$

where m_e/e is the mass-to-charge ratio of the electron and $Q(t) = Q_0 \cos(\omega t)$ as we had before. The *total* mass surrounding the atom in the center will therefore be given by $M(t) = m(t) + M_0$ where M_0 is the static background mass such that $M_0 >> m(t)$ at all times. Then the total mass as a function of time can be written as

$$M(t) = \frac{m_e Q_0}{e} \cos(\omega t) + M_0 \tag{1146}$$

As a consequence of this sinusoidally time-varying mass, there must also be a sinusoidally time-varying gravitational scalar potential given by

$$V_G(t) = \frac{4\pi G M(t)}{a} = \frac{M(t)}{a\varepsilon_G} \tag{1147}$$

at every point in the interior of the sphere.[110] Note that in the gravitational case, the sphere does *not* act like a Faraday cage in the sense of shielding the atom at the center from the gravitational field of the mass deposited on the surface of the sphere. However, the field will still be zero everywhere inside the conductor due to the particular geometry, namely, the spherical symmetry. This can be shown using Poisson's equation for Newtonian gravity

$$\nabla^2 V_G = \rho/\varepsilon_G \tag{1148}$$

[110] Note that we are considering a quasi-static limit where the propogation of the potential is much faster than the frequency of oscillation of the mass distribution and therefore Newton's law of gravity is valid to an extremely good approximation even in this time-varying case.

and applying the boundary conditions for the sphere. The field inside the conductor vanishes while the field outside the conductor is simply that of a point mass with mass $M(t)$. Finding the gravitational potential inside the cavity is then just a matter of integrating the field from infinity to the surface of the sphere.

$$V_G(t) = -\int_{\infty}^{a} \frac{GM(t)}{r'^2} dr' \tag{1149}$$

The result is that shown in (1147). Since the field is zero but the potential is constant inside the sphere, then the corresponding energy shifts of the atom are in fact a scalar gravitational AB effect. To describe the *gravitational* scalar AB effect, we now use a treatment directly analogous to the treatment used for the *electric* scalar AB effect. We may modify (3) so that the wavefunction of the atom will be phase modulated by the time-varying gravitational scalar potential (1147) in accordance with the time-dependent Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi = (H_0 + U(t) + U_G(t))\,\psi \tag{1150}$$

Here we have H is the total Hamiltonian, H_0 is the unperturbed Hamiltonian of the atom before the charge $Q(t)$ and corresponding mass $M(t)$ is injected onto the exterior surface of the conductor, and $U(t)$ is the *electric* potential energy and $U_G(t)$ is the *gravitational* potential energy of the atom at rest in the trap. Similar to (4) expressing the electric potential energy, likewise we can express the gravitational potential energy using (1146) and (1147).

$$U_G(t) = m_e V_G(t) = \frac{4\pi G m_e M(t)}{a} \tag{1151}$$

$$= \frac{4\pi G m_e}{a}\left(\frac{m_e Q_0}{e}\cos(\omega t) + M_0\right) \tag{1152}$$

Note that that the second term simply provides an overall "DC" shift in the energy levels while the first term will introduce an "FM phase modulation." We may modify (7) to write the phase shift of the wavefunction as

$$\varphi(t) = \frac{e}{\hbar}\int_0^t V(t)\,dt + \frac{m_e}{\hbar}\int_0^t V_G(t)\,dt \tag{1153}$$

$$= \frac{1}{\hbar}\int_0^t [eV(t) + m_e V_G(t)]\,dt \tag{1154}$$

Using (2) and (1152) to evaluate the integral gives

$$\varphi(t) = \frac{1}{\hbar}\left[\frac{eQ_0}{4\pi\varepsilon_0 a\omega}\sin(\omega t) + \frac{4\pi G m_e}{a}\left(\frac{m_e Q_0}{e\omega}\sin(\omega t) + M_0 t\right)\right] \tag{1155}$$

$$= \frac{eQ_0}{4\pi\varepsilon_0 a\hbar\omega}\sin(\omega t) + \frac{4\pi G m_e^2 Q_0}{ae\hbar\omega}\sin(\omega t) + \frac{4\pi G m_e M_0}{a\hbar}t \tag{1156}$$

We can define a gravitational "FM depth of modulation" parameter α_G similar to (9) as

$$\alpha_G = \frac{4\pi G m_e^2 Q_0}{ae\hbar\omega} \tag{1157}$$

We can also define

$$\beta_G = \frac{4\pi G m_e M_0}{a\hbar} \tag{1158}$$

so that the phase in (1156) can be written as

$$\varphi(t) = (\alpha + \alpha_G)\sin\omega t + \beta_G\, t \tag{1159}$$

Then we find that the hydrogen atom wavefunction in the presence of the *interior* scalar potentials $V(t)$ and $V_G(t)$ caused by the *exterior* charge $Q(t)$ and mass $M(t)$, respectively, will have the form

$$\psi(t) = \psi(0)\exp\left(-\frac{i}{\hbar}Et\right)\exp\left[-i(\alpha+\alpha_G)\sin\omega t - i\beta_G\, t\right] \tag{1160}$$

In principle, the gravitational potential introduces adjustments to the energy levels of the atom due to the gravitational potential. However, this gravitational scalar AB effect would be far weaker than the corresponding electric scalar AB effect as demonstrated by considering the ratio of the FM depth modulation parameters

$$\frac{\alpha_G}{\alpha} = \frac{16\pi^2 G\varepsilon_0 m_e^2}{e^2} \approx 3\times10^{-42} \tag{1161}$$

We can also determine the effective wave function for a “gravitational atom.” The Hamiltonian of an atom in the presence of a time-dependent potential is

$$\hat{H} = \hat{H}_0(\vec{p},\vec{x}) + U(t) \tag{1162}$$

where $\hat{H}_0(\vec{p},\vec{x})$ is the Hamiltonian of the ion before being exposed to the external potential, and $U(t) = mV(t)$ is the additional potential energy when the external time-dependent gravitational potential, $V(t)$, is introduced. For the case of a gravitational potential inside the sphere, we must have that $V(t)$ is uniform and therefore independent of position. In fact, for a potential that is produced by a sinusoidally time-varying mass, $M(t) = M_0\cos(\omega t)$, deposited on the surface of a conducting sphere with radius a surrounding the ion, we have

$$U(t) = qV(t) = \frac{m_e M_0\cos(\omega t)}{4\pi\varepsilon_G a} \tag{1163}$$

Then substituting (2345) into (194) gives

$$\hat{H} = \hat{H}_0(\vec{p},\vec{x}) + \frac{m_e M_0\cos(\omega t)}{4\pi\varepsilon_G a} \tag{1164}$$

The Hamiltonian for the hydrogenic atom *without* the presence of the external potential is

$$\hat{H}_0(\vec{p},\vec{x}) = \frac{\hat{p}^2}{2m} - \frac{m_e m_p}{4\pi\varepsilon_G r} \tag{1165}$$

which is simply the kinetic energy of the electron and its Newtonian interaction with the nucleus. We can substitute (1165) into (693) and make use of the operators $\hat{p}= -i\hbar\nabla$ and $\hat{H}_0 = i\hbar\partial_t$ to act on a wavefunction, $\Psi(\vec{x},t)$. This gives the following time-dependent Schrödinger wave equation.

$$i\hbar\partial_t\Psi(\vec{x},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 - \frac{m_e m_p}{4\pi\varepsilon_G r} + \frac{m_e M_0\cos(\omega t)}{4\pi\varepsilon_G a}\right]\Psi(\vec{x},t) \tag{1166}$$

Given the separation of space and time dependence in the wave equation, we can apply separation of variables to decompose the time-dependent wavefunction into a spatial function, $\psi(\vec{x})$, and a temporal function, $T(t)$. Then we have

$$\Psi(\vec{x},t) = \psi(\vec{x})\,T(t) \tag{1167}$$

Substituting (1167) into (1166) and distributing gives

$$i\hbar\psi(\vec{x})\,\partial_t T(t) = -\frac{\hbar^2}{2m}T(t)\nabla^2\psi(\vec{x}) - \frac{m_e m_p}{4\pi\varepsilon_G r}\Psi(\vec{x},t) + \frac{m_e M_0\cos(\omega t)}{4\pi\varepsilon_G a}\Psi(\vec{x},t) \tag{1168}$$

Dividing each term in (1168) by the wavefunction in (1167) and rearranging gives

$$\frac{i\hbar}{T(t)}\partial_t T(t) - \frac{mM_0\cos(\omega t)}{4\pi\varepsilon_G a} = -\frac{\hbar^2}{2m_e\psi(\vec{x})}\nabla^2\psi(\mathbf{x}) - \frac{m_e m_p}{4\pi\varepsilon_G r} \tag{1169}$$

Since the left side of (1169) is *only time*-dependent and the right side of (1169) is *only space*-dependent, then each side must be equal to a constant independently. Therefore we can write

$$\frac{i\hbar}{T(t)}\partial_t T(t) - \frac{m_e M_0 \cos(\omega t)}{4\pi\varepsilon_G a} = E \tag{1170}$$

and likewise

$$-\frac{\hbar^2}{2m\psi(\vec{x})}\nabla^2\psi(\vec{x}) - \frac{m_e m_p}{4\pi\varepsilon_G r} = E \tag{1171}$$

where we have used E for the separation constant. Notice that (1171) is simply the equation for a "gravitational hydrogen atom." The solution to this differential equation is also commonly found by applying separation of variables.[111] Using spherical coordinates (r,ϕ,θ), we can write $\psi(\mathbf{x}) = R(r)Y(\phi,\theta)$. It is well known that $R(r)$ can be written in terms of Laguerre polynomials,

$$L^p_{q-p}(x) \equiv (-1)^p(\partial_x)^q\left(e^{-x}x^q\right) \tag{1172}$$

It is also known that $Y(\phi,\theta)$ are the spherical harmonics,

$$Y^m_l(\theta,\phi) = \varepsilon\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}e^{im\phi}P^m_l(\cos\theta) \tag{1173}$$

where $\varepsilon = (-1)^m$ for $m \geq 0$ and $\varepsilon = 1$ for $m \leq 0$. Note that P^m_l is the associated Legendre function given by

$$P^m_l(x) \equiv \left(1-x^2\right)^{|m|/2}\left(\frac{d}{dx}\right)^{|m|}P_l(x) \tag{1174}$$

where $P_l(x)$ is the l^{th} Legendre polynomial. Then the time-independent, *normalized* wavefunction written in terms of the principle quantum numbers (n,l,m) is found to be

$$\psi_{nlm}(\mathbf{x}) = \sqrt{\left(\frac{2}{nr_0}\right)^3\frac{(n-l-1)!}{2n[(n+l)!]^3}}e^{-r/na}\left(\frac{2r}{nr_0}\right)^l L^{2l+1}_{n-l-1}\left(\frac{2r}{nr_0}\right)Y^m_l(\theta,\phi) \tag{1175}$$

Here we would use

$$r_0 \equiv \frac{4\pi\varepsilon_G\hbar^2}{m_e m_p} \tag{1176}$$

as a "gravitational Bohr radius." To find a solution to $T(t)$ in (1170), we first rearrange the equation as

$$T'(t) + \frac{iE}{\hbar}T(t) + \frac{im_e M_0}{4\pi\varepsilon_G a\hbar}\cos(\omega t)T(t) = 0 \tag{1177}$$

where $T'(t) = \partial_t T(t)$. We can let

$$A = \frac{iE}{\hbar} \qquad \text{and} \qquad B = \frac{im_e M_0}{4\pi\varepsilon_G a\hbar} \tag{1178}$$

so that (1177) can be written as

$$T'(t) + [A + B\cos(\omega t)]T(t) = 0 \tag{1179}$$

This is a first-order, linear differential equation. We can use an integrating factor written as

$$e^{\int[A+B\cos(\omega t)]dt} = e^{At+\frac{B}{\omega}\sin(\omega t)} \tag{1180}$$

[111] See [92] for a standard treatment of the hydrogen atom using separation of variables.

Multiplying each term in (1179) by the integrating factor in (1180) gives

$$e^{At+\frac{B}{\omega}\sin(\omega t)}T'(t)+e^{At+\frac{B}{\omega}\sin(\omega t)}\left[A+B\cos(\omega t)\right]T(t)=0 \tag{1181}$$

By the product rule this can be written as

$$\left[e^{At+\frac{B}{\omega}\sin(\omega t)}T(t)\right]'=0 \tag{1182}$$

Integrating both sides with respect to time from $t'=0$ to $t'=t$ gives

$$e^{At+\frac{B}{\omega}\sin(\omega t)}T(t)-T(0)=C(\vec{x}) \tag{1183}$$

where $C(\vec{x})$ would be a function only of position, not time. However, our use of separation of variables in (1167) required that $T(t)$ is a function only of time, so we must have that $C(\vec{x})$ is just a constant which we may simply call C. We can define another constant as $D=C+T(0)$ and then solve (1183) for $T(t)$. We can also substitute back in for A and B from (1178). This gives

$$T(t)=D\exp\left[-\frac{iEt}{\hbar}-\frac{im_eM_0}{4\pi\varepsilon_G a\hbar\omega}\sin(\omega t)\right] \tag{1184}$$

Note that from (1184) we see that $T(0)=D$. We also know from (1167) that $\Psi(\vec{x},0)=\psi_{nlm}(\vec{x})\,T(0)$ where $\psi_{nlm}(\vec{x})$ is already normalized. Thus for $\Psi(\vec{x},t)$ to also be normalized, we must have that $D=1$. Since D is obviously not a function of time, then this normalization is maintained even as time evolves. Therefore, we can now write the fully time-dependent wavefunction in (1167) as

$$\Psi(\vec{x},t)=\psi_{nlm}(\vec{x})\exp\left[-\frac{iEt}{\hbar}-\frac{im_eM_0}{4\pi\varepsilon_G a\hbar\omega}\sin(\omega t)\right] \tag{1185}$$

where $\psi_{nlm}(\vec{x})$ is given by (1175).

11 Interaction of gravito-electromagnetic (GEM) fields with superconductors

11.1 Overview of interaction of GEM fields with superconductors

Here we look at the interaction of gravitational fields with superconductors. First we develop London-like equations for the gravito-electromagnetic fields. It is found that the resulting differential equations do not allow for exponential decay solutions which would be associated with a Meissner-like effect. We find that there is effectively a *paramagnetism* that occurs rather than the *diamagnetic*-like behavior of the Meissner effect. We also consider the case when both electromagnetic and gravito-electromagnetic fields are present and show that the gravito-magnetic field provides an extremely small enhancement to the standard Meissner effect while the magnetic field provides a major enhancement to the paramagnetism of the gravito-magnetic field.

For the case of gravitational waves, we propose a constitutive equation which relates the transverse-traceless strain field of a gravitational wave to the transverse-traceless stress induced in matter by the wave. This leads to a dispersion relation and corresponding gravitational plasma frequency and penetration depth. Lastly, we consider a correction introduced by the Landau-Lifshitz pseudotensor which takes into account the self-gravitation of the gravitational wave which is necessary to correctly determine the true back-action of the matter on the wave.

11.2 Gravito-London equations for the GEM fields

To obtain gravito-London equations for the gravito-magnetic field (or Lense-Thirring field), we adopt the same process used in Appendix Q for the electromagnetic case. To do so, we begin by considering the appropriate approximation which yields a form for the gravitational Lorentz force that most closely resembles the Lorentz force from electromagnetism. In previous sections, we found numerous forms for the equation of motion according to the order of approximation used for the gravitational sources and the velocity of the test mass. The form which resembles the electromagnetic force most closely is given in (508) as

$$\underset{(low\text{-}velocity)}{\vec{a}_{(PN)}} = -\nabla\Phi_N - \dot{\vec{h}} + \left(\vec{v}\times\vec{B}_{g(PN)}\right), \qquad \text{for } \nabla\cdot\vec{h} = 0 \tag{1186}$$

From the metric in (176), we have $h_{0i} = \beta_i + \partial_i\alpha$, where h_{0i} is decomposed into a rotational component β_i and an irrotational component $\partial_i\alpha$. However, as described in Section 30, (508) is obtained only when $\alpha = 0$ so that h_{0i} is purely rotational. This means that $\nabla\cdot\vec{h} = 0$.

In Section 5 we defined the traditional gauge-*dependent* gravito-magnetic field as $\tilde{\tilde{\vec{B}}}_G = \nabla\times\vec{h}$ where $\vec{h} = c\,(h_{01}, h_{02}, h_{03})$. This means that

$$\tilde{\tilde{\vec{B}}}_G = \nabla\times\vec{h} = \nabla\times\vec{\beta} \tag{1187}$$

From (468) and (469) in Section 28, we also found that in the first-order post-Newtonian limit, the gauge-*invariant* gravito-magnetic field, $\vec{B}_G = \nabla\times\vec{\Xi}$ reduces to the gauge-*dependent* gravito-magnetic field, $\tilde{\tilde{\vec{B}}}_G = \nabla\times\vec{\beta}$. Therefore, $\vec{B}_{g(PN)} \approx \tilde{\tilde{\vec{B}}}_G$ and we can simply write (1186) using the notation $\tilde{\tilde{\vec{B}}}_G$ to emphasize that it is gauge-*dependent*.

In a previous section, the Newtonian gravito-electric field given by $\vec{E}_{g(N)} = -\partial_i\Phi_N$ is also gauge-*dependent*. Again, we will simply use the notation $\tilde{\tilde{\vec{E}}}_G$ to represent that it is gauge-*dependent*. Hence the entire formulation below is necessarily gauge-dependent. However, if we are to follow a London-like process involving a gravitational Lorentz force, then we cannot avoid gauge-dependence. This is because the geodesic equation of motion is necessarily gauge-dependent (at all orders of approximation) as shown in Section 31.[112]

If we neglect the gravito-magnetic force and use $\vec{a} = \partial\vec{v}/\partial t$ and $\tilde{\tilde{\vec{E}}}_G = -\partial_i\Phi_N - \dot{\vec{h}}$, then (1186) becomes

$$\partial_t\vec{v} = \tilde{\tilde{\vec{E}}}_G \tag{1188}$$

This implies that the electrons in a superconductor can be thought of as flowing with no resistance so that the supercurrent is effectively dissipationless. The "mass supercurrent" can be written as

$$\vec{J}_m = n_s m_e \vec{v} \tag{1189}$$

where n_s is the number density of superconducting carriers (electrons) and m_e is their mass. Then (1188) becomes what could be referred to as the gravito-*electric* London equation

$$\boxed{\partial_t\vec{J}_m = n_s m_e \tilde{\tilde{\vec{E}}}_G \qquad \textit{Gravito-electric London equation}} \tag{1190}$$

[112] In some sense this issue of gauge-dependence is still analogous to the formulation of the London equations in electromagnetism. This can be observed by recognizing that in the limit of steady currents, the continuity equation leads to $\nabla\cdot\vec{J}_c = 0$ and therefore the London equation given by (2931) necessarily "picks out" the Coulomb (or London) gauge as the required gauge for a superconductor with a steady-state current. However, in Section 25, we showed that the same condition is not required for gravitational field equations. This is because $\dot{\rho}_m = 0$ can be satisfied in (367) without requiring $\nabla\cdot\vec{J}_m = 0$.

We can take the curl of this equation and apply the gravitational Faraday law, $\nabla \times \tilde{\tilde{\vec{E}}}_G = -\partial_t \tilde{\tilde{\vec{B}}}_G$. This gives

$$\partial_t \nabla \times \vec{J}_m = -n_s m_e \partial_t \tilde{\tilde{\vec{B}}}_G \tag{1191}$$

Integrating both sides with respect to time and setting the integration constant to zero gives what could be referred to as the gravito-*magnetic* London equation.

$$\boxed{\nabla \times \vec{J}_m = -n_s m_e \tilde{\tilde{\vec{B}}}_G \qquad \textit{Gravito-magnetic London equation}} \tag{1192}$$

Then using $\tilde{\tilde{\vec{B}}}_G = \nabla \times \vec{h}$ gives

$$\nabla \times \left(\vec{J}_m + n_s m_e \vec{h}\right) = 0 \tag{1193}$$

The solution to this differential equation is

$$\vec{J}_m + n_s m_e \vec{h} + \nabla f(r,t) = 0 \tag{1194}$$

In the electromagnetic case shown in Appendix Q, it is argued that for a steady state current, we have $\dot{\rho}_m = 0$ and therefore $\nabla \cdot \vec{J}_m = 0$ by the continuity equation. Also, in (1186) we have $\nabla \cdot \vec{h} = 0$. Therefore, taking the divergence of (1194) as written with $\vec{J}_c$ and $\vec{A}$ for electromagnetism, requires that $\nabla^2 f(r,t) = 0$.

However, these arguments are not required in the gravitational case. Rather, we find that the condition $\nabla \cdot \vec{J}_m = 0$ is satisfied by virtue of the fact that $\vec{J}_m = n_s m_e \vec{v}$ and the superfluid is considered to have an "incompressible flow" so that $\nabla \cdot \vec{v} = 0$. This immediately requires $\nabla \cdot \vec{J}_m = 0$ without employing any continuity equation. Also, because $\vec{h} = c\,(h_{01}, h_{02}, h_{03})$ where $h_{0i} = \beta_i + \partial_i \alpha$, then $\nabla \cdot \vec{h} = 0$ simply by virtue of β_i being a purely rotational vector ($\partial_i \beta_i = 0$) and because we set $\alpha = 0$ in (1186).

Therefore, since $\nabla \cdot \left(\vec{J}_m + n_s m_e \vec{h}\right) = 0$ in (1194), and we assume that $\vec{J}_m$ and $\vec{h}$ go to zero as $r \to \infty$, then we must also have $\nabla f(r,t) \to 0$ as $r \to \infty$. This means $f(r,t) \to \textit{constant}$ as $r \to \infty$ and the unique solution of $\nabla^2 f(r,t) = 0$ is $f(r,t) = \textit{constant}$ everywhere. Therefore $\nabla f(r,t) = 0$ everywhere and we simply have

$$\boxed{\vec{J}_m = -n_s m_e \vec{h} \qquad \textit{Gravito-London constitutive equation}} \tag{1195}$$

The equation in (1195) can be thought of as a constitutive equation[113] which describes how the supercurrent in a superconductor responds to an external field $\vec{h}$. The equation in (1195) can also be thought of as a *single* gravito-London equation which combines the previous two gravito-London equations into one relationship. This can be observed from the fact that we if take the time derivative of (1195) and use $\tilde{\tilde{\vec{E}}}_G = -\partial_t \vec{h}$ then we get the gravito-electric London equation in (1190) but if we take the curl of (1195) and use $\tilde{\tilde{\vec{B}}}_G = \nabla \times \vec{h}$, then we get the gravito-magnetic London equation in (1192).

We may also derive (1195) by use of the minimal coupling rule found in (842). Using $q = -e$, we have

$$\vec{p}_{can} = m_e \vec{v} - e\vec{A} + m_e \vec{h} \tag{1196}$$

For a particle coupled only to a gravito-magnetic field, we only have $\vec{p}_{can} = m_e \vec{v} + m_e \vec{h}$. Since the superconducting state of a system is a zero-momentum state, then the canonical momentum becomes zero and we have $\vec{v} = -\vec{h}$. Substituting this into (1189) gives

$$\vec{J}_m = -n_s m_e \vec{h} \tag{1197}$$

This result again matches what we obtained in (1195).

[113]This would be similar to a gravitatioinal Ohm's law, $\vec{J}_m = \sigma_g \tilde{\tilde{\vec{E}}}_g$, where σ_g would be a "gravitational conductivity" that characterizes the response of a "gravitational conductor" to the gravito-electric field.

11.3 The absence of a gravito-Meissner effect for GEM fields

According to (431), the gravito-Ampere law is

$$\nabla \times \tilde{\tilde{\vec{B}}}_G = -2\mu_{g(SC)}\vec{J}_m \tag{1198}$$

where $\mu_{g(SC)}$ represents the gravitational permeability of the superconductor, versus μ_G in vacuum.[114]. Taking the curl of both sides of (1198) gives[115]

$$\nabla \times \nabla \times \tilde{\tilde{\vec{B}}}_G \quad = \quad -2\mu_{g(SC)}\nabla \times \vec{J}_m \tag{1199}$$

Since $\nabla \cdot \tilde{\tilde{\vec{B}}}_G = 0$, then the vector calculus identity $\nabla \times \nabla \times \tilde{\tilde{\vec{B}}}_G = \nabla\left(\nabla \cdot \tilde{\tilde{\vec{B}}}_G\right) - \nabla^2\tilde{\tilde{\vec{B}}}_G$ simply becomes $\nabla \times \nabla \times \tilde{\tilde{\vec{B}}}_G = -\nabla^2\tilde{\tilde{\vec{B}}}_G$. Then the equation above can be written as

$$\nabla^2\tilde{\tilde{\vec{B}}}_G = 2\mu_{g(SC)}\nabla \times \vec{J}_m \tag{1200}$$

Inserting (1192) into the right side of the equation above gives[116]

$$\nabla^2\tilde{\tilde{\vec{B}}}_G = -2\mu_{g(SC)}n_s m_e\tilde{\tilde{\vec{B}}}_G \tag{1201}$$

We can define

$$\alpha^{-2} \equiv 2\mu_{g(SC)}n_s m_e \tag{1202}$$

so that (1201) becomes

$$\boxed{\nabla^2\tilde{\tilde{\vec{B}}}_G + \frac{1}{\alpha^2}\tilde{\tilde{\vec{B}}}_G = 0} \tag{1203}$$

Notice that α^2 can never be negative since μ_G, n_s, and m_e are all necessarily positive. Therefore the differential equation above only allows real *sinusoidal* solutions, not real *exponential* solutions. Therefore we find that there is no exponential decay of the field and hence no associated penetration depth. Physically speaking, this implies that instead of a *diamagnetic* (Meissner) effect, there is essentially a *paramagnetic* effect.

[114] From (360) we have that $\mu_g = 4\pi G/c^2$ in vacuum. Since G and c are fundamental constants of nature, then we expect that μ_g would have the same value in matter as it does in vacuum. However, if there is an *effective* G in matter due to some kind of gravito-magnetic polarization in matter, then it is possible that $G_{eff} \neq G$ and therefore $\mu_{g\ (sc)} = 4\pi G_{eff}/c^2$ in the superconductor would differ from μ_g and vacuum. (This topic is discussed in further detail in Section 24.)

[115] Note that the curl of the gravito-Ampere law involves the *third* derivative of the metric and therefore it may be questioned whether keeping such terms is consistent with the approximations of linearized GR where we keep *second* derivatives of the metric. However, in Appendix A we show that taking higher derivatives is completely within the linearized approximation for a metric that has a very small amplitude (such as 10^{-20}) but a rapid variation (such as microwave frequencies). We assume these conditions to be the case here.

[116] Formally speaking, this substitution is not valid in linearized GR since (1192) was derived from the geodesic equation of motion and therefore cannot be substituted back into the field equation. See the end of Section 26 for details concerning this issue. Also, since this would be a higher order treatment, then we would need to introduce the Landau-Lifshitz pseudotensor to account for the higher order correction. A treatment of this kind is done in Section 75 for gravitational waves.

We can return to the gravito-Ampere law in (1198), substitute $\tilde{\vec{B}}_G = \nabla \times \vec{h}$ on the left side and substitute (1195) on the right side to obtain

$$\nabla \times \nabla \times \vec{h} \quad = \quad 2\mu_{g(SC)} n_s m_e \vec{h} \tag{1204}$$

Since $\nabla \cdot \vec{h} = 0$ in (1186), then the vector calculus identity given by $\nabla \times \nabla \times \vec{h} = \nabla \left(\nabla \cdot \vec{h} \right) - \nabla^2 \vec{h}$ becomes $\nabla \times \nabla \times \vec{h} = -\nabla^2 \vec{h}$. Then using (1202), the equation above becomes

$$\boxed{\nabla^2 \vec{h} + \frac{1}{\alpha^2} \vec{h} = 0} \tag{1205}$$

Once again we observe that there is no exponential decay of the gravito-vector potential and hence no associated penetration depth. Next, we can find a wave equation for the mass supercurrent density. Taking the curl of (1192) gives

$$\nabla \times \nabla \times \vec{J}_m \quad = \quad -n_s m_e \nabla \times \tilde{\vec{B}}_G \tag{1206}$$

Since $\dot{\rho}_m = 0$ for a steady state current, then by the continuity equation we have $\nabla \cdot \vec{J}_m = 0$. In that case, the vector calculus identity $\nabla \times \nabla \times \vec{J}_m = \nabla \left(\nabla \cdot \vec{J}_m \right) - \nabla^2 \vec{J}_m$ becomes $\nabla \times \nabla \times \vec{J}_m = -\nabla^2 \vec{J}_m$. Then the equation above can be written as

$$\nabla^2 \vec{J}_s \quad = \quad n_s m_e \nabla \times \tilde{\vec{B}}_G \tag{1207}$$

Substituting the gravito-Ampere law in (1198) and using $k^2 = 2\mu_{g(SC)} n_s m_e$ from (1202) gives

$$\boxed{\nabla^2 \vec{J}_s + \frac{1}{\alpha^2} \vec{J}_m = 0} \tag{1208}$$

Therefore we observe that there is no exponential decay of the mass supercurrent density and hence no associated penetration depth. This is consistent with the fact that there is no gravito-London penetration depth for the gravito-magnetic field which would drive the mass supercurrent. Taking the time derivative of (1208) and using the gravito-electric London equation from (1190) gives

$$\boxed{\nabla^2 \tilde{\vec{E}}_G + \frac{1}{\alpha^2} \tilde{\vec{E}}_G = 0} \tag{1209}$$

Therefore, there is no gravito-London penetration depth for the gravito-electric field. This result is consistent with the expectation that there would be no possibility of “shielding” the gravito-electric field (or in other words, “reflection” or “expulsion” of the Newtonian gravitational field by a superconductor). Otherwise, this would imply an “anti-gravity” effect which is not consistent with Newton’s law of gravitation

$$\tilde{\vec{E}}_G = -\frac{G m_1 m_2}{r^2} \tilde{r} \tag{1210}$$

and the fact that negative gravitational mass has never been observed.

For each of the differential equations found above $\left(\text{describing } \vec{\widetilde{B}}_G, \vec{h}, \vec{J}_m, \text{and } \vec{\widetilde{E}}_G\right)$, we can use a sinusoidal field $Ce^{i\vec{k}\cdot\vec{x}}$, where C is constant associated with the particular field, to obtain

$$\nabla^2 Ce^{i\vec{k}\cdot\vec{x}} + \frac{1}{\alpha^2} Ce^{i\vec{k}\cdot\vec{x}} = 0 \tag{1211}$$

$$-k^2 Ce^{i\vec{k}\cdot\vec{x}} + \frac{1}{\alpha^2} Ce^{i\vec{k}\cdot\vec{x}} = 0 \tag{1212}$$

$$-k^2 + \frac{1}{\alpha^2} = 0$$

Using α from (1202) gives

$$k = \frac{1}{2\mu_{g(SC)} n_s m_e} \tag{1213}$$

Using $k = 2\pi/\lambda$ gives a wavelength of

$$\boxed{\lambda = \frac{\mu_{g(SC)} n_s m_e}{\pi}} \tag{1214}$$

Examination of the reason for the absence of a penetration depth for gravito-electromagnetic fields

Here we compare the equations which describe electromagnetic and gravito-electromagnetic fields in a superconductor in order to examine the reason that there is a penetration depth (and therefore expulsion) for electromagnetic fields, but there is no penetration depth (and therefore no expulsion) for gravito-electromagnetic fields. The reason can be summarized by the fact that the electromagnetic and gravitational Ampere laws have opposite signs

$$\nabla \times \vec{B} = \mu \vec{J}_c \qquad \text{and} \qquad \nabla \times \vec{\widetilde{B}}_G = -2\mu_{g(SC)} \vec{J}_m \tag{1215}$$

while the constitutive equations which relate the mass and charge supercurrent densities ($\vec{J}_c$ and $\vec{J}_m$) to the their corresponding vector potentials ($\vec{A}$ and $\vec{h}$) have the *same* sign

$$\vec{J}_c = -\frac{n_s e}{m_e} \vec{A} \qquad \text{and} \qquad \vec{J}_m = -n_s m_e \vec{h} \tag{1216}$$

The combination of (1215) and (1216) is what leads to the discrepancy between the way a superconductor responds to electromagnetic fields versus gravitational fields. Specifically, from (1215) and (1216) we ultimately arrive at

$$\nabla^2 \vec{B} - \frac{1}{\lambda_L^2} \vec{B} = 0 \qquad \text{and} \qquad \nabla^2 \vec{\widetilde{B}}_G + k^2 \vec{\widetilde{B}}_G = 0 \tag{1217}$$

It is the *positive* sign in the electromagnetic Ampere law in (1215) which lead to a *Yukawa*-like differential equation (or screened Poisson equation) for the magnetic field in (1217). On the other hand, it is the *negative* sign in the gravitational Ampere law in (1215) which leads to the *Helmholtz*-like differential equation in (1203). All the other London equations (for electromagnetic fields and for gravito-electromagnetic fields) have the same sign. Therefore, it is the difference in signs for the Ampere laws in (1215) that also leads to the discrepancy when comparing the vector potential versus the gravito-vector potential, the electric versus gravito-electric field, and the charge current density versus the mass current density.

Let us compare side-by-side how the expressions in (1217) were developed for electromagnetic fields and gravito-electromagnetic fields to observe the difference. First, we note that the equations of motion for the electromagnetic case and the gravitational case have opposite signs.

$$\vec{a} \quad = \quad -\frac{e}{m_e}\vec{E} \qquad \text{and} \qquad \vec{a} = \tilde{\tilde{\vec{E}}}_G \tag{1218}$$

The opposite sign is due to $q = -e$ for an electron. The charge and mass current densities are also given, respectively, by

$$\vec{J}_c \quad = \quad -n_s e\vec{v} \qquad \text{and} \qquad \vec{J}_m = n_s m_e \vec{v} \tag{1219}$$

Again, the opposite sign is due to $q = -e$ for an electron. However, when combining (1218) and (1219), we find that the resulting equations of motion in a superconductor have the *same* sign.

$$\partial_t \vec{J}_c \quad = \quad \frac{n_s e^2}{m_e}\vec{E} \qquad \text{and} \qquad \partial_t \vec{J}_m = n_s m_e \tilde{\tilde{\vec{E}}}_G \tag{1220}$$

Then using $\vec{E} = -\partial_t \vec{A}$ and $\tilde{\tilde{\vec{E}}}_G = -\partial_t \vec{h}$, and integrating with time gives (1216). Therefore, we find that the difference in sign in the equations of motion (1218) and the difference in sign in the current densities (1219), together lead to the same sign in the supercurrents (1216). Likewise, the electromagnetic and gravitational Faraday laws also have the *same* sign.

$$\nabla \times E \quad = \quad -\partial_t B_G \qquad \text{and} \qquad \nabla \times \tilde{\tilde{\vec{E}}}_G = -\partial_t \tilde{\tilde{\vec{B}}}_G \tag{1221}$$

Therefore, taking the curl of (1220), inserting (1221) and integrating with time gives

$$\nabla \times \vec{J}_c \quad = \quad -\frac{n_s e^2}{m_e} B_G \qquad \text{and} \qquad \nabla \times \vec{J}_m = -n_s m_e \tilde{\tilde{\vec{B}}}_G \tag{1222}$$

Up to this point, the electromagnetic and gravitational case continue to have matching signs. The discrepancy in signs between the two cases is introduced by taking the curl of (1222) and combining with the the electromagnetic and gravitational Ampere laws in (1215) which gives (1217). Therefore, it is fundamentally the difference in signs in (1215) that leads to the electromagnetic fields having a London penetration depth while the gravito-electromagnetic fields do not have a corresponding gravito-London penetration depth.

Alternatively, we can observe how this discrepancy emerges by use of the canonical momentum. If we write (1196) for the electromagnetic and gravitational cases separately, we have

$$p_{can} \quad = \quad m_e \vec{v} - e\vec{A} \qquad \text{and} \qquad p_{can} = m_e \vec{v} + m_e \vec{h} \tag{1223}$$

Setting $p_{can} = 0$ in (1223) leads to $\vec{v} = e\vec{A}/m_e$ for electromagnetism and $\vec{v} = -\vec{h}$ for gravitation. Substituting these into the corresponding current densities in (1219) again leads to

$$\vec{J}_c \quad = \quad -\frac{n_s e}{m}\vec{A} \qquad \text{and} \qquad \vec{J}_m = -n_s m_e \vec{h} \tag{1224}$$

Therefore, we find that the difference in sign in the canonical momenta (1223) and the difference in sign in the current densities (1219), together lead to the same sign in the supercurrents (1216). Once again, the discrepancy in signs between the two cases is introduced by taking the curl of (1224) and combining with the the electromagnetic and gravitational Ampere laws in (1215) which gives (1217).

Comparison of these results with other authors

The result in (1203) is consistent with the results obtained by Ciubotariu and Agop in equation (32) of [68], aside from a disagreement in the prefactor[117] which they obtain as $k^2 = \frac{64}{3}\pi nm$. Similar to our conclusion here, they observe that there is no gravitational penetration depth and thus no gravitational Meissner effect.

On the other hand, our results contradict two papers by Lano [69, 70] which claim that there is a Meissner effect for the gravito-magnetic field. Upon close examination of the treatment in [69], it appears there is a sign error in going from equation (9) to (10) due to neglecting that the vector calculus identity given by $\nabla\times\nabla\times\vec{B} = \nabla\left(\nabla\cdot\vec{B}\right) - \nabla^2\vec{B}$ introduces a negative sign when substituting $\nabla\times\nabla\times\vec{B}$ with $-\nabla^2\vec{B}$. This same error also occurs in [70] in going from equations (24) to (25).

Our results also seem to contradict that of DeWitt in [42]. By assuming that we can treat the magnetic field and the gravito-magnetic field as effectively a *single* field, in his equation (5) he defines the single field[118] as $\vec{S} = e\vec{A} + m\vec{h}$. Consequently, he concludes that the vector

$$\vec{G} = \nabla\times\vec{S} = \left(e\nabla\times\vec{A}\right) + \left(m\nabla\times\vec{h}\right) \tag{1225}$$

must be expelled from the superconductor and the flux of $\vec{G}$ must be quantized in units of $\frac{1}{2}\hbar$. It seems that this result is stated purely based on the fact that $p_{can} - e\vec{A} - m\vec{h}$ appears in the kinetic term of the Hamiltonian, rather than the usual $p_{can} - e\vec{A}$. However, he does not mention anything about the Ampere and gravito-Ampere laws, or the associated differential equations which may or may not lead to a penetration depth. Instead, he proceeds into an example of a rotating superconducting ring. In this example, his equation (10) reads

$$\nabla\times\vec{h} = 16\pi\kappa\nabla^{-2}\vec{\nabla}\times(\rho\vec{v}) \tag{1226}$$

where he states that κ is the gravitation constant. (He also sets $c = 1$.) It appears that here he is implicitly using the gravito-Ampere law with the *correct* sign. This can be recognized by using the gravito-Ampere law from (58) to write

$$\nabla\times\nabla\times\vec{h} = \nabla\times\tilde{\vec{B}}_G = -8G\vec{J}_m \tag{1227}$$

where $\vec{J}_m = \rho\vec{v}$ and we set $c = 1$. Using the vector Calculus identity, $\nabla\times\nabla\times\vec{h} = \nabla\left(\nabla\cdot\vec{h}\right) - \nabla^2\vec{h}$, and the fact that $\nabla\cdot\vec{h} = 0$, we can write

$$\nabla\times\nabla\times\vec{h} = -\nabla^2\vec{h} \tag{1228}$$

Then equating (1227) and (1228) gives

$$\nabla^2\vec{h} = 16G\rho\vec{V} \tag{1229}$$

It appears that DeWitt then took the inverse of ∇^2 and then the curl of both sides to obtain (1226). The key point is that he used the *correct* sign for the gravito-Ampere law to obtain this result.

[117] Ciubotariu and Agop set $G = c = 1$ and therefore $\mu_g = 4\pi$. They also include a factor of $2/3$ due to the statistical fact that only two thirds of the superelectrons are moving in a plane perpendicular to $\vec{B}_g$.

[118] DeWitt uses $\vec{B}$ rather than $\vec{S}$ to define this single combined field, and he uses $\vec{H}$ for the magnetic field. However, since we have already been using $\vec{B}$ for the magnetic field, then we will continue do so and use $\vec{S}$ for the single combined field.

However, it appears that DeWitt did not recognize there is no Meissner effect for $\vec{h}$ because he did not formulate the differential equation which describes the behavior of $\vec{h}$. As a result, it was overlooked that $\vec{h}$ does not have exponentially decaying solutions and hence does not have an associated penetration depth. It therefore stands that the entire vector $\vec{G}$ in (1225) does *not* vanish inside a superconductor. Only $\left(e\nabla\times\vec{A}\right)$ vanishes but $\left(m\nabla\times\vec{h}\right)$ remains impervious to the superconductor.[119]

Perhaps the implied argument by DeWitt is that the charge and mass supercurrent densities in (1236) can be expressed together as a *particle* supercurrent density, $\vec{J}_n$, which describes the flow of the particle-density per unit time. This can be done by dividing $\vec{J}_c$ by e, dividing $\vec{J}_m$ by m_e, and adding them. From (1236) this gives

$$\vec{J}_n = -n_s\left(\vec{h}+\frac{e}{m_e}\vec{A}\right) \tag{1230}$$

Taking a curl and using $\vec{B}=\nabla\times\vec{A}$ and $\tilde{\vec{B}}_G=\nabla\times\vec{h}$ gives

$$\nabla\times\vec{J}_n = -n_s\left(\tilde{\vec{B}}_G+\frac{e}{m_e}\vec{B}\right) \tag{1231}$$

To relate this to the field equations, we can write the electromagnetic Ampere law using $\vec{J}_c = e\vec{J}_n$ and also write the gravito-Ampere law using $\vec{J}_m = m_e\vec{J}_n$. This gives

$$\nabla\times\vec{B}=\mu e\vec{J}_n \qquad \text{and} \qquad \nabla\times\tilde{\vec{B}}_G = -2\mu_{g(SC)}m_e\vec{J}_n \tag{1232}$$

Now curling (1232) and using $\nabla\times\nabla\times\vec{B}=-\nabla^2\vec{B}$ and $\nabla\times\nabla\times\tilde{\vec{B}}_G=-\nabla^2\tilde{\vec{B}}_G$ gives

$$-\nabla^2\vec{B}=\mu e\nabla\times\vec{J}_n \qquad \text{and} \qquad -\nabla^2\tilde{\vec{B}}_G = -2\mu_{g(SC)}m_e\nabla\times\vec{J}_n \tag{1233}$$

Finally, inserting (1231) gives

$$\nabla^2\vec{B}=n_s\mu e\left(\tilde{\vec{B}}_G+\frac{e}{m_e}\vec{B}\right) \qquad \text{and} \qquad \nabla^2\tilde{\vec{B}}_G = -2n_s\mu_{g(SC)}m_e\left(\tilde{\vec{B}}_G+\frac{e}{m_e}\vec{B}\right) \tag{1234}$$

This leads to coupled differential equations for $\vec{B}$ and $\tilde{\vec{B}}_G$ which must be decoupled and solved in order to determine if there is a penetration depth for each field. We follow a similar process in the following section and find that the magnetic field has a penetration depth which is slightly modified by the presence of the gravito-magnetic field. However, the gravito-magnetic field still does not have a penetration depth. Therefore, we conclude that the claim for a gravito-magnetic Meissner effect cannot be substantiated by the analysis of DeWitt in [42].

[119] DeWitt's example of the superconducting ring also does not seem to require the quantization of the flux of the vector $\vec{G}$ in (1225). Although he refers to this verbally in the paper, there is no explicit need for the quantization of the flux of $\vec{G}$ in the mathematical formulation of his example.

11.4 Gravito-London equations for combined EM and GEM fields

Next we consider the case of a magnetic and a gravito-magnetic field incident on the superconductor. The canonical momentum is given in (1196) as

$$\vec{p}_{can} = m_e\vec{v} + m_e\vec{h} - e\vec{A} \tag{1235}$$

Since the superconducting state of a system is a zero-momentum state, then the canonical momentum becomes zero and we have $\vec{v} = \frac{e}{m_e}\vec{A} - \vec{h}$. Substituting this into the *charge* and *mass* current densities in (1219) gives, respectively,

$$\vec{J}_c = -n_s e\left(\frac{e}{m_e}\vec{A} - \vec{h}\right), \qquad \vec{J}_m = n_s m_e\left(\frac{e}{m_e}\vec{A} - \vec{h}\right) \tag{1236}$$

Taking the curl of these equations and using $\vec{B} = \nabla \times \vec{A}$ and $\tilde{\tilde{\vec{B}}}_G = \nabla \times \vec{h}$ gives

$$\nabla \times \vec{J}_c = -n_s e\left(\frac{e}{m_e}\vec{B} - \tilde{\tilde{\vec{B}}}_G\right), \qquad \nabla \times \vec{J}_m = n_s m_e\left(\frac{e}{m_e}\vec{B} - \tilde{\tilde{\vec{B}}}_G\right) \tag{1237}$$

The static Ampere law and the static *gravito*-Ampere law are given, respectively, by

$$\nabla \times \vec{B} = \mu\vec{J}_c, \qquad \nabla \times \tilde{\tilde{\vec{B}}}_G = -2\mu_{g(SC)}\vec{J}_m \tag{1238}$$

where $\mu_{g(SC)}$ is the effective gravito-permeability of the superfluid in the superconductor. Taking the curl of both sides of these equations gives[120]

$$\nabla \times \nabla \times \vec{B} = \mu\nabla \times \vec{J}_c, \qquad \nabla \times \nabla \times \tilde{\tilde{\vec{B}}}_G = -2\mu_{g(SC)}\nabla \times \vec{J}_m \tag{1239}$$

Since $\nabla \cdot \vec{B} = \nabla \cdot \vec{B}_G = 0$, then the vector calculus identity $\nabla \times \nabla \times \vec{B} = \nabla\left(\nabla \cdot \vec{B}\right) - \nabla^2\vec{B}$ simply becomes $\nabla \times \nabla \times \vec{B} = -\,\nabla^2\vec{B}$ for both the magnetic and gravito-magnetic fields. Then the equations above can be written as

$$\nabla^2\vec{B} = -\mu\nabla \times \vec{J}_c, \qquad \nabla^2\tilde{\tilde{\vec{B}}}_G = 2\mu_{g(SC)}\nabla \times \vec{J}_m \tag{1240}$$

Inserting (1237) into these equations gives

$$\nabla^2\vec{B} = \mu n_s e\left(\frac{e}{m_e}\vec{B} - \tilde{\tilde{\vec{B}}}_G\right), \qquad \nabla^2\tilde{\tilde{\vec{B}}}_G = 2\mu_{g(SC)}n_s m_e\left(\frac{e}{m_e}\vec{B} - \tilde{\tilde{\vec{B}}}_G\right) \tag{1241}$$

To decouple these differential equations, we can start by solving the left equation for $\tilde{\tilde{\vec{B}}}_G$ and the right equation for $\vec{B}$ which gives

$$\tilde{\tilde{\vec{B}}}_G = -\frac{1}{\mu n_s e}\nabla^2\vec{B} + \frac{e}{m_e}\vec{B}, \qquad \vec{B} = \frac{1}{2\mu_{g(SC)}n_s e}\nabla^2\tilde{\tilde{\vec{B}}}_G + \frac{m_e}{e}\tilde{\tilde{\vec{B}}}_G \tag{1242}$$

[120] Note that the curl of the gravito-Ampere law involves the *third* derivative of the metric and therefore it may be questioned whether keeping such terms is consistent with the approximations of linearized GR where we keep *second* derivatives of the metric. However, in Appendix A we show that taking higher derivatives is completely within the linearized approximation for a metric that has a very small amplitude (such as 10^{-20}) but a rapid variation (such as microwave frequencies). We assume these conditions to be the case here.

Substituting $\tilde{\vec{B}}_G$ from (1242) into the second differential equation in (1241) gives

$$\nabla^2\left(-\frac{1}{\mu n_s e}\nabla^2\vec{B}+\frac{e}{m_e}\vec{B}\right)=2\mu_{g(SC)}n_s m_e\left(\frac{e}{m_e}\vec{B}+\frac{1}{\mu n_s e}\nabla^2\vec{B}-\frac{e}{m_e}\vec{B}\right) \tag{1243}$$

$$\nabla^2\left[\nabla^2\vec{B}+\left(2\mu_{g(SC)}n_s m_e-\frac{\mu n_s e^2}{m_e}\right)\vec{B}\right] = 0 \tag{1244}$$

Assuming $\nabla^2\vec{B}$ and $\vec{B}$ go to zero as $r\to 0$, then the solution to the outside differential equation is

$$\nabla^2\vec{B}-\left(\frac{n_s\mu e^2}{m_e}-2n_s\mu_{g(SC)}m_e\right)\vec{B} = 0 \tag{1245}$$

We can define $\lambda_{L\ (modified)}$ as the modified London penetration depth for the case where there is both a magnetic and gravito-magnetic field present. This can be expressed as

$$\frac{1}{\lambda^2_{L(modified)}}=\frac{n_s\mu e^2}{m_e}-2n_s\mu_{g(SC)}m_e \tag{1246}$$

so that we have

$$\boxed{\nabla^2\vec{B}-\frac{1}{\lambda^2_{L(modified)}}\vec{B}=0} \tag{1247}$$

The magnetic permeability of materials range, in general, from $\mu\sim 10^{-7}$ (for vacuum) to $\mu\sim 10^{-1}$ (for iron). Therefore $\mu e^2/m_e$ ranges between approximately 10^{-15} to 10^{-9} (SI units). However, $2\mu_G m_e\sim 10^{-56}$ (SI units) for vacuum. Therefore, it is obvious that $\lambda_{L(mod)}$ is always positive. Then the solution to the differential equation is $B(x)=B_0 e^{-x/\lambda_{L(mod)}}$ and $\lambda_{L(mod)}$ is the modified penetration depth for the magnetic field which also takes into account the effect of the superfluid coupling to the gravito-magnetic field. From (1246) we can write this modified penetration depth as

$$\boxed{\lambda_{L\ (modified)}=\sqrt{\frac{m_e}{n_s\mu e^2-2n_s\mu_{g(SC)}m_e^2}} \qquad \begin{array}{c}\textit{Modified London penetration depth}\\ \textit{due to the presence of a}\\ \textit{gravito-magnetic field}\end{array}} \tag{1248}$$

Notice that setting $\mu_{g(SC)}\approx 0$ makes this expression reduce to the London penetration depth for the purely magnetic case found in (2938) of Appendix Q. We can determine the correction introduced to the magnetic penetration depth due to the presence of a gravito-magnetic field by taking a ratio.

$$\frac{\lambda_{L\ (modified)}}{\lambda_L} = \sqrt{\frac{m_e}{n_s\mu e^2-2n_s\mu_{g(SC)}m_e^2}}\cdot\sqrt{\frac{n_s\mu e^2}{m_e}}=\sqrt{\frac{\mu e^2}{n_s\mu e^2-2n_s\mu_{g(SC)}m_e^2}} \tag{1249}$$

$$= \left(\frac{1}{1-2\mu_{g(SC)}m_e^2/\mu e^2}\right)^{1/2} \tag{1250}$$

To first order, we can approximate this result as

$$\frac{\lambda_{L\ (modified)}}{\lambda_L} \approx 1+\frac{\mu_{g(SC)}m_e^2}{\mu e^2} \tag{1251}$$

Therefore the change in the magnetic London penetration depth due to the presence of a gravito-magnetic field, $\Delta\lambda = \lambda_{L\ (modified)} - \lambda_L$, is given by

$$\Delta\lambda \quad = \quad \frac{\mu_{g(SC)} m_e^2}{\mu e^2}\lambda_L \tag{1252}$$

Returning to the coupled differential equations in (1241), we can also substitute $\vec{B}$ from (1242) into the first differential equation in (1241) which gives

$$-\nabla^2\left(-\frac{1}{2\mu_{g(SC)}n_s e}\nabla^2\tilde{\vec{B}}_G - \frac{m_e}{e}\tilde{\vec{B}}_G\right) \quad = \quad \mu n_s e\tilde{\vec{B}}_G + \frac{\mu n_s e^2}{m_e}\left(-\frac{1}{2\mu_{g(SC)}n_s e}\nabla^2\tilde{\vec{B}}_G - \frac{m_e}{e}\tilde{\vec{B}}_G\right) \tag{1253}$$

Simplifying this yields

$$\nabla^2\left[\nabla^2\tilde{\vec{B}}_G + 2\mu_{g(SC)}n_s e\left(\frac{m_e}{e} + \frac{\mu e}{2\mu_{g(SC)}m_e}\right)\tilde{\vec{B}}_G\right] \quad = \quad 0 \tag{1254}$$

Assuming $\nabla^2\tilde{\vec{B}}_G$ and $\tilde{\vec{B}}_G$ go to zero as $r \to 0$, then the solution to the outside differential equation is

$$\boxed{\nabla^2\tilde{\vec{B}}_G + \left(2\mu_{g(SC)}n_s m_e + \frac{n_s\mu e^2}{m_e}\right)\tilde{\vec{B}}_G = 0} \tag{1255}$$

Just as in (1203), again we find that there is no exponential decay solution for the gravito-magnetic field and hence no associated penetration depth. Instead, there is still a *paramagnetic* effect for the gravito-magnetic field. It is even enhanced by the magnetic field when they're both present.

<u>The magnetic vector potential and gravito-vector potential</u>

We can return to the static Ampere law and the static *gravito*-Ampere law given by (1238). On the left side of each equation, we can insert $\vec{B} = \nabla\times\vec{A}$ and $\tilde{\vec{B}}_G = \nabla\times\vec{h}$, respectively. This gives

$$\nabla\times\nabla\times\vec{A} \quad = \quad \mu\vec{J}_c, \qquad\qquad \nabla\times\nabla\times\vec{h} = -2\mu_{g(SC)}\vec{J}_m \tag{1256}$$

On the right side, we can insert the supercurrents from (1236) which gives

$$\nabla\times\nabla\times\vec{A} \quad = \quad -\mu n_s e\left(\frac{e}{m_e}\vec{A} - \vec{h}\right), \qquad \nabla\times\nabla\times\vec{h} = -2\mu_{g(SC)}n_s m_e\left(\frac{e}{m_e}\vec{A} - \vec{h}\right) \tag{1257}$$

Using the Coulomb (or London) gauge, $\nabla\cdot\vec{A} = 0$, then the vector calculus identity that is given by $\nabla\times\nabla\times\vec{A} = \nabla\left(\nabla\cdot\vec{A}\right) - \nabla^2\vec{A}$ becomes $\nabla\times\nabla\times\vec{A} = -\nabla^2\vec{A}$. Likewise, using $\nabla\cdot\vec{h} = 0$ from (1186), then the same vector calculus identity gives $\nabla\times\nabla\times\vec{h} = -\nabla^2\vec{h}$.

$$\nabla^2\vec{A} \quad = \quad \mu n_s e\left(\frac{e}{m_e}\vec{A} - \vec{h}\right), \qquad\qquad \nabla^2\vec{h} = 2\mu_{g(SC)}n_s m_e\left(\frac{e}{m_e}\vec{A} - \vec{h}\right) \tag{1258}$$

To decouple these differential equations, we can start by solving the left equation for $\vec{h}$ and the right equation for $\vec{A}$ which gives

$$\vec{h} \quad = \quad -\frac{1}{\mu n_s e}\nabla^2\vec{A} + \frac{e}{m_e}\vec{A}, \qquad\qquad \vec{A} = \frac{1}{2\mu_{g(SC)}n_s e}\nabla^2\vec{h} + \frac{m_e}{e}\vec{h} \tag{1259}$$

Substituting $\vec{h}$ from (1259) into the second differential equation in (1258) gives

$$\nabla^2\left(-\frac{1}{\mu n_s e}\nabla^2\vec{A}+\frac{e}{m_e}\vec{A}\right) = 2\mu_{g(SC)}n_s m_e\left(\frac{e}{m_e}\vec{A}+\frac{1}{\mu n_s e}\nabla^2\vec{A}-\frac{e}{m_e}\vec{A}\right) \tag{1260}$$

$$\nabla^2\left[\nabla^2\vec{A}+\left(2\mu_{g(SC)}n_s m_e-\frac{\mu n_s e^2}{m_e}\right)\vec{A}\right] = 0 \tag{1261}$$

Assuming $\nabla^2\vec{A}$ and $\vec{A}$ go to zero as $r \longrightarrow 0$, then the solution to the outside differential equation is

$$\nabla^2\vec{A}-\left(\frac{\mu n_s e^2}{m_e}-2\mu_{g(SC)}n_s m_e\right)\vec{A} = 0 \tag{1262}$$

Once again, we can use (1246) to write this result as

$$\boxed{\nabla^2\vec{A}-\frac{1}{\lambda_{L\ (modified)}}\vec{A}=0} \tag{1263}$$

It was shown earlier that $\mu n_s e^2/m_e >> 2n_s\mu_{g(SC)}m_e$ which means that $\lambda_{L\ (modified)}$ is always positive. Then the solution to the differential equation is $A(x)=A_0 e^{-x/\lambda_{L(mod)}}$ where $\lambda_{L(mod)}$ is the modified penetration depth for the magnetic vector potential which also takes into account the effect of the superfluid coupling to the gravito-magnetic field. Notice that setting $\mu_{g(SC)} \approx 0$ makes this expression reduce to the usual London penetration depth for the purely magnetic case.

Returning to the coupled differential equations in (1258), we can also substitute $\vec{A}$ from (1259) into the first differential equation in (1258) which gives

$$\nabla^2\left(\frac{1}{2\mu_{g(SC)}n_s e}\nabla^2\vec{h}+\frac{m_e}{e}\vec{h}\right) = \mu n_s e\left[\frac{e}{m_e}\left(\frac{1}{2\mu_{g(SC)}n_s e}\nabla^2\vec{h}+\frac{m_e}{e}\vec{h}\right)-\vec{h}\right] \tag{1264}$$

$$\nabla^2\left[\nabla^2\vec{h}+\left(2\mu_{g(SC)}n_s m_e-\frac{\mu n_s e^2}{m_e}\right)\vec{h}\right] = 0 \tag{1265}$$

Assuming $\nabla^2\vec{h}$ and $\vec{h}$ go to zero as $r \longrightarrow 0$, then the solution to the outside differential equation is

$$\nabla^2\vec{h}-\left(\frac{\mu n_s e^2}{m_e}-2\mu_{g(SC)}n_s m_e\right)\vec{h} = 0 \tag{1266}$$

Once again, we can use (1246) to write this result as

$$\boxed{\nabla^2\vec{h}-\frac{1}{\lambda_{L\ (modified)}}\vec{h}=0} \tag{1267}$$

Then the solution to the differential equation is $h(x)=h_0 e^{-x/\lambda_{L(mod)}}$ where $\lambda_{L(mod)}$ is the modified penetration depth for the gravito-vector potential which also takes into account the effect of the superfluid coupling to the magnetic field. This would seem contrary to the fact that the gravito-magnetic field, $\tilde{\vec{B}}_G = \nabla \times \vec{h}$, does *not* have an exponential decay solution in (1255). However, for a neutral superfluid, we can set $e = 0$ which leads to $\nabla^2\vec{h} + \frac{1}{\alpha^2}\vec{h} = 0$ where $\alpha^{-2} \equiv 2\mu_{g(SC)} n_s m_e$. This predicts *no* exponential decay solution for $\vec{h}$ and is consistent with the result found in (1205).

The charge and mass supercurrents

Taking the curl of both equations in (1237) gives

$$\nabla\times\nabla\times\vec{J}_c = -n_s e\left[\nabla\times\left(\frac{e}{m_e}\vec{B}-\tilde{\vec{B}}_G\right)\right], \qquad \nabla\times\nabla\times\vec{J}_m = n_s m_e\left[\nabla\times\left(\frac{e}{m_e}\vec{B}-\tilde{\vec{B}}_G\right)\right] \tag{1268}$$

Since $\dot{\rho}_c = \dot{\rho}_m = 0$ for a steady state current, then by the continuity equation for charge and mass, we have $\nabla\cdot\vec{J}_c = \nabla\cdot\vec{J}_m = 0$. In that case, the vector calculus identity $\nabla\times\nabla\times\vec{J} = \nabla\left(\nabla\cdot\vec{J}\right) - \nabla^2\vec{J}$ becomes $\nabla\times\nabla\times\vec{J} = -\nabla^2\vec{J}$ for both the charge current and the mass current. Then the equations above can be written as

$$\nabla^2\vec{J}_c = n_s e\left[\nabla\times\left(\frac{e}{m_e}\vec{B}-\tilde{\vec{B}}_G\right)\right], \qquad -\nabla^2\vec{J}_m = n_s m_e\left[\nabla\times\left(\frac{e}{m_e}\vec{B}-\tilde{\vec{B}}_G\right)\right] \tag{1269}$$

Substituting the static Ampere law and the static *gravito*-Ampere law (1238) gives

$$\nabla^2\vec{J}_c = n_s e\left(\frac{e\mu}{m_e}\vec{J}_c + 2\mu_{g(SC)}\vec{J}_m\right), \qquad -\nabla^2\vec{J}_m = n_s m_e\left(\frac{e\mu}{m_e}\vec{J}_c + 2\mu_{g(SC)}\vec{J}_m\right) \tag{1270}$$

To decouple these differential equations, we can start by solving the left equation for $\vec{J}_m$ and the right equation for $\vec{J}_c$ which gives

$$\vec{J}_m = \frac{1}{n_s e 2\mu_{g(SC)}}\nabla^2\vec{J}_c - \frac{e\mu}{2\mu_{g(SC)}m_e}\vec{J}_c, \qquad \vec{J}_c = -\frac{1}{\mu n_s e}\nabla^2\vec{J}_m - \frac{2\mu_{g(SC)}m_e}{\mu e}\vec{J}_m \tag{1271}$$

Substituting $\vec{J}_m$ from (1271) into the second differential equation in (1270) gives

$$\begin{aligned}-\nabla^2&\left(\frac{1}{2\mu_{g(SC)}n_s e}\nabla^2\vec{J}_c - \frac{e\mu}{2\mu_{g(SC)}m_e}\vec{J}_c\right)\\ &= n_s m_e\left[\frac{e\mu}{m_e}\vec{J}_c + 2\mu_{g(SC)}\left(\frac{1}{2\mu_{g(SC)}n_s e}\nabla^2\vec{J}_c - \frac{e\mu}{2\mu_{g(SC)}m_e}\vec{J}_c\right)\right]\end{aligned} \tag{1272}$$

$$\nabla^2\left[\nabla^2\vec{J}_c + \left(2\mu_{g(SC)}n_s m_e - \frac{\mu n_s e^2}{m_e}\right)\vec{J}_c\right] = 0 \tag{1273}$$

Assuming $\nabla^2\vec{J}_c$ and $\vec{J}_c$ go to zero as $r\longrightarrow 0$, then the solution to the outside differential equation is

$$\nabla^2\vec{J}_c - \left(\frac{\mu n_s e^2}{m_e} - 2\mu_{g(SC)}n_s m_e\right)\vec{J}_c = 0 \tag{1274}$$

Once again, we can use (1246) to write this result as

$$\boxed{\nabla^2\vec{J}_c - \frac{1}{\lambda_{L\ (modified)}}\vec{J}_c = 0} \tag{1275}$$

Then the solution to the differential equation is $J_c(x) = J_0 e^{-x/\lambda_{L(mod)}}$ where $\lambda_{L(mod)}$ is the modified penetration depth for the *charge* supercurrent. Notice that setting $\mu_{g(SC)} \approx 0$ makes this expression reduce to the London penetration depth for the purely magnetic case as expected.

Returning to the coupled differential equations in (1270), we can also substitute $\vec{A}$ from (1271) into the first differential equation in (1270) which gives

$$\nabla^2\left(-\frac{1}{\mu n_s e}\nabla^2\vec{J}_m - \frac{2\mu_{g(SC)}m_e}{\mu e}\vec{J}_m\right) = n_s e\left[\frac{e\mu}{m_e}\left(-\frac{1}{\mu n_s e}\nabla^2\vec{J}_m - \frac{2\mu_{g(SC)}m_e}{\mu e}\vec{J}_m\right) + 2\mu_{g(SC)}\vec{J}_m\right] \tag{1276}$$

$$\nabla^2\left[\nabla^2\vec{J}_m + \left(2\mu_{g(SC)}n_s m_e - \frac{\mu n_s e^2}{m_e}\right)\vec{J}_m\right] = 0 \tag{1277}$$

Assuming $\nabla^2\vec{J}_m$ and $\vec{J}_m$ go to zero as $r \to 0$, then the solution to the outside differential equation is

$$\nabla^2\vec{J}_m - \left(\frac{\mu n_s e^2}{m_e} - 2\mu_{g(SC)}n_s m_e\right)\vec{J}_m = 0 \tag{1278}$$

Once again, we can use (1246) to write this result as

$$\boxed{\nabla^2\vec{J}_m - \frac{1}{\lambda_{L\ (modified)}}\vec{J}_m = 0} \tag{1279}$$

Then the solution to the differential equation is $J_m(x) = J_0 e^{-x/\lambda_{L(mod)}}$ where $\lambda_{L(mod)}$ is the modified penetration depth for the *mass* supercurrent. This would seem contrary to the fact that the gravito-magnetic field, $\tilde{\vec{B}}_G = \nabla \times \vec{h}$, does *not* have an exponential decay solution in (1255). However, for a neutral superfluid, we can set $e = 0$ which leads to $\nabla^2 J_m + \frac{1}{\alpha^2}J_m = 0$ where $\alpha^{-2} \equiv 2\mu_{g(SC)}n_s m_e$. This predicts *no* exponential decay solution for J_m and is consistent with the result found in (1208). Also notice that the mass current, $\vec{J}_m$, and the charge current, $\vec{J}_c$,both decay exponentially with the *same* penetration depth, $\lambda_{L\ (modified)}$.

The electric and gravito-electric fields

Taking the time derivative of the supercurrents in (1236) and using $\vec{E} = -\partial_t\vec{A}$ and $\tilde{\vec{E}}_G = -\partial_t\vec{h}$ gives[121]

$$\partial_t\vec{J}_c = n_s e\left(\frac{e}{m_e}\vec{E} - \tilde{\vec{E}}_G\right), \qquad \partial_t\vec{J}_m = -n_s m_e\left(\frac{e}{m_e}\vec{E} - \tilde{\vec{E}}_G\right) \tag{1280}$$

[121] In Sections 4 and 5, there is a detailed discussion showing that the approximation of slow-moving sources requires $h_{ij} \approx 0$. Combining this with the harmonic gauge (which is required to obtain the gravito-electromagnetic "Maxwell-like" equations) requires that $\partial_t\vec{h} = 0$. However, in this section, we will simply assume that we have a fully relativistic formulation so that $\partial_t\vec{h} \neq 0$.

Also taking the time derivative of (1275) and (1279) gives

$$\nabla^2 \partial_t \vec{J}_c - \frac{1}{\lambda_{L\ (modified)}} \partial_t \vec{J}_c = 0, \qquad \nabla^2 \partial_t \vec{J}_m - \frac{1}{\lambda_{L\ (modified)}} \partial_t \vec{J}_m = 0 \tag{1281}$$

Inserting (1280) into (1281) gives

$$n_s e \nabla^2 \left(\frac{e}{m_e} \vec{E} - \tilde{\tilde{E}}_G \right) - \frac{n_s e}{\lambda_{L\ (modified)}} \left(\frac{e}{m_e} \vec{E} - \tilde{\tilde{E}}_G \right) = 0 \tag{1282}$$

and

$$-n_s m_e \nabla^2 \left(\frac{e}{m_e} \vec{E} - \tilde{\tilde{E}}_G \right) + \frac{n_s m_e}{\lambda_{L\ (modified)}} \left(\frac{e}{m_e} \vec{E} - \tilde{\tilde{E}}_G \right) = 0 \tag{1283}$$

The two equations above are redundant. For convenience, we can multiply through by m_e and use $q = -e$. This gives

$$\nabla^2 \left(q\vec{E} + m_e \tilde{\tilde{E}}_G \right) - \frac{1}{\lambda_{L\ (modified)}} \left(q\vec{E} + m_e \tilde{\tilde{E}}_G \right) = 0 \tag{1284}$$

The terms in parentheses are simply the Lorentz force, F. Therefore, we can write the equation as simply

$$\nabla^2 F - \frac{1}{\lambda_{L\ (modified)}} F = 0 \tag{1285}$$

Then the solution to the differential equation is $F(x) = F_0 e^{-x/\lambda_{L(mod)}}$ where $\lambda_{L(mod)}$ is the modified penetration depth for the total Lorentz force. This implies that it is the *combined* field $\left(q\vec{E} + m_e \tilde{\tilde{E}}_G \right)$ that decays exponentially inside the superconductor with a penetration depth given by $\lambda_{L\ (modified)}$.

12 Interaction of gravitational (GR) waves with superconductors

12.1 The London equation in the presence of GR waves

In this section, a constitutive equation is derived for the response of a superconductor to gravitational waves via a minimal coupling rule. It is known in electromagnetism that the minimal coupling rule for a charged particle in an electromagnetic field, $\vec{p}_{can} = m_e\vec{v} - e\vec{A}$. As shown in (2933) of Appendix Q, setting $\vec{p}_{can} = 0$ and solving for the velocity leads to $\vec{v} = e\vec{A}/m_e$. Then substituting this into the current density, $\vec{J}_s = -n_e e\vec{v}$, leads to the London equation written as

$$\vec{J}_s = -\frac{n_s e^2}{m_e}\vec{A} \tag{1286}$$

A similar procedure can be used for the case of gravitational waves interacting with a superconductor. The kinetic momentum in curves space-time for charged, revivalistic spinless massive particles (such as Cooper pairs) in the presence of electromagnetic fields was found in (657) to be

$$\pi_i = \gamma m \left(c g_{0i} + g_{ij} v^j\right) \tag{1287}$$

where the "Lorentz factor" in curved space-time is

$$\gamma = \left(-g_{00} - \frac{2}{c} g_{0k} v^k - \frac{1}{c^2} g_{kl} v^k v^l\right)^{-1/2} \tag{1288}$$

and π_i is expressed in terms of the canonical momentum and the vector potential as $\pi_i = p_i - qA_i$. For gravitational wave in the far-field, $g_{0i} = 0$ and $g_{jk} = \eta_{jk} + h^{\tau\tau}_{jk}$. This gives

$$\gamma = \left(1 - \frac{v^2}{c^2} - \frac{1}{c^2} h^{\tau\tau}_{kl} v^k v^l\right)^{-1/2} \tag{1289}$$

A first order approximation gives

$$\gamma \approx 1 + \frac{v^2}{2c^2} - \frac{1}{2c^2} h^{\tau\tau}_{kl} v^k v^l \tag{1290}$$

Inserting this into (1287), keeping to order v^2/c^2 in velocity, and using $g_{0i} = 0$ and $g_{jk} = \eta_{jk} + h^{\tau\tau}_{jk}$ gives

$$\pi_i = m\left(v_i + h^{\tau\tau}_{ij} v_j\right) \tag{1291}$$

Inserting $\pi_i = p_i - qA_i$ and promoting this equation to a quantum mechanical operator acting on a state ψ gives[122]

$$\left(\hat{p}_i - q\hat{A}_i\right)\psi = m\left(\hat{v}_i + h^{\tau\tau}_{ij}\hat{v}_j\right)\psi \tag{1292}$$

Since the Cooper pairs are in the zero-momentum eigenstate, then $\hat{p}_i\psi = 0$. Taking the expectation value of the equation above gives

$$\left\langle \hat{A}_i \right\rangle = -\frac{m}{q}\left\langle \hat{v}_i \right\rangle - \frac{m}{q} h^{\tau\tau}_{ij}\left\langle \hat{v}_j \right\rangle \tag{1293}$$

Applying Ehrenfest's theorem allows this equation to return to a *classical* equation of motion once again. Also, using $m = 2m_e$ and $q = 2e$ (for Cooper pairs) gives

$$A_i = -\frac{m_e}{e} v_i - \frac{m_e}{e} h^{\tau\tau}_{ij} v_j \tag{1294}$$

[122] This can be considered a semiclassical approach where the gravitational wave field, $h^{\tau\tau}_{ij}$, is still a *classical* field while $\hat{p}$, $\hat{v}$, and $\hat{A}$ are quantum operators that act on the Cooper pair state, ψ.

Solving for v_i in the first term gives $v_i = -\frac{e}{m_e}A_i + h_{ij}^{\tau\tau}v^j$. Inserting this into the second term of the the expression above gives

$$A_i = -\frac{m_e}{e}v_i - \frac{m_e}{e}h_{ij}^{\tau\tau}\left(-\frac{e}{m_e}A_j + h_{jk}^{\tau\tau}v^k\right) \tag{1295}$$

Staying to first order in the metric perturbation eliminates the last term above. Solving for v_i gives

$$v_i = -\frac{e}{m_e}\left(A_i + A_j h_{ij}^{\tau\tau}\right) \tag{1296}$$

Lastly, writing the expression in term of the current density, $J_s^i = n_s e v_i$, gives

$$\boxed{J_s^i = -\frac{n_s e^2}{m_e}\left(A^i + A_j h_{ij}^{\tau\tau}\right) \qquad \textit{Modified London constitutive equation in the presence of a gravitational wave}} \tag{1297}$$

This result shows that the usual London constitutive equation in (1286) is modified by an additional term involving $h_{ij}^{\tau\tau}A_j$ due to the presence of a gravitational wave. However, if the vector potential is set to zero, the supercurrent vanishes regardless of the presence of a gravitational wave. Therefore, this constitutive equation does not describe the supercurrents generated in a superconductor due to a gravitational wave alone. Rather, it involves a *correction* to the supercurrent which *already exists* in a superconductor. This correction is basically taking account the curved space-time due to the presence of a gravitational wave. To see this more clearly, note that (1297) could be obtained by writing the London constitutive equation in covariant form as

$$J_\mu = -\Lambda_L g_{\mu\nu}A^\nu \tag{1298}$$

Setting $\mu = i$ and summing over ν gives

$$J_i = -\Lambda_L\left(g_{i0}A^0 + g_{ij}A^j\right) \tag{1299}$$

Using $A^\mu = \left(\varphi/c, A^i\right)$ and $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ gives

$$J_i = -\Lambda_L\left(h_{i0}\varphi/c + A^i + h_{ij}A^j\right) \tag{1300}$$

Lastly, setting $h_{0i} = 0$ and $h_{ij} = h_{ij}^{\tau\tau}$, and using $\Lambda_L = n_s e^2/m_e$ leads directly to (1297). Therefore, it is shown here that the metric simply introduces a correction to the usual London constitutive equation due to the presence of curved space-time. However, the gravitational wave is not ultimately producing the current in this constitutive equation.

In Appendix Q, it is shown that taking the time-derivative and the curl of the London equation, $J^i = -\Lambda_L A^i$, leads to the electric and magnetic London equations, respectively. Using $\vec{E} = -\partial_t\vec{A}$ and $\vec{B} = \nabla\times\vec{A}$ gives

$$\partial_t\vec{J} = \Lambda_L\vec{E} \qquad \text{and} \qquad \nabla\times\vec{J} = -\Lambda_L\vec{B} \tag{1301}$$

where $\Lambda_L = n_s e^2/m_e$. The electric and magnetic London equations in the presence of a gravitational wave can also be developed by taking the time-derivative and the curl of (1297), respectively. The time-derivative gives

$$\partial_t J^i = \Lambda_L\left(E^i + E_j h_{ij}^{\tau\tau} - A_j\dot{h}_{ij}^{\tau\tau}\right) \tag{1302}$$

The first term is just the standard electric London equation. The second term and third terms are corrections associated the gravitational wave field. Similarly, taking the curl of (1297) and using $B_i = \varepsilon_{ijk}\partial_j A_k$ gives

$$\varepsilon_{ijk}\partial_j J^k = -\Lambda_L\left(B_i + \varepsilon_{ijk}\partial_j A_l h_{kl}^{\tau\tau}\right) \tag{1303}$$

Once again, the first term is the standard magnetic London equation. The second term term is a correction associated the gravitational wave field.

12.2 A modified penetration depth due to the presence of GR waves

To determine the modified electromagnetic penetration depth due to the presence of a gravitational wave, we begin with the modified London constitutive equation (for a supercurrent in the presence of a gravitational wave) found in (1297) as

$$J_s^i = -\Lambda_L \left(A^i + A^j h_{ij}^{\tau\tau}\right) \tag{1304}$$

where $\Lambda_L = n_s e^2/m_e$. Inserting this into Maxwell's field equations (in the Lorenz gauge) as shown in (1907) and taking the DC limit gives

$$\nabla^2 A^i - \mu_0 \Lambda_L \left(A^i + A^j h_{ij}^{\tau\tau}\right) = 0 \tag{1305}$$

A gravitational wave propagating in the z-direction has $h_{xx}^{\tau\tau} = -h_{yy}^{\tau\tau} = h_\oplus(z,t)$ for plus-polarization and $h_{xy}^{\tau\tau} = h_{yx}^{\tau\tau} = h_\otimes(z,t)$ for cross-polarization, with $h_{ij}^{\tau\tau} = 0$ for all other components. Therefore, summing over repeated indices in the expression above and separating components of A^i gives

$$\nabla^2 A_x - \mu_0 \Lambda_L (A_x + A_x h_\oplus + A_y h_\otimes) = 0 \tag{1306}$$

$$\nabla^2 A_y - \mu_0 \Lambda_L (A_y + A_x h_\otimes - A_y h_\oplus) = 0 \tag{1307}$$

$$\nabla^2 A_z - \mu_0 \Lambda_L A_z = 0 \tag{1308}$$

The London penetration depth for A_z remain unaffected by the gravitational wave since the wave is a *transverse* wave. For the other two components of A_i, consider a gravitational wave with plus-polarization. This leads to

$$\nabla^2 A_x - \mu_0 \Lambda_L (1 + h_\oplus) A_x = 0 \qquad \text{and} \qquad \nabla^2 A_y - \mu_0 \Lambda_L (1 - h_\oplus) A_y = 0 \tag{1309}$$

The solutions are

$$A_x = A_{x,0} e^{-z/\lambda_L^{(x)}} \qquad \text{and} \qquad A_y = A_{y,0} e^{-z/\lambda_L^{(y)}} \tag{1310}$$

where $A_{i,0}$ is the amplitude of the vector potential at the surface of the superconductor, and

$$\lambda_L^{(x)} = \frac{1}{\sqrt{\mu_0 \Lambda_L (1 + h_\oplus)}} \qquad \text{and} \qquad \lambda_L^{(y)} = \frac{1}{\sqrt{\mu_0 \Lambda_L (1 - h_\oplus)}} \tag{1311}$$

are the modified London penetration depths for the x-component and y-component of the vector potential, respectively. Using $\Lambda_L = n_s e^2/m_e$ gives

$$\boxed{\lambda_L^{(x)} = \sqrt{\frac{m_e}{\mu_0 n_s e^2 (1 + h_\oplus)}} \qquad \text{and} \qquad \lambda_L^{(y)} = \sqrt{\frac{m_e}{\mu_0 n_s e^2 (1 - h_\oplus)}}} \tag{1312}$$

Modified London penetration depths for the x- and y-components of the magnetic field in the presence of a plus-polarized gravitational wave

These results demonstrate that the presence of a plus-polarized gravitational wave will simply modify the penetration depth by one part in $1 \pm h_\oplus$.

12.3 A gravito-London constitutive equation for GR waves

In this section, a constitutive equation is given which does not come from the geodesic equation of motion or the canonical momentum for gravitational waves coupled to a superconductor. Rather, it can be developed using the free energy density in curved space-time. (The formal derivation will be shown in later sections.) In this section, we simply motivate the form of the constitutive equation for gravitational waves interacting with a superconductor by comparison to the form of the London constitutive equation for electromagnetism. Then applying the linearized Einstein equation for the transverse-traceless strain field (which contains the propagating degrees of freedom for gravitational waves), leads to a dispersion relationship, plasma frequency and penetration depth for gravitational waves incident on a superconductor. The results match that of Press in [73].

In (2931) of Appendix Q, the London equation is derived as

$$\vec{J}_s = -\Lambda_L \vec{A} \tag{1313}$$

where $\Lambda_L = n_s e^2/m_e$. The negative sign is responsible for the fact that electromagnetic fields are *expelled* from a superconductor rather than being admitted.[123] Similar to (1313) which predicts a linear response between $\vec{J}$ and $\vec{A}$, we may also conjecture a linear response between $h_{ij}^{\tau\tau}$ (the transverse-traceless strain for gravitational waves) and $T_{ij}^{\tau\tau}$ (the transverse-traceless stress). We know that $h_{ij}^{\tau\tau}$ is a dimensionless quantity since it is a metric component (and because it describes a change of length per unit length). We also know that $T_{ij}^{\tau\tau}$ has dimensions of ρv^2 since it is a stress tensor quantity. Therefore, we can propose a constitutive equation given by

$$\boxed{T_{ij}^{\tau\tau} = -\mu_{G(SC)} h_{ij}^{\tau\tau} \qquad \textit{Gravito-London constitutive equation for a superconductor}} \tag{1314}$$

where $\mu_{G(SC)}$ is an undetermined positive constant with the dimensions of energy density. It is referred to in this dissertation as the "gravitational shear modulus," or simply "gravitational modulus." Similar to the case with electromagnetism, this relationship is expected to follow as a direct result of assuming that particles within a superconductor (namely, Cooper pairs) undergo *dissipationless* acceleration due to the gravitational wave field. The negative sign relating $T_{ij}^{\tau\tau}$ and $h_{ij}^{\tau\tau}$ would be responsible for gravitational waves being *expelled* from a superconductor rather than being admitted.

The following are some observations concerning these results.

- The constitutive equation in (1314) can not be written covariantly as $T_{\mu\nu} = -\mu_{G(SC)} h_{\mu\nu}$ since this would imply *scalar* relation given by $T_{00} = -\mu_{G(SC)} h_{00}$. Using $T_{00} = \rho c^2$ and the gravito-scalar potential defined as $\varphi_G \equiv -c^2 \overline{h}_{00}/4$ would lead to $\rho c^2 = \mu_{G(SC)} h_{00}$. Since $h_{\mu\nu}$ is the *cause* and $T_{\mu\nu}$ is the *effect*, then this would imply that φ_G could "cause" ρc^2 which is clearly absurd since a gravito-scalar potential cannot *cause* a rest energy density to exist.[124]

[123] The negative sign relating $\vec{J}$ and $\vec{A}$ is also a necessary characteristic of *any* material responding to an external field according to a generalized Hook's law. This is required in order for the material to remain intact (via internal restoring forces), rather than diffuse more and more rapidly in a "run away" type of effect. Note that Ohm's law, $\vec{J} = \sigma_c \vec{E}$, also has a negative sign when $\vec{E} = -\partial_t \vec{A}$ is used to write it in terms of the vector potential, $\vec{J} = -\sigma_c \partial_t \vec{A}$.

[124] This is directly analogous to the fact London's law can not be written in a covariant form as $J^\mu = -\Lambda_L A^\mu$. Using $J^0 = c\rho_{charge}$, where ρ_{charge} is the charge density, and $A^0 = -\varphi_E/c$, where φ_E is the electric scalar potential, would lead to $\rho_{charge} = -\Lambda_L \varphi_E/c^2$. Since φ_E is the *cause* and ρ_{charge} is the *effect*, then this would imply that the electric scalar potential could "cause" a charge energy which is clearly absurd.

However, the constitutive equation *can* be applied to the *vector* relation $T_{0i} = -\mu_{G(SC)} h_{0i}$. Recall that $T_{0i} = \rho c v_i = c J_{m,i}$, where $J_{m,i}$ is the mass current density. Also, the gravito-vector potential is typically defined as $\vec{h} \equiv \frac{c}{4}(h_{01}, h_{02}, h_{03})$. These relations lead to $\vec{J}_m = -\mu_{G(SC)} \vec{h}/4$ which is just the gravito-London constitutive equation describing the mass current produced by a gravito-vector potential. In (1195), this constitutive equation was written as $\vec{J}_m = -n_s m_e \vec{h}$. However, the covariant expression, $T_{\mu\nu} = -\mu_{G(SC)} h_{\mu\nu}$, would still be a problem since it would imply that $\mu_{G(SC)} = 4 n_s m_e$ which is not consistent with the results found in the following sections. This is further indication that a covariant equation of the form $T_{\mu\nu} = -\mu_{G(SC)} h_{\mu\nu}$ is not valid.

- Unlike the spring constant, k, for a simple harmonic oscillator, the gravitational modulus, $\mu_{G(SC)}$, is not necessarily a constant. In fact, in later sections it will be found that $\mu_{G(SC)}$ can be a function of velocity, vector potential, and coherence length (as shown in (1530) for the case of the Cooper pairs), or a function of frequency, temperature, and spatial dimensions (as shown in (1696) for the case of the lattice ions). This should not be confused with the Equivalence Principle which predicts that all forms of mass-energy should respond to gravitation the same way.

12.4 A plasma frequency and penetration depth for GR waves

In this section, we develop a gravitational plasma frequency, index of refraction, penetration depth, and other quantities pertaining to the interaction of gravitational waves with a superconductor. The analysis is directly analogous to the electromagnetic case shown in Appendix R. Recall that the wave equation for gravitational waves is given in (361) as $\Box h_{ij}^{\tau\tau} = -2\kappa T_{ij}^{\tau\tau}$. Inserting the gravito-London constitutive equation, $T_{ij}^{\tau\tau} = -\mu_{G(SC)} h_{ij}^{\tau\tau}$, and expanding the box operator gives[125]

$$-\tfrac{1}{c^2}\partial_t^2 h_{ij}^{\tau\tau} + \nabla^2 h_{ij}^{\tau\tau} = 2\kappa\mu_G h_{ij}^{\tau\tau} \tag{1315}$$

A dispersion relation can be obtained by using a monochromatic, plane-fronted wave propagating in the z-direction given by

$$h_{ij}^{\tau\tau}(z,t) = h_{ij,0}^{\tau\tau} e^{i(kz-\omega t)} \tag{1316}$$

where $h_{ij,0}^{\tau\tau}$ is a constant amplitude tensor. Then the wave equation above gives[126]

$$-\frac{(-i\omega)^2}{c^2} h_{ij,0}^{\tau\tau} + (ik)^2 h_{ij,0}^{\tau\tau} = 2\kappa\mu_G h_{ij,0}^{\tau\tau} \tag{1317}$$

$$\left(k^2 - \frac{\omega^2}{c^2} + 2\kappa\mu_G\right) h_{ij,0}^{\tau\tau} = 0 \tag{1318}$$

For a non-trivial solution, $h_{ij,0}^{\tau\tau} \neq 0$, this gives

$$k^2 = \frac{\omega^2}{c^2} - 2\kappa\mu_G \tag{1319}$$

Factoring out ω^2/c^2 leads to the following dispersion relation for gravitational waves in a superconductor.[127]

$$\boxed{k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{2c^2\kappa\mu_G}{\omega^2}\right) \qquad \textit{Dispersion relation for gravitational waves in a superconductor}} \tag{1320}$$

This matches the result found by Press [73] in his equation (2) where $\mu_{G(SC)}$ is identified as an elastic shear modulus.[128] However, here $\mu_{G(SC)}$ will be referred to as a "*gravitational* shear modulus" (or "gravitational

[125] Note that the *full* transverse-tracess stress tensor within the superconductor is $T_{ij}^{\tau\tau} = -\mu_G h_{ij}^{\tau\tau} + T_{ij\,(superrcrrents)}^{\tau\tau}$ where $T_{ij\,(superrcurents)}^{\tau\tau}$ is given in (1923) and describes the stress produced in the superconductor as a result of electromagnetic fields driving supercurrents. The treatment in this section assumes there are no electromagnetic fields driving supercurrents, or equivalently, that the supercurrent is essentially a *neutral* superfluid.

[126] The frequency of the incoming gravitational wave (on the right) is assumed to have the *same* frequency as the outgoing gravitational wave (on the left).

[127] Typically the value of κ is determined by $\kappa = 8\pi G/c^4$. However, it is possible that a material (such as a superconductor) may have a relative *gravitational permeativity*, κ_r, such that $\kappa = \kappa_r\kappa_0$ where $\kappa_0 = 8\pi G/c^4$ in vacuum.

[128] Note that Press does not have a factor of c^2 appearing in the numerator which implies that he may have used $T_{ij} = \mu c^2 h_{ij}$. Also note that Press makes explicit reference to the transverse-traceless gauge in his formulation. However, this gauge is only valid in vacuum, not matter, as discussed in Appendix E. Therefore, h_{ij}^{TT} cannot be used in this formulation. Instead, we specifically work with the gauge-invariant transverse-traceless *part* of the metric, $h_{ij}^{\tau\tau}$, which satisfies a wave equation *in matter* as found using the Helmholtz Decomposition formulation.

modulus") since later it is shown to have important differences when compared to the usual *elastic* shear modulus from continuum mechanics. In [73], it is pointed out that (1320) resembles the electromagnetic equations of a dense plasma. For the case of a superconductor, it is expected that the charge-separation effect (which will be demonstrated later in the dissertation) will indeed cause the material to act as effectively a very dense plasma. In fact, a gravitational plasma frequency can be defined as

$$\boxed{\omega_G^2 \equiv 2c^2\kappa\mu_G \qquad \textit{Gravitational wave plasma frequency in a superconductor}} \tag{1321}$$

Then (1320) can also be written as

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_G^2}{\omega^2}\right) \tag{1322}$$

It is clear from (1322) that reflection or absorption will occur when $\omega \leq \omega_G$ which means k becomes imaginary. We can also write (1322) as

$$k^2 = \frac{\omega^2}{c^2} n_G^2(\omega) \tag{1323}$$

where we are using a "gravitational index of refraction" defined as

$$\boxed{n_G^2(\omega) \equiv \left(1 - \frac{\omega_G^2}{\omega^2}\right) \qquad \textit{Gravitational wave index of refraction in a superconductor}} \tag{1324}$$

This characterizes the reflection and refraction of gravitational waves in a superconductor. Next, a gravitational wave penetration depth will be determined. A complex wave number can be written as

$$k = K + i\alpha \tag{1325}$$

where K and κ_0 are real quantities. Inserting this into the plane wave of (1316) and separating the real and imaginary parts of the phase gives

$$h_{ij}^{\tau\tau}(z,t) = h_{ij,0}^{\tau\tau} e^{i[(K+i\alpha)z-\omega t]} \tag{1326}$$

$$= h_{ij,0}^{\tau\tau} e^{-\alpha z} e^{i(Kz-\omega t)} \tag{1327}$$

Here it is evident that the wave falls off exponentially with distance where α is the exponential decay factor. The square of the wave number in (1325) is

$$k^2 = K^2 - \alpha^2 + 2iK\alpha \tag{1328}$$

Since k^2 in (1319) is only real, then we must have either $K = 0$ or $\alpha = 0$ to eliminate the last cross term above. Setting $\alpha = 0$ and using (1319) gives

$$K^2 = \frac{\omega^2}{c^2} - 2c^2\kappa\mu_G \tag{1329}$$

Note that this condition is only valid for $\omega^2/c^2 \geq 2\kappa\mu_G$ since K is real. Using the gravitational plasma frequency in (1321), we can write this condition as $\omega \geq \omega_G$. Then the plane wave of (1327) becomes

$$\boxed{\begin{array}{c} h_{ij}^{\tau\tau}(z,t) = h_{ij,0}^{\tau\tau} e^{i(Kz-\omega t)} \qquad \text{where} \qquad K = \pm\sqrt{\frac{\omega^2}{c^2} - 2c^2\kappa\mu_G} \\ \textit{Propagating solution for a gravitational wave} \\ \textit{in a superconductor (for } \omega \geq \omega_G\textit{)} \end{array}} \tag{1330}$$

This condition corresponds to a wave propagating through the material with no attenuation. On the other hand, setting $K = 0$ and using (1319) and (1328) gives

$$\alpha^2 = 2c^2\kappa\mu_G - \frac{\omega^2}{c^2} \tag{1331}$$

Note that this condition is only valid for $\omega^2/c^2 \leq 2\kappa\mu_G$ since α is real. Using the gravitational plasma frequency in (1321), we can write this condition as $\omega \leq \omega_G$. Then the plane wave of (1327) becomes[129]

$$\boxed{\begin{array}{c} h_{ij}^{\tau\tau}(z,t) = h_{ij,0}^{\tau\tau} e^{-\alpha z} e^{-i\omega t} \qquad \text{where} \qquad \alpha = \sqrt{2c^2\kappa\mu_G - \frac{\omega^2}{c^2}} \\ \textit{Exponentially decaying solution for a gravitational wave} \\ \textit{in a superconductor for (} \omega \leq \omega_G\textit{)} \end{array}} \tag{1332}$$

Therefore, we can identify a characteristic frequency-dependent penetration depth as $\delta_G = 1/\alpha$. Using (1331) gives

$$\boxed{\delta_G^2 = \frac{c^2}{2c^2\kappa\mu_G - \omega^2} \qquad \begin{array}{c} \textit{Gravitational wave penetration depth} \\ \textit{in a superconductor} \end{array}} \tag{1333}$$

We can also write this in terms of the gravitational plasma frequency in (1321) as

$$\delta_G^2 = \frac{c^2}{\omega_G^2 - \omega^2} \tag{1334}$$

In this form, we see that as ω approaches ω_G, we have δ_G approaching infinity. This means that as the frequency approaches the plasma frequency, the wave is no longer attenuated with depth. On the other hand, for $\omega >> \omega_G$, it is evident that $\delta_G \rightarrow \lambda_G = c/\omega_G$. In fact, in the "DC" limit ($\omega \rightarrow 0$), the penetration depth is no longer frequency-dependent and using (1321) gives[130]

$$\boxed{\lambda_G = \frac{c}{\omega_G} = \frac{1}{\sqrt{2\kappa\mu_G}} \qquad \begin{array}{c} \textit{Gravitational wave penetration depth} \\ \textit{for a superconductor in the DC limit} \end{array}} \tag{1335}$$

[129] Formally, κ_0 should have a positive and negative root. However, if we consider the case of $z < 0$ representing the vacuum and $z > 0$ representing the superconductor, then we can write κ_0 with only the *positive* root in order to obtain an exponential *decay* solution and avoid a diverging exponential growth solution.

[130] Another way of arguing this is to consider that the wave can only penetrate the skin of the superconductor to a depth on the order of a wavelength. Then using $k = c/\omega_G$ and $\omega_G = \sqrt{16\pi G\mu_G}/c$ leads to (1335).

This matches the result obtained by Press in equation (3) of [73] for a gravitational wave mirror consisting of an elastic solid.[131] If we consider the DC limit of (1315), then we have a Yukawa-like equation given as

$$\nabla^2 h_{ij}^{\tau\tau} - 2\kappa\mu_G h_{ij}^{\tau\tau} = 0 \tag{1336}$$

The prefactor in (1336) can be written as $2\kappa\mu_G = 1/\lambda_G^2$ so the solution is $h_{ij}^{\tau\tau}(z) = A_{ij}^{\tau\tau} e^{-z/\lambda_G}$ where

$$\lambda_G = \frac{1}{\sqrt{2\kappa\mu_G}} \tag{1337}$$

This is the gravitational London penetration depth for a static gravitational strain field expelled from a superconductor. As expected, we find that $\lambda_G = c/\omega_G$ which is equivalent to the gravitational penetration depth found in (1335) in the "DC" limit.

Lastly, because the gravitational London-like constitutive equation given by $T_{ij}^{\tau\tau} = -\mu_{G(SC)} h_{ij}^{\tau\tau}$ is just a proportionality between $T_{ij}^{\tau\tau}$ and $h_{ij}^{\tau\tau}$, then $h_{ij}^{\tau\tau}$ can be replaced with $T_{ij}^{\tau\tau}$ all throughout the analysis above to obtain the same exact results for the dispersion relation and penetration depth of the sources. Therefore, the solutions in (1330) and (1332) can also be used to describe the stress tensor when the gravitational London-like constitutive equation applies. This means that $\omega \geq \omega_{EM}$ leads to

$$T_{ij}^{\tau\tau}(z,t) = T_{ij,0}^{\tau\tau} e^{i(Kz-\omega t)} \qquad \text{where} \qquad K = \pm\sqrt{\tfrac{\omega^2}{c^2} - 2c^2\kappa\mu_G} \tag{1338}$$

and $\omega \leq \omega_{EM}$ leads to

$$T_{ij}^{\tau\tau}(z,t) = T_{ij,0}^{\tau\tau} e^{-\alpha z} e^{-i\omega t} \qquad \text{where} \qquad \alpha = \sqrt{2c^2\kappa\mu_G - \tfrac{\omega^2}{c^2}} \tag{1339}$$

Here $T_{(0)ij}^{\tau\tau}$ is a constant amplitude tensor. Also, the penetration depth for the exponentially decaying stress in (1339) is given by (1333).

[131] Press also has c^2 appearing in the numerator. However, his dispersion equation implies that he used a plasma frequency defined as $\omega_G^2 \equiv 16\pi G$, and a constituent equation defined as $T_{ij} = -\mu h_{ij}$ where μ would have the units of mass density, not energy density. Thefore, dimensionally he should only have c in the numerator of his skin depth.

12.5 A gravito-Meissner effect for GR waves in the DC limit

In Appendix Q, it is shown that starting from the London constitutive equation, $\vec{J} = -\Lambda_L \vec{A}$ and then taking temporal and spatial derivatives leads to constitutive equations involving the electric and magnetic fields within a superconductor given as

$$\partial_t \vec{J} = \Lambda_L \vec{E} \qquad \text{and} \qquad \nabla \times \vec{J} = -\Lambda_L \vec{B} \tag{1340}$$

For a sinusoidal current density, $\partial_t J \propto \omega \vec{J}$. Therefore, in the DC limit $(\omega \to 0)$, the first equation above requires that $\vec{E} = 0$. This implies that in the DC limit, the electric field vanishes completely throughout the entire superconductor and only a magnetic field remains within the London penetration depth of the superconductor. The magnetic field drives the supercurrents, not via the Lorentz force $m\vec{a} = q\vec{v} \times \vec{B}$, but by the second constitutive equation in (1340). Furthermore, it is also shown in Appendix R that the magnetic field satisfies a Yukawa-like equation given as

$$\nabla^2 \vec{B} - \frac{1}{\lambda_L^2} \vec{B} = 0 \tag{1341}$$

The solution to this equation is $B(z) = B_0 e^{-z/\lambda_L}$ where z is the distance into the surface of the superconductor and λ_L is the London penetration depth found to be $\lambda_L^2 = \dfrac{m_e}{\mu_0 n_s e^2}$. This result implies that the magnetic field is expelled from the interior of the superconductor which is referred to as the Meissner effect.

In this section, an analogous approach is used to demonstrate a Meissner-like effect for the DC limit of gravitational waves in a superconductor. Recall that electric-like and magnetic-like tensor fields for gravitational waves are written in (354) as

$$\mathscr{E}_{ij} = -\partial_t h_{ij}^{\tau\tau}, \qquad \text{and} \qquad \mathscr{B}_{ijk} = \partial_k h_{ij}^{\tau\tau} \tag{1342}$$

Using these fields, the geodesic equation of motion was found to be

$$a^i = v^i \mathscr{E}_{ij} + v^i v^k \left(\tfrac{1}{2}\mathscr{B}_{jki} - \mathscr{B}_{ikj}\right) \tag{1343}$$

which demonstrates that the electric-like and magnetic-like tensor fields describe the actual physical motion of test particles in the presence of gravitational wave. These fields can be used to formulate a gravitational Meissner-like effect for gravitational waves in the DC limit. Recall that in (1314), the gravito-London constitutive equation for gravitational waves is given as $T_{ij}^{\tau\tau} = -\mu_{G(SC)} h_{ij}^{\tau\tau}$. Taking temporal and spatial derivatives of this equation and using (1342) leads to the following constitutive equations.

$$\partial_t T_{ij}^{\tau\tau} = \mu_{G(SC)} \mathscr{E}_{ij} \qquad \text{and} \qquad \partial_k T_{ij}^{\tau\tau} = -\mu_{G(SC)} \mathscr{B}_{ijk} \tag{1344}$$

These equations are directly analogous to the constitutive equations in (1340) for the electric and magnetic fields. For a sinusoidal stress tensor, $\partial_t T_{ij}^{\tau\tau} \propto \omega T_{ij}^{\tau\tau}$. Therefore, in the DC limit $(\omega \to 0)$, the first equation above requires that $\mathscr{E}_{ij} = 0$. This implies that in the DC limit, the electric-like tensor field vanishes completely throughout the entire superconductor and only the magnetic-like tensor field remains. (This is directly analogous to the electric field vanishing throughout the entire superconductor and only the magnetic field remaining.) The magnetic-like tensor field drives the supercurrents, not via the geodesic equation of motion (1343) but by the second constitutive equation in (1344). In (356), it was shown that the wave equation, $\Box h_{ij}^{\tau\tau} = -2\kappa T_{ij}^{\tau\tau}$, can be written in as

$$\partial_k \mathscr{B}_{ijk} = -\left(2\kappa T_{ij}^{\tau\tau} + \frac{1}{c^2} \partial_t \mathscr{E}_{ij}\right) \tag{1345}$$

This is a gravito-Ampere law in the sense that a spatial derivative of a magnetic-like tensor field is proportional to a source term plus a time-derivative of the electric-like tensor field. It was already stated that for sinusoidal fields and stresses, the DC limit requires $\mathscr{E}_{ij} = 0$. Taking a derivative, ∂_l, of (1345) and inserting the second equation of (1344) leads to

$$\partial_l \partial_k \mathscr{B}_{ijk} = 2\kappa\mu_G \mathscr{B}_{ijl} \tag{1346}$$

Using $\mathscr{B}_{ijk} = \partial_k h_{ij}^{\tau\tau}$ and $\partial_k \partial_k = \nabla^2$ gives

$$\nabla^2 \partial_l h_{ij}^{\tau\tau} = 2\kappa\mu_G \mathscr{B}_{ijl} \tag{1347}$$

Then inserting $\mathscr{B}_{ijl} = \partial_l h_{ij}^{\tau\tau}$, rearranging and changing "l" to "k" gives

$$\nabla^2 \mathscr{B}_{ijk} - 2\kappa\mu_G \mathscr{B}_{ijk} = 0 \tag{1348}$$

This is a Yukawa-like equation similar to (1341) which implies an exponential decay solution for $\mathscr{B}_{ijk}$ and therefore an associated penetration depth. The solution to this equation is $\mathscr{B}_{ijk}(z) = \mathscr{B}_{ijk,0}(z)\, e^{-z/\lambda_G}$ where z is the distance into the surface of the superconductor, and λ_G is the gravitational penetration depth in the DC limit which is found to be

$$\lambda_G = \frac{1}{\sqrt{2\kappa\mu_G}} \tag{1349}$$

This is consistent with the gravitational penetration depth in (1335) for a static gravitational strain field expelled from a superconductor. This result implies that the magnetic-like tensor field is expelled from the interior of the superconductor in a gravitational Meissner-like effect. This expulsion of the gravitational wave field is valid for *all* frequencies down to the DC limit. For an upper bound, it is expected that the BCS energy gap frequency would limit the maximum frequency permitted for this gravitational Meissner-like effect to occur since frequencies above this would break up the Cooper pairs and destroy the superconducting state.

12.6 Phase and group velocities for GR waves in a superconductor

Here we consider the phase velocity $\left(v_{phase} = \omega/k\right)$ and group velocity $(v_{group} = d\omega/dk)$ of the waves found in the previous section. Starting with (1319), dividing by ω^2 and getting a common denominator gives

$$\frac{k^2}{\omega^2} = \frac{\omega^2 c^2 - 2c^2\kappa\mu_G}{c^4\omega^2} \tag{1350}$$

Taking the reciprocal and using the gravitational plasma frequency, $\omega_G^2 = 2\kappa\mu_G$, gives

$$\frac{\omega^2}{k^2} = \frac{c^2}{1-\omega_G^2/\omega^2} \tag{1351}$$

Then solving for $v_{phase} = \omega/k$ gives

$$\boxed{v_{phase} = \frac{c}{\sqrt{1-\omega_G^2/\omega^2}} \qquad \textit{Phase velocity of gravitational waves in a superconductor}} \tag{1352}$$

For frequencies much greater than the gravitational plasma frequency, $\omega^2 >> \omega_G^2$, then we have $\omega_G^2/\omega^2 << 1$ which means we can use a binomial expansion (to first order) to obtain

$$v_{phase} \approx c\left(1+\frac{\omega_G^2}{2\omega^2}\right) \tag{1353}$$

This implies that the phase velocity will be superluminal. In fact, (1352) can be written as $v_{phase}^2 = c^2 + v_{phase}^2\omega_G^2/\omega^2$ which means that v_{phase}^2 always exceeds c^2 by an amount $v_{phase}^2\omega_G^2/\omega^2$. We can also see this from the gravitational index of refraction given in (1324) as $n_G(\omega) = \sqrt{1-\omega_G^2/\omega^2}$. For $\omega^2 >> \omega_G^2$, we have $n_G \lesssim 1$. Then $v = c/n_G$ implies that $v \gtrsim c$. Therefore the phase velocity is superluminal.

For frequencies just above the gravitational plasma frequency, $\omega^2 \gtrsim \omega_G^2$, we find that v_{phase} in (1352) becomes arbitrarily large. As ω approaches ω_G, then v_{phase} diverges to infinity This corresponds to n_G going to zero so that $v = c/n_G$ becomes infinite.

Lastly, when $\omega^2 < \omega_G^2$, then v_{phase} and n_G both become imaginary. This implies a complete expulsion of the wave such that there is no phase velocity of the wave in the superconductor. Since the supercurrent is completely dissipationless, then there can be no absorption of the wave at all. Rather, there must be perfect external reflection of the wave.

Next we consider the *group* velocity of the wave. Returning to (1351) and expressing ω^2 in terms of k^2 gives

$$\omega^2 = c^2k^2 + \omega_G^2 \tag{1354}$$

Taking the derivative with respect to k gives

$$2\omega\frac{d\omega}{dk} = 2c^2k \tag{1355}$$

Solving for $v_{group} = \dfrac{d\omega}{dk}$ gives

$$v_{group} = \frac{c^2}{\omega} k(\omega) \tag{1356}$$

Solving (1354) for $k(\omega)$ yields $k(\omega) = \dfrac{\omega}{c}\sqrt{1 - \omega_G^2/\omega^2}$. Inserting this above gives

$$\boxed{v_{group} = c\sqrt{1 - \omega_G^2/\omega^2} \qquad \textit{Group velocity of gravitational waves in a superconductor}} \tag{1357}$$

For frequencies much greater than the gravitational plasma frequency, $\omega^2 >> \omega_G^2$, then we have $\omega_G^2/\omega^2 << 1$ which means we can use a binomial expansion to first order to obtain

$$v_{group} \approx c\left(1 - \frac{\omega_G^2}{2\omega^2}\right) \tag{1358}$$

This implies that for an arbitrarily large ω, we can make v_{group} arbitrarily close to c. This would describe a wave that is almost completely unaffected by a medium and therefore propagates through it at almost the same speed it has in vacuum.

Notice that v_{group} is always subluminal. In fact, (1357) can be written as $v_{group}^2 = c^2 - c^2\omega_G^2/\omega^2$ which means that v_{phase}^2 always remains less than c^2 by an amount $c^2\omega_G^2/\omega^2$. As ω decreases, v_{group} in (1357) will decrease until it vanishes when $\omega = \omega_G$. For $\omega^2 < \omega_G^2$, then v_{group} becomes imaginary. Once again, this implies a complete expulsion of the wave at these frequencies such that there is no group velocity of the wave in the superconductor. These results collectively show that v_{group} is always subluminal which is expected since v_{group} is representative of the rate at which energy (and information) can be transported in the medium.

12.7 Equilibration time-scales for charge density and stress

First we consider the equation of motion for charge density within a normal conductor. Inserting Ohm's law, $\vec{J} = \sigma_c \vec{E}$ into Gauss's law, $\nabla \cdot \vec{E} = \rho/\varepsilon$, gives $\nabla \cdot \vec{J}/\sigma_c = \rho/\varepsilon$ where σ_c is the electrical conductivity. Then using the continuity equation, $\nabla \cdot \vec{J} = -\dot{\rho}$, gives

$$\dot{\rho} + \frac{\sigma_c}{\varepsilon}\rho = 0$$

We can define $\tau \equiv \varepsilon/\sigma_c$ so the general solution to the differential equation is $\rho = Ae^{-t/\tau} + Be^{t/\tau}$. To avoid having the charge density diverge with time, we set $B = 0$. Then letting $A = \rho_0$ be the charge density at $t = 0$ gives

$$\rho = \rho_0 e^{-t/\tau} \tag{1359}$$

It is clear that $\tau = \varepsilon/\sigma_c$ gives the characteristic time scale that describes how rapidly and net charge in the interior of the conductor will move to the surface. In other words, this time-scale determines how well a material behaves as an electrical *conductor*.[132] For a conductor like copper, this time scale is on the order of $10^{-19}s$. It is evident that as $\sigma_c \to \infty$, then $\tau \to 0$ and the material approaches the case of a "perfect" conductor.

For the case of a superconductor, we can use the *electric* London equation found in (2925) as

$$\partial_t \vec{J}_s = \frac{n_s e^2}{m_e}\vec{E} \tag{1360}$$

Taking the divergence and using Gauss's law, $\nabla \cdot \vec{E} = \rho/\varepsilon$, gives

$$\partial_t \nabla \cdot \vec{J}_s = \frac{n_s e^2}{m_e}\frac{\rho}{\varepsilon} \tag{1361}$$

Next, using the continuity equation, $\nabla \cdot \vec{J} = -\dot{\rho}$, gives

$$\ddot{\rho} + \frac{n_s e^2}{m_e \varepsilon}\rho = 0 \tag{1362}$$

We can use $\omega_p^2 = \dfrac{n_s e^2}{m_e \varepsilon}$ and write the solution as

$$\tilde{\rho} = \rho_0 e^{i\omega_p t} \tag{1363}$$

where $\rho = \mathrm{Re}\,\tilde{\rho}$. Notice that the charge density no longer decays exponentially but rather oscillates sinusoidally at the plasma frequency, ω_p.This is indicative of the *dissipationless* nature of the superconductor. It stems from the fact that the constitutive equation is no longer Ohm's law which is dissipative since it involves the conductivity σ_c derived from the Drude model.[133]. Rather, by using $\vec{E} = -\partial_t \vec{A}$ in (1360), we obtain the *dissipationless* London constitutive equation for a superconductor:

$$\vec{J}_s = -\frac{n_s e^2}{m_e}\vec{A} \tag{1364}$$

Nevertheless, we can find a time-scale using $n_s \approx \rho_{Niobium}/(82m_p)$ for Niobium and $\varepsilon_0 \approx \varepsilon$. Then finding the period from the electromagnetic plasma frequency gives

$$T = \frac{2\pi}{\omega_p} = \sqrt{\frac{m_e \varepsilon}{e^2 n_s}} \approx \sqrt{\frac{82 m_p m_e \varepsilon}{e^2 \rho_{Niobium}}} \approx 7.1 \times 10^{-17}s \tag{1365}$$

[132] See Griffiths [29], p. 393.

[133] See Griffiths [29], pp. 289-290.

Therefore, we can observe that a superconductor acts as essentially a "perfect" conductor. However, it is not due due to a particular value of an electrical property (such as particular values for σ_c or ε). Rather, it is due to a fundamentally different constitutive equation, namely, London's *dissipationless* constitutive equation versus Ohm's law which is a *dissipative* constitutive equation.

Now we carry over the same analysis for the case of gravitation. We can use our proposed constitutive equation

$$T_{ij}^{\tau\tau} = -\mu h_{ij}^{\tau\tau} \tag{1366}$$

where μ is an undetermined positive constant with the dimensions of energy density. Similar to the case with electromagnetism, this relationship is expected to follow as a direct result of assuming that particles within a superconductor (namely, Cooper pairs) undergo *dissipationless* acceleration due to the gravitational wave field. The wave equation for gravitational waves is given by (361) as $\Box h_{ij}^{\tau\tau} = -2\kappa T_{ij}^{\tau\tau}$. This time we will solve (1366) for $h_{ij}^{\tau\tau}$ and insert it into the wave equation to obtain

$$\Box T_{ij}^{\tau\tau} - 2\kappa\mu_G T_{ij}^{\tau\tau} = 0 \tag{1367}$$

This is clearly the same wave equation we had in (1315) for $h_{ij}^{\tau\tau}$. We can write the solution as a separable function using

$$T_{ij}^{\tau\tau}(\vec{x},t) = X(\vec{x})\,T(t) \tag{1368}$$

Then (1367) becomes

$$-\frac{1}{c^2}X(\vec{x})\,\partial_t^2 T(t) + T(t)\,\nabla^2 X(\vec{x}) - 2\kappa\mu X(\vec{x})\,T(t) \quad = \quad 0 \tag{1369}$$

Dividing through by $T_{ij}^{\tau\tau}(\vec{x},t) = X(\vec{x})\,T(t)$ and rearranging gives

$$-\frac{1}{c^2 T(t)}\partial_t^2 T(t) + \frac{1}{X(\vec{x})}\nabla^2 X(\vec{x}) \quad = \quad 2\kappa\mu \tag{1370}$$

Since $\kappa = 8\pi G/c^4$ and $\omega_G^2 = 16\pi G\mu_G/c^2$, then we can replace the right side with ω_G^2/c^2. Then using separation of variables gives

$$\partial_t^2 T(t) \quad = \quad -\omega_G^2 T(t) \qquad \text{and} \qquad \nabla^2 X(\vec{x}) = \frac{\omega_G^2}{c^2} X(\vec{x}) \tag{1371}$$

The solutions to these differential equations are

$$\tilde{T}(t) = T_0 e^{i\omega_G t} \qquad \text{and} \qquad X(\vec{x}) = X_0 e^{\frac{\omega_G}{c}\vec{x}} \tag{1372}$$

where $T(t) = \mathrm{Re}\,\tilde{T}(t)$. From the spatial solution, we can immediately identify a penetration depth given by $\delta = c/\omega_G$. From the temporal solution, we find that the stress oscillates at the gravitational plasma frequency given by $\omega_G^2 = 16\pi G\mu_G$. We can find a time-scale using

$$T = \frac{2\pi}{\omega_G} = \frac{2\pi}{\sqrt{16\pi G\mu_G/c^2}} = c\sqrt{\frac{\pi}{8G\mu_G}} \tag{1373}$$

12.8 The Landau-Lifshitz pseudotensor correction

The wave equation for gravitational waves is given by (361) as

$$\Box h_{ij}^{\tau\tau} = -2\kappa T_{ij}^{\tau\tau} \tag{1374}$$

where $T_{ij}^{\tau\tau}$ is the transverse-traceless *stress* of the matter-energy distribution that is producing $h_{ij}^{\tau\tau}$, the transverse-traceless *strain* field which propagates as a gravitational wave. In the previous section, we proposed a constitutive equation (1314) relating the transverse-traceless stress $T_{ij}^{\tau\tau}$ which is induced in a matter distribution due to the *incoming* gravitational wave $h_{ij}^{\tau\tau}$. We can write this as

$$T_{ij}^{\tau\tau} = -\mu_{G(SC)} h_{ij}^{\tau\tau} \tag{1375}$$

When determining the resulting *outgoing* gravitational wave, we must recognize that gravitation is a self-coupling field. Therefore, the incoming gravitational wave is also essentially a "source" of gravitation in addition to the stress tensor of the matter. To account for this, a common approach is to use the Landau-Lifshitz pseudotensor, $t_{\mu\nu}^{\text{L-L}}$, as a stress-energy pseudotensor for the gravitational field. As shown in (2668) of Appendix H, the resulting Einstein field equation for gravitational waves is

$$\Box h_{ij}^{\tau\tau\ (out)} = -2\kappa\left(T_{ij}^{\tau\tau} + t_{ij\ \text{L-L}}^{\tau\tau}\right) \tag{1376}$$

Here $h_{ij}^{\tau\tau\ (out)}$ is the outgoing gravitational wave[134] and $t_{ij\ \text{L-L}}^{\tau\tau}$ is the linearized transverse-traceless Landau-Lifshitz pseudotensor.[135] In (2679) of Appendix H, we found that

$$t_{ij\ \text{L-L}}^{\tau\tau} = \frac{1}{\kappa}\Box h_{ij}^{\tau\tau} \tag{1377}$$

Substituting (1375) and (1377) into (1376) gives

$$\boxed{\Box h_{ij}^{\tau\tau\ (out)} = 2\kappa\mu_G h_{ij}^{\tau\tau} - 2\Box h_{ij}^{\tau\tau}} \tag{1378}$$

This is the linearized wave equation which relates an incoming gravitational wave $h_{ij}^{\tau\tau}$ to an outgoing gravitational wave $h_{ij}^{\tau\tau\ (out)}$. The first term on the right side describes the linear response of the matter to the incoming wave (governed by the gravitational modulus, $\mu_{G(SC)}$, of the material). The second term on the right side takes into account the self-coupling of the gravitational field via the Landau-Lifshitz pseudotensor. To obtain a dispersion relation, we can use a plane wave solution $h_{ij}^{\tau\tau}(\vec{x},t) = \tilde{h}_{ij}^{\tau\tau} e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$ where $\tilde{h}_{ij}^{\tau\tau}$ is a constant amplitude tensor. Then evaluating the derivatives in (1378) gives

$$-\frac{(-i\omega)^2}{c^2}\tilde{h}_{ij}^{\tau\tau\ (out)} + (ik)^2\,\tilde{h}_{ij}^{\tau\tau\ (out)} = 2\kappa\mu_G\tilde{h}_{ij}^{\tau\tau} - 2\left[-\frac{(-i\omega)^2}{c^2}\tilde{h}_{ij}^{\tau\tau} + (ik)^2\,\tilde{h}_{ij}^{\tau\tau}\right] \tag{1379}$$

$$\left(\frac{\omega^2}{c^2} - k^2\right)\tilde{h}_{ij}^{\tau\tau\ (out)} = 2\kappa\mu_G\tilde{h}_{ij}^{\tau\tau} - 2\left(\frac{\omega^2}{c^2} - k^2\right)\tilde{h}_{ij}^{\tau\tau} \tag{1380}$$

[134]The Landau-Lifshitz pseudotensor in (2667) and the constituent equation in (1375) are both in terms of the *incoming* wave, $h_{ij}^{\tau\tau\ (in)}$. However, for brevity we drop the superscript "in" and use the convention that $h_{ij}^{\tau\tau}$ with no superscript refers only to the *incoming* wave.

[135]Note that we are working with the transverse-traceless *part* of the metric perturbation, $h_{ij}^{\tau\tau}$, not the transverse-traceless *gauge*, h_{ij}^{TT}, since the transverse-traceless gauge cannot be used in matter.

If the amplitude of the incoming wave and outgoing wave are the same (for dissipationless reflection), then we have

$$k^2 = \frac{\omega^2}{c^2} - \frac{2}{3}\kappa\mu_G \tag{1381}$$

We can factor out ω^2/c^2 and use the gravitational plasma frequency defined in (1321) as $\omega_G^2 = 2c^2\kappa\mu_G$. This gives

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_G^2}{3\omega^2}\right) \tag{1382}$$

Comparing this to (1322), we find that the inclusion of the Landau-Lifshitz pseudotensor simply modifies ω_G^2 by a factor of $1/3$. We can define a *modified* gravitational plasma frequency as

$$\boxed{\omega^2_{G\ (\text{modified})} = \frac{2c^2\kappa\mu_G}{3}} \tag{1383}$$

This modified gravitational plasma frequency takes into account the fact that the incoming gravitational wave is also effectively a "source" of gravitation.

12.9 An equation of motion derived from the constitutive equation

The full stress tensor for gravitational waves within a superconductor is given by

$$T_{ij}^{\tau\tau} = -\mu_{G(SC)} h_{ij}^{\tau\tau} + T_{ij\,(\text{supercurrents})}^{\tau\tau} \tag{1384}$$

where the first term is a gravito-London constitutive equation describing the interaction of the gravitational wave with the Cooper pairs and lattice ions, and $T_{ij\,(\text{supercurrents})}^{\tau\tau}$ is the stress tensor produced in the superconductor due to electromagnetic fields driving supercurrents. Using (1923) for $T_{ij\,(\text{supercurrents})}^{\tau\tau}$ gives

$$T_{ij}^{\tau\tau} = -\mu_{G(SC)} h_{ij}^{\tau\tau} + \frac{e^2\left(\rho + P/c^2\right)}{m_e^2}\left[A_iA_j + A_kA_jh_{ik}^{\tau\tau} + A_lA_ih_{jl}^{\tau\tau} - \tfrac{1}{3}\delta_{ij}\left(A^2 + 2A_kA_lh_{kl}^{\tau\tau}\right)\right] \tag{1385}$$

To simplify this expression, consider the case when $A^2 >> A_kA_jh_{ik}^{\tau\tau}$ for electromagnetic fields that dominate over the gravitational wave field. For convenience, a transverse-traceless "magnetic tensor potential" can be defined as

$$A_{ij}^{\tau\tau} \equiv A_iA_j - \tfrac{1}{3}\delta_{ij}A^2 \tag{1386}$$

Then (1917) becomes

$$T_{ij}^{\tau\tau} = -\mu_{G(SC)} h_{ij}^{\tau\tau} + \frac{e^2\left(\rho + P/c^2\right)}{m_e^2} A_{ij}^{\tau\tau} \tag{1387}$$

Finding the equation of motion requires taking a time derivative. Assuming $\dot{\rho} = \dot{P}/c^2 = 0$ gives

$$\dot{T}_{ij}^{\tau\tau} = -\mu \dot{h}_{ij}^{\tau\tau} + \frac{e^2\left(\rho + P/c^2\right)}{m_e^2} \dot{A}_{ij}^{\tau\tau} \tag{1388}$$

Using (1386), the expression can be written as

$$\dot{T}_{ij}^{\tau\tau} = -\mu \dot{h}_{ij}^{\tau\tau} + \frac{e^2\left(\rho + P/c^2\right)}{m_e^2}\left[\left(\dot{A}_iA_j - \tfrac{1}{3}\delta_{ij}\dot{A}_iA_j\right) - \left(A_i\dot{A}_j - \tfrac{1}{3}\delta_{ij}A_i\dot{A}_j\right)\right] \tag{1389}$$

$$= -\mu \dot{h}_{ij}^{\tau\tau} + \frac{e^2\left(\rho + P/c^2\right)}{m_e^2}\left(\dot{A}_iA_j - A_i\dot{A}_j\right) \tag{1390}$$

Since $E_i = -\dot{A}_i$ and $J_i = -\dfrac{n_s e^2}{m_e}A_i$, then we can also write this as

$$\dot{T}_{ij}^{\tau\tau} = -\mu \dot{h}_{ij}^{\tau\tau} + \frac{\left(\rho + P/c^2\right)}{n_s m_e}\left(E_iJ_j - J_iE_j\right) \tag{1391}$$

Now we can work on the left side. To obtain an expression for $\dot{T}_{ij}^{\tau\tau}$, we can use

$$T_{ij}^{\tau\tau} = \gamma^2\left(\rho_m + P/c^2\right)\left(v_iv_j - \tfrac{1}{3}\delta_{ij}v^2\right) \tag{1392}$$

Taking a time derivative of (1392) and recalling that $\partial_t\left(\rho_m + P/c^2 + \dfrac{m_e}{e}\rho_c\right) = 0$ gives

$$\begin{aligned}\dot{T}_{ij}^{\tau\tau} &= \left(\partial_t\gamma^2\right)\left(\rho_m + P/c^2\right)\left(v_iv_j - \tfrac{1}{3}\delta_{ij}v^2\right)\\ &\quad + \gamma^2\left(\rho_m + P/c^2\right)\left(a_iv_j + v_ia_j - \tfrac{2}{3}\delta_{ij}va\right)\end{aligned} \tag{1393}$$

From (421) we found that $\partial_t \gamma^2 = 2va\gamma^4/c^2$.

$$\dot{T}_{ij}^{\tau\tau} = \left(\rho_m + P/c^2\right)\left[\frac{2va\gamma^4}{c^2}\left(v_i v_j - \tfrac{1}{3}\delta_{ij}v^2\right) + \gamma^2\left(a_i v_j + v_i a_j - \tfrac{2}{3}\delta_{ij}va\right)\right] \tag{1394}$$

Since $\gamma^2 = \left(1 - v^2/c^2\right)$, then $\gamma^4 = \left(1 - v^2/c^2\right)^2 \approx \left(1 - 2v^2/c^2\right)$ to order v^2/c^2. This means that if we are only keeping terms to order v^2/c^2 in the expression above, we can eliminate the first term in the bracket and use $\gamma^2 \approx 1$ for the second term in the bracket. This gives

$$\dot{T}_{ij}^{\tau\tau} = \left(\rho_m + P/c^2\right)\left(a_i v_j + v_i a_j - \tfrac{2}{3}\delta_{ij}va\right) \tag{1395}$$

Now we can insert this into (1391) to obtain the equation of motion as

$$\left(\rho_m + P/c^2\right)\left(a_i v_j + v_i a_j - \tfrac{2}{3}\delta_{ij}va\right) = -\mu\dot{h}_{ij}^{\tau\tau} + \frac{\left(\rho_m + P/c^2\right)}{n_s m_e}\left(E_i J_j - J_i E_j\right) \tag{1396}$$

If we consider the case of no gravitational waves ($h_{ij}^{\tau\tau} = 0$), no internal pressures $(P = 0)$ and we write the mass density as $\rho_m = \dfrac{m_e}{e}\rho_c$, where ρ_c is the charge density, then we obtain

$$\frac{m_e}{e}\rho_c\left(a_i v_j + v_i a_j - \tfrac{2}{3}\delta_{ij}va\right) = -\mu\dot{h}_{ij}^{\tau\tau} + \frac{\rho_m}{n_s m_e}\left(E_i J_j - J_i E_j\right) \tag{1397}$$

Using $J_i = \rho_c v_i$ and $\rho_m = n_s m_e$ gives

$$\frac{m_e}{e}\left(a_i J_j + J_i a_j - \tfrac{2}{3}\delta_{ij}Ja\right) = -\mu\dot{h}_{ij}^{\tau\tau} + E_i J_j - J_i E_j \tag{1398}$$

Rearranging gives

$$\boxed{\left(\frac{m_e}{e}a_i - E_i\right)J_j + \left(\frac{m_e}{e}a_j + E_j\right)J_i - \frac{2m_e}{3e}\delta_{ij}Ja = -\mu\dot{h}_{ij}^{\tau\tau}} \tag{1399}$$

Equation of motion for the supercurrent density
in the presence of a gravitational wave

This is the equation of motion for the supercurrent density due to the presence of a gravitational wave and electromagnetic fields within the superconductor. Notice that it is still transverse (since $\partial_i J_i = \rho_c \partial_i v_i = 0$ and $\partial_i \dot{h}_{ij}^{\tau\tau} = 0$) and it is still traceless since taking a spatial trace will yield zero. For the case of no gravitational waves, $\dot{h}_{ij}^{\tau\tau} = 0$. Since the electromagnetic equation of motion (the Lorentz force) would give $\dfrac{m_e}{e}a_i - E_i = 0$, then (1399) would reduce to

$$\left(\frac{m_e}{e}a_j + E_j\right)J_i - \frac{2m_e}{3e}\delta_{ij}Ja = 0 \tag{1400}$$

Taking the spatial trace using δ_{ij} gives

$$-\frac{m_e}{e}Ja + EJ = 0 \tag{1401}$$

This expression is satisfied for the trivial case of no supercurrent, $J = 0$, or when

$$m_e a = eE \tag{1402}$$

This is simply the equation of motion based on the Lorentz force which is also used in the London formulation. Hence we recover the expected result from electromagnetism.

13 Interaction of gravitational (GR) waves with normal conductors

13.1 Overview of interaction of GR waves with normal conductors

In the sections that follow, we motivate the form of an Ohm-like gravitational constitutive equation for gravitational waves interacting with a normal conductor. We use the geodesic equation of motion and an analogy with Ohm's law from electromagnetism. Then applying the linearized Einstein equation for the transverse-traceless strain field (which contains the propagating degrees of freedom for gravitational waves), we develop a dispersion relationship, plasma frequency and penetration depth for gravitational waves incident on a normal conductor. Next we find the phase and group velocities, then a modified gravitational plasma frequency that takes into account the effect of the Landau-Lifshitz pseudotensor correction. Finally, we develop an electromagnetic Ohm's law with coupling to gravitational waves.

13.2 A gravito-Ohm constitutive equation for GR waves

In this section, a gravito-Ohm constitutive equation is developed for gravitational waves by using an analogy with electromagnetism. In equation (7.2) of [29], Griffiths states that the current density produced by electromagnetic fields in a conductor can be written as $\vec{J} = \sigma\left(\vec{E} + \vec{v} \times \vec{B}\right)$ where σ is the conductivity of the material. If the velocity is sufficiently small, then the second term can

be neglected[136] which simply gives

$$\vec{J} = \sigma\vec{E} \tag{1403}$$

which is Ohm's law. Since $\vec{E} = -\partial_t\vec{A}$, then $\vec{J} = -\sigma\partial_t\vec{A}$. For a sinusoidal vector potential, $A = A_0 e^{i(kz-\omega t)}$, we have $\partial_t\vec{A} = -i\omega\vec{A}$ and Ohm's law becomes

$$\vec{J} = i\omega\sigma\vec{A} \tag{1404}$$

We can follow a similar procedure using (1821). Recall that electric-like and magnetic-like tensor fields for gravitational waves are written in (354) as

$$\mathscr{E}_{ij} = -\partial_t h_{ij}^{\tau\tau} \qquad \text{and} \qquad \mathscr{B}_{ijk} = \partial_k h_{ij}^{\tau\tau} \tag{1405}$$

Using these fields, the geodesic equation of motion was found in () to be

$$a^i = v^i\mathscr{E}_{ij} + v^i v^k\left(\tfrac{1}{2}\mathscr{B}_{jki} - \mathscr{B}_{ikj}\right) \tag{1406}$$

which demonstrates that the electric-like and magnetic-like tensor fields describe the actual physical motion of test particles in the presence of gravitational wave. Again, for sufficiently small velocities, we can neglect the second term so that $a^i = v^i\mathscr{E}_{ij}$ to first order in the velocity. This implies that the contribution of the term involving the magnetic-like tensor field can be neglected in the constitutive equation. Therefore, a gravito-Ohm constitutive equation involving the stress tensor would be of the form

$$T_{ij}^{\tau\tau} = \eta_G\mathscr{E}_{ij} \tag{1407}$$

where η_G is a "gravitational conductivity" and $T_{ij}^{\tau\tau}$ is the transverse-traceless stress. This expression is directly analogous to the electromagnetic Ohm's law in (1403). Since $\mathscr{E}_{ij} = -\partial_t h_{ij}^{\tau\tau}$, then the constitutive equation can also be written $T_{ij}^{\tau\tau} = -\eta_G h_{ij}^{\tau\tau}$. For a sinusoidal gravitational wave field, we have $h_{ij}^{\tau\tau} = h_{ij,0}^{\tau\tau} e^{i(kz-\omega t)}$ where $h_{ij,0}^{\tau\tau}$ is a constant amplitude tensor. Then $\partial_t h_{ij}^{\tau\tau} = -i\omega h_{ij}^{\tau\tau}$ and the gravito-Ohm constitutive equation becomes

$$\boxed{T_{ij}^{\tau\tau} = i\eta_G\omega h_{ij}^{\tau\tau} \qquad \textit{Gravito-Ohm constitutive equation for a normal conductor}} \tag{1408}$$

Recall that $h_{ij}^{\tau\tau}$ is a dimensionless quantity since it is a metric component (and because it describes a change of length per unit length). Also, since $T_{ij}^{\tau\tau}$ has dimensions of energy density, then η_G has the dimensions of (energy density $\times$ time). Press [73] describes this quantity as "a shear viscosity" (versus $\mu_{G(SC)}$ which he refers to as an "elastic shear modulus.") He points out that "the constitutive relations required for a gravitational conductor are either (i) a very large *elastic shear modulus* μ, in which case the equations resemble the electromagnetic equations of a dense plasma, or (ii) a very large *shear viscosity* η, in which case the equations resemble those of a material of small resistivity." In this dissertation, the interpretation of η_G is consistent with Press in the sense of being a gravitational conductivity (which is the reciprocal of a resistivity).

[136] However, Griffiths points out that in plasmas, the magnetic contribution to the current density can be significant.

13.3 A plasma frequency and penetration depth for GR waves

In this section, we develop a gravitational plasma frequency, index of refraction, penetration depth, and other quantities pertaining to the interaction of gravitational waves with a normal conductor. The analysis is directly analogous to the electromagnetic case shown in Appendix R. Recall that he wave equation for gravitational waves is given in (361) as $\Box h_{ij}^{\tau\tau} = -2\kappa T_{ij}^{\tau\tau}$. Inserting the gravito-Ohm constitutive equation, $T_{ij}^{\tau\tau} = i\eta_G \omega h_{ij}^{\tau\tau}$, and expanding the box operator gives[137]

$$-\tfrac{1}{c^2}\partial_t^2 h_{ij}^{\tau\tau} + \nabla^2 h_{ij}^{\tau\tau} = -2i\kappa\eta_G \omega h_{ij}^{\tau\tau} \tag{1409}$$

A dispersion relation can be obtained by using a monochromatic, plane-fronted wave propagating in the z-direction given by

$$h_{ij}^{\tau\tau}(z,t) = h_{ij,0}^{\tau\tau} e^{i(kz-\omega t)} \tag{1410}$$

where $h_{ij,0}^{\tau\tau}$ is a constant amplitude tensor. Then the wave equation above gives[138]

$$-\frac{(-i\omega)^2}{c^2} h_{ij,0}^{\tau\tau} + (ik)^2 h_{ij,0}^{\tau\tau} = -2i\kappa\eta_G\omega h_{ij}^{\tau\tau} \tag{1411}$$

$$\left(k^2 - \frac{\omega^2}{c^2} - 2i\kappa\eta_G\omega h_{ij}^{\tau\tau}\right) h_{ij,0}^{\tau\tau} = 0 \tag{1412}$$

For a non-trivial solution, $h_{ij,0}^{\tau\tau} \neq 0$, this gives[139]

$$k^2 = \frac{\omega^2}{c^2} + 2i\kappa\eta_G\omega \tag{1413}$$

Factoring out ω^2/c^2 leads to the following dispersion relation for gravitational waves in a normal conductor.[140]

$$\boxed{k^2 = \frac{\omega^2}{c^2}\left(1 + i\frac{2c^2\kappa\eta_G}{\omega}\right) \qquad \textit{Dispersion relation for gravitational waves in a normal conductor}} \tag{1414}$$

This matches the result found by Press [73] in his equation (2) where η_G is identified as an elastic shear modulus.[141] However, here η_G will be referred to as a *gravitational conductivity*.

[137] Note that the *full* transverse-tracess stress tensor within the superconductor is $T_{ij}^{\tau\tau} = -\mu_G h_{ij}^{\tau\tau} + T_{ij\ (curents)}^{\tau\tau}$ where $T_{ij\ (currents)}^{\tau\tau}$ describes the stress produced in the conductor as a result of electromagnetic fields driving currents. The treatment in this section assumes there are no electromagnetic fields driving currents.

[138] The frequency of the incoming gravitational wave (on the right) is assumed to have the *same* frequency as the outgoing gravitational wave (on the left).

[139] This has the same form as the wave number expression in electromagnetism using Ohm's law. The expression is $k^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega$ as shown in equation (9.124) of [29].

[140] Typically the value of κ is determined by $\kappa = 8\pi G/c^4$. However, it is possible that a material (such as a conductor) may have a relative *gravitational permeativity*, κ_r, such that $\kappa = \kappa_r\kappa_0$ where $\kappa_0 = 8\pi G/c^4$ in vacuum.

[141] Note that Press does not have a factor of c^2 appearing in the denominator which implies that he must have used $T_{ij} = i\eta_G\omega c^2 h_{ij}$. Also, as mentioned before, Press makes explicit reference to the transverse-traceless gauge in his formulation. However, this gauge is only valid in vacuum, not matter, as discussed in Appendix E. Therefore, h_{ij}^{TT} cannot be used in this formulation. Instead, we specifically work with the gauge-invariant transverse-traceless *part* of the metric, $h_{ij}^{\tau\tau}$, which satisfies a wave equation *in matter* as found using the Helmholtz Decomposition formulation.

From (1414), it is evident that a gravitational plasma frequency can be defined as

$$\boxed{\omega_G \equiv 2c^2\kappa\eta_G \qquad \textit{Gravitational wave plasma frequency in a normal conductor}} \tag{1415}$$

Then (1414) can also be written as

$$k^2 = \frac{\omega^2}{c^2}\left(1+i\frac{\omega_G}{\omega}\right) \tag{1416}$$

It is clear from (1416) that reflection or absorption will occur when $\omega \leq \omega_G$ which means k becomes imaginary. This can also be written as

$$k^2 = \frac{\omega^2}{c^2}n_G^2(\omega) \tag{1417}$$

where a "gravitational index of refraction" is defined as

$$\boxed{n_G^2(\omega) \equiv \left(1+i\frac{\omega_G}{\omega}\right) \qquad \textit{Gravitational wave index of refraction in a normal conductor}} \tag{1418}$$

This characterizes the reflection and refraction of gravitational waves in a conductor. Next, a gravitational wave skin depth will be determined. A complex wave number can be written as

$$k = K + i\alpha \tag{1419}$$

where K and α are real quantities. Inserting this into the plane wave of (1410) and separating the real and imaginary parts of the phase gives

$$h_{ij}^{\tau\tau}(z,t) = h_{ij,0}^{\tau\tau}e^{i[(K+i\alpha)z-\omega t]} \tag{1420}$$

$$= h_{ij,0}^{\tau\tau}e^{-\alpha z}e^{i(Kz-\omega t)} \tag{1421}$$

Here it is evident that the wave falls off exponentially with distance where α is the exponential decay factor. The square of the wave number in (1419) is

$$k^2 = K^2 - \alpha^2 + 2iK\alpha \tag{1422}$$

The real and imaginary parts of k^2 can be identified by comparing the expression above to (1413). This gives

$$K^2 - \alpha^2 = \frac{\omega^2}{c^2} \quad \text{and} \quad K\alpha = \kappa\eta_G\omega \tag{1423}$$

These expressions can be solved algebraically for K and α. Solving the second equation for α and substituting into the first equation gives

$$K^2 - \left(\frac{\kappa\eta_G\omega}{K}\right)^2 = \frac{\omega^2}{c^2} \tag{1424}$$

Solving for K gives

$$K^4 - \frac{\omega^2}{c^2}K^2 - (\kappa\eta_G\omega)^2 = 0 \tag{1425}$$

This expression is quadratic in K^2 with the coefficients $A = 1$, $B = -\omega^2/c^2$, and $C = (\kappa\eta_G\omega)^2$. Using the quadratic equation to solve for K^2 gives

$$K^2 = \frac{\omega^2}{2c^2} \pm \frac{1}{2}\sqrt{\frac{\omega^4}{c^4} + 4(\kappa\eta_G\omega)^2} \tag{1426}$$

Taking the positive root, rearranging, and solving for K gives

$$K = \frac{\omega}{\sqrt{2}c}\left(\sqrt{1+\left(\frac{2\kappa c^2\eta_G}{\omega}\right)^2} + 1\right)^{1/2} \tag{1427}$$

Now solving the first equation in (1423) for K and inserting into the second equation in (1423) gives

$$\alpha\sqrt{\frac{\omega^2}{c^2} + \alpha^2} = \kappa\eta_G\omega \tag{1428}$$

Rearranging gives

$$\alpha^4 + \frac{\omega^2}{c^2}\alpha^2 - (\kappa\eta_G\omega)^2 = 0 \tag{1429}$$

This expression is quadratic in α^2 with the coefficients $A = 1$, $B = \omega^2/c^2$, and $C = -(\kappa\eta_G\omega)^2$. Using the quadratic equation to solve for α^2 gives

$$\alpha^2 = -\frac{\omega^2}{2c^2} \pm \frac{1}{2}\sqrt{\frac{\omega^4}{c^4} + 4(\kappa\eta_G\omega)^2} \tag{1430}$$

Taking the positive root, rearranging, and solving for α gives[142]

$$\alpha = \frac{\omega}{\sqrt{2}c}\left(\sqrt{1+\left(\frac{2\kappa c^2\eta_G}{\omega}\right)^2} - 1\right)^{1/2} \tag{1431}$$

---

[142]The expressions for K and α in (1427) and (1431), respectively, match Griffiths' results in equation (9.126) of [29] for the case of Ohm's law in electromagnetism. Griffiths uses k and κ, instead of K and α, respectively. His expressions are

$$k = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} + 1\right)^{1/2} \qquad \text{and} \qquad \alpha = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1\right)^{1/2}$$

These expressions become exactly the same as (1427) and (1431) by making the replacements $\varepsilon\mu \to 1/c^2$ and $\dfrac{\sigma}{\varepsilon} \to 2\kappa c^2\eta_G$. In fact, the similarity becomes even more striking by recalling that the gravitational permitivity and permeability are defined in (35) as $\varepsilon_G \equiv \dfrac{1}{4\pi G}$ and $\mu_G \equiv 4\pi G/c^2$, respectively. Also using $\kappa = 8\pi G/c^4$ makes the replacements become $\varepsilon\mu \to \varepsilon\mu$ and $\dfrac{\sigma}{\varepsilon} \to \dfrac{4}{c^2}\dfrac{\eta_G}{\varepsilon_G}$.

The expressions for K and α can also be written in terms of the gravitational plasma frequency for a normal conductor given in (1415) as $\omega_G = 2c^2\kappa\eta_G$. This gives

$$K = \frac{\omega}{\sqrt{2}c}\left(\sqrt{1+\left(\frac{\omega_G}{\omega}\right)^2}+1\right)^{1/2} \quad \text{and} \quad \alpha = \frac{\omega}{\sqrt{2}c}\left(\sqrt{1+\left(\frac{\omega_G}{\omega}\right)^2}-1\right)^{1/2} \tag{1432}$$

A frequency-dependent skin depth can be identified as $\delta_G = 1/\alpha$. Using α in the expression above gives

$$\delta_G^2 = \frac{2c^2}{\omega^2\left(\sqrt{1+\left(\frac{\omega_G}{\omega}\right)^2}-1\right)} \tag{1433}$$

Using $\omega_G = 2c^2\kappa\eta_G$ makes this expression become

$$\boxed{\delta_G^2 = \frac{2c^2}{\omega^2\left(\sqrt{1+\left(\frac{2c^2\kappa\eta_G}{\omega}\right)^2}-1\right)} \qquad \textit{Gravitational wave skin depth for a normal conductor}} \tag{1434}$$

Returning to (1433), notice that for $\omega >> \omega_G$, we have $\omega_G/\omega << 1$ and δ_G^2 becomes extremely large. This means that for frequencies well above the plasma frequency, the wave is no longer attenuated with depth. On the other hand, for $\omega << \omega_G$, we have $\omega_G/\omega >> 1$ and δ_G^2 becomes

$$\delta_G^2 = \frac{2c^2}{\omega_G\omega} \tag{1435}$$

We can use $\omega_G = 2c^2\kappa\eta_G$ and $\kappa = \kappa_r\kappa_0$, where $\kappa_0 = 8\pi G/c^4$ in vacuum. Then the expression becomes

$$\delta_G = \frac{c^2}{\sqrt{8\pi\kappa_r G\eta_G\omega}} \tag{1436}$$

This matches equation (3) of [73] for $\kappa_r = 1$. Notice that in the "DC" limit $(\omega \to 0)$, we have $\delta_G \to \infty$. This implies that there is gravitational wave skin depth. In other words, there is no gravito-Meissner effect for a normal conductor. This will also be demonstrated in the next section using the magnetic-like tensor field of a gravitational wave.

13.4 The absence of a gravito-Meissner effect for GR waves in the DC limit

In this section, it is shown that a gravito-Meissner effect for gravitational waves in the DC limit does not occurs for a normal conductor. In (356), it was shown that the wave equation, $\Box h_{ij}^{\tau\tau} = -2\kappa T_{ij}^{\tau\tau}$, can be written in as

$$\partial_k \mathscr{B}_{ijk} \quad = \quad -\left(2\kappa T_{ij}^{\tau\tau} + \frac{1}{c^2}\partial_t \mathscr{E}_{ij}\right) \tag{1437}$$

where $\mathscr{E}_{ij}$ and $\mathscr{B}_{ijk}$ are the electric-like and magnetic-like tensor fields, respectively. They are defined as

$$\mathscr{E}_{ij} = -\partial_t h_{ij}^{\tau\tau} \qquad \text{and} \qquad \mathscr{B}_{ijk} = \partial_k h_{ij}^{\tau\tau} \tag{1438}$$

For sinusoidal fields In the DC limit $(\omega \to 0)$, we have $\mathscr{E}_{ij} = -\partial_t h_{ij}^{\tau\tau} = -\omega h_{ij}^{\tau\tau} = 0$. Therefore, (1437) reduces to just

$$\partial_k \mathscr{B}_{ijk} = -2\kappa T_{ij}^{\tau\tau} \tag{1439}$$

The gravito-Ohm constitutive equation, $T_{ij}^{\tau\tau} = i\eta_G \omega h_{ij}^{\tau\tau}$, also vanishes in the DC limit which makes the expression above become $\partial_k \mathscr{B}_{ijk} = 0$. Therefore, we do not arrive at a Yukawa-like equation as we did in (1348) for a superconductor. However, we can use (1406) to write a modified gravito-Ohm constitutive equation that includes the magnetic-like tensor field, $\mathscr{B}_{ijk}$. Recall that the geodesic equation in (1406) was given as

$$a^i \quad = \quad v^i \mathscr{E}_{ij} + v^i v^k \left(\tfrac{1}{2}\mathscr{B}_{jki} - \mathscr{B}_{ikj}\right) \tag{1440}$$

From the second term above, we might expect the gravitational Ohm-like law to have the form

$$T_{ij}^{\tau\tau} = \eta_G \left(\mathscr{E}_{ij} + v^k \mathscr{B}_{ijk}\right) \tag{1441}$$

In the DC limit, we still have $\mathscr{E}_{ij} = -\partial_t h_{ij}^{\tau\tau} = -\omega h_{ij}^{\tau\tau} = 0$. Therefore, the expression above reduces to $T_{ij}^{\tau\tau} = \eta_G v^k \mathscr{B}_{ijk}$. Inserting this into (1439) and rearranging gives

$$\partial_k \mathscr{B}_{ijk} + 2\kappa \eta_G v^k \mathscr{B}_{ijk} = 0 \tag{1442}$$

Notice that we still do not obtain a Yukawa-like equation for the magnetic-like tensor Therefore, there is no gravito-Meissner effect expelling the magnetic-like tensor field in the DC limit for a normal conductor.

14 Interaction of gravitational (GR) waves with the Cooper Pairs of a superconductor

14.1 The Ginzburg-Landau free energy density in curved space-time

In this section, we will formulate the response of the Cooper pair density to a gravitational wave using the Ginzburg-Landau (G-L) free energy density in curved space-time. First, we begin with the non-relativistic G-L free energy density in flat space-time which is given in equation (4.2.11) of Zhou's text [33] as[143]

$$\mathscr{F}-\mathscr{F}_n = \frac{1}{2m}\left|\left(-i\hbar\nabla-q\vec{A}\right)\psi\right|^2+\alpha\left|\psi\right|^2+\frac{\beta}{2}\left|\psi\right|^4+\frac{\vec{B}^2}{2\mu_0} \tag{1443}$$

where $m=2m_e$ and $q=2e$. To embed this expression in curved space-time[144], we may consider the G-L Lagrangian density given in equation (4.1) of [38] written as[145]

$$\mathscr{L} = -\frac{1}{2}\sqrt{-g}\left[g^{\mu\nu}\left(D_\mu\phi\right)^*\left(D_\nu\phi\right)-\mu^2\left|\phi\right|^2+\frac{\lambda}{2}\left|\phi\right|^4\right]-\frac{1}{4}\sqrt{-g}g^{\mu\kappa}g^{\nu\lambda}F_{\mu\nu}F_{\kappa\lambda} \tag{1444}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is the electromagnetic field strength tensor and ϕ is a complex scalar field, $\phi=\phi_1+i\phi_2$. Also, D_μ is the gauge covariant derivative given by $D_\mu\equiv\nabla_\mu-\frac{iq}{\hbar}A_\mu$. Using a similar approach, we can generalize the G-L free energy density of (1443) to curved space-time. Since ψ is a scalar, then the covariant derivative of ψ is just a partial derivative: $\nabla_\mu\psi=\partial_\mu\psi$. Therefore the *gauge* covariant derivative becomes

$$D_\mu\equiv\partial_\mu-\frac{iq}{\hbar}A_\mu \tag{1445}$$

Also, in order to use this gauge covariant derivative and still recover (1443) in the non-relativistic limit, we need to introduce the appropriate prefactor. The kinetic term in (1443) can be written with index notation as

$$\left|\left(-i\hbar\nabla-q\vec{A}\right)\psi\right|^2=\left[\left(-i\hbar\partial_i-qA_i\right)\psi\right]^*\left(-i\hbar\partial_i-qA_i\right)\psi \tag{1446}$$

Factoring out $(i\hbar)^*(i\hbar)=\hbar^2$ and canceling negatives gives

$$\left|\left(-i\hbar\nabla-q\vec{A}\right)\psi\right|^2=\hbar^2\left[\left(\partial_i-\frac{iq}{\hbar}A_i\right)\psi\right]^*\left(\partial_i-\frac{iq}{\hbar}A_i\right)\psi \tag{1447}$$

Using (1445), we can write this as

$$\left|\left(-i\hbar\nabla-q\vec{A}\right)\psi\right|^2=\hbar^2\left(D_i\psi\right)^*\left(D_i\psi\right) \tag{1448}$$

Therefore, we find that a prefactor of $\hbar^2$ is needed to write (1443) in terms of D_μ. Using the notation $\mathscr{F}-\mathscr{F}_n=\mathscr{F}_{G\text{-}L}$, we can write $\mathscr{F}_{G\text{-}L}$ as[146]

$$\boxed{\mathscr{F}_{G\text{-}L}=\sqrt{-g}\left[\frac{\hbar^2}{2m}g^{\mu\nu}\left(D_\mu\psi\right)^*\left(D_\nu\psi\right)+\alpha\left|\psi\right|^2+\frac{\beta}{2}\left|\psi\right|^4-\frac{1}{4\mu_0}g_{\mu\rho}g_{\nu\sigma}F^{\mu\nu}F^{\rho\sigma}\right]} \tag{1449}$$

Ginzburg-Landau free energy density in curved space-time

[143] A similar form is given in eq. 4.1 of Tinkham's text [34] and eq. 6.15 of Timm's notes [35].

[144] For a *relativistic* covariant form of the Ginzburg-Landau equation in *flat* space-time, refer to Section 2.1 of [37].

[145] Note that $\hbar=c=1$ in [38], however, these constants will be left explicit in this formulation.

[146] The Lagrangian density in (1444) contains the determinant of the metic, $\sqrt{-g}$, to account for the fact that the *proper* volume is $d\mathscr{V}=\sqrt{-g}dV$ where V is the *coordinate* volume. Since the free energy *density* is ordinarily expressed as dF/dV, then expressing it in terms of *proper* volume gives $\mathscr{F}=\sqrt{-g}\,dF/d\mathscr{V}$.

This formulation is similar to that found in [36]. Using this formulation, we will now consider the special case of gravitational *waves* interacting with a superconductor.

It should actually be

$$\mathscr{L}=\sqrt{-g}\left[\left|\pi^{i}\psi\right|^{2}+\frac{1}{2m}h_{ij}^{\tau\tau}\left(\pi^{i}\psi\right)^{*}\left(\pi^{j}\psi\right)-\alpha\left|\psi\right|^{2}+\frac{\beta}{2}\left|\psi\right|^{4}\right] \tag{1450}$$

Try finding the Euler-Lagrange equations of motion from this.

The G-L free energy density in the presence of a gravitational wave

We can write the metric as a perturbation to flat space-time using $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$. Then summing over μ and ν in the kinetic term and summing over μ and ρ in the last term gives

$$\begin{aligned}\mathscr{F}_{\text{G-L}} &= \sqrt{-g}\left\{\frac{\hbar^{2}}{2m}\left[g^{00}\left(D_{0}\psi\right)^{*}\left(D_{0}\psi\right)+g^{0i}\left(D_{0}\psi\right)^{*}\left(D_{i}\psi\right)\right]\right.\\ &\quad+\frac{\hbar^{2}}{2m}\left[g^{i0}\left(D_{i}\psi\right)^{*}\left(D_{0}\psi\right)+g^{ij}\left(D_{i}\psi\right)^{*}\left(D_{j}\psi\right)\right]+\alpha\left|\psi\right|^{2}+\frac{\beta}{2}\left|\psi\right|^{4}\\ &\quad\left.-\frac{1}{4\mu_{0}}\left(g_{00}g_{\nu\sigma}F^{0\nu}F^{0\sigma}+g_{0i}g_{\nu\sigma}F^{0\nu}F^{i\sigma}+g_{i0}g_{\nu\sigma}F^{i\nu}F^{0\sigma}+g_{ij}g_{\nu\sigma}F^{i\nu}F^{j\sigma}\right)\right\}\end{aligned} \tag{1451}$$

We can also make use of the fact that the Helmholtz Decomposition formulation of linearized GR allows us to isolate the radiative degrees of freedom as $h_{ij}^{\tau\tau}$ which is a transverse-traceless, gauge-invariant quantity. Hence, for gravitational waves in the far-field, we can neglect all other components of the metric so that $h_{00}=h_{0i}=0$ and $h_{ij}=h_{ij}^{\tau\tau}$. In that case, we have $g_{00}=-1$ and we also have $g_{0i}=0$ which eliminates several terms. We also found in (2408) of Appendix A that the inverse metric (to first order) is $g^{\mu\nu}=\eta^{\mu\nu}-h^{\mu\nu}$. Lastly, summing over ν and σ in the bottom line gives

$$\begin{aligned}\mathscr{F}_{\text{G-L}} &= \sqrt{-g}\left\{\frac{\hbar^{2}}{2m}\left[-\left(D_{0}\psi\right)^{*}\left(D_{0}\psi\right)+\left(\eta^{ij}-h_{\tau\tau}^{ij}\right)\left(D_{i}\psi\right)^{*}\left(D_{j}\psi\right)\right]+\alpha\left|\psi\right|^{2}+\frac{\beta}{2}\left|\psi\right|^{4}\right.\\ &\quad\left.-\frac{1}{4\mu_{0}}\left[\left(g_{00}g_{00}F^{00}F^{00}+g_{00}g_{ij}F^{0i}F^{0j}\right)+\left(g_{ij}g_{00}F^{i0}F^{j0}+g_{ij}g_{kl}F^{ik}F^{jl}\right)\right]\right\}\end{aligned} \tag{1452}$$

Since $F^{\mu\nu}$ is anti-symmetric, then using $F^{0i}=-F^{i0}$ allows us to combine terms in the second line. We can also use $F^{00}=0$ as well as $g_{00}=-1$ and $g_{ij}=\eta_{ij}+h_{ij}^{\tau\tau}$ to obtain[147]

$$\begin{aligned}\mathscr{F}_{\text{G-L}} &= \sqrt{-g}\left\{\frac{\hbar^{2}}{2m}\left[-\left(D_{0}\psi\right)^{*}\left(D_{0}\psi\right)+\left(D_{i}\psi\right)^{*}\left(D_{i}\psi\right)-h_{ij}^{\tau\tau}\left(D_{i}\psi\right)^{*}\left(D_{j}\psi\right)\right]+\alpha\left|\psi\right|^{2}+\frac{\beta}{2}\left|\psi\right|^{4}\right.\\ &\quad\left.-\frac{1}{4\mu_{0}}\left[-2\left(\eta_{ij}+h_{ij}^{\tau\tau}\right)F^{0i}F^{0j}+\left(\eta_{ij}+h_{ij}^{\tau\tau}\right)\left(\eta_{kl}+h_{kl}^{\tau\tau}\right)F^{ik}F^{jl}\right]\right\}\end{aligned} \tag{1453}$$

[147] The spatial indices of the metric perturbation can be freely raised and lowered to first order in the metric.

Keeping only first order in $h_{\mu\nu}$ eliminates $h^{\tau\tau}_{ij}h^{\tau\tau}_{kl}$ in the last term. Also, using $F^{ik}F^{jl}=F^{ki}F^{lj}$ allows two terms to be combined. Then we have

$$
\begin{aligned}
\mathscr{F}_{G\text{-}L} \quad = \quad & \sqrt{-g}\left\{\frac{\hbar^2}{2m}\left[-\left(D_0\psi\right)^*\left(D_0\psi\right)+\left(D_i\psi\right)^*\left(D_i\psi\right)-h^{\tau\tau}_{ij}\left(D_i\psi\right)^*\left(D_j\psi\right)\right]+\alpha\left|\psi\right|^2+\frac{\beta}{2}\left|\psi\right|^4\right.\\
&-\frac{1}{4\mu}\left[-2\left(\eta_{ij}F^{0i}F^{0j}+h^{\tau\tau}_{ij}F^{0i}F^{0j}\right)\right]\\
&\left.-\frac{1}{4\mu_0}\left[\left(\eta_{ij}\eta_{kl}F^{ik}F^{jl}+\eta_{ij}h^{\tau\tau}_{kl}F^{ik}F^{jl}+h^{\tau\tau}_{ij}\eta_{kl}F^{ik}F^{jl}\right)\right]\right\}
\end{aligned}
\tag{1454}
$$

For a gravitational wave propagating in the z-direction, $h^{\tau\tau}_{i3}=0$ (since $h^{\tau\tau}_{ij}$ is transverse). Also $h^{\tau\tau}_{11}=-h^{\tau\tau}_{22}=h_{\oplus}$ where $h_{\oplus}=A_{\oplus}\cos\left(kz-\omega t\right)$ for plus polarization while $h^{\tau\tau}_{12}=h^{\tau\tau}_{21}=h_{\otimes}$ where $h_{\otimes}=A_{\otimes}\cos\left(kz-\omega t\right)$ for cross polarization.[148] Then summing over repeated indices gives[149]

$$
\begin{aligned}
\mathscr{F}_{G\text{-}L} \quad = \quad & \sqrt{-g^{\tau\tau}}\left\{\frac{\hbar^2}{2m}\left[-\left(D_0\psi\right)^*\left(D_0\psi\right)+\left(D_i\psi\right)^*\left(D_i\psi\right)\right]+\alpha\left|\psi\right|^2+\frac{\beta}{2}\left|\psi\right|^4\right.\\
&-\frac{\hbar^2}{2m}\left[\left(D_1\psi\right)^*\left(D_1\psi\right)-\left(D_2\psi\right)^*\left(D_2\psi\right)\right]h_{\oplus}\\
&-\frac{\hbar^2}{2m}\left[\left(D_1\psi\right)^*\left(D_2\psi\right)+\left(D_2\psi\right)^*\left(D_1\psi\right)\right]h_{\otimes}\\
&+\frac{1}{2\mu_0}\left[\left(F^{0i}\right)^2+h_{\oplus}\left(F^{01}\right)^2+2h_{\otimes}F^{01}F^{02}-h_{\oplus}\left(F^{02}\right)^2\right]\\
&+\frac{1}{2\mu_0}\left[-\frac{1}{2}F^{ij}F_{ij}-\frac{1}{2}h_{\oplus}\left(\left(F^{21}\right)^2+\left(F^{31}\right)^2\right)\right]\\
&\left.+\frac{1}{2\mu_0}\left[-h_{\otimes}\left(F^{31}F^{32}\right)+\frac{1}{2}h_{\oplus}\left(\left(F^{12}\right)^2+\left(F^{32}\right)^2\right)\right]\right\}
\end{aligned}
\tag{1455}
$$

Now we also use the components of the electromagnetic field strength tensor given by $F^{0i}=\frac{1}{c}E^i$, $F^{ij}=\varepsilon^{ijk}B^k$,

[148]By eliminating h_{00} and h_{0i}, we are effectively making the approximation that $h_{\mu\nu}$ of the superconductor is very small comapred to $h^{\tau\tau}_{ij}$ of the gravitational wave. According to (2471), this approximation is valid as long as $T^{\mu\nu}$ of the superconductor is small so that $h_{\mu\nu}$ is also small.

[149]Note that writing $\eta_{ij}\eta_{kl}F^{ik}F^{jl}=F^{il}F^{il}$ will lead to $F^{il}F^{il}=\vec{B}^2$. This will lead to $\frac{-1}{2\mu}\vec{B}^2$ which is the wrong sign for the magnetic field energy density. Therefore, we treat η_{ij} as a metric which lowers indices so that $\eta_{ij}\eta_{kl}F^{ik}F^{jl}=F^{il}F_{il}$ which leads to the corret sign for the magnetic field energy density.

and $F^{ii} = 0$ to obtain

$$\begin{aligned}
\mathscr{F}_{G\text{-}L} \quad = \quad & \sqrt{-g^{\tau\tau}} \left\{ \frac{\hbar^2}{2m} \left[-\left(D_0\psi\right)^* \left(D_0\psi\right) + \left(D_i\psi\right)^* \left(D_i\psi\right) \right] + \alpha \left|\psi\right|^2 + \frac{\beta}{2} \left|\psi\right|^4 \right. \\
& - \frac{\hbar^2}{2m} \left[\left(D_1\psi\right)^* \left(D_1\psi\right) - \left(D_2\psi\right)^* \left(D_2\psi\right) \right] h_{\oplus} \\
& - \frac{\hbar^2}{2m} \left[\left(D_1\psi\right)^* \left(D_2\psi\right) + \left(D_2\psi\right)^* \left(D_1\psi\right) \right] h_{\otimes} \\
& + \frac{1}{2\mu_0 c^2} \left(\vec{E}^2 + h_{\oplus} E_x^2 + 2\tilde{h}_{\otimes} E_x E_y - h_{\oplus} E_y^2 \right) \\
& \left. + \frac{1}{2\mu_0} \left[\vec{B}^{\,2} - \frac{1}{2} h_{\oplus} \left(B_z^2 + B_y^2 \right) + h_{\otimes} \left(B_y B_x \right) + \frac{1}{2} h_{\oplus} \left(B_z^2 + B_x^2 \right) \right] \right\}
\end{aligned} \tag{1456}$$

Lastly, we can use (2661) for the determinant of the metric in the case of a gravitational wave propagating in the z-direction. To first order in the metric we have $g^{\tau\tau} = -1 - (h_{\oplus} + h_{\otimes})$. Then using a binomial expansion to first order gives $\sqrt{-g^{\tau\tau}} = 1 + \dfrac{1}{2}h_{\oplus} + \dfrac{1}{2}h_{\otimes}$. We can insert this into (1449) and eliminate terms that are higher than linear order. Also using $1/c^2 = \mu_0 \varepsilon_0$ and rearranging gives

$$\boxed{\begin{aligned}
&\mathscr{F}_{G\text{-}L} = \left\{ \frac{\hbar^2}{2m} \left[-\left(D_0\psi\right)^* \left(D_0\psi\right) + \left(D_i\psi\right)^* \left(D_i\psi\right) \right] + \alpha \left|\psi\right|^2 + \frac{\beta}{2} \left|\psi\right|^4 \right\} \left(1 + \frac{1}{2} h_{\oplus} + \frac{1}{2} h_{\otimes} \right) \\
&\quad - \frac{\hbar^2}{2m} \left\{ \left[\left(D_1\psi\right)^* \left(D_1\psi\right) - \left(D_2\psi\right)^* \left(D_2\psi\right) \right] h_{\oplus} + \left[\left(D_1\psi\right)^* \left(D_2\psi\right) + \left(D_2\psi\right)^* \left(D_1\psi\right) \right] h_{\otimes} \right\} \\
&\quad + \frac{\varepsilon_0}{2} \vec{E}^2 + \frac{1}{2\mu_0} \vec{B}^{\,2} + \frac{1}{2} \left[\varepsilon_0 \left(E_x^2 - E_y^2 \right) - \frac{1}{2\mu_0} \left(B_x^2 - B_y^2 \right) \right] h_{\oplus} + \left(\varepsilon_0 E_x E_y + \frac{1}{2\mu_0} B_x B_y \right) h_{\otimes} \\
&\qquad\qquad \textit{G-L free energy density in the presence of a gravitational wave} \\
&\qquad\qquad \textit{propagating in the z-direction (to linear order in the metric)}
\end{aligned}} \tag{1457}$$

The first line contains the G-L free energy density in flat space-time. (Note that it is still a relativistically *covariant* form of the G-L equation with second order space and time derivatives. It is not the original G-L free energy density from (1443) which contains only spatial derivatives.) The second line describes the coupling of the gravitational wave to the quantum wave function (the G-L order parameter). The third line contains the electromagnetic energy density in flat space-time plus the energy density due to the gravitational wave coupling to the electromagnetic fields.

Utilizing the covariant derivative, canonical momentum and zero-momentum eigenstate

We now make use of the gauge covariant derivative given in (1445) as $D_\mu = \partial_\mu - \frac{iq}{\hbar}A_\mu$. Since the Cooper pairs are in a zero-momentum eigenstate $(p_0 = 0)$, then $\psi = Ce^{\left(\frac{i}{\hbar}\vec{p}_0 \cdot \vec{r}\right)} = C$. In that case, all the derivatives vanish and we can factor out $\psi^*\psi = C^*C = |C|^2$. Also recall that $|\psi|^2 = n_s$ which is the number density of

Cooper pairs. Therefore $|C|^2 = n_s$ and (1457) becomes

$$\mathscr{F}_{G\text{-}L} = \left\{ \frac{n_s q^2}{2m}\left[-(A_0)^2 + (A_i)^2\right] + n_s\alpha + \frac{n_s^2\beta}{2} + \frac{\varepsilon_0}{2}\vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^{\,2} \right\}\left(1 + \frac{1}{2}h_\oplus + \frac{1}{2}h_\otimes\right)$$

$$-\frac{n_s q^2}{2m}\left[(A_1)^2 h_\oplus - (A_2)^2 h_\oplus + 2A_1A_2h_\otimes\right]$$

$$+\frac{1}{2}\left[\varepsilon_0\left(E_x^2 - E_y^2\right) - \frac{1}{2\mu_0}\left(B_x^2 - B_y^2\right)\right]h_\oplus + \left(\varepsilon_0 E_x E_y + \frac{1}{2\mu_0}B_x B_y\right)h_\otimes \tag{1458}$$

In (657), we found the kinetic momentum for a charged, massive spinless relativistic particle (such as a relativistic Cooper pair) to be

$$\pi_i = \gamma m\left(c g_{0i} + g_{li}v_l\right) \tag{1459}$$

where $\pi_i = p_i - qA_i$ and $\gamma \equiv \left(-g_{00} - \frac{2}{c}g_{0j}v^j - \frac{1}{c^2}g_{jk}v^jv^k\right)^{-1/2}$. In our case here, we set $g_{0i} = 0$, $g_{li} = \eta_{li} + h_{li}^{\tau\tau}$, and $p_i = 0$. Therefore, we have

$$\gamma = \left(1 - \frac{v^2}{c^2} - \frac{1}{c^2}h_{jk}^{\tau\tau}v^jv^k\right)^{-1/2} \tag{1460}$$

and (1459) becomes

$$-qA_i = \left(1 - \frac{v^2}{c^2} - \frac{1}{c^2}h_{jk}^{\tau\tau}v^jv^k\right)^{-1/2} m\left(\eta_{li} + h_{li}^{\tau\tau}\right)v_l \tag{1461}$$

Keeping to order v^2/c^2 in velocity and using k for the repeated index gives

$$A_i = -\frac{m}{q}\left(v_i + h_{ki}^{\tau\tau}v_k\right) \tag{1462}$$

Since we will need $(A_i)^2$ in (1458), then to first order in the metric, (1462) gives

$$(A_i)^2 = \frac{m^2}{q^2}\left(v^2 + 2h_{ki}^{\tau\tau}v_iv_k\right) \tag{1463}$$

Summing over indices and using $h_{11}^{\tau\tau} = -h_{22}^{\tau\tau} = h_\oplus$ and $h_{12}^{\tau\tau} = h_{21}^{\tau\tau} = h_\otimes$ gives

$$(A_i)^2 = \frac{m^2}{q^2}\left[v^2 + 2h_\oplus(v_1)^2 - 2h_\oplus(v_2)^2 + 4h_\otimes v_1v_2\right] \tag{1464}$$

Since we will also need A_1 and A_2 in (1458), then using $h_{11}^{\tau\tau} = -h_{22}^{\tau\tau} = h_\oplus$ and $h_{12}^{\tau\tau} = h_{21}^{\tau\tau} = h_\otimes$ in (1462) gives

$$A_1 = -\frac{m}{q}\left(v_1 + h_\oplus v_1 + h_\otimes v_2\right) \qquad A_2 = -\frac{m}{q}\left(v_2 + h_\otimes v_1 - h_\oplus v_2\right) \tag{1465}$$

We also need A_1A_2 to first order in the metric which is

$$A_1A_2 = \frac{m^2}{q^2}\left[v_1v_2 + h_\otimes (v_1)^2 + h_\otimes (v_2)^2\right] \tag{1466}$$

Lastly, squaring A_1 and A_2 and remaining to first order in the metric gives

$$(A_1)^2 = \frac{m^2}{q^2}\left[(v_1)^2 + 2h_\oplus (v_1)^2 + 2h_\otimes v_1v_2\right] \tag{1467}$$

and

$$(A_2)^2 = \frac{m^2}{q^2}\left[(v_2)^2 - 2h_\oplus (v_2)^2 + 2h_\otimes v_1v_2\right] \tag{1468}$$

Now we can substitute (1464) – (1468) into (1458) and eliminate any terms higher than linear order.

$$\begin{aligned}
\mathscr{F}_{G\text{-}L} &= n_s\left[-\frac{q^2}{2m}(A_0)^2 + \frac{1}{2}mv^2\right]\left(1 + \frac{1}{2}h_\oplus + \frac{1}{2}h_\otimes\right) \\
&+ \frac{n_sm}{2}\left[2h_\oplus (v_1)^2 - 2h_\oplus (v_2)^2 + 4h_\otimes v_1v_2\right] \\
&+ \left(n_s\alpha + \frac{n_s^2\beta}{2} + \frac{\varepsilon_0}{2}\vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^{\,2}\right)\left(1 + \frac{1}{2}h_\oplus + \frac{1}{2}h_\otimes\right) \\
&- \frac{n_sm}{2}\left[(v_1)^2 h_\oplus - (v_2)^2 h_\oplus + 2v_1v_2h_\otimes\right] \\
&+ \frac{1}{2}\left[\varepsilon_0\left(E_x^2 - E_y^2\right) - \frac{1}{2\mu_0}\left(B_x^2 - B_y^2\right)\right]h_\oplus + \left(\varepsilon_0E_xE_y + \frac{1}{2\mu_0}B_xB_y\right)h_\otimes
\end{aligned} \tag{1469}$$

Rearranging and combining terms gives

$$\begin{aligned}
\mathscr{F}_{G\text{-}L} &= n_s\left[\frac{1}{2}mv^2 - \frac{q^2}{2m}(A_0)^2\right]\left(1 + \frac{1}{2}h_\oplus + \frac{1}{2}h_\otimes\right) \\
&+ \frac{n_sm}{2}\left(v_x^2h_\oplus - v_y^2h_\oplus + 2v_xv_yh_\otimes\right) \\
&+ \left(n_s\alpha + \frac{n_s^2\beta}{2} + \frac{\varepsilon_0}{2}\vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^{\,2}\right)\left(1 + \frac{1}{2}h_\oplus + \frac{1}{2}h_\otimes\right) \\
&+ \frac{1}{2}\left[\varepsilon_0\left(E_x^2 - E_y^2\right) - \frac{1}{2\mu_0}\left(B_x^2 - B_y^2\right)\right]h_\oplus + \left(\varepsilon_0E_xE_y + \frac{1}{2\mu_0}B_xB_y\right)h_\otimes
\end{aligned} \tag{1470}$$

Pressures and stresses in a superconductor in the presence of a gravitational wave strain

Recall that the thermodynamic work done on a system is given by $W = \int PdV$ which means $P = dW/dV$. By the work-energy theorem, the work done by the system must reduce the internal energy so that $U = -W$. Since the internal energy is described by the relation $dU = TdS - PdV$ and the Helmholtz free energy is $F = U - TS$, then it follows that the thermodynamic pressure can be expressed in terms of the the Helmholtz free energy as

$$P_{\text{thermodynamic}} = -\left(\frac{\partial F_n}{\partial V}\right)_T \tag{1471}$$

where F_n is the free energy of the normal (non-superconducting state) and the subscript "T" denotes that the derivative is for a fixed temperature. Also, since the total free energy is $F = F_n + F_{G\text{-}L}$, where $F_{G\text{-}L}$ is the Ginzburg-Landau free energy of the superconducting Cooper pair electrons, then a *quantum* pressure due to the quantum rigidity of the wave function can be found using

$$P_{\text{quantum}} = -\left(\frac{\partial F_{G\text{-}L}}{\partial V}\right)_T \tag{1472}$$

Note that the free energy *density* is the free energy per unit volume, $\mathscr{F} = dF/dV$, so we have

$$F_{G\text{-}L} = \int \mathscr{F}_{G\text{-}L} dV \tag{1473}$$

Then applying (1472) gives[150]

$$P_{\text{quantum}} = -\left(\frac{\partial}{\partial V}\int_0^V \mathscr{F}_{G\text{-}L} dV'\right)_T \tag{1474}$$

Now to consider the work done on a system due to a gravitational strain, h_{ij}, causing a stress, T_{ij}, we can use $\mathscr{W} = \int T^{ij} dh_{ij}$ where $\mathscr{W}$ is the work density (or work per unit volume). This means that $T^{ij} = d\mathscr{W}/dh_{ij}$. Once again, by the work-energy theorem, we can recognize that work done by the system must reduce the internal energy density so that $\mathscr{U} = -\mathscr{W}$. Also, we can use the fact that the internal energy satisfies the relation $dU = TdS - PdV$ and the Helmholtz free energy is $F = U - TS$. However, now we recognize that when stresses and strains are present, then the Helmholtz free energy also satisfies the relation

$$d\mathscr{F} = -\mathbb{S}dT + S^{ij}du_{ij} - T^{ij}dh_{ij} \tag{1475}$$

where $\mathbb{S}$ is the entropy density (or entropy per unit volume) Then it follows that the stress produced by a gravitational wave interacting with the Cooper pair density can be expressed in terms of the Ginzburg-Landau free energy density as[151]

$$T^{ij} = \left(\frac{\partial \mathscr{F}_{G\text{-}L}}{\partial h_{ij}}\right)_T \tag{1476}$$

[150] Note that in general, $\mathscr{F}_{G\text{-}L}$ is a function of volume since the normalization of the Ginzburg-Landau order parameter leads to $|C|^2 = 1/V$.

[151] Here we are using a formulation similar to [79], eq. 3.3, which gives $dF = -SdT + \sigma_{ij}du_{ij}$, where σ_{ij} is the material stress and u_{ij} is the material strain. It is also shown in eq. 3.6 that $\sigma_{ij} = (\partial F/\partial u_{ij})_T$ which leads analogously to our relation in (1476). Notice there is a *positive* sign in this relation versus the negative sign in $P = -(\partial F/\partial V)_T$. Also note that we are implicitly invoking *local* thermodynamic equilibrium so that the variation of the metric over the volume of the superconductor does not prevent the use of standard thermodynamic relations such (1471) and (1476).

For plus polarization, $h_{11}^{\tau\tau} = -h_{22}^{\tau\tau} = h_{\oplus}$ which corresponds to $T_{11}^{\tau\tau} = -T_{22}^{\tau\tau} = T_{\oplus}$. Since $h_{\oplus}$ and $h_{\otimes}$ are independent degrees of freedom[152], then we use[153]

$$T_{\oplus} = \left(\frac{\partial \mathscr{F}_{G\text{-}L}}{\partial h_{\oplus}}\right)_T \tag{1477}$$

Applying (1477) to (1470), and using $m = 2m_e$ and $q = 2e$ gives

$$\boxed{\begin{aligned} T_{\oplus} = n_s &\left[\frac{1}{2}m_e v^2 + m_e\left(v_x^2 - v_y^2\right) - \frac{e^2}{2m_e}\left(A_0\right)^2 + \frac{\alpha}{2} + \frac{\beta}{4}n_s\right] \\ &+ \frac{1}{4}\left(\varepsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^{\,2}\right) + \frac{1}{2}\left[\varepsilon_0\left(E_x^2 - E_y^2\right) - \frac{1}{2\mu_0}\left(B_x^2 - B_y^2\right)\right] \end{aligned}} \tag{1478}$$

The Cooper pair density stress due to a plus polarization gravitational wave in the z-direction

For cross polarization, we have $h_{12}^{\tau\tau} = h_{21}^{\tau\tau} = h_{\otimes}$ which corresponds to $T_{12}^{\tau\tau} = T_{21}^{\tau\tau} = T_{\otimes}$. In that case, we can use

$$T_{\otimes} = \left(\frac{\partial \mathscr{F}_{G\text{-}L}}{\partial h_{\otimes}}\right)_T \tag{1479}$$

Therefore, applying (1479) to (1470) and using $m = 2m_e$ and $q = 2e$ gives

$$\boxed{\begin{aligned} T_{\otimes} = n_s &\left[\frac{1}{2}m_e v^2 + 2m_e v_x v_y - \frac{e^2}{2m_e}\left(A_0\right)^2 + \frac{\alpha}{2} + \frac{\beta}{4}n_s\right] \\ &+ \frac{1}{4}\left(\varepsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^{\,2}\right) + \varepsilon_0 E_x E_y + \frac{1}{2\mu_0}B_x B_y \end{aligned}} \tag{1480}$$

The Cooper pair density stress due to a cross polarization gravitational wave in the z-direction

Using (1478) and (1480), we can construct a single stress tensor which incorporates both plus and cross

[152]Although $h_{\oplus}$ and $h_{\otimes}$ are independent degrees of freedom, they are not completely unrelated. Since $\partial_i h_{ij}^{\tau\tau} = 0$, then we must have $\partial_x h_{\oplus} + \partial_y h_{\otimes} = 0$ and $\partial_x h_{\otimes} - \partial_y h_{\oplus} = 0$. However, as shown at the end of Appendix E, these differential equations can be decoupled so that $h_{\oplus}$ and $h_{\otimes}$ are found to satisfy their own independent, second-order differential equations. Also, more relevant to the present context, $h_{\oplus}$ and $h_{\otimes}$ are independent in the sense that they are not functions of one another and therefore $\partial h_{\oplus}/\partial h_{\otimes} = \partial h_{\otimes}/\partial h_{\oplus} = 0$.

[153]Note that the derivative is with respect to the the full time-dependent function given by $h_{ij}^{\tau\tau} = A_{ij}^{\tau\tau}\cos\left(kz - \omega t\right)$, not just the *magnitude*, $A_{ij}^{\tau\tau}$. Therefore, we are considering a quasi-static approximation in (1476).

polarization waves. The tensor can be written as a 2×2 with $\{i,j\}$ running from 1 to 2.

$$T_{ij}^{\tau\tau} = \begin{pmatrix} T_{\oplus} & T_{\otimes} \\ T_{\otimes} & -T_{\oplus} \end{pmatrix} \tag{1481}$$

Inserting (1478) and (1480) and neglecting the contribution to the stress by the electromagnetic fields gives

$$\boxed{\begin{gathered} T_{ij}^{\tau\tau} = n_s\left[\frac{1}{2}m_e v^2 - \frac{e^2}{2m_e}(A_0)^2 + \frac{\alpha}{2} + \frac{n_s\beta}{4}\right]\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ +n_s\begin{pmatrix} m_e\left(v_x^2 - v_y^2\right) & 2m_e v_x v_y \\ 2m_e v_x v_y & -m_e\left(v_x^2 - v_y^2\right) \end{pmatrix} \\ \textit{Stress tensor of the Cooper pair density due to a gravitational wave} \end{gathered}} \tag{1482}$$

This expression describes the stress on a superconductor as a result of a gravitational strain field in the superconductor. The gravitational wave is propagating in the z-direction and is given by

$$h_{ij}^{\tau\tau} = \begin{pmatrix} A_{\oplus} & A_{\otimes} \\ A_{\otimes} & -A_{\oplus} \end{pmatrix}\cos(kz-\omega t) \tag{1483}$$

The following are some observations concerning these results.

- The result in (1482) is really an expectation value of a "quantum stress operator." This follows from the fact that the classical minimal coupling rule, $\pi_i = p_i - qA$, which was used to derive (1482), can be written as a *quantum* minimal coupling rule, $\hat{\pi}_i = \hat{p}_i - q\hat{A}$. This involves promoting the dynamical variables to operators so that the kinetic momentum, $\hat{\pi} = m\hat{v}$, and vector potential, $\hat{A}$, become *multiplicative* quantum operators and the canonical momentum, $\hat{p}_i = -i\hbar\partial_i$, becomes a *differentiating* quantum operator. Therefore, the expression for $T_{ij}^{\tau\tau}$ should really be considered as a *multiplicative* tensor operator. Furthermore, since $|\psi|^2 = n_s$ already appears explicitly, then in actuality, what we have in (1482) is $\left\langle \hat{T}_{ij}^{\tau\tau}\right\rangle$ where the expectation value has already been evaluated.

- For a numerical estimate of the stress, we can normalize the Ginzburg-Landau order parameter. Recall that $|\psi|^2 = n_s$ which is the number density of Cooper pairs. Then integrating over the volume of the superconductor gives
$$\int \psi^*\psi dV = \int n_s dV = n_s V = N \tag{1484}$$
where N is the number of Cooper pair in the superconductor. Therefore $n_s = N/V$ where $V = Ad$ with A being the surface area and d the thickness of the superconductor. For ultra-relativistic Cooper pairs, it is necessary that $v_{\max} = 2\sqrt{2}c/3$ (in order to avoid the production of particle-anti-particle pairs). Since the electron mass is $m_e \approx 10^{-30}kg$, then an upper bound on the stress is
$$\left\langle \hat{T}_{ij,\ \max}^{\tau\tau}\right\rangle \sim \frac{n_s m_e v_{\max}^2}{2} \sim \frac{N4m_e c^2}{3Ad} \tag{1485}$$

- The stress obtained in (1482) differs from the quantum stress tensor result of Section 50. There we found in (896) that the stress tensor for a "quantized ideal fluid" is

$$\hat{T}^{00} = \rho c^2, \qquad \hat{T}^{0i} = -icn\hbar\partial_i, \qquad \hat{T}^{ij} = -\frac{n\hbar^2}{m_0}\partial_i\partial_j + P\eta_{ij} \tag{1486}$$

  where n involves the particle number density of the material and m_0 is the mass of the constitutive particles. It is evident that the prefactor of $\hat{T}^{ij}$ involves $\hbar^2$ while the prefactor of (1482) does not. Rather, the prefactor in (1482) only involves n_s since $\left|\psi^2\right| = n_s$. This indicates that the role of quantum mechanics for the quantized ideal fluid occurs through the explicit appearance of $\hbar$ while the role of quantum mechanics for (1482) appears only through the normalization constant of the Ginzburg-Landau order parameter.

- In this treatment, by setting $h_{00} = h_{0i} = 0$ and $h_{ij} = h_{ij}^{\tau\tau}$, we have only considered the transverse-traceless strain field $h_{ij}^{\tau\tau}$. Therefore we have effectively neglected the contribution to the metric due to the mass density of the superconductor $\left(T_{00} = \rho_m c^2\right)$, the currents $\left(T_{0i} = \rho c v_i\right)$ and the other stress quantities in T_{ij}. However, in order to properly describe the *full* response of the Cooper pair density (all pressures, shears, and currents), we would need to use the full $h_{\mu\nu}$ metric perturbation and generalize (1476) to

$$T^{\mu\nu} = \left(\frac{\partial \mathscr{F}_{G\text{-}L}}{\partial h_{\mu\nu}}\right)_T \tag{1487}$$

- If we consider the charge-separation effect that is expected to occur in the superconductor, then it is critical to consider the role of the electromagnetic fields that would arise in the superconductor. In that case, it would be important to return to the expressions in (1478) and (1480) which describe the transverse-traceless stress in terms of both the Ginzburg-Landau supercurrents as well as the contribution of the electromagnetic fields to the resulting stress. In fact, since the supercurrents consist of electrons (which have very small mass), and the electric field due to the charge-separation effect is expected to be very large, then it is possible that the contribution to the stress due to the electromagnetic fields would even *exceed* the contribution due to the supercurrents.

  On the other hand, if the electromagnetic fields are absent, then $\vec{A} = 0$ which means that $\hat{p} = m\hat{v}$. However, for the zero momentum eigenstate, we have $\hat{p} = 0$ and therefore $m\hat{v} = 0$. This means that there would be no supercurrents and therefore $\hat{T}_{ij}^{\tau\tau}$ would reduce to just

$$\hat{T}_{ij}^{\tau\tau} = \frac{n_s}{2}\left(\alpha + \frac{\beta}{2}n_s\right)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{1488}$$

- The magnitude of $h_{ij}^{\tau\tau}$ does *not* affect the magnitude of $T_{ij}^{\tau\tau}$ since $h_{ij}^{\tau\tau}$ does not appear in the expression for $T_{ij}^{\tau\tau}$. This implies that the stress of the Cooper pair density (to linear order in the metric) is completely independent of the magnitude of the strain of the gravitational wave causing the stress. Physically, this is strange since it would imply that the gravitational field can be made arbitrarily large and yet the resulting stress would remain unchanged.

  It is important to note that the kinetic velocities in (1482) are not the *result* of the gravitational wave. Rather, these kinetic velocities were already present in the Ginzburg-Landau equation (as a kinetic energy term). They are simply *coupled* to the gravitational field due embedding the free energy density in curved space-time. Since these velocities were not *produced* by the gravitational wave, then the magnitude of the stress found in (1482) is truly independent of the strength of the gravitational wave.

- The result for $T_{ij}^{\tau\tau}$ implies that the stress is suddenly "switched on" the moment a gravitational wave is present. This follows from the fact that in the absence of a gravitational wave, we would have $h_{ij}^{\tau\tau} = 0$ and therefore (1476) clearly gives $T_{ij}^{\tau\tau} = 0$. However, when $h_{ij}^{\tau\tau} \neq 0$, then $T_{ij}^{\tau\tau} =$ constant regardless of the strength of $h_{ij}^{\tau\tau}$. Even if we *gradually* turn on the gravitational strain, we find that the stress undergoes a sudden jump from zero to some constant. This means that there is a "DC" offset in the stress that is suddenly "switched on" regardless of how small the gravitational wave strain is or how slowly it is turned on.

To further highlight this anomaly, we can compare it to the case of pressure or magnetization. In the case of pressure, we find that the derivative of the free energy with respect to volume must produce a result such that the pressure and volume varying inversely. In the case of magnetization, we find that the derivative of the free energy (which varies with B^2) leads to a linear response such that the magnetization varies linearly with the field.

However, in our case here, we find that the derivative of the free energy density with respect to strain is a constant so that the stress does not vary at all with strain. This is likely related to the fact that in [79], p. 10, we find the statement, "*In considering a deformed body at some temperature (constant throughout the body), we shall take the undeformed state to be the state of the body in the absence of external forces . . . Then, for $u_{ik} = 0$, the internal stresses are zero also, i.e. $\sigma_{ik} = 0$. Since $\sigma_{ij} = \partial F/\partial u_{ik}$, it follows that there is no linear term in the expansion of F in powers of u_{ik}. . . . Expanding F in powers of u_{ik}, we therefore have as far as terms of the second order*

$$F = F_0 + \tfrac{1}{2}\lambda u_{ii}^2 + \mu u_{ik}^2 \tag{1489}$$

Therefore, in our case here, in order to obtain an expression with a dependency on $h_{ij}^{\tau\tau}$ (that is, a constitutive equation relating $T_{ij}^{\tau\tau}$ and $h_{ij}^{\tau\tau}$), we would need an analysis that is *second* order in the metric. Then taking the derivative using (1476) would lead to a constitutive equation that is first order in the metric. This situation appears to be unique to gravitation.

- It is questionable whether the derivative of the free energy density with respect to the *gravitational* wave strain field gives the *material* stress. This approach is expressed by (1476) which is ordinarily applied to a *material* strain, not a *gravitational* strain.[154] Recall that the material strain, u_{ij}, is related to the stress, σ_{ij}, according to

$$\sigma_{ij} = -su_{ij} \tag{1490}$$

where s is the shear modulus of the material. In this equation, the stress, σ_{ij}, is the *cause* and the strain, u_{ij}, is the *effect*.[155] On the other hand, the *gravitational* strain, $h_{ij}^{\tau\tau}$, is related to the stress, $T_{ij}^{\tau\tau}$, according to the Einstein equation (in the linearized trace-reversed harmonic gauge) given by (1374) as

$$\Box h_{ij}^{\tau\tau} = -2\kappa T_{ij}^{\tau\tau} \tag{1491}$$

In this equation, the stress, $T_{ij}^{\tau\tau}$, is the *cause* and the strain, $h_{ij}^{\tau\tau}$, is the *effect*.

It is important to also recognize that u_{ij} is an *internal* strain of the *material* whereas $h_{ij}^{\tau\tau}$ is an *external* gravitational strain of *space* (which may propagate *through* the material). This distinction is evidenced further by the fact that the velocity of the internal material strain, u_{ij}, is given by $v = \sqrt{s/\rho}$ where s is the material shear modulus and ρ is the mass density. This velocity is similar to the velocity of sound (pressure waves) given by $v = \sqrt{\partial P/\partial \rho}$ where P is the pressure. This is in contrast to the

[154] The distinction between *gravitational* strain and *material* strain does not seem to be recognized by Press in [73] or Millette in [75]−[78]. However, it is clealy identified by Dyson in [74], eq. (2.25) and (2.26) where the material strain is written as z^m and the gravitational wave strain field is h^{mn}.

[155] In [79], eq. 3.6, it is shown that $\sigma_{ij} = (\partial F/\partial u_{ij})_T$, where σ_{ij} is the stress and u_{ij} is the strain. It is also shown in eq. 4.11, that when Hook's law is valid, the strain can be obtained from the stress using $u_{ij} = \partial F/\partial \sigma_{ij}$. This means that the relation $\sigma_{ij} = su_{ij}$ can be understood to describe either the stress as the cause of a strain, or the strain as the cause of a stress. It is analogous to $F = -k\Delta x$ which can be understood to describe the force as a cause of displacement, or the displacement as a cause of force.

velocity of the gravitational wave strain, $h_{ij}^{\tau\tau}$, which is given by a group velocity and phase velocity based on a particular constitutive equation. For example, for a constitutive equation given by

$$T_{ij}^{\tau\tau} = -\mu h_{ij}^{\tau\tau} \tag{1492}$$

we found the phase and group velocity in Section 76 to be

$$v_{phase} = \frac{c}{\sqrt{1-\omega_G^2/\omega^2}} \qquad \text{and} \qquad v_{group} = c\sqrt{1-\omega_G^2/\omega^2} \tag{1493}$$

Note that for the constitutive equation in (1492), the strain, $h_{ij}^{\tau\tau}$, is the *cause* and the stress, $T_{ij}^{\tau\tau}$, is the *effect* which is the *opposite* case of (1490).

In light of these considerations, we must determine whether the relation in (1476) is appropriate for a *gravitational* strain of space, $h_{ij}^{\tau\tau}$, or whether it is appropriate only for a *material* strain, u_{ij}. Since the material strain, u_{ij}, is an *internal* strain of the material (similar to the internal pressure and volume of the material), and because (1476) and (1471) arise from a similar derivation (involving the derivative of the free energy), then it would seem that (1476) is an appropriate formulation for the *internal* strain of the *material*, u_{ij}, not the *external gravitational* strain of *space*, $h_{ij}^{\tau\tau}$.

- It is unclear whether the quantum stress in (1482) can be used in the Einstein field equation to determine the resulting gravitational radiation that would be emitted by the Cooper pair density. Doing so may imply that we are using a semi-classical approach to gravitation as described by

$$G_{\mu\nu} = \kappa \left\langle \psi | \hat{T}_{\mu\nu} | \psi \right\rangle \tag{1494}$$

This introduces difficulties since the Einstein tensor is a *non-linear* function of the metric while the wave function and all operators obey standard *linear* quantum mechanical commutation relations. If one uses a semi-classical version of *linearized* GR, such as

$$\Box \overline{h}_{\mu\nu} = -2\kappa \langle \psi | \hat{T}_{\mu\nu} | \psi \rangle \tag{1495}$$

(which is in terms of the trace-reversed metric perturbation in the harmonic gauge), or a semi-classical equation for gravity in the Newtonian limit, such as

$$\nabla^2 \Phi_N = 4\pi G \langle \psi | \hat{T}_{00} | \psi \rangle \tag{1496}$$

there is still a problem with interpreting the meaning of a quantum measurement. In the common Copenhagen viewpoint, the effectively instantaneous "collapse" of the wave function upon making a quantum measurement will introduce a discontinuity in the gravitational field as it correspondingly "collapses" from a superposition state to a measured state.[156]

However, we should point out that because the Cooper pairs are in a zero-momentum eigenstate $(p_0 = 0)$ which leads to $\psi = C$, then the wavefunction is effectively already "collapsed." We are not introducing a superposition state to the classical Einstein equation. In fact, since this is the only state of the system, then the expectation value and the eigenvalue become the same and the quantum stress tensor essentially yields what could be considered a single, classical result. (In fact, one could argue that *any* classical source is really just the result of quantum decoherence which "collapses" the wave-function to a single eigenvalue. The only difference here is that there is only one eigenvalue to begin with, so there is no need for "collapse" of the wavefunction.) This process is very similar to the case of using the *classical* Maxwell equations (such as Ampere's law, $\nabla \times \vec{B} = \mu_0 \vec{J}$) with the Ginzburg-Landau current

$$\vec{J} = \frac{e}{m_e} \operatorname{Re}\left[\psi^* \left(-i\hbar\nabla - 2e\vec{A}\right)\psi\right] \tag{1497}$$

[156] For more discussion of these topics, see Wald [57], pp. 382-383, 410-411.

Therefore, although it may not be correct to write (1494) as a general equation for gravitation since ψ could be a *superposition* state, by contrast, it may be valid to write

$$G_{\mu\nu} = \kappa n_s \mathbb{T}_{\mu\nu} \tag{1498}$$

since $\langle\psi|\hat{T}_{\mu\nu}|\psi\rangle = |C|^2 \mathbb{T}_{\mu\nu} = n_s \mathbb{T}_{\mu\nu}$ where ψ has a single eigenstate with an eigenvalue of $\mathbb{T}_{\mu\nu}$. Then the linearized Einstein equation (1495) in terms of the transverse-traceless components, gives

$$\Box h_{ij}^{\tau\tau} = -2\kappa n_s \mathbb{T}_{ij}^{\tau\tau} \tag{1499}$$

where $\mathbb{T}_{ij}^{\tau\tau}$ can be considered a quantum stress eigenvalue given by (1482).

14.2 The Ginzburg-Landau free energy density to second order

We can return to (1449) and once again write the metric as a perturbation to flat space-time using $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. However, this time we do not approximate $\sqrt{-g} \approx 1$. As before, we sum over μ and ν in the kinetic term and sum over μ and ρ in the last term. In the electromagnetic strength tensor, we can use also use $F^{00} = 0$ and $F^{0i} = -F^{i0}$ again to simplify the expression.

$$\begin{aligned}\mathscr{F}_{G\text{-}L} &= \sqrt{-g}\left\{\frac{\hbar^2}{2m}\left[g^{00}\left(D_0\psi\right)^*\left(D_0\psi\right) + g^{0i}\left(D_0\psi\right)^*\left(D_i\psi\right)\right]\right.\\ &\quad + \frac{\hbar^2}{2m}\left[g^{i0}\left(D_i\psi\right)^*\left(D_0\psi\right) + g^{ij}\left(D_i\psi\right)^*\left(D_j\psi\right)\right] + \alpha\left|\psi\right|^2 + \frac{\beta}{2}\left|\psi\right|^4\\ &\quad \left. - \frac{1}{4\mu_0}\left[-2\left(\eta_{ij} + h_{ij}^{\tau\tau}\right)F^{0i}F^{0j} + \left(\eta_{ij} + h_{ij}^{\tau\tau}\right)\left(\eta_{kl} + h_{kl}^{\tau\tau}\right)F^{ik}F^{jl}\right]\right\}\end{aligned} \tag{1500}$$

For gravitational waves in the far-field, we can let $h_{00} = h_{0i} = 0$ so the inverse metric components become

$$g^{00} = -1, \qquad g^{0i} = 0, \qquad g^{ij} = \delta_{ij} - h_{ij} + h_{ki}h_{kj} \tag{1501}$$

We can use $h_{ij}^{\tau\tau} = A_{ij}^{\tau\tau}\cos\left(kz - \omega t\right)$ for a wave propagating in the z-direction. Then the last term in (1501) is

$$x^\sigma \partial_\sigma h_{ij}^{\tau\tau} = \left(zk - \omega ct\right) A_{ij}^{\tau\tau}\sin\left(kz - \omega t\right) \tag{1502}$$

For distance scales comparable to the wave length[157], then $z \approx \lambda$ which means $zk \approx 2\pi$. Also, averaging over a period gives $t \approx T = 2\pi/\omega$. Therefore, the entire expression in (1502) vanishes. Inserting (1501) into (1500) and using $F^{ik}F^{jl} = F^{ki}F^{lj}$ to combine two terms gives

$$\begin{aligned}\mathscr{F}_{G\text{-}L} &= \sqrt{-g^{\tau\tau}}\left\{\frac{\hbar^2}{2m}\left[-\left(D_0\psi\right)^*\left(D_0\psi\right) + \left(D_i\psi\right)^*\left(D_i\psi\right)\right]\right.\\ &\quad + \frac{\hbar^2}{2m}\left(-h_{ij}^{\tau\tau} + h_{ki}^{\tau\tau}h_{kj}^{\tau\tau}\right)\left(D_i\psi\right)^*\left(D_j\psi\right) + \alpha\left|\psi\right|^2 + \frac{\beta}{2}\left|\psi\right|^4\\ &\quad - \frac{1}{4\mu_0}\left[-2\left(\eta_{ij}F^{0i}F^{0j} + h_{ij}^{\tau\tau}F^{0i}F^{0j}\right)\right]\\ &\quad \left. - \frac{1}{4\mu_0}\left[\left(\eta_{ij}\eta_{kl}F^{ik}F^{jl} + \eta_{ij}h_{kl}^{\tau\tau}F^{ik}F^{jl} + h_{ij}^{\tau\tau}\eta_{kl}F^{ik}F^{jl} + h_{ij}^{\tau\tau}h_{kl}^{\tau\tau}F^{ik}F^{jl}\right)\right]\right\}\end{aligned} \tag{1503}$$

Since we are considering a wave propagating in the z-direction, then $h_{i3}^{\tau\tau} = 0$ (since $h_{ij}^{\tau\tau}$ is transverse). Also $h_{11}^{\tau\tau} = -h_{22}^{\tau\tau} = h_{\oplus}$ where $h_{\oplus} = h_{\oplus}\cos\left(kz - \omega t\right)$ for plus polarization while $h_{12}^{\tau\tau} = h_{21}^{\tau\tau} = h_{\otimes}$ where $h_{\otimes} =$

[157] We will be applying our results to a cavity with dimensions on the order of centimeters with microwave frequencies $\left(\lambda \approx 10^{-2}m\right)$. Therefore, this approximation will remain valid.

$h_{\otimes}\cos(kz-\omega t)$ for cross polarization.[158] We also sum over the indices in the last term and use $F^{ii}=0$.

$$\begin{aligned}
\mathscr{F}_{G\text{-}L} \quad = \quad & \sqrt{-g^{\tau\tau}}\Bigg\{\frac{\hbar^2}{2m}\left[-(D_0\psi)^*(D_0\psi)+(D_i\psi)^*(D_i\psi)\right]+\alpha|\psi|^2+\frac{\beta}{2}|\psi|^4 \\
& -\frac{\hbar^2}{2m}\left\{h_{\oplus}\left[(D_1\psi)^*(D_1\psi)-(D_2\psi)^*(D_2\psi)\right]+h_{\otimes}\left[(D_1\psi)^*(D_2\psi)+(D_2\psi)^*(D_1\psi)\right]\right\} \\
& +\frac{\hbar^2}{2m}\left(h_{\oplus}^2+h_{\otimes}^2\right)\left[(D_1\psi)^*(D_1\psi)+(D_2\psi)^*(D_2\psi)\right] \\
& +\frac{1}{2\mu_0}\left[\left(F^{0i}\right)^2+h_{\oplus}\left(F^{01}\right)^2+2\tilde{h}_{\otimes}F^{01}F^{02}-h_{\oplus}\left(F^{02}\right)^2\right] \\
& +\frac{1}{2\mu_0}\left[-\frac{1}{2}F^{ij}F_{ij}-\frac{1}{2}h_{\oplus}\left[\left(F^{21}\right)^2+\left(F^{31}\right)^2\right]\right] \\
& +\frac{1}{2\mu_0}\left[-h_{\otimes}\left(F^{31}F^{32}\right)+\frac{1}{2}h_{\oplus}\left[\left(F^{12}\right)^2+\left(F^{32}\right)^2\right]+\left(h_{\oplus}^2+h_{\otimes}^2\right)\left(F^{12}\right)^2\right]\Bigg\} \qquad (1504)
\end{aligned}$$

Now we also use the components of the electromagnetic field strength tensor given by $F^{0i}=\frac{1}{c}E^i$, $F^{ij}=\varepsilon^{ijk}B^k$, and $F^{ii}=0$ to obtain

$$\begin{aligned}
\mathscr{F}_{G\text{-}L} \quad = \quad & \sqrt{-g^{\tau\tau}}\Bigg\{\frac{\hbar^2}{2m}\left[-(D_0\psi)^*(D_0\psi)+(D_i\psi)^*(D_i\psi)\right]+\alpha|\psi|^2+\frac{\beta}{2}|\psi|^4 \\
& -\frac{\hbar^2}{2m}\left\{h_{\oplus}\left[(D_1\psi)^*(D_1\psi)-(D_2\psi)^*(D_2\psi)\right]+h_{\otimes}\left[(D_1\psi)^*(D_2\psi)+(D_2\psi)^*(D_1\psi)\right]\right\} \\
& +\frac{\hbar^2}{2m}\left(h_{\oplus}^2+h_{\otimes}^2\right)\left[(D_1\psi)^*(D_1\psi)+(D_2\psi)^*(D_2\psi)\right] \\
& +\frac{1}{2\mu_0c^2}\left(\vec{E}^2+h_{\oplus}E_x^2+2\tilde{h}_{\otimes}E_xE_y-h_{\oplus}E_y^2\right) \\
& +\frac{1}{2\mu_0}\left[\vec{B}^{\,2}-\frac{1}{2}h_{\oplus}\left(B_z^2+B_y^2\right)+h_{\otimes}\left(B_yB_x\right)\right] \\
& +\frac{1}{2\mu_0}\left[\frac{1}{2}h_{\oplus}\left(B_z^2+B_x^2\right)+\left(h_{\oplus}^2+h_{\otimes}^2\right)B_z^2\right]\Bigg\} \qquad (1505)
\end{aligned}$$

[158] Note that writing $\eta_{ij}\eta_{kl}F^{ik}F^{jl}=F^{il}F^{il}$ will lead to $F^{il}F^{il}=\vec{B}^2$. This will lead to $\frac{-1}{2\mu}\vec{B}^2$ which is the wrong sign for the magnetic field energy density. Therefore, we treat η_{ij} as a metric which lowers indices so that $\eta_{ij}\eta_{kl}F^{ik}F^{jl}=F^{il}F_{il}$ which leads to the corret sign for the magnetic field energy density.

Lastly, using $1/c^2 = \mu_0 \varepsilon_0$ and rearranging gives

$$\boxed{\begin{aligned}\mathscr{F}_{G\text{-}L} = \sqrt{-g^{\tau\tau}}\Bigg\{ & \frac{\hbar^2}{2m}\left[-\left(D_0\psi\right)^*\left(D_0\psi\right) + \left(D_i\psi\right)^*\left(D_i\psi\right)\right] + \alpha\left|\psi\right|^2 + \frac{\beta}{2}\left|\psi\right|^4 \\ & -\frac{\hbar^2}{2m}\left[\left(D_1\psi\right)^*\left(D_1\psi\right) - \left(D_2\psi\right)^*\left(D_2\psi\right)\right]h_{\oplus} \\ & -\frac{\hbar^2}{2m}\left[\left(D_1\psi\right)^*\left(D_2\psi\right) + \left(D_2\psi\right)^*\left(D_1\psi\right)\right]h_{\otimes} \\ & +\frac{\hbar^2}{2m}\left[\left(D_1\psi\right)^*\left(D_1\psi\right) + \left(D_2\psi\right)^*\left(D_2\psi\right)\right]\left(h_{\oplus}^2 + h_{\otimes}^2\right) \\ & +\frac{\varepsilon_0}{2}\vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^{\,2} + \frac{1}{2}\left[\varepsilon_0\left(E_x^2 - E_y^2\right) - \frac{1}{2\mu_0}\left(B_x^2 - B_y^2\right)\right]h_{\oplus} \\ & +\left(\varepsilon_0 E_x E_y + \frac{1}{2\mu_0}B_x B_y\right)h_{\otimes} + \frac{1}{2\mu_0}B_z^2\left(h_{\oplus}^2 + h_{\otimes}^2\right)\Bigg\} \\ & \textit{G-L free energy density in the presence of a gravitational wave} \\ & \textit{propagating in the z-direction (to second order in the metric)}\end{aligned}} \tag{1506}$$

Utilizing the covariant derivative, canonical momentum and zero-momentum eigenstate

We now make use of the gauge covariant derivative given in (1445) as $D_\mu = \partial_\mu - \frac{iq}{\hbar}A_\mu$. Since the Cooper pairs are in a zero-momentum eigenstate $(p_0 = 0)$, then $\psi = Ce^{\left(\frac{i}{\hbar}\vec{p}_0\cdot\vec{r}\right)} = C$. In that case, all the derivatives vanish and we can factor out $\psi^*\psi = C^*C = |C|^2$. Also recall that $|\psi|^2 = n_s$ which is the number density of Cooper pairs. Therefore $|C|^2 = n_s$ and (1457) becomes

$$\begin{aligned}\mathscr{F}_{G\text{-}L} &= \sqrt{-g^{\tau\tau}}\Bigg\{\frac{n_s q^2}{2m}\left[-\left(A_0\right)^2 + \left(A_i\right)^2\right] + n_s\alpha + \frac{n_s^2\beta}{2} + \frac{\varepsilon_0}{2}\vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^{\,2} \\ & +\frac{n_s q^2}{2m}\left[\left(A_1\right)^2\left(h_{\oplus}^2 + h_{\otimes}^2 - h_{\oplus}\right) + \left(A_2\right)^2\left(h_{\oplus}^2 + h_{\otimes}^2 + h_{\oplus}\right) - 2A_1A_2h_{\otimes}\right] \\ & +\frac{1}{2}\left[\varepsilon_0\left(E_x^2 - E_y^2\right) - \frac{1}{2\mu_0}\left(B_x^2 - B_y^2\right)\right]h_{\oplus} + \left(\varepsilon_0 E_x E_y + \frac{1}{2\mu_0}B_x B_y\right)h_{\otimes} \\ & +\frac{1}{2\mu_0}B_z^2\left(h_{\oplus}^2 + h_{\otimes}^2\right)\Bigg\}\end{aligned} \tag{1507}$$

In (657), we found the kinetic momentum for a charged, spinless particle (such as a Cooper pair) to be

$$\pi_i = \gamma m\left(cg_{0i} + g_{li}v_l\right) \tag{1508}$$

where $\pi_i = p_i - qA_i$ and $\gamma \equiv \left(-g_{00} - \frac{2}{c} g_{0j} v^j - \frac{1}{c^2} g_{jk} v^j v^k\right)^{-1/2}$. In our case here, we set $g_{0i} = 0$, $g_{li} = \eta_{li} + h_{li}^{\tau\tau}$, and $p_i = 0$. Therefore, we have

$$\gamma = \left(1 - \frac{v^2}{c^2} - \frac{1}{c^2} h_{jk}^{\tau\tau} v^j v^k\right)^{-1/2} \tag{1509}$$

and (1459) becomes

$$-qA_i = \left(1 - \frac{v^2}{c^2} - \frac{1}{c^2} h_{jk}^{\tau\tau} v^j v^k\right)^{-1/2} m\left(\eta_{li} + h_{li}^{\tau\tau}\right) v_l \tag{1510}$$

Keeping to order v^2/c^2 in velocity and using k for the repeated index gives

$$A_i = -\frac{m}{q}\left(v_i + h_{ki}^{\tau\tau} v_k\right) \tag{1511}$$

Since we will need $(A_i)^2$ in (1458), then to second order in the metric, (1462) gives

$$(A_i)^2 = \frac{m^2}{q^2}\left(v^2 + 2h_{ki}^{\tau\tau} v_i v_k + h_{ki}^{\tau\tau} h_{li}^{\tau\tau} v_k v_l\right) \tag{1512}$$

Summing over indices and using $h_{11}^{\tau\tau} = -h_{22}^{\tau\tau} = h_\oplus$ and $h_{12}^{\tau\tau} = h_{21}^{\tau\tau} = h_\otimes$ gives

$$(A_i)^2 = \frac{m^2}{q^2}\left[v^2 + 2h_\oplus (v_1)^2 - 2h_\oplus (v_2)^2 + 4h_\otimes v_1 v_2 + \left(h_\oplus^2 + h_\otimes^2\right)\left(v_1^2 + v_2^2\right)\right] \tag{1513}$$

Since we will also need A_1 and A_2 in (1458), then using $h_{11}^{\tau\tau} = -h_{22}^{\tau\tau} = h_\oplus$ and $h_{12}^{\tau\tau} = h_{21}^{\tau\tau} = h_\otimes$ in (1462) gives

$$A_1 = -\frac{m}{q}\left(v_1 + h_\oplus v_1 + h_\otimes v_2\right) \qquad A_2 = -\frac{m}{q}\left(v_2 + h_\otimes v_1 - h_\oplus v_2\right) \tag{1514}$$

We also need $A_1 A_2$ to second order in the metric which is

$$A_1 A_2 = \frac{m^2}{q^2}\left[v_1 v_2 + h_\otimes (v_1)^2 + h_\otimes (v_2)^2 + h_\otimes h_\oplus (v_1)^2 - h_\oplus h_\otimes (v_2)^2 + h_\otimes^2 v_1 v_2 - h_\oplus^2 v_1 v_2\right]$$

$$\begin{aligned} A_1 A_2 = \frac{m^2}{q^2}\Big[& v_1 v_2 + h_\otimes (v_1)^2 + h_\otimes (v_2)^2 + h_\otimes h_\oplus (v_1)^2 - h_\oplus h_\otimes (v_2)^2 + h_\otimes^2 v_1 v_2 - h_\oplus^2 v_1 v_2 \\ & - h_\oplus h_\otimes (v_2)^2 + h_\otimes^2 v_1 v_2 - h_\oplus^2 v_1 v_2\Big] \end{aligned} \tag{1515}$$

Lastly, squaring A_1 and A_2 and remaining to second order in the metric gives

$$(A_1)^2 = \frac{m^2}{q^2}\left[(v_1)^2 + 2h_\oplus (v_1)^2 + 2h_\otimes v_1 v_2 + h_\oplus^2 (v_1)^2 + h_\otimes^2 (v_2)^2 + 2h_\otimes h_\oplus v_1 v_2\right] \tag{1516}$$

and

$$(A_2)^2 = \frac{m^2}{q^2}\left[(v_2)^2 - 2h_\oplus (v_2)^2 + 2h_\otimes v_1 v_2 + h_\oplus^2 (v_2)^2 + h_\otimes^2 (v_1)^2 - 2h_\otimes h_\oplus v_1 v_2\right] \tag{1517}$$

Now we can substitute $(1513)-(1517)$ into (1507). Eliminating all terms that are higher than second order in the metric and simplifying gives

$$\begin{aligned}
\mathscr{F}_{G\text{-}L} \quad = \quad & \sqrt{-g^{\tau\tau}}\left\{\frac{n_s m}{2}v^2 - \frac{n_s q^2}{2m}(A_0)^2 + n_s\alpha + \frac{n_s^2\beta}{2} + \frac{\varepsilon_0}{2}\vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^{\,2}\right. \\
& + \frac{n_s m}{2}\left(h_\oplus v_x^2 - h_\oplus v_y^2 + 2h_\otimes v_x v_y\right) \\
& + \frac{1}{2}\left[\varepsilon_0\left(E_x^2 - E_y^2\right) - \frac{1}{2\mu_0}\left(B_x^2 - B_y^2\right)\right]h_\oplus + \left(\varepsilon_0 E_x E_y + \frac{1}{2\mu_0}B_x B_y\right)h_\otimes \Bigg\} \\
& \left. + \frac{1}{2\mu_0}B_z^2\left(h_\oplus^2 + h_\otimes^2\right)\right\} \qquad (1518)
\end{aligned}$$

Surprisingly, all second order terms cancel. Next we can use the determinant of the metric (to second order in the metric) found in (2661) as

$$g^{\tau\tau} = -1 - \left(h_\oplus + h_\otimes - h_\oplus^2 - h_\otimes^2\right) \qquad (1519)$$

Then we have

$$\sqrt{-g^{\tau\tau}} = \sqrt{1 + \left(h_\oplus + h_\otimes - h_\oplus^2 - h_\otimes^2\right)} \qquad (1520)$$

Using a binomial expansion to second order[159] gives

$$\sqrt{-g^{\tau\tau}} \quad = \quad 1 + \frac{1}{2}\left(h_\oplus + h_\otimes - h_\oplus^2 - h_\otimes^2\right) - \frac{1}{8}\left(h_\oplus + h_\otimes - h_\oplus^2 - h_\otimes^2\right)^2 \qquad (1521)$$

Multiplying out terms and remaining to second order in the metric gives

$$\sqrt{-g^{\tau\tau}} \quad = \quad 1 + \frac{1}{2}\left(h_\oplus + h_\otimes\right) - \frac{5}{8}\left(h_\oplus^2 + h_\otimes^2\right) - \frac{1}{4}h_\oplus h_\otimes \qquad (1522)$$

Inserting (1522) into (1518), distributing, and eliminating terms higher than second order gives

$$\begin{aligned}
\mathscr{F}_{G\text{-}L} \quad = \quad & \left(\frac{n_s m}{2}v^2 - \frac{n_s q^2}{2m}(A_0)^2 + n_s\alpha + \frac{n_s^2\beta}{2} + \frac{\varepsilon_0}{2}\vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^{\,2}\right) \\
& \cdot\left[1 + \frac{1}{2}\left(h_\oplus + h_\otimes\right) - \frac{5}{8}\left(h_\oplus^2 + h_\otimes^2\right) - \frac{1}{4}h_\oplus h_\otimes\right] \\
& + \frac{n_s m}{2}\left(h_\oplus v_x^2 - h_\oplus v_y^2 + 2h_\otimes v_x v_y\right) \\
& + \frac{n_s m}{4}\left[h_\oplus^2\left(v_x^2 - v_y^2\right) + 2h_\otimes^2 v_x v_y + h_\oplus h_\otimes\left(v_x^2 - v_y^2 + 2v_x v_y\right)\right] \\
& + \frac{1}{4}\left[\varepsilon_0\left(E_x^2 - E_y^2\right) - \frac{1}{2\mu_0}\left(B_x^2 - B_y^2\right)\right]\left(2h_\oplus + h_\oplus^2 + h_\oplus h_\otimes\right) \\
& + \frac{1}{2}\left(\varepsilon_0 E_x E_y + \frac{1}{2\mu_0}B_x B_y\right)\left(2h_\otimes + h_\oplus h_\otimes + h_\otimes^2\right) + \frac{1}{2\mu_0}B_z^2\left(h_\oplus^2 + h_\otimes^2\right) \qquad (1523)
\end{aligned}$$

[159]The binomial expansion to second order is $\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2$ for $x << 1$.

According to (1477), we can take a derivative of this expression with respect to $h_\oplus$ to obtain $T_\oplus$. However, this will result in $T_\oplus$ being expressed in terms of *both* $h_\oplus$ and $h_\otimes$. In order to obtain an expression for $T_\oplus$ that is purely in terms of $h_\oplus$, we can consider the cases of plus polarization and cross polarization waves independently. For a gravitational wave with purely *plus* polarization, we can set $h_\otimes = 0$ in (1523) to obtain

$$\begin{aligned}\mathscr{F}^{\oplus}_{G\text{-}L} &= \left(\frac{n_s m}{2}v^2 - \frac{n_s q^2}{2m}\left(A_0\right)^2|C|^2 + n_s\alpha + \frac{n_s^2\beta}{2} + \frac{\varepsilon_0}{2}\vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^{\,2}\right)\left(1+\frac{1}{2}h_\oplus - \frac{5}{8}h_\oplus^2\right)\\ &+\frac{n_s m}{2}\left(h_\oplus + \frac{1}{2}h_\oplus^2\right)\left(v_x^2 - v_y^2\right)\\ &+\frac{1}{4}\left[\varepsilon_0\left(E_x^2 - E_y^2\right) - \frac{1}{2\mu_0}\left(B_x^2 - B_y^2\right)\right]\left(h_\oplus^2 + 2h_\oplus\right) + \frac{1}{2\mu_0}B_z^2 h_\oplus^2 \qquad (1524)\end{aligned}$$

Similarly, for a gravitational wave with purely *cross* polarization, setting $h_\oplus = 0$ in (1523) gives

$$\begin{aligned}\mathscr{F}^{\otimes}_{G\text{-}L} &= \left(\frac{n_s m}{2}v^2 - \frac{n_s q^2}{2m}\left(A_0\right)^2 + n_s\alpha + \frac{n_s^2\beta}{2} + \frac{\varepsilon_0}{2}\vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^{\,2}\right)\left(1+\frac{1}{2}h_\otimes - \frac{5}{8}h_\otimes^2\right)\\ &+n_s m v_x v_y\left(h_\otimes + \frac{1}{2}h_\otimes^2\right)\\ &+\frac{1}{2}\left(\varepsilon_0 E_x E_y + \frac{1}{2\mu_0}B_x B_y\right)\left(h_\otimes^2 + 2h_\otimes\right) + \frac{1}{2\mu_0}B_z^2 h_\otimes^2 \qquad (1525)\end{aligned}$$

Pressures and stresses in a superconductor in the presence of a gravitational wave strain

Next we find the stress produced by a gravitational wave interacting with the Cooper pair density using the Ginzburg-Landau free energy density. From (1476) we have

$$T^{ij} = \left(\frac{\partial \mathscr{F}_{G\text{-}L}}{\partial h_{ij}}\right)_T \qquad (1526)$$

For plus polarization, we only have $h_{11}^{\tau\tau} = -h_{22}^{\tau\tau} = h_\oplus$ which corresponds to $T_{11}^{\tau\tau} = -T_{22}^{\tau\tau} = T_\oplus$. In that case, we can use

$$T_\oplus = \left(\frac{\partial \mathscr{F}^{\oplus}_{G\text{-}L}}{\partial h_\oplus}\right)_T \qquad (1527)$$

Applying (1527) to (1524) and using $m = 2m_e$ and $q = 2e$ gives

$$\begin{aligned} T_\oplus \quad = \quad & \left(n_s m_e v^2 - \frac{n_s e^2}{m_e}(A_0)^2 + \alpha |C|^2 + \frac{n_s^2 \beta}{2} + \frac{\varepsilon_0}{2}\vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^{\,2}\right)\left(\frac{1}{2} - \frac{5}{4}h_\oplus\right) \\ & + n_s m_e (1 + h_\oplus)\left(v_x^2 - v_y^2\right) + \frac{1}{2}\left[\varepsilon_0\left(E_x^2 - E_y^2\right) - \frac{1}{2\mu_0}\left(B_x^2 - B_y^2\right)\right](h_\oplus + 1) \\ & + \frac{1}{\mu_0} B_z^2 h_\oplus \end{aligned} \tag{1528}$$

We can separate terms that contain $h_\oplus$ and terms that do not. This gives

$$\begin{aligned} T_\oplus = n_s & \left[\frac{1}{2}m_e v^2 + m_e\left(v_x^2 - v_y^2\right) - \frac{e^2}{2m_e}(A_0)^2 + \frac{\alpha}{2} + \frac{n_s\beta}{4}\right] \\ & + \frac{1}{4}\left(\varepsilon_0\vec{E}^2 + \frac{1}{\mu_0}\vec{B}^{\,2}\right) + \frac{1}{2}\left[\varepsilon_0\left(E_x^2 - E_y^2\right) - \frac{1}{2\mu_0}\left(B_x^2 - B_y^2\right)\right] \\ & - \frac{5}{2}n_s\left[\frac{1}{2}m_e v^2 - \frac{2}{5}m_e\left(v_x^2 - v_y^2\right) - \frac{e^2}{2m_e}(A_0)^2 + \frac{\alpha}{2} + \frac{n_s\beta}{4}\right] h_\oplus \\ & - \frac{1}{2}\left[\frac{5}{4}\left(\varepsilon_0\vec{E}^2 + \frac{1}{\mu_0}\vec{B}^{\,2}\right) - \varepsilon_0\left(E_x^2 - E_y^2\right) + \frac{1}{2\mu_0}\left(B_x^2 - B_y^2 - 4B_z^2\right)\right] h_\oplus \end{aligned} \tag{1529}$$

Stress on the Cooper pair density due to a plus polarization gravitational wave propagating in the z-direction (to second order in the metric)

As expected, we find that the first two lines match (1478) where we found $T_\oplus$ to linear order in the metric. These stress terms are essentially a "DC background" associated with the Cooper pair density and the electromagnetic field. The second two lines come from working with the metric to second order and describe the *linear* response of the stress of the entire system (Cooper pair density and electromagnetic fields) to the gravitational wave strain field. If we consider a constitutive equation given by $T_\oplus = -\mu_\oplus h_\oplus$, where $\mu_\oplus$ is the "gravitational modulus" of the superconductor for plus polarization, then $\mu_\oplus$ can be found from the second two lines in the expression above.

$$\begin{aligned} \mu_\oplus = \frac{5}{2}n_s & \left[\frac{1}{2}m_e v^2 - \frac{2}{5}m_e\left(v_x^2 - v_y^2\right) - \frac{e^2}{2m_e}(A_0)^2 + \frac{\alpha}{2} + \frac{n_s\beta}{4}\right] \\ & + \frac{1}{2}\left[\frac{5}{4}\left(\varepsilon_0\vec{E}^2 + \frac{1}{\mu_0}\vec{B}^{\,2}\right) - \varepsilon_0\left(E_x^2 - E_y^2\right) + \frac{1}{2\mu_0}\left(B_x^2 - B_y^2 - 4B_z^2\right)\right] \end{aligned} \tag{1530}$$

Gravitational modulus of the Cooper pair density in response to a plus polarization gravitational wave propagating in the z-direction

This expression describes the effective "stiffness" of the superconductor (including the electromagnetic fields it contains) in response to the strain of a *plus* polarization gravitational wave. It also demonstrates a *linear* response of the induced stress as a result of the incident gravitational wave strain. Since the result is specifically for a *plus*-polarization wave, we have labeled the gravitational modulus with a corresponding subscript: $\mu_{\oplus}$. For cross polarization, we have $h_{12}^{\tau\tau} = h_{21}^{\tau\tau} = h_{\otimes}$ which corresponds to $T_{12}^{\tau\tau} = T_{21}^{\tau\tau} = T_{\otimes}$. In that case, we can use

$$T_{\otimes} = \left(\frac{\partial \mathscr{F}_{G\text{-}L}^{\otimes}}{\partial h_{\otimes}}\right)_T \tag{1531}$$

Then applying (1531) to (1524) and using $m = 2m_e$ and $q = 2e$ gives

$$\begin{aligned} T_{\otimes} &= \left(n_s m_e v^2 - \frac{n_s e^2}{m_e}(A_0)^2 + n_s\alpha + \frac{n_s^2\beta}{2} + \frac{\varepsilon_0}{2}\vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^{\,2}\right)\left(\frac{1}{2} - \frac{5}{4}h_{\otimes}\right) \\ &\quad + 2n_s m_e v_x v_y (1 + h_{\otimes}) + \left(\varepsilon_0 E_x E_y + \frac{1}{2\mu_0}B_x B_y\right)(1 + h_{\otimes}) + \frac{1}{\mu_0}B_z^2 h_{\otimes} \end{aligned} \tag{1532}$$

We can separate terms that contain $h_{\otimes}$ and terms that do not. This gives

$$\boxed{\begin{aligned} T_{\otimes} &= n_s\left[\frac{1}{2}m_e v^2 + 2m_e v_x v_y - \frac{e^2}{2m_e}(A_0)^2 + \frac{\alpha}{2} + \frac{n_s\beta}{4}\right] \\ &\quad + \frac{1}{4}\left(\varepsilon_0\vec{E}^2 + \frac{1}{\mu_0}\vec{B}^{\,2}\right) + \varepsilon_0 E_x E_y + \frac{1}{2\mu_0}B_x B_y \\ &\quad - \frac{5n_s}{2}\left[\frac{1}{2}m_e v^2 - \frac{4}{5}m_e v_x v_y - \frac{e^2}{2m_e}(A_0)^2 + \frac{\alpha}{2} + \frac{n_s\beta}{4}\right]h_{\otimes} \\ &\quad - \left[\frac{5}{8}\left(\varepsilon_0\vec{E}^2 + \frac{1}{\mu_0}\vec{B}^{\,2}\right) - \varepsilon_0 E_x E_y - \frac{1}{2\mu_0}\left(B_x B_y + 2B_z^2\right)\right]h_{\otimes} \end{aligned}} \tag{1533}$$

Stress on the Cooper pair density due to a cross polarization gravitational wave propagating in the z-direction (to second order in the metric)

As expected, we find that the first two lines match (1480) where we found $T_{\otimes}$ to linear order in the metric. Once again, we can notice that the first two lines contains stress terms which are essentially a "DC background" associated with the Cooper pair density and the electromagnetic field while the second two lines describe the linear response of the stress of the entire system (Cooper pair density and electromagnetic fields) to the gravitational wave strain field. If we consider a constitutive equation given by $T_{\otimes} = -\mu_{\otimes} h_{\otimes}$, where $\mu_{\otimes}$ is the "gravitational modulus" of the superconductor for cross polarization, then $\mu_{\otimes}$ can be found from the second two lines in the expression above.

$$\boxed{\begin{aligned}\mu_{\otimes} &= \frac{5n_s}{2}\left[\frac{1}{2}m_e v^2 - \frac{4}{5}m_e v_x v_y - \frac{e^2}{2m_e}(A_0)^2 + \frac{\alpha}{2} + \frac{n_s\beta}{4}\right] \\ &\quad + \left[\frac{5}{8}\left(\varepsilon_0\vec{E}^2 + \frac{1}{\mu_0}\vec{B}^{\,2}\right) - \varepsilon_0 E_x E_y - \frac{1}{2\mu_0}\left(B_x B_y + 2B_z^2\right)\right]\end{aligned}} \tag{1534}$$

Gravitational modulus of the Cooper pair density in response to a cross polarization gravitational wave propagating in the z-direction

Once again, this expression describes the effective "stiffness" of the superconductor (including the electromagnetic fields it contains) in response to the strain of a *cross* polarization gravitational wave. Since the result is specifically for a *plus*-polarization wave, we have labeled the gravitational modulus with a corresponding subscript: $\mu_{\otimes}$.

The following are some observations concerning these results.

- Unlike the usual definition of a shear modulus, the *gravitational* shear moduli in (1530) and (1534) are not necessarily positive and are not just an inherent property of the material (like the usual shear modulus and bulk modulus in continuum mechanics). Rather, the values of $\mu_{\oplus}$ and $\mu_{\otimes}$ depend on the supercurrent velocities and electromagnetic fields within the superconductor. This implies that the "stiffness" of the superconductor in response to gravitational waves can be affected by changing the electromagnetic fields in the superconductor.

  However, $\mu_{\oplus}$ and $\mu_{\otimes}$ still describe the effective "stiffness" of a superconductor (including the electromagnetic fields it contains) in response to the strain of a gravitational wave. They also demonstrate that there can be a *linear* response of the induced stress as a result of the incident gravitational wave strain.

- The derivation of $\mu_{\oplus}$ and $\mu_{\otimes}$ in (1530) and (1534), respectively, required using the metric to *second* order. The reason a second order treatment is required to derive this quantity is simply because the energy must necessarily be *second* order in $h_{ij}^{\tau\tau}$ in order to derive a *first* order (linear) Hooke's law. This topic is discussed in [92], pp. 31-32. It is shown that a Taylor expansion of an arbitrary potential about a maximum at $x = x_0$ is given by

  $$V(x) = V(x_0) + V'(x_0)(x - x_0) + \frac{1}{2}V''(x_0)(x - x_0)^2 + \cdots \tag{1535}$$

  It is then stated that $V(x_0)$ can be subtracted since a constant $V(x)$ term does not change the force. It is also recognized that $V'(x_0) = 0$ since $V(x)$ is a minimum at $x = x_0$. (There is no gradient of the potential, or net force, acting on the system.) Then dropping the higher order terms makes the potential become

  $$V(x) \approx \frac{1}{2}V''(x_0)(x - x_0)^2 \tag{1536}$$

  which describes simple harmonic oscillation about the point x_0 with an effective spring constant $k = V''(x_0)$. Note that $V''(x_0) \geq 0$ since x_0 is a minimum. If we choose our coordinate system so that $x_0 = 0$, then we obtain the usual simple harmonic potential, $V(x) \approx \frac{1}{2}kx^2$. Now applying the same treatment to our analysis here, we find that expressing the free energy density as a function of the strain gives the following Taylor expansion[160]

$$F\left(h_{ij}\right) \approx F\left(h_{ij\ (0)}\right) + F'\left(h_{ij\ (0)}\right)\left(h_{ij} - h_{ij\ (0)}\right) + \frac{1}{2}F''\left(h_{ij\ (0)}\right)\left(h_{ij} - h_{ij\ (0)}\right)^2 \tag{1537}$$

where $h_{ij\ (0)}$ is the internal strain when $F\left(h_{ij}\right)$ is a minimum and $F''\left(h_{ij\ (0)}\right)$ is the effective spring constant or "stiffness" of the system. In this context, we identify $F''\left(h^{\tau\tau}_{ij\ (0)}\right) = \mu_{G(SC)}$ which is the gravitational shear modulus, since $h_{ij} = h^{\tau\tau}_{ij}$ is a shear field. The corresponding Hooke's law then becomes $T^{\tau\tau}_{ij} = -\mu_{G(SC)} h^{\tau\tau}_{ij}$. In fact, if we consider the stress as a function of strain, then an expansion of the stress would give

$$T^{\tau\tau}_{ij}\left(h^{\tau\tau}_{ij}\right) = T^{\tau\tau}_{ij\ (0)} - \mu_{G(SC)} h^{\tau\tau}_{ij} + \cdots \tag{1538}$$

where the first term, $T^{\tau\tau}_{ij\ (0)}$, is an *internal* stress of the system in the absence of any *external* strain field $h^{\tau\tau}_{ij}$. The second term describes the first order (linear) response of the stress to the external strain field, $h^{\tau\tau}_{ij}$. We neglect higher order terms in the stress since this would involve a treatment that is *third* order in the metric.

What may be surprising is that our analysis shows that $T^{\tau\tau}_{ij\ (0)} \neq 0$. This implies that embedding the free energy density of the superconductor into curved space-time leads to the superconductor possessing an internal stress which is independent of the strength of the external strain field. The *physical* cause of this constant stress cannot be ascribed simply to electromagnetic or quantum mechanical reasons. It is necessarily associated with the gravitational field as is evidenced by the fact that $T_{\oplus\ (0)}$ and $T_{\otimes\ (0)}$ differ from one another. This can be seen by comparing the first two lines in (1529) and (1533). This means that the polarization of the gravitational wave affects the nature of this internal stress.

By comparison, recall that in (1535), we set $V'\left(x_0\right) = 0$ on the basis that $V\left(x\right)$ is a minimum at $x = x_0$. Because $V\left(x\right)$ is a function of x, then $V'\left(x_0\right) = 0$ means there is no *gradient* force on the system which would accelerate the system. However, for the case of the stress in a superconductor, we have described the free energy, $F\left(h_{ij}\right)$, as a function of the *strain* in the system, not the position. Therefore, having $F'\left(h_{ij}\right) \neq 0$ implies the free energy is *not* a minimum when the strain field is $h_{ij} = h_{ij\ (0)}$. It seems that a fixed background stress suddenly appears when any non-zero external strain field is introduced.

This stress term could be regarded as a "symmetry breaking" term in the superconductor since it appears only when an external strain field is applied but it does not depend on the strength of the external strain field. The final conclusion is that a superconductor subjected to a gravitational wave will exhibit a "DC" or static stress (a symmetry breaking term), as well as a stress which follows a *linear* response to $h^{\tau\tau}_{ij}$. Of course, there may also be other higher effects which we have not considered here.

- Since the derivation of $T_{\oplus}$ and $T_{\otimes}$ in (1529) and (1533), respectively, require using the metric to *second* order, it may be inquired whether it is legitimate to insert this result into the *linearized* Einstein equation. However, note that $T^{\tau\tau}_{ij} = -\mu_{G(SC)} h^{\tau\tau}_{ij}$ is still a *linear* relationship which is *first* order in the metric. Using a second order metric to analyze the free energy density (in order to obtain a linear response) does not imply that the field equations (relating $T^{\tau\tau}_{ij}$ to $h^{\tau\tau}_{ij}$) cannot also be linearized.

[160]Formally speaking, there is another second order term formed by the contraction of the metric perturbation that could be included which is $\left(\eta^{ij} h_{ij}\right)^2$. However, this term involves pressures (not shears) therefore to simplify the discussion, we only keep $h_{ij} h_{ij}$.

14.3 Properties of the Cooper pair density in response to a GR wave

In the previous section we derived a gravitational modulus for the Cooper pair density in a superconductor in response to a gravitational wave. This was done by starting with the Ginzburg-Landau free energy density in curved space-time and finding the second derivative with respect to the strain. Using this procedure, we found a gravitational modulus in (1530) and (1534) for plus and cross polarization waves, respectively. We can consider the case of no electromagnetic fields, $\vec{E},\vec{B}=0$, which also requires $\vec{A}=0$. This means that $\hat{p}=m\hat{v}-q\hat{A}$ reduces to just $\hat{p}=m\hat{v}$. However, for the zero momentum eigenstate, we have $\hat{p}=0$ and therefore $m\hat{v}=0$. This means that there would also be no supercurrents and both (1529) and (1533) reduce to just

$$T_{\oplus,\otimes} = \frac{n_s}{2}\left(\alpha+\frac{n_s\beta}{2}\right)-\frac{5n_s}{4}\left(\alpha+\frac{n_s\beta}{2}\right)h_{\oplus,\otimes} \tag{1539}$$

Correspondingly, using $T_{ij}^{\tau\tau}=-\mu_{G(SC)}h_{ij}^{\tau\tau}$, we find that the effective shear moduli become

$$\mu_{G(CP)}=\frac{e^2n_s}{m_e}A^2+\alpha n_s+\frac{n_s\beta}{2} \tag{1540}$$

$$\mu_{G(CP)}=\frac{5}{4}n_s\left(\alpha+\frac{n_s\beta}{2}\right) \tag{1541}$$

where "CP" represents the Cooper pairs. Since α and β are phenomenological parameters describing the superconductor, then this shear modulus also becomes a phenomenological parameter - an inherent property of the superconductor. It could be considered the irremovable quantum "stiffness" of the Cooper pair density in response to a gravitational wave of arbitrary polarization. It is an *irremovable* stiffness because it remains even in the absence of any supercurrents or electromagnetic fields which could contribute to the effective stiffness.)

It is interesting to compare the gravitational modulus, $\mu_{G(CP)}$, to the coherence length, ξ, which can also be thought of as a type of stiffness of the superfluid density, $|\psi|^2$, where

$$\xi=\frac{\hbar}{\sqrt{2m_e|\alpha|}} \tag{1542}$$

If ξ is small, then the energy cost of n_s (the particle density of Cooper pairs), varying from place to place will be small. If the order parameter has a homogeneous equilibrium value but is somehow changed from this equilibrium value by an external force at one point in space, then ξ specifies the length scale over which it is restored or "healed."[161] The equilibrium value of the Ginzburg-Landau order parameter in the absence of electromagnetic fields is[162]

$$|\psi_0|^2=-\frac{\alpha}{\beta} \tag{1543}$$

[161]This is similar to Hook's law, $F=-k\Delta x$, where k is the stiffness of the spring, Δx is the displacement from equilibrium, and $1/k$ is a characteristic length scale associated with the stiffness and the displacement from equilibrium.

[162]This follows from the fact that minimizing the free energy density in (1443) gives

$$0=\frac{1}{2m_e}\left(-i\hbar\nabla-2e\vec{A}\right)^2\psi+2\alpha\psi+2\beta|\psi|^2\psi+\frac{\vec{B}^2}{2\mu_0}$$

Therefore, when there are no electromagnetic fields, this reduces to just $0=2\alpha|\psi_0|+2\beta|\psi_0|^2\psi_0$. Then solving for $|\psi_0|^2$ gives (1543).

Therefore, it is intuitive that we find $\mu_{G(CP)}$ is also related to the parameters α and β just as the equilibrium value of the order parameter and the coherence length (which describes the stiffness of the superfluid density) are also related to the parameters α and β. In fact, $\mu_{G(CP)}$ can be expressed completely in terms of the coherence length. Using $|\psi_0|^2 = n_s$ in (1543) gives $\beta = -\alpha/n_s$. Then using (1542), we can express α and β as

$$\alpha = \frac{\hbar^2}{2m_e\xi^2} \qquad \text{and} \qquad \beta = -\frac{\hbar^2}{2m_e n_s \xi^2} \tag{1544}$$

We can insert these relations into (1541) to obtain

$$\mu_{G(CP)} = \frac{5\hbar^2 n_s}{16m_e\xi^2} \tag{1545}$$

If each atom contributes two conduction electrons, and only 10^{-3} of the conduction electrons are in a superconducting state, then $n_s \approx 2n\left(10^{-3}\right)$ where $n = \rho_m/m$ is the number density of atoms. For Niobium, the mass density is $\rho_m \approx 8.6\times10^3 kg/m^3$ and the mass per atom is $m \approx 1.5\times10^{-25} kg/atom$. Then the number density of atoms is $n \approx 5.7\times10^{28} m^{-3}$ and therefore the number density of Cooper pairs is

$$n_s \approx 2n\left(10^{-3}\right) \approx 1.1\times10^{26} m^{-3} \tag{1546}$$

Inserting this result in (1545) and using $\xi \approx 39nm$ for the coherence length of Niobium gives

$$\mu_{G(CP)} \approx 2.9\times10^2 J/m^3 \tag{1547}$$

Since the constitutive equation describing the stress in a superconductor due to a gravitational strain is $T_{ij}^{\tau\tau} = -\mu_{G(SC)} h_{ij}^{\tau\tau}$, then we see that for a given gravitational strain, the resulting stress is extremely small. In other words, the Cooper pair density demonstrates very little response to a gravitational wave. By equating these two constitutive equations, the *material* strain can be expressed in terms of the *gravitational* strain as

$$\boxed{u_{ij}^{\tau\tau} = \frac{\mu_{G(CP)}}{s} h_{ij}^{\tau\tau} \qquad \textit{Strain of matter in terms of gravitational strain of space}} \tag{1548}$$

For Niobium, $s \approx 38GPa = 3.8\times10^{10} J/m^3$. Therefore, using (1547) we find that the material strain is related to the gravitational strain by a factor of

$$\frac{\mu_{G(CP)}}{s} \approx 7.6\times10^{-9} \tag{1549}$$

If the amplitude of the gravitational wave is on the order of $\sim 10^{-20}$, then $u_{ij}^{\tau\tau} \sim 10^{-29}$. This demonstrates that the material strain caused by a gravitational wave acting on the Cooper pair density (for a Niobium superconductor with edges on the order of centimeters) is completely negligible. Of course, this result does not include the effects of electromagnetic fields and the associated supercurrents they induce. In fact, since the Cooper pair density will be effectively unresponsive while the lattice ions accelerate freely in the presence of the gravitational wave, then there may arise a charge-separation effect. This will lead to electromagnetic fields being setup (at least within the electromagnetic penetration depth of the super conductor) and therefore $\mu_{G(CP)}$ will need to be expressed in it's full form of (1530) or (1534). However, in the *incipient* state when this charge-separation effect is still forming, (1545) would describe the stiffness of the Cooper density.

Furthermore, we can also determine the time-scale for the stress tensor of the Cooper pair density to equilibrate in the presence of a gravitational wave. Inserting $\mu_{G(CP)} \approx 2.9 \times 10^2 J/m^3$ from (1547) into the time-scale given by (1373) gives

$$T = c\sqrt{\frac{\pi}{8G\mu_G}} \approx 1.3 \times 10^{12} s \approx 4.2 \times 10^4 \; years \tag{1550}$$

This tremendously large time-scale implies that the Cooper pair density is essentially a perfect "gravitational insulator." We can also consider the gravitational plasma frequency for the Cooper pair density. Inserting $\mu_{G(CP)} \approx 2.9 \times 10^2 J/m^3$ into (1321) gives

$$\omega_G = \sqrt{16\pi G\mu_G} \approx 3.3 \times 10^{-9} rad/s \tag{1551}$$

This incredibly small plasma frequency implies that the Cooper pair density is "stiff" all the way down to extremely low frequencies. Since reflection generally occurs for frequencies that satisfy $\omega < \omega_G$, then essentially no time-varying strain field would be expelled by the Cooper pair density. Lastly, if we consider the corresponding penetration depth, then using $\mu_{G(CP)} \approx 2.9 \times 10^2 J/m^3$ gives

$$\delta_G = \frac{c}{\omega_G} \approx 8.9 \times 10^{16} m \tag{1552}$$

This is about 9.6 thousand light years! Therefore, we see that the penetration depth is absurdly huge and there is essentially no attenuation of the gravitational wave field by the Cooper pair density.

We can also return to the expression for $\mu_{G(CP)}$ in (1530) and (1534) for plus and cross polarization, respectively, and evaluate the corresponding gravitational penetration depth. Let us consider a time-varying vector potential given by $A = A_0 \cos(\omega t)$. Then using $\vec{E} = -\partial_t \vec{A}$ gives $E = -\omega A$. Also, since $A_i = -\frac{m}{q} v_i$, then

$$v_i = \frac{e}{\omega m_e} E_i \tag{1553}$$

Inserting this into (1530) and dropping the magnetic field and scalar potential for simplicity gives

$$\begin{aligned}\mu_{\oplus} &= \frac{5}{2} n_s \left[\frac{e^2}{2\omega^2 m_e} E^2 - \frac{2e^2}{5\omega^2 m_e}\left(E_x^2 - E_y^2\right) + \frac{\alpha}{2} + \frac{n_s \beta}{4}\right] \\ &\quad + \frac{1}{2}\left[\frac{5}{4}\left(\varepsilon_0 \vec{E}^2\right) - \varepsilon_0\left(E_x^2 - E_y^2\right)\right]\end{aligned} \tag{1554}$$

We can confine our attention to a single hyperbolic trajectory and define $E_{\oplus}^2 \equiv E_x^2 - E_y^2$ which is constant on the path (although still time-dependent). Then the expression above becomes

$$\mu_{\oplus} = 16\pi G \frac{5}{2} n_s \left[\frac{e^2}{2m_e\omega^2}\left(E^2 - \frac{4}{5}E_{\oplus}^2\right) + \frac{\alpha}{2} + \frac{n_s\beta}{4}\right] + \frac{\varepsilon_0}{2}\left(\frac{5}{4}E^2 - E_{\oplus}^2\right) \tag{1555}$$

14.4 Upper bounds and approximations for fields and supercurrents

Here we consider some upper bounds concerning the relativistic supercurrents and the gravitational and electric fields that can exist in the superconductor without destroying the superconducting state of the system. Both a classical analysis as well as a quantum mechanical analysis is considered. We also examine some approximations that can be used to determine the electric field that would exist in the superconductor as a result of the charge-separation effect.

Classical versus quantum calculation of maximum gravitational wave due to BCS energy gap

The BCS energy gap is the binding energy associated with each Cooper pair due to the quantum mechanical self-coupling interactions that exist in the superconductor. This energy gap is described by the BCS theory of superconductivity as

$$E_{gap} = \tfrac{7}{2} k_B T_C \tag{1556}$$

where T_C is the critical temperature of the superconductor. For the case of Niobium, we have $T_C = 9.3$ K. Therefore to preserve the superconducting state of the Cooper pairs, the maximum power that can be delivered to the superconductor by an incident gravitational wave with frequency $\omega/2\pi$ is $P_{\max} = E_{gap}\omega/2\pi$. Using (1556) gives

$$P_{\max} = \frac{7}{4\pi}\omega k_B T_C \tag{1557}$$

This expression limits the power (and hence the wave amplitude) that is permitted for a gravitational wave incident on the superconductor without destroying the superconducting state of the system. We can use the Isaacson power flux formula as a means of relating the power to the strain field of the gravitational wave. The Isaacson power flux formula is given in [43] as

$$\mathscr{P} = \frac{c^3}{16\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle \tag{1558}$$

where $\mathscr{P}$ is the power per unit area (P/A) while h_+ and $h_\times$ are the plus and cross polarized strain fields. The polarization state is not relevant to our analysis here so we can simply use h_0 for the strain field. The time derivative for an oscillating field can be written as $\dot{h}_0 = \omega h_0$. We can also substitute $\mathscr{P} = P/A$ and solve for h_0. Doing so gives

$$h_0 = \sqrt{\frac{16\pi G P}{c^3\omega^2 A}} \tag{1559}$$

Inserting $P_{\max} = \frac{7}{4\pi}\omega k_B T_C$ from (1557) gives

$$h_{0,\ \max} = \sqrt{\frac{28\pi G k_B T_C}{c^3\omega A}} \tag{1560}$$

Using $T_C = 9.3$K, $\omega \sim 10^{10}$Hz, and $A \sim 10^{10}$ gives

$$h_{0,\max} \approx 10^{-31} \tag{1561}$$

This is the maximum strain field for a gravitational wave that can be incident on the superconductor without destroying any of the Cooper pair states.[163] Then we find that the maximum gravitational wave force on a Cooper pair is approximately

$$F_{\substack{GR\ wave \\ (\max\ -\ \text{classical})}} \approx m_e c h_0 \omega \sim 10^{-43} N \tag{1562}$$

[163] This value is comparable to the value obtained in equ. (40) of [91] which was 10^{-28} based on a 30 GHz gravitational wave with a milliwatt of power incident on an area of 1 cm^2.

On the other hand, if there is a gravitational photo-electric effect, then the energy of a gravitational wave that would destroy a Cooper pair would not be given by the amplitude of the wave but rather by the frequency of the wave using $E = \hbar\omega$. Equating this to (1556), and solving for ω gives

$$\omega = \frac{7k_B T_C}{2\hbar} \approx 7 \times 10^{11} Hz \tag{1563}$$

This implies that as long as the gravitational waves are below 70Ghz, then no Cooper pair states could be destroyed by the gravitational waves.

15 Interaction of gravitational (GR) waves with the lattice ions of a superconductor

15.1 The Debye free energy in the low-temperature limit

A treatment of the Debye model can be found in [80]-[83]. The Debye free energy can be derived by starting from the energy of N harmonic oscillators given by

$$E = \sum_{\alpha=1}^{N} \frac{p_\alpha^2}{2m} + \frac{1}{2}\sum_{\alpha,\beta}^{N} \frac{K}{2}\left|r_\alpha - r_\beta\right|^2 \tag{1564}$$

where α and β are indexing the N atoms being summed over, and K characterizes the harmonic electric potential between the atoms. This can be reformulated into a sum of quantum mechanical vibrational modes. We know that a quantum harmonic oscillator of the form

$$\left(\frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{q}^2\right)\varphi = \hat{E}\varphi \tag{1565}$$

has energy levels given by $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$, with $n = 0$ giving the ground state energy. Likewise, the N harmonic oscillators in (1564) can be written in terms of the energy levels as

$$E = \sum_{\alpha=1}^{3N} \hbar\omega_\alpha n_\alpha \tag{1566}$$

where α is now indexing the oscillators and n_α is the number of phonons with frequency ω_α. (We have dropped the ground state energy, $\frac{1}{2}\hbar\omega$.) Note that the upper limit of the summation is $3N$ to account for the three spatial degrees of freedom for the oscillators. Then the Debye free energy is found to be

$$F = \frac{3k_BTV}{2\pi^2}\int_0^{k_D} \ln\left(1 - e^{-\beta\hbar vk}\right)k^2 dk \tag{1567}$$

where the Debye wave vector and frequency are, respectively,

$$k_D = \left(\frac{6\pi^2 N}{V}\right)^{1/3} \quad \text{and} \quad \omega_D = v\left(\frac{6\pi^2 N}{V}\right)^{1/3} \tag{1568}$$

with N being the number of atoms. Using $k = \omega/v$ and $\beta = (k_BT)^{-1}$ makes (1567) become

$$F = \frac{3V}{2\pi^2\beta v^3}\int_0^{\omega_D} \ln\left(1 - e^{-\beta\hbar\omega}\right)\omega^2 d\omega \tag{1569}$$

We now consider the low temperature limit. For $T \approx 10^{-2}K$, we have $\beta\hbar \approx 1.5 \times 10^{-59}s$. From the graph below, we can see that the integral in (1569) has almost no contribution past $\beta\hbar\omega \approx 0.2$. After that, the graph is asymptotically zero. From (1585) we see that $\omega_D \approx 10^{13}Hz$ and therefore $\beta\hbar\omega_D \approx 1.5 \times 10^{-46}$. This means that to a very good approximation, the integral can be cut off well before every reaching the upper limit of ω_D. Stating it another way, we can extend the upper limit to infinity and still have a very good approximation to the integral.

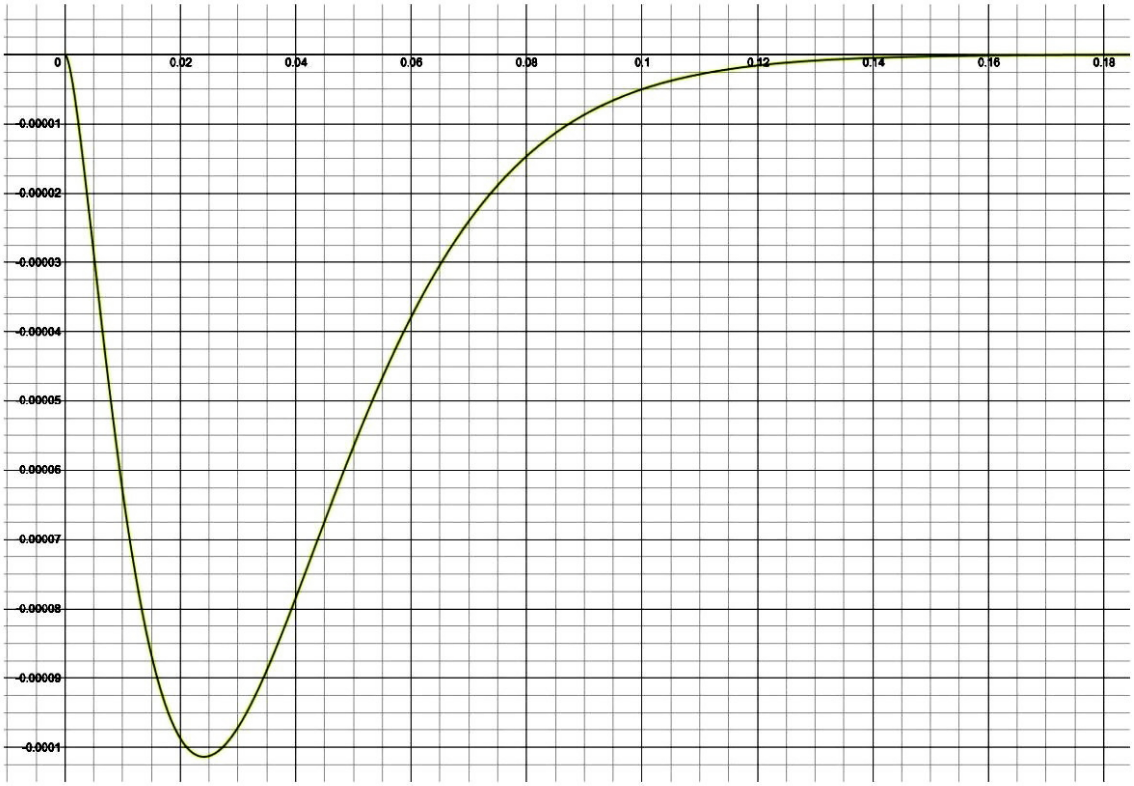

Figure 9: A graph of the free energy of the Debye model in the low-temperature limit.

To evaluate the integral in (1569), we can use integration by parts with

$$U = \ln\left(1 - e^{-\beta\hbar\omega}\right) \qquad dV = \omega^2 d\omega$$
$$dU = \frac{\beta\hbar}{e^{\beta\hbar\omega} - 1} d\omega \qquad V = \frac{1}{3}\omega^3 \tag{1570}$$

Then the integral in (1569) becomes

$$\int_0^{\omega_D} \ln\left(1 - e^{-\beta\hbar\omega}\right)\omega^2 d\omega = \left[\frac{1}{3}\omega^3 \ln\left(1 - e^{-\beta\hbar\omega}\right)\right]_0^{\omega_D} - \frac{\beta\hbar}{3}\int_0^{\omega_D} \frac{\omega^3}{e^{\beta\hbar\omega} - 1} d\omega \tag{1571}$$

We can use a change of variable given by

$$x = \beta\hbar\omega \qquad \text{and} \qquad dx = (\beta\hbar)\, d\omega \tag{1572}$$

Then the upper bound of the integral on the right side of (1571) becomes $x_D = \beta\hbar\omega_D$ and we have[164]

$$\frac{\beta\hbar}{3}\int_0^{\omega_D} \frac{\omega^3}{e^{\beta\hbar\omega} - 1} d\omega = \frac{1}{3\beta^3\hbar^3}\int_0^{x_D} \frac{x^3}{e^x - 1} dx \tag{1573}$$

[164] The Debye temperature is also defined as $T_D \equiv \hbar\omega_D/k_B$, so we could also express the integral in terms of an upper bound given by $x_D = T_D/T$. At very low temperatures where $T << T_D$, only long wavelength acoustic modes are thermally excited. These are modes that can be treated as an elastic continuum with macroscopic elastic constants. The energy of those short wavelength modes are too high to be populated significantly at low temperatures.

The exponential in the denominator becomes extremely large well before x reaches the upper limit of the integral[165]. This means the integrand is extremely small near the upper limit and the integral can be approximated by increasing the limit to infinity. Then we can make use of the standard integral[166]

$$\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15} \tag{1574}$$

Therefore, the free energy in (1569) becomes

$$F \approx \frac{V}{2\pi^2 \beta v^3} \left\{ \left[\omega^3 \ln\left(1 - e^{-\beta\hbar\omega}\right) \right]_0^{\omega_D} - \frac{\pi^4}{15\beta^3\hbar^3} \right\} \tag{1575}$$

Evaluating the first term at $\omega = 0$ requires taking a limit. Rearranging that term and using L'Hospital's rule gives

$$\lim_{\omega\to 0} \frac{\ln\left(1 - e^{-\beta\hbar\omega}\right)}{1/\omega^3} \overset{LH}{=} \lim_{\omega\to 0} \frac{\beta\hbar e^{-\beta\hbar\omega}}{-3\left(1 - e^{-\beta\hbar\omega}\right)\omega^{-4}} = -\frac{\beta\hbar}{3} \lim_{\omega\to 0} \frac{\omega^4}{e^{\beta\hbar\omega} - 1} \tag{1576}$$

Applying L'Hospital's rule again gives

$$-\frac{\beta\hbar}{3} \lim_{\omega\to 0} \frac{\omega^4}{e^{\beta\hbar\omega} - 1} \overset{LH}{=} -\frac{4}{3} \lim_{\omega\to 0} \frac{4\omega^3}{e^{\beta\hbar\omega}} = 0 \tag{1577}$$

Then (1575) becomes

$$F \approx \frac{V}{2\pi^2 \beta v^3} \left[\omega_D^3 \ln\left(1 - e^{-\beta\hbar\omega_D}\right) - \frac{\pi^4}{15\beta^3\hbar^3} \right] \tag{1578}$$

Inserting ω_D from (1568) and distributing gives

$$\boxed{\begin{array}{c} F \approx \dfrac{3N}{\beta} \ln\left(1 - e^{-\beta\hbar\omega_D}\right) - \dfrac{\pi^2 V}{30\beta^4\hbar^3 v^3} \\ \\ \textit{Debye free energy for a low-temperature lattice} \end{array}} \tag{1579}$$

The number of atoms, N, can be found from

$$N = \frac{M}{m} = \frac{\rho V}{m} \tag{1580}$$

where M is the total mass, m is the mass per atom, and ρ is the mass density. Using (1580) we can also write ω_D from (1568) as

$$\omega_D = v \left(\frac{6\pi^2 \rho}{m} \right)^{1/3} \tag{1581}$$

[165] We found in (1585) that the Debye frequency is $\omega_D \approx 3.1 \times 10^{13} Hz$ for Niobium. This means that for $T \approx 10^{-2} K$, we have $x_D = \beta\hbar\omega_D \approx 2.4 \times 10^4$. Therefore $1/e^x$ becomes vanishingly small well before reaching the upper bound.

[166] This procedure is described on p. 157 of [81], p. 200 of [82], and p. 13 of [83].

For a shear wave in Niobium, we can use the material shear modulus, s, to obtain a wave speed.

$$v = \sqrt{\frac{s}{\rho}} \tag{1582}$$

Inserting $(1580)-(1582)$ into (1579) and factoring out the volume gives

$$\boxed{\begin{gathered} F \approx V\left[\frac{3\rho}{\beta m}\ln\left(1-e^{-\beta\hbar\omega_D}\right) - \frac{\pi^2}{30\beta^4\hbar^3}\left(\frac{\rho}{s}\right)^{3/2}\right] \\ \\ \textit{Debye free energy for a low-temperature lattice} \\ \textit{in terms of mass density and material shear modulus} \end{gathered}} \tag{1583}$$

The free energy for low temperature Niobium

For the case of Niobium, the material shear modulus is $s \approx 38GPa = 3.8\times 10^{10}J/m^3$, so the speed of a shear wave according to (1582) is

$$v \;=\; \sqrt{\frac{s}{\rho}} \approx 2.1\times 10^3 m/s \tag{1584}$$

Also, for Niobium we have $m \approx 1.54\times 10^{-25}kg/atom$ and $\rho \approx 8.6\times 10^3 kg/m^3$. Then (1580) and (1581) give

$$\frac{N}{V} \;=\; \frac{\rho}{m} \approx 5.6\times 10^{28}m^{-3}, \qquad k_D \approx 1.5\times 10^{10}m^{-1}, \qquad \omega_D \approx 3.1\times 10^{13}s^{-1} \tag{1585}$$

Note that k_D corresponds to a wavelength of $\lambda = 2\pi/k_D \approx 4.2\times 10^{-10}m$ which is on the order of the interatomic spacing as expected for modes in the Debye model. Also notice that ω_D is above microwave frequencies and therefore our approximation here is valid for microwave frequency oscillations induced in the lattice. For $T \approx 10^{-2}K$, we also have

$$\beta\hbar \;\approx\; 7.6\times 10^{-10}s \qquad \text{and} \qquad \beta\hbar\omega_D \approx 2.4\times 10^4 \tag{1586}$$

We can now evaluate the free energy for Niobium. Using $(1584)-(1586)$, we find that the first and second terms in the bracket of (1583) are

$$\frac{3\rho}{\beta m}\ln\left(1-e^{-\beta\hbar\omega_D}\right) \;\approx\; 0 \qquad \text{and} \qquad \frac{\pi^2}{30\beta^4\hbar^3 v^3} \approx 10^{-8}J/m^3 \tag{1587}$$

Note that the first term is vanishingly small because $\beta\hbar\omega_D$ in (1586) is so large. Therefore (1583) becomes $F \approx -\left(10^{-8}J/m^3\right)V$. This implies that for a superconductor with dimensions on the order of centimeters ($V \approx 10^{-6}m^3$), the total free energy of the lattice is $F \approx -10^{-14}J \approx -100keV$.

15.2 Quantum harmonic oscillators (QHO) coupled to GR waves

To describe the interaction of a gravitational wave with the quantum harmonic oscillators, we can begin with (790) which gives the Hamiltonian for coupling to gravitational waves to first order in the metric.[167]

$$H = mc^2 + \frac{p^2}{2m} - \frac{h_{ij}^{\tau\tau} p^i p^j}{2m} \tag{1588}$$

This Hamiltonian is similar to the interaction Hamiltonian obtained by Rothman and Boughn (RB) in equation (5.6) of [54] and on page 8 of [55]. However, RB employ the transverse-traceless *gauge* which is only valid in vacuum. Instead, we specifically work with the transverse-traceless *part* of the metric, $h_{ij}^{\tau\tau}$, which is the radiative degrees of freedom of the metric. The important feature to note about $h_{ij}^{\tau\tau}$ is that it is a gauge-*invariant* quantity that satisfies a wave equation *in matter* as found using the Helmholtz Decomposition formulation of linearized GR. Therefore, $h_{ij}^{\tau\tau}$ can be used to describe the interaction of gravitational waves with quantum harmonic oscillators *inside the matter*.

Using (1588), we can modify (1564) to write the Hamiltonian for relativistic harmonic oscillators coupled to a gravitational wave as

$$H = \sum_{\alpha=1}^{N} \left(m_\alpha c^2 + \frac{p_\alpha^2}{2m_\alpha} - \frac{h_{ij}^{\tau\tau} \left(p^i p^j\right)_\alpha}{2m_\alpha} \right) + \frac{1}{2} \sum_{\alpha,\beta}^{N} \frac{K}{2} \left| r_\alpha - r_\beta \right|^2 \tag{1589}$$

If we quantize the momentum, $p^i \to \hat{p}^i = -i\hbar\partial^i$, then the term involving the gravitational wave becomes[168]

$$\frac{h_{ij}^{\tau\tau} \hat{p}^i \hat{p}^j}{2m} = \frac{\hbar^2}{2m} h_{ij}^{\tau\tau} \partial^i \partial^j \tag{1590}$$

For a harmonic oscillator, we know that $\hat{p}^i = m\hbar\omega\hat{x}^i$. This means we have

$$\hat{p}^i \hat{p}^j = m^2 \hbar^2 \omega^2 \hat{x}^i \hat{x}^j = m\hbar^2 \omega^2 \hat{D}^{ij} \tag{1591}$$

We can define a “mass quadrupole moment operator” as

$$\hat{D}^{ij} \equiv m\hat{x}^i \hat{x}^j \tag{1592}$$

so that we have

$$\hat{p}^i \hat{p}^j = m\hbar^2 \omega^2 \hat{D}^{ij} \tag{1593}$$

Then (1590) becomes

$$\frac{h_{ij}^{\tau\tau} \hat{p}^i \hat{p}^j}{2m} = \frac{1}{2} \hbar^2 \omega^2 h_{ij}^{\tau\tau} \hat{D}^{ij} \tag{1594}$$

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[167]Another alternative is to use an effective field theory similar to Blencowe in [84]. The Hamiltonian obtained is

$$H = \hbar\omega_0 a^\dagger a \left(1 + \sum_i \lambda_i \frac{q_i}{\Delta_i}\right) + \sum_i \left(\frac{p_i^2}{2m_i} + \frac{1}{2} m_i \omega_i^2 q_i^2\right)$$

Here the coupling to gravity comes in through the term involving λ_i. Blencowe uses this Hamiltonian to describe a gravitational decoherence due to a cosmic gravitational wave background. However, this result involves a second quantization approach where the scalar field has also been quantized (hence the creation and annihilation operators). By contrast, the Debye model does not necessitate a second quantization approach.

[168]Note that here we are taking a semiclassical approach where the gravitational wave field, $h_{ij}^{\tau\tau}$, is a *classical* field while $\hat{p}$ is a quantum operator describing the matter.

Also defining $\hat{x}^i_{\alpha\beta} \equiv \left|\hat{r}_\alpha - \hat{r}_\beta\right|^i$ as the displacement in the i-direction between atom number α and atom number β, we can extend the definition of the "mass quadrupole moment operator" to be

$$\hat{D}^{ij}_{\alpha\beta} \quad \equiv \quad m\left|\hat{r}_\alpha - \hat{r}_\beta\right|^i\left|\hat{r}_\alpha - \hat{r}_\beta\right|^j \tag{1595}$$

This is the mass quadrupole moment between atom number α and atom number β, where m is the mass of either atom.[169] Now we can write the Hamiltonian in (1589) as

$$\boxed{\begin{array}{c}\hat{H} = \sum\limits_{\alpha,\beta}^{N}\left(m_\alpha c^2 + \dfrac{\left(\hat{p}^i_\alpha\right)^2}{2m_\alpha} - \dfrac{\hbar^2\omega^2_\alpha}{2m_\alpha}h^{\tau\tau}_{jk}\hat{D}^{jk}_{\alpha\beta} + \dfrac{K_i}{4}\left(\hat{x}^i_{\alpha\beta}\right)^2\right) \\ \\ \textit{Hamiltonian for } N \textit{ quantum harmonic oscillators} \\ \textit{coupled to a gravitational wave (first order in the metric)}\end{array}} \tag{1596}$$

Here K_i characterizes the harmonic electric potential in the i direction. This expression gives a compact form for expressing the Hamiltonian in terms of the quadrupole moments formed by N quantum harmonic oscillators due to a gravitational wave. Notice that the free index (i) runs from 1 to 3 which means that there is a Hamiltonian for each value of i. In order to develop the Debye free energy density, we will need to consider each of these three Hamiltonian separately. In the case of no gravitational wave, the three Hamiltonians are identical and therefore the entire set of $3N$ quantum harmonic oscillators can be written in terms of a single sum as given in (1566). However, with the introduction of a gravitational wave coupled to the oscillators, this symmetry is broken and we must deal with each Hamiltonian separately for the x-direction, y-direction, and z-direction.

[169] Here we are assuming that the quadrupole consists of identical atoms. For a lattice with differing types of atoms, the expression becomes more complicated.

15.3 Quasi-energies of QHO coupled to gravitational waves

We can return to (1589) and use $x^i \equiv \left|r_\alpha - r_\beta\right|^i$ as the distance between harmonic oscillators. If we consider a single oscillator, then we can drop the summation. In the non-relativistic limit, we can also drop the rest mass energy.[170] Then promoting the Hamiltonian and canonical momentum to quantum operators gives[171]

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{h_{ij}^{\tau\tau}\hat{p}^i\hat{p}^j}{2m} + \frac{K_i}{4}\hat{x}_i^2 \tag{1597}$$

For a wave propagating in the z-direction, $h_{xx}^{\tau\tau} = -h_{yy}^{\tau\tau} = h_\oplus(z,t)$ for plus-polarization as well as $h_{xy}^{\tau\tau} = h_{yx}^{\tau\tau} = h_\otimes(z,t)$ for cross-polarization, with $h_{ij}^{\tau\tau} = 0$ for all other components. Then summing over repeated indices in the expression above gives

$$\hat{H} = (1+h_\oplus)\frac{\hat{p}_x^2}{2m} + (1-h_\oplus)\frac{\hat{p}_y^2}{2m} + h_\otimes\frac{\hat{p}_x\hat{p}_y}{m} + \frac{\hat{p}_z^2}{2m} + \frac{K_i}{4}\hat{x}_i^2 \tag{1598}$$

We can act the Hamiltonian on a state $\Psi(\vec{x},t)$ and use the quantum operator for the Hamiltonian, $\hat{H} = i\hbar\partial_t$, and for the canonical momentum, $\hat{p}_i = -i\hbar\partial_i$.

$$i\hbar\partial_t\Psi(\vec{x},t) = -\frac{\hbar^2}{2m}\left[(1+h_\oplus)\partial_x^2 + (1-h_\oplus)\partial_y^2 + 2h_\otimes\partial_x\partial_y + \partial_z^2\right]\Psi(\vec{x},t) + \frac{K_i}{4}\hat{x}_i^2\Psi(\vec{x},t) \tag{1599}$$

We can consider a separable solution given by $\Psi(\vec{x},t) = \psi(\vec{x})\varphi(t)$. Inserting this in the Hamiltonian and dividing by $\Psi(\vec{x},t) = \psi(\vec{x})\varphi(t)$ gives

$$i\hbar\frac{1}{\varphi(t)}\partial_t\varphi(t) = -\frac{\hbar^2}{2m}\frac{1}{\psi(\vec{x})}\left[(1+h_\oplus)\partial_x^2 + (1-h_\oplus)\partial_y^2 + 2h_\otimes\partial_x\partial_y + \partial_z^2\right]\psi(\vec{x}) + \frac{K_i}{4}\hat{x}_i^2 \tag{1600}$$

We can equate the left side to $E(t)$ which is the energy as a function of time. Then we have

$$i\hbar\frac{1}{\varphi(t)}\partial_t\varphi(t) = E(t) \qquad \Rightarrow \qquad \partial_t\varphi(t) = -\frac{iE(t)}{\hbar}\varphi(t) \tag{1601}$$

If we define

$$\varepsilon(t) \equiv \int_0^t E(t')\,dt' \tag{1602}$$

[170] It was found in (778) that remaining to order p^2 requires remaining to first order in $h_{ij}^{\tau t}$. Going to second order in $h_{ij}^{\tau t}$ would lead to terms involve p^4 which is beyond the non-relativistic limit considered here. Furthermore, we will being using a formulation involving the energy eigenvalues of a harmonic oscillator which necessitates remianing to order p^2 in momentum.

[171] We are taking a semiclassical approach where the gravitational wave field, $h_{ij}^{\tau\tau}$, is a *classical* field while $\hat{p}$ is a quantum operator describing the matter. Also note that the *internal* electromagnetic field between atoms is accounted for by the harmonic potential term, $\frac{K}{4}\hat{x}_i^2$. However, if there is also an *external* electromagnetic field, then we would need to account for this by using the minimal coupling rule, $\pi^i = p^i - qA^i$. This leads to a more complicated formulation which is handled in a later section.

then the solution to the differential equation above has the form

$$\varphi(t) = e^{-i\varepsilon(t)t/\hbar} \tag{1603}$$

Hence we find that the gravitational wave will introduce a phase shift to the quantum wave function[172] determined by $\varepsilon(t)$. Returning to (1600), we can equate the right side to $E(t)$ and multiply through by $\psi(\vec{x})$.

$$\begin{aligned} E(t)\,\psi(\vec{x}) &= -\frac{\hbar^2}{2m}\left[(1+h_\oplus)\,\partial_x^2 + (1-h_\oplus)\,\partial_y^2 + 2h_\otimes(z,t)\,\partial_x\partial_y + \partial_z^2\right]\psi(\vec{x}) \\ &\quad + \frac{K_i}{4}\hat{x}_i^2\,\psi(\vec{x}) \end{aligned} \tag{1604}$$

Notice that this is *not* a time-*independent* Schrödinger equation. The states are time-independent, but the coefficients, $h_\oplus(z,t)$ and $h_\otimes(z,t)$, make the Hamiltonian time-*dependent*. We can separate the time-dependent terms of the Hamiltonian by writing the expression above as

$$\begin{aligned} E(t)\,\psi(\vec{x}) &= \left[-\frac{\hbar^2}{2m}\left(\partial_x^2+\partial_y^2+\partial_z^2\right) + \frac{K_i}{4}\hat{x}_i^2\right]\psi(\vec{x}) \\ &\quad -\frac{\hbar^2}{2m}\left[h_\oplus\partial_x^2 - h_\oplus\partial_y^2 + 2h_\otimes\partial_x\partial_y\right]\psi(\vec{x}) \end{aligned} \tag{1605}$$

The first line can now be considered as an *unperturbed* Hamiltonian, $\hat{H}_0$, while the second line is a time-*dependent* perturbation, $\hat{H}_1(t)$, so that we have

$$\hat{H}(t) = \hat{H}_0 + \hat{H}_1(t) \tag{1606}$$

where

$$\hat{H}_0 = -\frac{\hbar^2}{2m}\left(\partial_x^2+\partial_y^2+\partial_z^2\right) + \frac{K_i}{4}\hat{x}_i^2 \tag{1607}$$

and

$$\hat{H}_1(t) = -\frac{\hbar^2}{2m}\left[h_\oplus\partial_x^2 - h_\oplus\partial_y^2 + 2h_\otimes(z,t)\,\partial_x\partial_y\right] \tag{1608}$$

Ordinarily, we would need to apply time-dependent perturbation theory to this problem. However, for a gravitational wave with *periodic* time-dependence (such as a standing or traveling sinusoidal wave), then the full Hamiltonian (with the perturbation) has a *periodic* behavior. This means that the Hamiltonian satisfies the condition

$$\hat{H}(t+T) = \hat{H}(t) \tag{1609}$$

Therefore, we can applying Floquet's theorem leads to obtain quasi-energy eigenvalues as described by Zel'dovich [86] and later with further detail by Sambe [87].

[172] This was also pointed out by Stodolsky [64] and Sorge [85].

Zel'dovich points out that for a Hamiltonian satisfying the condition in (1609), we can find periodic solutions given by[173]

$$\varphi_\alpha(t+T) = e^{-i\alpha}\varphi_\alpha(t) \tag{1610}$$

Then the corresponding quasi-energy eigenvalues are given by[174]

$$E = \hbar\alpha/T, \qquad T = 2\pi/\omega \tag{1611}$$

In our case, we know that ω will be the frequency of the gravitational wave which is causing the time-dependent perturbation.[175] We can define the quasi-energy eigenstates of the system as $\Psi_\alpha(\vec{x},t) \equiv \psi_\alpha(\vec{x})\varphi_\alpha(t)$ where $\varphi_\alpha(t)$ satisfies the condition required by (1610). Now the quasi-energy eigenvalues in (1611) simply require that we determine α such that the condition given by (1610) is satisfied. Evaluating (1603) at $t+T$ gives

$$\varphi(t+T) = e^{-i\varepsilon(t+T)/\hbar} \tag{1612}$$

To evaluate $\varepsilon(t+T)$, we first need to determine $\varepsilon(t)$ using (1602). This means we need a function for $E(t)$. Using (1330) ,we can consider the case of standing gravitational waves in the z-direction given by

$$h_\oplus(z,t) = A_\oplus\cos(kz)\sin(\omega t) \qquad \text{and} \qquad h_\otimes(z,t) = A_\otimes\cos(kz)\sin(\omega t) \tag{1613}$$

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[173]Zel'dovich states, "With respect to the operator $H(t)$, states of this kind play the same role as the stationary states do for a constant Hamiltonian. In this case however, the expansion coefficients of an arbitrary function $\psi(x)$ with respect to the states $\psi_\alpha(x,t)$ will depend on the time t, and their absolute values will exhibit a periodic time dependence (with period T). On the contrary, an arbitrary exact solution $\psi(x,t)$ of the Schrödinger equation with the Hamiltonian $H(t)$ will have constant expansion coefficients when expanded with respect to the states $\psi_\alpha(x,t)$." The key point being made here is that by choosing to express our solutions in terms of $\psi_\alpha(x,t)$ in (1610), we pay the price by the fact that any general solution expanded with respect to $\psi_\alpha(x,t)$ will have time-*dependent* coefficients. In a sense, we have shifted the time dependence from the energy eigen-values to the expansion coefficients.

[174]The validity of (1611) can be easily verified by considering the trivial case where $E(t)$ is constant in time. In that case, $\varepsilon(t) = \int E(t)\,dt = Et$ and (1603) becomes $\varphi(t) = e^{-iEt/\hbar}$. If we separate the unperturbed energy, E_0, from the perturbation energy, $E_1(t)$, as shown in (1606), then we have

$$\varphi(t) = e^{-i(E_0+E_1(t))t/\hbar} = e^{-iE_0t/\hbar}e^{-iE_1(t)t/\hbar}$$

Evaluating $\varphi(t+T)$ gives

$$\varphi_\alpha(t+T) = e^{-iE_0(t+T)/\hbar}e^{-iE_1(t+T)t/\hbar} = e^{-i(E_0+E_1)T/\hbar}e^{-i(E_0+E_1)t/\hbar} = e^{-i\alpha}\varphi_\alpha(t)$$

where $\alpha = (E_0+E_1)T/\hbar$ and $\varphi_\alpha(t) = e^{-i(E_0+E_1)t/\hbar}$. Therefore we have satisfied the form required in (1610). From (1611), this means that the quasi-energy eigenvalues are simply $E = E_0 + E_1$. This is precisely what would be expected from (1606).

[175]Zel'dovich makes the assertion that "The totality of all linearly independent solutions with different quasi-energies forms a complete set of functions (at any given instant of time)." This fact suits our purpose here since we are interested in ultimately determining the free energy of the system *at a particular instant of time* so that we can take the derivative with respect to the strain (at that same instant of time) and find the gravitational shear modulus. We do not ultimately need a free energy that is a function of time.

Here we are using $A_{\oplus}$ and $A_{\otimes}$ for the amplitudes for *plus*-polarization and *cross*-polarization waves, respectively.[176] We can insert (1613) into (1598) and write the expression in terms of the eigenvalues of the energy and the eigenvalues of momenta.

$$\begin{aligned} E(t) &= (1+A_{\oplus}\cos(kz)\sin(\omega t))\frac{p_x^2}{2m} + (1-A_{\oplus}\cos(kz)\sin(\omega t))\frac{p_y^2}{2m} \\ &\quad + A_{\otimes}\cos(kz)\sin(\omega t)\frac{p_x p_y}{m} + \frac{p_z^2}{2m} + \frac{K_i}{4}x_i^2 \end{aligned} \tag{1614}$$

Then evaluating $\varepsilon(t) = \int_0^t E(t')\,dt'$ gives

$$\begin{aligned} \varepsilon(t) &= \left[t - \frac{A_{\oplus}}{\omega}\cos(kz)\cos(\omega t)\right]\frac{p_x^2}{2m} + \left[t + \frac{A_{\oplus}}{\omega}\cos(kz)\cos(\omega t)\right]\frac{p_y^2}{2m} \\ &\quad - \frac{A_{\otimes}}{\omega}\cos(kz)\cos(\omega t)\frac{p_x p_y}{m} + \left(\frac{p_z^2}{2m} + \frac{K_i}{4}x_i^2\right)t \\ &\quad - \frac{\cos(kz)}{2m\omega}\left(A_{\oplus}p_x^2 - A_{\oplus}p_y^2 - 2A_{\otimes}p_x p_y\right) \end{aligned} \tag{1615}$$

Now evaluating $\varepsilon(t+T)$ and using the fact that $\cos[\omega(t+T)] = \cos(\omega t)$ gives

$$\begin{aligned} \varepsilon(t+T) &= \left[t + T - \frac{A_{\oplus}}{\omega}\cos(kz)\cos(\omega t)\right]\frac{p_x^2}{2m} \\ &\quad + \left[t + T + \frac{A_{\oplus}}{\omega}\cos(kz)\cos(\omega t)\right]\frac{p_y^2}{2m} \\ &\quad - \frac{A_{\otimes}}{\omega}\cos(kz)\cos(\omega t)\frac{p_x p_y}{m} + \left(\frac{p_z^2}{2m} + \frac{K_i}{4}x_i^2\right)(t+T) \\ &\quad + \frac{\cos(kz)}{2m\omega}\left(A_{\oplus}p_x^2 - A_{\oplus}p_y^2 + 2A_{\otimes}p_x p_y\right) \end{aligned} \tag{1616}$$

[176] Alternatively, it is also possible to use a gravitational wave given by (1332) which could be written as

$$h_{\oplus}(z,t) = A_{\oplus}e^{-z/\delta_G}\sin(\omega t) \qquad \text{and} \qquad h_{\otimes}(z,t) = A_{\otimes}e^{-z/\delta_G}\sin(\omega t)$$

where δ_G is the gravitational penetration depth.

To satisfy the form required by (1610), we need to exponentiate this result and separate the time-dependent part from the time-independent part. We can define the time-dependent part as

$$
\begin{aligned}
f(t) \quad \equiv \quad & \left[t - \frac{A_{\oplus}}{\omega}\cos(kz)\cos(\omega t)\right]\frac{p_x^2}{2m} + \left[t + \frac{A_{\oplus}}{\omega}\cos(kz)\cos(\omega t)\right]\frac{p_y^2}{2m} \\
& - \frac{A_{\otimes}}{\omega}\cos(kz)\cos(\omega t)\frac{p_x p_y}{m} + \left(\frac{p_z^2}{2m} + \frac{K_i}{4}x_i^2\right)t
\end{aligned} \tag{1617}
$$

We can also define the time-independent part as

$$
g(T) \quad \equiv \quad \left[\frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + \frac{K_i}{4}x_i^2\right]T + \frac{\cos(kz)}{2m\omega}\left[A_{\oplus}\left(p_x^2 - p_y^2\right) + 2A_{\otimes}p_x p_y\right] \tag{1618}
$$

Then exponentiating (1616) and expressing it in terms of $f(t)$ and $g(t)$ gives

$$
e^{\varepsilon(t+T)} = e^{g(T)+f(t)} = e^{g(T)}e^{f(t)} \tag{1619}
$$

Therefore, we can write (1612) as

$$
\varphi(t+T) \quad = \quad e^{-ig(T)/\hbar}e^{-if(t)/\hbar} \tag{1620}
$$

Comparing this to $\varphi_{\alpha}(t+T) = e^{-i\alpha}\varphi_{\alpha}(t)$ from (1610), we find that

$$
\alpha \quad = \quad g(T)/\hbar \qquad \text{and} \qquad \varphi_{\alpha}(t) = e^{-if(t)/\hbar} \tag{1621}
$$

We now have an expression for α using (1618). We can insert α into the quasi-energy eigenvalues given in (1611) as $E = \hbar\alpha/T$ where $T = 2\pi/\omega$. This gives

$$
\boxed{E = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + \frac{K_i}{4}x_i^2 + \frac{\cos(kz)}{4\pi m}\left[A_{\oplus}\left(p_x^2 - p_y^2\right) + 2A_{\otimes}p_x p_y\right]} \tag{1622}
$$

Quasi-energy eigenvalues for a quantum harmonic oscillator in the presence of a gravitational wave in the z-direction

The expression above has a form that can be interpreted as $E = E_0 + E_1$. This is consistent with (1606) which gives the full Hamiltonian as $\hat{H} = \hat{H}_0 + \hat{H}_1(t)$, where $\hat{H}_0$ is the unperturbed Hamiltonian and $\hat{H}_1(t)$ is the time-dependent perturbation. Also notice that the energy eigenvalues have a periodic dependence on z which is expected since the gravitational wave is in the z-direction.

There are essentially two options for writing the entire energy of the system. One option is to recognize that the unperturbed energy, E_0, has the form of a spherically symmetric 3-dimensional quantum harmonic oscillator, and then use the known energy eigenvalues for such a system. In that case, the perturbation energy, E_1, will simply remain expressed in terms of the momenta. However, this will complicate the process of applying the Debye model which ordinarily begins with the energy eigenvalue of each quantum harmonic oscillator, $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$, and then sums over all N oscillators in the x, y, and z directions as shown in (1566). If part of the energy is expressed in terms of n and part of it is expressed in terms of momentum eigenvalues, then we will not be able to follow the usual process for applying the Debye model.

Therefore, we consider a second option which is to express the entire energy, $E = E_0 + E_1$, as a single *modified* 3-dimensional quantum harmonic oscillator. The *plus*-polarization wave, $A_\oplus$, will have the effect of changing the spherical symmetry to an elliptical symmetry. The *cross*-polarization wave, $A_\otimes$, will have the effect of coupling the momentum in the x-direction and the y-direction. In both cases, we find that the factor of $\cos(kz)$ will prevent the entire equation from being separable into three independent differential equations for x, y, and z. However, we can try a similar approach to the one used above which made use of the *temporal* periodicity of the energy. Instead, we would make use of the *spatial* periodicity of the energy in the z-direction. Specifically, Block's theorem states that for a periodic potential

$$V(z+a) = V(z) \tag{1623}$$

where a is a lattice spacing, there are periodic solutions given by

$$\psi(z) = e^{ik_z z}u(z) \qquad \text{where} \qquad u(z+a) = u(z) \tag{1624}$$

It is evident from (1622) that the periodicity of the energy in the z-direction can be described by $k_{GR}z \rightarrow k_{GR}z + 2\pi n$, or equivalently, $z \rightarrow z + 2\pi n/k_{GR}$, where k_{GR} is the wave number of the gravitational wave. Therefore, we can identify the "lattice spacing" due to the periodic form of the gravitational wave as

$$a = 2\pi/k_{GR} = \lambda_{GR} \tag{1625}$$

where λ_{GR} is the wavelength of the gravitational wave. Writing (1624) in a form that is more analogous to the quasi-energy eigenstates in (1610) gives

$$\psi_\beta(z+\lambda_{GR}) = e^{i\beta}\psi_\beta(z) \tag{1626}$$

If we also evaluate (1624) for $z+\lambda_{GR}$ and use the fact that $u(z+\lambda_{GR}) = u(z)$, then we have

$$\psi(z+\lambda_{GR}) = e^{ik_z(z+\lambda_{GR})}u(z) \tag{1627}$$

Also analogous to the quasi-energy eigenvalues in (1611), we might anticipate that the corresponding quasi-momentum eigenvalues in the z-direction would be given by

$$p_z = \hbar\beta/\lambda_{GR}, \qquad \lambda_{GR} = 2\pi/k_{GR} \tag{1628}$$

Returning to (1622), we can consider the case of a thin film[177] where $z << \lambda$ so that $kz << 1$ and therefore $\cos(kx) \approx 1$. We can also insert $(\hat{x}_i)^2 = \hat{x}^2 + \hat{y}^2 + \hat{z}^2$ and use K_x, K_y and K_z to characterize the harmonic electric potential in the x, y, and z-directions, respectively. Then writing (1622) as an operator equation and acting on a state $\psi(\vec{x})$ gives

$$\begin{aligned} E\psi(\vec{x}) \quad &= \quad \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2\right)\psi(\vec{x}) + \frac{1}{4}\left(K_x\hat{x}^2 + K_y\hat{y}^2 + K_z\hat{z}^2\right)\psi(\vec{x}) \\ &\quad + \frac{1}{4\pi m}\left[A_\oplus\left(\hat{p}_y^{2x} - \hat{p}_y^2\right) + 2A_\otimes\hat{p}_x\hat{p}_y\right]\psi(\vec{x}) \end{aligned} \tag{1629}$$

We can use $\hat{p}_i = -i\hbar\partial_i$ and insert a separable solution, $\psi(\vec{x}) = \psi_x(x)\,\psi_y(y)\,\psi_z(z)$. After distributing, we

[177]Note that the gravitational waves in (1613) could also be written with a factor e^{-z/δ_G}, where δ_G is a gravitational penetration depth. In that case, it would be necessary to impose the additional condition $z << \delta_G$ in order to approximate $e^{-z/\delta_G} \approx 1$ and hence obtain a result that has the form of quantum harmonic oscillator.

have

$$E\psi_x\psi_y\psi_z = -\frac{\hbar^2}{2m}\left(\psi_y\psi_z\partial_x^2\psi_x + \psi_x\psi_z\partial_y^2\psi_y + \psi_x\psi_y\partial_z^2\psi_z\right)$$

$$+\frac{1}{4}\left(K_x\hat{x}^2 + K_y\hat{y}^2 + K_z\hat{z}^2\right)\psi_x\psi_y\psi_z$$

$$-\frac{\hbar^2}{4\pi m}\left[A_\oplus\left(\psi_y\psi_z\partial_x^2\psi_x - \psi_x\psi_z\partial_y^2\psi_y\right) + 2A_\otimes\psi_z\partial_x\partial_y\left(\psi_x\psi_y\right)\right] \tag{1630}$$

We can divide by $\psi_x\psi_y\psi_z$ and group terms to identify harmonic oscillators in x, y, and z.

$$E = \left[-\frac{\hbar^2}{2m}\left(1+\frac{A_\oplus}{2\pi}\right)\frac{1}{\psi_x}\partial_x^2\psi_x + \frac{1}{4}K_x\hat{x}^2\right]$$

$$+\left[-\frac{\hbar^2}{2m}\left(1-\frac{A_\oplus}{2\pi}\right)\frac{1}{\psi_y}\partial_y^2\psi_y + \frac{1}{4}K_y\hat{y}^2\right]$$

$$+\left[-\frac{\hbar^2}{2m}\frac{1}{\psi_z}\partial_z^2\psi_z + \frac{1}{4}K_z\hat{z}^2\right]$$

$$-\frac{\hbar^2 A_\otimes}{2\pi m}\frac{1}{\psi_x\psi_y}\partial_x\partial_y\left(\psi_x\psi_y\right) \tag{1631}$$

In the last term we find there is a coupling of the differential equations in x and y. To avoid this, we can choose to consider the case of a *plus*-polarization gravitational wave so we can set $A_\otimes = 0$. Then we can separate the differential equation into the following three differential equations.

$$E_x = -\frac{\hbar^2}{2m}\left(1+\frac{A_\oplus}{2\pi}\right)\frac{1}{\psi_x}\frac{\partial^2\psi_x}{\partial x^2} + \frac{K_x}{4}\hat{x}^2 \tag{1632}$$

$$E_y = -\frac{\hbar^2}{2m}\left(1-\frac{A_\oplus}{2\pi}\right)\frac{1}{\psi_y}\frac{\partial^2\psi_y}{\partial y^2} + \frac{K_y}{4}\hat{y}^2 \tag{1633}$$

$$E_z = -\frac{\hbar^2}{2m}\frac{1}{\psi_z}\frac{\partial^2\psi_z}{\partial z^2} + \frac{K_z}{4}\hat{z}^2 \tag{1634}$$

For brevity, we can define

$$A_\oplus^\pm \equiv 1 \pm \frac{A_\oplus}{2\pi} \tag{1635}$$

Also rewriting the energies above in terms of momentum operators and rearranging gives

$$\frac{E_x}{A_\oplus^+}\psi_x = \frac{\hat{p}_x^2}{2m}\psi_x + \frac{K_x}{4A_\oplus^-}\hat{x}^2\psi_x \tag{1636}$$

$$\frac{E_y}{A_\oplus^-}\psi_y = \frac{\hat{p}_y^2}{2m}\psi_y + \frac{K_y}{4A_\oplus^+}\hat{y}^2\psi_y \tag{1637}$$

$$E_z\psi_z = \frac{\hat{p}_z^2}{2m}\psi_z + \frac{K_z}{4}\hat{z}^2\psi_z \tag{1638}$$

We know that a quantum harmonic oscillator has the form

$$\hat{H}\psi = \left(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right)\psi \tag{1639}$$

Therefore, to put $(1636)-(1638)$ into the form of a quantum harmonic oscillator, we can define $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ as[178]

$$\frac{1}{2}m\omega_x^2\tilde{x}^2 \equiv \frac{K}{4A_\oplus^+}\hat{x}^2, \qquad \frac{1}{2}m\omega_y^2\tilde{y}^2 \equiv \frac{K}{4A_\oplus^-}\hat{y}^2, \qquad \frac{1}{2}m\omega_z^2\tilde{z}^2 \equiv \frac{K}{4}\hat{z}^2 \tag{1640}$$

Then $(1636)-(1638)$ become

$$\frac{E_x}{A_\oplus^+}\psi_x = \frac{\hat{p}_x^2}{2m}\psi_x + \frac{1}{2}m\omega_x^2\tilde{x}^2\psi_x \tag{1641}$$

$$\frac{E_y}{A_\oplus^-}\psi_y = \frac{\hat{p}_y^2}{2m}\psi_y + \frac{1}{2}m\omega_y^2\tilde{y}^2\psi_y \tag{1642}$$

$$E_z\psi_z = \frac{\hat{p}_z^2}{2m}\psi_z + \frac{1}{2}m\omega_y^2\tilde{z}^2\psi_z \tag{1643}$$

For a quantum harmonic oscillator, we need to insure $[\tilde{x},\hat{p}_x] = [\tilde{y},\hat{p}_y] = [\tilde{z},\hat{p}_z] = i\hbar$. First, we can solve (1640) for $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ which gives

$$\tilde{x} = \sqrt{\frac{K}{2m\omega_x^2A_\oplus^+}}\hat{x}, \qquad \tilde{y} = \sqrt{\frac{K}{2m\omega_y^2A_\oplus^-}}\hat{y}, \qquad \tilde{z} = \sqrt{\frac{K}{2m\omega_z^2}}\hat{z} \tag{1644}$$

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[178]Note that we can drop the subscripts on K_x, K_y, and K_z which were used to distinguish the different effective "spring constants" in the x, y, and z directions due to the effect of the gravitational wave. Now we can use a single constant K for all directions since the effect of the gravitational wave in the x, y, and z directions is shown explicitly in (1640) in terms of $A_\oplus^-$ and $A_\oplus^+$. The subscripts of ω_x, ω_y, and ω_z still represent the fact that the frequencies are modulated differently in the x, y, and z directions due to the gravitational wave.

Then evaluating $[\tilde{x},\hat{p}_x]$, $[\tilde{y},\hat{p}_y]$, and $[\tilde{z},\hat{p}_z]$ gives

$$[\tilde{x},\hat{p}_x] = \sqrt{\frac{K}{2m\omega_x^2 A_\oplus^+}}i\hbar, \qquad [\tilde{y},\hat{p}_y] = \sqrt{\frac{K}{2m\omega_y^2 A_\oplus^-}}i\hbar, \qquad [\tilde{z},\hat{p}_z] = \sqrt{\frac{K}{2m\omega_z^2}}i\hbar \tag{1645}$$

If we require $[\tilde{x},\hat{p}_x] = [\tilde{y},\hat{p}_y] = [\tilde{z},\hat{p}_z] = i\hbar$, then the left side of each equation above is $i\hbar$. We can define the *unmodulated* frequency in the absence of the gravitational wave as[179]

$$\omega = \sqrt{\frac{K}{2m}} \tag{1646}$$

Then the expressions in (1645) become

$$\omega_x = \frac{\omega}{\sqrt{A_\oplus^+}}, \qquad \omega_y = \frac{\omega}{\sqrt{A_\oplus^-}}, \qquad \omega_z = \omega \tag{1647}$$

Since a quantum harmonic oscillator has energy levels given by $E_n = \left(n+\frac{1}{2}\right)\hbar\omega$, with $n=0$ giving the ground state energy, then the left side of each equation in (1641) – (1643) is playing the role of E_n. Therefore we have

$$E_{n_x} = A_\oplus^+\left(n_x+\tfrac{1}{2}\right)\hbar\omega_x, \qquad E_{n_y} = A_\oplus^-\left(n_y+\tfrac{1}{2}\right)\hbar\omega_y, \qquad E_{n_z} = \left(n_z+\tfrac{1}{2}\right)\hbar\omega_z \tag{1648}$$

where n_x, n_y, and n_z are numbering the modes with frequencies ω_x, ω_y, and ω_z in the x, y, and z directions, respectively.[180] We can sum over N oscillators, with n_α being the number of phonons with frequency ω_a. This gives[181]

$$E_{n_x} = \sum_\alpha^N A_\oplus^+\hbar\omega_{x,\alpha}\left(n_{x,\alpha}+\tfrac{1}{2}\right) \tag{1649}$$

$$E_{n_y} = \sum_\alpha^N A_\oplus^-\hbar\omega_{y,\alpha}\left(n_{y,\alpha}+\tfrac{1}{2}\right) \tag{1650}$$

$$E_{n_z} = \sum_\alpha^N \hbar\omega_{z,\alpha}\left(n_{z,\alpha}+\tfrac{1}{2}\right) \tag{1651}$$

[179] As in the case of the standard Debye model (with no coupling to gravitational waves), we assume that K is the same in each direction for an isotropic lattice (in the absence of a gravitational wave).

[180] Note that the energy eigenvalues of the quantum harmonic oscillators in the z-direction are not affected by the presence of a gravitational wave propagating in the z-direction. This is expected since a gravitational wave is a *transverse* wave and therefore only the energy of the oscilaltors in the x-direction and y-direction should be affected.

[181] We could also drop the zero-point energy, $\frac{1}{2}\hbar\omega$, since it does not affect the thermodynamic quantities that are derived from the partition function (such as the entropy, specific heat, etc.) However, because the zero-point energy does *not* vanish from the *free energy* (which we will be interested in calculating later), then we opt to keep it in the expression for the energy eigenvalues.

We can write the total energy as $E = E_{n_x} + E_{n_y} + E_{n_z}$. Inserting (1647) and factoring out ω_α gives

$$\boxed{\begin{array}{c} E_{\left(n_x,n_y,n_z\right)} = \hbar \sum_{\alpha}^{N} \omega_\alpha \left[\sqrt{A_{\oplus}^{+}} \left(n_{x,\alpha} + \frac{1}{2}\right) + \sqrt{A_{\oplus}^{-}} \left(n_{y,\alpha} + \frac{1}{2}\right) + \left(n_{z,\alpha} + \frac{1}{2}\right) \right] \\ \textit{Quasi-energy eigenvalues in terms of phonon modes for} \\ \textit{N quantum harmonic oscillators in the presence of a gravitational wave} \\ \textit{with plus-polarization, propagating in the z-direction} \end{array}} \tag{1652}$$

Ordinarily, an isotropic ensemble of 3-D harmonic oscillators has $\omega_x = \omega_y = \omega_z = \sqrt{K/m}$. This allows the total energy to be written simply as $\hbar\omega_\alpha \left(n_\alpha + \frac{1}{2}\right)$ summed over $3N$ oscillators as shown in (1566). However, here we find that the coupling to a gravitational wave breaks the isotropy and prevents this simplification. In particular, (1647) shows $\omega_x \neq \omega_y \neq \omega_z$ which means there is no isotropy and therefore we cannot simplify (1652).

The gravitational wave (propagating in the z-direction) essentially squeezes/stretches space in the x and y directions. This effectively shortens/lengthens the boundary conditions for the quantum harmonic oscillators in the x and y directions, and as a result, modulates the frequencies in the x and y directions. Therefore, the factors, $\sqrt{A_{\oplus}^{+}}$ and $\sqrt{A_{\oplus}^{-}}$, can be considered as essentially "gravitational modulation factors" which determine the modulation of the frequencies (and corresponding energies) for the quantum harmonic oscillators in the x and y directions.

Since there are different prefactors in the x, y, and z directions (which are $\sqrt{A_{\oplus}^{+}}$, $\sqrt{A_{\oplus}^{-}}$, and unity, respectively), it is evident that the gravitational wave also breaks the spatial isotropy of the ionic lattice. In fact, because the gravitational wave couples to the zero-point energy of the oscillators ($\frac{1}{2}\hbar\omega$), this implies that the gravitational wave also breaks the isotropy of the *vacuum* as well, and therefore introduces anisotropy to the ground state vacuum energy of the ionic lattice.

Since the gravitational wave is dynamic, then it *dynamically* modulates the zero-point energies of the phonon modes. Therefore, this can be thought of as a type of dynamical Casimir effect which is driven by a gravitational wave. (This is analogous to the mechanical oscillation of conducting plates which leads to the standard electromagnetic dynamical Casimir effect.) Since the zero-point energy is that of lattice phonons, then effect might be referred to as "dynamical gravito-phonon Casimir effect."

The physical meaning of this dynamical Casimir effect is that an increase in the occupation number of the phonon modes of the lattice is predicted to occur in the presence of the gravitational wave. This effect could be interpreted as a quantum-mechanical analog of a "Weber-bar effect" where the amplitude of the sound waves in the lattice grows in the classical limit due to the coupling of energy from the gravitational wave into energy in the modes of the lattice. However, in this case, the gravitational wave energy incident upon the superconductor is coupled to the *vacuum energy* of the ionic lattice and then converted into sound wave energy in the lattice.

15.4 The Debye free energy in curved space-time

We can now evaluate the free energy using the energy eigenvalues found in (1652). Recall that the partition function for a canonical ensemble is given by

$$Z = \sum_n \exp\left(-\beta E_n\right) \tag{1653}$$

where E_n are the energy eigenvalues values of the system. Then substituting (1652) into the partition function gives[182]

$$Z = \sum_n \exp\left\{-\beta\hbar\sum_\alpha^N \omega_\alpha\left[\sqrt{A^-_\oplus}\left(n_{x,\alpha}+\tfrac{1}{2}\right)+\sqrt{A^+_\oplus}\left(n_{y,\alpha}+\tfrac{1}{2}\right)+\omega_{z,\alpha}\left(n_{z,\alpha}+\tfrac{1}{2}\right)\right]\right\} \tag{1654}$$

We can also simplify the expression by first summing over the zero-point energies of all N oscillators in the x, y, and z direction. Then the zero-point energy of the entire system can be described by a partition function, Z_0, found to be[183]

$$Z_0 = e^{-\beta E_0} \qquad \text{where} \qquad E_0 = \tfrac{1}{2}\hbar N\omega\left(\sqrt{A^-_\oplus}+\sqrt{A^+_\oplus}+1\right) \tag{1655}$$

Here we are using ω_0 to represent the zero-point energy frequency. Factoring Z_0 out of (1654) gives

$$Z = Z_0\sum_n \exp\left[-\beta\hbar\sum_\alpha^N \omega_\alpha\left(\sqrt{A^-_\oplus}n_{x,\alpha}+\sqrt{A^+_\oplus}n_{y,\alpha}+n_{z,\alpha}\right)\right] \tag{1656}$$

We would like to simplify the expression so that it does not contain two summations. Since the exponential of a sum is the product of exponentials, then we have

$$Z = Z_0\sum_{n_1,n_2,\cdots n_\alpha,\cdots n_N}\prod_\alpha \exp\left[-\beta\hbar\omega_\alpha\left(\sqrt{A^-_\oplus}n_{x,\alpha}+\sqrt{A^+_\oplus}n_{y,\alpha}+n_{z,\alpha}\right)\right] \tag{1657}$$

Here we are essentially taking a product of the partition function for each phonon state (numbered by α) and then summing over the number of phonons that occupy each state (given by n_α). However, if we move the sum past the product, then we would first sum over the phonons, n_α, that occupy a given state α (where n_α goes from 1 to ∞ since there can be an unlimited number of phonons in a given state) and then take the product over all the phonon states numbered by α.

$$Z = Z_0\prod_\alpha\sum_{n_\alpha=1}^\infty \exp\left[-\beta\hbar\omega_\alpha\left(\sqrt{A^-_\oplus}n_{x,\alpha}+\sqrt{A^+_\oplus}n_{y,\alpha}+n_{z,\alpha}\right)\right] \tag{1658}$$

[182] Here we are essentially applying the concept of "periodic thermodynamics" as described by Kohn in [88] and more recently by Langemeyer and Holthaus in [89].

[183] Notice that the gravitational field also couples to the ground state energy and therefore breaks the isotropy of the zero-point energy in the lattice.

We can take the log of both sides and use the fact that the log of a product is the sum of logs, then we have

$$\begin{aligned}
\ln(Z) \quad &= \quad \ln Z_0 + \ln \sum_{n_1=1}^{\infty} \exp\left[-\beta\hbar\omega_0\left(\sqrt{A_{\oplus}^{-}}n_{x,0} + \sqrt{A_{\oplus}^{+}}n_{y,0} + n_{z,0}\right)\right] \\
&\quad + \ln \sum_{n_2=1}^{\infty} \exp\left[-\beta\hbar\omega_1\left(\sqrt{A_{\oplus}^{-}}n_{x,1} + \sqrt{A_{\oplus}^{+}}n_{y,1} + n_{z,1}\right)\right] \\
&\quad + \cdots + \ln \sum_{n_\alpha=\alpha}^{\infty} \exp\left[-\beta\hbar\omega_\alpha\left(\sqrt{A_{\oplus}^{-}}n_{x,\alpha} + \sqrt{A_{\oplus}^{+}}n_{y,\alpha} + n_{z,\alpha}\right)\right] + \cdots
\end{aligned} \tag{1659}$$

Notice that if we perform the summation from 1 to ∞ in each term above, then all the terms will be identical. Since the terms are being added, and because there is an infinite number of them, then we can write them all in a single summation numbered by n_α going from 1 to ∞. We also use the fact that the sum of logs is a log of the product to pull the log outside again.

$$\ln(Z) \quad = \quad \ln\left\{Z_0 \sum_{n_\alpha=1}^{\infty} \exp\left[-\beta\hbar\omega_\alpha\left(\sqrt{A_{\oplus}^{-}}n_{x,\alpha} + \sqrt{A_{\oplus}^{+}}n_{y,\alpha} + n_{z,\alpha}\right)\right]\right\} \tag{1660}$$

We have now reduced the expression to a single summation over the number of phonons in each state n_α. Again we can make use of the fact that the exponential of a sum is the product of exponentials in order to separate the exponentials involving n_x, n_y, and n_z. Then we have

$$\ln(Z) \quad = \quad \ln\left[Z_0 \sum_{n_\alpha=1}^{\infty} \exp\left(-\beta\hbar\sqrt{A_{\oplus}^{-}}\omega_\alpha n_{x,\alpha}\right) \sum_{n_\alpha=1}^{\infty} \exp\left(-\beta\hbar\sqrt{A_{\oplus}^{+}}\omega_\alpha n_{y,\alpha}\right)\right. \tag{1661}$$

$$\left. \cdot \sum_{n_\alpha=1}^{\infty} \exp\left(-\beta\hbar\omega_\alpha n_{z,\alpha}\right)\right] \tag{1662}$$

Each summation is now an infinite geometric sum[184] which is given by

$$\sum_{n_\alpha=0}^{\infty} ar^{n_\alpha} = \frac{a}{1-r} \qquad \text{if} \qquad |r| < 1 \tag{1663}$$

In each summation of (1661) we have $a = 1$. In the first sum, we have $r = \exp\left(-\beta\hbar\sqrt{A_{\oplus}^{-}}\omega_\alpha\right)$. In the second sum, we have $r = \exp\left(-\beta\hbar\sqrt{A_{\oplus}^{+}}\omega_\alpha\right)$. In the last sum, we have $r = \exp(-\beta\hbar\omega_\alpha)$. Then using the formula for an infinite geometric sum in (1663), we find that (1661) becomes

$$\ln(Z) \quad = \quad \ln\left[Z_0\left(\frac{1}{1-e^{-\beta\hbar\sqrt{A_{\oplus}^{-}}\omega_\alpha}}\right)\left(\frac{1}{1-e^{-\beta\hbar\sqrt{A_{\oplus}^{+}}\omega_\alpha}}\right)\left(\frac{1}{1-e^{-\beta\hbar\omega_\alpha}}\right)\right] \tag{1664}$$

[184] Note that because $\omega_n = \pi n v/L$, then we can start each of the summations at $n = 0$ without changing the value of the summation.

Finally, exponentiating both sides gives

$$Z = Z_0 \left(\frac{1}{1 - e^{-\beta\hbar\sqrt{A_{\oplus}^{-}}\omega_\alpha}} \right) \left(\frac{1}{1 - e^{-\beta\hbar\sqrt{A_{\oplus}^{+}}\omega_\alpha}} \right) \left(\frac{1}{1 - e^{-\beta\hbar\omega_\alpha}} \right) \tag{1665}$$

Each of the factors in parentheses is effectively a partition function for a set of oscillators with frequencies given by ω_α, where α numbers the oscillators from $\alpha = 1$ to N. Then the Helmholtz free energy, $F = -k_B T \ln(Z)$, for the sum of oscillators in the x, y, and z directions are, respectively,

$$F_x = -k_B T \sum_{\alpha=1}^{N} \ln \left(\frac{1}{1 - e^{-\beta\hbar\sqrt{A_{\oplus}^{-}}\omega_\alpha}} \right) \tag{1666}$$

$$F_y = -k_B T \sum_{\alpha=1}^{N} \ln \left(\frac{1}{1 - e^{-\beta\hbar\sqrt{A_{\oplus}^{+}}\omega_\alpha}} \right) \tag{1667}$$

$$F_z = -k_B T \sum_{\alpha=1}^{N} \ln \left(\frac{1}{1 - e^{-\beta\hbar\omega_\alpha}} \right) \tag{1668}$$

Since the Helmholtz free energy is additive, then the combined free energy in the x, y, and z directions can be found by adding the expressions above. We also include the zero-point energy as

$$F_0 = -k_B T \ln Z_0 = -k_B T \ln e^{-\beta E_0} = E_0 \tag{1669}$$

Lastly, using the log property, $\log a^{-1} = -\log a$, leads to a total free energy given by

$$\boxed{\begin{array}{c} F = E_0 + k_B T \sum_{\alpha=1}^{N} \ln\left(1 - e^{-\beta\hbar\sqrt{A_{\oplus}^{-}}\omega_\alpha}\right) + k_B T \sum_{\alpha=1}^{N} \ln\left(1 - e^{-\beta\hbar\sqrt{A_{\oplus}^{+}}\omega_\alpha}\right) \\ + k_B T \sum_{\alpha=1}^{N} \ln\left(1 - e^{-\beta\hbar\omega_\alpha}\right) \end{array}} \tag{1670}$$

Helmholtz free energy for N quantum harmonic oscillators in the presence of a gravitational wave with plus-polarization in the z-direction

Notice that dropping the zero-point energy and setting $A_{\oplus}^{-} = A_{\oplus}^{+} = 1$, makes all three summations become identical and the free energy reduces to the known expression for $3N$ quantum harmonic oscillators. Using (1566), it is found in equation 4.3.7 of [80] to be

$$F = k_B T \sum_{\alpha=1}^{3N} \ln\left(1 - e^{-\beta\hbar\omega_\alpha}\right) \tag{1671}$$

Next we apply the Debye model to the free energy in (1670). Each three-dimensional oscillator is characterized by a wave vector $\vec{k}$. If the crystal lattice is a volume $V = L_x L_y L_z$, then the allowed values of the wavelength in any direction is $\lambda_n/2 = nL$, with n being nonnegative integers. This means that the components of $k = \lambda/2\pi$ are

$$k_x = \frac{\pi n_x}{L_x}, \qquad k_y = \frac{\pi n_y}{L_y}, \qquad k_z = \frac{\pi n_z}{L_z} \tag{1672}$$

Debye's assumption is that $\omega = vk$, where v is a constant speed in the material for all modes. Then we can write (1670) as

$$F_D = E_0 + k_B T \sum_{n_x=1}^{N} \ln\left(1 - e^{-\beta\hbar\sqrt{A_{\oplus}^{-}}vk}\right) + k_B T \sum_{n_y=1}^{N} \ln\left(1 - e^{-\beta\hbar\sqrt{A_{\oplus}^{+}}vk}\right) + k_B T \sum_{n_z=1}^{N} \ln\left(1 - e^{-\beta\hbar vk}\right) \tag{1673}$$

Successive terms in the summations correspond to k's which differ by π/L. When L is macroscopically large, then the difference between successive terms of $\exp(-\beta\hbar vk)$ in a given sum will be extremely small.[185] Therefore the summations can be approximated by integrals. Also using (1672), we see that $dn = \frac{L}{\pi}dk$ which means that we can write the integrals in terms of dk as

$$F_D = E_0 + k_B T\frac{L_x}{\pi}\int \ln\left(1 - e^{-\beta\hbar\sqrt{A_{\oplus}^{-}}v_x k_x}\right) dk_x + k_B T\frac{L_y}{\pi}\int \ln\left(1 - e^{-\beta\hbar\sqrt{A_{\oplus}^{+}}v_x k_y}\right) dk_y + k_B T\frac{L_z}{\pi}\int \ln\left(1 - e^{-\beta\hbar v_z k_z}\right) dk_z \tag{1674}$$

Ordinarily in the Debye model, the material is assumed to be isotropic so that the three summations in (1673) are identical and we can simply write

$$F_D = E_0 + 3k_B T \sum_{n_x,n_y,n_z}^{N} \ln\left(1 - e^{-\beta\hbar vn/L}\right) \tag{1675}$$

Then instead of obtaining the three integrals in (1674), we find instead that using $d^3n = \frac{L^3}{\pi^3}d^3k$ with $V = L^3$ gives a single integral.

$$F_D = E_0 + \frac{3k_B T V}{\pi^3}\int \ln\left(1 - e^{-\beta\hbar vk}\right) d^3k \tag{1676}$$

This makes it possible to integrate over a sphere in k-space which can be done by integrating $d^3k = 4\pi k^2 dk$ from 0 to $k_D = \omega_D/v$ where ω_D is the Debye cut-off frequency. However, in our case here, we found in (1647) that $\omega_x \neq \omega_y \neq \omega_z$. Therefore, $\omega = vn\pi/L$ is *not* identical in each direction. This means that we

[185] For example, using $T \approx 10^{-2}K$, a macroscopic length scale of centimeters $\left(L \approx 10^{-2}m\right)$, and a shear velocity for Niobium given by (1584) as $v_s \approx 10^3 m/s$, we find that $\beta\hbar v_s k = \frac{\hbar v_s}{k_B T}\frac{\pi n}{L} \approx 10^{-6}n$. The difference between the terms with $n = 1$ and $n = 2$ will be $\ln\left(1 - e^{-\beta\hbar v_s k_2}\right) - \ln\left(1 - e^{-\beta\hbar v_s k_1}\right) \approx 0.7$. At the upper limit of n, the difference beteween the terms with $n = N - 1$ and $n = N$ can be found using $N = V\rho/m \approx 10^{22}$ from (1585). This gives $\ln\left(1 - e^{\beta\hbar v_s k_{N-1}}\right) - \ln\left(1 - e^{\beta\hbar v_s k_N}\right) \approx 0$.

cannot write (1673) as a single summation and hence obtain a single spherically symmetric integral in k-space. Returning to (1674), we can apply integration by parts to each integral. Focusing on the first integral, we can let

$$\begin{aligned} U &= \ln\left(1 - e^{-\beta\hbar\sqrt{A_{\oplus}^{-}}vk}\right), & dV &= dk \\ dU &= \frac{\beta\hbar\sqrt{A_{\oplus}^{-}}v}{e^{\beta\hbar\sqrt{A_{\oplus}^{-}}vk} - 1}dk, & V &= k \end{aligned} \tag{1677}$$

Then the integral becomes

$$\int \ln\left(1 - e^{-\beta\hbar\sqrt{A_{\oplus}^{-}}vk}\right) dk = k\ln\left(1 - e^{-\beta\hbar\sqrt{A_{\oplus}^{-}}vk}\right) - \beta\hbar\sqrt{A_{\oplus}^{-}}v\int \frac{k}{e^{\beta\hbar\sqrt{A_{\oplus}^{-}}vk} - 1}dk \tag{1678}$$

As shown in (1587), we know that in the low-temperature limit, the first term on the right will become vanishingly small. Also, the integral on the right can be approximated extremely well by using bounds from 0 to infinity. We can also use the following change of variables

$$x = \left(\beta\hbar\sqrt{A_{\oplus}^{-}}v\right)k, \qquad dx = \left(\beta\hbar\sqrt{A_{\oplus}^{-}}v\right)dk \tag{1679}$$

Then the integral on the right side of (1678) becomes a known integral

$$-\beta\hbar\sqrt{A_{\oplus}^{-}}v\int_0^{\infty} \frac{k}{e^{\beta\hbar\sqrt{A_{\oplus}^{-}}vk} - 1}dk = -\frac{1}{\beta\hbar\sqrt{A_{\oplus}^{-}}v}\int_0^{\infty}\frac{x}{e^x - 1}dx = -\frac{1}{\beta\hbar\sqrt{A_{\oplus}^{-}}v}\frac{\pi^2}{6} \tag{1680}$$

Similarly, applying integration by parts to the second and third integrals in (1674) will lead to a form analogous to (1678) in each case. Then similar to (1680), we will obtain

$$\int \ln\left(1 - e^{-\beta\hbar\sqrt{A_{\oplus}^{+}}vk}\right) dk \approx -\frac{1}{\beta\hbar\sqrt{A_{\oplus}^{+}}v}\frac{\pi^2}{6} \tag{1681}$$

and

$$\int \ln\left(1 - e^{-\beta\hbar vk}\right) dk \approx -\frac{1}{\beta\hbar v}\frac{\pi^2}{6} \tag{1682}$$

Now we can insert (1680) – (1682) into (1674), insert the expression for E_0 from (1655), and use $\beta = (k_B T)^{-1}$. This gives

$$F_D = \frac{1}{2}\hbar N\omega\left(\sqrt{A_{\oplus}^{-}} + \sqrt{A_{\oplus}^{+}} + 1\right) - \frac{\pi}{6\hbar\beta^2 v}\left(\frac{L_x}{\sqrt{A_{\oplus}^{-}}} + \frac{L_y}{\sqrt{A_{\oplus}^{+}}} + L_z\right) \tag{1683}$$

We can also use the following expression for $A_{\oplus}^{\pm}$ from (1635) which is

$$A_{\oplus}^{\pm} \equiv 1 \pm \frac{A_{\oplus}}{2\pi} \tag{1684}$$

In (1683), we can expand $\left(A_{\oplus}^{\pm}\right)^{1/2}$ and $\left(A_{\oplus}^{\pm}\right)^{-1/2}$ to first order as[186]

$$\left(A_{\oplus}^{\pm}\right)^{1/2} \quad \approx \quad \left(A_{\oplus}^{\pm}\right)^{-1/2} \approx 1 \pm \frac{A_{\oplus}}{4\pi} \tag{1685}$$

Inserting this into (1683) and eliminating terms higher than first order in $A_{\oplus}$ gives

$$F_D = \frac{3}{2}\hbar N\omega - \frac{\pi}{6\hbar\beta^2 v}L_x\left(1-\frac{A_{\oplus}}{4\pi}\right) - \frac{\pi}{6\hbar\beta^2 v}L_y\left(1+\frac{A_{\oplus}}{4\pi}\right) - \frac{\pi}{6\hbar\beta^2 v}L_z \tag{1686}$$

Debye free energy for a low-temperature lattice in the presence of a gravitational wave with plus-polarization in the z-direction

We can now formulate the Debye free energy density in curved space-time by taking the derivative of (1683) with respect to *proper* volume. Recall that the coordinate-dependent volume can be expressed in terms of the proper volume as $dV = dV_{proper}/\sqrt{-g}$. We can use $g_{\oplus}^{\tau\tau}$ to represent the determinant of the metric when the only perturbation to flat space-time is $h_{\oplus}$. We can also define the Debye free energy *density* with respect to proper volume (in the presence of a plus-polarized gravitational wave) as

$$\mathscr{F}_D \quad \equiv \quad \frac{dF_D}{dV} = \sqrt{-g_{\oplus}^{\tau\tau}}\frac{dF_D}{dV_{proper}} \tag{1687}$$

To evaluate the dF_D/dV, we need to express F_D in (1686) in terms of the volume. We can do this by inserting $N = nV$ where n is the number density of atoms in the lattice, and by using $V = L_xL_yL_z$. This gives

$$F_D \quad = \quad \frac{3\hbar\omega nV}{2} - \frac{\pi V}{6\hbar\beta^2 vL_yL_z}\left(1-\frac{A_{\oplus}}{4\pi}\right) - \frac{\pi V}{6\hbar\beta^2 vL_xL_z}\left(1+\frac{A_{\oplus}}{4\pi}\right) - \frac{\pi V}{6\hbar\beta^2 vL_xL_y} \tag{1688}$$

Evaluating dF_D/dV and inserting into (1687) gives the Debye free energy density with respect to proper volume in curved space-time as

$$\mathscr{F}_D \quad = \quad \sqrt{-g_{\oplus}^{\tau\tau}}\left[\frac{3\hbar\omega n}{2} - \frac{\pi}{6\hbar\beta^2 vL_yL_z}\left(1-\frac{A_{\oplus}}{4\pi}\right) - \frac{\pi}{6\hbar\beta^2 vL_xL_z}\left(1+\frac{A_{\oplus}}{4\pi}\right) - \frac{\pi}{6\hbar\beta^2 vL_xL_y}\right] \tag{1689}$$

In (2661) of Appendix G, it was found that the determinant of the metric (in terms of the transverse-traceless metric perturbation) is $g^{\tau\tau} = -1 + h_{\oplus}^2 + h_{\otimes}^2$. Setting $h_{\otimes}$ set to zero for plus-polarized waves and using a binomial expansion to first order[187] gives

$$\sqrt{-g^{\tau\tau}} \approx 1 + \frac{1}{2}h_{\oplus} \tag{1690}$$

Recall that we have been working with a standing gravitational wave in the z-direction given by (1613) as $h_{\oplus}(z,t) = A_{\oplus}\cos(kz)\sin(\omega t)$. Also recall that we are considering the case of a thin film where $z << \lambda$ so that $kz << 1$ and therefore $\cos(kx) \approx 1$. For approximation purposes, we can also choose $\omega t = \pi/2$ so that $\sin(\omega t) = 1$. Then (1690) can be expressed just in terms of the amplitude, $A_{\oplus}$. We can insert (1690) into

[186] The expansions used here are $(1 \pm x)^{1/2} \approx (1 \pm x)^{-1/2} \approx 1 \pm \frac{1}{2}x$ for $|x| << 1$.
[187] The binomial expansion to second order is $\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2$ for $x << 1$.

(1689), distribute, and eliminate terms that are higher than second order in $A_\oplus$.

$$\mathscr{F}_D = \frac{3\hbar\omega n}{2} + \frac{3\hbar\rho\omega}{4m}A_\oplus - \frac{\pi}{6\hbar\beta^2 v L_y L_z}\left(1 - \frac{A_\oplus}{4\pi}\right) - \frac{\pi}{12\hbar\beta^2 v L_y L_z}A_\oplus$$

$$- \frac{\pi}{6\hbar\beta^2 v L_x L_z}\left(1 + \frac{A_\oplus}{4\pi}\right) - \frac{\pi}{12\hbar\beta^2 v L_x L_z}A_\oplus - \frac{\pi}{6\hbar\beta^2 v L_x L_y}\left(1 + \frac{1}{2}A_\oplus\right)$$

Grouping terms according to order in $A_\oplus$ gives

$$\mathscr{F}_D = \frac{3\hbar\omega n}{2} - \frac{\pi}{6\hbar\beta^2 v}\left(\frac{1}{L_y L_z} + \frac{1}{L_x L_z} + \frac{1}{L_x L_y}\right)$$
$$+ \frac{3\hbar\omega n}{4}A_\oplus + \frac{1}{24\hbar\beta^2 v}\left(\frac{1}{L_y L_z} - \frac{1}{L_x L_z}\right)A_\oplus$$
$$- \left[\frac{\pi}{12\hbar\beta^2 v}\left(\frac{1}{L_y L_z} + \frac{1}{L_x L_z} + \frac{1}{L_x L_y}\right)\right]A_\oplus \qquad (1691)$$

Debye free energy density with respect to proper volume in curved space-time for a low-temperature lattice in the presence of a gravitational wave with plus-polarization in the z-direction

Note that the expression reduces considerably if we drop the zero-point energy which is given by the expressions involving ω. Then we have

$$\mathscr{F}_D = -\frac{\pi}{6\hbar\beta^2 v}\left(\frac{1}{L_y L_z} + \frac{1}{L_x L_z} + \frac{1}{L_x L_y}\right) + \frac{1}{12\hbar\beta^2 v}\left(\frac{1-2\pi}{2L_y L_z} - \frac{1+2\pi}{2L_x L_z} - \frac{\pi}{L_x L_y}\right)A_\oplus \qquad (1692)$$

The zero-point energy is ordinarily neglected because it does not contribute to macroscopic thermodynamic quantities (such as entropy, specific heat, etc.). However, in this context we find that the strain field of the gravitational wave is coupled to the zero-point energy. This will be relevant in the next section where we will take the derivative of the free energy with respect to the strain field to find the stress induced by a gravitational wave acting on the ionic lattice of a superconductor. This means that if we remove the zero-point energy at this stage, we would be neglecting its contribution to the stress in the following section.

15.5 A gravitational shear modulus for the ionic lattice

To consider the work done on a system due to a gravitational strain, h_{ij}, causing a stress, T_{ij}, we can use $\mathscr{W} = \int T^{ij} dh_{ij}$ where $\mathscr{W}$ is the work density (or work per unit volume). This means that $T^{ij} = d\mathscr{W}/dh_{ij}$. By the work-energy theorem, we can recognize that work done by the system must reduce the internal energy density so that $\mathscr{U} = -\mathscr{W}$. Also, we can use the fact that the internal energy satisfies the relation $dU = TdS - PdV$ and the Helmholtz free energy is $F = U - TS$. However, now we recognize that when stresses and strains are present, then the Helmholtz free energy also satisfies the relation $d\mathscr{F} = -\mathbb{S}dT + T^{ij}dh_{ij}$, where $\mathbb{S}$ is the entropy density (or entropy per unit volume). It follows that the stress produced by a gravitational wave interacting with the Cooper pair density can be expressed in terms of the Debye free energy density as[188]

$$T^{ij} = \left(\frac{\partial \mathscr{F}_D}{\partial h_{ij}}\right)_T \tag{1693}$$

For plus polarization, we have $h_{11}^{\tau\tau} = -h_{22}^{\tau\tau} = h_{\oplus}$ which corresponds to $T_{11}^{\tau\tau} = -T_{22}^{\tau\tau} = T_{\oplus}$. In that case, we can use[189]

$$T_{\oplus} = \left(\frac{\partial \mathscr{F}_D}{\partial h_{\oplus}}\right)_T \tag{1694}$$

Therefore, applying (1694) to (1691) gives

$$\boxed{\begin{aligned} T_{\oplus} = {} & \frac{3\hbar\omega n}{4} + \frac{1}{24\hbar\beta^2 v}\left(\frac{1}{L_yL_z} - \frac{1}{L_xL_z}\right) \\ & - \left[\frac{\pi}{12\hbar\beta^2 v}\left(\frac{1}{L_yL_z} + \frac{1}{L_xL_z} + \frac{1}{L_xL_y}\right)\right] \\ & - \frac{\hbar\omega n}{8}\left(\frac{1}{2\pi^2} + 15\right)A_{\oplus} + \frac{1}{24\hbar\beta^2 vL_yL_z}\left[5\pi + 1 - \left(\frac{8\pi^2+3}{4\pi}\right)\right]A_{\oplus} \\ & + \frac{1}{24\hbar\beta^2 vL_xL_z}\left[5\pi - 1 + \left(\frac{8\pi^2-3}{4\pi}\right)\right]A_{\oplus} + \frac{5\pi}{24\hbar\beta^2 vL_xL_y}A_{\oplus} \end{aligned}} \tag{1695}$$

Stress on a low-temperature lattice due to a gravitational wave with plus-polarization in the z-direction

The first two lines are stress terms that are essentially a "background stress" due to the metric perturbation to *first* order. The last two lines come from the metric to *second* order and describe the *linear* response of the stress in the ionic lattice to the strain field of the gravitational wave. If we consider a constitutive equation given by $T_{\oplus} = -\mu_{\oplus} h_{\oplus}$, where $\mu_{\oplus}$ is the "gravitational modulus" of the ionic lattice for a plus-polarization

[188] Here we are using a formulation similar to [79], eq. 3.3, which gives $dF = -SdT + \sigma_{ij}du_{ij}$, where σ_{ij} is the material stress and u_{ij} is the material strain. It is also shown in eq. 3.6 that $\sigma_{ij} = (\partial F/\partial u_{ij})_T$ which leads analogously to our relation in (1476). Notice there is a *positive* sign in this relation versus the negative sign in $P = -(\partial F/\partial V)_T$.

[189] Note that the derivative is with respect to the the full time-dependent function given by $h_{ij}^{\tau\tau} = A_{ij}^{\tau\tau}\cos(kz - \omega t)$, not just the *magnitude*, $A_{ij}^{\tau\tau}$. However, for approximation purposes, we have eliminated the time-dependence in the free energy density and therefore the derivative can be considered as being with respect to the amplitude $A_{\oplus}$.

wave, then $\mu_\oplus$ can be found from the last two lines in the expression above.

$$\boxed{\begin{gathered}\mu_\oplus = \frac{\hbar\rho\omega}{8m}\left(\frac{1}{2\pi^2}+15\right) - \frac{1}{24\hbar\beta^2 \nu L_y L_z}\left[5\pi+1-\left(\frac{8\pi^2+3}{4\pi}\right)\right] \\ -\frac{1}{24\hbar\beta^2 \nu L_x L_z}\left[5\pi-1+\left(\frac{8\pi^2-3}{4\pi}\right)\right] - \frac{5\pi}{24\hbar\beta^2 \nu L_x L_y} \\ \textit{Gravitational modulus for a low-temperature lattice in response} \\ \textit{to a gravitational wave with plus-polarization in the z-direction}\end{gathered}} \tag{1696}$$

This expression describes the effective "stiffness" of the ionic lattice in response to the strain of a plus-polarization gravitational wave. It also demonstrates a *linear* response of the induced stress as a result of the incident gravitational wave strain. Since the result is specifically for a *plus*-polarization wave, we have labeled the gravitational modulus with a corresponding subscript. Note that if $L_x \approx L_y$, then we can simplify the result to

$$\mu_\oplus \approx \frac{\hbar\omega n}{8}\left(\frac{1}{2\pi^2}+15\right) - \frac{5\pi}{24\hbar\beta^2 \nu L_z L_x^2}\left(2L_x + L_z\right) \tag{1697}$$

We can also approximate $\left(\frac{1}{2\pi^2}+15\right) \approx 15$ which simplifies the first term. Also, since we have been considering the case of a thin film where $L_z << \lambda$, then we would also expect that $L_z << L_x$ which simplifies the second term. Then the result above can be reduced further to

$$\mu_\oplus \approx \frac{15\hbar\omega n}{8} - \frac{5\pi}{12\hbar\beta^2 \nu L_x L_z} \tag{1698}$$

The first term can be referred to as the "zero-point energy" term since it originates from the contribution of the zero-point energy to the free energy of the system. The second term can be referred to as the "sum of modes" term since it originates from the contribution of the sum of all the other modes of the system. There are several important differences between these two terms.

- The "zero-point energy" term involves $\hbar$ in the numerator while the "sum of modes" term involves $\hbar$ in the denominator. The $\hbar$ shifted to the denominator in the process of utilizing the Debye model and evaluating the integral over all frequencies in $(1680)-(1682)$.

- The "sum of modes" term varies with T^2 which means that as the temperature approaches zero, this term vanishes. However, the "zero-point energy" term would always remain.

- The "zero-point energy" term is an *intensive* property of the system since subdividing the system would not change the uniform number density of atoms, n, or the zero-point energy frequency, ω. On the other hand, the "sum of modes" term is an *extensive* property of the system since subdividing the system would change the length (either L_x, L_y, or L_z) and therefore would change the contribution to $\mu_\oplus$ from the second term in the expression above.

- The "zero-point energy" term is always a *positive* value while the "sum of modes" term is always a *negative* value. Since the constitutive equation is $T_\oplus = -\mu_\oplus h_\oplus$, then this means that the "zero-point energy" term follows the generalized Hooke's law relationship which predicts that an external field $h_\oplus$ would produce a stress $T_\oplus$ that opposes the external field. However, the "sum of modes" term does *not* follow the generalized Hooke's law relationship and would therefore contribute to a "run away" effect if the "sum of modes" term were dominant.

For an upper bound on the "zero-point energy" term, we can use the Debye frequency, ω_D, which is the cut-off frequency in the Debye model. For a Niobium superconductor, we found in (1584) that $n = N/V \approx 5.6 \times 10^{28} m^{-3}$ and $\omega_D \approx 3.1 \times 10^{13} s^{-1}$. For the "sum of modes" term, we can use a thickness for the film on the order of millimeters $\left(L_z \approx 10^{-3} m\right)$ and a surface with edges on the order of centimeters $\left(L_x \approx 10^{-2} m\right)$. From (1584) we also have $v = \sqrt{s/\rho} \approx 2.1 \times 10^3 m/s$. We can also use $T \approx 10^{-2} K$ for the temperature of the superconductor. Using all of these values, we find the gravitational modulus has a value given by

$$\mu_{\oplus} \approx 3.4 \times 10^8 J/m^3 - 1.4 \times 10^{-13} J/m^3 \tag{1699}$$

Here we find that the "zero-point energy" term produces a result that is 21 orders of magnitude greater than the "sum of modes" term. This is obviously due to the factor of n appearing in the "zero-point energy" term. In order for the "zero-point energy" term to be comparable to the "sum of modes" term, then the two terms in (1698) we would need to be comparable.

$$\frac{15\hbar\omega n}{8} \approx \frac{5\pi}{12\hbar\beta^2 v L_z L_x} \tag{1700}$$

This leads to a frequency of

$$\omega \approx \frac{2\pi}{9\hbar^2\beta^2 n v L_z L_x} \approx 1.3 \times 10^{-8} s^{-1} \tag{1701}$$

This corresponds to a period of $T = 2\pi/\omega \approx 4.8 \times 10^8 s \approx 15\ years$. This is clearly an unrealistic period and therefore we see that the "sum of modes" term would never be comparable to the "zero-point energy" term. However, if we solve this for the temperature, then we have

$$T \approx \frac{3\hbar}{k_B}\sqrt{\frac{\omega n v L_z L_x}{2\pi}} \tag{1702}$$

Using the frequency in (1701) gives $T \approx 3.5 \times 10^{-2} K$. This is a reasonable temperature that can be achieved in the lab. On the other hand, if we use the upper limit of the zero-point energy given by the Debye frequency and still require the "sum of modes" term to be comparable to the "zero-point energy" term then we have $T \approx 1.7 \times 10^9 K$. This is obviously far beyond lab-scale temperatures.

15.6 Quantifying the charge-separation effect

In previous sections, a constitutive equation, $T_{ij}^{\tau\tau} = -\mu_{G(SC)} h_{ij}^{\tau\tau}$, was developed which describes the linear response of the transverse-traceless stress in a superconductor, $T_{ij}^{\tau\tau}$, caused by a transverse-traceless *gravitational strain* of space, $h_{ij}^{\tau\tau}$, where $\mu_{G(SC)}$ is the *gravitational* shear modulus. From continuum mechanics, it is also possible to develop a constitutive equation describing the transverse-traceless *material strain*, $u_{ij}^{\tau\tau}$, caused by a transverse-traceless stress applied to an object. In equation (4.6) of Landau and Lifshitz's *Theory of Elasticity* [79], the stress tensor is related to the material strain tensor as[190]

$$T_{ij} = Ku\delta_{ij} + 2s\left(u_{ij} - \tfrac{1}{3}u\delta_{ij}\right) \tag{1703}$$

where K is the bulk modulus, s is the shear modulus, and $u = \delta^{ij}u_{ij}$ is the trace of u_{ij}. Taking the trace of this expression, $\delta^{ij}T_{ij}$, gives

$$T = 3Ku \tag{1704}$$

The transverse-traceless stress tensor can be defined as

$$T_{ij}^{\tau\tau} \equiv T_{ij} - \tfrac{1}{3}T\delta_{ij} \tag{1705}$$

where $\partial_i T_{ij} = 0$ in order for $T_{ij}^{\tau\tau}$ to be transverse as well as traceless. Inserting (1703) and (1704) into (1705) gives

$$T_{ij}^{\tau\tau} = 2s\left(u_{ij} - \tfrac{1}{3}u\delta_{ij}\right) \tag{1706}$$

The transverse-traceless material strain can be defined as

$$u_{ij}^{\tau\tau} \equiv u_{ij} - \tfrac{1}{3}u\delta_{ij} \tag{1707}$$

where $\partial_i u_{ij} = 0$ in order for $u_{ij}^{\tau\tau}$ to be transverse as well as traceless. Then the material constitutive equation in (1706) can be written as

$$\boxed{T_{ij}^{\tau\tau} = 2su_{ij}^{\tau\tau} \qquad \textit{Transverse-traceless constitutive equation for the material strain}} \tag{1708}$$

This relationship has the same form as $T_{ij}^{\tau\tau} = -\mu_{G(SC)} h_{ij}^{\tau\tau}$ which is the constitutive equation describing the response of a superconductor to a gravitational wave. However, there is an important distinction between $h_{ij}^{\tau\tau}$ (the strain of *space*) and $u_{ij}^{\tau\tau}$ (the strain of *matter*). If an object is in the presence of a gravitational strain of *space*, it is possible that the resulting strain of *matter* in the material is not equal to the strain of space. The reason is because the material can essentially *resist* the strain of space due to an internal rigidity (caused by electromagnetic, quantum mechanical, or other internal interactions within the material). Therefore, it should not be assumed that $h_{ij}^{\tau\tau} = u_{ij}^{\tau\tau}$ for a given object. In fact, this would only be true of free particles.

Ordinarily, if the space between particles stretches/squeezes, then the particles will move *with* space so that the distance between them increases/decreases accordingly. This means $u_{ij}^{\tau\tau} = h_{ij}^{\tau\tau}$ which can be associated with "motion *with* space" in the presence of a gravitational wave. By contrast, if the particles that make up an object are fixed rigidly with respect to each other (such as in a crystal lattice structure or an incompressible fluid), then the distance between the particles will *not* increase/decrease in a manner that is equal to the stretching/squeezing of the space between them. Since space would be stretching/squeezing relative to the particles, then the particles could be considered as effectively moving *through* space. Therefore, $u_{ij}^{\tau\tau} \neq h_{ij}^{\tau\tau}$ can be associated with "motion *through* space" in the presence of a gravitational wave. This distinction between moving *with* space and moving *through* space can be compared to the distinction between a *Doppler* redshift (due to motion of objects *through* space) and a *Cosmological* redshift (due to motion of objects *with* space).

[190] Landau and Lifshitz use σ_{ik} for the stress tensor, u_{ll} for the trace, and μ for the shear modulus.

Now comparing the stress found in the *material* constitutive equation, $T^{\tau\tau}_{ij} = su^{\tau\tau}_{ij}$, and the stress found in the *gravito-London* constitutive equation, $T^{\tau\tau}_{ij} = -\mu_{G(SC)}h^{\tau\tau}_{ij}$, requires careful consideration of the difference between an *internal* stress within a material versus an *external* stress applied to a material. It should be recognized that there is a sign difference between these two stress quantities. This can be understood by recognizing that in each constitutive equation, there is an *external* quantity causing an *internal* effect. For the material constitutive equation, there is an *external* stress acting on the material which causes an *internal* strain. Hence, writing the equation in the form "*internal* effect caused by *external* source" leads to $u^{\tau\tau}_{ij\,(internal)} = \left(\frac{1}{2s}\right) T^{\tau\tau}_{ij\,(external)}$. This is consistent with the fact that s is interpreted as the "stiffness" of a material to an external stress, which means $1/s$ could be interpreted as the "responsiveness" of a material to an external stress. In contrast to this, the gravito-London constitutive equation, $T^{\tau\tau}_{ij} = -\mu_{G(SC)}h^{\tau\tau}_{ij}$, involves an *external* gravitational strain field causing an *internal* stress. Once again, writing the equation in the form "*internal* effect caused by *external* source" leads to $T^{\tau\tau}_{ij\,(internal)} = -\mu_{G(SC)}h^{\tau\tau}_{ij\,(external)}$.

Therefore, to relate the two constitutive equations, it is important to recognize that mechanical equilibrium requires that an *external* stress acting on a material must be equal and opposite to the *internal* stress that exists within the material. This means $T^{\tau\tau}_{ij\,(external)} = -T^{\tau\tau}_{ij\,(internal)}$. Then using the two constitutive equations leads to the following relationship.[191]

$$\boxed{u^{\tau\tau}_{ij} = \frac{\mu_{G(SC)}}{2s}h^{\tau\tau}_{ij} \qquad \textit{Strain of matter in terms of gravitational strain of space}} \tag{1709}$$

This gives a cause-effect relationship between $h^{\tau\tau}_{ij}$ (an external gravitational wave strain field acting as the *cause*) and $u^{\tau\tau}_{ij}$ (the internal strain of an object which is the *effect*). As expected, the material strain is in the *same* direction as the gravitational wave strain field. For example, if a gravitational strain is expanding space (at a given instant), then the material strain should also be an expansion, not a contraction.

Also notice that the proportionality constant relating $u^{\tau\tau}_{ij}$ and $h^{\tau\tau}_{ij}$ in (1709) has an intuitive interpretation. It is $\mu_{G(SC)}$ (the responsiveness of a material to a gravitational wave) divided by s (the stiffness of the material to an external stress). A larger responsiveness (and/or smaller stiffness) will lead to a larger material strain, $u^{\tau\tau}_{ij}$, for a given gravitational wave strain, $h^{\tau\tau}_{ij}$. In fact, the free particle limit would be $\mu_{G(SC)}/s = 1$ so that $u^{\tau\tau}_{ij} = h^{\tau\tau}_{ij}$. This implies that free particles move in manner that is completely slaved to the gravitational wave. The other extreme case would be an object with very low responsiveness (small $\mu_{G(SC)}$) and/or a very high stiffness (large s). Such an object would experience very little material strain in response to a gravitational strain of space. Therefore, from the discussion above, it must be the case that

$$0 < \frac{\mu_{G(SC)}}{s} \leq 1 \tag{1710}$$

where the lower bound is for a very stiff material, and the upper bound is for free particles.[192]

[191]The relation $T^{\tau\tau}_{ij\,(external)} = -T^{\tau\tau}_{ij\,(internal)}$ is exactly the same physical relation given in equation (2.10) of [74] as $T_{jk} = -S_{jk}$, where T_{jk} is the *external* stress tensor and S_{jk} is the *internal* Cauchy stress tensor of the material.

[192]This result can be compared to Dyson's equation (2.31) in [74] which gives a boundary expression expressed as

$$\lambda N_j z^{m,m} + \mu N_k \left(z^{j,k} + z^{k,j} - h^{jk}\right) = 0$$

where z_j describes the elastic motions of the solid, and N_j is a vector normal at any point on the surface. For a transverse symmetric strain, $z^{m,m} = 0$ and $z^{j,k} = z^{k,j}$, which leads to $z^{j,k} = \frac{1}{2}h^{jk}$. Matching this to (1709) implies that $\mu_G = s$.

Dyson's formulation is based on taking a standard Lagrangian density found in non-relativistic mechanics to describe the motion of a solid, and adding an additional interaction term, $\mathscr{L} = -\frac{1}{2}h^{jk}S_{jk}$, to describe the

To formulate the charge-separation effect in a superconductor, $u_{ij(LI)}^{\tau\tau}$ can be used to describe the strain of the ionic lattice, and $u_{ij(CP)}^{\tau\tau}$ can be used to describe the strain of the Cooper pairs density. Then using (1709) leads to

$$u_{ij(CP)}^{\tau\tau} = \frac{\mu_{G(CP)}}{2s} h_{ij}^{\tau\tau} \qquad \text{and} \qquad u_{ij(LI)}^{\tau\tau} = \frac{\mu_{G(LI)}}{2s} h_{ij}^{\tau\tau} \tag{1711}$$

The *material* shear modulus, s, must be the same for the Cooper pair density and the ionic lattice. This is due to the fact that s is determined experimentally by applying a mechanical stress to a material and then measuring the corresponding strain. In doing so, the Cooper pair density and ionic lattice will both expand/contract *together* due to the electric force which binds them and preserves charge neutrality throughout the superconductor. If this were not the case, then applying a mechanical stress on a superconductor would lead to a charge-separation effect, or in other words, an electric polarization.

However, the *gravitational* shear moduli, $\mu_{G(LI)}$ and $\mu_{G(CP)}$, are very different from one another due to the difference in the quantum mechanical nature of the ionic lattice and the Cooper pair density. For the Cooper pair density, the coupling to gravity was due to a term of the form $h^{ij}T_{ij}$ which appeared in the energy expression after expanding the relativistic four-momentum invariant: $g_{\mu\nu}p^{\mu}p^{\nu} = -m^2c^2$. The energy was evaluated (to second order in velocity) and a partition function was formulated. The Cooper pairs are modeled as effectively a *single* quantum particle in the ground state, rather than an ensemble of particles in multiple states. Therefore, the energy becomes equivalent to the free energy, and the Ginzburg-Landau free energy density can be written by introducing the coupling terms involving α and β.

From the free energy density, it is apparent that the coupling $h^{ij}T_{ij}$ involves a quantum stress of the form $T^{ij} = \frac{1}{2m}\left(\pi^i\psi\right)^*\left(\pi^j\psi\right)$. The free energy density can be written in curved space-time by introducing a Jacobian factor, $\sqrt{-g}$. Expanding to second order in the metric and taking a derivative of the free energy density with respect to the gravitational strain field leads to an interaction stress tensor with an expression for the gravitational modulus. Using an upper bound imposed by the BCS energy gap, and using the coherence length of the superconductor, it was shown in (1547) that $\mu_{G(CP)} \approx 10^4 J/m^3$ for niobium. It is ultimately the coherence length that determines the value of $\mu_{G(CP)}$.

On the other hand, the ionic lattice was modeled as an ensemble of quantum harmonic oscillators coupled to a gravitational wave. The interaction term in the Hamiltonian has the form $h_{ij}^{\tau\tau}(z,t)\,T^{ij}$, where $T^{ij} = \hat{p}^i\hat{p}^j/2m$. The periodic time-dependence of the Hamiltonian leads to quasi-energy eigenvalues which can be used to construct a partition function. Summing over all the phonon modes in the lattice, evaluating the Helmholtz free energy, applying the Debye model in the low-temperature limit, and introducing the Jacobian factor, $\sqrt{-g}$, leads to a free energy density in curved space-time. Then taking the derivative of the free energy density with respect to the strain field leads to a stress tensor with an expression for the gravitational modulus. For a niobium slab with an area of $1cm^2$, it was shown in (1699) that $\mu_{G(LI)} \approx 10^8 J/m^3$. It is ultimately the ground state energy of the lattice phonons that determines the value of $\mu_{G(LI)}$.

Therefore, it is clear that the interaction of the gravitational wave occurs in a different way for the Cooper pair density than it does for the ionic lattice, and consequently the results for $\mu_{G(LI)}$ and $\mu_{G(CP)}$ are very

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coupling of the metric perturbation to the stress tensor. Then the Euler-Lagrange equation is used to obtain an equation of motion involving the material strain and the gravitational strain (subject to appropriate boundary conditions and conservation laws). In this formulation, the only parameter describing the response of matter to a gravitational wave is the *material* shear modulus (since $\mu_G = s$). Therefore, this result treats $h_{ij}^{\tau\tau}$ and u_{ij} as essentially the same type of strain. There is no distinction between a *gravitational* strain and a *material* strain in the equation of motion. However, this cannot be the case since the *gravitational* strain , $h_{ij}^{\tau\tau}$, is coupled to the system not just through an interaction Lagrangian, but also through a Jacobian factor, $\sqrt{-g^{\tau\tau}}$, which descibes all densitiy quantities in terms of the invariant proper volume. This is fundamentally different from way that the material strain enters in the formulation. In other words, $h_{ij}^{\tau\tau}$ introduces a *curvature to space-time* itself, unlike $u_{ij}^{\tau\tau}$ which simply describes the strain of a material.

different. This leads to a charge-separation effect that can now be quantified. The *relative* strain between the Cooper pairs and lattice ions can be defined as

$$u^{\tau\tau}_{ij(relative)} \equiv u^{\tau\tau}_{ij(LI)} - u^{\tau\tau}_{ij(CP)} \tag{1712}$$

If the responses to a gravitational wave by the Cooper pair density and the ionic lattice were the same, then $u^{\tau\tau}_{ij(LI)} = u^{\tau\tau}_{ij(CP)}$. This would lead to $u^{\tau\tau}_{ij(relative)} = 0$ which means that the Cooper pairs and lattice ions would simply co-move together in response to the gravitational wave. There would be no relative strain between them and hence there would be no charge-separation effect. However, since the response of the Cooper pair density and the ionic lattice are *different*, then inserting (1711) into (1712) gives

$$\boxed{u^{\tau\tau}_{ij(relative)} = \frac{\mu_{G(LI)} - \mu_{G(CP)}}{2s} h^{\tau\tau}_{ij} \qquad \textit{Relative strain between the lattice ions and Cooper pairs of a superconductor due to a gravitational wave}} \tag{1713}$$

Since $\mu_{G(CP)} \approx 10^4 J/m^3$ for the Cooper pairs and $\mu_{G(LI)} \approx 10^8 J/m^3$ for the lattice ions, then $\mu_{G(LI)} >> \mu_{G(CP)}$ which means

$$u^{\tau\tau}_{ij(relative)} \approx \frac{\mu_{G(LI)}}{2s} h^{\tau\tau}_{ij} \tag{1714}$$

This shows that the charge-separation effect can be described simply in terms of the response of the ionic lattice. The Cooper pair density can be considered as relatively unresponsive to the gravitational wave. For niobium, the material shear modulus is $s \approx 10^{10} J/m^3$. This leads to a relative strain between the Cooper pair density and ionic lattice that is given by

$$u^{\tau\tau}_{ij(relative)} \approx 0.01 h^{\tau\tau}_{ij} \tag{1715}$$

Hence the charge-separation effect proposed conceptually in [7] has now been quantified. For a $1cm^2$ slab of superconducting niobium, the relative strain between the Cooper pair density and the ionic lattice is approximately 1% of the gravitational wave strain field. However, it should be noted that the charge separation effect will produce an electric field which will act as a *restoring force* between the ionic lattice and Cooper pair density. This force will *oppose* the relative strain between the the ionic lattice and Cooper pair density. Therefore, including this force in the analysis would *reduce* the relative strain. In other words, the result in (1715) was obtained by calculating the response of the Cooper pair density and the response of the ionic lattice *independently*, then simply taking the difference of their strains to determine the charge separation effect. However, a more realistic model would include the restoring electric force which increases along with the relative strain in (1713), and hence would lead to a smaller coefficient in (1715).

16 Interaction of gravitational (GR) waves with electromagnetic fields

16.1 The electromagnetic free energy density in curved space-time

In this section, the response of the electromagnetic fields to a gravitational wave is formulated using the electromagnetic free energy density in curved space-time. In general, the four-momentum can be expressed in terms of the stress tensor as[193]

$$p^{\mu} = \int \sqrt{-g} T^{\mu\nu} dS_{\nu} \tag{1716}$$

where dS_{ν} is a differential four-surface with ν indicating the direction of the normal vector. The energy can be obtained by setting $\mu = 0$ so that $p^0 = E/c$. Raising the index of the four-surface using a metric and summing over repeated indices gives

$$E = c \int \sqrt{-g} g_{\rho\nu} T^{0\nu} dS^{\rho} \tag{1717}$$

$$= c \int \sqrt{-g} \left(g_{00} T^{00} dS^0 + g_{0j} T^{0j} dS^j \right) \tag{1718}$$

Note that $dS^0 = dV \cdot d\hat{x}^0$ is a differential volume with a time-like normal unit vector $d\hat{x}^0 = \frac{1}{c} d\hat{t}$. Also note that $dS^i = \left(dA \cdot \frac{1}{c} dt \right) d\hat{x}^i$ is a differential area multiplied by a differential time with a space-like normal unit vector. The metric can be expanded as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. For gravitational waves in the far-field, $h_{ij} = h_{ij}^{\tau\tau}$ and $h_{0\mu} = 0$. Also, the Jacobian for the transverse-traceless metric can be written as $g^{\tau\tau}$. Then the total energy can now be written as

$$E = \int \sqrt{-g} \left[T^{00} dV + T^{0j} \left(dA \cdot dt \right) \right] \tag{1719}$$

It is now evident that the first term in the integral represents the energy density, T^{00}, integrated over a spatial volume. The second term in the integral represents an energy flux density (or moment flow density), T^{0j}, integrated over an area and time. Evaluating the energy at a single instant in time means $dt = 0$ and therefore the second term in the integral vanishes. In (2661) of Appendix G, it was found that $g^{\tau\tau} = -1 + h_{\oplus}^2 + h_{\otimes}^2$. Then applying a first order approximation to the metric gives

$$E = \int \left(1 - \tfrac{1}{2} h_{\oplus}^2 - \tfrac{1}{2} h_{\otimes}^2 \right) T^{00} dV \tag{1720}$$

Determining T^{00} for electromagnetic fields can be done by starting with the electromagnetic energy-momentum tensor.

$$T_{(EM)}^{\mu\nu} = \frac{1}{\mu_0} \left(F^{\mu\alpha} F^{\nu}_{\ \alpha} - \tfrac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \right) \tag{1721}$$

Using the metric to write the electromagnetic strength tensor in terms of upper indices only gives

$$T_{(EM)}^{\mu\nu} = \frac{1}{\mu_0} \left(g_{\alpha\rho} F^{\mu\alpha} F^{\nu\rho} - \tfrac{1}{4} g^{\mu\nu} g_{\alpha\rho} g_{\beta\sigma} F^{\rho\sigma} F^{\alpha\beta} \right)$$

The metric can be expanded as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Once again, for gravitational waves in the far-field, $h_{ij} = h_{ij}^{\tau\tau}$ and $h_{0\mu} = 0$. Evaluating $T_{(EM)}^{00}$, summing over repeated indices, and using $\eta_{\mu 0} = (-1, 1, 1, 1)$ gives

$$\begin{aligned} T_{(EM)}^{00} &= \frac{1}{\mu_0} \left(-F^{00} F^{00} + g_{ij} F^{0i} F^{0j} \right) \\ &\quad - \frac{1}{4\mu_0} \left(F^{00} F^{00} - g_{ij} F^{0i} F^{0j} - g_{ij} F^{i0} F^{j0} + g_{ij} g_{kl} F^{jl} F^{ik} \right) \end{aligned} \tag{1722}$$

[193]Note that the integral in terms of *proper* volume since the proper volume is $d\mathscr{V} = \sqrt{-g} dV$, where V is the *coordinate* volume and $\sqrt{-g}$ is the Jacobian expressed in terms of the determinant of the metric, g.

The components of the electromagnetic field tensor can be written in terms of the electric and magnetic fields as

$$F^{0i} = \frac{1}{c}E^i, \qquad F^{ij} = \varepsilon^{ijk}B^k, \qquad F^{\mu\mu} = 0, \qquad \text{where } F^{\mu\nu} = -F^{\nu\mu} \tag{1723}$$

Using these relations makes the stress tensor become

$$T^{00}_{(EM)} = \frac{1}{\mu_0 c^2} g_{ij} E^i E^j - \frac{1}{4\mu_0}\left(\frac{2}{c^2} g_{ij} E^i E^j - g_{ij} g_{kl} \varepsilon^{jlm} B^m \varepsilon^{ikn} B^n\right) \tag{1724}$$

The spatial metric can now be written as $g_{ij} = \delta_{ij} + h^{\tau\tau}_{ij}$. Separating out terms involving $h^{\tau\tau}_{ij}$, staying to first order in $h^{\tau\tau}_{ij}$, and using $\varepsilon_0 = 1/\left(\mu_0 c^2\right)$ gives

$$\begin{aligned} T^{00}_{(EM)} = {} & \frac{\varepsilon_0}{2}E^2 + \frac{1}{4\mu_0}\varepsilon^{ikm}\varepsilon^{ikn}B^m B^n + \frac{\varepsilon_0}{2} h^{\tau\tau}_{ij} E^i E^j \\ & + \frac{1}{4\mu_0}\left(h^{\tau\tau}_{ij}\delta_{kl} + \delta_{ij} h^{\tau\tau}_{kl}\right)\varepsilon^{jlm}\varepsilon^{ikn} B^m B^n \end{aligned} \tag{1725}$$

Using $\varepsilon^{ikm}\varepsilon^{ikn} = 2\delta^{mn}$ as well as

$$\varepsilon^{jlm}\varepsilon^{ikn} = \delta^{ij}\left(\delta^{kl}\delta^{mn} - \delta^{ln}\delta^{mk}\right) - \delta^{jk}\left(\delta^{il}\delta^{mn} - \delta^{ln}\delta^{mi}\right) - \delta^{jn}\left(\delta^{li}\delta^{mk} - \delta^{kl}\delta^{mi}\right) \tag{1726}$$

gives[194]

$$\begin{aligned} T^{00}_{(EM)} = {} & \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) + \frac{\varepsilon_0}{2} E^i E^j h^{\tau\tau}_{ij} \\ & + \frac{1}{2\mu_0}\left[3B^i B^j - \delta^{kl}\delta^j{}_k \delta^i{}_l B^2 + \delta^{kl}\delta^j{}_k B_l B^i - \delta^{kl}\delta^i{}_l B_k B^j\right] h^{\tau\tau}_{ij} \end{aligned} \tag{1727}$$

The first term is the standard result for the electromagnetic energy density in flat space-time. The second and third terms include the coupling to a gravitational wave field, $h^{\tau\tau}_{ij}$. Inserting (1727) into (1720) and staying to second order in the metric gives

$$E = \frac{1}{2}\int \left(1 - \tfrac{1}{2}h^2_{\oplus} - \tfrac{1}{2}h^2_{\otimes}\right)\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) dV \tag{1728}$$

Evaluating this integral requires explicit functions for the gravitational wave fields, $h_{\oplus}$ and $h_{\otimes}$, as well as the electromagnetic fields, E and B. Using (1332), the gravitational wave fields can be written as

$$h_{\oplus}(z,t) = A_{\oplus} e^{-z/\delta_G}\cos(\omega t) \qquad \text{and} \qquad h_{\otimes}(z,t) = A_{\otimes} e^{-z/\delta_G}\cos(\omega t) \tag{1729}$$

where δ_G is the frequency-dependent gravitational penetration depth, and $A_{\oplus}$ and $A_{\otimes}$ are amplitudes for *plus*-polarization and *cross*-polarization waves, respectively. To determine the electromagnetic fields inside the superconductor, we consider the case of a plus-polarized gravitational wave with normal incidence on a planar superconductor. As described in [7], it is expected that the edges of the superconductor will exhibit a charge-separation effect due to a difference in the way the Cooper pair density and the ionic lattice respond to the gravitational wave. This difference produces a surface charge density at the edges of the superconductor while the bulk of the superconductor remains electrically neutral. (The total charge is conserved since the surface charge on adjacent faces have different signs.)

[194] Note that upper indices which match lower indices imply a summation over indices, such as the upper and lower indices in $\delta^{kl}\delta^j{}_k = \delta^{1l}\delta^j{}_1 + \delta^{2l}\delta^j{}_2 + \delta^{3l}\delta^j{}_3$. However, lower indices that match are understood to be simply Kronecker deltas, such as $\delta_{ij}\delta_{ik} = \delta_{jk}$.

The following diagram (found in [7]) shows the charge accumulation on the edges of the planar superconductor at a particular instant.

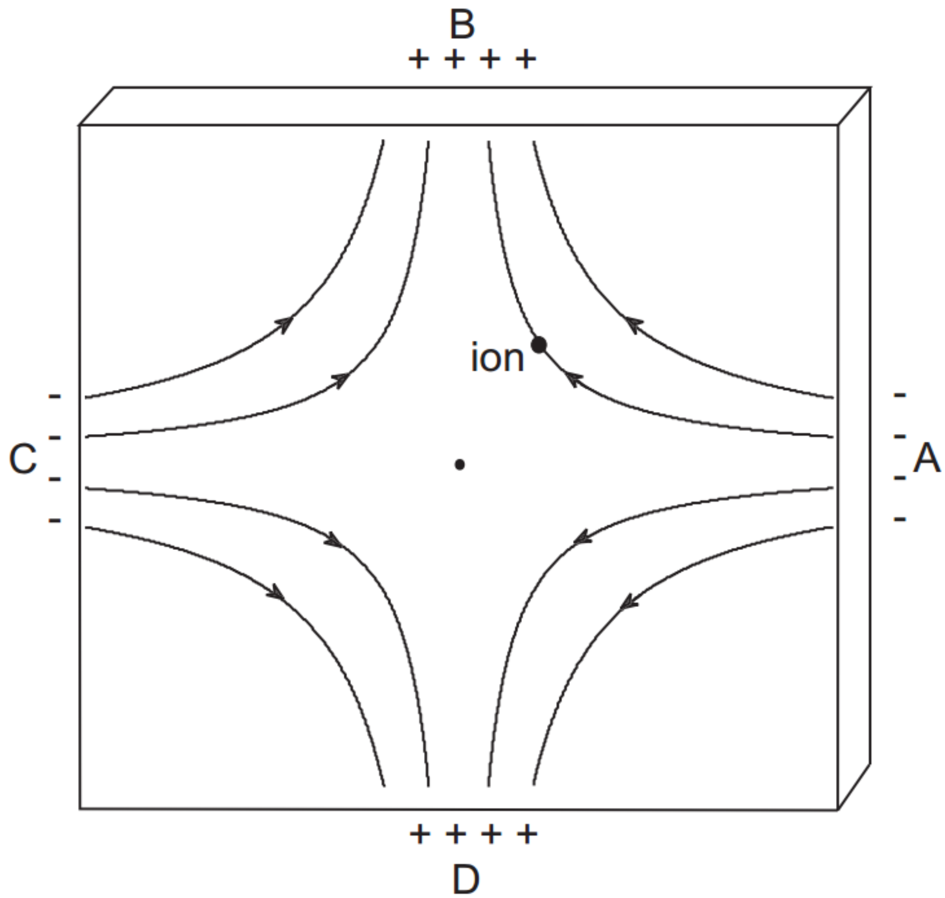

Figure 10: The charge separation effect in a superconductor (for a plus-polarized gravitational wave).

Since the supercurrents follow hyperbolic trajectories, it is expected that the associated electric field (due to a plus-polarized gravitational wave) will have a form given by

$$\vec{E} = \frac{E_0}{L}\left(-x\hat{x} + y\hat{y}\right) e^{-z/\lambda_L} \cos\left(\omega t\right) \tag{1730}$$

where L is the length of one side of a superconducting slab with a square face, and λ_L is the London penetration depth.[195] Inserting (1729) and (1730) into (1728), setting $t = 0$, and omitting the magnetic field gives[196]

$$E = \frac{\varepsilon_0 E_0^2}{2L^2} \int_0^L \int_0^L \int_0^d \left(1 - \tfrac{1}{2} A_{\oplus}^2 e^{-2z/\lambda_G}\right) \left(y^2 + x^2\right) e^{-2z/\lambda_L} dx dy dz \tag{1731}$$

[195] Using a *cross*-polarized gravitational wave would lead to an electric field induced in the superconductor given by $\vec{E} = \frac{E_0}{L}\left(y\hat{x} + x\hat{y}\right) e^{-z/\lambda_L} \cos\left(\omega t\right)$, which leads to the same result for the energy found in (1735).

[196] In the "DC limit," the frequency-dependent gravitational penetration depth, δ_G, becomes the frequency-independent value, λ_G.

Here d is the thickness of the superconducting slab. The integral can be written in cylindrical coordinates using $dA = dxdy = 2\pi r dr$ and $x^2 + y^2 = r^2$. Also, the upper bounds ($x = L$ and $y = L$) lead to $r = \sqrt{2}L$. Then the integral becomes

$$E = \frac{\pi\varepsilon_0 E_0^2}{L^2}\int_0^{\sqrt{2}L} r^3 dr \int_0^d \left(e^{-2z/\lambda_L} - \tfrac{1}{2}A_\oplus^2 e^{-2z(1/\lambda_G + 1/\lambda_L)}\right) dz \tag{1732}$$

Evaluating the integrals gives

$$E = \pi L^2 \varepsilon_0 E_0^2 \left[\left(-\frac{\lambda_L}{2}e^{-2z/\lambda_L} + \frac{1}{4}\left(\frac{\lambda_G\lambda_L}{\lambda_G+\lambda_L}\right)A_\oplus^2 e^{-2z(1/\lambda_G+1/\lambda_L)}\right)\right]_0^d \tag{1733}$$

If the thickness of the superconducting slab is much greater than the London penetration depth and gravitational penetration depth, then $d >> \lambda_L, \lambda_G$ which means $e^{-2d/\lambda_L} \approx e^{-2d/\lambda G} \approx 0$. This gives

$$E = \frac{\pi L^2 \lambda_L \varepsilon_0 E_0^2}{2}\left[\left(1 - \frac{\lambda_G}{2(\lambda_G+\lambda_L)}A_\oplus^2\right)\right] \tag{1734}$$

Since the amplitude of the gravitational wave is typically very small, $A_\oplus << 1$, then the second term can be neglected which leaves

$$E = \tfrac{1}{2}\pi L^2 \lambda_L \varepsilon_0 E_0^2 \tag{1735}$$

This is the total energy due to the electric field induced in the superconducting slab. It is essentially the energy density, $\frac{1}{2}\varepsilon_0 E_0^2$, multiplied by a volume, $\pi L^2 \lambda_L$. The Helmholtz free energy, $F = -k_B T \ln(Z)$, can now be evaluated use the partition function for a canonical ensemble, $Z = \sum \exp(\beta E_n)$, with E_n being the energy modes of the system, and $\beta = (k_B T)^{-1}$ where k_B is the Boltzmann constant, and T is the temperature. Since the electromagnetic field is induced by the incident gravitational wave, then the frequency of the electromagnetic field oscillation will be that same as the frequency of the gravitational wave. In that case, there is only a single mode, and the free energy is the same as the energy of the system.

$$F = -k_B T \ln(Z) = -k_B T \ln\left(e^{-\beta E}\right) = E \tag{1736}$$

Therefore, the electromagnetic free energy is just

$$F_{EM} = \tfrac{1}{2}\pi L^2 \lambda_L \varepsilon_0 E_0^2 \tag{1737}$$

The electromagnetic free energy *density* in curved space-time can be found by taking the derivative of the free energy with respect to *proper* volume. Recall that the coordinate-dependent volume can be expressed in terms of the proper volume as $dV = d\mathscr{V}/\sqrt{-g^{\tau\tau}}$, where $\sqrt{-g_\oplus^{\tau\tau}}$ is the Jacobian, and $g_\oplus^{\tau\tau}$ is the determinant of the metric containing only the transverse-traceless part of the metric perturbation for a plus-polarized gravitational wave. Then the electromagnetic free energy density can be expressed in terms of proper volume as

$$\mathscr{F}_{EM} \equiv \frac{dF_{EM}}{d\mathscr{V}} = \frac{1}{\sqrt{-g^{\tau\tau}}}\frac{dF_{EM}}{dV} \tag{1738}$$

Evaluating dF_{EM}/dV requires expressing (1737) in terms of the volume, $V = L^2 d$, where d is the thickness of the superconducting slab. This gives

$$F_{EM} = \frac{\pi V \lambda_L \varepsilon_0 E_0^2}{2d} \tag{1739}$$

Taking the derivative with respect to volume and inserting into (1738) gives the electromagnetic energy density with respect to proper volume in curved space-time.

$$\mathscr{F}_{EM} = \frac{1}{\sqrt{-g^{\tau\tau}}}\frac{\pi \lambda_L \varepsilon_0 E_0^2}{2d} \tag{1740}$$

Using $g_{\oplus}^{\tau\tau} = -1 + h_{\oplus}^2$ and applying a binomial approximation gives

$$\mathscr{F}_{EM} = \left(1 + \frac{1}{2}h_{\oplus}^2\right)\frac{\pi\lambda_L\varepsilon_0 E_0^2}{2d} \tag{1741}$$

The magnitude of the electric field induced in a superconductor can be determined using the charge-separation effect. For an order of magnitude approximation, the adjacent sides of a superconducting slab can be modeled as a parallel plate capacitor so that

$$E = \frac{\sigma}{\varepsilon_0} \tag{1742}$$

The surface charge density, $\sigma = Q/A$, can be determined using $Q = \rho V$, where ρ is volume charge density occupying a volume $V = A\Delta L$, where ΔL is the separation distance between the Cooper pair density and the ionic lattice. Also using $\rho = n_s e$, where n_s is the Cooper pair density, gives

$$\sigma = \frac{Q}{A} = \frac{\rho V}{A} = \frac{\rho A\Delta L}{A} = n_s e\Delta L \tag{1743}$$

The separation distance in the x-direction and y-direction can be found, respectively, using

$$\Delta L_x = u_{xx}L_x \qquad \text{and} \qquad \Delta L_y = u_{yy}L_y \tag{1744}$$

where L_x and L_y are the reference lengths of the superconductor along the x-direction and the y-direction, respectively, before a strain u_{ij} is applied. Arbitrarily choosing the x-direction or the y-direction, and using (1714) gives

$$\Delta L = \frac{\mu_{G(LI)}}{s}h_{\oplus}L \tag{1745}$$

Inserting this result into (1743) and putting the result into (1742) gives

$$E = \frac{n_s e\mu_{G(LI)}h_{\oplus}L}{\varepsilon_0 s} \tag{1746}$$

Finally, using $\mu_{G(LI)} = \frac{3}{2}\hbar\omega_D n$, where ω_D is the Debye frequency and n is the number density of atoms, gives

$$\boxed{E = \frac{3Ln_s en\hbar\omega_D}{2\varepsilon_0 s}h_{\oplus}} \tag{1747}$$

Electric field due to the charge-separation effect in a superconductor induced by a plus-polarized gravitational wave

As would be expected, the amplitude of the induced electric field in the superconductor is determined by the strength of the gravitational wave, $h_{\oplus}$. Inserting (1747) into (1741) and staying to second order in the metric gives

$$\boxed{\mathscr{F}_{EM} = \frac{9\pi L^2\lambda_L n_s^2 e^2 n^2\hbar^2\omega_D^2}{8d\varepsilon_0 s^2}h_{\oplus}^2} \tag{1748}$$

Electromagnetic free energy density in curved space-time in the presence of a plus-polarized gravitational wave

16.2 A gravitational shear modulus for the electromagnetic fields

Since $h_{11}^{\tau\tau} = -h_{22}^{\tau\tau} = h_{\oplus}$ for a plus-polarized gravitational wave, then the stress induced by this wave can be written correspondingly as $T_{11}^{\tau\tau} = -T_{22}^{\tau\tau} = T_{\oplus}$. Similar to (1477), $T_{\oplus}$ can then be found using

$$T_{\oplus} = -\left(\frac{\partial \mathscr{F}_{EM}}{\partial h_{\oplus}}\right)_T \tag{1749}$$

Applying this to (1748) gives

$$\boxed{T_{\oplus} = -\frac{9\pi L^2 \lambda_L n_s^2 e^2 n^2 \hbar^2 \omega_D^2}{8d\varepsilon_0 s^2} h_{\oplus}} \tag{1750}$$

Stress induced in the electromagnetic fields of a superconductor due to a plus-polarized gravitational wave

A constitutive equation can be written as $T_{\oplus} = -\mu_{G\ (EM)} h_{\oplus}$, where $\mu_{G\ (EM)}$ is the “gravitational modulus” of the electromagnetic fields. Then $\mu_{G\ (EM)}$ can be identified above as

$$\boxed{\mu_{G\ (EM)} = \frac{9\pi L^2 \lambda_L n_s^2 e^2 n^2 \hbar^2 \omega_D^2}{8d\varepsilon_0 s^2}} \tag{1751}$$

Gravitational modulus for the electromagnetic fields of a superconductor in response to a plus-polarized gravitational wave

This expression describes the effective “responsiveness” of the electromagnetic fields in response to the strain of a plus-polarized gravitational wave. It also demonstrates a linear response of the induced stress as a result of the incident gravitational wave strain. Since the energy found in (1735) is valid for both plus-polarization and cross-polarization, then the gravitational modulus found above is also valid for both polarizations.

For an order of magnitude approximation, consider a niobium superconducting slab ($n_s \approx 10^{26} m^{-3}$, $n_s \approx 6 \times 10^{28} m^{-3}$, $s \approx 4 \times 10^{10} J/m^3$, and $\lambda_L \approx 4 \times 10^{-8} m$) with centimeter edges and micrometer thickness. Also using $\omega_D \approx 3 \times 10^{13} s^{-1}$ gives

$$\mu_{G\ (EM)} \approx 10^{17} J/m^3 \tag{1752}$$

This value is clearly much greater than the gravitational modulus for the Cooper pairs ($\mu_{G\ (CP)} \approx 10^4 J/m^3$) or the lattice ions ($\mu_{G\ (LI)} \approx 10^8 J/m^3$).

16.3 The full gravitational shear modulus for a superconductor

The full gravitational shear modulus for a superconductor can be written as

$$\mu_{G\ (SC)} = \mu_{G\ (CP)} + \mu_{G\ (LI)} + \mu_{G\ (EM)} \tag{1753}$$

It was found that these gravitational shear moduli can be expressed as

$$\mu_{G\ (CP)} = \frac{mn_s v^2}{2} + \frac{\hbar^2 n_s}{4m_e \xi^2} \tag{1754}$$

$$\mu_{G\ (LI)} = \frac{3}{2}\hbar\omega_D n + \frac{\pi k_B^2 T^2}{3\hbar L^2}\sqrt{\frac{\rho}{s}} \tag{1755}$$

$$\mu_{G\ (EM)} = \frac{9\pi L^2 \lambda_L n_s^2 n^2 e^2 \hbar^2 \omega_D^2}{8d\varepsilon_0 s^2} \tag{1756}$$

For a niobium superconductor, at a temperature $T \approx 10^{-2}K$, with centimeter edges ($L \approx 10^{-2}m$) and micrometer thickness ($d \approx 10^{-2}m$), it was found that the maximum values for the gravitational shear moduli (in J/m^3) are

$$\mu_{G\ (CP)} \approx 10^4, \qquad \mu_{G\ (LI)} \approx 10^8, \qquad \mu_{G\ (EM)} \approx 10^{12} \tag{1757}$$

Recall that the gravitational penetration depth was found in (1337) to be

$$\lambda_G = \frac{1}{\sqrt{2\kappa\mu_{G\ (SC)}}} \tag{1758}$$

where $\kappa = \kappa_r \kappa_0$ and $\kappa_0 = 8\pi G/c^4$. Since $\mu_{G(EM)}$ dominates over all the other terms, then $\mu_{G\ (SC)} \approx \mu_{G\ (EM)}$. Therefore, using (1756) makes the gravitational penetration depth become

$$\boxed{\lambda_G = \frac{c^2 s}{6\pi L n_s e n \hbar\omega_D}\sqrt{\frac{2d\varepsilon_0}{G\kappa_r\lambda_L}}} \tag{1759}$$

Gravitational penetration depth of a superconductor

Hence it is found that the electromagnetic energy density due to the charge-separation effect ultimately determines the gravitational penetration depth. Since $\mu_{G\ (EM)} \approx 10^{17} J/m^3$, then the gravitational penetration depth (in meters) is

$$\lambda_G \approx \frac{10^{12}}{\sqrt{\kappa_r}} \tag{1760}$$

16.4 The Maxwell stress tensor in the presence of GR waves

The linearized Einstein field equation for gravitational waves (using the Helmholtz Decomposition approach) was found in (333) to be $\Box h_{ij}^{\tau\tau} = -2\kappa T_{ij}^{\tau\tau}$, where the source term involves the *transverse-traceless* stress tensor. Therefore, in this section the transverse-traceless part of the electromagnetic stress tensor, $T_{ij\,(EM)}^{\tau\tau}$, is evaluated starting from the full electromagnetic stress tensor in curved space-time given by

$$T_{(EM)}^{\mu\nu} = \frac{1}{\mu_0}\left(F^{\mu\alpha}F^{\nu}_{\alpha} - \frac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\right) \tag{1761}$$

where $F^{\mu\nu}$ is the electromagnetic field tensor. Using the metric to write the electromagnetic field tensor with raised indices gives

$$T_{(EM)}^{\mu\nu} = \frac{1}{\mu_0}\left(g_{\alpha\beta}F^{\mu\alpha}F^{\nu\beta} - \frac{1}{4}g^{\mu\nu}g_{\alpha\rho}g_{\beta\sigma}F^{\rho\sigma}F^{\alpha\beta}\right) \tag{1762}$$

The metric can be expanded as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. For gravitational waves in the far-field, $h_{ij} = h_{ij}^{\tau\tau}$ and $h_{0\mu} = 0$. Evaluating the spatial components of the stress tensor, $T_{(EM)}^{ij}$, summing over repeated indices, and using $\eta_{\mu 0} = (-1,1,1,1)$ gives

$$\begin{aligned} T_{(EM)}^{ij} &= \frac{1}{\mu_0}\left(-F^{i0}F^{j0} + g_{kl}F^{ik}F^{jl}\right) \\ &\quad -\frac{1}{4\mu_0}g^{ij}\left(F^{00}F^{00} - g_{kl}F^{0l}F^{0k} - g_{kl}F^{l0}F^{k0} + g_{kl}g_{mn}F^{lm}F^{kn}\right) \end{aligned} \tag{1763}$$

The components of the electromagnetic field tensor can be written in terms of the electric and magnetic fields as

$$F^{0i} = \frac{1}{c}E^i, \qquad F^{ij} = \varepsilon_{ijk}B^k, \qquad F^{\mu\mu} = 0, \qquad \text{where } F^{\mu\nu} = -F^{\nu\mu} \tag{1764}$$

Using these makes the stress tensor become

$$\begin{aligned} T_{(EM)}^{ij} &= \frac{1}{\mu_0}\left(-\frac{1}{c^2}E^iE^j + g_{kl}\varepsilon_{ikm}B^m\varepsilon_{jln}B^n\right) \\ &\quad +\frac{1}{4\mu_0}g^{ij}\left(\frac{2}{c^2}g_{kl}E^kE^l - g_{kl}g_{mn}\varepsilon_{lmp}B^p\varepsilon_{knq}B^q\right) \end{aligned} \tag{1765}$$

The metric can be written as $g_{ij} = \delta_{ij} + h_{ij}^{\tau\tau}$. Separating out terms involving $h_{ij}^{\tau\tau}$, staying to first order in $h_{ij}^{\tau\tau}$, and using $\varepsilon_0 = 1/\left(\mu_0 c^2\right)$ gives

$$\begin{aligned} T^{ij}_{(EM)} &= -\varepsilon_0 E^i E^j + \frac{1}{\mu_0}\varepsilon_{ikm}\varepsilon_{jkn}B^m B^n + \delta^{ij}\frac{\varepsilon_0}{2}E^2 - \frac{1}{4\mu_0}\delta^{ij}\varepsilon_{lnp}\varepsilon_{lnq}B^p B^q \\ &+\delta^{ij}h^{\tau\tau}_{kl}\varepsilon_0 E^k E^l - \frac{1}{4\mu_0}\delta^{ij}\left(h^{\tau\tau}_{kl}\varepsilon_{lmp}\varepsilon_{kmq} + h^{\tau\tau}_{mn}\varepsilon_{kmp}\varepsilon_{knq}\right)B^p B^q \\ &+h^{ij}_{\tau\tau}\left(\frac{1}{2}\varepsilon_0 E^2 - \frac{1}{4\mu_0}\varepsilon_{knp}\varepsilon_{knq}B^p B^q\right) + \frac{1}{\mu_0}h^{\tau\tau}_{kl}\varepsilon_{ikm}\varepsilon_{jln}B^m B^n \end{aligned} \quad (1766)$$

Using $\varepsilon_{kim}\varepsilon_{kjn} = \delta_{ij}\delta_{mn} - \delta_{in}\delta_{mj}$ and $\varepsilon_{lnp}\varepsilon_{lnq} = 2\delta_{pq}$ leads to

$$T^{ij}_{(EM)} = -\varepsilon_0 E^i E^j - \frac{1}{\mu_0}B^i B^j + \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)\delta^{ij} \quad (1767)$$

$$\begin{aligned} &+\delta^{ij}h^{\tau\tau}_{kl}\varepsilon_0 E^k E^l - \frac{1}{2\mu_0}\delta^{ij}\left(h^{\tau\tau}_{kl}\delta_{lk}B^2 - h^{\tau\tau}_{kl}B^k B^l\right) \\ &+h^{ij}_{\tau\tau}\left(\frac{1}{2}\varepsilon_0 E^2 - \frac{1}{2\mu_0}B^2\right) + \frac{1}{\mu_0}h^{\tau\tau}_{kl}\varepsilon_{ikm}\varepsilon_{jln}B^m B^n \end{aligned} \quad (1768)$$

The last term can be evaluated using

$$\varepsilon_{ikm}\varepsilon_{jln} = \delta_{ij}\left(\delta_{kl}\delta_{mn} - \delta_{kn}\delta_{ml}\right) - \delta_{il}\left(\delta_{kj}\delta_{mn} - \delta_{kn}\delta_{mj}\right) - \delta_{in}\left(\delta_{kj}\delta_{ml} - \delta_{kl}\delta_{mj}\right) \quad (1769)$$

This gives

$$\boxed{\begin{aligned} T^{ij}_{(EM)} &= -\varepsilon_0 E^i E^j - \frac{1}{\mu_0}B^i B^j + \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)\delta^{ij} \\ &+\left[\varepsilon_0 E^k E^l - \frac{1}{2\mu_0}\left(\delta_{lk}B^2 - B^k B^l\right)\right]\delta^{ij}h^{\tau\tau}_{kl} + \left(\frac{1}{2}\varepsilon_0 E^2 - \frac{1}{2\mu_0}B^2\right)h^{ij}_{\tau\tau} \\ &+\frac{1}{\mu_0}\left[\delta_{ij}\left(\delta_{kl}B^2 - B^k B^l\right) - \delta_{il}\left(\delta_{kj}B^2 - B^j B^k\right) - \delta_{kj}B^l B^i + \delta_{kl}B^j B^i\right]h^{\tau\tau}_{kl} \end{aligned}} \quad (1770)$$

Maxwell stress tensor with coupling to a gravitational wave

The top line is the standard result for the Maxwell stress tensor in flat space-time. The second and third lines have a similar form except they include coupling to the gravitational wave field, $h^{\tau\tau}_{ij}$. The transverse-traceless part of (1770) can be found by using

$$T^{\tau\tau}_{ij\ (EM)} = T_{ij\ (EM)} - \tfrac{1}{3}T_{(EM)}\delta_{ij} \quad (1771)$$

where $T_{(EM)}$ is the trace of $T_{ij\ (EM)}$. To first order in the metric, we can use $T_{(EM)} = \delta^{ij}T_{ij\ (EM)}$. Working to this order requires neglecting the terms in (1770) that involve $h^{\tau\tau}_{ij}$ so that it becomes simply

$$T^{ij}_{(EM)} = -\varepsilon_0 E^i E^j - \frac{1}{\mu_0}B^i B^j + \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)\delta^{ij} \quad (1772)$$

Then taking the trace gives

$$T_{(EM)} = -\varepsilon_0 E^2 - \frac{1}{\mu_0} B^2 + \frac{3}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) \tag{1773}$$

Putting this into (1392) and using (1772) gives

$$\boxed{T^{\tau\tau}_{ij\,(EM)} = -\varepsilon_0 E^i E^j - \frac{1}{\mu_0} B^i B^j + \frac{1}{3}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)\delta_{ij}} \tag{1774}$$

Transverse-traceless Maxwell stress tensor

Note that in order for this tensor to be transverse, it requires that $\partial_i B^i = 0$ (which is always true due to the absence of magnetic monopoles) and $\partial_i E^i = 0$. As shown in Appendix Q, the London formulation requires that $\nabla\varphi = 0$ so that $\vec{E} = -\partial_t \vec{A}$. Also, the London gauge requires $\partial_i A^i = 0$ in a superconductor, which therefore leads to $\partial_i E^i = 0$. Hence $T^{\tau\tau}_{ij\,(EM)}$ is in fact transverse as well as traceless.

17 The charge-separation effect and superconductors as “gravitational mirrors”

17.1 Equations of motion due to the charge-separation effect

In this section, equations of motion are found for the Cooper pair density and the ionic lattice in response to a plus-polarized gravitational wave with normal incidence on a planar superconductor. As described in previous sections, it is expected that the edges of the superconductor will exhibit a charge-separation effect due to a difference in the way the Cooper pair density and the ionic lattice respond to the gravitational wave. This difference produces a surface charge density at the edges of the superconductor while the bulk of the superconductor remains electrically neutral. (The total charge is conserved since the surface charge on adjacent faces have different signs.) The following figure shows the charge accumulation on the edges of the planar superconductor at a particular instant.

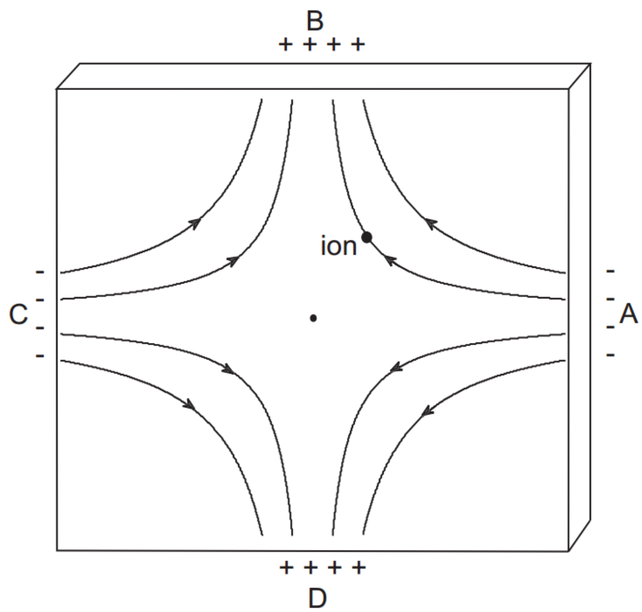

Figure 11: The charge separation effect in a superconductor (for a plus-polarized gravitational wave).

A Hooke's law response due to the combined electric and gravitational forces on the ions

Since the charge-separation effect will set up an electric field within the superconductor, then the positively charged lattice ions will experience an electric force as well as a gravitational force due to the gravitational wave. If the gravitational force on the lattice ions is described as $\vec{F}_G^{(ion)}$ and the electric force on the lattice ions is $\vec{F}_E^{(ion)}$, then the equation of motion for a lattice ions is

$$\vec{F}_G^{(ion)} + \vec{F}_E^{(ion)} = m^{(ion)}\vec{a}^{(ion)} \tag{1775}$$

The electric force will oppose the gravitational wave force at every point in the superconductor. This means that along a trajectory in any one of the quadrants of the figure above, the equation of motion in terms of magnitudes is

$$F_G^{(ion)} - F_E^{(ion)} = m^{(ion)}a^{(ion)} \tag{1776}$$

Furthermore, the lattice ions will have an oscillatory motion which implies that the total force acting on them is essentially a Hooke's law restoring force. Therefore, the net force (due to both the gravitational wave and the electric field) can be written as

$$F_G^{(ion)} - F_E^{(ion)} = -kf(\vec{x}) \qquad \text{where} \qquad -kf(\vec{x}) = m^{(ion)}a^{(ion)} \tag{1777}$$

Here k is an effective "spring constant" which characterizes the stiffness of the superconductor and $f(\vec{x})$ is a linear function of the position of the lattice ions which leads to a simple harmonic motion for the lattice ions.[197]

<u>The gravitational wave field, electric field, and acceleration field acting on ions</u>

A plus-polarized gravitational wave field acting on an ion (represented as a vector for a single particle in one quadrant) will have the form

$$\vec{h}_{\oplus} = \frac{A_{\oplus}}{\sqrt{2}L}\left(-x\hat{x}+y\hat{y}\right)\cos\left(kz-\omega t\right) \tag{1778}$$

where $A_{\oplus}$ is the amplitude of the gravitational wave, and $\vec{k} = k\hat{z}$ is the wave vector for a gravitational wave propagating in the z-direction.[198] Note that x and y are measured from the center of the superconducting slab, and $2L$ is the length of the superconducting slab in the x-direction or y-direction. Therefore, $x, y \leq L$ so that $\sqrt{x^2+y^2}$ is confined to the dimensions of the superconductor. It is evident that $\vec{h}_{\oplus}$ has a minimum value of zero at the center of the superconducting slab, and a maximum value at the corners of the superconductor where $x, y = \pm L$.

When the gravitational wave field is at a maximum, the charge separation effect will also be at a maximum which means that the resulting electric field must be in phase with the gravitational wave field. This implies that the electric field will have the form

$$\vec{E} = \frac{E_0}{\sqrt{2}L}\left(-x\hat{x}+y\hat{y}\right)\cos\left(kz-\omega t\right) \tag{1779}$$

where E_0 is the amplitude of the electric field. Once again, $\vec{E}$ has a minimum value of zero at the center of the superconductor, and a maximum value at the corners. The resulting acceleration field along a trajectory in the diagram above (in any one of the four quadrants) will have the form

$$\vec{a} = \frac{a_{\max}}{\sqrt{2}L}\left(-x\hat{x}+y\hat{y}\right)\cos\left(kz-\omega t\right) \tag{1780}$$

Then the magnitude of the acceleration is

$$a = \sqrt{\vec{a}^2} = \frac{a_{\max}}{\sqrt{2}L}\sqrt{(x^2+y^2)}\cos\left(kz-\omega t\right) \tag{1781}$$

From this expression, it is clear that the acceleration is zero at the center where $x = y = 0$. On the other hand, the acceleration is a maximum when $x, y = \pm L$. Once again, this corresponds to the corners of the

[197] Note that the geodesic equation leads to a *velocity*-dependent equation of motion. As shown in (2765), to lowest order in velocity, the equation of motion is $a^i = -\dot{h}^{\tau\tau}_{ij}v^j$. However, the geodesic equation is a *gauge-dependent* equation, in contrast to the the geodesic *deviation* equation which can be expressed in terms of the Riemann tensor and is therefore *gauge-invariant*. As shown in (527), to lowest order in velocity, the geodesic deviation equation is $\ddot{L}_i = \frac{1}{2}\ddot{h}^{\tau\tau}_{ij}L^j$ which is *not* velocity-dependent. It is linearly dependent on the displacement between two test masses in the presence of a gravitational wave. Therefore, it is consistent with (1777) which implies a Hook's law force that is displacement-dependent.

[198] Note that there is a limit to the value for $A_{\oplus}$ for a superconductor acting as a mirror for gravitationl waves. Press states in [73] that the condition for a mirror to be possible is $\mu_G \gtrsim \rho c^2$, where μ_G is the gravitational shear modulus and ρ is the mass density. A material that satisfies this condition will violate the dominant energy condition if μ_G remains constant up to strains of order unity. However, a material that is very rigid (such that $\mu_G \gtrsim \rho c^2$ only over a very small range of strain), will not violate the dominant energy condition.

superconductor.[199] Relating the acceleration field to the forces in (1776) implies that a maximum acceleration must correspond to a maximum in the *difference* between the gravitational wave force and the electric force. Since this occurs at the corners of the superconductor, then is is necessary to relate $a_{\max}$, $A_{\oplus}$, and E_0 at the corners of the slab.

The relative acceleration between lattice ions and Cooper pairs

The equation of motion for Cooper pairs has some important differences from the equation of motion for the lattice ions. This is due to two reasons: (i) the Cooper pair density has a negligible response to the gravitational wave compared to the ionic lattice[200] ($F_G^{(CP)} \approx 0$); and (ii) the Cooper pair density has the opposite charge of the lattice ion density. This means that the electric forces are related by

$$\vec{F}_E^{(CP)} = -\vec{F}_E^{(ion)} \tag{1782}$$

If we consider Cooper pairs on the same trajectory as the lattice ions, then the equation of motion has the same form as (1776) except the gravitational force is set to zero and the electric force is reversed. In terms of magnitudes, the force on Cooper pairs is

$$F_E^{(CP)} = m^{(CP)} a^{(CP)} \tag{1783}$$

Using (1782) gives

$$F_E^{(ion)} = -m^{(CP)} a^{(CP)} \tag{1784}$$

The relative acceleration between the ionic lattice and the Cooper pair density can be expressed as

$$a_{relative} \equiv a^{(ion)} - a^{(CP)} \tag{1785}$$

Using (1776) and (1784) gives

$$a_{relative} = \frac{F_G^{(ion)} - F_E^{(ion)}}{m^{(ion)}} + \frac{F_E^{(ion)}}{m^{(CP)}} \tag{1786}$$

Since $m^{(ion)} = 41\,(m_{proton} + m_{neutron})$ and $m^{(CP)} = 2m_e$, then $m^{(ion)}$ is greater than $m^{(CP)}$ by a factor of $\sim 10^3$. This would imply that the relative acceleration is dominated by the second term $-$ the acceleration of the Cooper pairs due to the electric field. However, if the superconductor is very "*stiff*" in response to gravitational waves, then k in (1777) could be a very large value. This could also make $f_G^{(ion)} - f_E^{(ion)}$ very large, depending on the magnitude of $f(\vec{x})$. A more formal analysis is required to determine the relative magnitude of these values.

Also, since the Cooper pairs are only accelerated by the electric field while the lattice ions are accelerated by the electric field as well as the gravitational field, then there is no reason to assume that the Cooper pairs and lattice ions will have the same amplitude of motion. It is expected that they should have the same frequency (assuming a linear response of all charges/masses to their corresponding fields). However, the difference in amplitude could make the relative motion of the Cooper pairs and lattice ions more complicated.

[199] For a *cross*-polarized gravitational wave, the acceleration field, gravitational wave field, and electric field would have $(y\hat{x} + x\hat{y})$ replacing $(-x\hat{x} + y\hat{y})$. The magnitude of the fields would all stay the same, however, the maxima would occur at points A, B, C, and D on the diagram, rather than at the corners.

[200] It was shown in (1713) that the relative strain between the lattice ions and Cooper pairs due to a gravitational wave is

$$u_{ij(relative)}^{\tau\tau} = \frac{\mu_{G(LI)} - \mu_{G(CP)}}{2s} h_{ij}^{\tau\tau}$$

Since $\mu_{G(CP)} \approx 10^4 J/m^3$ for the Cooper pairs and $\mu_{G(LI)} \approx 10^8 J/m^3$ for the lattice ions, then $\mu_{G(LI)} >> \mu_{G(CP)}$ which means the Cooper pair density can be considered as relatively unresponsive to the gravitational wave compared to the ionic lattice.

17.2 Concerning superluminal supercurrents in the "Mirrors" paper

In the paper, "Do gravitational mirrors exist?" (Minter, Wegter-McNelly, Chiao) [7] (henceforth referred to as the "Mirrors" paper), it is hypothesized that superconductors may act as mirrors for gravitational fields. The process for such reflection is said to require the presence of superluminal supercurrents.[201] Here we examine the details of this conclusion to determine if superluminal supercurrents are in fact predicted by the analysis in that paper.

A summary of the Mirrors paper analysis which led to the impression of superluminal supercurrents

The key equation in the Mirrors paper that is used to argue that the induced supercurrents are superluminal is equation (101). This gives the velocity of the induced supercurrents as

$$\left|\frac{\mathbf{v}}{c}\right| = \frac{1}{c}\frac{\Xi}{\Xi-1}\left|\mathbf{h}\right| \tag{1787}$$

where $\mathbf{h}$ is the gravitational vector potential and Ξ is a proportionality constant defined in equation (76) as

$$\mathbf{F}_{\text{tot}} = \Xi q\mathbf{E} \tag{1788}$$

This constant Ξ is referred to as the "fractional correction factor" of the total force acting upon a given Cooper pair relative to a purely electrical force acting on the same pair. The total force on a given Cooper pair is also stated in equation (74) as

$$\mathbf{F}_{\text{tot}} = q\mathbf{E} + m\mathbf{E}_G \tag{1789}$$

where $\mathbf{E}$ is the electric field and $\mathbf{E}_G$ is the gravito-electric field (which is essentially the Newtonian gravitational field). Equating (1788) and (1789) and solving for $\mathbf{E}$ gives

$$\mathbf{E} = \frac{1}{(\Xi-1)}\frac{m}{q}\mathbf{E}_G \tag{1790}$$

which is given in equation (77) of the paper. In equation (96) it is found that Ξ can be expressed as

$$\Xi = 1 - \frac{4\pi\varepsilon_0 G m_e^2}{e^2} \approx 1 - \frac{1}{4.2\times 10^{42}} \tag{1791}$$

In equation (69), the kinetic velocity of the quantum current is found to be

$$\mathbf{v} = -\frac{q}{m}\mathbf{A} - \mathbf{h} \tag{1792}$$

As shown in the Mirrors paper, we can take a time-derivative of (1792) and use equation (71) given as[202]

$$\mathbf{E} \;=\; -\frac{\partial}{\partial t}\mathbf{A} \qquad \text{and} \qquad \mathbf{E}_G = -\frac{\partial}{\partial t}\mathbf{h} \tag{1793}$$

to obtain equation (70) which is effectively

$$m\mathbf{a} = q\mathbf{E} + m\mathbf{E}_G \tag{1794}$$

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[201] The same discussion occurs in Stephen Minter's dissertation [9], p. 124.

[202] In Sections 4 and 5, there is a detailed discussion showing that the approximation of slow-moving sources (which is required to obtain the gravito-electromagnetic "Maxwell-like" equations) requires that $\partial_t\vec{h} = 0$. However, in this section, we will simply overlook this issue.

The acceleration for an oscillating electron can be written as $\mathbf{a} = \omega\mathbf{v}$ so that we obtain equation (73) in the Mirrors paper given as

$$m\omega\mathbf{v} = q\mathbf{E} + m\mathbf{E}_G \tag{1795}$$

Using this expression, the Mirrors paper points out the issue of a possible superluminal supercurrent. This is shown by substituting (1790) into (1795) which gives

$$m\omega\mathbf{v} = \frac{1}{\Xi - 1} m\mathbf{E}_G + m\mathbf{E}_G \tag{1796}$$

$$\omega\mathbf{v} = \mathbf{E}_G \left(\frac{1}{\Xi - 1} + 1 \right) \tag{1797}$$

$$\mathbf{v} = \frac{\mathbf{E}_G}{\omega} \left(\frac{\Xi}{\Xi - 1} \right) \tag{1798}$$

Since (1791) gives $\Xi \approx 1 - 10^{-43}$, then the expression above becomes

$$\mathbf{v} = \frac{\mathbf{E}_G}{\omega} \left(\frac{1 - 10^{-43}}{10^{-43}} \right) \tag{1799}$$

Since $1 - 10^{-43} \approx 1$, then we simply have

$$\mathbf{v} \approx \frac{\mathbf{E}_G}{\omega} 10^{43} \tag{1800}$$

The enormous value of 10^{43} is pointed out in the Mirrors paper as requiring the supercurrent to be superluminal in order to have a non-negligible gravitational field. For example, for microwave frequencies ($\omega \approx 10^{10} Hz$) and extremely small values of $\mathbf{E}_G$ (such as 10^{-24} SI units), the velocity will still be superluminal ($v \sim 10^9$ m/s).

Although the Mirrors paper refers specifically to (1795) as the equation which leads to a superluminal result, it can be shown that this same result follows from *any* form of the gravito-electromagnetic Lorentz force given by (1789), (1790), or (1795) which are equations 74, 77, and 73, respectively, in the Mirrors paper. However, next we will show that the result in (1800) simply follows from the fact that the entire analysis in the Mirrors paper assumes that $\mathbf{E}_G$ is extremely small. In fact, we will show that the paper implicitly treats $\mathbf{E}_G$ and $\mathbf{E}$ as essentially the gravitational and electric fields, respectively, between two electrons. Because the mass-to-charge ratio of the electron is so small, then naturally $\mathbf{E}_G$ is extremely small compared to $\mathbf{E}$. However, the process used to arrive at $\mathbf{v}$ in (1800) gives the impression that $\mathbf{E}_G$ can be made larger by allowing $\mathbf{v}$ to be made larger. This seems to imply that it is possible to have a non-negligible value for $\mathbf{E}_G$ by using a superluminal value for $\mathbf{v}$. This false impression caused by (1800) can be clarified by considering the following more concise approach to the analysis.

A streamlined analysis showing that superluminal supercurrents are *not* required

The most streamlined way to show the relevant relations is to consider the Coulomb force and the Newtonian force on a given electron due to another electron. Since they will point in opposite directions, then the total force is

$$\mathbf{F}_{\text{tot}} = (F_C - F_N)\hat{F} \tag{1801}$$

where F_C and F_N are the Coulomb and Newtonian forces, respectively. Using Coulomb's law and Newton's law of gravitation gives

$$\mathbf{F}_{\text{tot}} = \left(\frac{Ke^2}{r^2} - \frac{Gm_e^2}{r^2}\right)\hat{F} \tag{1802}$$

$$= \frac{Ke^2}{r^2}\left(1 - \frac{Gm_e^2}{Ke^2}\right)\hat{F} \tag{1803}$$

$$= e\mathbf{E}\left(1 - \frac{Gm_e^2}{Ke^2}\right) \tag{1804}$$

Also, using $q = e$ in (1788) gives $\mathbf{F}_{\text{tot}} = \Xi e\mathbf{E}$. Matching this to (1804), we see immediately that $\Xi = 1 - 4\pi\varepsilon_0 Gm_e^2/e^2$. From this result it is evident that Ξ can be expressed in terms of the ratio of the Coulomb electric force and the Newtonian gravitational force for the case of two electrons (or Cooper pairs) separated by an arbitrary distance.[203] Dividing the Newtonian gravitational force by the Coulomb electric force gives

$$\frac{F_N}{F_C} = \frac{Gm_e^2}{Ke^2} = \frac{4\pi\varepsilon_0 Gm_e^2}{e^2} \tag{1805}$$

where we have used $K = \dfrac{1}{4\pi\varepsilon_0}$. Therefore, (1791) can actually be written as

$$\Xi = 1 - \frac{F_N}{F_C} \tag{1806}$$

We can also define

$$\varepsilon \equiv \frac{F_N}{F_C} \approx 2.4 \times 10^{-43} \tag{1807}$$

so that $\Xi = 1 - \varepsilon$. This method for expressing Ξ is not explicitly shown in the Mirrors paper, however, it will prove useful as we examine the claim that the supercurrents are predicted to be superluminal. Specifically, this approach is instructive in demonstrating that the value of Ξ already assumes that $\mathbf{E}_G$ and $\mathbf{E}$ are essentially the gravitational and electric fields, respectively, between two electrons. This already *requires* $\mathbf{E}_G$ to be extremely small. We cannot change the value of $\mathbf{E}_G$ by requiring the supercurrents to be superluminal as (1800) would seem to suggest.

To highlight this point further, we can return to the expression in (1801) and show that this expression alone is all that is needed to examine the claim of superluminal supercurrents. First, we can write the total force for an oscillating electron as $\mathbf{F}_{\text{tot}} = m\mathbf{a} = m\omega\mathbf{v}$, which is similarly done in the Mirrors paper. Then the magnitude of (1801) becomes

$$m\omega v = F_C - F_N \tag{1808}$$

This is essentially the same as (1795) which is equation (73) in the Mirrors paper. The key is to recognize that there are two ways to rearrange this relation. We can either choose to factor out F_N (which leads to the false impression that superluminal supercurrents are necessary) or we can choose to factor out F_C (which does *not* lead to the impression that superluminal supercurrents are necessary). Here we show the two cases

[203] This ratio is discussed in [91], pp. 3-7. Specifically, when $F_N = F_C$, then the resulting "criticality" charge-to-mass ratio is $\dfrac{q}{m} = \sqrt{4\pi\varepsilon_0 G}$. This leads to an equality of quadrupolar EM and GR radiation.

side by side. In both cases, we make use of $\varepsilon = \dfrac{F_N}{F_C}$ from (1807).

$$\begin{aligned} m\omega v &= F_N\left(\frac{F_C}{F_N}-1\right) & \qquad m\omega v &= F_C\left(1-\frac{F_N}{F_C}\right) \\ &= F_N\left(\frac{1-\varepsilon}{\varepsilon}\right) & &= F_C\left(1-\varepsilon\right) \end{aligned} \tag{1809}$$

Here we can use $F_N = m_e E_G$ and $F_C = eE$. Then solving each expression for v gives

$$v = \frac{E_G}{\omega}\left(\frac{1-\varepsilon}{\varepsilon}\right) \qquad\qquad v = \frac{eE}{m_e\omega}\left(1-\varepsilon\right) \tag{1810}$$

Since $\alpha \approx 10^{-43}$, then $(1-\varepsilon) \approx 1$ and we have

$$v = \frac{E_G}{\omega\varepsilon} \approx \frac{E_G}{\omega}\left(10^{43}\right) \qquad\qquad v \approx \frac{eE}{m_e\omega} \tag{1811}$$

The first expression in (1811) is the same as the expression found in (1800) which led to the impression in the Mirrors paper that superluminal supercurrents are required for a non-negligible E_G field. However, the second expression in (1811) does *not* indicate the need for a superluminal supercurrent. In fact, the second expression in (1811) is simply the Lorentz force on an electron which is oscillating in the presence of an oscillating electric field. If we require the velocity to be superluminal, then setting $v > c$ in the second expression of (1811) gives

$$\frac{eE}{m_e\omega} > c \qquad \Rightarrow \qquad E > \frac{m_e\omega c}{e} \tag{1812}$$

For microwave frequencies ($\omega \approx 10^{10} Hz$), this implies $E > 1.07 \times 10^9$ N/C which is clearly an enormous electric field. This demonstrates that in order to maintain consistency in the formulation of the Mirrors paper, it is necessary for E_G to remain a very small value. Increasing the value of E_G by increasing the value of v (even to large but subluminal values), will require that E is also increased accordingly. It can be misleading to express v in terms of just E_G (without E appearing) as is done in (1811). This gives the impression that v and E_G can be adjusted *independently* of E. However, this is not the case since we find that v, E_G, and E are all related by (1795).

The fundamental reason for this interdependence of v, E_G, and E is that the same particle (the electron) plays the role of both the *mass*-carrier as well as the *charge*-carrier. It is not possible to change the velocity and the gravito-electric field independently of changing the electric field. Also, the electric field will always vastly dominate the dynamics compared to the gravito-electric field. Therefore, the velocity of the electron is effectively "slaved" to the electric field. Attempting to change the velocity and gravito-electric field will quickly lead to huge values of the electric field.

Some additional comments concerning footnotes [33] and [34] of the Mirrors paper

From (1810) we pointed out that the equation which gave the impression that superluminal supercurrents are required for a non-negligible gravitational field is

$$|\mathbf{v}| = \frac{|\mathbf{E}_G|}{\omega}\left|\frac{1-\alpha}{\alpha}\right| \tag{1813}$$

Since (1793) gives $\mathbf{E}_G = -\partial_t \mathbf{h}$, then for a sinusoidally oscillating field, we have $\mathbf{E}_G = \omega \mathbf{h}$. Also using $|1-\alpha| = \Xi$ and dividing by c gives

$$\left|\frac{\mathbf{v}}{c}\right| = \frac{1}{c}\frac{\Xi}{\Xi - 1}\left|\mathbf{h}\right| \tag{1814}$$

This relation is shown in equation (101) of the Mirrors paper. However, equation (101) also includes an additional equality. It is given as

$$\left|\frac{\mathbf{v}}{c}\right| = \frac{1}{c}\frac{\Xi}{\Xi-1}|\mathbf{h}| = \frac{1}{2}\frac{\Xi}{\Xi-1}|h_{+}| \tag{1815}$$

The last part of this equation involves h_+ which is the strain field for a gravitational wave with plus polarization. The relationship between the gravito-vector potential, $\mathbf{h}$, and the strain field, h_+, is found in footnote [33] of the Mirrors paper. The derivation uses equation (7) from Saulson's paper [31] which gives the gravitational wave flux as

$$\mathscr{F} = \frac{1}{32\pi}\frac{c^3}{G}\omega^2 h_+^2 \tag{1816}$$

where h_+ is the amplitude of the gravitational wave.[204] Although Saulson does not show this explicitly, (1816) comes from the Isaacson power flux formula which gives the power flux of a gravitational wave. It is given in [43] as

$$\mathscr{P}_{Isaacson} = \frac{c^3}{16\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle \tag{1817}$$

where P is the power per unit area while h_+ and $h_\times$ are the plus and cross polarization strain fields. Using a sinusoidal strain field for plus polarization, $h_{xx} = h_+ \sin\left(\vec{k}\cdot\vec{x} - \omega t\right)$, the power flux becomes

$$\mathscr{P}_{Isaacson} = \frac{c^3}{16\pi G}\left\langle \omega^2 h_+^2 \sin^2\left(\vec{k}\cdot\vec{x} - \omega t\right)\right\rangle = \frac{c^3\omega^2 h_+^2}{32\pi G} \tag{1818}$$

which matches Saulson's expression in (1816).

Therefore, Saulson used a correct representation of the gravitational wave as a *strain* field (which is a *tensor* wave by nature), and a power given by the Isaacson power flux formula. In the Mirrors paper, the gravitational wave is described as a *vector* wave consisting of oscillating gravito-electromagnetic fields. Accordingly, there is a "time-averaged Poynting vector" written in the first line of equation (151) as

$$\langle S\rangle = \frac{1}{2}\hat{\mathbf{k}}\cdot\mathrm{Re}\left(\mathbf{E}^*_{\text{G-incident}} \times \mathbf{H}_{\text{G-incident}}\right) \tag{1819}$$

where, presumably, $\mathbf{H}_{\text{G-incident}}$ is the (auxiliary) gravito-magnetic field. In the second line of equation (151) of the Mirrors paper, we find

$$\langle S\rangle = \frac{1}{2}\frac{1}{Z_G}\mathrm{Re}\left(\mathbf{E}^*_{\text{G-incident}} \times \mathbf{E}_{\text{G-incident}}\right) = \frac{1}{2Z_G}\left|\mathbf{E}^*_{\text{G-incident}}\right|^2 \tag{1820}$$

This implies that the relation $\mathbf{H}_{\text{G-incident}} = Z_G\mathbf{E}_{\text{G-incident}}$ was used. However, this relation can *not* be obtained from the gravito-electromagnetic field equations. This is discussed in detail in Section 23. The reason is because we do not have the relation $\mathbf{B}_G = \mathbf{E}_G/c$ for gravitation. The analogous relationship in electromagnetism, $\mathbf{B} = \mathbf{E}/c$, occurs as a result of considering plane *wave* solutions to the *vector* wave equations involving $\mathbf{E}$ and $\mathbf{B}$. Such waves do not exist for gravitation. Nevertheless, continuing on from equation (152) of the Mirrors paper, we find that the time-averaged energy flux of the gravito-electric field is given as[205]

$$\langle S\rangle = \frac{1}{2Z_G}\langle E_{\text{G-incident}}\rangle^2 \tag{1821}$$

[204] Saulson uses h_0 for the wave amplitude. However, here we use h_+ to follow the notation of the Mirrors papers.

[205] This expression treats $\mathbf{E}_G$ as completely analogous to $\mathbf{E}$ by assuming that just as the time-average power flux of $\mathbf{E}$ is $\langle S_E\rangle = \frac{1}{2Z_0}|\mathbf{E}|^2$, so also the time-average power flux of $\mathbf{E}_G$ is $\langle S_{E_G}\rangle = \frac{1}{2Z_G}|\mathbf{E}_G|^2$. However, in General Relativity, there is a long standing issue with attempting to describe a local energy density of a gravitational field. This is due to the fact that the Equivalence Principle allows the field (and hence the energy) to vanish by transforming to a local Lorentz (freely-falling) frame. (For a discussion of this issue, see MTW [11], pp. 466-467).

From (1793), we have $|\mathbf{E}_G| = |\partial_t \mathbf{h}| = \omega\, |\mathbf{h}|$ for a sinusoidally oscillating field. Also, using the gravitational impedance, $Z_G = 4\pi G/c$, then (1821) becomes

$$\langle S \rangle = \frac{c}{8\pi G}\omega^2\, |\mathbf{h}|^2 \tag{1822}$$

At this point the assumption is made that the gravitational wave flux found by Saulson in (1816) is the same as the gravitational wave flux for a gravito-electric field found in (1822). Therefore, these equations are set equal to obtain

$$\frac{|\mathbf{h}|}{c} = \frac{|h_+|}{2} \tag{1823}$$

This relation is what leads to the last equality in (1815). From there the Mirrors paper proceeds with the issue of superluminal supercurrents. However, because $\mathbf{v}$ is now expressed in terms of h_+, then the argument is described in terms of the gravitational *wave* field, rather than the *gravito-electric* field or *gravito-vector potential* field. Nevertheless, this alternative form does not affect the question of whether there are superluminal supercurrents. In fact, the additional steps to express $\mathbf{v}$ in terms of h_+ (instead of $\mathbf{E}_G$ or $\mathbf{h}$) may only cloud the issue. The issue is shown simply in (1811) where we find that expressing $\mathbf{v}$ in terms of $\mathbf{E}_G$ (without $\mathbf{E}$ appearing) gives the impression that a non-negligible $\mathbf{E}_G$ can be achieved via a superluminal supercurrent.

Since $\mathbf{E}_G$ and $\mathbf{h}$ are not radiation fields (but rather near-zone fields), then they cannot be related to h_+ by equating a time-averaged energy density of $\mathbf{E}_G$ (1822) to the Isaacson power flux of h_+ (1816). Rather, these two fields can be related (at least approximately) by their relationship to a common source. Specifically, we know from (437) that the components of the trace-reversed metric perturbation are related by

$$\overline{h}_{0i} \sim \overline{h}_{00}\left(\frac{v_s}{c}\right) \qquad \text{and} \qquad \overline{h}_{ij} \sim \overline{h}_{00}\left(\frac{v_s}{c}\right)^2 \tag{1824}$$

where v_s is the characteristic speed of the sources of the gravitational fields. Also, in (32) we wrote the gravito-scalar potential and gravito-vector potential each in terms of the trace-reversed metric perturbation as, respectively,

$$\varphi_G = -\frac{c^2}{4}\overline{h}^{00} \qquad \text{and} \qquad \mathbf{h} = \frac{c}{4}\left(\overline{h}_{01},\, \overline{h}_{02},\, \overline{h}_{03}\right) \tag{1825}$$

These are the relationships that lead to the "Maxwell-like" gravito-electromagnetic equations in (58) which are similar to those used in the Mirrors paper. Also, the gravito-electric field is defined in (46) as $\mathbf{E}_G = -\nabla\varphi_G$. For sinusoidally varying fields, we can use $\nabla\varphi_G = k\varphi_G$ where $k = \omega/c$ in vacuum. Combining this with $\varphi_G = -\frac{c^2}{4}\overline{h}^{00}$ from (1825) gives

$$|\mathbf{E}_G| = \frac{\omega c}{4}\overline{h}_{00} \tag{1826}$$

Then using $\overline{h}_{00} \approx \overline{h}_{0i}\left(\frac{c}{v_s}\right)$ from (1824) and $\mathbf{h} = \frac{c}{4}\left(\overline{h}_{01},\, \overline{h}_{02},\, \overline{h}_{03}\right)$ from (1825) gives

$$|\mathbf{E}_G| \approx \frac{\omega c}{4}\left(\frac{c}{v_s}\right)\left(\frac{4}{c}\,|\mathbf{h}|\right) = \frac{\omega c}{v_s}\,|\mathbf{h}| \tag{1827}$$

Inserting this into (1813), using $\Xi = 1-\alpha$, and dividing by c gives

$$\left|\frac{\mathbf{v}}{c}\right| = \frac{1}{v_s}\left(\frac{\Xi}{\Xi - 1}\right)|\mathbf{h}| \tag{1828}$$

This would be the appropriate expression for $|\mathbf{v}|$ rather than (1814) which relies on $\mathbf{E}_G = -\partial_t \mathbf{h}$, which is a relationship we have shown in Section 4 is not valid in this approximation. Next we can combine the two expressions in (1824) to obtain $\overline{h}_{ij} \approx \overline{h}_{0i}\left(\frac{v_s}{c}\right)$. Then using $\mathbf{h} = \frac{c}{4}\left(\overline{h}_{01},\, \overline{h}_{02},\, \overline{h}_{03}\right)$ from (1825) gives

$\bar{h}_{ij} \approx \frac{4v_s}{c^2}|\mathbf{h}|$. In the transverse-traceless gauge, we have $\bar{h}_{xx} = h_+$ for a plus-polarization wave. This means that

$$|\mathbf{h}|\left(\frac{v_s}{c^2}\right) = \frac{|h_+|}{4} \tag{1829}$$

This clearly differs from the result found in the Mirrors paper as shown in (1823). Solving (1829) for $|\mathbf{h}|$ and inserting into (1828) gives

$$\left|\frac{\mathbf{v}}{c}\right| = \frac{1}{v_s}\left(\frac{\Xi}{\Xi-1}\right)|\mathbf{h}| = \frac{c^2}{4v_s^2}\left(\frac{\Xi}{\Xi-1}\right)|h_+| \tag{1830}$$

We have now obtained a complete replacement for equation (101) in the Mirrors paper. This new expression gives a more valid relationship[206] between $|\mathbf{v}|$, $|\mathbf{h}|$, and $|h_+|$. If we consider the motion of the test mass, $|\mathbf{v}|$, to be comparable to the motion of the gravitational sources, v_s, then $|\mathbf{v}| \approx v_s$. In that case, multiplying (1830) through by v/c gives

$$\left|\frac{\mathbf{v}}{c}\right|^2 = \left(\frac{\Xi}{\Xi-1}\right)\frac{|\mathbf{h}|}{c} = \frac{c}{4v}\left(\frac{\Xi}{\Xi-1}\right)|h_+| \tag{1831}$$

Relating $|\mathbf{v}|$ on the far left to $|h_+|$ on the far right gives

$$\frac{v^3}{c^3} = \frac{1}{4}\left(\frac{\Xi}{\Xi-1}\right)|h_+| \tag{1832}$$

Using $\left(\frac{\Xi}{\Xi-1}\right) \approx 10^{43}$ gives

$$\frac{v}{c} \approx \left(1.4\times 10^{14}\right)|h_+|^{1/3} \tag{1833}$$

In this case, we find that $v < c$ when $|h_+| < 3.6\times 10^{-43}$. In fact, the velocity is 10% of the speed of light when $|h_+| < 3.6\times 10^{-46}$.

As a final note, we point out that the issue of group velocity versus phase velocity (as discussed in footnote [34] of the Mirrors paper) is not applicable to the gravito-Lorentz force which is a force acting on a *point particle*. One must recall that the gravito-Lorentz force is derived from the geodesic equation of motion (or the Lorentz force in curved space-time if electromagnetic fields are also included). The geodesic equation describes a *single* worldline of a completely localized particle. In fact, the geodesic equation can even be derived directly from the Equivalence Principle (see Weinberg's text, [32]) which is only valid for a point-like particle, not for an extended object such as a wave. This is further demonstration that phase velocity is not applicable to the analysis using the gravito-Lorentz force.

[206]Formally speaking, the expression in (1830) is still not legitimate because the gravito-electromagnetic "Maxwell-like" equations involve the approximation that $h_{ij} \approx 0$ which means that there are no gravitational waves. Therefore, the appearance of h_+ in (1830) is technically not consistent.

17.3 Alternative formulations for the "Mirrors" paper

There are three critical issues in the "mirrors" paper that need to be addressed: 1. The coupling rule; 2. The field equation; 3. The equation of motion. The following are possible replacements. The Newtonian field equation, $\nabla \cdot \mathbf{E}_G = -\rho/\varepsilon_G$, could be replaced with the field equation for the gauge-invariant, transverse-traceless strain field.

$$\Box h_{ij}^{\tau\tau} = -\frac{16\pi G}{c^4} T_{ij}^{\tau\tau} \tag{1834}$$

The non-relativistic equation of motion in terms of the gravito-electric field, $m\vec{a} = q\vec{E} + m\vec{E}_G$, could be replaced with the geodesic equation of motion (to lowest order in velocity) in terms of the gravitational wave field.

$$ma_i = qE_i - mv^j \dot{h}_{ij}^{\tau\tau} \tag{1835}$$

For the coupling rule, the DeWitt coupling rule in terms of the gravito-vector potential, $p_i \implies p_i - mch_{0i}$, can be replaced with the coupling rule for a gravitational wave field given by

$$p^2 \implies p^2 - p^i p^j h_{ij} \tag{1836}$$

The equation of motion in terms of a factional correction factor

Using $ma_i = \Xi qE_i$ to express (1835) in terms of a fractional correction factor, Ξ, and solving for the electric field gives

$$\Xi qE_i = qE_i - mv^j \dot{h}_{ij}^{\tau\tau} \tag{1837}$$

$$E_i = \frac{1}{1-\Xi}\frac{m}{q} v^j \dot{h}_{ij}^{\tau\tau} \tag{1838}$$

Assuming a sinusoidal solution for the acceleration gives

$$m\omega v_i = qE_i - mv^j \dot{h}_{ij}^{\tau\tau} \tag{1839}$$

$$v_i = \frac{1}{\omega}\left(\frac{\Xi}{1-\Xi}\right) v^j \dot{h}_{ij}^{\tau\tau} \tag{1840}$$

It is evident that we can not simply solve for the velocity because it is contracted with the metric on the right side. We could sum over the indices and solve for the separate components of the velocity, but we will not be able to get a single vector relationship. Also note that it's not clear if this equation permits sinusoidal solutions.

$$m\frac{\partial^2}{\partial t^2} x_i(t) = qE_i(t) - m\dot{h}_{ij}^{\tau\tau}(t)\, v^j(t) \tag{1841}$$

We could assume that the gravitational field, the electric field, and the particle motion all have the same frequency and vary sinusoidally with the same phase. However, instead we will just assume that $x_i = A_i \cos(\omega t)$ is a valid solution for the position and leave the frequency and phase of $h_{ij}^{\tau\tau}$ and E_i unspecified. Then we have

$$-m\omega^2 A_i \cos(\omega t) = qE_i(t) - m\omega \dot{h}_{ij}^{\tau\tau}(t) A^j \sin(\omega t) \tag{1842}$$

At $t = 0$, we have

$$-m\omega^2 A_i = qE_{i(0)} \tag{1843}$$

This means that $x_i(t)$ and $E_i(t)$ must be in phase otherwise the amplitude of motion would be zero. Similarly, a half cycle later at $t = T/2$

$$0 = qE_{i(t=T/2)} - m\omega \dot{h}^{\tau\tau}_{ij(t=T/2)} A^j \tag{1844}$$

This means that the electric field and the gravitational field can also be in phase. Therefore, it appears that we can have the displacement, electric field, and gravitational field all in phase. We can write $E_i(t) = E_{i,0}\cos(\omega t)$ and $h^{\tau\tau}_{ij}(t) = h^{\tau\tau}_{ij,0}\cos(\omega t)$ where $h^{\tau\tau}_{ij,0}$ is a constant transverse-traceless amplitude tensor. Then the equation of motion becomes

$$-m\omega^2 A_i \cos(\omega t) = qE_{i,0}\cos(\omega t) - m\omega^2 A^j h^{\tau\tau}_{ij,0}\sin^2(\omega t) \tag{1845}$$

Notice that from this result we cannot proceed to a simplified expression because the terms above do not all have the same sinusoidal dependence. In fact, the last term (which describes the force due to the gravitational wave) appears as a squared sinusoid. This is a second harmonic behavior which is similar to the second harmonic behavior found in Appendix S for a single mass oscillator generating gravitational waves. Also notice that the amplitude of motion is summed over the amplitude tensor of the gravitational wave. As a result of these complications, it is not possible to write a linear expression relating the displacement to the electric field by a proportionality constant.

17.4 A new fractional correction factor for gravitational waves

The factional correction factor in (1806) is shown to be $\Xi = 1 - \varepsilon$, where ε is given by (1807) as $\varepsilon = F_N/F_C$. This is the ratio of the Newtonian gravitational force to the Coulomb electric force between two elections. We can define the reciprocal of this quantity as

$$\varepsilon \equiv \frac{F_N}{F_C} = \frac{Gm_e^2}{Ke^2} \approx 2.4 \times 10^{-43} \tag{1846}$$

This is described in the Mirrors paper as a huge enchancement factor in the response of a superconductor to an incident gravitational wave. The justification is that the charge-separation effect produced in a superconductor in response to a gravitational wave will cause an electric force on the Cooper pairs that is greater than the gravitational force by a factor of $1/\varepsilon \approx 10^{42}$, as evidenced by the value in (1846). Therefore, the electric force acts as an extremely stiff Hooke's law back-action that leads to the reflection of the gravitational wave.

Notice that (1846) involves the *Newtonian* gravitational force rather than the gravitational *wave* force. Therefore, we would like to derive an alternative to (1846) that compares the electric force to the gravitational wave force. We can define this quantity as

$$\alpha \equiv \frac{F_{GR\ wave}}{F_C} \tag{1847}$$

Before evaluating an expression for this quantity, we first note some important features about (1846) that do not occur for (1847). First, we find that (1846) only involves fundamental constants of nature (K and G) and *static* properties of the Cooper pair particles ($q = 2e$ and $m = 2m_e$). This fact is due to the symmetry between the Coulomb electric potential and the Newtonian gravitational potential. The potential field equations are given by

$$\nabla^2\varphi = \rho_c/\varepsilon_0 \qquad \text{and} \qquad \nabla^2\varphi_G = \rho_m/\varepsilon_G \tag{1848}$$

Using Green's function solutions, these lead to

$$\varphi(\vec{r}) = K\int \frac{\rho_c(\vec{r}')}{|\vec{r}-\vec{r}'|}dV' \qquad \text{and} \qquad \varphi_G(\vec{r}) = G\int \frac{\rho_m(\vec{r}')}{|\vec{r}-\vec{r}'|}dV' \tag{1849}$$

Since φ and φ_G are both *static* fields with inverse square laws, then the $1/r^2$ dependence vanishes in the ratio given in (1846) and there is no dependence on velocity, frequency, or any other dynamic quantity. There is also symmetry in the equations of motion which are

$$\vec{a} = -\frac{q}{m}\nabla\varphi \qquad \text{and} \qquad \vec{a} = -\nabla\varphi_G \tag{1850}$$

Note that in both cases, the acceleration is given by a type of charge-to-mass ratio multiplying a field. For the electric case, the charge-to-mass ratio is q/m which multiplies the electric field. For the gravitational case, the ratio of "gravitational charge" to the mass is unity by the Equivalence Principle. All of these characteristics make the the Coulomb electric force and the Newtonian gravitational force similar which leads to the result in (1846). In fact, from (1849) and (1850), it is clear that we can express (1846) as

$$\varepsilon = \frac{m\nabla\varphi_G}{q\nabla\varphi} \tag{1851}$$

Now we consider the case for the ratio of the electric force to the gravitational wave force which we defined in (1847) as

$$\alpha \equiv \frac{F_{GR\ wave}}{F_C} \tag{1852}$$

Note that the gravitational wave field is essentially a *potential* field (in the sense that its *derivative* appears in the equation of motion, similar to φ and $\vec{A}$ for EM). Therefore, to draw a proper comparison, we need to compare the electric *potential* to the gravitational wave *potential* similar to the comparison in (1849).

Note that the field equation for the transverse-traceless strain field is $\Box h_{ij}^{\tau\tau} = -2\kappa T_{ij}^{\tau\tau}$. The Green's function solution is therefore

$$h_{ij}^{\tau\tau}(t,\vec{r}) = \frac{\kappa}{2\pi}\int \frac{T_{ij}^{\tau\tau}(t_r,\vec{r}')}{|\vec{r}-\vec{r}'|}dV' \tag{1853}$$

where t_r is the retarded time. To draw a closer comparison between the electric potential and the gravitational wave potential, we can take the static limit for the gravitational wave field so that the box operator becomes a Laplacian. If we consider the case of relativistic dust, then (431) shows that the transverse-traceless stress is $T_{ij}^{\tau\tau} = \rho_m\left(v_iv_j - \frac{1}{3}\delta_{ij}v^2\right)$. Then we can write a comparison between the electric potential and the gravitational wave potential similar to the comparison given by (1849). Using $\kappa = 8\pi G/c^4$, we have

$$\varphi(\vec{r}) = K\int \frac{\rho_c(\vec{r}')}{|\vec{r}-\vec{r}'|}dV' \qquad \text{and} \qquad h_{ij}^{\tau\tau}(\vec{r}) = \frac{4G}{c^4}\int \frac{\rho_m\left(v_iv_j - \frac{1}{3}\delta_{ij}v^2\right)}{|\vec{r}-\vec{r}'|}dV' \tag{1854}$$

If we consider just the $h_{xy}^{\tau\tau}$ component for a cross-polarization gravitational wave, with $v_x = v_y$, then

$$\left(v_iv_j - \tfrac{1}{3}\delta_{ij}v^2\right) = v_xv_y - \tfrac{1}{3}\delta_{ij}v^2 = \tfrac{2}{3}v^2 \tag{1855}$$

For a spherically symmetric *charge* distribution, the electric potential reduces to the familiar expression for a point charge. If we also assume a uniform *mass* distribution, then we can also approximate the gravitational wave potential. Therefore the expressions above become

$$\varphi(\vec{r}) = \frac{Kq}{r} \qquad \text{and} \qquad A_{\otimes}(\vec{r}) \approx \frac{G}{c^4}\frac{mv_s^2}{r} \tag{1856}$$

where $A_{\oplus}$ represents the amplitude of a cross-polarization gravitational wave and v_s is the speed of the gravitational source. Next we use the geodesic equation of motion (1835) for charged particles in the presence of gravitational waves and an electric field. Writing it in terms of the electric potential and a *plus*-polarization gravitational wave, $h_{ij}^{\tau\tau}(t) = h_{ij,0}^{\tau\tau}\cos(\omega t)$ where $h_{12} = h_{21} = A_{\otimes}$, gives

$$ma_i = -q\partial_i\varphi - 2mA_{\otimes}\omega v_p \tag{1857}$$

where v_p is the speed of the particles in the presence of a gravitational wave. Therefore, we can write the magnitude of the gravitational wave force as $F_{GR\ wave} = mv_p\omega A_{\oplus}$. We can also use $A_{\otimes}$ from (1856). Inserting these expressions into (1852) gives

$$\alpha = \frac{mv_p\omega\left(\frac{G}{c^4}\frac{mv_s^2}{r}\right)}{Kq^2/r^2} = \frac{Gm^2}{Kq^2}\frac{\omega v_pv_s^2}{c^4}r \tag{1858}$$

Since we are considering a situation where the particles that are oscillated by the gravitational wave are also the same particles that will produce the outgoing gravitational wave, then we have $v = v_p = v_s$. Also, since we are working in the near-field zone (where the electric field is predominantly the Coulomb field), then we can approximate $k \approx 1/r$, where $k = \omega/v_{wave}$ with v_{wave} being the speed of the gravitational wave. Also, using $\kappa = 8\pi G/c^4$ as well as $q = 2e$ and $m = 2m_e$ gives

$$\alpha = \frac{Gm_e^2}{Ke^2}\frac{v^3v_{wave}}{c^4} \tag{1859}$$

If the gravitational wave is in vacuum or a dispersionless medium, then $v_{wave} = c$ and we have

$$\boxed{\begin{array}{c} \alpha = \left(\dfrac{Gm_e^2}{Ke^2}\right)\left(\dfrac{v^3}{c^3}\right) \\ \\ \textit{Fractional correction factor for gravitational wave force in vacuum} \\ \textit{relative to Coulomb force for a dispersionless medium} \end{array}} \tag{1860}$$

Comparing this to (1846), we find that ε is simply multiplied by a factor of v^3/c^3. Therefore, we can write the expression above as

$$\alpha = \varepsilon\left(\frac{v^3}{c^3}\right) \tag{1861}$$

Although we have shown the formal details for obtaining this result, we could have also obtained it quickly by simply making use of the relation $h_{00} \approx h_{ij}\left(c^2/v^2\right)$ as given in (437). Since $\varphi_G \sim h_{00}$, then it is evident that (1846) can be written in terms h_{ij} by simply introducing a factor of v^2/c^2. Also, notice that the gravitational wave force in (1857) involves a factor of ωv while the ratio of $\nabla\varphi_G \sim 1/r^2$ and $A_\otimes \sim 1/r$ from (1854) will involve a factor of $1/r \approx k$ (in the near-field zone). This gives an overall factor of $k/(\omega v) = c/v$ for gravitational waves in vacuum. Lastly, since the field equation in (1854) has a coupling constant of $2\kappa = 16\pi G/c^4$, rather than $1/\varepsilon_G = 4\pi G$, then putting it all together would lead to

$$\alpha = \varepsilon\left(\frac{h_{00}}{h_{ij}}\right)\left(\frac{\omega v}{k}\right)(2\kappa\varepsilon_G) \approx \varepsilon\left(\frac{v^3}{c^3}\right) \tag{1862}$$

Notice that the ultra-relativistic limit, $v \approx c$, leads to $\alpha \approx \varepsilon$. In this limit, the gravitational wave force becomes comparable to the Newtonian gravitational force since the source of each field would become comparable: $T_{ij} = \rho_m v_i v_j \approx \rho_m c^2 = T_{00}$. Also, writing (1852) as $F_{GR\ wave} = \alpha F_C$ and inserting (1860) gives

$$F_{GR\ wave} = F_C\left(\frac{Gm_e^2}{Ke^2}\right)\left(\frac{v^3}{c^3}\right) \tag{1863}$$

Here we find that as $v \to 0$, then $F_{GR\ wave} \to 0$ which is exactly what we would expect due to the velocity dependence of the gravitational wave *force* in (1857) as well as the velocity dependence of the gravitational wave *field* in (1856).

Returning to (1859), note that if the gravitational wave is propagating in matter which causes a dispersion, then we do not simply have $v_{wave} = c$. Instead, the velocity of the wave can be expressed as $v_{wave} = c/n_G$, where n_G is a gravitational index of refraction. For a superconductor, (1324) gives $n_G^2(\omega) \equiv \left(1 - \omega_G^2/\omega^2\right)$ so that $c/v_{wave} = n_G = \sqrt{1 - \omega_G^2/\omega^2}$. Then (1859) becomes

$$\boxed{\begin{array}{c} \alpha = \left(\dfrac{Gm_e^2}{Ke^2}\right)\left(\dfrac{v^3}{c^3}\right)\left(1 - \dfrac{\omega_G^2}{\omega^2}\right)^{-1/2} \\ \\ \textit{Fractional correction factor for gravitational wave force} \\ \textit{relative to Coulomb force in a medium with dispersion} \end{array}} \tag{1864}$$

A fractional correction factor from the geodesic *deviation* equation

We could also consider a fractional correction factor derived from the geodesic *deviation* equation rather than the geodesic equation of motion as we have done above. We can define this fractional correction factor as

$$\beta \quad \equiv \quad \frac{F_C}{F_{GR\ wave\ (deviation)}} \tag{1865}$$

From (540) we can write the geodesic deviation force as $F_{GR\ wave\ (deviation)} = m\ddot{L}_i \approx m\ddot{h}_{ij}^{\tau\tau} L^j$. Once again, we can use $\ddot{h}_{ij}^{\tau\tau} = \omega^2 h_{ij}^{\tau\tau} \sim \omega^2 A_\otimes$ for cross-polarization. Then using (1856) gives

$$F_{GR\ wave\ (deviation)} \quad = \quad m\omega^2 \left(\frac{G}{c^4} \frac{mv^2}{r} \right) L = \frac{Gm^2 \omega^2 v^2 L}{c^4 r} \tag{1866}$$

where v as the speed of the source of gravitational waves. Inserting this into (1865) and using $F_C = Kq^2/r^2$ gives

$$\beta \quad = \quad \left(\frac{Gm^2}{Kq^2} \right) \left(\frac{\omega^2 v^2 L r}{c^4} \right) \tag{1867}$$

In the near-field zone, we can set $L = r \approx 1/k$ and use $k = \omega/v_{wave}$. Also using $q = 2e$ and $m = 2m_e$ gives

$$\beta \quad = \quad \left(\frac{Gm_e^2}{Ke^2} \right) \left(\frac{v^2 v_{wave}^2}{c^4} \right) \tag{1868}$$

If the gravitational wave is in vacuum or a dispersionless medium, then $v_{wave} = c$ and we have

$$\beta = \left(\frac{Gm_e^2}{Ke^2} \right) \left(\frac{v^2}{c^2} \right) \tag{1869}$$

Fractional correction factor for gravitational wave deviation force (in vacuum or a dispersionless medium) relative to Coulomb force

As before, we find that in the ultra-relativistic limit, $v \approx c$, we have $\beta \approx \alpha$. Also, taking the static limit using (1866) leads to $F_{GR\ wave\ (deviation)} \rightarrow 0$ as $v \rightarrow 0$ and therefore (1869) does not diverge. Comparing (1869) to (1860) we find that the result obtained from the geodesic *deviation* is smaller than the the result obtained from the geodesic equation of motion by a factor of v/c. This follows from the fact that the geodesic deviation equation does *not* have a velocity dependence while the geodesic equation of motion does.

Returning to (1868), note that if the gravitational wave is propagating in matter which causes a dispersion, again we can use $c^2/v_{wave}^2 = 1 - \omega_G^2/\omega^2$. Then (1859) becomes

$$\beta = \left(\frac{Gm_e^2}{Ke^2} \right) \left(\frac{v^2}{c^2} \right) \left(1 - \frac{\omega_G^2}{\omega^2} \right)^{-2} \tag{1870}$$

Fractional correction factor for Coulomb force relative to gravitational wave deviation force in a medium with dispersion

Once again, we find that the fractional correction factor is frequency dependent for the case of dispersion. However, comparing to (1864), we find that the result obtained from the geodesic *deviation* varies with $\left(1 - \omega_G^2/\omega^2\right)^{-2}$ while the the result obtained from the geodesic equation of motion varies with $\left(1 - \omega_G^2/\omega^2\right)^{-1/2}$.

17.5 Energy conservation and linear response by a superconductor

Constitutive equations

Here we consider the maximum electric fields that could be produced by the charge-separation effect in a superconductor. By energy conservation, we can equate the energy of the incident gravitational wave to the energy of the electric fields produced in a superconductor. The Isaacson power flux formula can be used to relate the gravitational wave power to the strain field of the gravitational wave. It is given in [43] as

$$\mathscr{P} = \frac{c^3}{16\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle \tag{1871}$$

Here $\mathscr{P}$ is the power per unit area $(\mathscr{P} = P/A)$ while h_+ and $h_\times$ are the plus and cross polarization strain fields. The particular polarization is not relevant to the analysis here so we can simply use h_0 for the amplitude and write a sinusoidal strain field as

$$h = h_0 \sin\left(\vec{k}\cdot\vec{x} - \omega t\right) \tag{1872}$$

Inserting this in the Isaacson power flux formula (1971) and using $\mathscr{P} = P_{in}/A$ for an incoming gravitational wave gives

$$P_{in} = \frac{c^3}{16\pi G} A \left\langle \omega^2 h_0^2 \sin^2\left(\vec{k}\cdot\vec{x} - \omega t\right)\right\rangle = \frac{c^3}{32\pi G} A h_0^2 \omega_{wave}^2 \tag{1873}$$

where ω_{wave} is the angular frequency of the wave. The energy density of the electric field is $U = E^2/\varepsilon_0$. We can multiply this by the volume, $V = A\delta_L$, where δ_L is the London penetration depth, and divide by the period of the wave, $T = 2\pi/\omega$, to obtain the average power of the electric field.

$$P_{Electric} = \frac{E^2 A \delta_L \omega_{wave}}{\varepsilon_0 2\pi} \tag{1874}$$

Equating (1873) and (1874), and solving for E gives

$$E = h_0 \sqrt{\frac{c^3 \varepsilon_0 \omega_{wave}}{16 G \delta_L}} \tag{1875}$$

This can be considered as a constitutive equation which predicts the maximum electric field that could be induced in a superconductor in response to a gravitational wave. Note that it gives a *linear* response of the electric field to the gravitational wave field. We can write this relationship as

$$E = \Xi h_0 \qquad \text{where} \qquad \Xi \equiv \sqrt{\frac{c^3 \varepsilon_0 \omega_{wave}}{16 G \delta_L}} \tag{1876}$$

A linear constitutive equation for the electric field induced in a superconductor due to an incident gravitational wave

The London penetration depth from (2938) can be written as $\lambda_L = c\sqrt{\frac{\varepsilon_0 m_e}{ne^2}}$. Therefore, we can also write Ξ as

$$\Xi = c\left(\frac{\varepsilon_0 n e^2 \omega_{wave}^2}{256 G^2 m_e}\right)^{1/4} \tag{1877}$$

Using microwave gravitational waves ($\omega_{wave} \approx 10^{10} Hz$) and the London penetration depth for Niobium ($\lambda_L \approx 39nm$), we find that (1876) gives

$$\Xi \equiv \sqrt{\frac{c^3 \varepsilon_0 \omega_{wave}}{16 G \delta_L}} \approx 2.4 \times 10^{20} N/C \tag{1878}$$

This implies the possibility of a large electric field for a relatively small gravitational wave strain field. For an order of magnitude approximation of the electric field, we need a value for h_0. We can solve (1873) for h_0 and consider a milliwatt of gravitational wave power ($P_{GR\ wave} \approx 10^{-3} W$) with microwave frequencies $\left(\omega \approx 10^{10} Hz\right)$ incident on a square centimeter $\left(A \approx 10^{-4} m\right)$. This gives

$$h_0 = \sqrt{\frac{16\pi G}{c^3} \frac{P_{GR\ wave}}{A \omega_{wave}^2}} \approx 3.5 \times 10^{-27} \tag{1879}$$

Inserting this into (1876) and using the value of Ξ in (1878) gives

$$E = \Xi h_0 \approx 8.5 \times 10^{-7} N/C \tag{1880}$$

Now we can use the Lorentz force to describe the non-relativistic equation of motion for lattice ions or Cooper pairs.[207]

$$ma = qE \tag{1881}$$

We can use this to obtain a relationship between the non-relativistic[208] supercurrents induced in the superconductor and the gravitational wave field. For particles with sinusoidal motion, we can use $a = v\omega_{particles}$. Then inserting $E = \Xi h_0$ from (1876) into the Lorentz force and solving for v gives

$$v = \frac{q}{m\omega_{particles}} \Xi h_0 \tag{1882}$$

The expression in (1882) gives an effective supercurrent velocity. Inserting this into the charge supercurrent ($J = nqv$) and mass supercurrent ($J = nmv$) gives

$$J = \frac{nq^2}{m\omega_{particles}} \Xi h_0 \qquad \text{and} \qquad J_m = \frac{nq}{\omega_{particles}} \Xi h_0 \tag{1883}$$

where n is the number density of the charge/mass carriers which have charge and mass given by q and m, respectively. Once again, we have a linear relationship between the supercurrents and the gravitational

[207] Note that the Lorentz force can be applied to either the Cooper pairs or the lattice ions. Therefore, we can use $m = 2m_e$ for Cooper pairs or we can use $m \approx m_{Niobium}$ for lattice ions where $m_{Niobium}$ is the mass of a Niobium atom. The mass used in the Lorentz force should not be confused with the mass used in the London penetration depth which is necessarily the mass of the Cooper pairs, $m = 2m_e$. This is because it is the only Cooper pair supercurrent that determines the depth to which electromagnetic fields can penetrate a superconductor.

[208] The velocities are certainly non-relativistic as can be confirmed from (1882) by inserting the value for Ξ from (1876), using microwave frequencies ($\omega \approx 10^{10} Hz$), and using the strain field in (1879). For lattice ions ($m \approx m_{Niobium} \approx 1.5 \times 10^{-25} kg$), we find $v \approx 10^{-10} m/s$. For Cooper pairs ($m \approx m_e$), we find $v \approx 10^{-5} m/s$. In fact, solving (1882) for h_0, we find that in order for the Cooper pairs to have a velocity that is 10% of the speed of light ($v = .1c$), we would need microwave gravitational waves with a strain field on the order of 10^{-23}. Using the Isaacson power formula (1873), this would correspond to a power flux (P/A) on the order of $10^{27} W/m^2$ which is clearly absurd.

wave field. To emphasize the linear relationship, we can absorb the constants in (1883) into newly defined constants. Using Ξ from (1876), we have[209]

$$\Xi_q \equiv \frac{nq^2}{m}\sqrt{\frac{c^3\varepsilon_0}{16G\delta_L\omega}} \qquad \text{and} \qquad \Xi_m \equiv nq\sqrt{\frac{c^3\varepsilon_0}{16G\delta_L\omega}} \tag{1884}$$

Then the supercurrents in (1883) simply become

$$\boxed{\begin{gathered} J = \Xi_q h_0 \qquad \text{and} \qquad J_m = \Xi_m h_0 \\ \\ \textit{Linear constitutive equations for the charge/mass supercurrents} \\ \textit{in a superconductor due to an incident gravitational wave} \end{gathered}} \tag{1885}$$

Re-radiated gravitational waves calculated from the Einstein field equation

Now we can consider the gravitational wave power that can be emitted back out from the superconductor. We can use the linearized Einstein equation for the transverse-traceless strain using the Helmholtz Decomposition formulation. From (361) we have

$$\Box h_{ij}^{\tau\tau} = -\frac{16\pi G}{c^4}T_{ij}^{\tau\tau} \tag{1886}$$

The retarded Green's function solution is

$$h_{ij}^{\tau\tau}(t,\vec{x}) = \frac{4G}{c^4}\int\frac{T_{ij}^{\tau\tau}(t_r,\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x' \tag{1887}$$

where $\vec{x}'$ is the spatial coordinate of each infinitesimal element of $T_{ij}^{\tau\tau}$ occupying a differential volume element d^3x. Also, $T_{ij}^{\tau\tau}(t_r,\vec{x}')$ is the stress-energy-momentum contribution at $\vec{x}'$ evaluated at a retarded time t_r and located at a distance $|\vec{x}-\vec{x}'|$ from the field point where $h_{ij}^{\tau\tau}$ is measured. We can therefore express the retarded time as $t_r = t - |\vec{x}-\vec{x}'|/c$. For simplicity, we can consider the case of relativistic dust (to order v^2/c^2) which we found in (431) gives

$$T_{ij}^{\tau\tau} = \rho\left(v_iv_j - \tfrac{1}{3}\delta_{ij}v^2\right) \tag{1888}$$

For plus polarization, we have $T_\oplus = \dfrac{2}{3}\rho v_x^2 = -\dfrac{2}{3}\rho v_y^2$. Likewise, for cross polarization, we have $T_\otimes = \rho v_xv_y$. Therefore, we can simply approximate $T \approx \rho v^2$ in (2356) and use h_{out} to represent the *outgoing* gravitational wave. This gives

$$h_{out}(t,\vec{x}) = \frac{4G}{c^4}\int\frac{\rho v^2}{|\vec{x}-\vec{x}'|}d^3x' \tag{1889}$$

where ρ and v^2 can be functions of $(t_r,\vec{x}')$. For small distance and time scales, we can neglect the dependence of ρ and v^2 on the retarded time and also consider them approximately uniform over a volume V. Let us also consider the distance from the source to the field point as remaining approximately a constant distance given by d. Then we have

$$h_{out} \approx \frac{4G\rho v^2V}{c^4d} \tag{1890}$$

[209]Here we are assuming that the frequency of oscillation of the particles, $\omega_{particles}$, is comparable to the frequency of the gravitational wave, ω_{wave}, so that we can work with a single frequency, ω.

We can use $V = Ad$ where A is the surface area that the gravitational wave is incident upon. We can also insert the velocity found in (1882) to obtain

$$h_{out} \approx \frac{4GA\rho v^2 q^2}{m^2 c^4 \omega^2} \Xi^2 h_0^2 \tag{1891}$$

where h_0 is still the amplitude of the *incoming* gravitational wave field. We can use the number density, $n = \rho/m$, and $q^2 = 4e^2$ (for lattice ions or Cooper pairs). We can also insert the expression for Ξ from (1876). This gives[210]

$$h_{out} \approx \left(\frac{e^2 \varepsilon_0}{4c}\right)\left(\frac{Ah_0^2}{\delta_L \omega}\right)\left(\frac{n}{m}\right) \tag{1892}$$

The first parentheses only involves constants of nature and is on the order of $\sim 10^{-58}$ in SI units. The second parentheses involves the properties of the incident gravitational wave (h_0 and ω) and the properties of the superconductor (A and λ_L). The last parentheses depends on whether we choose to consider the lattice ions or the Cooper pairs as the particles which re-radiate the gravitational waves. This follows from the fact that m is the mass of the particles being accelerated by the Lorentz force in (1881) and $n = \rho/m$ where ρ is the mass density of the particles radiating gravitational wave according to (1889).

To obtain a numeric value for h_{out}, we can use a surface with dimensions on the order of centimeters ($A \approx 10^{-4}m$) made of Niobium ($\lambda_L \approx 39nm$) and consider a gravitational wave with microwave frequency ($\omega \approx 10^{10}Hz$) and a power of 1mW ($h_0 \approx 3.5 \times 10^{-27}$) as given by (1879). First we can consider the case of *lattice ions* re-radiating gravitational waves. Then $m \approx 1.5 \times 10^{-25}kg$ and $n = \rho/m \approx 5.7 \times 10^{28}m^{-3}$ where we have used $\rho \approx 8.6 \times 10^3 kg/m^3$ for Niobium. Using these values in (1892) gives $h_{out} \approx 10^{-63}$. We can use the Isaacson formula (1873) to obtain a power flux for the outgoing gravitational wave.

$$\boxed{\mathscr{P}_{out} \approx 10^{-73} W/m^2} \tag{1893}$$

Power flux of GR waves re-radiated by lattice ions from a Niobium superconductor with incident microwave gravitational waves

On the other hand, if we consider the case of *Cooper pairs* re-radiating gravitational waves, then $m \approx 9.1 \times 10^{-31}kg$. We would still have $n \approx 5.7 \times 10^{28}m^{-3}$ for an electrically neutral sample.[211] Using these values in (1892) gives $h_{out} \approx 10^{-57}$. We can insert this result into the Isaacson power formula (1873) to obtain a power flux for the outgoing gravitational wave.

$$\boxed{\mathscr{P}_{out} \approx 10^{-61} W/m^2} \tag{1894}$$

Power flux of GR waves re-radiated by Cooper pairs from a Niobium superconductor with incident microwave gravitational waves

The results for the lattice ions in (1893) and the Cooper pairs in (1894) are obviously both miniscule. However, it is interesting to note that the power flux by the Cooper pairs is 12 orders of magnitude larger than for the lattice ions. This follows from the fact that the smaller mass of the Cooper pairs makes the velocity of the Cooper pairs in (1882) much greater than the velocity for lattice ions. Then in (1890) we find that the

[210] It is interesting to note that the outgoing strain field, h_{out}, varies according to the *square* of the incoming strain field, h_0. This can be traced back to (1882) where the velocity of the particles varies with the incoming strain field, $v \sim h_0$, and (1889) where the outgoing strain field varies with the *square* of the velocity, $h_{out} \sim v^2$.

[211] Technically, by using this approximation, we are using *all* the electrons of the sample for determing the gravitional waves re-radiated This means we are using the electrons that are in a superconducting Cooper pair state as well as the normal electrons.

re-radiated field varies linearly with the mass but quadratically with the velocity. Therefore, the field for the Cooper pairs has an overall factor of $m_{Niobium}/m_e \approx 10^6$ compared to the field of the lattice ions. Since the power varies with the square of the field, then the power flux for Cooper pairs is expected to be 12 orders of magnitude greater than the lattice ions.

We can summarize the procedure and results of this analysis as follows: The energy of an incoming gravitational wave was assumed to be completely converted into the energy of electric fields in the superconductor. The Lorentz force (on lattice ions or Cooper pairs) gives a velocity for the supercurrents induced in the superconductor. The velocity is then used in the Einstein field equation to determine the outgoing gravitational wave field. The Isaacson power formula is utilized to determine the power of the outgoing gravitational wave.

This procedures leads to the conclusion that for an incoming gravitational wave with a strain of 10^{-27}, the lattice ions produce an outgoing wave with a strain of 10^{-63} and the Cooper pairs produce an outgoing wave with a strain of 10^{-57}. These results can also be expressed in terms of power flux. A milliwatt of incoming gravitational wave power incident on a square centimeter of Niobium will produces an outgoing gravitational wave power flux of $10^{-61} W/m^2$ at most.[212]

Re-radiated gravitational wave power from a high Q cavity

Recall that in obtaining (1874), we divided the energy of the electric field by the period the gravitational wave to obtain a power. However, for a high Q cavity, the appropriate approach may be to divide the energy of the electric field by the *ring down time* of the cavity, $\tau = Q/\omega$, instead of the period of the gravitational wave. In that case, (1874) becomes

$$P_{Electric} = \frac{E^2 A \delta_L \omega}{\varepsilon_0 Q} \tag{1895}$$

For a cavity with $Q \approx 10^9$, we find that (1878) would become

$$\Xi \equiv \sqrt{\frac{c^3 \varepsilon_0 Q \omega}{32\pi G \delta_L}} \approx 3.0 \times 10^{24} N/C \tag{1896}$$

and this would ultimately lead to (1892) becoming

$$h_{out} \approx \left(\frac{e^2 \varepsilon_0}{c}\right) \left(\frac{Q A h_0^2}{2\pi \delta_L \omega}\right) \left(\frac{n}{m}\right) \tag{1897}$$

Inserting this into the Isaacson power flux formula (1873) and using $\lambda_L = c\sqrt{\frac{\varepsilon_0 m_e}{n e^2}}$ gives

$$\boxed{\mathscr{P}_{out} = \left(\frac{e^6 \varepsilon_0}{128\pi^3 G c m_e}\right) \frac{Q^2 A^2 n^3 h_0^4}{m^2}} \tag{1898}$$

Power flux of GR waves re-radiated by a cavity with quality factor Q

The prefactor in parentheses is $\sim 10^{-96}$ in SI units. Using (1897) and (1898), we find that lattice ions give a strain field of $h_{out} \approx 10^{-54}$ for the outgoing gravitational wave and a corresponding power flux of approximately $10^{-55} W/m^2$. For Cooper pairs, we have a strain field of $h_{out} \approx 10^{-48}$ and a corresponding power flux of approximately $10^{-43} W/m^2$. Therefore, we find that the benefit of a high Q cavity is not sufficient to make the outgoing gravitational wave power significant.

[212] It would be helpful to verify these results by calculating the outgoing gravitational wave power predicted by the Einstein quadrupole formula (1975) for the same system analyzed here.

In this analysis we have arbitrarily set the power of the incoming gravitational wave to a milliwatt. However, we could consider a laser-like system where parametric amplification can spontaneously occur from quantum vacuum fluctuations through a gravitational Casimir-like effect. In that case, we would need to start with a gravitational wave that originates from the quantum ground state of the cavity. The gravitational wave strain due to the ground state energy of a cavity with dimensions on the order of centimeters and a quality factor of $Q \approx 10^9$ was found in (2354) to be

$$h_{ground\ state} = \sqrt{\frac{8Gh}{AQc^3}} \approx 1.45 \times 10^{-37} \tag{1899}$$

Using this value for h_0, the *incoming* gravitational wave strain in the analysis above, leads to an *outgoing* gravitational wave strain field of $h_{out} \approx 10^{-54}$ for the lattice ions and a corresponding power flux of approximately $10^{-55} W/m^2$. For Cooper pairs, we obtain $h_{out} \approx 10^{-48}$ and a corresponding power flux of approximately $10^{-43} W/m^2$.

18 Reflection/Expulsion of gravitational (GR) waves by a superconductor

18.1 Conditions for the reflection of gravitational fields

In this section we consider the conditions required for to reflect gravitational fields. First we evaluate the condition that would be required for the reflection of a time-varying gravito-electric field due to a planar slab of mass with time-varying mass currents. We perform a similar calculation for the a time-varying gravito-magnetic field. Then we consider the case a gravitational wave using the geodesic equation of motion for the test masses making up the material. Similarly, we consider the Lorentz force in curved space-time (which is the geodesic equation with an additional contribution due to electromagnetic fields) to examine the same situation involving a gravitational wave but with the inclusion of electric forces. Lastly, we examine the results obtained by using the geodesic *deviation* equation.

Reflection of the gravito-electric field

Solving Newton's law of gravitation for the case of a planar slab of mass gives

$$\nabla \cdot \vec{E}_G = 4\pi G\rho \qquad \Rightarrow \qquad E_G = \sigma/\varepsilon_G \tag{1900}$$

where $\varepsilon_G = 1/(4\pi G)$. We can consider a sinusoidally time-varying mass density σ which would produce a corresponding sinusoidally time-varying gravito-electric field, E_G. We can also consider the effect of this field on a second planar slab which is parallel to the first slab and consists of freely moving massive particles. If we use the geodesic equation with only the gravito-electric field, $a = E_G$, and assume a sinusoidal acceleration, $a = v\omega$, then the equation of motion of the particles in the second slab is given by

$$v\omega = \sigma/\varepsilon_G \tag{1901}$$

Let us describe the lowest order (linear) response of the second slab by a constitutive equation analogous to Ohm's law:

$$\vec{J} = k\vec{E}_G \tag{1902}$$

where $\vec{J} = \rho\vec{v}$ is the mass current density per unit volume and k is a material parameter characterizing the response of the material to the gravitational field (analogous to an electrical conductivity). We can use $\rho = \sigma'/\delta_G$ where δ_G is a characteristic depth to which the gravito-electric field penetrates the slab and σ' is the mass density of the second slab. Then we can write $J = \sigma' v/\delta_G$. Inserting this into the left side of (1902) and using (1900) on the right side gives

$$\frac{\sigma' v}{\delta_G} = \frac{k\sigma}{\varepsilon_G} \tag{1903}$$

In order to have reflection, we need E_G and E'_G (the field due to each slab) to be comparable. Therefore, from (1900) we find that the slabs must also have comparable mass densities, $\sigma' = \sigma$. We can also use $\delta_G = c/\omega_G$, where ω_G is a gravitational plasma frequency. Then we obtain

$$k = \varepsilon_G \omega_G \frac{v}{c} \tag{1904}$$

Solving this for v and inserting into (1901) gives

$$\sigma = kc\frac{\omega}{\omega_G} \tag{1905}$$

Notice that to generate the largest possible mass current in (1902), we would want k to be as large as possible. Therefore, as an upper limit, we can set $v = c$ in (1904) which means $k = \varepsilon_G \omega_G$ and therefore (1905) becomes $\sigma = \varepsilon_G c\omega$. Finally, using $\varepsilon_G = 1/(4\pi G)$ gives

$$\boxed{\sigma = \frac{c}{4\pi G}\omega \qquad \textit{Minimum surface mass density required for reflection of the gravito-electric field}} \tag{1906}$$

The prefactor $c\varepsilon_G$ is on the order of 10^{17} (SI units). Therefore, using microwave frequencies, $\omega \approx 10^{10} Hz$, leads to $\sigma \approx 10^{28} kg/m^2$ which is on the order of an earth mass per square centimeter. On the other hand, if we use laboratory-scale mass densities, such as a Niobium cavity $\left(\rho \approx 8.6 \times 10^3 kg/m^3\right)$ with thickness $d \approx 10^{-2} m$, then $\sigma \approx \rho d \approx 86\ kg/m^2$. In that case, (1906) gives $\omega \approx 10^{-16} Hz$. This corresponds to a time scale on the order of 84 million years! However, in the following sections it is shown that for the gravitational wave tensor field, $h_{ij}^{\tau\tau}$, this the mass density is not the determining factor of whether reflection can occur. Rather, it is the gravitational shear modulus, $\mu_{G(SC)}$, and the relative gravitational permeativity, κ_r.

18.2 Relating the GR and EM penetration depths by proportionalities

In this section, a gravitational penetration depth is developed in relation to the London penetration depth from electromagnetism. The analysis starts with Maxwell's field equations for electromagnetism (in the Lorenz gauge) as well as the transverse-traceless linearized Einstein field equations for gravitation (using the Helmholtz Decomposition formulation). These are, respectively,

$$\Box A^i = -\mu_0 J^i \qquad \text{and} \qquad \Box h_{ij}^{\tau\tau} = -2\kappa T_{ij}^{\tau\tau} \tag{1907}$$

By using Fourier transforms or applying a Green's function solution, it can be shown that

$$A^i \propto J^i \qquad \text{and} \qquad h_{ij}^{\tau\tau} \propto T_{ij}^{\tau\tau} \tag{1908}$$

Furthermore, in a superconductor, the London constitutive equation and the gravitational London-like constitutive equation are, respectively,[213]

$$J^i = -\Lambda_L A^i \qquad \text{and} \qquad T_{ij}^{\tau\tau} = -\mu_{G(SC)} h_{ij}^{\tau\tau} \tag{1909}$$

Inserting these into the corresponding field equations in (1907), and taking the DC limit, leads to the following Yukawa-like equations

$$\nabla^2 A^i - \Lambda_L A^i = 0 \qquad \text{and} \qquad \nabla^2 h_{ij}^{\tau\tau} - 2\kappa\mu_G h_{ij}^{\tau\tau} = 0 \tag{1910}$$

These differential equations have solutions given, respectively, by

$$A^i \propto e^{-z/\lambda_L} \qquad \text{and} \qquad h_{ij}^{\tau\tau} \propto e^{-z/\lambda_G} \tag{1911}$$

where λ_L is the London penetration depth, and λ_G is the gravitational penetration depth. Using (1908) also implies that

$$J^i \propto e^{-z/\lambda_L} \qquad \text{and} \qquad T_{ij}^{\tau\tau} \propto e^{-z/\lambda_G} \tag{1912}$$

Relating the current density and stress tensor to the supercurrent velocity of the charge/mass carriers of the superconductor (namely, the Cooper pairs) gives

$$J^i \propto v^i \qquad \text{and} \qquad T_{ij}^{\tau\tau} \propto v_i v_j \tag{1913}$$

Comparing the expressions for J^i found in (1912) and (1913) requires that

$$v^i \propto e^{-z/\lambda_L} \tag{1914}$$

Therefore, (1908), (1911), and (1914) can be used to summarize the relationship between the gravitational wave field, the stress tensor, and the supercurrent velocity as

$$h_{ij}^{\tau\tau} \quad \propto \quad T_{ij}^{\tau\tau} \propto v_i v_j \propto e^{-2z/\lambda_L} \propto e^{-z/\lambda G} \tag{1915}$$

From the last proportionality in (1915), it is found that the gravitational penetration depth is half of the London penetration depth, $\lambda_G = \frac{1}{2}\lambda_L$. Using the expression for the London penetration depth found in (2983) leads to

$$\lambda_G = \sqrt{\frac{m_e}{4\mu_0 n_s e^2}} \tag{1916}$$

[213]From (1297), the London constituent equation in the presence of a gravitational wave is $J_s^i = -\Lambda_L\left(A^i + A_j h_{ij}^{\tau\tau}\right)$ where $\Lambda_L = n_s e^2/m_e$. However, since the electromagnetic fields will completely dominate over the gravitational wave field in driving the supercurrent, then $A^i >> A_j h_{ij}^{\tau\tau}$ and therefore $A_j h_{ij}^{\tau\tau}$ can be neglected.

Conceptually, this result follows from the fact that any supercurrents generated within the superconductor by a gravitational wave would also produce electromagnetic fields. However, since the supercurrents and electromagnetic fields must decay according to the London penetration depth, likewise, the stress tensor and gravitational wave field must also decay within a gravitational penetration depth that is comparable to the London penetration depth.

However, it is troubling that the *gravitational* penetration depth in (1916) would depend only on *electromagnetic* properties, namely, the magnetic permeability of free space. It does not include the Einstein constant, κ, which is found in the Einstein field equations and determines the strength of the gravitational field that is produced by a given stress tensor.[214] Also, the result in (1916) does not include the gravitational modulus, $\mu_{G(SC)}$, which characterizes the response of a superconductor to a gravitational wave. These factors have been omitted due to the use of proportional reasoning which bypasses the use of Einstein's equation or a gravitational constitutive equation for the superconductor. In a sense, the only field equation that is implicitly utilized is the Maxwell field equation, and the only constitutive equation that is implicitly utilized is the London constitutive equation. The only connection to gravity is through the proportionality $h_{ij}^{\tau\tau} \propto T_{ij}^{\tau\tau}$ and $T_{ij}^{\tau\tau} \propto v_i v_j$. In fact, the gravitational side of equations (1909) and (1910) is never even used in the proportional reasoning argument that leads to (1916). Therefore, a more formal approach is to construct the complete stress tensor for a superconductor (which incorporates the interaction of the gravitational wave with the superconductor as well as the presence of supercurrents and electromagnetic fields), then apply the linearized Einstein field equations in order to arrive at a result for the gravitational penetration depth. This analysis is carried out in the following sections.

[214]Typically the value of κ is determined by $\kappa = 8\pi G/c^4$. However, it is possible that a material may have a relative gravitational permeativity, κ_r, such that $\kappa = \kappa_r \kappa_0$ where $\kappa_0 = 8\pi G/c^4$ in vacuum. Since the gravitational penetration depth is found in (1335) to be $\lambda_G = 1/\sqrt{2\kappa c^2 \mu_G}$, then it is possible that a large enough value for κ_r would lead to λ_G being comparable to λ_L, the London penetration depth.

18.3 A complete stress tensor in the presence of GR waves

A complete stress tensor for a superconductor in the presence of gravitational waves can be written as

$$T_{ij}^{\tau\tau} = -\mu_{G(SC)} h_{ij}^{\tau\tau} + T_{ij\,(EM)}^{\tau\tau} + T_{ij\,(supercurrents)}^{\tau\tau} \tag{1917}$$

The first term here is the gravito-London constitutive equation describing the interaction of the gravitational wave with the Cooper pairs and lattice ions, the second term is the stress tensor associated with electromagnetic fields within the superconductor, and the third term is the stress tensor produced in the superconductor due to the presence of supercurrents. For the electromagnetic stress tensor, we can assume that the electromagnetic fields dominate over the gravitational wave field so that $B^2 >> B_k B_j h_{ik}^{\tau\tau}$ and $E^2 >> E_k E_j h_{ik}^{\tau\tau}$. This leads to (1774) which gives

$$T_{ij\,(EM)}^{\tau\tau} \quad = \quad -\varepsilon_0 E^i E^j - \frac{1}{\mu_0} B^i B^j + \frac{1}{3}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)\delta_{ij} \tag{1918}$$

For a superconductor, the London constitutive equations were found in Appendix Q as

$$\partial_t \vec{J}_s \quad = \quad \frac{n_s e^2}{m_e}\vec{E} \qquad \text{and} \qquad \nabla \times \vec{J}_s = -\frac{n_s e^2}{m_e}\vec{B} \tag{1919}$$

In the DC limit, the first constitutive equation requires $\vec{E} = 0$. For convenience, a transverse-traceless "magnetic field tensor" can be defined as

$$B_{ij}^{\tau\tau} \equiv B_i B_j - \tfrac{1}{3}\delta_{ij} B^2 \tag{1920}$$

Then in the DC limit, (1918) simply becomes

$$T_{ij\,(EM)}^{\tau\tau} \quad = \quad -\frac{1}{\mu_0} B_{ij}^{\tau\tau} \tag{1921}$$

For the supercurrent stress tensor,

It is also possible to develop a constitutive equation for the transverse-traceless stress tensor within a superconductor. In (429) it is shown that the source of gravitational waves is given by the transverse-traceless stress, $T_{ij}^{\tau\tau} = \left(\rho + P/c^2\right)\left(v_i v_j - \frac{1}{3}\delta_{ij} v^2\right)$, for an ideal fluid (to second order in velocity). Substituting in (1296) for each velocity term gives

$$T_{ij}^{\tau\tau} \quad = \quad \frac{e^2\left(\rho + P/c^2\right)}{m_e^2}\left[\left(A_i + h_{ik}^{\tau\tau} A_k\right)\left(A_j + h_{jl}^{\tau\tau} A_l\right) - \tfrac{1}{3}\delta_{ij}\left(A_k + h_{kl}^{\tau\tau} A_l\right)^2\right] \tag{1922}$$

Remaining to first order in $h_{ij}^{\tau\tau}$ gives[215]

$$\boxed{T_{ij\,(supercurrents)}^{\tau\tau} = \frac{e^2\left(\rho + P/c^2\right)}{m_e^2}\left[A_i A_j + A_k A_j h_{ik}^{\tau\tau} + A_l A_i h_{jl}^{\tau\tau} - \tfrac{1}{3}\delta_{ij}\left(A^2 + 2A_k A_l h_{kl}^{\tau\tau}\right)\right]} \tag{1923}$$

Transverse-traceless stress tensor due to supercurrents
in the presence of a gravitational wave

[215] Note that $T_{ij\,(superrcurrents)}^{\tau\tau}$ is transverse by virtue of the fact that taking a divergence will lead to terms involving $\partial_i A_i = 0$ which is essentially the London gauge condition that is known to exist within a superconductor.

This is the stress tensor produced inside a superconductor due to supercurrents in the presence of a gravitational wave field. Once again, if the vector potential is set to zero, the stress tensor vanishes regardless of the presence of a gravitational wave. Therefore, this constitutive equation does not describe the stress generated in a superconductor due to a gravitational wave alone. Rather, it involves terms of the form $A_l A_i h^{\tau\tau}_{jl}$ which are *corrections* to the stress tensor that would already exist in a superconductor due to electromagnetic fields. This stress tensor will be used in a later section (in conjunction with the Einstein field equation) to find a gravitational penetration depth due to the supercurrents produced in a superconductor by electromagnetic fields in the presence of a gravitational wave.

Note that (1923) gives

$$T^{\tau\tau}_{ij\,(supercurrents)} \quad = \quad \frac{e^2\left(\rho+P/c^2\right)}{m_e^2}\left[A_iA_j + A_kA_jh^{\tau\tau}_{ik} + A_lA_ih^{\tau\tau}_{jl} - \tfrac{1}{3}\delta_{ij}\left(A^2+2A_kA_lh^{\tau\tau}_{kl}\right)\right] \tag{1924}$$

To simplify this expression, again we consider the case when the electromagnetic fields dominate the gravitational fields so that $A^2 >> A_kA_jh^{\tau\tau}_{ik}$. For convenience, a transverse-traceless "magnetic tensor potential" can be defined as

$$A^{\tau\tau}_{ij} \equiv A_iA_j - \tfrac{1}{3}\delta_{ij}A^2 \tag{1925}$$

Then (1924) becomes

$$T^{\tau\tau}_{ij\,(supercurrents)} = \frac{e^2\left(\rho+P/c^2\right)}{m_e^2}A^{\tau\tau}_{ij} \tag{1926}$$

Inserting (1921) and (1926) into (1917) gives

$$T^{\tau\tau}_{ij} \quad = \quad -\mu_{G(SC)}h^{\tau\tau}_{ij} + \frac{e^2\left(\rho+P/c^2\right)}{m_e^2}A^{\tau\tau}_{ij} - \frac{1}{\mu_0}B^{\tau\tau}_{ij} \tag{1927}$$

This is the full stress tensor for the superconductor in the DC limit,

18.4 A gravitational penetration depth from the complete stress tensor

Next, the *gravitational* penetration depth can be determined using a stress tensor given by

Inserting this into the transverse-traceless linearized Einstein field equations given in (1907) and taking the DC limit of the wave equation leads to a Poisson equation.

$$\nabla^2 h_{ij}^{\tau\tau} = 2\kappa\mu_G h_{ij}^{\tau\tau} - \frac{2\kappa e^2\left(\rho + P/c^2\right)}{m_e^2} A_{ij}^{\tau\tau} + \frac{2\kappa}{\mu_0} B_{ij}^{\tau\tau} \tag{1928}$$

In this approximation (with electromagnetic fields dominating the gravitational wave field), the solution to Maxwell's equation is $A_i = A_{i,0}e^{-z/\lambda_L}$ and $B_i = B_{i,0}e^{-z/\lambda_L}$. Inserting these above gives

$$\nabla^2 h_{ij}^{\tau\tau} = 2\kappa\mu_G h_{ij}^{\tau\tau} - \frac{2\kappa e^2\left(\rho + P/c^2\right)}{m_e^2} A_{ij,0}^{\tau\tau} e^{-2z/\lambda_L} + \frac{2\kappa}{\mu_0} B_{ij,0}^{\tau\tau} e^{-2z/\lambda_L} = 0 \tag{1929}$$

The following is an ansatz solution to the differential equation.

$$h_{ij}^{\tau\tau} = h_{ij,0(incident\ GR\ wave)}^{\tau\tau} e^{-z/\lambda_G} + \frac{\kappa\lambda_L^2}{2}\left(\frac{1}{\mu_0} B_{ij,0}^{\tau\tau} - \frac{e^2\left(\rho + P/c^2\right)}{m_e^2} A_{ij,0}^{\tau\tau}\right) e^{-2z/\lambda_L} \tag{1930}$$

where $h_{ij,0(incident\ GR\ wave)}^{\tau\tau}$ is the amplitude of the gravitational wave that is incident on the superconductor. From the second term in (1930), it is evident that the amplitude of the gravitational wave produced by the supercurrents and electromagnetic fields is given by

$$h_{ij,0(due\ to\ EM\ fields\ and\ supercurrents)}^{\tau\tau} = \frac{\kappa\lambda_L^2}{2}\left(\frac{1}{\mu_0} B_{ij,0}^{\tau\tau} - \frac{e^2\left(\rho + P/c^2\right)}{m_e^2} A_{ij,0}^{\tau\tau}\right) \tag{1931}$$

Then the solution in (1930) can be written as

$$h_{ij}^{\tau\tau} = h_{ij,0(incident\ GR\ wave)}^{\tau\tau} e^{-z/\lambda_G} - h_{ij,0(due\ to\ EM\ fields\ and\ supercurrents)}^{\tau\tau} e^{-2z/\lambda_L} \tag{1932}$$

From these results, it is found that there are effectively *two* gravitational penetration depths. One penetration depth, which may be called $\lambda_{G(incident\ GR\ wave)}$, is associated with the *incident* gravitational wave field which interacts with the Cooper pairs and lattice ions according to the coupling parameter $\mu_{G(SC)}$. The other penetration depth, which may be called $\lambda_{G(due\ to\ EM\ fields\ and\ supercurrents)}$, is associated with the gravitational wave field produced by the electromagnetic fields and the supercurrents in the superconductor. The two penetration depths can be written as

$$\boxed{\lambda_{G(incident\ GR\ wave)} = \frac{1}{\sqrt{2\kappa\mu_G}} \quad \text{and} \quad \lambda_{G(due\ to\ EM\ fields\ and\ supercurrents)} = \lambda_L/2} \tag{1933}$$

Penetration depths for a gravitational wave in a superconductor
(first depth is due to a gravito-London constitutive equation
and second depth is due to supercurrents and EM fields)

The first expression describes the depth that an incident gravitational wave will penetrate into a superconductor. It is determined by Einstein's constant, κ, and the gravitational modulus, $\mu_{G(SC)}$. The second expression gives the penetration depth of the gravitational wave that is produced by the electromagnetic fields and the supercurrents driven by electromagnetic fields. In the absence of electromagnetic fields ($A_i = 0$ and $B_i = 0$), the second term in (1930) would vanish. In that case, the only penetration depth remaining is the first one given in (1933).

Lastly, it should be noted that the amplitude of the gravitational wave that is produced by the electromagnetic fields (and the supercurrents driven by electromagnetic fields) is extremely small. To determine a value, assume that $\rho >> P/c^2$ and $\rho = n_s m_e$ so that (1931) simplifies to

$$h^{\tau\tau}_{ij,0(\text{due to EM fields})} = \frac{\kappa\lambda_L^2}{2}\left(\frac{1}{\mu_0}B^{\tau\tau}_{ij,0} - \frac{e^2 n_s}{m_e}A^{\tau\tau}_{ij,0}\right) \tag{1934}$$

Using the definition of $A^{\tau\tau}_{ij}$ and $B^{\tau\tau}_{ij}$ in (1925) and (1920), respectively, along with the London constitutive equation, $J^i_s = -\frac{n_s e^2}{m_e}A^i$, and the current density, $J^i_s = n_s e v^i$, gives

$$h^{\tau\tau}_{ij,0(\text{due to EM fields and supercurrents})} = \frac{\kappa\lambda_L^2}{2}\left[\frac{1}{\mu_0}\left(B_iB_j - \tfrac{1}{3}\delta_{ij}B^2\right) - n_s m_e\left(v_{i,0}v_{j,0} - \tfrac{1}{3}\delta_{ij}v_0^2\right)\right] \tag{1935}$$

Notice that $n_s m_e v_0^2$ is essentially the kinetic energy density of the Cooper pairs. To obtain a numerical result, we consider that each atom contributes two conduction electrons, and only 10^{-3} of the conduction electrons are in a superconducting state [34], then $n_s \approx 2n\left(10^{-3}\right)$ where $n = \rho_m/m$ is the number density of atoms. For Niobium, the mass density is $\rho_m \approx 8.6\times10^3 kg/m^3$ and the mass per atom is $m \approx 1.5\times10^{-25} kg/atom$. Then the number density of atoms is $n \approx 5.7\times10^{28} m^{-3}$ and therefore the number density of Cooper pairs is $n_s \approx 2n\left(10^{-3}\right) \approx 1.1\times10^{26} m^{-3}$. For an upper limit on the maximum kinetic energy density of the Cooper pairs, note that the superconducting state is only preserved up to the BCS energy gap, $E_{gap} = \frac{7}{2}k_B T_c$, where k_B is the Boltzmann constant and T_c is the critical temperature. For niobium, $T_c = 9.3K$, so the energy density of the BCS energy gap is $E_{gap}n_s \approx 5\times10^4 J/m^3$. Using this value for $n_s m_e v_0^2$, as well as $\kappa = 8\pi G/c^4 \approx 10^{-43} m/J$ and $\lambda_L \approx 40nm$ for niobium, gives the following dimensionless amplitude of the gravitational field.

$$h^{\tau\tau}_{ij,0(\text{due to EM fields and supercurrents})} \approx \kappa\lambda_L^2 E_{gap} n_s \approx 10^{-53} \tag{1936}$$

This is the maximum amplitude of the gravitational field that can be produced by the supercurrents in the superconductor without exceeding the BCS energy gap. Any higher amplitude would require supercurrent velocities with a kinetic energy that exceeds the BCS energy gap and therefore breaks up the Cooper pairs and destroys the superconducting state of the system.

18.5 GR wave reflectivity and transmissivity for linear media

In this section, reflection and transmission coefficients are developed for a gravitational wave that is normally incident on the interface between two linear media.

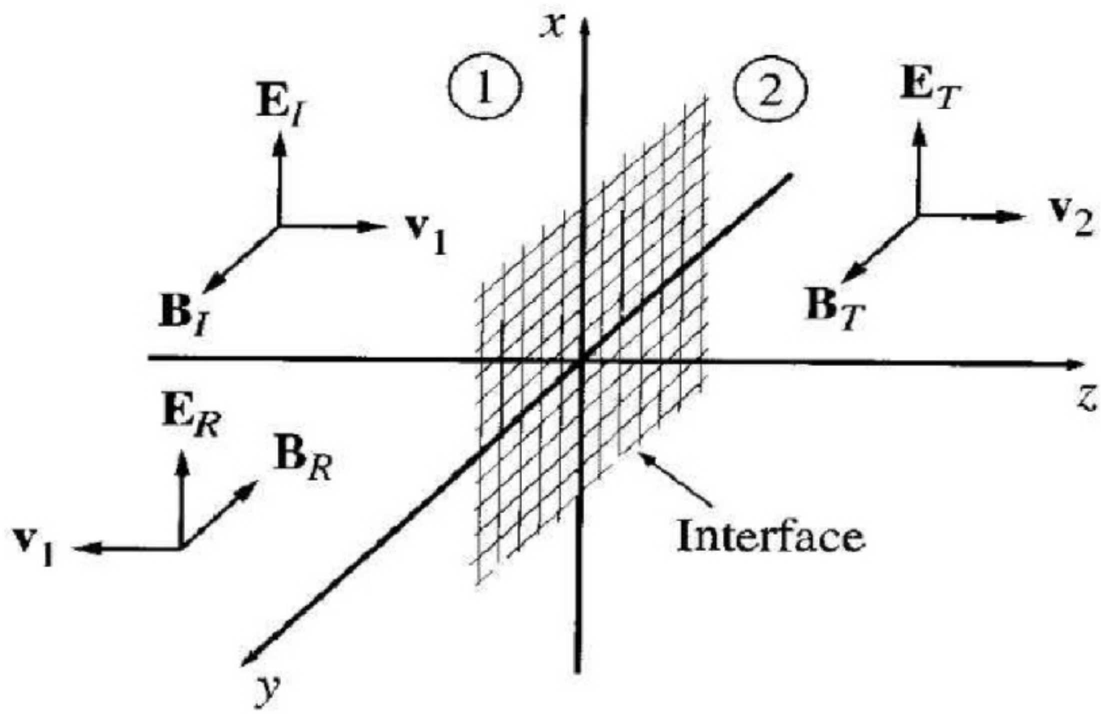

Figure 12: An incident, reflected, and transmitted gravitational waves at the interface of two linear media.

The reflected and transmitted gravitational wave fields can be shown to satisfy

$$\mathscr{E}_{ij}^{(R)} = \left(\frac{Z_G^{(2)} - Z_G^{(1)}}{Z_G^{(1)} + Z_G^{(2)}}\right)\mathscr{E}_{ij}^{(I)} \qquad \text{and} \qquad \mathscr{E}_{ij}^{(T)} = \left(\frac{2Z_G^{(2)}}{Z_G^{(1)} + Z_G^{(2)}}\right)\mathscr{E}_{ij}^{(I)} \tag{1937}$$

We can write these expressions in terms of the *amplitude* reflection and transmission coefficients, r and t, respectively as

$$\mathscr{E}_{ij}^{(R)} = r\mathscr{E}_{ij}^{(I)} \qquad \text{and} \qquad \mathscr{E}_{ij}^{(T)} = t\mathscr{E}_{ij}^{(I)} \tag{1938}$$

Then we have

$$r = \left(\frac{Z_G^{(2)} - Z_G^{(1)}}{Z_G^{(1)} + Z_G^{(2)}}\right) \qquad \text{and} \qquad t = \left(\frac{2Z_G^{(2)}}{Z_G^{(1)} + Z_G^{(2)}}\right) \tag{1939}$$

These can also be written as

$$r = \left(\frac{1 - Z_G^{(1)}/Z_G^{(2)}}{1 + Z_G^{(1)}/Z_G^{(2)}}\right) \qquad \text{and} \qquad t = \left(\frac{2}{1 + Z_G^{(1)}/Z_G^{(2)}}\right) \tag{1940}$$

This form emphasizes that the ratio $Z_G^{(1)}/Z_G^{(2)}$ determines the reflection and transmission of the wave.

For complex impedance values, the *power* reflection and tramission coefficients can be written as

$$R = r^* r \qquad \text{and} \qquad T = t^* t \tag{1941}$$

If the impedance values are real, then using (1940) makes these become[216]

$$R = \left(\frac{1 - Z_G^{(1)}/Z_G^{(2)}}{1 + Z_G^{(1)}/Z_G^{(2)}} \right)^2 \qquad \text{and} \qquad T = \left(\frac{2}{1 + Z_G^{(1)}/Z_G^{(2)}} \right)^2 \tag{1942}$$

Now consider the case of medium 1 being vacuum, and medium 2 being a superconductor. Then

$$Z_G^{(1)} = Z_G^{(vac)} = \frac{4\pi G}{c} \qquad \text{and} \qquad Z_G^{(2)} = Z_G^{(SC)} = \frac{4\pi G/c}{\sqrt{1 - 2c^2 \kappa \mu_{G(SC)}/\omega^2}} \tag{1943}$$

Then using the gravitational plasma frequency for a superconductor, $\omega_G = \sqrt{2c^2 \kappa \mu_{G(SC)}}$, we can define a dimensionless relative impedance as

$$Z_r^{(SC)} \equiv \frac{Z_G^{(vac)}}{Z_G^{(SC)}} = \sqrt{1 - \omega_G^2/\omega^2} \tag{1944}$$

Rearranging gives

$$Z_r^{(SC)} = \frac{\sqrt{\omega^2 - \omega_{G(SC)}^2}}{\omega} \tag{1945}$$

Notice that $Z_r^{(SC)}$ is purely imaginary if $\omega < \omega_{G(SC)}$. In fact, it can be written as

$$Z_r^{(SC)} = i \frac{\sqrt{\omega_{G(SC)}^2 - \omega^2}}{\omega} \tag{1946}$$

In that case, the reflection coefficient in (1941) becomes

$$R = \left(\frac{1 - Z_r^{(SC)}}{1 + Z_r^{(SC)}} \right)^* \left(\frac{1 - Z_r^{(SC)}}{1 + Z_r^{(SC)}} \right) = \left(\frac{1 + Z_r^{(SC)}}{1 - Z_r^{(SC)}} \right) \left(\frac{1 - Z_r^{(SC)}}{1 + Z_r^{(SC)}} \right) = 1 \tag{1947}$$

This means that there is total reflection of the wave. On the other hand, the transmission coefficient in (1941) becomes

$$T = \left(\frac{2}{1 + Z_r^{(SC)}} \right)^* \left(\frac{2}{1 + Z_r^{(SC)}} \right) = \frac{4}{\left(1 - Z_r^{(SC)}\right)\left(1 + Z_r^{(SC)}\right)} = \frac{4}{1 - \left(Z_r^{(SC)}\right)^2} \tag{1948}$$

Using (1946) and simplifying gives

$$T = \frac{4\omega^2}{\omega_{G(SC)}^2} \tag{1949}$$

However, we would expect that $T = 0$ when $R = 0$.

[216]The results in (1942) have the same form as equation (9.109) of Griffiths [29].

If $\omega \gtrsim \omega_{G(SC)}$, then $Z_G^{(SC)}$ is real but much greater than $Z_G^{(vac)}$. This leads to $R \approx 1$. (Note that the expression for the transmission coefficient is not valid in these cases.) However, if $\omega > \omega_G(SC)$, then $Z_G^{(SC)}$ is real which leads to values for R and T. In the limit that $\omega >> \omega_G$, we find that $Z_G^{(vac)}/Z_G^{(SC)} \approx 1$ which leads to $R \approx 0$ and $T \approx 1$. This means that there is almost perfect transmission.

For a normal conductor (NC), the gravitational impedance is

$$Z_G^{(NC)} = \frac{4\pi G/c}{\sqrt{1+i2c^2\kappa\eta_{G(NC)}/\omega}} \qquad (1950)$$

Once again, we can let $Z_G^{(1)} = Z_G^{(vac)}$ and $Z_G^{(2)} = Z_G^{(NC)}$. For a perfect gravitational conductor, $\eta_{G(NC)} \to \infty$, which means that $Z_G^{(NC)} \to 0$ and (1941) predicts $R = 1$ and $T = 0$, which implies perfect reflection. On the other hand, for an extremely poor gravitational conductor, $\eta_{G(NC)} \to 0$, which means $Z_G^{(NC)} \to Z_G^{(vac)}$ and (1941) predicts $R = 0$ and $T = 1$, which implies perfect transmission.

18.6 Summary of electromagnetic quantities and gravitational analogs

In this section, a summary is provided of electromagnetic quantities and their corresponding gravitational analogs as developed in this dissertation.

Electromagnetic quantities	**Electromagnetic equations**		**Gravitational quantities**	**Gravitational equations**
Electric Permittivity $\varepsilon_0 = \frac{1}{4\pi K}$ (in vacuum) $\varepsilon = \varepsilon_r\, \varepsilon_0$ (in matter)	Gauss law $\nabla \cdot \bar{E} = \rho_{charge} / \varepsilon$ Electric auxiliary field $\bar{D} = \varepsilon \bar{E}$		Gravito-electric permittivity $\varepsilon_{G,0} = \frac{1}{4\pi G}$ (in vacuum) $\varepsilon_G = \varepsilon_{G,r}\, \varepsilon_{G,0}$ (in matter)	Gravito-Gauss law $\nabla \cdot \bar{E}_G = \rho_{mass} / \varepsilon_G$ Gravito-electric auxiliary vector field $\bar{D}_G = \varepsilon \bar{E}_G$
Magnetic permeability $\mu_0 = 4\pi\ K/c^2$ (in vacuum) $\mu = \mu_r\, \mu_0$ (in matter) $\varepsilon\, \mu = 1/v^2$	Magneto-static Ampere law $\nabla \times \bar{B} = \mu \bar{J}_{charge}$ Magnetic auxiliary field $\bar{B} = \mu \bar{H}$		Gravito-magnetic permeability $\mu_{G,0} = 4\pi G/c^2$ (in vacuum) $\mu_G = \mu_{G,r}\, \mu_{G,0}$ (in matter) $\varepsilon_G\, \mu_G = 1/v^2$	Gravito-magneto-static Ampere law $\nabla \times \bar{B}_G = \mu_G \bar{J}_{mass}$ Gravito-magnetic auxiliary field $\bar{B}_G = \mu \bar{H}_G$
			Gravitational wave permeativity $\kappa_0 = \frac{8\pi G}{c^4}$ (in vacuum) $\kappa = \kappa_r\, \kappa_0$ (in matter)	Gravitational wave equation for transverse-traceless strain field $\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right) h_{ij}^{\tau\tau} = -2\kappa T_{ij}^{\tau\tau}$ or $\partial_k B_{ijk} = -2\kappa T_{ij}^{\tau\tau} + \frac{1}{c^2}\partial_t E_{ij}$ where $E_{ij} = -\partial_t h_{ij}^{\tau\tau}$ and $B_{ijk} = \partial_k h_{ij}^{\tau\tau}$ are electric-like and magnetic-like gravitational wave tensor fields, respectively.

Electromagnetic quantities	**Electromagnetic equations**		**Gravitational quantities**	**Gravitational equations**
Electric susceptibility $\varepsilon = \varepsilon_0\,(1+\chi_E)$	Electric polarization $\bar{P} = \chi_E \varepsilon_0 \bar{E}$		Susceptibility for the gravito-electric *vector* field $\varepsilon_G = \varepsilon_{G,0}\,(1+\chi_{G\text{-}E})$	Gravito-electric *dipole* polarization $\bar{P}_{mass} = \chi_{G-E}\,\varepsilon_{G,0}\bar{E}_G$
			Susceptibility for the electric-like gravitational wave tensor field $\kappa_G = \kappa_{G,0}\,(1+\chi_{GW\text{-}E})$	Gravito-electric *quadrupole* polarization $Q_{ij}^{(mass)} = \chi_{GW\text{-}E}\,\varepsilon_{G,0}\,E_{ij}$
			Susceptibility for the *tendex* field $\varepsilon_{G\text{-}T} = \varepsilon_{G\text{-}T,0}\,(1+\chi_{G\text{-}T})$	Gravito-electric *quadrupole* polarization $Q_{ij}^{(mass)} = \chi_{G\text{-}T}\,\varepsilon_{G,0}\,E_{ij}^{(tendex)}$
Magnetic susceptibility $\varepsilon = \varepsilon_0\,(1+\chi_E)$	Magnetic polarization $\bar{M} = \chi_M \bar{H}$		Susceptibility for the gravito-magnetic *vector* field $\mu_G = \mu_{G,0}\,(1+\chi_{G\text{-}M})$	Gravito-magnetic *dipole* polarization $\bar{M}_{mass} = \chi_{G-M}\bar{H}_G$
			Susceptibility for the magnetic-like gravitational wave tensor field $\kappa_G = \kappa_{G,0}\,(1+\chi_{GW\text{-}E})$	Gravito-magnetic *quadrupole* polarization $Q_{ij}^{(mass\text{-}current)} = \chi_{GW\text{-}M}\,H_{ij}$ where $H_{ij} = B_{ijk}\hat{n}_k / \mu_G$
			Susceptibility for the *vortex* field $\varepsilon_{G\text{-}T} = \varepsilon_{G\text{-}T,0}\,(1+\chi_{G\text{-}T})$	Gravito-magnetic *quadrupole* polarization $Q_{ij}^{(mass\text{-}current)} = \chi_{G\text{-}V}\,B_{ij}^{(vortex)}$
Impedance using permittivity and permeability $Z = \sqrt{\frac{\mu}{\varepsilon}} = \mu v$ $Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \mu_0 c$	Impedance using ratio of fields $Z = \frac{E}{H} = \frac{\omega\mu}{k} = \mu v$		Gravitational impedance using gravito-permittivity and gravito-permeability $Z_G = \sqrt{\frac{\mu_G}{\varepsilon_G}} = \mu_G v$ $Z_{G,0} = \sqrt{\frac{\mu_{G,0}}{\varepsilon_{G,0}}} = \mu_{G,0} c = \frac{4\pi G}{c}$	Gravitational impedance using ratio of tensor gravitational wave fields $Z_G = \frac{E_{ij}}{H_{ij}} = \frac{\omega\mu_G}{k} = \mu_G v$
Electrical resistance $R = \varphi_E / I_{charge}$	Units of electrical resistance $[\Omega] = [J][s] / [C]^2$		Gravitational resistance $R_G = \varphi_G / I_{mass}$	Units of gravitational resistance

Electromagnetic quantities	**Electromagnetic equations**		**Gravitational quantities**	**Gravitational equations**
Electrical conductivity $\sigma = \sigma_1 + i\sigma_2$	Ohm law and London law $\boldsymbol{J}_{\text{charge}} = \sigma \boldsymbol{A}$		Gravito-conductivity for *vector* fields $\sigma_{\text{G}} = \sigma_{\text{G},1} + i\sigma_{\text{G},2}$	Gravito-Ohm law and Gravito-London law for *vector* fields $\boldsymbol{J}_{\text{mass}} = \sigma_{\text{G}}\, \boldsymbol{h}$
			Gravito-conductivity for gravitational wave *tensor* fields $\Sigma_G = \eta_{\text{G}} + i\mu_{\text{G}}$	Gravito-Ohm law and Gravito-London law for gravitational wave *tensor* fields $T_{\text{ij}}^{\tau\tau} = \Sigma_G\, h_{\text{ij}}^{\tau\tau}$

19 Gravitational wave boundary conditions and power output

19.1 Gravitational wave boundary conditions and waveguides

Here we consider gravitational waves confined to the interior of a waveguide which is assumed to be a perfect "gravitational conductor." We can make use of the fact that the Helmholtz Decomposition formulation of linearized GR allows us to isolate the radiative degrees of freedom as $h_{ij}^{\tau\tau}$ which is a transverse-traceless, gauge-invariant quantity. Using this formalism, we find the boundary conditions and wave modes that are possible in an arbitrarily shaped waveguide.

Gravitational wave field constraints, sources and field equations

As shown in $(330)-(333)$, it was found that Φ, Θ, and Ξ_i satisfy Poisson equations while $h_{ij}^{\tau\tau}$ satisfies a wave equation. Therefore, the solutions for Φ, Θ, and Ξ_i will fall off as $1/r^2$, while the solution for $h_{ij}^{\tau\tau}$ will fall off as $1/r$. For that reason, $h_{ij}^{\tau\tau}$ can be considered as describing gravitational waves that propagate out to the far-field region, while Φ, Θ, and Ξ_i are all confined to the near-field region. Therefore, the treatment in this section will be focused on characterizing the boundary conditions for $h_{ij}^{\tau\tau}$. Recall from (181) and (182) that $h_{ij}^{\tau\tau}$ is both transverse and traceless so that

$$\partial_i h_{ij}^{\tau\tau} \quad = \quad 0 \qquad \text{and} \qquad \delta^{ij} h_{ij}^{\tau\tau} = 0 \tag{1951}$$

The wave equation in (361) for $h_{ij}^{\tau\tau}$ can be written as

$$\Box h_{ij}^{\tau\tau} = -2\kappa T_{ij}^{\tau\tau} \tag{1952}$$

The retarded Green's function solution to the D'Alembert (wave) operator is $\dfrac{-\delta\left(t-|\vec{x}-\vec{x}'|/c\right)}{4\pi|\vec{x}-\vec{x}'|}$. Therefore, we have

$$h_{ij}^{\tau\tau}(t,\vec{x}) \quad = \quad \frac{4G}{c^4}\int \frac{T_{ij}^{\tau\tau}(t_r,\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' \tag{1953}$$

where $\vec{x}'$ is the spatial coordinate of each infinitesimal element of $T_{ij}^{\tau\tau}$ occupying a differential volume element d^3x. Also, $T_{ij}^{\tau\tau}(t_r,\vec{x}')$ is the stress-energy-momentum contribution at $\vec{x}'$ evaluated at a retarded time t_r and located at a distance $|\vec{x}-\vec{x}'|$ from the field point where $h_{ij}^{\tau\tau}$ is measured. We can therefore express the retarded time as $t_r = t-|\vec{x}-\vec{x}'|/c$. In (429) we found that the transverse-traceless stress tensor for an ideal fluid is

$$T_{ij}^{\tau\tau} \quad = \quad \left(\rho+\frac{P}{c^2}\right)\gamma^2\left(v_iv_j-\tfrac{1}{3}\delta_{ij}v^2\right) \tag{1954}$$

For the case of relativistic dust (to order v^2/c^2), we found in (431) that this reduces to

$$T_{ij}^{\tau\tau} \quad = \quad \rho\left(v_iv_j-\tfrac{1}{3}\delta_{ij}v^2\right) \tag{1955}$$

Therefore, we can consider $T_{ij}^{\tau\tau}$ as effectively describing mass-currents (of a tensor nature) in a gravitational conductor.

Gravitational wave boundary conditions

If we consider a "Gaussian pillbox" extending just slightly from the vacuum (region 1) to the inside of the conductor (region 2) and apply Gauss's theorem, we find that the normal component of $h_{ij}^{\tau\tau}$ is continuous since $T_{ij}^{\tau\tau}$ is purely transverse.[217]

$$h_{ij\ 1\perp}^{\tau\tau} - h_{ij\ 2\perp}^{\tau\tau} \quad = \quad 0 \tag{1956}$$

This is consistent with the fact that (1951) gives $\partial_i h_{ij}^{\tau\tau} = 0$ which is analogous to the magnetic field which satisfies $\partial_i B_i = 0$. This is no surprise since B_i and $h_{ij}^{\tau\tau}$ are both transverse fields which are generated from electric currents and mass currents, respectively. They are both unlike E_i which can be generated from static charges.

We can also use a rectangular "Amperian loop" of length $\vec{l}$ that is straddling the surface of the conductor and therefore "threaded" by a free surface *mass*-current density, $\vec{K}_f$, on the conductor.Note that $\hat{n}$ is a unit

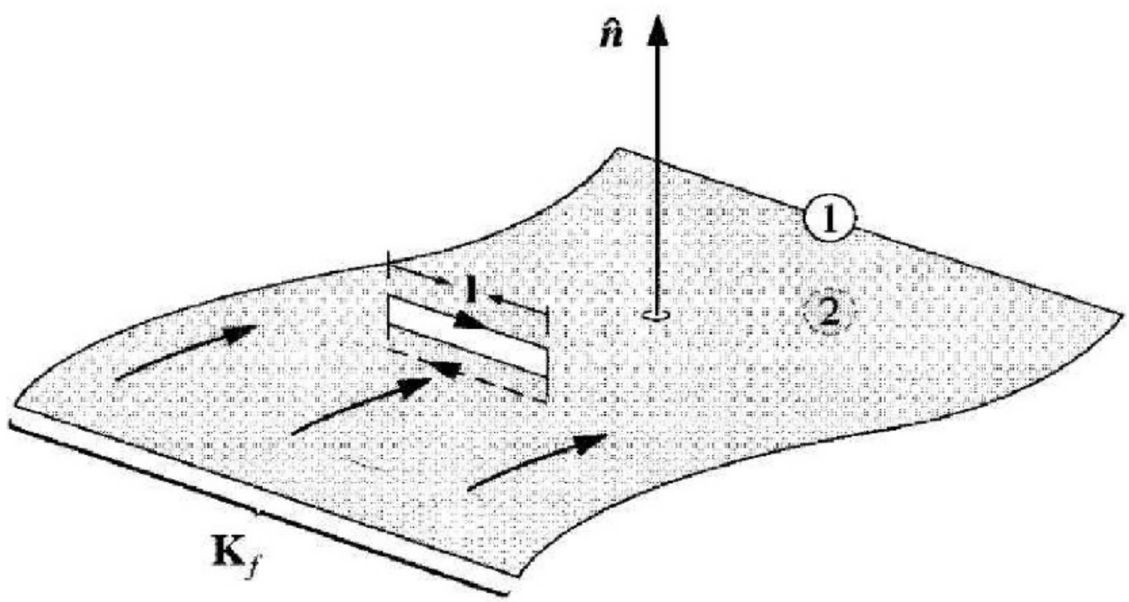

Figure 13: An Amperian loop straddling the surface of a gravitational conductor with a surface mass-current density, $\vec{K}_f$.

vector normal to the conductor (pointing from region 2 to region 1). Therefore $\hat{n} \times \vec{l}$ is normal to the Amperian loop and the free mass-current on the conductor is given by

$$I_{free} = \vec{K}_f \cdot \left(\hat{n} \times \vec{l}\right) = K_f \hat{z} \cdot l\hat{z} = K_f l \tag{1957}$$

In this case, (1953) reduces to effectively just a line-integral and therefore $h_{ij\ \parallel}^{\tau\tau}$ is discontinuous across the boundary between regions 1 and 2. From (1953) we obtain

$$h_{ij\ 1,\parallel}^{\tau\tau}(t,\vec{x}) - h_{ij\ 2,\parallel}^{\tau\tau}(t,\vec{x}) = \frac{4G}{c^4}\int_0^l \frac{T_{ij}^{\tau\tau}}{|x-x'|}dx' \tag{1958}$$

[217]Formally, to show this we would need to apply a time-like Killing vector to $T_{\mu\upsilon}^{\tau\tau}$ to turn it into a rank-1 tensor (four-vector) $J^{\mu} = \left(c\rho, J_m^i\right)$. Then we can properly apply Gauss's law in curved space-time. We can also obtain the free surface mass-current density $\vec{K}_f$ by taking a line-integral of $\vec{J}_m$.

Gravitational waves propagating along the axis of a waveguide of arbitrary geometry

We can begin with a waveguide of arbitrary geometry as shown in the diagram below.Since the interior

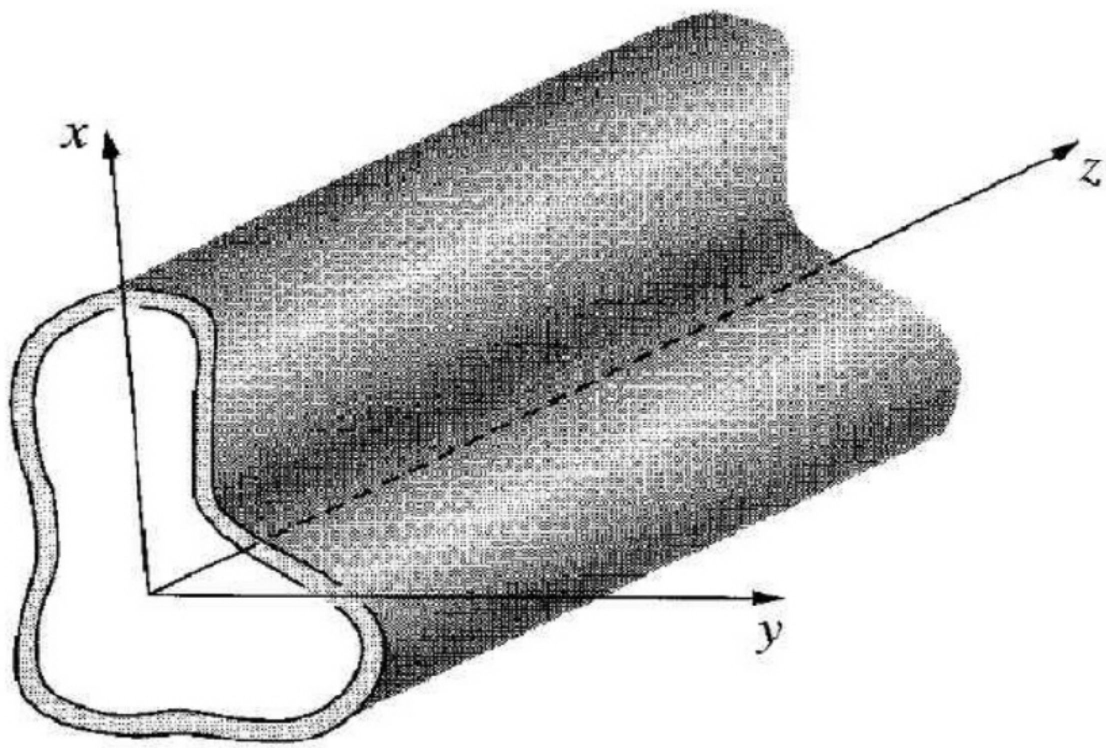

Figure 14: A gravitational wave guide with arbitrary geometry

of the waveguide is vacuum, then in that region we simply have

$$\Box h_{ij}^{\tau\tau} = 0 \tag{1959}$$

Note that $h_{ij}^{\tau\tau}$ has *six* components which are all associated with gravitational radiation. However, within the six components there are really only two physical degrees of freedom due to the constraint equations given by (1951) where $\partial_i h_{ij}^{\tau\tau} = 0$ involves three constraints (for transversality) and $\delta^{ij} h_{ij}^{\tau\tau} = 0$ is one additional constraint (for tracelessness). For a plane wave propagating in the z-direction, the wave vector is $k^i = (0,0,k)$. We can write the gravitational wave as $h_{ij}^{\tau\tau}(\vec{x},t) = \mathrm{Re}\,\tilde{h}_{ij}^{\tau\tau}(\vec{x},t)$ where

$$\tilde{h}_{ij}^{\tau\tau}(\vec{x},t) = A_{ij}^{\tau\tau}(x,y)\, e^{i(kz-\omega t)} \tag{1960}$$

Applying $\partial^j h_{ij}^{\tau\tau} = k^j A_{ij}^{\tau\tau} = 0$ requires $A_{i3} = 0$. Also, applying $\delta^{ij} h_{ij}^{\tau\tau} = 0$ requires $A_{11} = -A_{22}$. We can use the notation $A_{11} = h_{\oplus}$ and $A_{12} = h_{\otimes}$ to write $A_{ij}^{\tau\tau}$ in a form similar to the transverse-traceless gauge. Then we have

$$A_{ij}^{\tau\tau}(x,y) = \begin{pmatrix} h_{\oplus} & h_{\otimes} \\ h_{\otimes} & -h_{\oplus} \end{pmatrix} \tag{1961}$$

where $h_{\oplus}$ and $h_{\otimes}$ are functions of x and y such that $A_{ij}^{\tau\tau}(x,y) \to 0$ as $r \to \infty$ in order to satisfy the conditions necessary for the Helmholtz Decomposition formulation. Also, it is shown in (2637) and (2638) of Appendix F, that because $h_{ij}^{\tau\tau}$ is transverse $\left(\partial_i h_{ij}^{\tau\tau} = 0\right)$ and traceless $\left(\delta^{ij} h_{ij}^{\tau\tau} = 0\right)$, we also must satisfy the following relations.

$$\partial_x h_{\oplus} + \partial_y h_{\otimes} = 0 \qquad \text{and} \qquad \partial_x h_{\otimes} - \partial_y h_{\oplus} = 0 \tag{1962}$$

Inserting (1960) into (1959) and using $\nabla_{\perp}^{2} = \partial_{x}^{2} + \partial_{y}^{2}$ as the transverse Laplacian gives

$$\left(\nabla_{\perp}^{2} + \partial_{z}^{2} - \frac{1}{c^{2}}\partial_{t}^{2}\right) A_{ij}^{\tau\tau}(x,y)\, e^{i(kz-\omega t)} = 0 \tag{1963}$$

Although this is formally a tensor wave equation, we can express it as two scalar wave equations since $A_{ij}^{\tau\tau}(x,y)$ has only two degrees of freedom. Using $h_{\oplus}$ and $h_{\otimes}$ from (1961) gives

$$\left(\nabla_{\perp}^{2} - k^{2} + \frac{\omega^{2}}{c^{2}}\right) h_{\oplus}(x,y) = 0 \qquad \text{and} \qquad \left(\nabla_{\perp}^{2} - k^{2} + \frac{\omega^{2}}{c^{2}}\right) h_{\otimes}(x,y) = 0 \tag{1964}$$

Notice that this is mathematically identical to the case for *electromagnetic* waves in the z-direction. According to Griffiths [29], equation (9.181), the electric and magnetic fields are given by

$$\left(\nabla_{\perp}^{2} - k^{2} + \frac{\omega^{2}}{c^{2}}\right) E_{z}(x,y) = 0 \qquad \text{and} \qquad \left(\nabla_{\perp}^{2} - k^{2} + \frac{\omega^{2}}{c^{2}}\right) B_{z}(x,y) = 0 \tag{1965}$$

The difference is that for the case of electromagnetism, the boundary conditions for a perfect conductor are given in [29], equation (9.175) as $\mathbf{E}^{\parallel} = 0$, $\mathbf{B}^{\perp} = 0$. However, in the case of gravitational waves, we find from (1956) that $h_{\oplus}^{\perp} = h_{\otimes}^{\perp} = 0$. This means that gravitational radiation degrees of freedom have boundary conditions that are similar only to the magnetic field in electromagnetism. (As noted earlier, this is due to the fact that both B_i and $h_{ij}^{\tau\tau}$ are transverse and are generated only by currents, not static sources.)

We can also notice that in the case of electromagnetism, it is possible to have $E_z \neq 0$ or $B_z \neq 0$ for a wave propagating in the z-direction. In other words, it is possible to have a *longitudinal* degree of freedom for the fields. However, in the case of gravitation, a wave propagating in the z-direction can only have fields given by $h_{xx} = -h_{yy}$ and/or $h_{xy} = h_{yx}$. This means that there are absolutely no longitudinal degrees of freedom. This is consistent with the fact that $h_{ij}^{\tau\tau}$ is transverse and *traceless*.

Furthermore, in the case of electromagnetism, there can be three wave modes: the TE (transverse electric) mode which has $E_z = 0$; the TM (transverse magnetic) mode which has $B_z = 0$; and the TEM (transverse electromagnetic) mode which has $E_z = B_z = 0$. However, in the case of gravitation, these modes don't seem to exist. In a sense, there is only a single mode (which could be called the "transverse gravitational" mode). This is due to the fact that setting $h_{\oplus}$ or $h_{\otimes}$ to zero (analogous to setting E_z or B_z to zero) doesn't lead to different *modes* but rather to different *polarizations*.

Also, setting $h_{\oplus} = h_{\otimes} = 0$ implies there is no gravitational wave. This is clearly in contrast to the case for electromagnetism where the choice of setting $E_z = B_z = 0$ simply leads to the TEM mode.[218] This is a consequence of the fact that $\vec{E}$ and $\vec{B}$ satisfy wave equations that are fundamentally derived from *first-order* differential equations, namely, the Maxwell equations. Therefore, if E_z and B_z are both zero, it is still possible to return to the Maxwell equations to describe the other components of $\vec{E}$ and $\vec{B}$. (See footnote 18 on p. 407 of [29] concerning this issue.)

This is also related to the fact that $\vec{E}$ and $\vec{B}$ can be near-fields as well as radiation fields. (In other words, they satisfy first-order differential equations involving sources and they also satisfy second-order wave equations.) In contrast to this, $h_{ij}^{\tau\tau}$ is purely a radiation field. It only satisfies a second-order wave equation. In fact, as shown by the Helmholtz Decomposition formulation, $h_{ij}^{\tau\tau}$ is uniquely projected out from the rest of the metric as the only quantity that satisfies a wave equation.

Yet another factor related to this point is the proof for showing that the TEM mode is not supported in a waveguide. It is shown on p. 407 of [29] that due to Gauss's law and Faraday's law, it is impossible to have an

[218] Although the TEM mode is not supported by a waveguide, it is certainly valid in vacuum. This still demonstrates the point that setting $E_z = B_z = 0$ can lead to EM waves, whereas setting $h_{\oplus} = h_{\otimes} = 0$ leads to *no* GR wave.

electromagnetic wave if $E_z = B_z = 0$. However, for the case of gravitational waves, there is no gravitational Gauss's law or Faraday's law for $h_{ij}^{\tau\tau}$. Therefore there is no corresponding restrictions that would apply to $h_{ij}^{\tau\tau}$.

Concerning polarizations, we find for EM waves that setting $E_z = 0$ or $B_z = 0$ in (1965) doesn't determine the polarization of the waves. For example, for $E_z = 0$ (the TE mode), we still have E_x and E_y remaining unspecified. Only if we set $E_x = 0$ or $E_y = 0$ do we have the polarization specified. This is in contrast to gravitational waves in (1964) where setting $h_\oplus$ or $h_\otimes$ to zero will determine the polarization of the gravitational wave. In fact, setting $h_\otimes = 0$ gives a purely "plus" polarization wave, setting $(h_\oplus = 0)$ gives a purely "cross" polarization wave, and letting $(h_\otimes = h_\oplus \neq 0)$ gives a "mixed" polarization which can be a "circular" or "elliptical" polarization, depending on the relative phase of $h_\oplus(x,y)$ and $h_\otimes(x,y)$.

Lastly, in (2637) and (2638) of Appendix F, it was found that the transverse Laplacian of $h_\oplus$ or $h_\otimes$ is zero. Therefore, (1964) reduces to just

$$\left(-k^2 + \frac{\omega^2}{c^2}\right) h_\oplus(x,y) = 0 \qquad \text{and} \qquad \left(-k^2 + \frac{\omega^2}{c^2}\right) h_\otimes(x,y) = 0 \tag{1966}$$

As a result, the boundary conditions in the x and y directions don't play a role and no modes or cut-off frequency can be obtained. This is a surprising result since it would be expected that a gravitational waveguides would have a cut-off frequency for the gravitational waves that it can support. However, it follows mathematically from the fact that $h_{ij}^{\tau\tau}$ is transverse $(\partial_i h_{ij}^{\tau\tau} = 0)$ and traceless $(\delta^{ij} h_{ij}^{\tau\tau} = 0)$ that $\nabla_\perp^2 h_\oplus(x,y) = \nabla_\perp^2 h_\otimes(x,y) = 0$.

The role of near fields and conservation laws for determining boundary conditions

In the treatment above, boundary conditions for $h_{ij}^{\tau\tau}$ were developed based on the transversality of the field (1956) and the transversality of $T_{ij}^{\tau\tau}$ in (1958). However, it should be noted that $h_{ij}^{\tau\tau}$ is also related to other near-fields by the Bianchi identities. In previous sections, the following relations were developed.

$$\partial_j S_{ik} - \partial_k S_{ij} = -\partial_t U_{ijk} \qquad \text{and} \qquad \partial_k V_{lij} - \partial_l V_{kij} = -\partial_t W_{ijkl} \tag{1967}$$

Here we have used the following tensor field definitions.

$$\begin{aligned} &S_{ij} \equiv \delta_{ij}\ddot{\Theta} + \ddot{h}_{ij}^{\tau\tau}, && U_{ijk} \equiv \partial\left[_k \delta_{ij}\dot{\Theta}\right] - \partial_i \partial\left[_k \Xi_j\right] + \partial\left[_k \dot{h}_{ij}^{\tau\tau}\right] \\ &V_{ijk} \equiv \partial\left[_k \delta_{ij}\dot{\Theta}\right] + \partial\left[_k \dot{h}_{ij}^{\tau\tau}\right], && W_{ijkl} \equiv \partial_k \partial\left[_j \left(h_{il}^{\tau\tau} + \delta_{ij}\Theta\right)\right] - \partial_i \partial\left[_j \left(h_{ik}^{\tau\tau} + \delta_{ik}\Theta\right)\right] \end{aligned} \tag{1968}$$

These relations imply that spatial and temporal changes in Θ and Ξ_i can induce spatial and temporal changes in $h_{ij}^{\tau\tau}$. (This is similar to the case in electromagnetism where spatial and temporal changes in $\vec{E}$ can induce spatial and temporal changes in $\vec{B}$, and vice versa.) Therefore, inside a wave guide, it is expected that $h_{ij}^{\tau\tau}$ would be determined not just by the source $T_{ij}^{\tau\tau}$, but by changes in Θ and Ξ_i due to changed in their associated sources as well.

Note that the field equations for Φ, Θ, Ξ_i, and $h_{ij}^{\tau\tau}$ were fond in (330) – (333) to be

$$\begin{aligned} &\nabla^2\Phi = 4\pi G\left(\rho + \frac{3}{c^2}\left(\mathbb{P} - \dot{I}\right)\right), && \nabla^2\Theta = -\frac{8\pi G}{c^2}\rho \\ &\nabla^2\Xi_i = -\frac{16\pi G}{c^2}R_i, && \Box h_{ij}^{\tau\tau} = -\frac{16\pi G}{c^4}T_{ij}^{\tau\tau} \end{aligned} \tag{1969}$$

These equations would seem to imply that $h_{ij}^{\tau\tau}$ could also be indirectly coupled to Φ,Θ, and Ξ_i through conservation of the stress-energy tensor, $\partial^\nu T_{\mu\nu} = 0$, which would relate the sources of Φ,Θ, and Ξ_i to the source of $h_{ij}^{\tau\tau}$. However, the conversation laws were found in (283), (289), and (292) to be, respectively,

$$\dot{\rho} = \nabla^2 I, \qquad \tfrac{2}{3}\nabla^2 L = \dot{I} - \mathbb{P}, \qquad \nabla^2 r_i = \dot{R}_i \tag{1970}$$

Since $T_{ij}^{\tau\tau}$ is not related to the other stress tensor sources by any conservation law, then $h_{ij}^{\tau\tau}$ is not related to Φ,Θ, and Ξ_i through conservation of the stress tensor. Hence we conclude that the boundary conditions for $h_{ij}^{\tau\tau}$ are determined only by the transversality of the field (1956), the transversality of $T_{ij}^{\tau\tau}$ in (1958), and the Bianchi identities that relate $h_{ij}^{\tau\tau}$ to Θ and Ξ_i.

19.2 Ratio of output to input GR wave power (scattering cross-section)

The Isaacson power flux formula can be used to relate the gravitational wave power to the strain field of the gravitational wave. It is given in [43] as

$$\mathscr{P} = \frac{c^3}{16\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle \tag{1971}$$

Here $\mathscr{P}$ is the power per unit area $(\mathscr{P} = P/A)$ while h_+ and $h_\times$ are the plus and cross polarization strain fields. The particular polarization is not relevant to the analysis here so we can simply use h_0 for the amplitude and write a sinusoidal strain field as

$$h = h_0 \sin\left(\vec{k}\cdot\vec{x} - \omega t\right) \tag{1972}$$

Inserting this in the Isaacson power flux formula (1971) and using $\mathscr{P} = P_{in}/A$ for an incoming gravitational wave gives

$$P_{in} = \frac{c^3}{16\pi G} A \left\langle \omega^2 h_0^2 \sin^2\left(\vec{k}\cdot\vec{x} - \omega t\right)\right\rangle \tag{1973}$$

$$= \frac{c^3}{32\pi G} A h_0^2 \omega_{wave}^2 \tag{1974}$$

where ω_{wave} is the angular frequency of the wave. If this power is deposited on a mass distribution that re-radiates gravitationally, then the outgoing gravitational radiation power can be found using Einstein's gravitational quadrupole power formula given in [91] as

$$P_{out} = \frac{G}{45c^5}\left\langle \dddot{D}_{ij}^2 \right\rangle \tag{1975}$$

Here $\left\langle \dddot{D}_{ij}^2 \right\rangle$ is the square of the third time-derivative of the mass quadrupole-moment which is time-averaged over at least a period. We can divide (1975) by (1971) to determine the ratio of outgoing to incoming power. This gives

$$\frac{P_{out}}{P_{in}} = \left(\frac{32\pi G^2}{45c^8}\right)\left\langle \frac{\dddot{D}_{ij}^2}{\dot{h}_+^2 + \dot{h}_\times^2}\right\rangle \tag{1976}$$

The prefactor here is $\sim 10^{-89}$ SI units. This is obviously an extremely small value which would imply that there is no appreciable reflected gravitational wave. However, we can examine the remaining part of the expression to determine if the ratio of outgoing to incoming power must necessarily be miniscule. Using (1974), we can rewrite the ratio as

$$\frac{P_{out}}{P_{in}} = \left(\frac{32\pi G^2}{45c^8}\right)\frac{\left\langle \dddot{D}_{ij}^2 \right\rangle}{A h_0^2 \omega_{wave}^2} \tag{1977}$$

For an order of magnitude calculation, we can consider a quadrupole moment similar to the form shown in Appendix T.[219] The following is a diagram of the system.

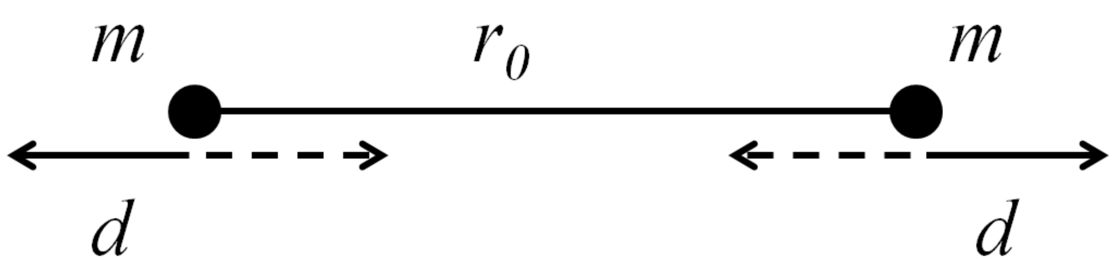

Figure 15: A time-varying mass-quadrupole moment consisting of two particles of mass m which are separated by an average distance r_0 and oscillate with an amplitude of motion given by d. The solid arrows represent the half-cycle when the particles are moving *away* from each other, and the dashed arrows represent the half-cycle when the particles are moving *toward* each other.

If ω_{mass} is the angular frequency describing the sinusoidal oscillation of the masses, then from (2200) we find

$$\left\langle \dddot{D}^2 \right\rangle = 4m^2 d^2 \omega_{mass}^6 \left(16d^2 + \frac{r_0^{\,2}}{2} \right) \tag{1978}$$

To simplify the calculation for each of the four cases, we will always consider the frequency of oscillation of the masses to be comparable to the frequency of the gravitational wave so that we can work with a single frequency: $\omega \approx \omega_{mass} \approx \omega_{wave}$. We can also express the mass of the quadrupole moment in terms of a mass density using $m = \rho V$. The volume containing the Niobium atoms (or Cooper pairs) that re-radiate can be described by the surface area of the cavity, A, multiplied by the gravitational penetration depth, δ_G, which characterizes the depth to which the transverse-traceless stress current will exist in the walls of the superconductor. Therefore, we have $m = \rho A \delta_G$ and (1978) becomes

$$\left\langle \dddot{D}^2 \right\rangle = 4\rho^2 A^2 \delta_G^2 d^2 \omega^6 \left(16d^2 + \frac{r_0^{\,2}}{2} \right) \tag{1979}$$

There are four cases that we can consider:

1. A single large quadrupole moment occupying the face of the material $\left(A \approx r_0^2\right)$ and a *large* strain field which oscillates particles across the face of the material $(r_0 \approx d)$;

2. A single large quadrupole moment occupying the face of the material $\left(A \approx r_0^2\right)$ and a *small* strain field which oscillates particles a small distance $(r_0 >> d)$;

3. An array of small quadrupole moments with small average spacing between them $\left(A >> r_0^2\right)$ which are driven *in phase*.

4. An array of small quadrupole moments with small average spacing between them $\left(A >> r_0^2\right)$ and a *random phase distribution*.

We will determine the gravitational modulus, gravitational plasma frequency, and associated penetration depth for each of the cases above.

[219]This model could be referred to as a ”*gravitational* Lorentz oscillator” since it is similar to the standard Lorentz oscillator which consists of an electron in harmonic motion relative to an atomic nucleus. The difference here is that we consider the two masses of the oscillator to be equal and we neglect the charge.

1. A single large quadrupole $\left(A \approx r_0^2\right)$ with a *large* strain field $(r_0 \approx d)$

If the entire ensemble of particles on the surface of the material form a single quadrupole, then for plus-polarization and cross-polarization, respectively, the surface would have acceleration fields shown below.Since we are considering the case when $r_0 \approx d$, then (1979) gives

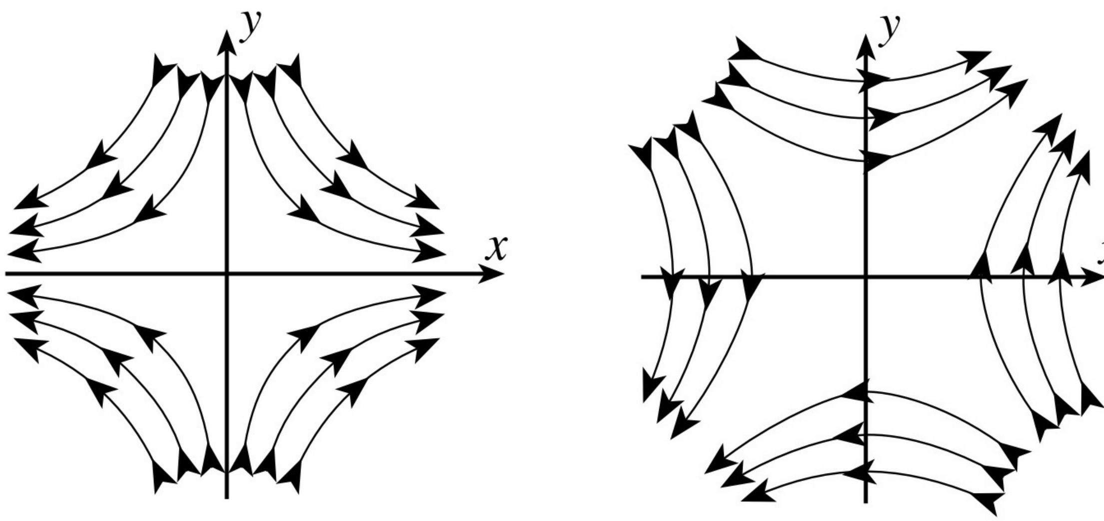

Figure 16: Plus-polarized and cross-polarized gravitational wave acceleration fields.

$$\left\langle \dddot{D}^2 \right\rangle = 66\rho^2 A^2 \delta_G^2 d^4 \omega^6 \tag{1980}$$

Inserting this into (1977) gives

$$\frac{P_{out}}{P_{in}} = \left(\frac{704\pi G^2}{15c^8}\right)\frac{\rho^2 A \delta_G^2 d^4 \omega^4}{h_0^2} \tag{1981}$$

From the geodesic deviation equation in (577) or (578), we can consider that $\ddot{r} \approx \frac{1}{2} r_0 h_0 \omega^2$. Also, from (2072) and (2073) we have $\ddot{r} \approx 2d\omega^2$. Combining these gives[220]

$$d \approx \frac{r_0 h_0}{4} \tag{1982}$$

Note that if we are considering $r_0 \approx d$, then we must have $h_0 \approx 4$. We are also considering the case when $A \approx r_0^2$. Lastly, we can set the frequency to the gravitational plasma frequency, $\omega_G^2 = 16\pi G \mu_G$, where $\mu_{G(SC)}$ is the gravitational modulus, and also use $\delta_G = c/\omega_G$. Then (1981) becomes

$$\frac{P_{out}}{P_{in}} = \left(\frac{352\pi^2 G^3}{15c^6}\right)\rho^2 r_0^6 \mu_{G(SC)} \tag{1983}$$

[220]Note that rearranging (1982) gives $h_0 \approx 4d/r_0$. This is consistent with the fact that h_0 is a strain field which can be expressed as $h_0 = \Delta L/L$, where ΔL is the change in length per unit length L. However, notice that if $r_0 \approx d$, then this would imply that the strain field, $h_0 \approx 4d/r_0$, is greater than unity. This is far from realistic since (1879) predicts that a milliwatt of power will only produce $h_0 \approx 10^{-27}$.

For nearly perfect reflection, we have $P_{out}/P_{in} \approx 1$. Then solving for the gravitational modulus gives

$$\mu_{G(SC)} \approx \left(\frac{15c^6}{704\pi^2 G^3}\right)\frac{1}{\rho^2 r_0^6} \tag{1984}$$

For a *Niobium* superconductor with edges on the order of centimeters, then $\rho \approx 8.6\times 10^3 kg/m^3$ and $r_0 \approx 10^{-2}m$. This gives $\mu_{G(SC)} \approx 7\times 10^{82} J/m^3$. This implies that the gravitational plasma frequency, $\omega_G = \sqrt{16\pi G\mu_G}$, and penetration depth, $\delta_G = c/\omega_G$, are

$$\boxed{\begin{array}{c}\omega_G \approx 2\times 10^{37} s^{-1} \qquad \text{and} \qquad \delta_G \approx 2\times 10^{-29} m \\[1em] \textit{Gravitational plasma frequency and penetration depth for} \\ \textit{a single large quadrupole with } A \approx r_0^2 \approx d^2\end{array}} \tag{1985}$$

The reason for these extreme results is due to the approximation that $A \approx r_0^2 \approx d^2$ which implies that the gravitational strain field greater than unity and the surface of the material forms a single large quadrupole.

2. A single large quadrupole $\left(A \approx r_0^2\right)$ with a *small* strain field $(r_0 >> d)$

If the entire ensemble of particles on the surface of the material form a single quadrupole, then for plus and cross polarization, respectively, we still have the diagrams given above. However, now since we are considering the case when $r_0 >> d$, then we can eliminate the first term in parentheses in (1979) which gives

$$\left\langle \dddot{D}^2 \right\rangle = 2\rho^2 A^2 \delta_G^2 d^2 \omega^6 r_0^2 \tag{1986}$$

Inserting this into (1977) gives

$$\frac{P_{out}}{P_{in}} = \left(\frac{64\pi G^2}{45c^8}\right)\frac{\rho^2 A \delta_G^2 d^2 \omega^4 r_0^2}{h_0^2} \tag{1987}$$

Again we can use $d \approx r_0 h_0/4$ from (1982) and $A \approx r_0^2$. We can also set the frequency to the gravitational plasma frequency, $\omega_G^2 = 16\pi G\mu_G$, and use $\delta_G = c/\omega_G$. Then we have

$$\frac{P_{out}}{P_{in}} = \left(\frac{64\pi^2 G^3}{45c^6}\right)\rho^2 r_0^6 \mu_{G(SC)} \tag{1988}$$

This result is almost identical to the previous result in (1983) except for a slightly different numeric prefactor. For nearly perfect reflection, again we have $P_{out}/P_{in} \approx 1$. Then solving for the gravitational modulus gives

$$\mu_{G(SC)} \approx \left(\frac{45c^6}{64\pi^2 G^3}\right)\frac{1}{\rho^2 r_0^6} \tag{1989}$$

For a *Niobium* superconductor with edges on the order of centimeters, then $\rho \approx 8.6\times 10^3 kg/m^3$ and $r_0 \approx 10^{-2}m$. This gives $\mu_{G(SC)} \approx 2.3\times 10^{84} J/m^3$. This implies that the gravitational plasma frequency, $\omega_G = \sqrt{16\pi G\mu_G}$, and penetration depth, $\delta_G = c/\omega_G$, are

$$\boxed{\begin{array}{c}\omega_G \approx 9\times 10^{37} s^{-1} \qquad \text{and} \qquad \delta_G \approx 3\times 10^{-30} m \\[1em] \textit{Gravitational plasma frequency and penetration depth for} \\ \textit{a single large quadrupole with } A \approx r_0^2 \textit{ and } r_0 >> d^2\end{array}} \tag{1990}$$

The reason for these extreme results is again due to the approximation that $A \approx r_0^2$ which implies that the surface of the material forms a single large quadrupole. Comparing this with (1985) shows that the small strain field, $r_0 >> d$, does not end up playing a significant role.

Note that for a gravitational plasma frequency that is on the order of microwaves, we would need the gravitational modulus,

$$\mu_{G(SC)} \quad = \quad \frac{\omega_G^2}{16\pi G} \approx 3 \times 10^{28} J/m^3 \tag{1991}$$

Using (1989), we find that this would require

$$r_0 \quad \approx \quad \left[\left(\frac{45c^6}{32\pi^2 G^3} \right) \frac{1}{\rho^2 \mu_{G(SC)}} \right]^{1/6} \approx 5 \times 10^{15} m \tag{1992}$$

This means the gravitational wave would need to be incident on a surface with an area on the order of $A \approx 2.5 \times 10^{31} m^2$. If we insist on using $A \approx r_0^2 \approx 10^{-4} m$ and microwave frequencies (with $\mu_{G(SC)} \approx 3 \times 10^{28}$), then (1988) gives

$$\frac{P_{out}}{P_{in}} \quad = \quad \left(\frac{64\pi^2 G^3}{45c^6} \right) \rho^2 \mu_{G(SC)} r_0^6 \approx 5 \times 10^{-57} \tag{1993}$$

This means that there is effectively no output power scattered by the material.

3. An array of small quadrupole moments $\left(A >> r_0^2\right)$ which are *in phase*.

Now we consider the case where the surface of the material consists of an array of small quadrupole moments. We can represent plus and cross polarization quadrupole moments with the following diagrams.

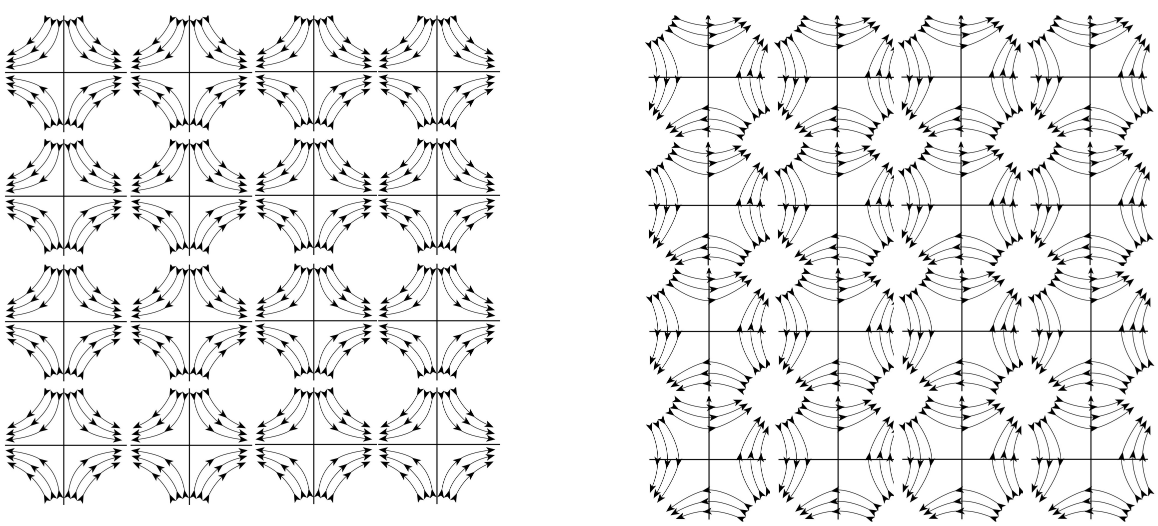

Figure 17: Arrays of small plus-polarized and cross-polarized quadrupole moments.

Since the quadrupole moments are all *in phase*, then they can be considered as a *coherent* array and we can sum over $\dddot{D}_i$ for all the quadrupole moments. Summing over an Avogadro number of quadrupoles gives

$$\dddot{D} = \sum_{i=1}^{N_A} \dddot{D}_i = N_A \dddot{D}_i \tag{1994}$$

where $\dddot{D}_i$ applies to any of the small identical quadrupole momentums and N_A is Avogadro's number. Again we can use (2200) and assume $r_0 >> d$ for a small strain field which means we neglect the term involving d^2. Then we have[221]

$$\left\langle \dddot{D}^2 \right\rangle = N_A^2 \left\langle \dddot{D}_i^2 \right\rangle = 2N_A^2 m_i^2 d^2 \omega^6 r_0^2 \tag{1995}$$

where m_i of a single quadrupole moment. Inserting this into (1977) gives

$$\frac{P_{out}}{P_{in}} = \left(\frac{64\pi G^2}{45c^8}\right) \frac{N_A^2 m_i^2 d^2 \omega^4 r_0^2}{A h_0^2} \tag{1996}$$

Notice that the output power scales with N_A^2, not with N_A. However, instead of using Avogadro's number, we can simply use n as the number of particles occupying a penetration depth with volume $V = A\delta_G$. Then we can express this in terms of a mass density using $nm_i = \rho A \delta_G$. Then we have

$$\frac{P_{out}}{P_{in}} = \left(\frac{64\pi G^2}{45c^8}\right) \frac{\rho^2 A \delta_G^2 d^2 \omega^4 r_0^2}{h_0^2} \tag{1997}$$

This matches (1987), however, here we do not have $A \approx r_0^2$. Instead, r_0 is essentially the lattice spacing between the individual quadrupole moments. We can also use $d \approx r_0 h_0/4$ from (1982) as well as set the frequency to the gravitational plasma frequency, $\omega_G^2 = 16\pi G \mu_G$, and use $\delta_G = c/\omega_G$. Then we have

$$\frac{P_{out}}{P_{in}} = \left(\frac{64\pi^2 G^3}{45c^6}\right) A \rho^2 r_0^4 \mu_{G(SC)} \tag{1998}$$

For nearly perfect reflection, again we have $P_{out}/P_{in} \approx 1$. Then solving for the gravitational modulus gives

$$\mu_{G(SC)} = \left(\frac{45c^6}{64\pi^2 G^3}\right) \frac{1}{\rho^2 A r_0^4} \tag{1999}$$

If this spacing of the quadrupole moments is on the order of the wavelength of the gravitational wave, then $r_0 \approx \lambda = c/\omega_G$ and we have

$$\mu_{G(SC)} = \left(\frac{G}{180c^2}\right) \rho^2 A \tag{2000}$$

For a *Niobium* superconductor with edges on the order of centimeters, then $\rho \approx 8.6 \times 10^3 kg/m^3$ and $A \approx 10^{-4} m$. This gives $\mu_{G(SC)} \approx 3.0 \times 10^{-26} J/m^3$. This implies that the gravitational plasma frequency, $\omega_G = \sqrt{16\pi G \mu_G}$, and penetration depth, $\delta_G = c/\omega_G$, are

$$\boxed{\begin{array}{c} \omega_G \approx 1 \times 10^{-17} s^{-1} \qquad \text{and} \qquad \delta_G \approx 3 \times 10^{25} m \\[1em] \textit{Gravitational plasma frequency and penetration depth for an array of quadrupoles} \\ \textit{that are in phase and separated by a gravitational wavelength } (r_0 \approx c/\omega_G) \end{array}} \tag{2001}$$

[221] Since (1879) predicts that a milliwatt of power will only produce $h_0 \approx 10^{-27}$, then clearly we should assume a small strain field. Then (1982) requires that $r_0 >> d$.

Here we find a drastically different result for the gravitational plasma frequency and penetration depth. There is clearly a significant difference between having a *single* large quadrupole occupy the material versus an array of independent quadrupoles. Even having all of the small quadrupoles *in phase* does not lead to a result that is comparable to a single large quadrupole.

In (1991) we found that for a gravitational plasma frequency that is on the order of microwaves, we would need the gravitational modulus to be $\mu_{G(SC)} \approx 3 \times 10^{28}$. Using (2000), this implies that we would need

$$A = \left(\frac{180c^2}{G}\right)\frac{\mu_{G(SC)}}{\rho^2} \approx 10^{50} m^2 \tag{2002}$$

Alternatively, if the area is kept to $A \approx 10^{-4} m$, then (1999) requires the lattice spacing between the quadrupole moments would be

$$r_0 = \left[\left(\frac{45c^6}{64\pi^2 G^3}\right)\frac{1}{\rho^2 A \mu_G}\right]^{1/4} \approx 9 \times 10^{11} m \tag{2003}$$

If we insist on using $A \approx 10^{-4} m$, and a lattice spacing on the order of the wavelength of the gravitational wave, $r_0 \approx \lambda = c/\omega_G$, and microwave frequencies for the wave ($\mu_{G(SC)} \approx 3 \times 10^{28}$), then (1998) gives

$$\frac{P_{out}}{P_{in}} = \left(\frac{G}{180c^2}\right)\frac{\rho^2 A}{\mu_{G(SC)}} \approx 10^{-54} \tag{2004}$$

This means that there is effectively no output power scattered by the material.

If we consider the quadrupoles to consist of neighboring atoms, then $r_0^3 \approx 8m_i/\rho$ where $m_i \approx 1.5 \times 10^{-25} kg$ for the mass of a single Niobium atom.[222] In that case, we have $r_0 \approx 5.2 \times 10^{-10} m$. Then using (1998) with $A \approx 10^{-4} m$ and microwave frequencies for the wave ($\mu_{G(SC)} \approx 3 \times 10^{28}$) gives

$$\frac{P_{out}}{P_{in}} = \left(\frac{64\pi^2 G^3}{45c^6}\right) A \rho^2 r_0^4 \mu_{G(SC)} \approx 9.5 \times 10^{-86} \tag{2005}$$

If we do not require microwave frequencies, then (1999) becomes

$$\mu_{G(SC)} = \left(\frac{45c^6}{64\pi^2 G^3}\right)\frac{1}{\rho^2 A r_0^4} \approx 3.2 \times 10^{47} J/m^3 \tag{2006}$$

Then the gravitational plasma frequency, $\omega_G = \sqrt{16\pi G \mu_G}$, and penetration depth, $\delta_G = c/\omega_G$, are

$$\boxed{\begin{array}{c} \omega_G \approx 3 \times 10^{19} s^{-1} \qquad \text{and} \qquad \delta_G \approx 9 \times 10^{-12} m \\ \\ \textit{Gravitational plasma frequency and penetration depth for an array of quadrupoles} \\ \textit{that are in phase and separated by a lattice length } \left(r_0^3 \approx 8m_i/\rho\right) \end{array}} \tag{2007}$$

[222] We obtain $r_0^3 \approx m_i/\rho$ by considering that the mass density for a lattice cube (with one atom at each corner and sides of length r_0) is $\rho \approx 8m_i/r_0^3$ since there are 8 corners. Therefore $r_0^3 \approx 8m_i/\rho$.

4. An array of small quadrupole moments $\left(A >> r_0^2\right)$ with a *random phase distribution*.

Now we consider the case where the surface of the material consists of an array of small quadrupole moments which have a random distribution of phases. This can be considered as an *incoherent* array. The diagrams given above still apply, however, now we cannot *coherently* add $\dddot{D}$ for each quadrupole. Instead, we must consider the power produced by each quadrupole individually, P_i, and then sum them. Using (1975) to find the power of each quadrupole moment and then summing over an Avogadro number of them gives

$$P_{out} = \sum_{i=1}^{N_A} P_i = \sum_{i=1}^{N_A} \frac{G}{45c^5} \left\langle \dddot{D}^2 \right\rangle_i = \frac{G}{45c^5} N_A \left\langle \dddot{D}^2 \right\rangle_i \tag{2008}$$

Again we can use (2200) and assume $r_0 >> d$ for a small strain field which means we neglect the term involving d^2.

$$P_{out} = \frac{2G}{45c^5} N_A m_i^2 d^2 \omega^6 r_0^2 \tag{2009}$$

Notice that now the output power scales with N_A, not with N_A^2 as it did for a *coherent* array.[223] Instead of using Avogadro's number, we can simply use n as the number of particles occupying a penetration depth with volume $V = A\delta_G$. Then we can express this in terms of a mass density using $nm_i = \rho A \delta_G$. Then using (1974) to write the ratio of output power to input power gives

$$\frac{P_{out}}{P_{in}} = \left(\frac{64\pi G^2}{45c^8}\right) \frac{nm_i^2 d^2 \omega^4 r_0^2}{Ah_0^2} \tag{2010}$$

We can express this in terms of a mass density using $nm_i = \rho V$ and $V = A\delta_G$.

$$\frac{P_{out}}{P_{in}} = \left(\frac{64\pi G^2}{45c^8}\right) \frac{m_i \rho \delta_G d^2 \omega^4 r_0^2}{h_0^2} \tag{2011}$$

Again we can use $d \approx r_0 h_0/4$ from (1982) as well as set the frequency to the gravitational plasma frequency, $\omega_G^2 = 16\pi G \mu_G$, and use $\delta_G = c/\omega_G$. Then we have

$$\frac{P_{out}}{P_{in}} = \left(\frac{256\pi^{5/2} G^{7/2}}{45c^7}\right) m_i \rho r_0^4 \mu_{G(SC)} \tag{2012}$$

For nearly perfect reflection, again we have $P_{out}/P_{in} \approx 1$. Then solving for the gravitational modulus gives

$$\mu_{G(SC)} = \left[\left(\frac{45}{256}\right)^2 \frac{c^{14}}{\pi^5 G^7} \frac{1}{m_i^2 \rho^2 r_0^8}\right]^{1/3} \tag{2013}$$

If the spacing is on the order of the wavelength of the gravitational wave, then $r_0 \approx \lambda = c/\omega_G$ and we have

$$\mu_{G(SC)} = \frac{1}{2025} \frac{\pi G^3}{c^6} m_i^2 \rho^2 \tag{2014}$$

[223] This issue of coherent versus incoherent sources (which leads to a factor of N_A^2 or N_A, respectively) is dealt with in more detail in [102].

For a *Niobium* superconductor, $\rho \approx 8.6 \times 10^3 kg/m^3$ and $m_i \approx 1.5 \times 10^{-25} kg$. This gives $\mu_{G(SC)} \approx 1.0 \times 10^{-126} J/m^3$. This implies that the gravitational plasma frequency, $\omega_G = \sqrt{16\pi G \mu_G}$, and penetration depth, $\delta_G = c/\omega_G$, are

$$\boxed{\begin{array}{c} \omega_G \approx 6 \times 10^{-67} s^{-1} \qquad \text{and} \qquad \delta_G \approx 5 \times 10^{74} m \\ \\ \textit{Gravitational plasma frequency and penetration depth for an array} \\ \textit{of quadrupoles with a random distribution of phases} \\ \textit{and separated by a gravitational wavelength } (r_0 \approx c/\omega_G) \end{array}} \tag{2015}$$

Here we find a gravitational plasma frequency and penetration depth that is even more extreme. There is clearly a huge difference between having an array of quadrupoles that are in phase versus a random distribution of phases.

If we consider the quadrupoles to consist of neighboring atoms, then $r_0^3 \approx 8m_i/\rho$ where $m_i \approx 1.5 \times 10^{-25} kg$ for the mass of a single Niobium atom. In that case, we have $r_0 \approx 5.2 \times 10^{-10} m$. Then using (2012) with $A \approx 10^{-4} m$ and microwave frequencies for the wave ($\mu_{G(SC)} \approx 3 \times 10^{28}$) gives

$$\frac{P_{out}}{P_{in}} = \left(\frac{256\pi^{5/2} G^{7/2}}{45c^7} \right) m_i \rho r_0^4 \mu_{G(SC)} \approx 5.5 \times 10^{-109} \tag{2016}$$

If we do not require microwave frequencies, then (2013) becomes

$$\mu_{G(SC)} = \left[\left(\frac{45}{256} \right)^2 \frac{c^{14}}{\pi^5 G^7} \left(\frac{\rho^2}{m_i^{14}} \right)^{1/3} \right]^{1/3} \approx 4.5 \times 10^{100} J/m^3 \tag{2017}$$

Then the gravitational plasma frequency, $\omega_G = \sqrt{16\pi G \mu_G}$, and penetration depth, $\delta_G = c/\omega_G$, are

$$\boxed{\begin{array}{c} \omega_G \approx 3 \times 10^{45} s^{-1} \qquad \text{and} \qquad \delta_G \approx 1 \times 10^{53} m \\ \\ \textit{Gravitational plasma frequency and penetration depth for an array} \\ \textit{of quadrupoles with a random distribution of phases} \\ \textit{and separated by a lattice length } \left(r_0^3 \approx 8m_i/\rho\right) \end{array}} \tag{2018}$$

Notice that for an array of quadrupole moments (cases 3 and 4), the result is very sensitive to the spacing between quadrupole moments. For example, for the coherent array, if the spacing is on the order of a wavelength, then the plasma frequency is on the order of $10^{-17} s^{-1}$. However, if the spacing is on the order of a lattice separation, then the plasma frequency is on the order of $10^{19} s^{-1}$. That's a factor of 36 orders of magnitude! The situation is even more extreme with the incoherent array. If the spacing is on the order of a wavelength, then the plasma frequency is on the order of $10^{-67} s^{-1}$. However, if the spacing is on the order of a lattice separation, then the plasma frequency is on the order of $10^{45} s^{-1}$. That's a factor of 112 orders of magnitude!

19.3 Ratio of output EM to input GR wave power (GR to EM transduction)

In the previous section, we found the ratio of output GR wave power to input GR wave power for a "gravitational Lorentz oscillator" which was essentially a time-varying quadrupole consisting of two equal masses oscillating relative to each other. In this section, we will analyze the same system again except now we consider the two equal masses to also be *charged* with equal and opposite charges.[224] The following is a diagram of the system.

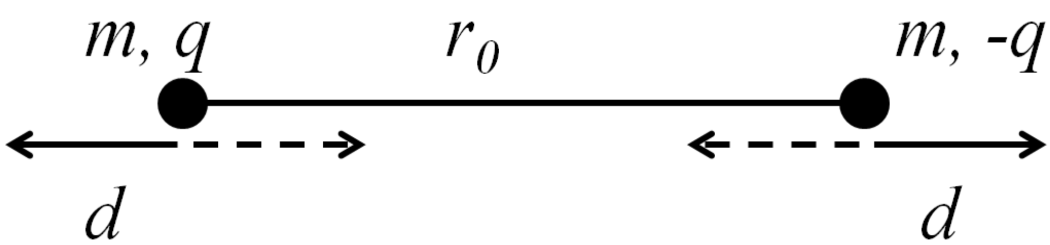

Figure 18: A modified Lorentz oscillator consists of two particles with equal mass m, and opposite charges, q and $-q$. The particles oscillate with an amplitude of motion given by d. They are separated by an average distance $r_0 >> d$. Note that the solid arrows represent the half-cycle when the particles are moving *away* from each other and the dashed arrows represent the half-cycle when the particles are moving *toward* each other.

As in the previous section, we consider the masses to be driven into motion by a gravitational wave. However we will now determine the ratio of outgoing *electromagnetic* (EM) wave power relative to the incoming gravitational (GR) wave power. In this sense, we are effectively determining the efficiency of GR to EM wave transduction by the system. Once again, we begin with the Isaacson power flux formula to relate the gravitational wave power to the strain field of the gravitational wave. It is given in [43] as

$$\mathscr{P} = \frac{c^3}{16\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle \tag{2019}$$

Here $\mathscr{P}$ is the power per unit area $(\mathscr{P} = P/A)$ while h_+ and $h_\times$ are the plus and cross polarization strain fields. The particular polarization is not relevant to our analysis here so we can simply use h_0 for the strain field. The time derivative for an oscillating field can be written as $\dot{h}_0 = \omega h_0$. Then taking the time average gives $\left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle \approx \frac{1}{2}\omega^2 h_0^2$. We can also substitute $\mathscr{P} = P_{in}/A$ for an incoming gravitational wave. This gives

$$P_{in}^{(GR)} = \frac{c^3}{32\pi G} A h_0^2 \omega_{wave}^2 \tag{2020}$$

where ω_{wave} is the angular frequency of the gravitational wave. If the gravitational wave oscillates the charges in the modified Lorentz oscillator, then they will re-radiate electromagnetic radiation. We can consider four possible cases:

1. An array of dipole moments driven in phase.
2. An array of dipole moments with a random phase distribution.
3. An array of quadrupole moments driven in phase.
4. An array of quadrupole moments with a random phase distribution.

[224] This system could be referred to as a "*modified* Lorentz oscillator" since it is similar to the standard Lorentz oscillator which consists of an electron in harmonic motion relative to an atomic nucleus. The difference here is that we consider the two masses of the oscillator to be equal. Hypothetically, this could be considered an "*electron-positron* Lorentz oscillator."

1. An array of dipole moments driven in phase.

If the charges in the Lorentz oscillator have *opposite sign*, then Appendix T shows that the lowest order radiation will be *dipole* radiation. The outgoing dipolar EM radiation can be found using the Larmor power formula[225] given by Griffiths [29] (equ. 11.70) as[226]

$$P_{out}^{(EM)} = \frac{1}{6\pi\varepsilon_0 c^3}\left\langle \ddot{p}^2 \right\rangle \tag{2021}$$

where $\left\langle \ddot{p}^2 \right\rangle$ is a time-average of $\ddot{p}^2$ over a period. The second time-derivative of the dipole was found in (3027) as

$$\ddot{p} = 2q\ddot{r}_0 \tag{2022}$$

If ω_{mass} is the angular frequency describing the sinusoidal oscillation of the masses, then from (2072) or (2073) we find that the acceleration of the particles relative to each other given as $\ddot{r}_0^2 = 4d^2\omega_{mass}^4 \sin^2(\omega_{mass}t)$. Taking a time average and using $\left\langle \sin^2(\omega t) \right\rangle = 1/2$ gives

$$\left\langle \ddot{r}_0^2 \right\rangle = 2d^2\omega_{mass}^4 \tag{2023}$$

Then using this in (2022) gives $\left\langle \ddot{p}^2 \right\rangle = 8q^2d^2\omega_{mass}^4$. We can insert this into (2021) and divide by (2020) to determine the ratio of outgoing EM wave power to incoming GR wave power. This gives

$$\frac{P_{out}^{(EM\ dipole)}}{P_{in}^{(GR)}} = \left(\frac{128G}{3\varepsilon_0 c^6}\right)\frac{q^2d^2\omega_{mass}^4}{Ah_0^2\omega_{wave}^2} \tag{2024}$$

The prefactor in parentheses has a value of $\sim 10^{-49}$ in SI units. This is obviously an extremely small value which would imply that there is no appreciable transduction of GR wave power to dipolar EM wave power. However, we can examine the remaining part of the expression to determine if the ratio of outgoing to incoming power must necessarily be miniscule.

To simplify the calculation, we can consider the frequency of oscillation of the masses to be comparable to the frequency of the gravitational wave so that we can work with a single frequency: $\omega \approx \omega_{mass} \approx \omega_{wave}$. Also, if the charge carriers are either Cooper pairs or lattice ions in a superconductor, then $q = 2e$. Therefore (2024) becomes

$$\frac{P_{out}^{(EM\ dipole)}}{P_{in}^{(GR)}} = \left(\frac{512G}{3\varepsilon_0 c^6}\right)\frac{e^2d^2\omega^2}{Ah_0^2} \tag{2025}$$

[225] Note that for *relativistic* charges, we would need to use Liénard's generalization of the Larmor formula given in Griffiths [29] (equ. 11.73) as $P = \frac{\mu_0 q^2 a^2 \gamma^6}{6\pi c}\left(a^2 - \left|\vec{v}\times\vec{a}\right|^2/c^2\right)$.

[226] The expression in Griffiths is actually $P = \frac{\mu_0 q^2 a^2}{6\pi c}$. In (2021) we have used $\mu_0 = \frac{1}{\varepsilon_0 c^2}$ and written q^2a^2 as $\left\langle \ddot{p}^2 \right\rangle$ for the case of a time-varying acceleration.

From the geodesic deviation equation in (577) or (578), we have $\ddot{r} \approx \frac{1}{2} r_0 h_0 \omega^2$. Also, from (2072) or (2073), we already have $\ddot{r} \approx 2d\omega^2$. Combining these gives[227]

$$d \approx \frac{r_0 h_0}{4} \tag{2026}$$

Inserting this into (2025) gives

$$\frac{P_{out}^{(EM\ dipole)}}{P_{in}^{(GR)}} = \left(\frac{32G}{3\varepsilon_0 c^6}\right) \frac{e^2 r_0^2 \omega^2}{A} \tag{2027}$$

From Griffiths [29] (pp. 454-457), we find that the Larmor power formula in (2021) is derived under the following three conditions: (1) $r_0 << r$ where r_0 is the length scale of the source and r is the distance from the source to the field point; (2) $r_0 << \lambda$ where λ is the wavelength of the EM wave; (3) $1/r^2 << 1/r$ which implies that the power is evaluated in the far-field so that near-fields can be neglected. In our case here, we can consider the dipoles as involving pairs of neighboring atoms so that r_0 is the lattice spacing of Niobium. Using $r_0^3 \approx 8m/\rho$ for the lattice spacing[228] with $m \approx 1.5 \times 10^{-25} kg$ for the mass of a single Niobium atom and $\rho \approx 8.6 \times 10^3 kg$ for the mass density gives $r_0 \approx 5.2 \times 10^{-10} m$. We can also consider microwave radiation ($\omega \approx 10^{10} s^{-1}$) and use an area $A = r^2$ where $r \approx 1cm$ for the walls of a cavity with centimeter dimensions.

$$\boxed{\begin{array}{c} \dfrac{P_{out}^{(EM\ dipole)}}{P_{in}^{(GR)}} = \left(\dfrac{32G}{3\varepsilon_0 c^6}\right) \dfrac{e^2 r_0^2 \omega^2}{A} \approx 7.6 \times 10^{-82} \\ \\ \textit{Transduction efficiency of GR waves} \\ \textit{to dipolar EM waves by a single Lorentz oscillator} \end{array}} \tag{2028}$$

An array of dipole moments that are all *in phase* can be considered as a *coherent* array. Therefore, we can add $\ddot{p}_i$ for all the dipole moments in the array. Summing over an Avogadro number of dipoles gives

$$\ddot{p} = \sum_{i=1}^{N_A} \ddot{p}_i = N_A \ddot{p}_i = N_A q \ddot{d}_i \tag{2029}$$

where N_A is Avogadro's number. Then using the same analysis as above, we find $\langle \ddot{p}^2 \rangle = 8 N_A^2 q^2 d^2 \omega^4$ and therefore (2028) will have an extra factor of N_A^2.

$$\boxed{\begin{array}{c} \dfrac{P_{out}^{(EM\ dipole)}}{P_{in}^{(GR)}} = \left(\dfrac{32G}{3\varepsilon_0 c^6}\right) \dfrac{N_A^2 e^2 r_0^2 \omega^2}{A} \approx 2.8 \times 10^{-34} \\ \\ \textit{Transduction efficiency of GR waves} \\ \textit{to dipolar EM waves by a coherent array of Lorentz oscillators} \end{array}} \tag{2030}$$

We can compare this to (2005) from the previous section to determine how much of the re-radiation will be

[227] Note that rearranging (2026) gives $h_0 \approx 4d/r_0$. This is consistent with the fact that h_0 is a strain field which can be expressed as $h_0 = \Delta L/L$, where ΔL is the change in length per unit length L.

[228] We obtain $r_0^3 \approx m_i/\rho$ by considering that the mass density for a lattice cube (with one atom at each corner and sides of length r_0) is $\rho \approx 8m_i/r_0^3$ since there are 8 corners. Therefore $r_0^3 \approx 8m_i/\rho$.

electromagnetic and how much will be gravitational.[229]

$$\frac{P_{out}^{(EM\ dipole)}}{P_{out}^{(GR)}} = \left(\frac{15}{8\pi^2\varepsilon_0 G^2}\right)\frac{N_A^2 e^2\omega^2}{A^2\rho^2 r_0^2 \mu_{G(SC)}} \approx \frac{2.8\times 10^{-34}}{9.5\times 10^{-86}} \approx 2.9\times 10^{51} \tag{2031}$$

This implies that the electromagnetic re-radiation is 51 orders of magnitude greater than the gravitational re-radiation from an array of dipole oscillators that are in phase.

2. An array of dipoles moments with a random phase distribution.

Now we consider the case of dipole moments which have a random distribution of phases. This can be considered as an *incoherent* array and therefore we cannot add $\ddot{p}$ for each dipole. Instead, we must consider the power produced by each dipole individually, P_i, and then sum them. Using (2021) to find the power of each dipole moment and then summing over an Avogadro number of them gives

$$P_{out}^{(EM\ dipole)} = \sum_{i=1}^{N_A} P_i = \sum_{i=1}^{N_A} \frac{1}{6\pi\varepsilon_0 c^3}\ddot{p}_i^2 = \frac{1}{6\pi\varepsilon_0 c^3} N_A \ddot{p}_i^2 \tag{2032}$$

This will lead to a result that is identical to (2028) except for a factor of N_A. Once again, we can use $r_0 \approx 5.2\times 10^{-10} m$ as the lattice spacing of Niobium, $q = 2e$ for each charge in the dipole, $A \approx 10^{-4} m$ for the walls of a cavity with centimeter dimensions, and $\omega \approx 10^{10} s^{-1}$ for microwave radiation. Therefore, we have

$$\boxed{\frac{P_{out}^{(EM\ dipole)}}{P_{in}^{(GR)}} = \left(\frac{8G}{3\varepsilon_0 c^6}\right)\frac{N_A e^2 r_0^2\omega^2}{A} \approx 4.6\times 10^{-58}} \tag{2033}$$

Transduction efficiency of GR waves to dipolar EM waves by an incoherent array of Lorentz oscillators with a random distribution of phases

We can compare this result to (2016) to determine how much of the re-radiation will be electromagnetic and how much will be gravitational.

$$\frac{P_{out}^{(EM\ dipole)}}{P_{out}^{(GR)}} = \frac{15c}{32\varepsilon_0 \pi^{5/2} G^{5/2}}\frac{N_A e^2 r_0^2\omega^2}{A^2 m_i \rho r_0^4 \mu_{G(SC)}} \approx \frac{4.6\times 10^{-58}}{5.5\times 10^{-109}} \approx 8.4\times 10^{50} \tag{2034}$$

Similar to (2031), we find that the EM re-radiation is 50 orders of magnitude greater than the GR re-radiation from an array of dipole oscillators with a random distribution of phases.

3. An array of quadrupole moments driven in phase.

If the charges in the Lorentz oscillator have the *same sign*, then Appendix T shows that the lowest order radiation will be *quadrupole* radiation. The outgoing EM radiation can be found using the quadrupole power

[229] Note that in the previous section, we considered four cases. However, two cases involved $A \approx r_0^2$ which is not permitted in the approximation required to obtain the Larmor power formula. Therefore, we are only comparing the results here to cases 3 and 4 in the previous section.

formula found in Jackson, [40] (equ. 9.49) as

$$P_{out}^{(EM\ quad)} = \frac{1}{1440\pi\varepsilon_0 c^5}\left\langle \dddot{Q}_{ij}^2 \right\rangle \tag{2035}$$

where $\left\langle \dddot{Q}_{ij}^2 \right\rangle$ is a time-average of $\dddot{Q}_{ij}^2$ over a period. The third time-derivative of the quadrupole moment was found in (3027) as

$$\dddot{Q} = 2q\left(3\dot{\vec{r}}\ddot{\vec{r}} + \vec{r}\cdot\dddot{\vec{r}}\right) \tag{2036}$$

If the amplitude of the oscillation is small relative to the distance between the oscillators,[230] then $r_0 >> A$. In that case, we can use the same process that led to (2200) and we would similarly have

$$\left\langle \dddot{Q}^2 \right\rangle = 8q^2 d^2 {r_0}^2 \omega_{mass}^6 \tag{2037}$$

We can insert (2037) into (2035) and divide by (2020) to determine the ratio of outgoing EM wave power to incoming GR wave power. This gives

$$\frac{P_{out}^{(EM\ quad)}}{P_{in}^{(GR)}} = \left(\frac{8G}{45\varepsilon_0 c^8}\right)\frac{q^2 d^2 {r_0}^2 \omega_{mass}^6}{A h_0^2 \omega_{wave}^2} \tag{2038}$$

The prefactor in parentheses has a value of $\sim 10^{-68}$ in SI units. This is even smaller than the case for dipolar EM radiation. Once again, such a small value would imply that there is no appreciable transduction of GR wave power to quadrupolar EM wave power. However, we can examine the remaining part of the expression to determine if the ratio of outgoing to incoming power must necessarily be miniscule. To simplify the calculation, once again we can consider the frequency of oscillation of the masses to be comparable to the frequency of the gravitational wave so that we can work with a single frequency: $\omega \approx \omega_{mass} \approx \omega_{wave}$. Also, if the charge carriers are either Cooper pairs or lattice ions in a superconductor, then $q = 2e$. We can also use $d \approx r_0 h_0/4$ from (2026). Therefore (2038) becomes

$$\frac{P_{out}^{(EM\ quad)}}{P_{in}^{(GR)}} = \left(\frac{2G}{45\varepsilon_0 c^8}\right)\frac{e^2 {r_0}^4 \omega^4}{A} \tag{2039}$$

Once again, we can use $r_0 \approx 5.2 \times 10^{-10} m$ as the lattice spacing of Niobium, $A \approx 10^{-4} m$ for the walls of a cavity with centimeter dimensions, and $\omega \approx 10^{10} s^{-1}$ for microwave radiation. Then we have

$$\boxed{\frac{P_{out}^{(EM\ quad)}}{P_{in}^{(GR)}} = \left(\frac{2G}{45\varepsilon_0 c^8}\right)\frac{e^2 {r_0}^4 \omega^4}{A} \approx 9.6 \times 10^{-100}} \tag{2040}$$

Transduction efficiency of GR waves
to quadrupolar EM waves by a single Lorentz oscillator

Once again, an array of quadrupole moments that are all *in phase* can be considered as a *coherent* array. Therefore, we can add $\dddot{Q}_i$ for all the quadrupole moments in the array. Summing over an Avogadro number

[230] This is equivalent to requiring a *linear response* between fields and sources. In other words, since the power is proportional to the square of the field $\left(\mathscr{P}_{GR} \sim \vec{h}^2 \text{ and } \mathscr{P}_{EM} \sim \vec{E}^2\right)$ and the fields are proportional to the amplitude of motion ($d \sim h_0$ and $d \sim E$ for GR and EM waves, respectively) then we should also have the power proportional to the square of the amplitude of motion $\left(\mathscr{P}_{GR} \sim d^2 \text{ and } \mathscr{P}_{EM} \sim d^2\right)$.

of quadrupoles gives

$$\dddot{Q} \quad = \quad \sum_{i=1}^{N_A} \dddot{Q}_i = N_A \dddot{Q}_i \tag{2041}$$

Using the same analysis as above, we find that (2037) becomes $\left\langle \dddot{Q}^2 \right\rangle = 8N_A^2 q^2 d^2 {r_0}^2 \omega^6$ and therefore (2040) will have an extra factor of N_A^2.

$$\boxed{\begin{array}{c} \dfrac{P_{out}^{(EM\ quad)}}{P_{in}^{(GR)}} = \left(\dfrac{2G}{45\varepsilon_0 c^8}\right) \dfrac{N_A^2 e^2 {r_0}^4 \omega^4}{A} \approx 3.5 \times 10^{-52} \\ \\ \textit{Transduction efficiency of GR waves to} \\ \textit{quadrupolar EM waves by a coherent array of Lorentz oscillators} \end{array}} \tag{2042}$$

We can compare this result to (2005) from the previous section to determine how much of the re-radiation will be electromagnetic and how much will be gravitational.[231]

$$\frac{P_{out}^{(EM\ quad)}}{P_{out}^{(GR)}} \quad = \quad \left(\frac{1}{32\pi^2 \varepsilon_0 c^2 G^2}\right) \frac{N_A^2 e^2 \omega^4}{A^2 \rho^2 \mu_{G(SC)}} \approx \frac{3.5 \times 10^{-52}}{9.5 \times 10^{-86}} \approx 3.7 \times 10^{33} \tag{2043}$$

This implies that the EM re-radiation is 33 orders of magnitude greater than the GR re-radiation from an array of quadrupole oscillators that are in phase.

<u>4. An array of quadrupole moments with a random phase distribution.</u>

Now we consider the case of quadrupole moments which have a random distribution of phases. This can be considered as an *incoherent* array and therefore we cannot add $\dddot{Q}_i$ for each quadrupole. Instead, we must consider the power produced by each quadrupole individually, P_i, and then sum them. Using (2035) to find the power of each dipole moment and then summing over an Avogadro number of them gives

$$P_{out}^{(EM\ quad)} \quad = \quad \sum_{i=1}^{N_A} P_i = \sum_{i=1}^{N_A} \frac{1}{1440\pi\varepsilon_0 c^5} \dddot{Q}_i^2 = \frac{1}{1440\pi\varepsilon_0 c^5} N_A \dddot{Q}_i^2 \tag{2044}$$

This will lead to a result that is identical to (2040) except for a factor of N_A. Once again, we can use $r_0 \approx 5.2 \times 10^{-10} m$ as the lattice spacing of Niobium, $q = 2e$ for each charge in the dipole, $A \approx 10^{-4} m$ for the walls of a cavity with centimeter dimensions, and $\omega \approx 10^{10} s^{-1}$ for microwave radiation. Therefore, we have

$$\boxed{\begin{array}{c} \dfrac{P_{out}^{(EM\ quad)}}{P_{in}^{(GR)}} = \left(\dfrac{2G}{45\varepsilon_0 c^8}\right) \dfrac{N_A e^2 {r_0}^4 \omega^4}{A} \approx 5.8 \times 10^{-75} \\ \\ \textit{Transduction efficiency of GR waves} \\ \textit{to quadrupolar EM waves by an incoherent array} \\ \textit{of Lorentz oscillators with a random distribution of phases} \end{array}} \tag{2045}$$

[231] In the previous section, we considered four cases. However, two cases involved $A \approx r_0^2$. Here we have specifically required that $A >> r_0^2$ for a linear response approximation. Therefore, we are only comparing the results here to cases 3 and 4 in the previous section.

We can compare this to (2016) to determine how much of the re-radiation will be electromagnetic and how much will be gravitational.

$$\frac{P_{out}^{(EM\ quad)}}{P_{out}^{(GR)}} = \left(\frac{1}{32\pi^2\varepsilon_0 c^2 G^2}\right)\frac{N_A e^2\omega^4}{A^2\rho^2\mu_{G(SC)}} \approx \frac{5.8\times10^{-75}}{5.5\times10^{-109}} \approx 1.1\times10^{34} \tag{2046}$$

Similar to (2031), we find that the EM re-radiation is 34 orders of magnitude greater than the GR re-radiation from an array of quadrupole oscillators with a random distribution of phases. As a final observation, we can see from (2046) that in order for the outgoing GR wave power to be comparable to the outgoing EM wave power, we can set $P_{out}^{(EM\ quad)}/P_{(out)}^{GR} \approx 1$. The only free parameters in the expression are ω, A, and $\mu_{G(SC)}$. For microwave frequencies ($\omega \approx 10^{10}s^{-1}$) and centimeter dimensions ($A \approx 10^{-4}m$), we can solve for $\mu_{G(SC)}$ to obtain

$$\mu_{G(SC)} \approx \left(\frac{N_A e^2}{32\pi^2\varepsilon_0 c^2 G^2}\right)\frac{\omega^4}{A^2\rho^2} \approx 1.4\times10^{36} J/m^3 \tag{2047}$$

Using (1321), this would correspond to a gravitational plasma frequency given by

$$\omega_G = \sqrt{16\pi G\mu_G} \approx 6.7\times10^{13} Hz \tag{2048}$$

Therefore, we find that a material with a value for $\mu_{G(SC)}$ that allows GR waves to be scattered with the same efficiency as EM waves would correspond to a gravitational plasma frequency on the order of terahertz. This means that an incident GR wave at gigahertz frequencies (microwaves) could produce comparable scattered EM and GR waves. However, even in such a situation, both the EM and GR waves will still be 52 orders of magnitude weaker than the incident GR wave as shown by (2042) for a *coherent* array of quadrupoles, and 75 orders of magnitude weaker than the incident GR wave as shown by (2042) for an *incoherent* array of quadrupoles.

20 Models for coupling gravitation to quantum matter

20.1 Overview of models of coupling gravitation to quantum matter

Here we consider the detection and generation of gravitational waves via low-temperature quantum-coherent matter. Equating the Compton wavelength and Schwarzschild radius of a particle can be used as a model for describing matter that is completely quantum coherent while also exhibiting extremely high gravitation. We discuss Planck quantities (length, time, energy, and power scales) which give an upper bound for highly gravitational, quantum coherent matter. Because the Planck scale is experimentally inaccessible, we consider how low-temperature quantum states may allow an extended object to continue exhibiting quantum coherence even at mesoscopic length scales. In particular, we examine the possibilities of Bose-Einstein condensates, Fermi-pair condensates, superfluids and superconductors as candidates for low-temperature quantum coherent matter that can be coupled to gravitational waves.

Gravitational waves are predicted by General Relativity to exist, however, because the coupling strength to matter is so weak, detecting gravitational waves is extremely difficult.[22][57] Therefore, it has long been believed that generating any detectable gravitational waves in the laboratory is effectively impossible.[11] In fact, only astrophysical sources have been considered to have enough mass moving at high enough velocities to generate detectable gravitational waves.[90] As a result, the focus has been to build extremely large interferometers (such as LIGO) in the attempt to detect gravitational waves.

We propose an alternative approach which utilizes the quantum characteristics of low-temperature matter to effectively couple gravitational waves to quantum-coherent material.[91] Doing so would ultimately provide a way of performing extremely delicate quantum mechanical experiments for the purpose of detecting gravitational waves. In addition to such detection, one could also in principle *generate* gravitational waves.[4] The ability to both detect and generate gravitational waves would completely revolutionize various areas of physics and technology, in particular, the fields of communication, astronomy, and cosmology.

20.2 The Compton wavelength for relativistic quantum particles

The total relativistic energy of a particle is given by

$$E^2 = m_0^2c^4 + p^2c^2 \tag{2049}$$

where the particle's rest energy is $E_0 = mc^2$, and the kinetic energy is $KE = pc$. A particle becomes highly relativistic when the kinetic energy approaches the rest mass energy, $KE \approx E_0$, so $pc \approx m_0c^2$. This means that[232]

$$p \approx m_0c \tag{2050}$$

Substituting this into the Heisenberg Uncertainty Principle, $\Delta x \Delta p \geq \hbar/2$, and solving for Δx gives

$$\Delta x \geq \frac{\hbar}{2m_0c} \tag{2051}$$

Aside from a factor of $\frac{1}{2}$, it is evident that Δx is the reduced Compton wavelength of a particle. [92]

$$\lambda_C = \frac{\hbar}{m_0c} \tag{2052}$$

This sets the ideal length-scale for which a particle will exhibit quantum mechanical, wave-like behavior. Objects occupying length-scales significantly *larger* than λ_C will not typically exhibit quantum mechanical behavior due quantum decoherence (in accordance with the correspondence principle). On the other hand, length-scales significantly *smaller* than λ_C will have a larger momentum according to the de Broglie wavelength, $\lambda = \dfrac{h}{p}$. This means that the momentum will be greater than $p \approx mc$ in (2050) and thus the kinetic energy will exceed the rest energy, $KE >> E_0$.

Therefore, if a particle reaches speeds this high, then it must be properly treated by the use of *relativistic* quantum mechanics, or more importantly, quantum field theory (QFT). An important prediction of QFT is that particles with such high kinetic energy can spontaneously "decay" into other particles which emerge from the vacuum. This occurs when the large kinetic energy of the original relativistic particle is "deposited" into the vacuum and effectively transformed into the rest mass energy of the new particles. Therefore, although quantum mechanics does not forbid a particle from occupying a length-scale that is smaller than the Compton wavelength, in order to maintain quantum coherent matter, it is necessary that particles would *not* occupy a length scale below their Compton wavelength. Otherwise their associated kinetic energy will be so high as to cause pair-creation events which would disrupt the existence of the quantum-coherent state.

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[232] Note that the relativistic momentum of a particle is $p = \gamma m_0 v$ where $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$. Therefore, if $p = m_0c$, as assumed in (2050), then $\gamma v = c$. Using the definition for γ and solving for v gives $v = \frac{c}{\sqrt{2}} \approx .71c$. This means that the particle is going over 71% of the speed of light which is indeed highly relativistic.

20.3 Planck quantities at the interface of gravity and quantum mechanics

It can also be said that the Schwarzschild radius [11]

$$R_s = \frac{2mG}{c^2} \tag{2053}$$

sets the length scale for which the particle will exhibit strong gravitational fields. Setting these two lengths equal ($\lambda = R_s = l$) turns (2052) and (2053) into $l = \dfrac{2\pi\hbar}{mc}$ and $l = \dfrac{2mG}{c^2}$, respectively. If we use these equations to solve for the mass m (ignoring a factor of $\sqrt{\pi}$), then we obtain the *Planck mass*.[93]

$$m_p = \sqrt{\frac{\hbar c}{G}} \qquad \textit{Planck mass} \tag{2054}$$

The numeric value (in SI units) obtained for this mass is $\sim$2.2x10^{-8}kg. However, having an object with this mass is not sufficient to determine if it will be highly gravitational, quantum coherent matter. In this analysis, there was the critical requirement that this mass is localized to a length scale determined by the Compton wavelength and Schwarzschild radius which were set equal. Substituting the Planck mass into (2052) (again ignoring a factor of $\sqrt{\pi}$) gives the *Planck length.*

$$l_p = \sqrt{\frac{G\hbar}{c^3}} \qquad \textit{Planck length} \tag{2055}$$

The numeric value (in SI units) obtained for this length is $\sim$1.6x10^{-35}m. This length is considered to be the smallest physical length possible before the very concept of space and time break down. At this scale, the quantum fluctuations of the vacuum (as a result of the Heisenberg Uncertainty principle) are believed to produce a turbulent "quantum space-time foam" (as John Wheeler described it). In addition, a particle possessing a Planck mass contained within a region as small as a Planck length (or, more accurately, as small as a *Planck volume*, $V_p \approx l_p^3 \approx 4.2x10^{-105}m^3$) would generate such a massive gravitational field in such an infinitesimal region of space that there would be no way to understand such a system without a fully developed theory of quantum gravity.

It is interesting to note that the value of the Planck mass ($\sim$2.2x10^{-8}kg) would give the impression that this highly-gravitational quantum matter is within the realm of experimental possibilities. However, because this Planck mass must be considered in conjunction with the Planck length (or volume) it occupies, it is typically considered well beyond the scope of any modern day experimentation. However, as will be shown later, this assumption is not necessarily true when low-temperature, quantum coherent matter is considered.

It is also helpful to note the *Planck time* which corresponds to this Planck scale. Since a photon in vacuum will travel at the maximum speed of light, then to travel a distance of one Planck length would require a time interval given by

$$t_p = \frac{l_p}{c} = \sqrt{\frac{G\hbar}{c^3}}\frac{1}{c} = \sqrt{\frac{G\hbar}{c^5}} \qquad \textit{Planck time} \tag{2056}$$

The numeric value (in SI units) of the Planck time is approximately $\sim$5.4x10^{-44}s. This is considered to be the smallest "quantum of time." Since the Planck length is considered the shortest conceivable distance, and a photon in vacuum is the fastest conceivable speed, then this time duration would be the shortest possible time duration that can exist. We now have the three fundamental Planck quantities that can be used to determine all other Planck quantities as well. The *Planck energy* scale is derived from the Planck mass as

$$E_p = m_p c^2 = \sqrt{\frac{\hbar c}{G}}c^2 = \sqrt{\frac{\hbar c^5}{G}} \qquad \textit{Planck energy} \tag{2057}$$

This energy has a value of approximately $\sim$2.0x10^{9}J or $\sim$1.2x10^{28}eV. This is far above the maximum energy that has been achieved in high energy physics which is $\sim$6x10^{12} eV. This further illustrates the fact that probing the Planck scale is far from technologically achievable at this time.

Finally, we can consider a *Planck power* using the Planck quantities determined above. In general, power is $P = dE/dt$, therefore we can describe the Planck power as a Planck energy per Planck time.

$$P_p = \frac{E_p}{t_p} = \frac{\sqrt{\hbar c^5/G}}{\sqrt{G\hbar/c^5}} = \frac{c^5}{G} \qquad \textit{Planck power} \tag{2058}$$

This power has a value of $\sim$3.6x10^{52}W. This is a tremendously large power which would be found only in an astrophysical context, not in a laboratory environment. It is interesting that the Planck power only involves c and G, constants from General Relativity, and not $\hbar$ from Quantum Mechanics. In that sense, the Planck power is not necessarily associated with Quantum Mechanics but rather with General Relativity (incorporated here via the Schwarzschild radius). To find the Planck power directly from General Relativity, we can consider the Einstein field equations given by [11]

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{2059}$$

Here the coupling constant, $\kappa = \dfrac{8\pi G}{c^4}$, couples the curvature of space-time, $G_{\mu\nu}$, to the stress-energy-momentum tensor, $T_{\mu\nu}$. If we define $f = 1/\kappa$, then we obtain the "Einstein force," $f_{\mathrm{E}} = \dfrac{c^4}{8\pi G}$ which is $\sim$4.8x10^{42}N. We can find a power associated with this force if we consider the force acting on a mass moving nearly the speed of light. Then the "Einstein power" is $P_{\mathrm{E}} = f_{\mathrm{E}} \cdot c$.

$$P_{\mathrm{E}} = \frac{c^5}{8\pi G} \qquad \textit{Einstein power} \tag{2060}$$

This is again the same as the Planck power (neglecting a factor of 8π). The value of P_{E} is $\sim$1.4x10^{51}W which is obviously comparable to the Planck power found above.[233] It is remarkable that this amount of power would be associated with a mass that is only a Planck mass ($\sim$2.2x10^{-8}kg). The reason for this tremendous power is due to the fact that we have considered the Planck mass as occupying only a Planck length ($\sim$1.6x10^{-35}m). This extreme compactification results in an extreme mass density, namely, the *Planck mass density.*

$$\rho_p = \frac{m_p}{V_p} = \frac{m_p}{l_p^3} = \frac{\sqrt{\hbar c/G}}{\left(\sqrt{G\hbar/c^3}\right)^3} = \frac{c^5}{G^2\hbar} \qquad \textit{Planck mass density} \tag{2061}$$

The value of ρ_p is $\sim$5.2x10^{98}kg/m^3, once again an enormous value. Since the Planck length is far too small to be probed directly, and the Planck density is far too large to achieve experimentally, we have no choice but to consider a much larger length scale (with a correspondingly lower mass density). However, it should be noted that in order to detect and possibly generate gravitational waves, it may *not* be necessary to have such an extremely small length scale with such a large mass density. The Planck mass and Planck length essentially give the *upper limit* where the effects of Quantum Mechanics and General Relativity would both be tremendous. This is the most extreme case of a "Planck particle" that is compressed to nearly a black hole, occupying the smallest conceivable volume of space, and yet remains within it's own Compton wavelength as it moves at nearly the speed of light. We can obviously consider a far less extreme state of matter that would still exhibit quantum mechanical properties and also be coupled to gravity with sufficient strength that gravitational waves could be detected and possibly generated.

[233] Due to the fact that the Planck power (the maximum power in nature) is the prefactor found in the equation for General Relativity (GR), it has been said that perhaps GR could be considered a theory of *maximum power* just as Special Relativity is a theory of *maximum speed* and Quantum Mechanics may be considered a theory of *minimum action*.

If we do not require the Compton wavelength to be equal to the Schwarzschild radius, then we have some freedom to consider different mass values and determine the quantum behavior and gravitational coupling such matter would still have. Notice that the Compton wavelength relation motivates us to use the *smallest mass* possible to raise the wavelength value up from the Planck scale. On the other hand, gravitation motivates us to use the *largest mass* possible to produce the greatest mass density and hence the greatest gravitational effects. Therefore, there is a tension concerning the choice of mass scale.

There is *not* such a tension concerning the choice of length scale. Quantum mechanics motivates us to use the *smallest length* possible in order to main quantum coherence. With a larger length scale, the object will no longer occupy the space of only a single compton wavelength and therefore will have less quantum coherence. The quantum properties of the object will diminish according to the correspondence principle. In fact, larger objects must necessarily be composite objects consisting of many constitutive particles. The more particles present, the less quantum coherence is likely. Likewise, gravitation also motivates us to use the *smallest length* possible to produce the greatest mass density and hence the greatest gravitational effects. We can summarize these observations with the following four possibilities:

1. Small mass & large length $\Longrightarrow$ Low gravitation and low quantum coherence (common objects)

1. Small mass & small length $\Longrightarrow$ Low gravitation but high quantum coherence (subatomic particles)

2. Large mass & large length $\Longrightarrow$ High gravitation but no quantum coherence (black holes, neutron stars)

3. Large mass & small length $\Longrightarrow$ High gravitation and high quantum coherence (“Planck particles” which are ideal for quantum coherent, highly gravitational matter)

The following graph shows length versus mass for a “gravity curve” (the Schwarzschild radius) and a “quantum curve” (the Compton wavelength). The four possibilities described above (for different mass and length scales) is illustrated.

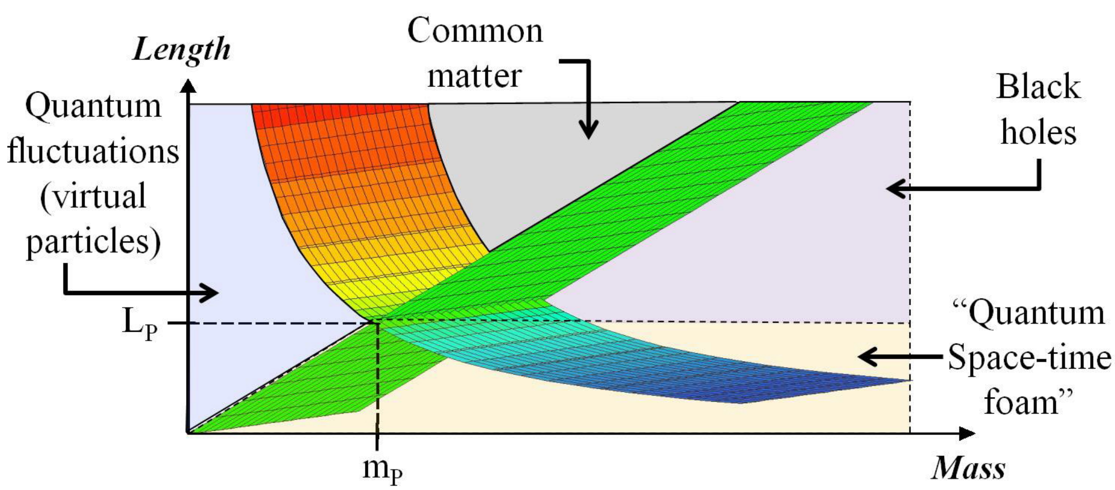

Figure 19: The “quantum curve” (multi-colored) is the Compton wavelength relationship (2052) while the “gravity curve” (green) is the Schwarzchild radius relationship (2053). The graphs are in natural units: $c = G = \hbar = 1$.

Notice that the closer the curves are to each other in the figure above, the closer the system is to an ideal “quantum gravitational system.” Therefore, the curves intersect at the Planck mass and Planck length. The further these curves are apart, the more the system exhibits mainly quantum mechanical characteristics or mainly gravitational characteristics.

20.4 Coupling ultra-cold quantum systems to gravitation

For the purpose of our analysis, we can fix the mass quantity to a single Planck mass since this is an experimentally realistic, mesoscopic value ($\sim$2.2x10^{-8}kg). However, because the Planck length is not an experimentally tenable value ($\sim$1.6x10^{-35}m), we must consider how to increase the length scale while still preserving the quantum coherence of the system. This leads to the consideration of low-temperature, quantum coherent states. In these states, there is the theoretical possibility of having an extended, mesoscopic object which maintains quantum coherence. In that case, the restriction posed by the Compton wavelength for a single quantum particle is no longer a limitation.

One example of achieving such a state is through ultra-cold states of matter which behave quantum mechanically. These include Bose-Einstein condensates and Fermi-pair condensates leading to phenomena such as superconductors and super-fluids. Such materials have the unique characteristic of behaving quantum mechanically even while possessing a size and mass significantly larger than single quantum particles. In fact, these objects can consist of a large ensemble of particles which are in an identical quantum state thereby forming an extended quantum coherent object. Such materials offer the possibility of exhibiting detectable gravitational coupling to quantum coherent matter.

A Bose-Einstein condensate (BEC) occurs when an ensemble of bosons is brought to a temperature low enough that the chemical potential becomes effectively zero and a large number of particles fall together to the ground state. Since these particles have an identical quantum state, they behave as a single quantum particle with a single de Broglie wavelength. In this way, an extended object, composed of multiple particles, can behave quantum mechanically as though it is a single quantum particle.

A Fermi-pair condensate can be considered as a more sophisticated form of a BEC, where fermion pairs are coupled in such a way as to behave as bosons. As $T \to 0$, there is a critical temperature at which a type of *spontaneous symmetry breaking* occurs and particles suddenly drop to the lowest energy state of the system $(\varepsilon = 0)$. The occupation of this state is effectively a phase transition which can consist of a macroscopically large population of particles. When this occurs, the sample is said to form a Bose-Einstein condensate (BEC).[234][98]

Since the BEC allows an extended, macroscopic ensemble to behave as effectively a single quantum particle, then experiments could be conducted without concern for classical particle interactions. Instead the entire ensemble can be treated as a single wave. In the case of superconductivity, it is pairs of electrons, referred to as Cooper pairs, that behave as bosons.[94] In the case of superfluidity, it is pairs of atomic nuclei that are coupled together and behave as bosons.[95] A detailed treatment of such topics requires more advanced theories, such as BCS theory for superconductivity.[96][97] However, as a simple model, consider the relativistic energy of the condensate taken as a single Planck mass object.

$$E^2 = m^2c^4 + p^2c^2 \tag{2062}$$

The momentum is related to the kinetic energy by $p^2 = 2mK$. We can write the kinetic energy as a thermal energy, $K \approx k_BT$ (neglecting prefactors) so combining these relations gives $p^2 \approx mk_BT$. Substituting this into the relativistic energy in (2062) yields

$$E^2 = m^2c^4 + (mk_BT)\,c^2 \tag{2063}$$

For a quantum particle, the energy is also $E = \hbar\omega = h\dfrac{c}{\lambda}$. Substituting this into (2063) and solving for λ gives

$$\lambda = \frac{h}{\sqrt{m^2c^2 + mk_BT}} \tag{2064}$$

[234] It is interesting to note that this mechanism can be likened to the Higgs mechanism in high-energy particle physics which effectively gives mass to particles in the Standard Model. Although the Higgs mechanism is purely quantum mechanical, it gives rise to inertial mass which (according to the Equivalence Principle) is equivalent to gravitational mass which of course is the source of gravitation.

Thus we see that the length scale for quantum coherence is modified according to the temperature. Note that as $T \to 0$ in (2064) we recover the Compton wavelength, $\lambda_c = \frac{h}{mc}$. For very large T, we would have $m_p k_B T >> m_p^2 c^2$ which means $T >> \frac{m_p c^2}{k_B}$. In that case the thermal energy would be much greater than the rest mass energy giving the thermal de Broglie wavelength.[235]

$$\lambda_{dB} = \frac{h}{\sqrt{m k_B T}} \tag{2065}$$

The original graph in the previous figure would then be modified to give the graph below.[236]

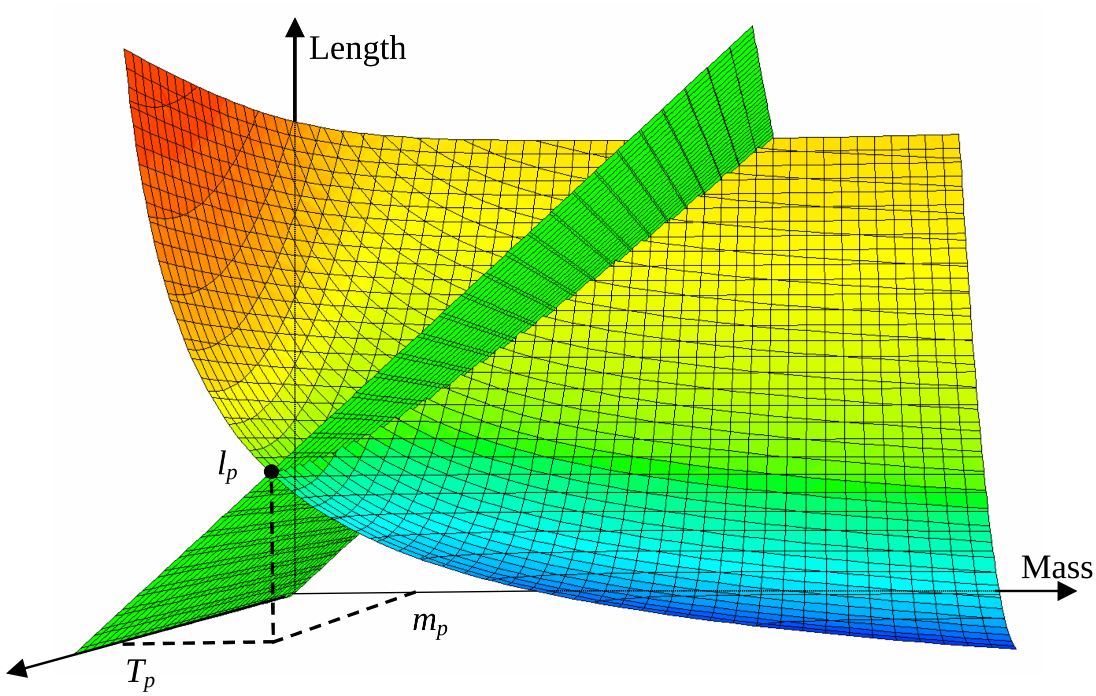

Figure 20: The "quantum curve" (multi-colored) is the modified wavelength (eq.2064) while the "gravity curve" (green) is the Schwarzchild radius relationship (eq.2053). The graphs are in Planck units, $c = G = \hbar = k_B = 1$.

Notice that introducing the temperature in this figure has the effect of "bending" the "quantum surface" so that it now intersects with the "gravity surface" for a variety of mass and length values. Notice in particular that there is now an intersection of the curves for mass values and length values greater than the Planck mass and Planck length. Therefore, it should be experimentally possible to examine systems that exhibit both quantum coherence and strong gravitational coupling provided the system can be brought to a low enough temperature to produce a BEC.

[235] For a mass as small as the Planck mass, $m_p = \sqrt{\frac{\hbar c}{G}}$, a thermal de Broglie wavelength would require an enormously large temperature, namely the *Planck temperature*: $T_p = \frac{m_p c^2}{k_B} = \sqrt{\frac{\hbar c^5}{G k_B^2}} \approx 1.4x10^{32} K$.

[236] Note that the curves are shown in the graph to extend below the Planck length. This is done only to allow the shape of the curves to be viewed. However, it is generally believed that there is no meaningful spatial length shorter than the Planck length.

When two such BEC ensembles interact, then instead of particle collisions, there will be wave interference as shown in the following figure. Although achieving a BEC as been very difficult (requiring temperatures $\sim$1.7nK), current research has shown that a BEC could be formed at higher temperatures, even as high as room temperature.[99]

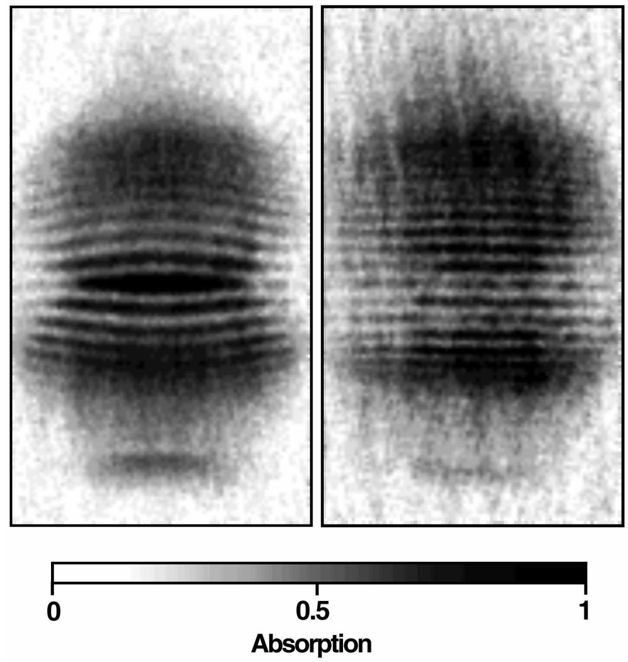

Figure 21: Wave interference between two BEC clouds has been observed. The diagram above can be found at http://cua.mit.edu/ketterle_group/Projects_1997/Interference/Interference_BEC.htm

Another benefit of such low-temperature systems is the fact that all internal degrees of freedom are essentially "frozen out" so that only the center of mass of the entire quantum system can respond to an external impulse. Since it is impossible for a crystal lattice to respond to a fraction of a phonon, the entire macroscopic system must recoil as a single unit. (For example, this is demonstrated by the Mossbauer effect[100] when an entire macroscopic mass can recoil due to the emission of a gamma ray by a single nucleus.) Such quantum coherence in a macroscopic object could be instrumental in detecting and generating gravitational waves.

From this examination, we find that it is possible in principle to have matter in a quantum coherent state while also exhibiting large gravitational coupling. The ideal case was shown to be the "Planck particle" which would maintain quantum coherence (being confined to a space the size of its Compton wavelength) and also exhibit extremely strong gravitation (being nearly a black hole). However, such particles are shown to be experimentally untenable due to the extremely small length scales and extremely large mass densities required. Instead, we considered the possibility of using ultra low-temperature material which can exist at larger length scales without losing quantum coherence. Such materials hold the promise of strongly coupling to gravity while still maintaining quantum coherence. There only remains the challenge of maintaining the system in such a low temperature that quantum coherence is preserved. It is left to experimental innovation to demonstrate if this can be realistically accomplished so that gravitational waves can be detected and generated via such a mechanism.

21 Levitated charged spheres at the foci of a superconducting ellipsoid

21.1 Description of the model

In this section we consider a pair of Planck-mass SC spheres levitated by means of microwaves focused at the two foci F_1 and F_2 of an ellipsoidal microwave SC cavity. The two spheres are charged to "criticality", i.e., so that the Newtonian force of attraction is balanced against the Coulomb force of repulsion between them.[237] In this model, the ellipsoid is a superconducting cavity with a transverse-magnetic (TM) standing

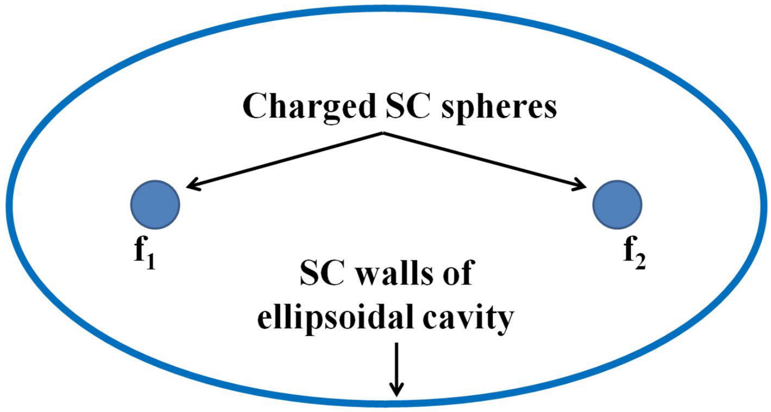

Figure 22: A pair of Planck-mass SC spheres is levitated by means of microwaves focused at the two foci F_1 and F_2 of an ellipsoidal microwave SC cavity. The two spheres are charged to "criticality", i.e., so that the Newtonian force of attraction is balanced against the Coulomb force of repulsion between them.

wave mode within. This standing wave oscillates each charged SC sphere about its corresponding focus of the ellipsoid with a motion that is along the line between the spheres. It is expected that this motion will cause each sphere to generate both electromagnetic (EM) and gravitational (GR) radiation which are necessarily quadrupolar to lowest order. This is understood plainly from the symmetric geometry of the system. A formal proof is also provided in Appendix T. The two identical superconducting (SC) Planck mass spheres are negatively charged such that the Coulomb electrostatic force and Newtonian gravitational force are equal in magnitude and opposite in direction, $\vec{F}_{Coulomb} = -\vec{F}_{Newton}$. This means that

$$\frac{1}{4\pi\varepsilon_0}\frac{|q|^2}{r^2} = G\frac{m^2}{r^2} \tag{2066}$$

Solving this for the charge-to-mass ratio gives

$$\boxed{\left|\frac{q}{m}\right| = \sqrt{4\pi\varepsilon_0 G}} \tag{2067}$$

This condition is referred to as the "criticality" condition. Since each mass is a *Planck* mass, then

$$m_p = \sqrt{\frac{\hbar c}{G}} \tag{2068}$$

Substituting this mass into (2067) gives the charge on each sphere.

$$|q| = G\sqrt{\frac{4\pi\varepsilon_0}{\hbar c}} \tag{2069}$$

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[237] This ratio is discussed in [91], pp. 3-7. Specifically, when $F_N = F_C$, then the resulting "criticality" charge-to-mass ratio is $\dfrac{q}{m} = \sqrt{4\pi\varepsilon_0 G}$. This leads to an equality of quadrupolar EM and GR radiation.

21.2 Motion of the charged spheres due to the standing TM wave

For the purpose of this analysis, we consider the charged spheres to be effectively point charges: q_1 at focus F_1 and q_2 at focus F_2. The distance between the foci is r_0. If the wavelength of the standing TM wave is adjusted such that

$$r_0 = \left(n+\tfrac{1}{2}\right)\lambda, \qquad n = 0,1,2,3,... \tag{2070}$$

then the electric field will point in opposite directions at the two foci. The charges will therefore accelerate sinusoidally out of phase along the line between them so that the motion of the two charges is anti-symmetric.[238] The relative displacement between the charges can be given by[239]

$$r = r_0 + 2A\sin(\omega t) \tag{2071}$$

where A is the amplitude of oscillation of each charge about its corresponding focus (with $2A < r_0$ to avoid collision). Also, ω is the angular frequency of the standing TM wave that is oscillating the charges. Note that the use of $\sin(\omega t)$ means that at $t = 0$ the charges are positioned at their corresponding foci and moving *away* from each other. Then the velocity[240] and acceleration of q_1 relative to q_2 will be, respectively,

$$\vec{v}_1 = -2A\omega\cos(\omega t)\hat{z} \qquad \text{and} \qquad \vec{a}_1 = 2A\omega^2\sin(\omega t)\hat{z} \tag{2072}$$

Also, because the motion of the charges is *anti*-symmetric, then $\vec{v}_2 = -\vec{v}_1$ and $\vec{a}_2 = -\vec{a}_1$.

$$\vec{v}_2 = 2A\omega\cos(\omega t)\hat{z} \qquad \text{and} \qquad \vec{a}_2 = -2A\omega^2\sin(\omega t)\hat{z} \tag{2073}$$

Ignoring for the moment the interaction between the charges, we know that each charge is only being oscillated by the standing wave in the TM mode. Therefore, the acceleration of each charge due to this wave can be found from Newton's Second law.

$$\vec{F}_z(t) = q\vec{E}_{wave}(t) = m\vec{a}(t) \tag{2074}$$

[238] A standing TM wave inside an ellipsoidal conductor will have a vanishing magnetic field on the axis between the foci. Therefore, only the electric force needs to be considered.

[239] I have choosen to use $2A$ instead of A for the coefficient of $\sin(\omega t)$ so that A can represent the amplitude of a *single* point charge about its corresponding focus. Since r is the *relative* displacement between the charges, then the maximum displacement between the charges will be $r_0 + 2A$.This is because the motion of each charge will add an amount A to the distance r_0 between them. Likewise, the minimum distance between the charges will be $r_0 - 2A$.

[240] The *magnitudes* of $\mathbf{v}_1$ and $\mathbf{v}_2$ is found from the time-derivative of r in (2071) However, the *direction* of $\mathbf{v}_1$ and $\mathbf{v}_2$ are put in "by hand" based on the fact that at $t = 0$ the charges are moving *away* from each other. Therefore, we know that we must have $\mathbf{v}_1$ pointing in the $-\hat{z}$ direction and $\mathbf{v}_2$ point in the $+\hat{z}$ direction. The accelerations $\mathbf{a}_1$ and $\mathbf{a}_2$ are found by simply taking the time-derivatives of $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively.

The acceleration of a single Planck mass (relative to the cavity) will be $\vec{a} = A\omega^2 \sin(\omega t)\hat{z}$. Since (2074) requires that $q\vec{E}_{wave}(t)$ and $\vec{a}_1(t)$ are in phase, then we must have[241]

$$q\vec{E}_{wave}(t) = qE_0 \sin(\omega t)\hat{z} \tag{2075}$$

where E_0 is the amplitude of the standing wave's electric field. Then (2074) gives

$$qE_0 = mA\omega^2 \tag{2076}$$

Therefore, we must have that the amplitude of oscillation is

$$A = \frac{qE_0}{m\omega^2} \tag{2077}$$

Formally speaking, Newton's Second Law should take into account *all* the forces acting on the charge. For example, the total force acting on the *left* sphere (with charge q_1 and mass m_1) will be due to the electric field of the standing TM wave, as well as the electromagnetic and gravitational forces of the other sphere (with charge q_2 and mass m_1).

However, the static Coulomb force and Newton force cancel by the "criticality" condition. The remaining *dynamic* contributions from the electromagnetic and gravitational forces are $q\frac{\partial \vec{A}_2}{\partial t}$ and $4m\frac{\partial \mathbf{h}_2}{\partial t}$, respectively. It will be shown later that these forces point in the *same* direction and therefore could alter the amplitude and motion of the spheres. However, if the charged masses have low velocities and weak fields (compared to the field of the standing TM wave), then their effects on the spheres can be considered small perturbations that are relatively negligible.

[241] Note that $\mathbf{a}(t)$ is in phase with $q\mathbf{E}_{wave}(t)$, not with $\mathbf{E}_{wave}(t)$. Since q is negative, this means that we actually have $\mathbf{E}_{wave}(t) = -E_0 \sin(\omega t)\hat{z}$.

21.3 Electromagnetic power

The two moving charges will have their own electric and magnetic fields associated with their motion. If the velocity of q_1 relative to q_2 is given by $\vec{v}_1$, then the electromagnetic Lorentz force on q_1 due to q_2 will be

$$\underset{(EM)}{\vec{F}_{2\rightarrow 1}} = q_1\left(\vec{E}_2+\vec{v}_1\times\vec{B}_2\right) \tag{2078}$$

where $\vec{E}_2$ and $\vec{B}_2$ are the electric and magnetic fields of q_2, respectively. They can be written in terms of potentials as

$$\vec{E}_2 = -\nabla\varphi_2-\frac{\partial\vec{A}_2}{\partial t} \qquad \text{and} \qquad \vec{B}_2=\nabla\times\vec{A}_2 \tag{2079}$$

Substituting $\vec{E}_2$ into the Lorentz force (2078) gives the force of q_2 acting on q_1.

$$\underset{(EM)}{\vec{F}_{2\rightarrow 1}} = q_1\left[-\nabla\varphi_2-\frac{\partial\vec{A}_2}{\partial t}+\left(\vec{v}_1\times\vec{B}_2\right)\right] \tag{2080}$$

Now we can dot this with $\vec{v}_1$ (the velocity of q_1 relative to q_2) to find the electromagnetic power, $\mathscr{P}_{EM,\,1}$, delivered to q_1 due to the electromagnetic fields of q_2.

$$\mathscr{P}_{1,\,EM} = \underset{(EM)}{\vec{F}_{2\rightarrow 1}}\cdot\vec{v}_1 \tag{2081}$$

$$= q_1\left[-\nabla\varphi_2-\frac{\partial\vec{A}_2}{\partial t}+\left(\vec{v}_1\times\vec{B}_2\right)\right]\cdot\vec{v}_1 \tag{2082}$$

$$= \left(-q_1\nabla\varphi_2\cdot\vec{v}_1\right)+\left(-q_1\frac{\partial\vec{A}_2}{\partial t}\cdot\vec{v}_1\right)+\left[q_1\left(\vec{v}_1\times\vec{B}_2\right)\cdot\vec{v}_1\right] \tag{2083}$$

The first term is essentially $\vec{F}_{(Coulomb)}\cdot\vec{v}_1$. This is the power due to the *electrostatic* Coulomb field. The second term is the power due to the *electrodynamic* field $\dfrac{\partial\vec{A}_2}{\partial t}$. The third term is clearly zero due to orthogonality, which is expected since the magnetic field can not do work. So the electromagnetic power in (2083) has two non-zero terms:

$$\mathscr{P}_{1,\,EM} = \underset{(Coulomb)}{\vec{F}_{2\rightarrow 1}}\vec{v}_1-q_1\frac{\partial\vec{A}_2}{\partial t}\cdot\vec{v}_1 \tag{2084}$$

We can describe these two terms as the "*semi-static*" term and the "*dynamic*" term.[242]

$$\mathscr{P}_{1,\,EM} = \underset{(semi-static)}{\mathscr{P}_{1,\,EM}}+\underset{(dynamic)}{\mathscr{P}_{1,\,EM}} \tag{2085}$$

[242]The "*semi-static*" term is not truly static since there is still a velocity, $\vec{v}_1$, associated with this power. However, the terms are meant to distinguish between the power due to electro*static* Coulomb force, $\vec{F}_{static}=-q\nabla\varphi$, and the power due to the electro*dynamic* force, $\vec{F}_{dynamic}=-q\,\dfrac{\partial\vec{A}}{\partial t}$.

where

$$\underset{(semi-static)}{\mathscr{P}_{1,\,EM}} = \underset{(Coulomb)}{\vec{F}_{2\to 1}}\,\vec{v}_1 \tag{2086}$$

and

$$\underset{(dynamic)}{\mathscr{P}_{1,\,EM}} = -q_1\frac{\partial \vec{A}_2}{\partial t}\cdot\vec{v}_1 \tag{2087}$$

First we will look at $\underset{(semi-static)}{\mathscr{P}_{1,\,EM}}$. The Coulomb force due to q_2 acting on q_1 is always pointing to the left so

$$\underset{(Coulomb)}{\vec{F}_{2\to 1}} = -\frac{1}{4\pi\varepsilon_0}\frac{|q_1q_2|}{r^2}\hat{z} \tag{2088}$$

We can substitute this into (2086) and also use (2071) and (2072) for r and $\vec{v}_1$, respectively. Then we have

$$\underset{(semi-static)}{\mathscr{P}_{1,\,EM}} = \left(-\frac{1}{4\pi\varepsilon_0}\frac{|q_1q_2|}{[r_0+2A\sin(\omega t)]^2}\hat{z}\right)\cdot 2A\omega\cos(\omega t)(-\hat{z}) \tag{2089}$$

$$= \frac{|q_1q_2|A\omega}{2\pi\varepsilon_0}\frac{\cos(\omega t)}{[r_0+2A\sin(\omega t)]^2} \tag{2090}$$

We can use $A = \dfrac{qE_0}{m\omega^2}$ from (2077) and also the fact that the charge on the two spheres is identical, $q_1 = q_2 = q$.

$$\underset{(semi-static)}{\mathscr{P}_{EM}} = \frac{q^2\omega}{2\pi\varepsilon_0}\left(\frac{qE_0}{m\omega^2}\right)\frac{\cos(\omega t)}{\left[r_0+2\left(\frac{qE_0}{m\omega^2}\right)\sin(\omega t)\right]^2} \tag{2091}$$

$$= \frac{q^2\omega}{4\pi\varepsilon_0 r_0}\left(\frac{2qE_0}{m\omega^2 r_0}\right)\frac{\cos(\omega t)}{\left[1+\left(\frac{2qE_0}{m\omega^2 r_0}\right)\sin(\omega t)\right]^2} \tag{2092}$$

We can define a dimensionless parameter[243] as

$$\gamma \equiv \frac{2qE_0}{m\omega^2 r_0} \tag{2093}$$

Using this definition for γ gives

$$\boxed{\underset{(semi-static)}{\mathscr{P}_{EM}} = \left(\frac{1}{4\pi\varepsilon_0}\frac{q^2\omega\gamma}{r_0}\right)\frac{\cos(\omega t)}{[1+\gamma\sin(\omega t)]^2}} \tag{2094}$$

[243] Note that this means that $\gamma = 2A/r_0$. Since we stated that $r_0 > 2A$ to avoid a collision of the spheres, then we must always have $\gamma < 1$. This fact will be useful later in the analysis.

Next we will look at $\underset{(dynamic)}{\mathscr{P}_{1,\,EM}}$ from (2087) which contains $\frac{\partial \vec{A}_2}{\partial t}\cdot\vec{v}_1$ due to the *electrodynamic* field. To visualize this dot product, consider the following diagram.

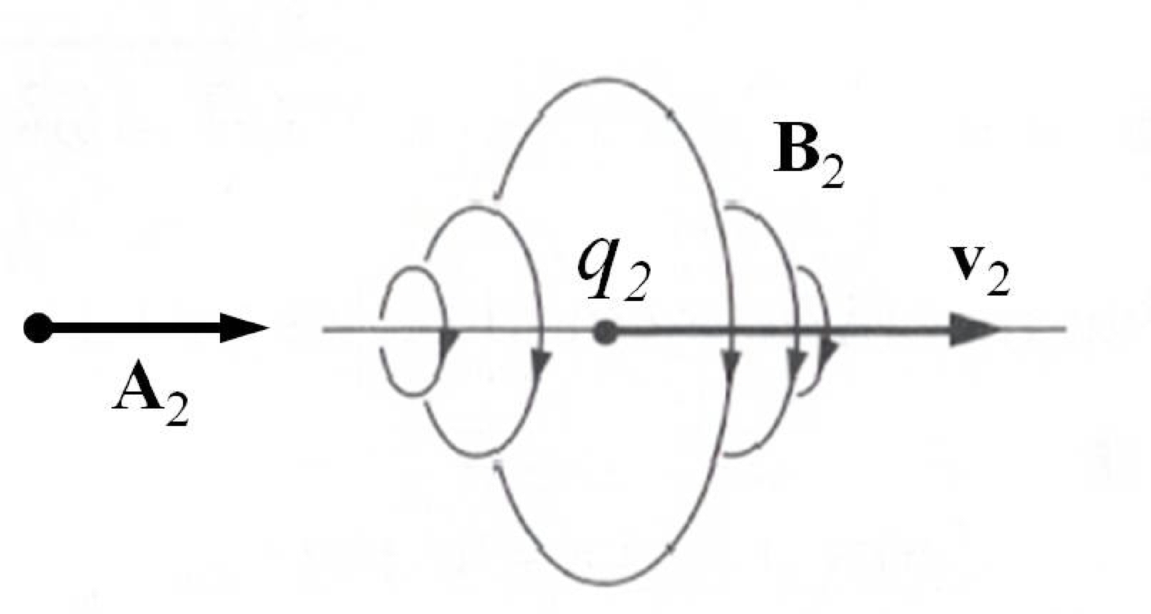

Figure 23: Point charge, q_2, moving to the right with velocity $\vec{v}_2$ relative to q_1. This generates a magnetic field, $\vec{B}_2$, and an associated magnetic vector potential, $\vec{A}_2$.

It can be seen that the magnetic field, $\vec{B}_2$, circulates around the line of motion while $\vec{A}_2$ points in the same direction as $\vec{v}_2$. Since the two particles move anti-symmetrically, then $\vec{v}_1$ points in the opposite direction of $\vec{A}_2$. Also, since q_2 is in motion, then the time-derivative of $\vec{A}_2$ must be non-zero. Therefore, it is evident that

$$\frac{\partial \vec{A}_2}{\partial t}\cdot\vec{v}_1 \neq 0. \tag{2095}$$

We can write the magnetic vector potential (in vacuum) in terms of the current density[244]

$$\vec{A}_2 = \frac{\mu_0}{4\pi}\int\frac{\vec{J}_2}{r}dV \tag{2096}$$

where $\vec{r}$ is the vector from the current density $\vec{J}_2$ (occupying a differential volume dV) to the field point where $\vec{A}_2$ is evaluated. Since $\vec{J}=\rho\vec{v}$, then for a point charge we have $\int\vec{J}dV = q\vec{v}$. So we can write (2096) as[245]

$$\vec{A}_2 = \frac{\mu_0}{4\pi}\frac{q_2\vec{v}_2}{r} \tag{2097}$$

where r is the distance between q_1 and q_2, while $\vec{v}_2$ is the velocity of q_2 relative to q_1.

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[244]See Giffiths [29], equation (5.63).

[245]This is approximately true for the non-relativistic case when $v << c$, which is the case throughout all of this treatment. Otherwise, the potentials are given by the relativistic Liénard-Wiechert potentials for a moving point charge. These are shown in Griffiths [29], equations 10.39 and 10.40 whcih are, respectively, $V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{qc}{(rc-\mathbf{r}\cdot\mathbf{v})}$ and $\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{qc\mathbf{v}}{(rc-\mathbf{r}\cdot\mathbf{v})}$.

The expression in (2084) requires evaluating $\frac{\partial \vec{A}_2}{\partial t}$.

$$\frac{\partial \vec{A}_2}{\partial t} = \frac{\mu_0 q_2}{4\pi} \frac{\partial}{\partial t} \left(\frac{\vec{v}_2}{r} \right) \tag{2098}$$

$$= \frac{\mu_0 q_2}{4\pi} \left(\frac{r\,\vec{a}_2 - \vec{v}_2 \dot{r}}{r^2} \right) \tag{2099}$$

where $\vec{a}_2$ is the acceleration of q_2 relative to q_1. Substituting (2099) back into (2087) gives

$$\underset{(dynamic)}{\mathscr{P}_{1,\,EM}} = -\frac{\mu_0 q_1 q_2}{4\pi r^2} \left(r\,\vec{a}_2 - \vec{v}_2 \dot{r} \right) \cdot \vec{v}_1 \tag{2100}$$

We can use (2071) – (2073) to substitute for r, $\vec{v}_2$, and $\vec{a}_2$. Then $(r\,\vec{a}_2 - \vec{v}_2 \dot{r})$ from (2100) becomes

$$\left(r\,\vec{a}_2 - \vec{v}_2 \dot{r} \right) = \left[r_0 + 2A \sin(\omega t) \right] \left[-2A\omega^2 \sin(\omega t)\,\hat{z} \right] - \left[2A\omega \cos(\omega t)\,\hat{z} \right] \left[2A\omega \cos(\omega t) \right] \tag{2101}$$

$$= -2r_0 A\omega^2 \sin(\omega t)\,\hat{z} - 4A^2\omega^2 \sin^2(\omega t)\,\hat{z} - 4A^2\omega^2 \cos^2(\omega)\,\hat{z} \tag{2102}$$

$$= -\left[2r_0 A\omega^2 \sin(\omega t) + 4A^2\omega^2 \right] \hat{z} \tag{2103}$$

We can use the result in (2103) along with $\vec{v}_1$ from (2072) to write the dot product, $(r\,\vec{a}_2 - \vec{v}_2 \dot{r}) \cdot \vec{v}_1$, from (2100) as

$$\left(r\,\vec{a}_2 - \vec{v}_2 \dot{r} \right) \cdot \vec{v}_1 = -\left[2r_0 A\omega^2 \sin(\omega t) + 4A^2\omega^2 \right] \hat{z} \cdot \left[-2A\omega \cos(\omega t) \right] \hat{z} \tag{2104}$$

$$= 4r_0 A^2 \omega^3 \sin(\omega t) \cos(\omega t) + 8A^3\omega^3 \cos(\omega t) \tag{2105}$$

$$= 4A^2\omega^3 \cos(\omega t) \left[r_0 \sin(\omega t) + 2A \right] \tag{2106}$$

Substituting this result as well as $r = r_0 + 2A \sin(\omega t)$ from (2071) into $\underset{(dynamic)}{\mathscr{P}_{1,\,EM}}$ in (2100) gives

$$\underset{(dynamic)}{\mathscr{P}_{1,\,EM}} = -\frac{\mu_0 q_1 q_2}{4\pi \left[r_0 + 2A \sin(\omega t) \right]^2} 4A^2 \omega^3 \cos(\omega t) \left[r_0 \sin(\omega t) + 2A \right] \tag{2107}$$

$$= -\frac{\mu_0 q_1 q_2}{\pi} A^2 \omega^3 \frac{r_0 \sin(\omega t) + 2A}{\left[r_0 + 2A \sin(\omega t) \right]^2} \cos(\omega t) \tag{2108}$$

Finally, we can use $A = \frac{qE_0}{m\omega^2}$ from (2077) and the fact that the charge on the two spheres is identical, $q_1 = q_2 = q$. This leads to

$$\underset{(dynamic)}{\mathscr{P}_{EM}} = -\frac{\mu_0 q^2}{\pi} \left(\frac{qE_0}{m\omega^2} \right)^2 \omega^3 \frac{r_0 \sin(\omega t) + 2\left(\frac{qE_0}{m\omega^2} \right)}{\left[r_0 + 2\left(\frac{qE_0}{2m\omega^2} \right) \sin(\omega t) \right]^2} \cos(\omega t) \tag{2109}$$

Rearranging gives

$$\underset{(dynamic)}{\mathscr{P}_{EM}} = -\frac{\mu_0 q^2\omega^3 r_0}{4\pi}\left(\frac{2qE_0}{m\omega^2 r_0}\right)^2 \frac{\left(\frac{2qE_0}{m\omega^2 r_0}\right)+\sin(\omega t)}{\left[1+\left(\frac{2qE_0}{m\omega^2 r_0}\right)\sin(\omega t)\right]^2}\cos(\omega t) \tag{2110}$$

Once again, using the dimensionless constant, $\gamma = \dfrac{2qE_0}{m\omega^2 r_0}$, gives

$$\boxed{\underset{(dynamic)}{\mathscr{P}_{EM}} = -\left(\frac{\mu_0}{4\pi}q^2\omega^3 r_0\gamma^2\right)\frac{\gamma+\sin(\omega t)}{[1+\gamma\sin(\omega t)]^2}\cos(\omega t)} \tag{2111}$$

We can write a single combined expression for the total electromagnetic power delivered by one charge to the other using (2094) and (2111).

$$\mathscr{P}_{EM} = \underset{(semi-static)}{\mathscr{P}_{EM}} + \underset{(dynamic)}{\mathscr{P}_{EM}} \tag{2112}$$

$$= \left(\frac{1}{4\pi\varepsilon_0}\frac{q^2\omega\gamma}{r_0}\right)\frac{\cos(\omega t)}{[1+\gamma\sin(\omega t)]^2} - \left(\frac{\mu_0}{4\pi}q^2\omega^3 r_0\gamma^2\right)\frac{\gamma+\sin(\omega t)}{[1+\gamma\sin(\omega t)]^2}\cos(\omega t) \tag{2113}$$

Using $\mu_0 = \dfrac{1}{\varepsilon_0 c^2}$, we can write this as

$$\mathscr{P}_{EM} = \left(\frac{1}{4\pi\varepsilon_0}\frac{q^2\omega}{r_0}\right)\frac{\gamma\cos(\omega t)}{[1+\gamma\sin(\omega t)]^2} - \left(\frac{1}{4\pi\varepsilon_0 c^2}q^2\omega^3 r_0\right)\frac{\gamma^3+\gamma^2\sin(\omega t)}{[1+\gamma\sin(\omega t)]^2}\cos(\omega t) \tag{2114}$$

$$\boxed{\mathscr{P}_{EM} = \left(\frac{1}{4\pi\varepsilon_0}\frac{q^2\omega}{r_0}\right)\frac{\cos(\omega t)}{[1+\gamma\sin(\omega t)]^2}\left\{\gamma - \left(\frac{\omega^2 r_0^2}{c^2}\right)\left[\gamma^3+\gamma^2\sin(\omega t)\right]\right\}} \tag{2115}$$

We now have an explicit function for the *total* electromagnetic power in terms of just the properties of the TM standing wave (ω and E_o) and the properties of the SC Planck mass spheres (q and m).

Observations concerning the EM power

For the purpose of making some observations about $\underset{(semi-static)}{\mathscr{P}_{EM}}$ and $\underset{(dynamic)}{\mathscr{P}_{EM}}$, we can substitute $\gamma = \dfrac{2qE_0}{m\omega^2 r_0}$ back into just the prefactors of (2094) and (2111). Then we have

$$\underset{(semi-static)}{\mathscr{P}_{EM}} = \left(\frac{q^3 E_0}{2\pi\varepsilon_0 m\omega r_0^2}\right)\frac{\cos(\omega t)}{[1+\gamma\sin(\omega t)]^2} \tag{2116}$$

and

$$\underset{(dynamic)}{\mathscr{P}_{EM}} = -\left(\frac{\mu_0 q^4 E_0{}^2}{\pi m^2 \omega r_0}\right) \frac{\gamma + \sin(\omega t)}{[1+\gamma \sin(\omega t)]^2} \cos(\omega t) \tag{2117}$$

Using (2116) and (2117) we can make the following observations.

1. The *semi-static* power scales *linearly* with E_0 (the magnitude of the electric field of the TM wave). This seems to emerge from the linear coupling of the charged mass with the electric force of the TM standing wave which was utilized to find the amplitude of the motion in (2074). The *dynamic* power scales *quadratically* with E_0. This could be related to the fact that the energy of an electromagnetic wave also scales *quadratically* with the *magnitude of the electric field*: $U = \varepsilon_0 E_0{}^2$.

2. The *semi-static* power scales as $r_0{}^{-2}$ where r_0 is the initial separation distance of the charges masses before oscillation. This is consistent with the fact that this "semi-static" power emerges from Coulomb's law which has an inverse square dependence. The *dynamic* power scales as $r_0{}^{-1}$ which is consistent with the fact that radiation should fall off as r^{-1} in the far-field.

3. Both the *semi-static* and *dynamic* power scale as ω^{-1} where ω the angular frequency of the standing TM wave. This is surprising since it would be expected that the more rapidly the spheres are oscillated, the more rapidly energy could be transferred in or out of the two-sphere system and hence the higher the power values. However, the ω^{-1} dependency arises from the following:

   - $\underset{(semi-static)}{\mathscr{P}_{EM}} \sim A\omega$ from (2090) and $A \sim \omega^{-2}$ from (2077).
   - $\underset{(dynamic)}{\mathscr{P}_{EM}} \sim A^2\omega^3$ from (2108) and $A \sim \omega^{-2}$ from (2077).

   Therefore, the reason that the power *decreases* with higher angular frequency is because the *amplitude decreases* with higher angular frequency. Basically, if the spheres oscillate faster then they move through a smaller distance and therefore deliver and receive less power.

4. The *semi-static* power scales as m^{-1} which is consistent with the fact that a heavier mass will have greater inertia and therefore will not accelerate as easily by the standing TM wave. The *dynamic* power is suppressed even further by the mass since it scales as m^{-2}.

5. The *semi-static* power scales as q^3 while the *dynamic* power scales as q^4. These dependencies arise from the fact that

   - $\underset{(semi-static)}{\mathscr{P}_{EM}} \sim q^2A$ from (2090) and $A \sim q$ from (2077).
   - $\underset{(dynamic)}{\mathscr{P}_{EM}} \sim q^2A^2$ from (2108) and $A \sim q$ from (2077).

   From these results, we can see that both the *semi-static* power and *dynamic* power are affected more by the *charge* on the spheres than the *mass* of the spheres. This means that if both the charge and mass are increased together linearly according to the "criticality" condition in (2067), the result will be *more* power, not less. In other words, even though the increased mass will introduce more inertia and hence slow down the motion of the spheres, nevertheless, the higher amount of charge will still increase the power delivered and received.

6. Both the *semi-static* power and *dynamic* power fluctuate in sign ($+$ or $-$) due to the $\cos(\omega t)$ term. This is consistent with the fact that the power is associated with the sinusoidal motion of the moving charges.

   - At $t = 0$, when the charges are a distance r_0 apart and moving *away* from each other, the power is a *positive maximum*. The spheres have the fastest rate of EM energy *received*.

- A quarter cycle later, when they are a distance $r_0 + 2A$ apart and not moving, there is *no* energy transfer so the power is zero.
- Another quarter cycle later, when they are a distance r_0 apart again but moving *toward* each other, the power is a negative maximum. The spheres have the fastest rate of EM energy *extracted*.
- Finally, another quarter cycle later, when they are a distance $r_0 - 2A$ apart and not moving, there is *no* energy transfer so the power is zero.

  These results are consistent with the fact that as the charges move *toward* each other, they experience stronger fields due to their proximity and hence have a *positive* energy transfer. Likewise, as the charges move *away* from each other, they experience weaker fields due to their distance and hence have a *negative* energy transfer. Also, when the charges are moving their fastest (when they are a distance r_0 apart) they have a maximum energy transfer (either positive or negative).

7. Both the *semi-static* power and *dynamic* power demonstrate a "sloshing" behavior.

   - For the *semi-static* power, the "sloshing" is due to the term $\dfrac{1}{[1+\gamma\sin(\omega t)]^2}$.
   - For the *dynamic* power, the "sloshing" is due to the term $\dfrac{\gamma+\sin(\omega t)}{[1+\gamma\sin(\omega t)]^2}$.

     This "sloshing" may be due to the near-field effects by sinusoidally moving charges. The speeding up and slowing down of the charges at sinusoidal rates in the proximity of each other will cause a "sloshing" behavior in the transfer of energy.

   - Notice that when $\gamma << 1$, then $\dfrac{1}{[1+\gamma\sin(\omega t)]^2} \approx 1$ and $\dfrac{\gamma+\sin(\omega t)}{[1+\gamma\sin(\omega t)]^2} \approx 1$.

     This means that the "sloshing" behavior of the total power vanishes and there is only a "smooth" $\cos(\omega t)$ fluctuation in the energy transfer.

   - We defined $\gamma \equiv \dfrac{2qE_0}{m\omega^2 r_0}$ which means that to have $\gamma << 1$ we need $m\omega^2 r_0 >> 2qE_0$ or $\dfrac{\omega^2 r_0}{2E_0} >> \dfrac{q}{m}$. Using the "criticality" condition, $\dfrac{q}{m} = \sqrt{4\pi\varepsilon_0 G}$, means we need $\dfrac{\omega^2 r_0}{E_0} >> 2\sqrt{4\pi\varepsilon_0 G}$. Thus, for given values of ω and E_0 of the standing TM wave, the "sloshing" can be reduced if the distance between the spheres, r_0, is made large.

8. Mathematically, there is the possibility for the power to diverge if $[1+\gamma\sin(\omega t)] = 0$ in the denominator. Physically this is due to the fact that the denominator contains the displacement vector r between the two charges. If this becomes zero at any time then it implies that the two charges are completely "overlapped." Power is the transfer of energy per time, therefore if there is no separation, then the transfer will be instantaneous and hence the power will become infinite.

   To avoid a divergence we must have $|1+\gamma\sin(\omega t)| > 0$ for all t values. Since $\sin(\omega t)$ has a minimum of -1, then we must make $\gamma < 1$. This means we need $\dfrac{2qE_0}{m\omega^2 r_0} < 1$ or $\dfrac{q}{m} < \dfrac{\omega^2 r_0}{2E_0}$. Again, we can use the "criticality" condition, $\dfrac{q}{m} = \sqrt{4\pi\varepsilon_0 G}$, so the condition to avoid divergence becomes $\dfrac{\omega^2 r_0}{2E_0} > \sqrt{4\pi\varepsilon_0 G}$.

21.4 Gravitational power

In the gravito-electromagnetic framework, the gravito-electric field, $\vec{E}_G$, and gravito-magnetic field, $\vec{B}_G$, can be defined in terms of the gravitational scalar potential, φ_G, and gravito-magnetic vector potential, $\vec{h}$, as follows:

$$\vec{E}_G = -\nabla\varphi_G - 4\frac{\partial\vec{h}}{\partial t} \qquad \text{and} \qquad \vec{B}_G = \nabla\times\vec{h} \tag{2118}$$

The *gravito*-electromagnetic Lorentz force on m_1 (the mass of charge q_1) due to m_2 (the mass of charge q_2) is[246]

$$\underset{(GR)}{\vec{F}_{2\to1}} = m_1\left[-\nabla\varphi_{G,2} - 4\frac{\partial\vec{h}_2}{\partial t} + 4\left(\vec{v}_1\times\vec{B}_{G,2}\right)\right] \tag{2119}$$

Now we can dot this with $\vec{v}_1$ (the velocity of m_1 relative to m_2) to find the gravitational power, $\mathscr{P}_{1,\,GR}$, delivered to m_1 due to the gravitational fields of m_2.

$$\mathscr{P}_{1,\,GR} = \underset{(GR)}{\vec{F}_{2\to1}}\cdot\vec{v}_1 \tag{2120}$$

$$= m_1\left[-\nabla\varphi_{G,2} - 4\frac{\partial\vec{h}_2}{\partial t} + 4\left(\vec{v}_1\times\vec{B}_{G,2}\right)\right]\cdot\vec{v}_1 \tag{2121}$$

$$= \left(-m_1\nabla\varphi_{G,2}\cdot\vec{v}_1\right) + \left[-4m_1\frac{\partial\vec{h}_2}{\partial t}\cdot\vec{v}_1\right] + \left[4m_1\left(\vec{v}_1\times\vec{B}_{G,2}\right)\cdot\mathbf{v}_1\right] \tag{2122}$$

The first term is essentially $\vec{F}_{(Newton)}\cdot\vec{v}_1$. This is the power due to the *gravito-electrostatic* Newtonian field. The second term is the power due to the *gravito-electrodynamic* field $\frac{\partial\vec{h}_2}{\partial t}$. The third term is clearly zero due to orthogonality, which is expected since the gravito-magnetic field can not do work. So the gravitational power in (2122) has two non-zero terms:

$$\mathscr{P}_{1,\,GR} = \underset{(Newton)}{\vec{F}_{2\to1}}\cdot\vec{v}_1 - 4m_1\frac{\partial\vec{h}_2}{\partial t}\cdot\vec{v}_1 \tag{2123}$$

We can describe these two terms as the "*semi-static*" term and the "*dynamic*" term.[247]

$$\mathscr{P}_{1,\,GR} = \underset{(semi-static)}{\mathscr{P}_{1,\,GR}} + \underset{(dynamic)}{\mathscr{P}_{1,\,GR}} \tag{2124}$$

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[246]Note that a factor of 4 appears in the terms containing $4\frac{\partial\vec{h}_2}{\partial t}$ and $4\vec{v}_1\times\vec{B}_{G,2}$. This arises from the linearized geodesic equation, $\frac{du^\mu}{d\tau} = -\Gamma^{\mu(1)}_{\alpha\beta}u^\alpha u^\beta$ where $-\Gamma^{\mu(1)}_{\alpha\beta}$ is linear in $h_{\mu\nu}$, the perturbation metric. This also assumes that we only keep terms that are first order in v/c.

[247]The "*semi-static*" term is not truly static since there is still a velocity, $\vec{v}_1$, associated with this power. However, the terms are meant to distinguish between the power due to the gravito-electro*static* Newtonian force, $\vec{F}_{static} = -m\nabla\varphi_G$, and the power due to the gravito-electro*dynamic* force, $\vec{F}_{dynamic} = -4m\frac{\partial\vec{h}}{\partial t}$.

where

$$\underset{(semi-static)}{\mathscr{P}_{1,\ GR}} = \underset{(Newton)}{\vec{F}_{2\to 1}} \cdot \vec{v}_1 \tag{2125}$$

and

$$\underset{(dynamic)}{\mathscr{P}_{1,\ GR}} = -4m_1 \frac{\partial \vec{h}_2}{\partial t} \cdot \vec{v}_1 \tag{2126}$$

First we will look at $\underset{(semi-static)}{\mathscr{P}_{1,\ GR}}$. The Newton force due to m_2 acting on m_1 is always pointing to the right so

$$\underset{(Newton)}{\vec{F}_{2\to 1}} = G\frac{m_1 m_2}{r^2}\hat{z} \tag{2127}$$

We can substitute this into (2125) and also use (2071) and (2072) for r and $\mathbf{v}_1$, respectively. Then we have

$$\underset{(semi-static)}{\mathscr{P}_{1,\ GR}} = \left(G\frac{m_1 m_2}{\left[r_0 + 2A\sin(\omega t)\right]^2}\hat{z}\right) \cdot 2A\omega\cos(\omega t)\,(-\hat{z}) \tag{2128}$$

$$= -2Gm_1 m_2 A\omega \frac{\cos(\omega t)}{\left[r_0 + 2A\sin(\omega t)\right]^2} \tag{2129}$$

We can use $A = \dfrac{qE_0}{m\omega^2}$ from (2077) and also the fact that the mass of the two spheres is identical, $m_1 = m_2 = m$.

$$\underset{(semi-static)}{\mathscr{P}_{GR}} = -\left(2Gm^2\omega\right)\left(\frac{qE_0}{m\omega^2}\right)\frac{\cos(\omega t)}{\left[r_0 + 2\left(\frac{qE_0}{m\omega^2}\right)\sin(\omega t)\right]^2} \tag{2130}$$

$$= -\left(\frac{Gm^2\omega}{r_0}\right)\left(\frac{2qE_0}{m\omega^2 r_0}\right)\frac{\cos(\omega t)}{\left[1 + \left(\frac{2qE_0}{m\omega^2 r_0}\right)\sin(\omega t)\right]^2} \tag{2131}$$

Using the dimensionless parameter defined in (2093) as

$$\gamma \equiv \frac{2qE_0}{m\omega^2 r_0} \tag{2132}$$

leads to

$$\boxed{\underset{(semi-static)}{\mathscr{P}_{GR}} = -\left(\frac{Gm^2\omega\gamma}{r_0}\right)\frac{\cos(\omega t)}{\left[1+\gamma\sin(\omega t)\right]^2}} \tag{2133}$$

Next we will look at $\underset{(dynamic)}{\mathscr{P}_{1,\,GR}}$ from (2126) which contains $\frac{\partial \vec{h}_2}{\partial t} \cdot \vec{v}_1$ due to the gravito-*electrodynamic* field. To visualize this dot product, consider the following diagram.

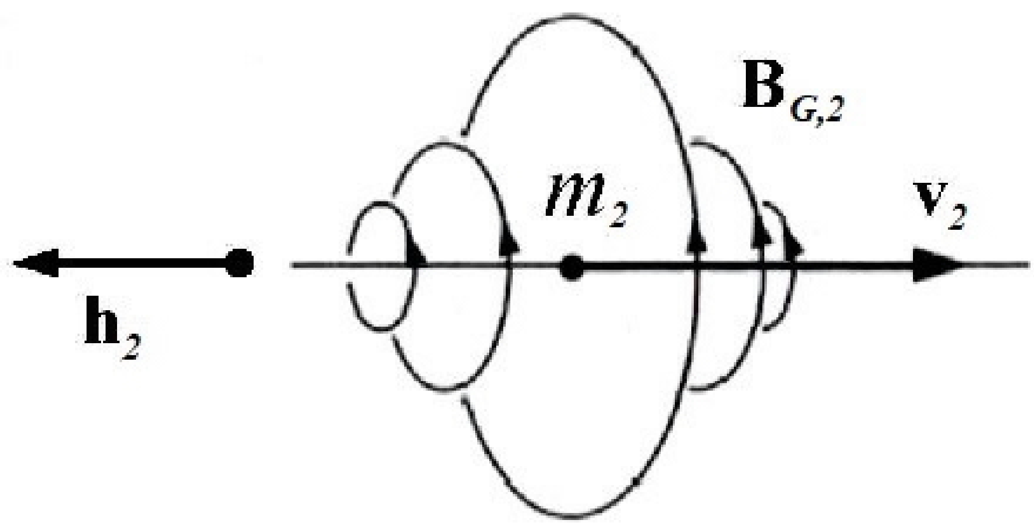

Figure 24: The point mass, m_2, moving to the right with velocity $\vec{v}_2$ relative to m_1. This generates a gravito-magnetic field, $\vec{B}_{G,2}$, and an associated gravito-magnetic vector potential, $\vec{h}_2$.

It can be seen that the gravito-magnetic field, $\vec{B}_{G,2}$, circulates around the line of motion, however, it is in the *opposite* direction of the right-hand rule due to the negative sign in the static gravito-electromagnetic Ampere's law, $\nabla \times \vec{B}_G = -\mu_G \vec{J}_m$. Also notice that $\vec{h}_2$ points in the *opposite* direction of $\vec{v}_2$ due to defining the gravito-magnetic field in terms of the vector potential as $\vec{B}_G \equiv \nabla \times \vec{h}$, which leads to $\nabla \times \left(\nabla \times \vec{h}\right) = -\mu_G \vec{J}_m$. Since the two particles move anti-symmetrically, then $\mathbf{v}_1$ points in the *same* direction as $\vec{h}_2$. Also, since m_2 is in motion, then the time-derivative of $\vec{h}_2$ must be non-zero. Therefore, it is evident that

$$\frac{\partial \vec{h}_2}{\partial t} \cdot \vec{v}_1 \neq 0 \tag{2134}$$

The gravito-magnetic vector potential can be written in terms of the mass-current density[248] by analogy to (2096). This gives

$$\vec{h}_2 = -\frac{\mu_G}{4\pi} \int \frac{\vec{J}_{m,2}}{r} dV \tag{2135}$$

where $\vec{r}$ is the vector from the mass-current density $\vec{J}_{m,2}$ (occupying a differential volume dV) to the field point where $\vec{h}_2$ is evaluated. Since $\vec{J}_m = \rho_m \vec{v}$ (where ρ_m is the mass density), then for a point mass we have $\int \vec{J}_m dV = m\vec{v}$. So we can write (2135) as

$$\vec{h}_2 = -\frac{\mu_G}{4\pi} \frac{m_2 \vec{v}_2}{r} \tag{2136}$$

where r is the distance between m_2 and m_1, while $\vec{v}_2$ is the velocity of q_2 relative to q_1.

[248] See [10], equation 4.52.

Since (2123) requires that we evaluate $\frac{\partial \vec{h}_2}{\partial t} \cdot \vec{v}_1$, then by analogy to the electromagnetic case in (2099), we will have

$$\frac{\partial \vec{h}_2}{\partial t} = -\frac{\mu_G m_2}{4\pi}\left(\frac{r\,\vec{a}_2 - \vec{v}_2 \dot{r}}{r^2}\right) \tag{2137}$$

where $\vec{a}_2$ is the acceleration of m_2 relative to m_1. Substituting (2137) back into (2126) gives

$$\underset{(dynamic)}{\mathscr{P}_{1,\,GR}} = \frac{\mu_G m_1 m_2}{\pi r^2}\left(r\,\vec{a}_2 - \vec{v}_2 \dot{r}\right) \cdot \vec{v}_1 \tag{2138}$$

From (2106) and (2071) we can use

$$\left(r\,\vec{a}_2 - \vec{v}_2 \dot{r}\right) \cdot \vec{v}_1 = 4A^2\omega^3 \cos(\omega t)\left[r_0 \sin(\omega t) + 2A\right] \qquad \text{and} \qquad r = r_0 + 2A\sin(\omega t)$$

to write (2138) in a form analogous to (2108).

$$\underset{(dynamic)}{\mathscr{P}_{1,\,GR}} = \frac{\mu_G m_1 m_2}{\pi\left[r_0 + 2A\sin(\omega t)\right]^2} 4A^2\omega^3 \cos(\omega t)\left[r_0 \sin(\omega t) + 2A\right] \tag{2139}$$

$$= \frac{4\mu_G m_1 m_2}{\pi} A^2\omega^3 \frac{r_0 \sin(\omega t) + 2A}{\left[r_0 + 2A\sin(\omega t)\right]^2} \cos(\omega t) \tag{2140}$$

Finally, we can use $A = \frac{qE_0}{m\omega^2}$ from (2077) and the fact that the mass of the two spheres is identical, $m_1 = m_2 = m$.

$$\underset{(dynamic)}{\mathscr{P}_{GR}} = \frac{4\mu_G m^2}{\pi}\left(\frac{qE_0}{m\omega^2}\right)^2 \omega^3 \frac{r_0 \sin(\omega t) + 2\left(\frac{qE_0}{m\omega^2}\right)}{\left[r_0 + 2\left(\frac{qE_0}{m\omega^2}\right)\sin(\omega t)\right]^2} \cos(\omega t) \tag{2141}$$

Rearranging gives

$$\underset{(dynamic)}{\mathscr{P}_{GR}} = \; = \frac{\mu_G m^2 \omega^3 r_0}{\pi}\left(\frac{2qE_0}{m\omega^2 r_0}\right)^2 \frac{\left[\left(\frac{2qE_0}{m\omega^2 r_0}\right) + \sin(\omega t)\right]}{\left[1 + \left(\frac{2qE_0}{m\omega^2 r_0}\right)\sin(\omega t)\right]^2} \cos(\omega t) \tag{2142}$$

Once again, using the dimensionless constant, $\gamma = \frac{2qE_0}{m\omega^2 r_0}$, gives

$$\boxed{\underset{(dynamic)}{\mathscr{P}_{GR}} = \left(\frac{\mu_G}{\pi} m^2 \omega^3 r_0 \gamma^2\right) \frac{\gamma + \sin(\omega t)}{\left[1 + \gamma\sin(\omega t)\right]^2} \cos(\omega t)} \tag{2143}$$

We can write a single combined expression for the total gravitational power delivered by one charge to the other using (2133) and (2143).

$$\mathscr{P}_{GR} = \underset{(semi-static)}{\mathscr{P}_{GR}} + \underset{(dynamic)}{\mathscr{P}_{GR}} \tag{2144}$$

$$= -\left(\frac{Gm^2\omega\gamma}{r_0}\right) \frac{\cos(\omega t)}{\left[1 + \gamma\sin(\omega t)\right]^2} + \left(\frac{\mu_G}{\pi} m^2 \omega^3 r_0 \gamma^2\right) \frac{\gamma + \sin(\omega t)}{\left[1 + \gamma\sin(\omega t)\right]^2} \cos(\omega t) \tag{2145}$$

Using $\mu_G = \dfrac{4\pi G}{c^2}$, we can write this as

$$\mathscr{P}_{GR} \quad = \quad -\left(\frac{Gm^2\omega}{r_0}\right)\frac{\gamma\cos(\omega t)}{[1+\gamma\sin(\omega t)]^2}+\left(\frac{4\pi G}{\pi c^2}m^2\omega^3 r_0\right)\frac{\gamma^3+\gamma^2\sin(\omega t)}{[1+\gamma\sin(\omega t)]^2}\cos(\omega t)$$

$$\boxed{\mathscr{P}_{GR} = -\left(\frac{Gm^2\omega}{r_0}\right)\frac{\cos(\omega t)}{[1+\gamma\sin(\omega t)]^2}\left\{\gamma-\left(\frac{4\omega^2 r_0^2}{c^2}\right)\left[\gamma^3+\gamma^2\sin(\omega t)\right]\right\}} \tag{2146}$$

where $\gamma = \dfrac{2qE_0}{m\omega^2 r_0}$. We now have explicit functions for the *total* gravitational power in terms of just the properties of the TM standing wave (ω and E_o) and the properties of the SC Planck mass spheres (q and m).

Observations concerning the EM power

For the purpose of making some observations about $\underset{(semi-static)}{\mathscr{P}_{GR}}$ and $\underset{(dynamic)}{\mathscr{P}_{GR}}$, we can substitute $\gamma = \dfrac{2qE_0}{m\omega^2 r_0}$ back into just the prefactors of (2133) and (2143). Then we have

$$\underset{(semi-static)}{\mathscr{P}_{GR}} \quad = \quad -\left(\frac{2GmqE_0}{r_0^2\omega}\right)\frac{\cos(\omega t)}{[1+\gamma\sin(\omega t)]^2} \tag{2147}$$

and

$$\underset{(dynamic)}{\mathscr{P}_{GR}} \quad = \quad \left(\frac{4\mu_G}{\pi\omega r_0}q^2 E_0^2\right)\frac{\gamma+\sin(\omega t)}{[1+\gamma\sin(\omega t)]^2}\cos(\omega t) \tag{2148}$$

Almost all the observations made concerning $\mathscr{P}_{EM}$ can likewise apply to $\mathscr{P}_{GR}$. Specifically, in Section 79, "Observations Concerning the EM Power," points 1-3 and 6-8 apply to both $\mathscr{P}_{EM}$ and $\mathscr{P}_{GR}$ so they will not be repeated here. However, points 4-5 (which deal with the way the power scales with mass and charge) are not the same for both $\mathscr{P}_{EM}$ and $\mathscr{P}_{GR}$, therefore they will be discussed below. Also, there are a number of other differences that will be highlighted when comparing (2116) and (2117) for EM power to (2147) and (2148) for GR power.

1. The *semi-static* power scales linearly with m. This is due to the fact that mass plays the role of "gravitational charge" as well as a measure of the inertia of an object. Therefore, although increasing the mass of the spheres would make them more difficult to move and thus could decrease gravitational radiation, on the other hand, increasing the mass also increases the *source* of gravitational radiation and hence could increase the radiation. It turns out that the net result for the *semi-static* power is an *increase* in radiation.

   It is interesting that the *dynamic* power does *not* scale with m at all. In this case, an increase in "gravitational charge" would perfectly cancel an increase in inertia so that the net result is that the *dynamic* power does not depend on the mass at all.

   Note that these results assume the Equivalence Principle holds absolutely true since it treats inertial mass and gravitational mass as equivalent and allows them to "cancel out." The inertial mass only enters into the calculation via Newton's Second Law when finding the amplitude of oscillation in (2077).

As a result, it can be seen by backtracking through the calculation that every mass that appears in (2142) is an inertial mass except for the squared mass appearing in the first term, $\left(\frac{\mu_G}{\pi} m^2 \omega^3 r_0\right)$, which is a gravitational mass coming from the gravitational Lorentz force (2119) and the gravito-magnetic vector potential (2136). If this gravitational mass cancels with the inertial mass appearing in the term $\left(\frac{qE_0}{m\omega^2 r_0}\right)^2$, then the Equivalence Principle predicts that $\underset{(semi-static)}{\mathscr{P}_{GR}}$ in (2147) scales linearly with m and $\underset{(dynamic)}{\mathscr{P}_{GR}}$ in (2148) does not scale with the mass at all.

2. The *semi-static* power scales linearly with q while the *dynamic* power scales quadratically with q^2. This means that if both the charge and mass are increased together linearly according to the "criticality" condition in (2067), the result will be *more* power, not less. In other words, even though the increased mass will introduce more inertia and hence slow down the motion of the spheres, nevertheless, the higher amount of charge will still increase the power delivered and received. So the power is affected more by the *charge* on the spheres than the *mass* of the spheres. This is similar to the case with electromagnetic power.

3. Both the *semi-static* and *dynamic* gravitational power expressions have a minus sign when compared to the corresponding electromagnetic power expressions. For the *semi-static* power, the minus sign is due to the Coulomb force being in the opposite direction from the Newtonian force. For the *dynamic* power, the minus sign is due to the minus sign in the static gravitational Ampere's law, $\nabla \times \vec{B}_G = -\mu_G \mathbf{J}_m$ which does not appear in the electromagnetic Ampere's law, $\nabla \times \vec{B} = \mu_0 \mathbf{J}$.

   This overall minus sign for the gravitational power compared to the electromagnetic power means that the fluctuation of each type of power will be perfectly out of phase. Therefore, when electromagnetic energy is being lost, gravitational energy will be gained, and vice versa.

21.5 “Criticality” condition for semi-static versus dynamic power and forces

Semi-static versus dynamic *power*

The total EM and GR power from (2115) and (2146) are, respectively,

$$\mathscr{P}_{EM} = \left(\frac{1}{4\pi\varepsilon_0}\frac{q^2\omega}{r_0}\right)\frac{\cos(\omega t)}{[1+\gamma\sin(\omega t)]^2}\left\{\gamma-\left(\frac{\omega^2 {r_0}^2}{c^2}\right)\left[\gamma^3+\gamma^2\sin(\omega t)\right]\right\} \tag{2149}$$

and

$$\mathscr{P}_{GR} = -\left(\frac{Gm^2\omega}{r_0}\right)\frac{\cos(\omega t)}{[1+\gamma\sin(\omega t)]^2}\left\{\gamma-\left(\frac{4\omega^2 {r_0}^2}{c^2}\right)\left[\gamma^3+\gamma^2\sin(\omega t)\right]\right\} \tag{2150}$$

In either expression above, the first term in the braces (the single factor of γ) is the *semi-static* power term. The second term in the braces (which has an extra factor of 4 for the GR power) is the *dynamic* power term. First we can consider a ratio of magnitudes of the *semi-static* power terms.

$$\left|\frac{\underset{(semi-static)}{\mathscr{P}_{EM}}}{\underset{(semi-static)}{\mathscr{P}_{GR}}}\right| = \frac{\left(\frac{1}{4\pi\varepsilon_0}\frac{q^2\omega}{r_0}\right)\gamma}{\left(\frac{Gm^2\omega}{r_0}\right)\gamma} = \frac{1}{4\pi\varepsilon_0 G}\frac{q^2}{m^2} \tag{2151}$$

If we require the EM and GR *semi-static* power to be equal, then $\left|\frac{\underset{(semi-static)}{\mathscr{P}_{EM}}}{\underset{(semi-static)}{\mathscr{P}_{GR}}}\right| = 1$ so we have $\frac{1}{4\pi\varepsilon_0 G}\frac{q^2}{m^2} = 1$. This leads to

$$\left|\frac{q}{m}\right| = \sqrt{4\pi\varepsilon_0 G} \tag{2152}$$

This is precisely the “criticality” condition obtained in (2077) by equating the Coulomb electrostatic force, $\vec{F}_{(Coulomb)}$, and the Newtonian gravitational force, $\vec{F}_{(Newton)}$. Therefore, we find that the the “criticality” condition makes the *semi-static* EM and GR power equal. Next we can consider the ratio of magnitudes of the *dynamic* power terms.

$$\left|\frac{\underset{(dynamic)}{\mathscr{P}_{EM}}}{\underset{(dynamic)}{\mathscr{P}_{GR}}}\right| = \frac{\left(\frac{1}{4\pi\varepsilon_0}\frac{q^2\omega}{r_0}\right)\left(\frac{\omega^2 {r_0}^2}{c^2}\right)\left[\gamma^3+\gamma^2\sin(\omega t)\right]}{\left(\frac{Gm^2\omega}{r_0}\right)\left(\frac{4\omega^2 {r_0}^2}{c^2}\right)\left[\gamma^3+\gamma^2\sin(\omega t)\right]} = \frac{1}{4\pi\varepsilon_0 G}\frac{q^2}{4m^2} \tag{2153}$$

If we require the EM and GR *dynamic* power to be equal, then we will have $\left|\frac{\underset{(dynamic)}{\mathscr{P}_{EM}}}{\underset{(dynamic)}{\mathscr{P}_{GR}}}\right| = 1$ which gives $\frac{1}{4\pi\varepsilon_0 G}\frac{q^2}{4m^2} = 1$. This leads to

$$\left|\frac{q}{m}\right| = 2\sqrt{4\pi\varepsilon_0 G} \tag{2154}$$

Notice that this is exactly *twice* the "criticality" condition. Therefore, we find that there is no single "criticality" condition that will make the *total* EM and GR power equal. It is only when $\gamma << 1$ that we will have γ^3 and γ^2 become very small and therefore the total power in (2149) and (2150) will become approximately just the *semi-static* power. Then the "criticality" condition will make the EM and GR power equal. Since we defined $\gamma = \dfrac{2qE_0}{m\omega^2 r_0}$ and the criticality conditions is $\left|\dfrac{q}{m}\right| = \sqrt{\pi\varepsilon_0 G}$, then having $\gamma << 1$ requires that $\dfrac{\omega^2 r_0}{E_0} >> 2\sqrt{4\pi\varepsilon_0 G}$.

Semi-static versus dynamic *forces*

The "criticality" condition, $\left|\dfrac{q}{m}\right| = \sqrt{4\pi\varepsilon_0 G}$, could be described as a "static criticality" condition since only the electro*static* Coulomb force and the gravitationally *static* Newtonian force are made equal by it. However, because the spheres are in motion, then we must consider the full *dynamic* forces acting on the spheres. This leads to what could be referred to as a "dynamic criticality" condition. The *static* and *dynamic* criticality conditions can then be compared.

First, let us recall that in the static case, the net force on q_1 (with mass m_1) due to q_2 (with mass m_2) is given by

$$\underset{(Static)}{\vec{F}_{2\rightarrow 1}} = \underset{(Coulomb)}{\vec{F}_{2\rightarrow 1}} + \underset{(Newton)}{\vec{F}_{2\rightarrow 1}} \tag{2155}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2}{r^2}(-\hat{z}) + G\frac{m^2}{r^2}\hat{z} \tag{2156}$$

Applying the static criticality condition, $\left|\dfrac{q}{m}\right| = \sqrt{4\pi\varepsilon_0 G}$ will give

$$\underset{(Static)}{\vec{F}_{2\rightarrow 1}} = 0 \tag{2157}$$

This is expected, of course, since the static criticality condition was designed to yield this result. In a similar way, we can consider the *dynamic* case by adding the *total* EM and GR Lorentz forces from (2080) and (2119), respectively. This gives

$$\underset{(Total)}{\vec{F}_{2\rightarrow 1}} = \underset{(EM)}{\vec{F}_{2\rightarrow 1}} + \underset{(GR)}{\vec{F}_{2\rightarrow 1}} \tag{2158}$$

$$= q_1\left[-\nabla\varphi_2 - \frac{\partial\vec{A}_2}{\partial t} + \left(\vec{v}_1\times\vec{B}_2\right)\right] + m_1\left[-\nabla\varphi_{G,2} - 4\frac{\partial\vec{h}_2}{\partial t} + 4\left(\vec{v}_1\times\vec{B}_{G,2}\right)\right] \tag{2159}$$

If we impose the static criticality condition, then the first term in each bracket will cancel each other. As described before, the third term in each bracket is zero since the motion of spheres is along the line between them and therefore $\vec{B}_2$ and $\vec{B}_{G,2}$ is zero along that line. This leaves the *dynamic* force as

$$\underset{(Dynamic)}{\vec{F}_{2\rightarrow 1}} = -q_1\frac{\partial\vec{A}_2}{\partial t} - 4m_1\frac{\partial\vec{h}_2}{\partial t} \tag{2160}$$

From (2099) and (2137) we had, respectively,

$$\frac{\partial \vec{A}_2}{\partial t} = \frac{\mu_0 q_2}{4\pi}\left(\frac{r\,\vec{a}_2 - \vec{v}_2 \dot{r}}{r^2}\right) \qquad \text{and} \qquad \frac{\partial \vec{h}_2}{\partial t} = -\frac{\mu_G m_2}{4\pi}\left(\frac{r\,\vec{a}_2 - \vec{v}_2 \dot{r}}{r^2}\right) \tag{2161}$$

We can substitute these into (2160). Also, since the charge and mass of the two spheres are identical, then we can use $q_1 = q_2 = q$ and $m_1 = m_2 = m$.

$$\vec{F}_{(Dynamic)} = \left[-\frac{\mu_0 q^2}{4\pi} + \frac{4\mu_G m^2}{4\pi}\right]\left(\frac{r\,\vec{a}_2 - \vec{v}_2 \dot{r}}{r^2}\right) \tag{2162}$$

In order to make $\vec{F}_{(Dynamic)} = 0$ we would need a *dynamic* criticality condition which makes $-\mu_0 q^2 + 4\mu_G m^2 = 0$. This leads to

$$\frac{q}{m} = \sqrt{\frac{4\mu_G}{\mu_0}} \tag{2163}$$

Making use of $\mu_0 = \dfrac{1}{c^2 \varepsilon_0}$ and $\mu_G = \dfrac{4\pi G}{c^2}$ gives

$$\left|\frac{q}{m}\right| = 2\sqrt{4\pi\varepsilon_0 G} \tag{2164}$$

Once again, we find that *dynamic* criticality condition is *twice* the charge-to-mass ratio of the *static* criticality condition. This is directly attributable to the factor of 4 that is found in the dynamic part of the GR Lorentz force.

21.6 Behavior of the dynamic force

We can return to (2162) and determine an explicit function for $\vec{F}_{(Dynamic)}$ in the case of *static* criticality. Substituting in $\left|\frac{q}{m}\right| = \sqrt{4\pi\varepsilon_0 G}$ as well as the relations $\mu_0 = \frac{1}{\varepsilon_0 c^2}$ and $\mu_G = \frac{4\pi G}{c^2}$ gives

$$\vec{F}_{(Dynamic)} = \frac{1}{4\pi}\left[-\left(\frac{1}{\varepsilon_0 c^2}\right)\left(m\sqrt{4\pi\varepsilon_0 G}\right)^2 + 4\left(\frac{4\pi G}{c^2}\right)m^2\right]\left(\frac{r\vec{a}_2 - \vec{v}_2\dot{r}}{r^2}\right) \tag{2165}$$

$$= \frac{3Gm^2}{c^2}\left(\frac{r\vec{a}_2 - \vec{v}_2\dot{r}}{r^2}\right) \tag{2166}$$

From (2103) we have $(r\vec{a}_2 - \vec{v}_2\dot{r}) = -\left[2r_0 A\omega^2\sin(\omega t) + 4A^2\omega^2\right]\hat{z}$. Also, from (2071) we have $r = r_0 + 2A\sin(\omega t)$. Substituting these into (2166) gives

$$\vec{F}_{(Dynamic)} = -\frac{3Gm^2}{c^2}\left(\frac{2r_0 A\omega^2\sin(\omega t) + 4A^2\omega^2}{\left[r_0 + 2A\sin(\omega t)\right]^2}\right)\hat{z} \tag{2167}$$

Using $A = \frac{qE_0}{m\omega^2}$ from (2077) gives

$$\vec{F}_{(Dynamic)} = -\frac{3Gm^2}{c^2}\left(\frac{2r_0\left(\frac{qE_0}{m\omega^2}\right)\omega^2\sin(\omega t) + 4\left(\frac{qE_0}{m\omega^2}\right)^2\omega^2}{\left[r_0 + 2\left(\frac{qE_0}{m\omega^2}\right)\sin(\omega t)\right]^2}\right)\hat{z} \tag{2168}$$

$$= -\frac{3Gm^2}{c^2}\left(\frac{r_0^2\left(\frac{2qE_0}{m\omega^2 r_0}\right)\omega^2\sin(\omega t) + r_0^2\left(\frac{2qE_0}{m\omega^2 r_0}\right)^2\omega^2}{r_0^2\left[1 + \left(\frac{2qE_0}{m\omega^2 r_0}\right)\sin(\omega t)\right]^2}\right)\hat{z} \tag{2169}$$

Just as before, we define $\gamma = \frac{2qE_0}{m\omega^2 r_0}$. Then we have

$$\boxed{\vec{F}_{(Dynamic)} = -\frac{3Gm^2\omega^2\gamma}{c^2}\left\{\frac{\gamma + \sin(\omega t)}{\left[1 + \gamma\sin(\omega t)\right]^2}\right\}\hat{z}} \tag{2170}$$

As discussed before, we must have $\gamma < 1$ in order for the spheres not to collide and produce a divergence in the power. However, if $\gamma \approx 1$, then the numerator, $\gamma + \sin(\omega t)$, is positive for most values of t. This means that the force spends most of the time pointing in the negative $\hat{z}$ direction for q_1 (and the the positive $\hat{z}$ direction for q_2). This would result in the spheres being pushed further and further apart by the "DC" component of the force. The only compensating force that could keep the spheres oscillating about the foci would be due to the standing TM wave mode which would have nodal regions at the foci. These nodal regions would effectively provide a negative pressure that could help to keep the motion of the spheres centered at the foci.

On the other hand, γ can also be made arbitrarily small. For the case when $\gamma << 1$, then the force approaches

$$\vec{F}_{(Dynamic)} \approx -\frac{3Gm^2\omega^2}{c^2}\gamma\sin(\omega t)\hat{z} \tag{2171}$$

This force does not have a "DC" component at all. However, it could serve to either drive or dampen the motion of the spheres depending on its phase relationship with the oscillatory motion of the spheres. From (2074) we know the force of the TM standing wave is

$$\vec{F}_z(t) = q\vec{E}_{wave}(t) = qE_0\sin(\omega t)\hat{z} \tag{2172}$$

Since q is negative, then the force due to the standing wave will be

$$\vec{F}_z(t) = -|q|E_0\sin(\omega t)\hat{z} \tag{2173}$$

This means that $\vec{F}_{(Dynamic)}$ will be *in phase* with $\vec{F}_z(t)$ and therefore will serve to *drive* the system, not dampen it. This could be beneficial for the purpose of generating more radiation. However, if the spheres are driven too hard relative to any compensating damping force (such as the negative pressure of the nodal regions at the foci), then the oscillation amplitude could grow indefinitely and even lead to collision of the spheres.

21.7 Summary of ideal experimental specifications

Based on all of the analysis shown above, we can summarize the ideal specifications that would be needed to optimize this experiment.

1. To keep the spheres from colliding, we must satisfy the condition $\dfrac{\omega^2 r_0}{2E_0} > \sqrt{4\pi\varepsilon_0 G}$.

   This means the following:

   - The distance between the foci of the ellipse must satisfy $r_0 > \dfrac{2E_0\sqrt{4\pi\varepsilon_0 G}}{\omega^2}$.
   - The amplitude of the electric field of the standing wave must satisfy $E_0 < \dfrac{\omega^2 r_0}{2\sqrt{4\pi\varepsilon_0 G}}$.
   - The angular frequency of the standing wave must satisfy $\omega^2 > \dfrac{2E_0\sqrt{4\pi\varepsilon_0 G}}{r_0}$.

2. To reduce "sloshing" in the transfer of energy, we need $\dfrac{\omega^2 r_0}{E_0} >> 2\sqrt{4\pi\varepsilon_0 G}$. This means it is ideal to have the highest angular frequency possible and lowest power possible for the standing TM wave. It is also ideal to have r_0 as large as possible which means using the largest possible ellipsoid with the greatest eccentricity.

3. To produce the greatest amount of EM and GR radiation, the spheres should be charged as much as possible. This can be seen from the way that the EM power and GR power both scale with the charge.

   - $\underset{(semi-static)}{\mathscr{P}_{EM}} \sim q^3$ and $\underset{(dynamic)}{\mathscr{P}_{EM}} \sim q^4$
   - $\underset{(semi-static)}{\mathscr{P}_{GR}} \sim q$ and $\underset{(dynamic)}{\mathscr{P}_{GR}} \sim q^2$

4. Keep the charge-to-mass ratio constant by using more mass in proportion to using more charge. The EM and GR power scale with lower powers of the mass than they do with charge. This means the additional inertia that would inhibit motion will be superseded by the additional radiation from the added charge. This can be seen from the way that the EM power and GR power both scale with the mass.

   - $\underset{(semi-static)}{\mathscr{P}_{EM}} \sim m^{-1}$ and $\underset{(dynamic)}{\mathscr{P}_{EM}} \sim m^{-2}$
   - $\underset{(semi-static)}{\mathscr{P}_{GR}} \sim m$ and $\underset{(dynamic)}{\mathscr{P}_{GR}}$ doesn't scale with mass

   Note also that mass is essentially "gravitational charge" which means that the greater the mass, the greater the gravitational radiation.

5. The charge-to-mass ratio should satisfy the "criticality" condition, $\left|\dfrac{q}{m}\right| = \sqrt{4\pi\varepsilon_0 G}$ to insure the strongest coupling to EM radiation while also producing the greatest GR radiation. This condition will make the EM and GR power approximately equal. Since using the greatest amount of charge ideal, the only limitation would be the amount of mass that could be levitated at the foci of the ellipsoid and kept stable while oscillating.

6. If the *dynamic* force (EM and GR) acting on the spheres has a “DC” component that is too large, then it will push the spheres further and further apart. In that case, try to make $\frac{\omega^2 r_0}{E_0} >> 2\sqrt{4\pi\varepsilon_0 G}$.

7. If the *dynamic* force is too close to a pure sinusoid that is in phase with the standing TM wave, then it will drive the spheres in resonance which will increase the amplitude of the motion and could cause the spheres to collide. In that case, try to reduce $\frac{\omega^2 r_0}{E_0}$ so that it is closer to $2\sqrt{4\pi\varepsilon_0 G}$.

21.7 Numerical results of power delivered between the charged masses

The following are some numerical estimates based on the relations developed in this memo. All calculations assume that a charge-to-mass ratio satisfying criticality. The suggested arrangement is an ellipsoid with a 1cm spacing between the foci and a milliwatt of microwave (30GHz) power for the standing TM wave. First we can check if these parameters are compatible with the restrictions of the system as described previously in this memo.

- Allowable TM wave power

  For a microwave frequency of 30GHz , the *angular* frequency will be

$$\omega = 2\pi f = 2\pi \left(30 \times 10^9\right) \approx 1.88 \times 10^{11} rad/s \tag{2174}$$

  For a separation distance between the foci of $1cm$, the amplitude of the electric field of the standing wave must satisfy

$$E_0 \quad < \quad \frac{\omega^2 r_0}{2\sqrt{4\pi\varepsilon_0 G}} \approx \frac{\left(1.88 \times 10^{11}\right)^2 \left(10^{-2}\right)}{2\left(8.61 \times 10^{-11}\right)} N/C \approx 1.32 \times 10^{27} N/C \tag{2175}$$

  This corresponds to a power of

$$P \quad = \quad \varepsilon_0 E^2 c \approx \left(8.85 \times 10^{-12}\right)\left(1.32 \times 10^{27}\right)\left(3.00 \times 10^8\right) W \approx 4.62 \times 10^{51} W \tag{2176}$$

  This is obviously *far* above any laboratory power. Therefore, there is no risk of using too much power to drive the spheres. Using a milliwatt of power is far below what this system can tolerate.

- Allowable frequency of the TM wave

  For a milliwatt of TM wave power, the corresponding electric field amplitude is

$$E_0 \quad = \quad \sqrt{\frac{P}{\varepsilon_0 c}} \approx \sqrt{\frac{10^{-6}}{(8.85 \times 10^{-12})(3.00 \times 10^8)}} N/C \approx 1.94 \times 10^{-2} N/C \tag{2177}$$

  Using a separation distance between the foci of $1cm$, the angular frequency of the the standing wave must satisfy

$$\omega \quad > \quad \sqrt{\frac{2E_0\sqrt{4\pi\varepsilon_0 G}}{r_0}} \approx \sqrt{\frac{2\left(1.94 \times 10^{-2}\right)\left(8.61 \times 10^{-11}\right)}{10^{-2}}} \approx 1.83 \times 10^{-5} rad/s \tag{2178}$$

  This corresponds to a frequency of

$$f = \frac{\omega}{2\pi} \approx 2.87 \times 10^{-5} Hz \tag{2179}$$

  This is about 9 cycles per hour which is obviously *far* slower than any frequency of interest. Therefore, there is no risk of driving the system too slowly.

- Allowable distance between the ellipsoid foci

  For a milliwatt of 30GHz microwave power, the distance between the foci of the ellipse must satisfy

$$r_0 \quad > \quad \frac{2E_0\sqrt{4\pi\varepsilon_0 G}}{\omega^2} \approx \frac{2\left(1.94\times10^{-2}\right)\left(8.61\times10^{-11}\right)}{\left(1.88\times10^{11}rad/s\right)^2}m \approx 1.47\times10^{-31}m \tag{2180}$$

  This is obviously *far* smaller than any laboratory length scale. Therefore, there is no risk of having a cavity that is too small. The smaller the cavity can be made, the better the results will be.

We have now established that the use of one milliwatt of microwave power at 30GHz with a separation distance of 1cm is completely within the bounds permitted by this system. Next we can calculate the maximum displacement, speed, and acceleration of the spheres. We can also calculate the maximum EM and GR power between the spheres.

- Maximum displacement of the spheres

  The amplitude of oscillation for the spheres was found in (2077) to be

$$A \quad = \quad \frac{qE_0}{m\omega^2} \tag{2181}$$

  Using the criticality condition, $\left|\frac{q}{m}\right| = \sqrt{4\pi\varepsilon_0 G}$, this becomes

$$A \quad = \quad \frac{E_0\sqrt{4\pi\varepsilon_0 G}}{\omega^2} \tag{2182}$$

  For a milliwatt TM wave power, the corresponding electric field was found above to be $E_0 \approx 1.94\times10^{-2}N/C$. Also, for a microwave frequency of 30GHz, the *angular* frequency was found to be $1.88\times10^{11}rad/s$. Then the amplitude of oscillation is

$$A \quad \approx \quad \frac{\left(1.94\times10^{-2}\right)\left(8.61\times10^{-11}\right)}{\left(1.88\times10^{11}rad/s\right)^2}m \approx 4.73\times10^{-35}m \tag{2183}$$

  Since the spheres will move through a distance of $2A$, then the maximum displacement will be $9.45\times10^{-35}m$. This is just about the Planck length.

- Maximum speed of the spheres

  From (2072) we know that the maximum speed of either sphere[249] is $|\vec{v}| = A\omega$. Using the amplitude found above, we have

$$|\vec{v}| \quad \approx \quad 2\left(7.52\times10^{-34}\right)\left(1.88\times10^{11}\right)m/s \approx 2.82\times10^{-22}m/s \tag{2184}$$

[249] We do not include the factor of 2 here because this factor was associated with the velocity and acceleration of one sphere *with respect to the other sphere*. In this calculation we are interested in the speed and acceleration with respect to the cavity (or lab frame).

- <u>Maximum acceleration of the spheres</u>

  From (2072) we know that the maximum acceleration of either sphere is $|\vec{a}| = A\omega^2$. Using the amplitude found above, we have

$$|\vec{v}| \approx 2\left(7.52\times10^{-34}\right)\left(1.88\times10^{11}\right)^2 m/s^2 \approx 5.32\times10^{-11} m/s^2 \tag{2185}$$

These results are obviously extremely small. However, it was shown on the previous page that this system would permit much higher power from the TM standing wave, or a much lower frequency, or a much larger separation between the foci. Making any of these adjustments would increase the displacement, speed, and acceleration of the spheres. However, we will continue with the values selected above for the calculation of the maximum EM or GR power.

- <u>Maximum EM or GR power due to oscillation of the spheres</u>

  The total EM and GR power from (2115) and (2146) are, respectively,

$$\mathscr{P}_{EM} = \left(\frac{1}{4\pi\varepsilon_0}\frac{q^2\omega}{r_0}\right)\frac{\cos(\omega t)}{\left[1+\gamma\sin(\omega t)\right]^2}\left\{\gamma - \left(\frac{\omega^2 r_0{}^2}{c^2}\right)\left[\gamma^3+\gamma^2\sin(\omega t)\right]\right\} \tag{2186}$$

and

$$\mathscr{P}_{GR} = -\left(\frac{Gm^2\omega}{r_0}\right)\frac{\cos(\omega t)}{\left[1+\gamma\sin(\omega t)\right]^2}\left\{\gamma - \left(\frac{4\omega^2 r_0{}^2}{c^2}\right)\left[\gamma^3+\gamma^2\sin(\omega t)\right]\right\} \tag{2187}$$

where $\gamma = \dfrac{2qE_0}{m\omega^2 r_0}$. If we use the criticality condition, $\left|\dfrac{q}{m}\right| = \sqrt{4\pi\varepsilon_0 G}$, then we have

$$\gamma = \frac{2E_0\sqrt{4\pi\varepsilon_0 G}}{\omega^2 r_0} \approx \frac{2\left(1.94\times10^{-2}\right)\left(8.61\times10^{-11}\right)}{\left(1.88\times10^{11} rad/s\right)^2\left(10^{-2}\right)} \approx 9.45\times10^{-33} \tag{2188}$$

Since $\gamma << 1$, then γ^2 and γ^3 are negligible and we only need the first term in the braces of (2186) or (2187). In other words, the total power will become approximately just the *semi-static* power. It was also shown that for the criticality condition, the *semi-static* EM and GR power are equal so we can just calculate one of them. Using the expression for $\mathscr{P}_{GR}$, we then have

$$\mathscr{P}_{GR} = -\left(\frac{Gm^2\omega}{r_0}\right)\frac{\gamma\cos(\omega t)}{\left[1+\gamma\sin(\omega t)\right]^2} \tag{2189}$$

For $\gamma << 1$, the denominator of this expression becomes $\left[1+\gamma\sin(\omega t)\right]^2 \approx 1$. Also, to find the maximum power, we can take $\cos(\omega t) = 1$. This gives

$$\mathscr{P}_{GR,\,max} \approx \frac{Gm^2\omega\gamma}{r_0} \tag{2190}$$

If the spheres are chosen to be a *Planck* mass, then we can use (2068) to substitute $m = \sqrt{\dfrac{\hbar c}{G}}$. This gives

$$\mathscr{P}_{GR,\,max} \approx \frac{\hbar c\omega\gamma}{r_0} \approx \frac{\left(1.05\times10^{-34}\right)\left(3.00\times10^{8}\right)\left(1.88\times10^{11}rad/s\right)\left(9.45\times10^{-33}\right)}{10^{-2}}W$$

$$\mathscr{P}_{GR,\,max} \approx 5.60\times10^{-45}W \tag{2191}$$

This is obviously a negligible result. It represents the total power fluctuations (either EM or GR) that would occur between the two spheres.

22 Transmitter and receiver cavities as a GR wave communication system

22.1 Configuring “transmitter” and “receiver” ellipsoid cavities

We now consider the case of *two* ellipsoid configurations that can communicate with each other – one acting as a “transmitter” and the other as a “receiver.” We will *not* assume a charge-to-mass ratio satisfying the “criticality” condition. We also do not need to assume that the ellipsoids are necessarily identical. In fact, it will be shown that the specifications for the *transmitter* must be different from the specifications for the *receiver*.For the case of the *transmitter* cavity, by exciting a TM standing wave, the charged spheres are

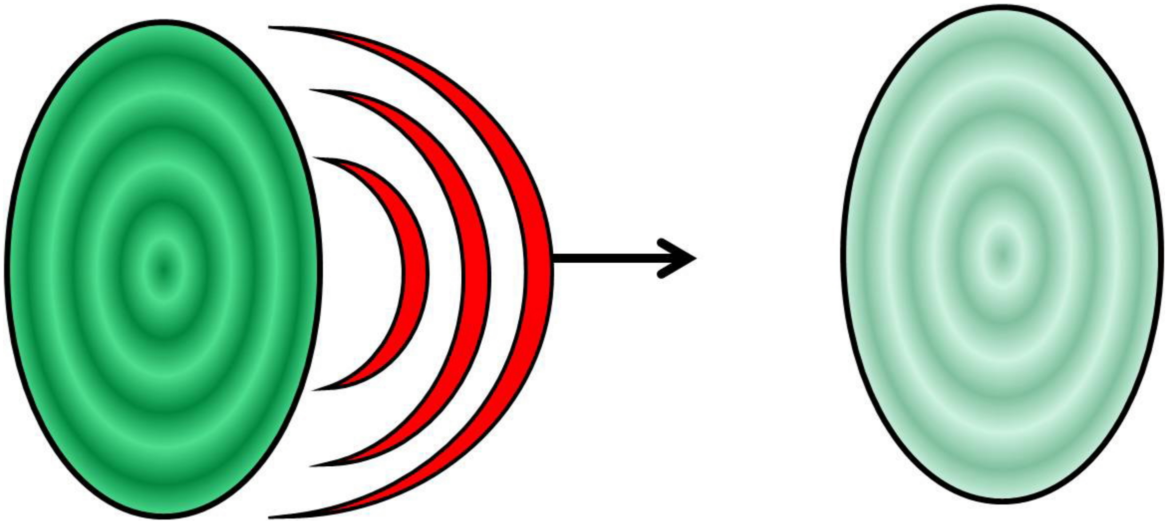

Figure 25: A transmitter-receiver gravitational wave communication system. The dark green pattern in the transmitter cavity depicts the standing TM wave that is excited in the cavity. The charged masses at the foci are driven into oscillation as shown by the arrows representing their motion. (Note the *solid* arrows and *dotted* arrows represent the fact that the motion is anti-symmetric. This quadrupole motion of the masses generates GR waves. The wave fronts are shown in red as they propagate toward the receiver cavity. The energy falls off as $1/r^2$ but remains sufficient to drive the charged masses at the foci of the *receiver* cavity into oscillation. As these charged masses oscillate, they will produce small but detectable EM radiation which can be detected in the receiver cavity. This EM radiation is depicted as a lighter-colored green pattern in the receiver cavity.

driven into quadrupolar motion which allows them to radiate gravitationally. In that sense, the spheres could be considered as a transducer that is powered by an EM wave and produces GR waves of the same frequency. The GR waves will propagate out of the transmitter cavity and travel toward the *receiver* cavity. In this process, the energy of a GR wave would diffuse over a spherical shell which would cause the energy to scale by the inverse square law as it propagates away from the *transmitter* cavity.

When the GR wave enters the *receiver* cavity, the acceleration field of the GR wave will oscillate the two spherical masses at the foci of the ellipsoid with the same frequency as the wave. The spheres will then generate EM and GR waves inside the *receiver* cavity. In other words, the spheres will be acting as a transducer which is powered by an incident GR wave and produces EM waves as a result. We could then measure the resulting EM radiation that is produced in the receiver cavity.

Power Delivered by Transmitter Cavity

Inside the “transmitter” cavity, a standing TM wave is used to oscillate the charged spheres. The spheres would then produce GR waves given again by Einstein’s gravitational quadrupolar formula.[91]

$$\mathscr{P}_{GR}^{(quad)} = \frac{G}{45c^5}\left\langle \dddot{D}_{ij}^2 \right\rangle \tag{2192}$$

where $\left\langle \dddot{D}_{ij}^2 \right\rangle$ is the square of the third time-derivative of the mass quadrupole-moment tensor time-averaged over a period. In (3025) from Appendix T, the third time-derivative of the mass quadrupole moment for the system of two point masses oscillating in anti-symmetric directions along the line between them was found to be

$$\dddot{D} = 2m\left(3\dot{\vec{r}}\cdot\ddot{\vec{r}}+\vec{r}\cdot\dddot{\vec{r}}\right) \tag{2193}$$

where $\vec{r}$ is the displacement between the spheres. Using (2071) for the displacement between the spheres, we have

$$r = r_0+2A\sin(\omega t), \qquad \dot{r}=2A\omega\cos(\omega t), \qquad \ddot{r}=-2A\omega^2\sin(\omega t), \qquad \dddot{r}=-2A\omega^3\cos(\omega t) \tag{2194}$$

Substituting these into (2193) gives

$$\dddot{D} = 2m\left\{3\left[2A\omega\cos(\omega t)\right]\left[-2A\omega^2\sin(\omega t)\right]+\left[r_0+2A\sin(\omega t)\right]\left[-2A\omega^3\cos(\omega t)\right]\right\} \tag{2195}$$

$$= -2m\left[12A^2\omega^3\cos(\omega t)\sin(\omega t)+2r_0A\omega^3\cos(\omega t)+4A^2\omega^3\sin(\omega t)\cos(\omega t)\right] \tag{2196}$$

$$= -4mA\omega^3\left[8A\cos(\omega t)\sin(\omega t)+r_0\cos(\omega t)\right] \tag{2197}$$

Squaring this gives

$$\dddot{D}^2 = 16m^2A^2\omega^6\left[64A^2\cos^2(\omega t)\sin^2(\omega t)+r_0^2\cos^2(\omega t)+8Ar_0\cos^2(\omega t)\sin(\omega t)\right] \tag{2198}$$

Next we take the time average over a period. Since $\left\langle\sin^2(\omega t)\right\rangle=\left\langle\cos^2(\omega t)\right\rangle=1/2$ and also $\langle\sin(\omega t)\rangle=0$, then we have

$$\left\langle\dddot{D}^2\right\rangle = 16m^2A^2\omega^6\left[16A^2+r_0^2/2\right] \tag{2199}$$

If the amplitude of the oscillation is small relative to the distance between the spheres in the cavity, then $r_0 >> A$ which gives[250]

$$\left\langle\dddot{D}^2\right\rangle = 8m^2A^2r_0^2\omega^6 \tag{2200}$$

We can substitute this result back into (2192) to obtain the GR wave power from the transmitter cavity.

$$\underset{(Transmitter\ ouput)}{\mathscr{P}_{GR}} = \frac{2G}{45c^5}m_T^2A_T^2r_0^2\omega^6 \tag{2201}$$

[250]Notice that this approximation removes the term with A^4 and leaves only the term with A^2. This is equivalent to requiring a *linear response* between fields and sources. In other words, since the power is proportional to the square of the field $\left(\mathscr{P}_{GR}\sim\vec{h}^2 \text{ and } \mathscr{P}_{EM}\sim\vec{E}^2\right)$ and the fields are proportional to the amplitude of motion ($A_{GR}\sim g$ and $A_{EM}\sim E$ by Newton's Second Law as seen in (2202) and (2216) for GR and EM waves, respectively) then we should also have the power proportional to the square of the amplitude of motion $\left(\mathscr{P}_{GR}\sim A_{EM}^2 \text{ and } \mathscr{P}_{EM}\sim A_{GR}^2\right)$, not the *fourth power* of the amplitude of motion.

where m_T is the mass of either sphere in the transmitter cavity. According to (2077), the amplitude of the standing TM wave, E_0, determines the amplitude of oscillation for the spheres. So we obtain

$$A_T = \frac{q_T E_0}{m_T \omega^2} \tag{2202}$$

where q_T is the charge on a sphere in the transmitter cavity. Substituting (2202) into (2201) gives

$$\underset{(Transmitter\ ouput)}{\mathscr{P}_{GR}} = \frac{2G}{45c^5} q_T^2 {E_0}^2 {r_0}^2 \omega^2 \tag{2203}$$

If the power injected in the cavity is $\mathscr{P}_{TM}$, then the power delivered to the charged spheres at the foci will be $Q\mathscr{P}_{TM}$ where Q is the quality factor of the resonator. This power is related to the electric field by $Q\mathscr{P}_{TM} = \varepsilon_0 c {E_0}^2 A_\perp$, where $A_\perp = 2\left(\pi a_T^2\right)$ is the total cross-sectional area receiving the power, with the factor of 2 taking into account *both* transmitter spheres. Then we have

$${E_0}^2 = \frac{Q\mathscr{P}_{TM}}{2\pi\varepsilon_0 c a_T^2} \tag{2204}$$

Then we can write (2203) as

$$\underset{(Transmitter\ ouput)}{\mathscr{P}_{GR}} = \frac{G}{45\pi\varepsilon_0 c^6} \left(\frac{r_0 \omega q_T}{a_T}\right)^2 Q\mathscr{P}_{TM} \tag{2205}$$

Power Loss Between the Cavities

Between the cavities, the GR wave would be traveling the distance that the cavities are separated by. If we call this distance, d, then the GR power could be considered as spread over a spherical shell with area $4\pi d^2$. The power incident on the receiver spheres (each of radius a_R) would be incident on one side of *both* spherical surfaces, which means over a total area of $2\left(\pi a_R^2\right)$. Therefore, the power incident on the two spheres due to the wave propagating from the transmitter spheres to the receiver spheres can be found by proportionately.

$$\underset{(Receiver\ input)}{\mathscr{P}_{GR}} = \underset{(Transmitter\ ouput)}{\mathscr{P}_{GR}} \frac{2\pi a_R^2}{4\pi d^2} \tag{2206}$$

$$= \underset{(Transmitter\ ouput)}{\mathscr{P}_{GR}} \left(\frac{a_R^2}{2d^2}\right) \tag{2207}$$

Substituting in (2205) gives

$$\underset{(Receiver\ input)}{\mathscr{P}_{GR}} = \frac{G}{90\pi\varepsilon_0 c^6} \left(\frac{r_0 \omega a_R q_T}{a_T d}\right)^2 Q\mathscr{P}_{TM} \tag{2208}$$

Power Detected by Receiver Cavity

The gravitational wave incident on the *receiver* cavity will oscillate the charged spheres and cause them to produce quadrupolar EM radiation. The formula for the power of quadrupolar EM radiation is[91]

$$\mathscr{P}_{EM}^{(quad)} = \frac{K}{45c^5} \left\langle \dddot{Q}_{ij}^2 \right\rangle \tag{2209}$$

where $K=\frac{1}{4\pi\varepsilon_0}$ and $\left\langle \dddot{Q}_{ij}^2 \right\rangle$ is the square of the third time-derivative of the *charge* quadrupole-moment tensor time-averaged over a period. In (3027) from Appendix T, the third time-derivative of the charge quadrupole-moment for this system was found to be

$$\dddot{Q}_m \quad = \quad 2q\left(3\dot{\vec{r}}\ddot{\vec{r}}+\vec{r}\cdot\dddot{\vec{r}}\right) \tag{2210}$$

Using the same process that led to (2200), we would similarly have

$$\left\langle \dddot{Q}^2 \right\rangle \quad = \quad 8q^2A^2{r_0}^2\omega^6 \tag{2211}$$

We can substitute this result back into (2209) to obtain the EM wave power from the receiver cavity.

$$\underset{(Receiver\ output)}{\mathscr{P}_{EM}} \quad = \quad \frac{8K}{45c^5}q_R^2A_R^2{r_0}^2\omega^6 \tag{2212}$$

where q_R is the charge on either sphere in the receiver cavity.[251] To find the amplitude of motion, A_R, we can ignore the small interaction between the charges and consider the spheres as being oscillated only by the GR wave. Therefore, the acceleration of each charge due to this wave can be found from Newton's Second law.

$$\vec{F}_z(t)=m_R\mathbf{g}(t)=m_R\vec{a}_R(t) \tag{2213}$$

where $\mathbf{g}(t)$ is the acceleration field[252] of the GR wave at the receiver cavity. The acceleration of one sphere is

$$\vec{a}_R(t) \quad = \quad A_R\omega^2\sin(\omega t)\hat{z} \tag{2214}$$

Since $\mathbf{g}(t)$ and $\vec{a}_R(t)$ must be in phase, then we must have

$$\mathbf{g}(t)=g\sin(\omega t)\hat{z} \tag{2215}$$

where g is the amplitude of the GR wave at the receiver cavity. Matching $\mathbf{g}(t)$ and $\vec{a}_R(t)$ requires that we have $A_R\omega^2=g$. Then the amplitude of oscillation of the receiver spheres is

$$A_R \quad = \quad \frac{g}{\omega^2} \tag{2216}$$

Substituting (2216) into (2212) gives

$$\underset{(Receiver\ output)}{\mathscr{P}_{EM}} \quad = \quad \frac{8K}{45c^5}q_R^2g^2{r_0}^2\omega^2 \tag{2217}$$

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[251]This is consistent with [11], p. 978 where it is stated that the gravitational wave "luminosity" is roughly the square of the reduce quadrupole moment multiplied by G/c^5.

[252]This is identical to the *gravito-electric field*, E_g, from the gravito-electromagnetic framework described in Part I of this memo.

We can relate the wave amplitude g to the power of the GR wave using $\mathscr{P}_{GR} = \varepsilon_G c g^2 A$ where A is the cross-sectional area receiving the power analogous to $\mathscr{P}_{EM} = \varepsilon_0 c E_0^2 A$ for electromagnetism.[253] Here we are using $\varepsilon_G = \dfrac{1}{4\pi G}$ as the gravitational analog to the permittivity of free space. Since the power is delivered to one side of each of the receiver spheres (each of radius a_R), then there is a total area of approximately $A = 2\left(2\pi a_R^2\right)$. So the wave amplitude is

$$g^2 = \frac{G}{c a_R^2} \underset{(Receiver\ input)}{\mathscr{P}_{GR}} \tag{2218}$$

and (2217) becomes

$$\underset{(Receiver\ output)}{\mathscr{P}_{EM}} = \frac{2KG}{45c^6}\left(\frac{r_0 \omega q_R}{a_R}\right)^2 \underset{(Receiver\ input)}{\mathscr{P}_{GR}} \tag{2219}$$

To relate this back to the initial EM wave power provided to the transmitter cavity we can substitute for $\mathscr{P}_{GR\ (Transmitter\ ouput)}$ using (2208). Also, using $K = \dfrac{1}{4\pi\varepsilon_0}$ gives

$$\underset{(Receiver\ output)}{\mathscr{P}_{EM}} = \frac{G}{90\pi\varepsilon_0 c^6}\left(\frac{r_0 \omega q_R}{a_R}\right)^2 \left[\frac{G}{90\pi\varepsilon_0 c^6}\left(\frac{r_0 \omega a_R q_T}{a_T d}\right)^2 Q \mathscr{P}_{TM}\right] \tag{2220}$$

$$\boxed{\underset{(Receiver\ output)}{\mathscr{P}_{EM}} = \frac{G^2}{8100\varepsilon_0^2 c^{12}} {r_0}^4 \omega^4 \left(\frac{q_R\, q_T}{a_T d}\right)^2 Q P_{TM}} \tag{2221}$$

This result now describes the power of the EM wave detected in the *receiver* cavity in terms of the power of the standing TM wave in the *transmitter* cavity. Note that this equation is only valid for $r_0 >> A$ which means sufficiently small g and/or large ω since $A = g/\omega^2$.Observations concerning receiver power expression

We can make the following observations concerning the output power detected in the receiver cavity as shown in (2241) to be

$$\underset{(Receiver\ output)}{\mathscr{P}_{EM}} = \frac{G^2}{8100\varepsilon_0^2 c^{12}} {r_0}^4 \omega^4 \left(\frac{q_R\, q_T}{a_T d}\right)^2 Q P_{TM} \tag{2222}$$

- Power scales linearly with QP_{TM}.

  This is expected for a linearly driven system. A linear response of the oscillation to the field amplitude was assumed when applying Newton's Second Law in (2074) and (2213).

[253]In describing the power of the gravitational wave, we are choosing the lab frame where the receiver masses are accelerating. In principle, it is always possible to transform to the local inertial frame of these masses where an observer is co-accelerating with the masses in "freefall." In that frame there is no gravitational field and hence no gravitational wave power. (This is emphasized in MTW [11], pp. 466-468.) However, in that frame, we would also find that the amplitude of the electromagnetic wave vanishes as well as since (2077) relates the amplitude of the wave to the accleration of the sphere which would be measured to be zero in its local inertial frame.

- Power scales quadratically with the charge of the transmitter/receiver spheres, q_T and q_R.

  This is due to the fact that more charge on the transmitter spheres will cause the spheres to be coupled more strongly to the standing EM wave in the transmitter cavity and hence they will oscillate with a greater amplitude. This will in turn generate stronger gravitational waves which will oscillate the receiver spheres with a greater amplitude as well and hence produce more EM power in the receiver cavity. Also, the more charge on the receiver spheres, the more EM wave power they will produce for a given oscillation amplitude. Therefore, this will also produce more EM power in the receiver cavity.

- Power scales as the inverse square of the radius of the transmitter spheres, a_T.

  This is a result of (2204) which shows that $a_T^2 E_0 \sim \mathscr{P}_{TM}$. Therefore, with a given input power $\mathscr{P}_{TM}$, a smaller radius a_T will require a larger wave amplitude E_0 to keep the same input power $\mathscr{P}_{TM}$. Since there will be a larger wave amplitude E_0, then the amplitude of motion of the transmitter spheres will be larger and hence will produce GR waves with greater power.

- Power does not depend on the radius of the receiver spheres, a_R.

  This is a result of (2204) which shows that $a_R^2 g \sim \mathscr{P}_{GR}$. Therefore, with a given input power $\mathscr{P}_{GR}$, a smaller radius a_R will require a larger wave amplitude g to keep the same input power $\mathscr{P}_{GR}$. Since there will be a larger wave amplitude g, then the amplitude of motion of the transmitter spheres will be larger and hence will produce GR waves with greater power. This would lead to a inverse square dependence $\left(1/a_R^2\right)$ like we have for a_T. However, from (2207) we see that the power of the input GR wave that is incident on the receiver spheres depends on a_R^2 therefore the net result is that the receiver output power does not depend on a_R at all.

- Power depends on the radius of the transmitter spheres.

  This is due to the fact that the larger the radius of the spheres, the more surface there is to be driven by a wave, whether an EM or a GR wave. Therefore, having large transmitter or receiver spheres results in more EM power in the receiver cavity.

- Power does not depend on the mass of the transmitter spheres.

  This is due to the fact that the mass was eliminated from the expression in (2201) for the gravitational wave power emitted by the transmitter spheres. The GR wave power scales as $\mathscr{P}_{GR} \sim m_T^2 A_T^2$ and the amplitude of motion in (2202) due to the spheres being driven by the TM wave scales as $A \sim 1/m_T$. The net result is that the GR wave power is not dependent on the mass of the transmitter spheres.

- The power does not depend on the mass of the transmitter spheres.

  This is due to the fact that the mass was eliminated from the expression in (2201) for the gravitational wave power emitted by the transmitter spheres. The GR wave power scales as $\mathscr{P}_{GR} \sim m_T^2 A_T^2$ and the amplitude of motion in (2202) due to the spheres being driven by the TM wave scales as $A \sim 1/m_T$. The net result is that the GR wave power is not dependent on the mass of the transmitter spheres.

- The power does not depend on the mass of the receiver spheres.

  This is due to the fact that the expression in (2212) for the EM power emitted by the receiver spheres does not contain the mass. Also the amplitude of motion in (2216) due to the spheres being driven by the GR wave also does not depend on mass. This is a result of the Equivalence Principle which was implicitly used when the gravitational mass of the spheres was canceled with the inertial mass of the spheres in going from (2213) to (2216). In other words, when the receiver spheres experience “free fall” motion in the presence of the gravitational field of the GR wave, the mass of the spheres becomes irrelevant to the motion. The net result is that the GR wave power is not dependent on the mass of the transmitter spheres. Therefore, the final EM power in the receiver cavity does not depend on the mass of either the transmitter spheres or the receiver spheres.

- The receiver power does *not* scale with the charge-to-mass ratio of either set of spheres.

  This is obviously because the receiver power does not depend on the mass of any of the spheres at all. This indicates that the "criticality" charge-to-mass ratio given by (2067) is not necessarily the ideal choice for maximizing the receiver power. Rather, the greater the charge that is used (regardless of the mass of the spheres), the more the final power produced in the receiver cavity will be.

- The power scales with the squared inverse of the distance between the cavities, $\left(1/d^2\right)$.

  This is simply a result of (2207) where we use the fact that the inverse-square law for energy propagates away from the source and diffuses over the surface of a sphere which has a surface area $4\pi r^2$.

- The power scales with the fourth power of the angular frequency, ω^4.

  This is intuitive since it is expected that the more rapidly the standing TM wave oscillates the transmitter spheres, the greater the final signal will be in the receiver cavity. Mathematically, this result arises in going from (2201) to (2203) where it can be seen that the amplitude scales as $A \sim \omega^{-2}$ and the GR wave power scales as $\mathscr{P}_{GR} \sim A^2\omega^6$ so that the net result is that $\mathscr{P}_{GR} \sim \omega^2$. Once again it is due to the Equivalence Principle which relates the amplitude of oscillation to the amplitude of the GR wave by $A = \dfrac{h_0}{\omega^2}$.

22.2 The receiver power in terms of experimental parameters

We can reformulate the receiver power expression from (2221) in terms of experimental parameters that would be relevant to an actual laboratory arrangement. First, we can write the receiver power expression in terms of ρ, the *mass density* of the spheres, where $m_T = \rho V_T = \rho\left(\frac{4}{3}\pi a_T^3\right)$. Then (2221) becomes

$$\underset{(Receiver\ output)}{\mathscr{P}_{EM}} = \frac{QP_{TM}}{90\pi\varepsilon_0^2 c^6}(q_R r_0 \omega)^2 \left(\frac{3q_T}{4\pi\rho a_T^3}\right)^2 \left(\frac{a_R}{d}\right)^2 \tag{2223}$$

$$= \frac{QP_{TM}}{160\pi^3\varepsilon_0^2 c^6}\left(\frac{r_0\omega a_R q_R q_R}{\rho d a_T^3}\right)^2 \tag{2224}$$

We can also convert the charges into voltages by considering the energy of a uniformly charged spherical shell of radius a and charge q. Using equation (2.45) from [29] we have the electrostatic work associated with assembling a total charge q is

$$W = \frac{\varepsilon_0}{2}\int_{all\ space} E^2 dV \tag{2225}$$

where E is the electric field associated with the charge distribution. Since the field inside a conductor is zero and the field outside a spherical charged conductor is $E = \dfrac{q}{4\pi\varepsilon_0 r^2}$, then the work required to assemble the charge q is

$$W = \frac{\varepsilon_0}{2}\int_a^\infty \left(\frac{q}{4\pi\varepsilon_0 r^2}\right)^2 4\pi r^2 dr = \frac{\varepsilon_0}{2}\frac{4\pi q^2}{(4\pi\varepsilon_0)^2}\int_a^\infty \frac{1}{r^2}dr = \frac{\varepsilon_0}{2}\frac{4\pi q^2}{(4\pi\varepsilon_0)^2}\left[-\frac{1}{r}\right]_a^\infty = \frac{q^2}{8\pi\varepsilon_0 a} \tag{2226}$$

Since the energy of this configuration can also be written as $W = qV$, then we have

$$qV = \frac{q^2}{8\pi\varepsilon_0 a} \qquad \Longrightarrow \qquad q = 8\pi\varepsilon_0 a V \tag{2227}$$

Now we can write the charge in terms of the voltage for the transmitter spheres and receiver spheres, respectively, as

$$q_T = 8\pi\varepsilon_0 a_T V_T \qquad \text{and} \qquad q_R = 8\pi\varepsilon_0 a_R V_R \tag{2228}$$

Substituting these into (2224) gives

$$\underset{(Receiver\ output)}{\mathscr{P}_{EM}} = \frac{QP_{TM}}{160\pi^3\varepsilon_0^2 c^6}\left[\frac{r_0\omega a_R (8\pi\varepsilon_0 a_R V_R)(8\pi\varepsilon_0 a_T V_T)}{\rho d a_T^3}\right]^2 \tag{2229}$$

Simplifying gives

$$\boxed{\underset{(Receiver\ output)}{\mathscr{P}_{EM}} = \kappa\left(\frac{r_0\omega}{\rho d}\frac{a_R^2}{a_T^2}V_r V_t\right)^2 QP_{TM}} \tag{2230}$$

where $\kappa = \dfrac{8\pi\varepsilon_0^2}{5c^6} \approx \dfrac{8\pi\left(8.85\times 10^{-12}\right)^2}{5\left(3.00\times 10^8\right)^6} \approx 5.40\times 10^{-73}$ in SI units.

It is a challenge to find numerical values for the parameters of this system that are experimentally tenable and yet satisfy the expression in (2230). The parameters can be classified in terms of “hard” parameters which are necessary constraints defined by limits which cannot be exceeded, and "soft" parameters which can be adjusted to allow the system to operate the most effectively. (This is classification is somewhat artificial but is helpful in determining what combination of numeric values can lead to a realizable experiment.) The "hard" parameters are dictated by the following requirements:

Condition 1. The oscillation amplitude and separation distance must satisfy $A << r_0$**.**

For the *linear* driving response approximation that was used, the amplitude of oscillation must be significantly smaller than the separation distance of the spheres. The amplitude is given in (2202) as $A = \frac{q_T E_0}{m_T \omega^2}$. We can write this in terms of experimental parameters using $m_T = \rho V_T = \rho\left(\frac{4}{3}\pi a_T^3\right)$ as well as ${E_0}^2 = \frac{QP_{TM}}{\varepsilon_0 c}$ from (2204) and $q_T = 4\pi\varepsilon_0 a_T V_T$ from (2228). This gives

$$A = \frac{q_T E_0}{m_T \omega^2} = \frac{(4\pi\varepsilon_0 a_T V_T)}{\rho\left(\frac{4}{3}\pi a_T^3\right)\omega^2}\sqrt{\frac{QP_{TM}}{\varepsilon_0 c}} \tag{2231}$$

$$A = \frac{3V_t}{\rho a_T^2 \omega^2}\left(\frac{\varepsilon_0}{c} QP_{TM}\right)^{1/2} \tag{2232}$$

Condition 2. The sphere size and separation distance must satisfy $a << r_0$**.**

The size of the spheres must be significantly smaller than the separation distance of the spheres. This is required for the point-mass and point-charge approximations that were used.

Condition 3. Parameters must be physically realistic.

All the values of the parameters must be technologically possible and, more strictly, within the engineering that is available to our laboratory.

<u>Numerical results for a transmitter-receiver communication system</u>

The following are the "hard" parameters dictated by the requirements described in the previous section.

- **The density of the spheres:** $\rho \approx 8.57\ kg/m^3$.

  This assumes that we must use niobium as the superconducting material for the spheres.

- **The highest standing TM wave must be in the microwave range:** $\omega \approx 2\pi \times 10^{10} Hz$**.**

  The receiver power in (2230) scales with ω^2 so the higher the frequency, the greater the receiver power will be. However, the maximum frequency of the standing TM wave must be in the microwave range for a superconducting cavity to remain below the BCS energy gap.

- **The smallest receiver power:** $\underset{(Receiver\ output)}{\mathscr{P}_{EM}} \approx 10^{-20} W$**.**

The minimum EM power that can be detected[254] is on the order of 10 zeptowatts (where $1zW = 10^{-21}W$) assuming a bandwidth limited to only $1Hz$ at room temperature ($\sim 300\ K$).

Using the values stated above in (2230) gives

$$\underset{(Receiver\ output)}{\mathscr{P}_{EM}} = \kappa\left(\frac{r_0\omega}{\rho d}\frac{a_R^2}{a_T^2}V_RV_T\right)^2 QP_{TM} \tag{2233}$$

$$10^{-20} \approx \left(5.40\times10^{-73}\right)\left[\frac{2\pi\times10^{10}}{8.57}\left(\frac{r_0a_R^2V_RV_T}{da_T^2}\right)\right]^2 QP_{TM} \tag{2234}$$

$$\frac{r_0a_R^2V_RV_T}{a_T^2d}\sqrt{QP_{TM}} \approx 1.86\times10^{16}\ \text{ in SI units} \tag{2235}$$

We must now satisfy this relationship with the reaming “soft” parameters. It is evident that we need the largest possible values in the numerator of (2235) and the smallest possible values in the denominator. The following are values that can be considered.

- **For long-range communication we need at least $d \approx 1m$.**

  To test a long-range communication system, the receiver and transmitter should be at least one meter apart.

- **The cavity could have a size on the order of decimeters: $r_0 \approx 10^{-1}m$.**

  The receiver power in (2230) scales with r_0^2 where r_0 is the separation distance of the spheres. Therefore, the larger the separation distance, the greater the receiver power will be. However, the separation distance dictates the cavity size and it is not within our engineering options to construct a cavity larger than $\sim 10cm$.

- **The radius of the transmission spheres can be $a_T \approx 10^{-8}m$.**

  The size of the transmission spheres should not be smaller than this since it is approaching the size of a single atom which is $\sim 10^{-10}m$. A radius of $10^{-8}m$ means that we are still working with spheres that are at least hundreds of atoms and can be modeled classically.

- **The radius of the receiver spheres can be $a_R \approx 10^{-3}m$.**

  Since r_0 is on the order of centimeters and we must have $a << r_0$, then the radius of the spheres must be on the order of millimeters at most. Otherwise we will violate this condition.

- **The voltage used to charge the spheres can be $V \approx 10^2V$.**

  The highest voltage that can be realistically applied in a standard laboratory setting is in the hundred-volt range.

[254]This limit is due to the Johnson-Nyquist thermal noise power, $P = k_BT\Delta f$, where k_B is Boltzmann's constant, T is the ambient temperature, and Δf is the bandwidth of sensitivity.

- **The effective TM wave power (due to resonance with high Q):** $QP_{TM} \approx 10^6 W$.

  The receiver power in (2230) scales linearly with P_{TM} so the higher the TM wave power, the higher the receiver power will be. The TM wave power supplied to the transmitter cavity could be on the order of hundreds of milliwatts ($10^{-4}W$) and the superconducting cavity could have a quality factor of $Q \approx 10^{10}$. Then the effective power driving the spheres would be $QP_{TM} \approx \left(10^{10}\right)\left(10^{-4}W\right) = 10^6 W$.

Substituting the values stated above into (2235) gives

$$\frac{r_0 a_R^2 V_R V_T}{d a_T^2}\sqrt{QP_{TM}} \quad \approx \quad 10^{16} \tag{2236}$$

This matches the order of magnitude required in (2235). This means that we have successfully satisfied the expression for the receiver power given in (2230). We must also check that the values chosen above will satisfy conditions 1 through 3 stated earlier which gave the "hard" parameters. Condition 1 requires that $A << r_0$ where the amplitude was given in (2232) as $A = \dfrac{3V_T}{\rho a_T^2 \omega^2}\left(\dfrac{\varepsilon_0}{c} QP_{TM}\right)^{1/2}$. Using the values given above, we have $A \approx 1.52 \times 10^{-11} m$. Since r_0 is on the order of $10^{-1} m$, then we see that $A << r_0$ so Condition 1 is clearly satisfied.

To satisfy Condition 2, we must have $a << r_0$. Since the radii of the spheres are $a_T \approx 10^{-8} m$ and $a_R = 10^{-3}$ while the separation distance is $r_0 \approx 10^{-1} m$, then we clearly have $a << r_0$ for both sets of spheres. Condition 2 is therefore satisfied.

Lastly, we have satisfied Condition 3 by carefully choosing all values with consideration of actual engineering and laboratory limitations. Hence we see that even with a system of classical masses and charges, there is a realistic possibility of producing an EM to GR wave transducer (and vice-versa). The only factors in this system that are related to quantum mechanics are the superconductivity of the cavity (required for high Q) and the possible superconductivity of the charged spheres (which may be necessary to "freeze out" all internal degrees of freedom so that only the center of mass and center of charge would co-move together rigidly as assumed in this model).

Note that Misner, Thorne, and Wheeler [11] give an example in their text of a massive rotating steel beam as a generator of GR waves. They conclude that the power produced by such an object would only be on the order of $10^{-29} W$. To compare, we can also calculate the GR wave power that would be generated by the system of superconducting ellipsoids with oscillating charged masses at the foci. The GR wave power is given in (2201) as

$$\underset{(Transmitter\ output)}{\mathscr{P}_{GR}} = \frac{2G}{45c^5}\left(m_T A \omega^3 r_0\right)^2 \tag{2237}$$

The mass of each transmitter sphere is

$$m_T = \rho V_T = \rho\left(\frac{4}{3}\pi a_T^3\right) \approx 3.59 \times 10^{-23} kg \tag{2238}$$

So the GR wave power is

$$\underset{(Transmitter\ output)}{\mathscr{P}_{GR}} \approx 2.23 \times 10^{-58} W \tag{2239}$$

Although this power is many orders of magnitude weaker than the experiment described by MTW, it has been shown here that it is still possible to detect this signal indirectly by using the strong coupling of charge to EM waves. This concept of a GR-to-EM wave transducer for detecting GR waves is completely absent from the argument by MTW.

22.3 The receiver power in terms of Planck masses

An alternative way of expressing the receiver power expression is to consider the case when the spheres in the transmitter cavity are each a Planck mass. From (2068), the Planck mass is

$$m_p = \sqrt{\frac{\hbar c}{G}} \tag{2240}$$

From (2221), the receiver power expression is

$$\underset{(Receiver\ output)}{\mathscr{P}_{EM}} = \frac{QP_{TM}}{90\pi\varepsilon_0^2 c^6}(q_R r_0 \omega)^2 \left(\frac{q_T}{m_T}\right)^2 \left(\frac{a_R}{d}\right)^2 \tag{2241}$$

Substituting the Planck mass (2240) into the receiver power (2241) gives

$$\underset{(Receiver\ output)}{\mathscr{P}_{EM}} = \frac{QP_{TM}}{90\pi\varepsilon_0^2 c^6}(q_R r_0 \omega)^2 \left(q_T\sqrt{\frac{G}{\hbar c}}\right)^2 \left(\frac{a_R}{d}\right)^2 \tag{2242}$$

$$= \frac{GQP_{TM}}{90\pi\varepsilon_0^2 \hbar c^7}\left(\frac{q_R q_T r_0 \omega a_R}{d}\right)^2 \tag{2243}$$

Substituting in $q_T = 4\pi\varepsilon_0 a_T V_T$ and $q_R = 4\pi\varepsilon_0 a_R V_R$ from (2228) gives

$$\underset{(Receiver\ output)}{\mathscr{P}_{EM}} = \frac{GQP_{TM}}{90\pi\varepsilon_0^2 \hbar c^7}\left(\frac{(4\pi\varepsilon_0 a_R V_R)(4\pi\varepsilon_0 a_T V_T) r_0 \omega a_R}{d}\right)^2 \tag{2244}$$

$$= \frac{128\pi^3\varepsilon_0^2 G}{45\hbar c^7}\left(\frac{V_R V_T a_T a_R^2}{d} r_0 \omega\right)^2 QP_{TM} \tag{2245}$$

We can define the prefactor as $\Lambda = \dfrac{128\pi^3\varepsilon_0^2 G}{45\hbar c^7}$ so

$$\boxed{\underset{(Receiver\ output)}{\mathscr{P}_{EM}} = \Lambda\left(\frac{V_R V_T a_T a_R^2}{d} r_0 \omega\right)^2 QP_{TM}} \tag{2246}$$

where

$$\Lambda \approx \frac{128\pi^3\left(8.85\times10^{-12}\right)^2\left(6.74\times10^{-11}\right)}{45\left(1.05\times10^{-34}\right)\left(3.00\times10^{8}\right)^7} \approx 2.03\times10^{-56} \text{ in SI units} \tag{2247}$$

First we can consider the result of using the same parameters used before.

- Distance between cavities: $d \approx 10m$
- Distance between spheres inside cavity: $r_0 \approx 10^{-1}m$
- Radius of transmission spheres: $a_T \approx 10^{-8}m$
- Radius of receiver spheres: $a_R \approx 10^{-3}m$
- Voltage to charge spheres: $V \approx 10^2 V$
- Standing TM wave angular frequency: $\omega \approx 2\pi \times 10^{10} Hz$
- Effective TM wave power (due to resonance with high Q): $QP_{TM} = 10^6 W$

Using these values in (2246), we obtain

$$\underset{(Receiver\ output)}{\mathscr{P}_{EM}} = \Lambda \left(\frac{V_R V_T a_T a_R^2}{d} r_0 \omega \right)^2 QP_{TM} \approx 8.01 \times 10^{-53} W \tag{2248}$$

This is *many* orders of magnitude below the minimum value of detectable EM power which is $\sim 10^{-20} W$. This is not surprising if we consider how the mass density of the transmitter spheres has been changed by using a Planck mass. The new mass density would be

$$\rho = \frac{m_p}{V} = \frac{\sqrt{\hbar c/G}}{\frac{4}{3}\pi a_T^3} = \frac{3\sqrt{\hbar c}}{4\pi\sqrt{G} a_T^3} \tag{2249}$$

Substituting in values with $a_T \approx 10^{-8}m$ gives

$$\rho \approx 5.16 \times 10^{15} kg/m^3 \tag{2250}$$

This is an enormous mass density. The elements with the greatest density are only on the order of $10^4 kg/m^3$ (such as for osmium or iridium). If we insist on using a Planck mass and we use the densest element possible for the sphere, then the minimum radius of the sphere can be found by solving for a_T in (2249).

$$a_t = \left(\frac{3\sqrt{\hbar c}}{4\pi\sqrt{G}\rho} \right)^{1/3} \approx 8.02 \times 10^{-5} m \tag{2251}$$

This shows that we cannot use the radius of $a_T \approx 10^{-8}m$ that we used before. We must use $a_T \approx 8.02 \times 10^{-5}m$ instead. Now that we have a value for the mass density and the radius of the transmitter sphere, we can return to the original receiver power expression that we had in (2230).

$$\underset{(Receiver\ output)}{\mathscr{P}_{EM}} = \kappa \left(\frac{r_0 \omega}{\rho a_T^2} \frac{a_R^2}{d} V_R V_T \right)^2 QP_{TM} \qquad \text{where } \kappa \approx 5.40 \times 10^{-73} \text{ in SI units} \tag{2252}$$

We can again attempt to choose values for the parameters that will satisfy this equation. Notice that because the mass density is much greater than before ($10^4 kg/m^3$ versus 8.57 kg/m^3), then we must attempt to compensate for this. The values in the numerator of (2252) must be made as large as reasonably possible while the values in the denominator must be made as small as reasonably possible. First we will begin with using some of the "hard" parameters that are *required* in the system. We can use the following values.

- Density of transmission spheres: $\rho \approx 10^4 kg/m^3$

- Radius of transmission spheres: $a_T = 8.02 \times 10^{-5} m$
- Standing TM wave angular frequency: $\omega \approx 2\pi \times 10^{10} Hz$
- Minimum detectable EM power in receiver cavity: $\underset{(Receiver\ output)}{\mathscr{P}_{EM}} \approx 10^{-20} W$

Using these values in (2252), we obtain

$$10^{-20} W \quad \approx \quad \left(5.40 \times 10^{-73}\right) \left[\frac{2\pi \times 10^{10}}{(10^4)\,(8.02 \times 10^{-5})^2} \left(\frac{r_0 a_R^2 V_R V_T}{d} \right) \right]^2 QP_{TM} \tag{2253}$$

$$\frac{r_0 a_R^2 V_R V_T}{d} \sqrt{QP_{TM}} \quad \approx \quad 1.44 \times 10^{11} \text{ in SI units} \tag{2254}$$

We must choose values that satisfy (2254) and also satisfy Conditions 1 and 2 from Section 89. These conditions require $a << r_0$ and $A << r_0$. We can try the following values

- Distance between cavities: $d \approx 10m$
- Distance between spheres inside cavity: $r_0 \approx 10^{-1} m$
- Radius of receiver spheres: $a_R \approx 10^{-2} m$
- Voltage to charge spheres: $V \approx 10^6 V$
- Effective TM wave power with $Q = 10^{10}$ and $1W$ injected: $QP_{TM} \approx 10^{10} W$

Using these values in (2254) gives

$$\frac{r_0 a_R^2 V_R V_T}{d} \sqrt{QP_{TM}} \quad \approx \quad 10^{11} \tag{2255}$$

This matches the order of magnitude required in (2254). This means that we have successfully satisfied the expression for the receiver power given in (2252). However, notice that these parameters are even *more extreme* than we had earlier in Section 90. Specifically, the spheres must now be charged to *thousands* of kilovolts, and the power injected into the transmitter cavity must be on the order of watts rather than hundreds of milliwatts.

We must also check that Conditions 1 and 2 are still satisfied. Condition 1 requires that $A << r_0$ where $A = \frac{3V_t}{\rho a_t^2 \omega^2} \left(\frac{\varepsilon_0}{c} P_{TM} \right)^{1/2}$. Using the values given above, we have $A \approx 1.52 \times 10^{-14} m$. Since r_0 is on the order of $1m$, then we see that $A << r_0$ so Condition 1 is clearly satisfied.

To satisfy Condition 2, we must have $a << r_0$. Since the radii of the spheres are $a_T \approx 10^{-5} m$ and $a_R = 10^{-2}$ while the separation distance is $r_0 \approx 1m$, then we clearly have $a << r_0$ for both spheres. Condition 2 is therefore satisfied. Condition 3 is clearly *not* satisfied since many of the values are not within the range of actual engineering and laboratory limitations. Therefore, we see that using Planck masses for the transmitter spheres will greatly reduce the likelihood that this GR wave communication system could be realistically operational.

22.4 Efficiency factors

There are three efficiency factors that can be considered for this GR wave transmitter-receiver system.

1. "Transmitter-Receiver Efficiency"

The most important efficiency factor of this system can be referred to as the "Transmitter-Receiver Efficiency" which relates the final output power from the receiver to the initial input power of the transmitter. It is given by

$$\eta_{\substack{Transmitter \\ -\,Receiver}} = \frac{\mathscr{P}_{EM\ (Receiver\ output)}}{\mathscr{P}_{TM\ (Transmiter\ input)}} \tag{2256}$$

Here we have $\mathscr{P}_{EM\ (Receiver\ output)}$ is the final electromagnetic output power that would be detected in the receiver. It is given in (2221) as

$$\mathscr{P}_{\substack{EM \\ (Receiver\ output)}} = \frac{4G^2}{2050\varepsilon_0^2 c^{12}} \omega^4 {r_0}^4 \left(\frac{a_T\ a_R\ q_R\ q_T}{d}\right)^2 QP_{TM} \tag{2257}$$

Also, we define $\mathscr{P}_{TM\ (Transmiter\ input)} = QP_{TM}$ where P_{TM} is the electromagnetic power injected in the transmitter cavity and Q is the quality factor of the transmitter cavity. So we can write the "Transmitter-Receiver Efficiency" as

$$\boxed{\eta_{\substack{Transmitter \\ -\,Receiver}} = \frac{4G^2}{2050\varepsilon_0^2 c^{12}} \omega^4 {r_0}^4 \left(\frac{a_T\ a_R\ q_R\ q_T}{d}\right)^2} \tag{2258}$$

Notice that this efficiency factor takes into account all the relevant parameters that exist in *both* the transmitter and receiver cavities. Therefore, this efficiency factor alone provides all the information necessary to work on improving the efficiency of the entire communication system. Also, this efficiency factor is dependent on the distance between the cavities as well. Therefore, it is always possible to increase this efficiency by bringing the cavities closer to each other if needed. Lastly, note that we could use $\mathscr{P}_{EM\ (Receiver\ output)}$ in terms of experimental parameters as found in (2230). Then the "Transmitter-Receiver Efficiency" would be

$$\eta_{\substack{Transmitter \\ -\,Receiver}} = \left(\frac{8\pi\varepsilon_0^2}{5c^6}\right)\left(\frac{r_0\omega}{\rho}\frac{a_R^2}{a_T^2}V_R V_T\right)^2 \tag{2259}$$

In a previous section, we found that for the receiver cavity to give an output power of $10^{-20}W$, then we needed to inject $10^{-4}W$ into the transmitter cavity with a quality factor of 10^{10} so that $P_{TM\ (Transmiter\ input)} = QP_{TM} = \left(10^{10}\right)\left(10^{-4}\right) = 10^6 W$. This gives a "Transmitter-Receiver Efficiency" of

$$\eta_{\substack{Transmitter \\ -\,Receiver}} = \frac{\mathscr{P}_{EM\ (Receiver\ output)}}{QP_{TM}} \approx \frac{10^{-20}W}{10^{10}10^{-4}W} \approx 10^{-26} \tag{2260}$$

It is important to note that in Section 90 we used the most extreme parameters possible to maximize this efficiency while still insuring that the system is still experimentally tenable. The two other efficiency factors below will help to explain the reason that this "Transmitter-Receiver Efficiency" is so small.

2. <u>“EM-to-GR Transducer Efficiency”</u>

If we isolate our attention to just the transmitter cavity, we can determine an “EM-to-GR Transducer Efficiency.” This is the efficiency by which the transmitter cavity can convert EM wave power to GR wave power.

$$\eta_{\substack{EM-to-GR\\Transducer}} = \frac{\mathscr{P}_{GR\ (Transmitter\ ouput)}}{P_{TM\ (Transmiter\ input)}} \tag{2261}$$

Here we have $\mathscr{P}_{GR\ (Transmitter\ ouput)}$ is the GR wave power that is emitted from the transmitter cavity when the cavity is filled with an EM wave power given by $P_{TM\ (Transmiter\ input)}$. The GR wave power is given in (2205) as

$$\underset{(Transmitter\ ouput)}{\mathscr{P}_{GR}} = \frac{2G}{45c^6\varepsilon_0} q_T^2 \omega^2 r_0^2 \underset{(Transmitter\ input)}{P_{TM}} \tag{2262}$$

Again, we define $\underset{(Transmiter\ input)}{P_{TM}} = QP_{TM}$ where P_{TM} is the electromagnetic power injected in the transmitter cavity and Q is the quality factor of the transmitter cavity. So the “EM-to-GR Transducer Efficiency” is

$$\boxed{\eta_{\substack{EM-to-GR\\Transducer}} = \frac{2G}{45c^6\varepsilon_0} q_T^2 \omega^2 r_0^2} \tag{2263}$$

Notice that this efficiency only depends on the charge of the transmitter spheres and not the mass. This is a surprising result since this efficiency is describing the amount of GR wave power produced and therefore it would be assumed that more mass would produce more GR radiation. However, this is not the case. The reason can be seen by looking at the way (2205) was derived. Since the gravitational quadrupole wave power goes as $\mathscr{P}_{GR}^{(quad)} \sim A^2 m^2$ and the amplitude goes as $A \sim m^{-2}$, then the masses cancel (according to the Equivalence Principle). Therefore the gravitational power does not depend on the mass, and likewise neither does the “EM-to-GR Transducer Efficiency.”

Conceptually, this can be understood by recognizing that having more mass will produce more GR wave power, and yet having *less* mass will decrease the inertia and allow the spheres to oscillate faster which produces more GR radiation. The net result is a perfect balance of these two effects so that the mass does not determine the GR wave power or the “EM-to-GR Transducer Efficiency” at all. Only the charge plays a role since it determines how strongly the charged spheres will be coupled to the EM wave driving their motion.

Again we can use the parameters given in Section 90 to find a value for this efficiency. In (2239) we already found that $\mathscr{P}_{GR\ (Transmitter\ ouput)} \approx 2.23 \times 10^{-58} W$. For the input power we already used $P_{TM\ (Transmiter\ input)} = QP_{TM} = \left(10^{10}\right)\left(10^{-4}\right) = 10^6 W$. This gives an “EM-to-GR Transducer Efficiency” of

$$\eta_{\substack{EM-to-GR\\Transducer}} = \frac{\mathscr{P}_{GR\ (Transmitter\ ouput)}}{QP_{TM}} \approx \frac{2.23 \times 10^{-58} W}{10^{10} 10^{-4} W} \approx 2.23 \times 10^{-64} \tag{2264}$$

Obviously this is an extremely small efficiency factor. Therefore, when driving the charged, massive spheres into oscillation by an EM wave, the resulting GR wave power is extremely small. It is only because the *opposite* procedure will take place in the *receiver* cavity that it is still possible to indirectly detect these GR waves.

3. <u>"GR-to-EM Transducer Efficiency"</u>

Lastly, we can isolate our attention to just the receiver cavity to determine a "GR-to-EM Transducer Efficiency." This is the efficiency by which the receiver cavity can convert EM wave power to GR wave power.

$$\eta_{\substack{GR-to-EM\\Transducer}} = \frac{\mathscr{P}_{EM\ (Receiver\ output)}}{\mathscr{P}_{GR\ (Receiver\ input)}} \tag{2265}$$

Here we have $\mathscr{P}_{EM\ (Receiver\ output)}$ is the EM wave power that is emitted from the receiver cavity when a GR wave is incident on the receiver cavity. The power of this output EM wave is given in (2217) in terms of the GR wave amplitude as

$$\mathscr{P}_{\substack{EM\\(Receiver\ output)}} = \frac{4G^2}{2050\varepsilon_0^2c^{12}}\omega^4{r_0}^4\left(\frac{a_T\ a_R\ q_R\ q_T}{d}\right)^2 QP_{TM} \tag{2266}$$

The power of the *input* GR wave is given in (2208) as

$$\mathscr{P}_{\substack{GR\\(Receiver\ input)}} = \frac{8\pi G}{45\varepsilon_0 c^6}\left(\omega r_0 a_T\ q_T\right)^2\left(\frac{a_R^2}{2d^2}\right)Q\mathscr{P}_{TM} \tag{2267}$$

So the "GR-to-EM Transducer Efficiency" is

$$\boxed{\eta_{\substack{GR-to-EM\\Transducer}} = \frac{9G}{410\pi\varepsilon_0 c^6}\left(\omega r_0 q_R\right)^2} \tag{2268}$$

Notice that this efficiency is determined only by the *charge* of the *receiver* spheres which determines the amount of EM wave power that is produced by the spheres. So we expect to see it play a role in the "GR-to-EM Transducer Efficiency." The *mass* of either spheres does *not* play a role once again due to the Equivalence Principle.

Once more we can use the parameters given in Section 90 to find a value for this efficiency. We already know from (2239) that the GR wave power produced by the transmitter is $\mathscr{P}_{GR\ (Transmitter\ ouput)} \approx 2.23\times10^{-58}W$. However, since the wave diffuses in 3-dimensional space as it travels to the receiver cavity, then we must multiply this by $(a_R/d)^2$ so we have

$$\mathscr{P}_{\substack{GR\\(Receiver\ output)}} = \mathscr{P}_{\substack{GR\\(Transmitter\ ouput)}}\left(\frac{a_R}{d}\right)^2 \approx 2.23\times10^{-58}W\left(\frac{10^{-3}m}{1m}\right)^2 \tag{2269}$$

$$\mathscr{P}_{\substack{GR\\(Receiver\ input)}} \approx 2.23\times10^{-64}W \tag{2270}$$

The out EM power was determined to be $\mathscr{P}_{\substack{EM\\(Receiver\ output)}} \approx 10^{-20}W$ so the "GR-to-EM Transducer Efficiency" is

$$\eta_{\substack{GR-to-EM\\Transducer}} = \frac{\mathscr{P}_{EM\ (Receiver\ output)}}{\mathscr{P}_{GR\ (Receiver\ input)}} \approx \frac{10^{-20}W}{2.23\times10^{-58}W} \approx 4.48\times10^{37} \tag{2271}$$

Therefore, we find that the "GR-to-EM Transducer Efficiency" can be extremely large. This is the reason that although there is a tremendous "loss" in the "EM-to-GR Transducer Efficiency" $\left(\sim 10^{-64}\right)$ and there is also a "diffusion factor" of $(a_R/d)^2 \approx 10^{-6}$, it is still possible to recover a detectable amount of EM power in the receiver cavity because of a very large “GR-to-EM Transducer Efficiency” $\left(\sim 10^{37}\right)$. The net result, as shown above, is a “Transmitter-Receiver Efficiency” of $\sim 10^{-26}$. Since EM power on the order of $10^6 W$ can be produced at the foci of a high-Q superconducting resonator, and because EM power as low as $10^{-20} W$ can be detected, then this overall communication system is physically realizable.

As a final consideration, we can recognize that the “Transmitter-Receiver Efficiency” can be obtained by multiplying the “GR-to-EM Transducer Efficiency,” the “EM-to-GR Transducer Efficiency,” and the “diffusion factor.”

$$\eta_{\substack{Transmitter \\ -\ Receiver}} = \eta_{\substack{EM-to-GR \\ Transducer}} \left(\frac{a_R}{d}\right)^2 \eta_{\substack{GR-to-EM \\ Transducer}} \tag{2272}$$

Substituting the efficiency expressions from (2258), (2263), and (2268) gives

$$\frac{(q_R r_0 \omega)^2}{90\pi\varepsilon_0^2 c^6}\left(\frac{q_T}{m_T}\right)^2\left(\frac{a_R}{d}\right)^2 = \left(\frac{2Gq_T^2\omega^2 r_0^2}{45c^6\varepsilon_0}\right)\left(\frac{a_R}{d}\right)^2\left(\frac{Kq_R^2}{Gm_T^2}\right) \tag{2273}$$

$$\frac{1}{90\pi\varepsilon_0^2 c^6}\left(\frac{r_0\omega q_R q_T a_R}{m_T d}\right)^2 = \frac{2}{45c^6\varepsilon_0}\left(\frac{r_0\omega q_R q_T a_R}{m_T d}\right)^2\left(\frac{1}{4\pi\varepsilon_0}\right) \tag{2274}$$

$$\frac{1}{90\pi\varepsilon_0^2 c^6}\left(\frac{r_0\omega q_R q_T a_R}{m_T d}\right)^2 = \frac{1}{90\pi\varepsilon_0^2 c^6}\left(\frac{r_0\omega q_R q_T a_R}{m_T d}\right)^2 \tag{2275}$$

Therefore we see that we have algebraic consistency. We can also substitute into (2272) the values of the efficiencies from (2260), (2264), and (2271) to confirm there is numeric consistency. We find that both give a value of 10^{-26}. Alternatively, we can express (2272) in terms of wave power. This gives

$$\frac{\mathscr{P}_{EM\ (Receiver\ output)}}{P_{TM\ (Transmiter\ input)}} = \frac{\mathscr{P}_{GR\ (Transmitter\ ouput)}}{P_{TM\ (Transmiter\ input)}}\left(\frac{a_R}{d}\right)^2\frac{\mathscr{P}_{EM\ (Receiver\ output)}}{\mathscr{P}_{GR\ (Receiver\ input)}} \tag{2276}$$

Once again, we can confirm numeric consistency by substituting in the value for each power. Once again, this leads to 10^{-26} on each side.

Note that in the entire discussion of this section, the use of the term “efficiency” is different from the standard use of the terms in physics. Typically, the efficiency of a system gives a ratio of the input energy to the *usable* output energy of a system which can do work. The difference between the input and output energy is understood to be irrecoverably lost. It is associated with a non-reversible process which requires that any type of energy exchange cannot be 100% efficient because some energy must be lost as heat and thereby contribute to the ever-increasing entropy of the universe. So the efficiency is a *thermodynamic* property of the system and can never be greater than unity.

However, the use of “efficiency” here is very different. In fact, it is clearly *not* limited to a maximum value of one. This is because the conversion of input to output energy is in fact a reversible process and therefore the “efficiency” described here is *not* describing the energy that is lost to entropy. In this model, we have assumed that *all* energy is conserved. So we are simply interested in how much of it *transitions* from one form to another. We have neglected the issue of thermodynamic efficiency since the charged spheres can be considered as levitated in vacuum so that there is effectively no loss to any kind of friction. Therefore, any thermal loss is completely negligible.

In this context, the term “efficiency” is understood to mean the effective conversion of one *type* of energy to another, namely from EM energy to GR energy or vice-versa. This conversion is determined by the way a source couples to its corresponding field, that is, the way that charge couples to EM fields and the way mass couples to GR fields. Since charge couples much more strongly to EM fields than mass does to GR fields, then we expect that a “GR-to-EM transducer” will be highly “*efficient*” while an “EM-to-GR transducer” will be highly “inefficient.” In other words, when a charged, massive system is driven into oscillation, then it will produce relatively large amounts of EM radiation and relatively small amounts of GR radiation. As a result, we anticipate that there will be a “loss” at the transmitter cavity (where we use EM waves to generate GR waves), but there will be a “gain” at the receiver cavity (where we use GR waves to generate EM waves). Since there is also a diffusion of the energy due to the inverse-square law as the wave propagates in 3-dimensional space between the cavities, then there is an additional “loss” due to the corresponding dissipation factor. So the net result is that the “loss” at the transmitter and the “loss” during propagation must be compensated for by the “gain” at the receiver as much as possible.

23 Electromagnetic and gravitational Casimir Effects in a parallel-plate waveguide

23.1 The TEM mode in a parallel-plate waveguide

The following figure shows a parallel-plate waveguide with two perfectly conducting, infinitesimally thin plates that extend infinitely in the z-direction. The width of the plates is W and the separation distance is d with $W >> d$ (so that fringing effects can be neglected). The upper plate is at $y = d$ while the lower plate is at $y = 0$ (the x-z plane).The fundamental mode in such a waveguide is the TEM mode. In this mode, both the

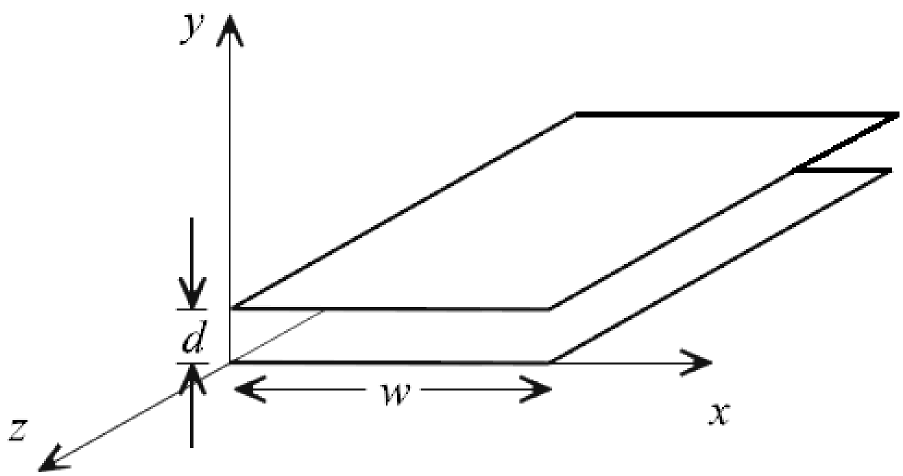

Figure 26: A parallel-plate wave guide.

electric and magnetic fields only have components that are transverse to the direction of propagation.[29][40] For a wave propagating in the positive or negative z-direction, this means $E_z = 0$ and $B_z = 0$. Note that for uncharged conductors, we must also satisfy the boundary conditions at the conductor required by Faraday's Law and the Ampere-Maxwell Law.

$$E_{\|}\big|_{y=0,d} = 0 \qquad \text{and} \qquad B_{\perp}\big|_{y=0,d} = 0 \tag{2277}$$

To satisfy these, we can consider a plane-polarized EM wave with the electric field polarized in the y-direction, which means the magnetic field is in the x-direction. For a wave propagating in the *positive* z-direction, the fields can be described as

$$\vec{E}_+ = E_0 \cos(kz - \omega t)\,\hat{y} \qquad \text{and} \qquad \vec{B}_+ = \frac{E_0}{c}\cos(kz - \omega t)\,\hat{x} \tag{2278}$$

For a wave propagating in the *negative* z-direction (with the same polarization) we have

$$\vec{E}_- = E_0 \cos(kz + \omega t)\,\hat{y} \qquad \text{and} \qquad \vec{B}_- = -\frac{E_0}{c}\cos(kz + \omega t)\,\hat{x} \tag{2279}$$

These traveling waves (propagating in opposite directions) can be combined to describe a single standing wave.

$$\vec{E} = \vec{E}_+ + \vec{E}_- = E_0\left[\cos(kz - \omega t) + \cos(kz + \omega t)\right]\hat{y} \tag{2280}$$

$$\vec{B} = \vec{B}_+ + \vec{B}_- = \frac{E_0}{c}\left[\cos(kz - \omega t) - \cos(kz + \omega t)\right]\hat{x} \tag{2281}$$

It is shown in Appendix U that applying trigonometric identities can simplify these expressions to

$$\boxed{\vec{E} = 2E_0 \cos(kz)\cos(\omega t)\,\hat{y} \qquad \text{and} \qquad \vec{B} = \frac{2E_0}{c}\sin(kz)\sin(\omega t)\,\hat{x}} \tag{2282}$$

23.2 Currents and magnetic forces on the plates

By Ampere's law, the magnetic field will have an associated current on each plate. We can use a rectangular Amperian loop of length $\vec{l}$ that is straddling the surface of the conductor and therefore "threaded" by the free surface current density, $\vec{K}_f$, on the conductor.Note that $\hat{n}$ is a unit vector normal to the conductor

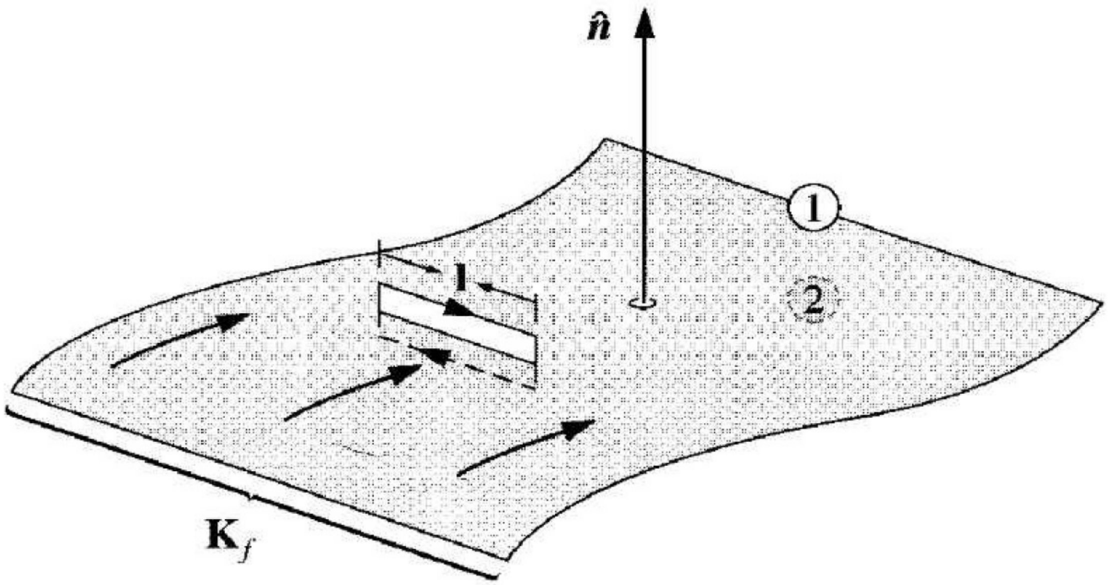

Figure 27: An Amperian loop straddling the surface of a conductor with a surface current density $\vec{K}_f$.

(pointing from 2 to 1 for the lower plate and pointing from 1 to 2 for the upper plate). Then $\hat{n}\times\vec{l}$ is normal to the Amperian loop. So the free current on the conductor is given by

$$I_{free} = \vec{K}_f \cdot \left(\hat{n}\times\vec{l}\right) = K_f\hat{z}\cdot l\hat{z} = K_f l \tag{2283}$$

If we let the Amperian loop have the same width as the conductor, W, then the total current is $I_{free} = K_f W$. Because $\vec{B}$ is uniform in the x-direction, then K_f uniform is as well and the current I is a constant within the Amperian loop. Also, since the magnetic field of the EM wave only exists on *one* side of the conductor, then $\oint \vec{B}\cdot d\vec{l} = BW$. So Ampere's law gives

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enclosed} \tag{2284}$$

$$BW = \mu_0 I \tag{2285}$$

$$I = \frac{BW}{\mu_0} \tag{2286}$$

Substituting for the magnetic field from (2282) gives

$$I = \frac{2E_0W}{\mu_0 c}\sin(kz)\sin(\omega t) \begin{cases} \text{in the } \textit{negative} \text{ z-direction for the } \textit{lower} \text{ plate} \\ \text{in the } \textit{positive} \text{ z-direction for the } \textit{upper} \text{ plate} \end{cases} \tag{2287}$$

Equivalently, we could also define the currents on the lower and upper plates in terms of the positive z-direction as

$$I_{lower} = -\frac{2E_0W}{\mu_0 c}\sin(kz)\sin(\omega t) \qquad \text{and} \qquad I_{upper} = \frac{2E_0W}{\mu_0 c}\sin(kz)\sin(\omega t) \tag{2288}$$

which shows explicitly that we always have $I_{lower} = -I_{upper}$ at any given time and location along the z-direction.

The *direction* of the current was be determined by the right-hand-rule. For the lower plate, when the magnetic field points in the positive x-direction, the current points in the negative z-direction. For the upper plate, when the magnetic field points in the positive x-direction, the current points in the positive z-direction. This means the currents directly across from one another will always be in opposite directions and therefore will cause a *repulsive* force between the two plates.

From (2287) we see that the current will alternate between the positive and negative z-direction depending both on time and location along the z-direction. It is essentially a longitudinal oscillation of charge along the length of the waveguide. Since the oscillation is effectively a standing wave, then no net charge is transferred.

The force on each plate is actually due to the magnetic field of the EM wave interacting with the currents on the plates. From (2282) and (2287) it can be seen that the magnetic field and currents are in phase. As stated earlier, when the magnetic field points in the positive x-direction, the current points in the negative z-direction for the lower plate. Then the right-hand rule gives a force in the negative y-direction for the lower plate.

The magnetic force, $\vec{f}_M$, along a one-dimensional infinitesimally thin strip of the lower plate (in the z-direction) from $z = 0$ to $z = L$ is

$$\vec{f}_M = \int_0^L I d\vec{l} \times \vec{B} \tag{2289}$$

where the direction of $d\vec{l}$ is determined by the current in the lower plate. According to(2287) this means $d\vec{l} = -d\vec{z}$ for the lower plate so the force on the strip of the lower plate becomes

$$\vec{f}_M = \int_0^z I\left(-d\vec{z}'\right) \times B\hat{x} = -\int_0^z IBdz'\hat{y} \tag{2290}$$

Substituting into (2290) the expressions for I and B from (2287) and (2282), respectively, gives

$$\vec{f}_M = -\int_0^z \left[\frac{2E_0W}{\mu_0 c}\sin\left(kz'\right)\sin(\omega t)\right]\left[\frac{2E_0}{c}\sin\left(kz'\right)\sin(\omega t)\right]dz'\hat{y} \tag{2291}$$

$$= -\frac{4E_0^2W}{\mu_0 c^2}\sin^2(\omega t)\int_0^z \sin^2\left(kz'\right)dz'\hat{y} \tag{2292}$$

$$= -\frac{4E_0^2W}{\mu_0 c^2}\sin^2(\omega t)\int_0^z \frac{1}{2}\left[1-\cos\left(2kz'\right)\right]dz'\hat{y} \tag{2293}$$

Evaluating the integral gives

$$\vec{f}_M = -\frac{2E_0^2 W}{\mu_0 c^2} \sin^2(\omega t) \left[z' - \frac{1}{2k} \sin(2kz') \right]_0^z \hat{y} \tag{2294}$$

$$= -\frac{2E_0^2 W}{\mu_0 c^2} \sin^2(\omega t) \left[z - \frac{1}{2k} \sin(2kz) \right] \hat{y} \tag{2295}$$

Since this is the force on a single strip with infinitesimal width dx and length L, then the total force on the lower plate can be found by integrating the force on each strip, $\vec{f}_M$, from $x = 0$ to $x = W$ and then dividing by W. This gives

$$\vec{F}_M = \frac{1}{W} \int_0^W \vec{f}_M dx \tag{2296}$$

Since $\vec{f}_M$ does not vary in the x-direction, then the integral simply gives W. So we arrive back at the same expression for the magnetic force on the lower plate.

$$\boxed{\vec{F}_M = -\frac{2E_0^2 W}{\mu_0 c^2} \sin^2(\omega t) \left[z - \frac{1}{2k} \sin(2kz) \right] \hat{y}} \tag{2297}$$

The magnetic force on the upper plate is equal and opposite.

23.3 The charge and electric forces on the plates

To find the *electric* force on the lower plate, we must determine the charge distribution on the plate. Since $I = \dfrac{dq}{dt}$, then we can use the current on the lower plate from (2288) to find the charge on the lower plate at a given time t.

$$q = \int_0^t I_{lower} dt' = \int_0^t \left[-\frac{2E_0 W}{\mu_0 c} \sin(kz) \sin\left(\omega t'\right) \right] dt' \tag{2298}$$

$$= \frac{2E_0 W}{\mu_0 c \omega} \sin(kz) \left[\cos(\omega t) - 1\right] \tag{2299}$$

Note that according to (2288) we always have $I_{lower} = -I_{upper}$ at any given time and location along the z-direction. Therefore the result for q will also have *opposite* signs for the charge on the upper and lower plates. This means that an electric force will always be *attractive* between the plates at any location along the waveguide. To find the electric force on the bottom plate, we can consider the charge per unit length, λ_q, on an infinitesimally thin strip of the plate from $z' = 0$ to $z' = z$.

$$\lambda_q = \frac{dq}{dz} = \frac{d}{dz} \left\{ \frac{2E_0 W}{\mu_0 c \omega} \sin(kz) \left[\cos(\omega t) - 1\right] \right\} \tag{2300}$$

$$= \frac{2E_0 W}{\mu_0 c^2} \left[\cos(kz)\right] \left[\cos(\omega t) - 1\right] \tag{2301}$$

Since the electric force is given by $\vec{f}_E = q\vec{E}$, then the force due to the electric field acting on a charge q on the strip from $z = 0$ to $z = L$ is

$$\vec{f}_E = \int_0^q dq' \vec{E} = \int_0^z \lambda_q dz' \vec{E} \tag{2302}$$

Substituting into (2302) the expressions for λ_q and $\vec{E}$ from (2301) and (2282), respectively, gives

$$\vec{f}_E = \int_0^z \frac{2E_0 W}{\mu_0 c^2} \left[\cos\left(kz'\right)\right] \left[\cos(\omega t) - 1\right] \left[2E_0 \cos\left(kz'\right) \cos(\omega t) \hat{y}\right] dz' \tag{2303}$$

$$= \frac{4E_0^2 W}{\mu_0 c^2} \left[\cos(\omega t) - 1\right] \cos(\omega t) \int_0^z \cos^2\left(kz'\right) dz' \hat{y} \tag{2304}$$

$$= \frac{4E_0^2 W}{\mu_0 c^2} \left[\cos^2(\omega t) - \cos(\omega t)\right] \int_0^z \frac{1}{2} \left[1 + \cos\left(2kz'\right)\right] dz' \hat{y} \tag{2305}$$

$$= \frac{2E_0^2 W}{\mu_0 c^2} \left[\cos^2(\omega t) - \cos(\omega t)\right] \left[z' + \frac{1}{2k} \sin\left(2kz'\right)\right]_0^z \hat{y} \tag{2306}$$

$$= \frac{2E_0^2 W}{\mu_0 c^2} \left[\cos^2(\omega t) - \cos(\omega t)\right] \left[z + \frac{1}{2k} \sin(2kz)\right] \hat{y} \tag{2307}$$

Since this is the force on a single strip with infinitesimal width dx and length L, then the total force on the plate can be found by integrating the force on each strip, $\vec{f}_E$, from $x = 0$ to $x = W$ and then dividing by W. This gives

$$\vec{F}_E = \frac{1}{W}\int_0^W \vec{f}_E dx \tag{2308}$$

However, $\vec{f}_E$ does not vary in the x-direction so the integral simply gives W. Then we simply arrive back at the same expression for the electric force on the lower plate.

$$\boxed{\vec{F}_E = \frac{2E_0^2 W}{\mu_0 c^2}\left[\cos^2(\omega t) - \cos(\omega t)\right]\left[z + \frac{1}{2k}\sin(2kz)\right]\hat{y}} \tag{2309}$$

The electric force on the upper plate is equal and opposite.

23.4 The full electromagnetic force on the plates

Adding the electric and magnetic forces from (2297) and (2309), respectively, gives the full electromagnetic force on the lower plate.

$$\vec{F}_{EM} = \vec{F}_E + \vec{F}_M \tag{2310}$$

$$= \frac{2E_0^2 W}{\mu_0 c^2}\left\{\left[\cos^2(\omega t) - \cos(\omega t)\right]\left[z + \frac{1}{2k}\sin(2kz)\right] - \sin^2(\omega t)\left[z - \frac{1}{2k}\sin(2kz)\right]\right\}\hat{y} \tag{2311}$$

Using $k = 2\pi/\lambda$ and rearranging gives

$$\vec{F}_{EM} = \frac{2E_0^2 W}{\mu_0 c^2}\left\{\left[\cos^2(\omega t) - \cos(\omega t)\right]\left[\frac{\lambda}{4\pi}\sin\left(\frac{4\pi z}{\lambda}\right) + z\right]\right.$$
$$\left. + \sin^2(\omega t)\left[\frac{\lambda}{4\pi}\sin\left(\frac{4\pi z}{\lambda}\right) - z\right]\right\}\hat{y} \tag{2312}$$

$$= \frac{2E_0^2 W}{\mu_0 c^2}\left\{\left[\frac{\lambda}{4\pi}\cos^2(\omega t)\sin\left(\frac{4\pi z}{\lambda}\right) - \frac{\lambda}{4\pi}\cos(\omega t)\sin\left(\frac{4\pi z}{\lambda}\right) + z\cos^2(\omega t) - z\cos(\omega t)\right]\right.$$
$$\left. + \frac{\lambda}{4\pi}\sin^2(\omega t)\sin\left(\frac{4\pi z}{\lambda}\right) - z\sin^2(\omega t)\right\}\hat{y} \tag{2313}$$

Using $\cos^2\phi + \sin^2\phi = 1$ and $\cos^2\phi - \sin^2\phi = \cos(2\phi)$ gives

$$\vec{F}_{EM} = \frac{2E_0^2 W}{\mu_0 c^2}\hat{y}\left\{\frac{\lambda}{4\pi}\sin\left(\frac{4\pi z}{\lambda}\right) - \frac{\lambda}{4\pi}\cos(\omega t)\sin\left(\frac{4\pi z}{\lambda}\right) + z\cos(2\omega t) - z\cos(\omega t)\right\} \tag{2314}$$

$$\boxed{\vec{F}_{EM} = \frac{2E_0^2 W}{\mu_0 c^2}\hat{y}\left\{z\cos(2\omega t) - z\cos(\omega t) - \frac{\lambda}{4\pi}\sin\left(\frac{4\pi z}{\lambda}\right)\cos(\omega t) + \frac{\lambda}{4\pi}\sin\left(\frac{4\pi z}{\lambda}\right)\right\}} \tag{2315}$$

This is the full electromagnetic force on the lower plate as a function of time t. The electromagnetic force on the upper plate is equal and opposite. Notice that the last term is a constant with respect to time. This indicates that it should contribute a "DC" offset to the sinusoidal oscillation of the force. The graph on the following page shows $\vec{F}_{EM}$ with respect to time with $z >> \lambda$. The graph is over two periods from $t = -T$ to $t = T$, where the period is $T = 2\pi$.

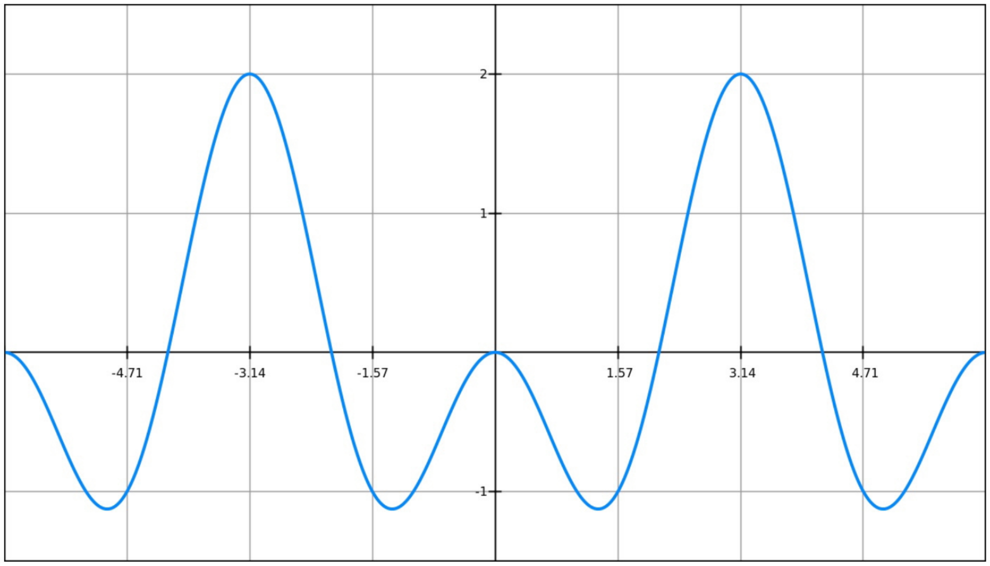

Figure 28: Graph of electromagnetic force (with respect to time) on the walls of a parallel-plate waveguide due to a TEM wave.

Notice that because the force expression in (2315) contains sinusoids of different frequencies, the result is a "sloshing" behavior in time. There is still, however, an overall "DC" force in the negative direction. In fact, if we take a time average of (2315), then we obtain

$$\left\langle \vec{F}_{EM} \right\rangle = \frac{\lambda E_0^2 W}{2\pi\mu_0 c^2}\hat{y}\sin\left(\frac{4\pi z}{\lambda}\right) \tag{2316}$$

Using $\sin\left(\frac{4\pi z}{\lambda}\right) = 1 - 2\sin\left(\frac{2\pi z}{\lambda}\right)$, we can write this as

$$\left\langle \vec{F}_{EM} \right\rangle = \frac{\lambda E_0^2 W}{2\pi\mu_0 c^2}\hat{y}\left[1 - 2\sin\left(\frac{2\pi z}{\lambda}\right)\right] \tag{2317}$$

The modes that lead to the greatest *repulsive* force on the plates occur when $\sin(2\pi z/\lambda) = 1$ which means $2\pi z/\lambda = \pi/2 + 2\pi n$ or

$$\frac{z}{\lambda} = \frac{1}{4} + n = \frac{1+4n}{4} \qquad \Longrightarrow \qquad \lambda = \frac{4z}{4n+1}, \qquad n = 1,2,3... \tag{2318}$$

On the other hand, modes for the greatest *attractive* force occur when $\sin(2\pi z/\lambda) = -1$ which means $2\pi z/\lambda = -\pi/2 + 2\pi n$ or

$$\frac{z}{\lambda} = -\frac{1}{4} + n = \frac{4n-1}{4} \qquad \Longrightarrow \qquad \lambda = \frac{4z}{4n-1}, \qquad n = 1,2,3... \tag{2319}$$

If we consider the case when the length of the waveguide is significantly longer than the wavelength of the electromagnetic wave, then we can have $z >> \lambda$ so $z << k$. since $k = 2\pi/\lambda$. In that case, from (2312) we see that the second term in brackets (the magnetic force) is slightly greater than the first term in brackets (the electric force) by a factor of z. This means that for large length values along the stripline, the cause for the DC offset will be the magnetic field. Lastly, if $\left\langle \vec{F}_{EM} \right\rangle$ is in the lowest repulsive mode ($n = 1$), then (2316) can be written as

$$\boxed{\left|\left\langle \vec{F}_{EM} \right\rangle\right| = -\frac{zE_0^2 W}{5\pi\mu_0 c^2}\hat{y}} \tag{2320}$$

23.5 Quantizing the electromagnetic waveguide energy

The average energy density in the waveguide due to the standing EM wave is

$$\langle u\rangle = \frac{1}{2}\varepsilon_0 E_0^2 \tag{2321}$$

The volume of the waveguide from $z' = 0$ to $z' = z$ is

$$V = zWd \tag{2322}$$

Then the total EM energy in that volume is

$$\begin{aligned} Energy &= \langle u\rangle V &(2323)\\ &= \frac{1}{2}\varepsilon_0 E_0^2 zWd &(2324)\end{aligned}$$

If the standing EM wave is due to quantum vacuum fluctuations, then the ground state (zero-point) energy is

$$Energy = \frac{1}{2}\hbar\omega = \frac{hc}{2\lambda} \tag{2325}$$

The fundamental mode between the plates is when $d = \lambda/2$ so $\lambda = 2d$ and the energy is

$$Energy = \frac{hc}{4d} \tag{2326}$$

Equating the EM energy in (2324) with the quantum ground state energy in (2326) gives

$$\frac{1}{2}\varepsilon_0 E_0^2 zWd = \frac{hc}{4d} \tag{2327}$$

Solving for the electric field amplitude gives

$$E_0^2 = \frac{hc}{2\varepsilon_0 zWd^2} \tag{2328}$$

Substituting this result into the force expression found in (2320) and using $c^2 = \dfrac{1}{\varepsilon_0\mu_0}$ gives

$$\left\langle \vec{F}_{EM}\right\rangle = -\frac{zW}{5\pi\mu_0 c^2}\left(\frac{hc}{2\varepsilon_0 zWd^2}\right)\hat{y} \tag{2329}$$

$$\boxed{\left\langle \vec{F}\right\rangle_{EM} = -\frac{hc}{10\pi d^2}\hat{y}} \tag{2330}$$

This result can be understood to be an effective Casimir force as it is the force experienced by a conducting plate due to an electromagnetic quantum vacuum fluctuation. The commonly accepted Casimir force for a parallel-plate Fabry-Perot resonator (as shown in the figure below) is given by[255]

$$\vec{F} = \frac{hc\pi^2 A}{240 d^4}\hat{y} \tag{2331}$$

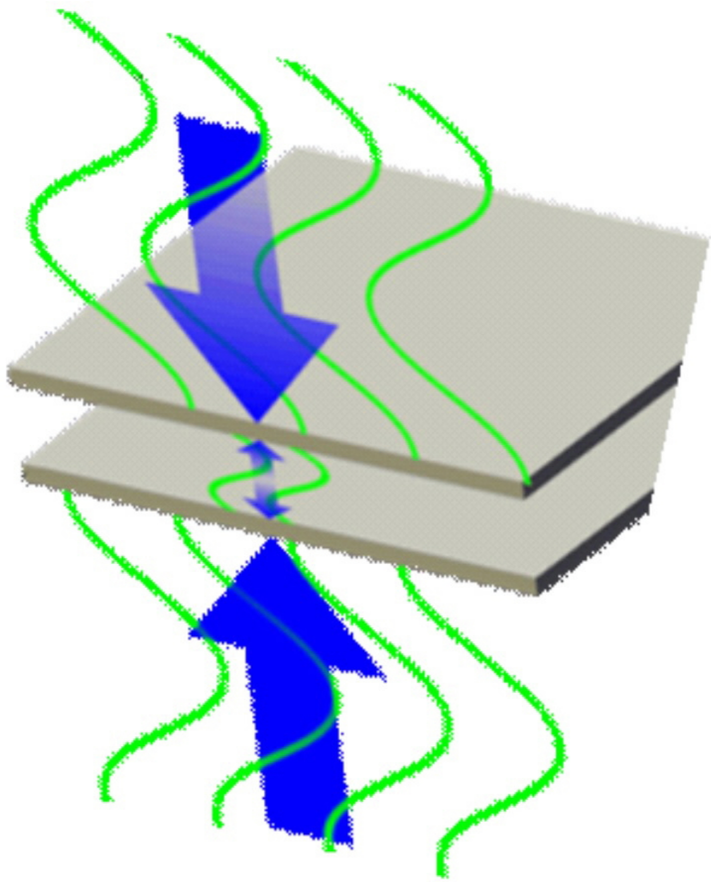

Figure 29: The *attractive* Casimir force that is known to occur between two plates due to the difference in standing wave modes of quantum energy fluctuations. These standing waves are normal to the cavity (such as in a Fabry-Perot resonator arrangement) rather than transverse (such as in the paralle-plate wave guide arrangement).

We can note the following differences between the "Waveguide Casimir force" in (2330) and the common "Fabry-Perot Casimir force" in (2331).

1. The force is repulsive rather than attractive.
2. The force falls off as $1/d^2$ rather than $1/d^4$.
3. The force is independent of the area of the plates.

[255] See J. Garrison and R. Chiao, *Quantum Optics* [101], pp. 32-38, 60-65.

23.6 The full gravito-electromagnetic force on the plates

We now consider the gravitational analogue of the electromagnetic force derived above. To do so, we use the gravito-electromagnetic (GEM) fields as derived in previous sections. The gravitational "Lorentz-like" force was is given by (105) as

$$\vec{F}_{GEM} = m\left(\vec{E}_G + 4\vec{v}\times\vec{B}_G\right) \tag{2332}$$

Therefore, the gravitational analogue to the magnetic force in (2289) is

$$\vec{F}_{GM} = 4\int_0^L I_m d\vec{l}\times\vec{B}_G \tag{2333}$$

By direct analogy with (2297), we can recognize that the gravito-magnetic force on the bottom plate will be

$$\boxed{\vec{F}_{GM} = -\frac{8E_{g,0}^2 W}{\mu_G c^2}\sin^2(\omega t)\left[L - \frac{1}{2k}\sin(2kL)\right]\hat{y}} \tag{2334}$$

where $E_{g,0}$ is the amplitude of oscillation[256] of the gravito-electric field, $\vec{E}_G$. Also, by direct analogy with (2309), we can recognize that the gravito-electric force on the bottom plate will be

$$\vec{F}_{GE} = \frac{2E_{g,0}^2 W}{\mu_G c^2}\left[\cos^2(\omega t) - \cos(\omega t)\right]\left[L + \frac{1}{2k}\sin(2kL)\right]\hat{y} \tag{2335}$$

Combining the gravito-electric and gravito-magnetic forces gives the full gravito-electromagnetic (GEM) force on the lower plate.

$$\begin{aligned}\vec{F}_{GEM} \quad = \quad & \left\{\frac{8E_0^2 W}{\mu_0 c^2}\left[\cos^2(\omega t) - \cos(\omega t)\right]\left[L + \frac{1}{2k}\sin(2kL)\right]\right. \\ & \left. + -\frac{2E_0^2 W}{\mu_0 c^2}\sin^2(\omega t)\left[L - \frac{1}{2k}\sin(2kL)\right]\right\}\hat{y}\end{aligned} \tag{2336}$$

Notice that because the gravito-magnetic force expression has a factor of 4, it is not possible to simplify the full GEM force as we did for the full EM force. Instead, we will immediately consider the time-average of the force to obtain an expression analogous to (2316). Since

$$\langle\cos(\omega t)\rangle = \langle\sin(\omega t)\rangle = 0 \qquad \text{and} \qquad \left\langle\cos^2(\omega t)\right\rangle = \left\langle\sin^2(\omega t)\right\rangle = 1/2 \tag{2337}$$

then we obtain

$$\boxed{\left\langle\vec{F}_{GEM}\right\rangle = \frac{3LE_0^2 W}{\mu_0 c^2}\hat{y}} \tag{2338}$$

This is the full GEM force on the lower plate as a function of time t. The GEM force on the upper plate is equal and opposite. Notice that the result is similar to the electromagnetic case in (2320).

[256] We are careful not to refer to these gravito-electromagnetic fields as "waves" since gravitational waves *tensor* fields arising from quadrupolar soures (versus *vector* fields arising from dipolar soures, as is the case for electromagnetism). Therefore, we will refer to the gravito-electromagnetic fields here as simply "oscillating fields."

23.7 Quantizing the gravitational waveguide energy

Here we use an order-of-magnitude calculation to show that microwave gravitational radiation in a quantum ground state is associated with mass-energy densities that are within laboratory scales. For the sake of discussion, consider a parallel-plate configuration similar to that used for the Casimir force as shown in the figure below.[257] Each plate is composed of niobium in a superconducting state.

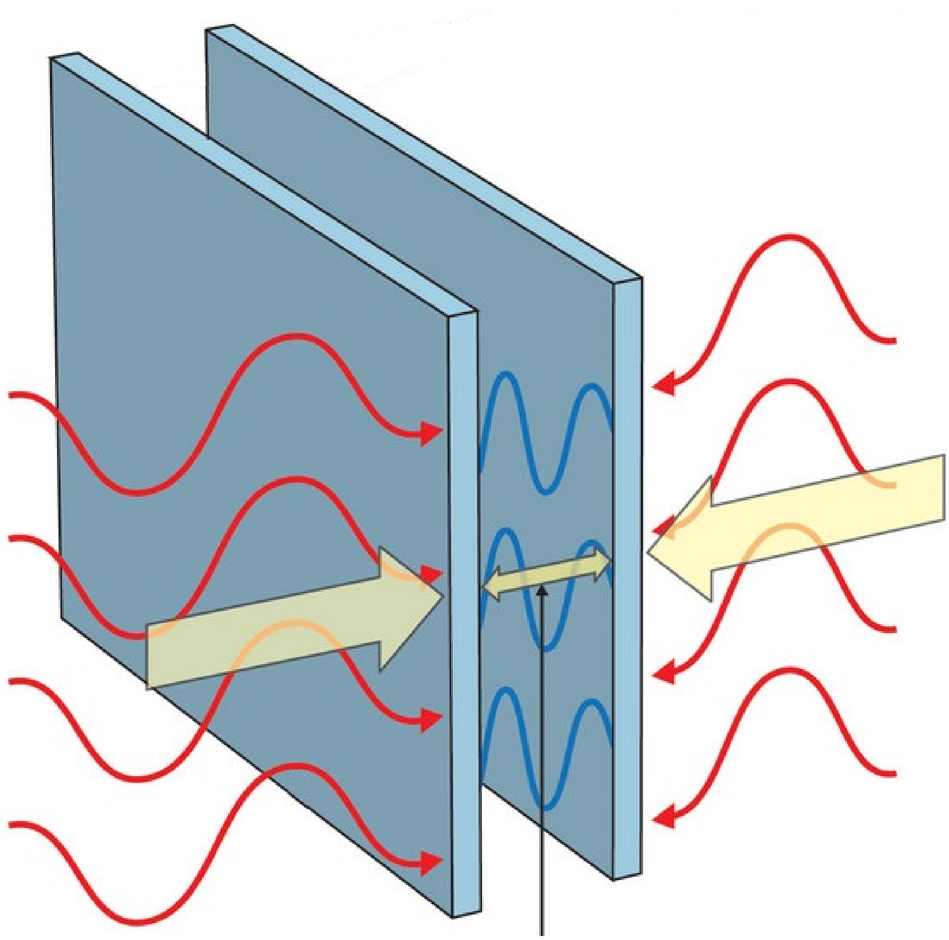

Figure 30: Casimir force on a superconducting parallel-plate configuration

Let us assume that super-conductors can act as mirrors for gravitational waves as described in [7]. Then quantum mechanically, the lowest energy mode for an oscillating gravitational field is given by the ground state energy (zero-point energy) due to quantum vacuum fluctuations.

$$U = \frac{1}{2}\hbar\omega \tag{2339}$$

To examine the gravitational fields associated with this energy, we can begin with the Einstein Field equations.

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{2340}$$

where $\kappa = 8\pi G/c^4$ is the Einstein coupling constant. Since a gravitational field in a quantum ground state would be extremely weak, we can use the linearized, weak-field limit of the Einstein equations. (See Appendix A for details.) In the non-relativistic, weak-field limit, we have $T_{\mu\nu} \approx 0$ for $\mu, \nu \neq 0$ and the lowest order

[257] A gravitational Casimir force is considered in [103] by Quach.

contribution to the energy-momentum tensor is

$$T_{00} \approx \rho c^2 \tag{2341}$$

where ρ is the volume mass density of the plates. The metric in the weak-field limit can be considered as a small perturbation about a flat Minkowski space-time metric: $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu} << 1$. The Newtonian limit gives us $g_{00} \simeq -1 + h_{00}$ where $h_{00} = -2\varphi_G/c^2$ and φ_G is the gravitational potential. Therefore we can use

$$h_{00} = -\frac{2\varphi_G}{c^2} \tag{2342}$$

As shown in Appendix A, the linearized, weak-field limit of the Einstein equations leads to

$$\nabla^2 h_{00} \approx -\kappa T_{00} \tag{2343}$$

Substituting (2341) and (2342) into (2343) gives

$$\nabla^2 \left(\frac{2\varphi_G}{c^2}\right) = \kappa\rho c^2 \tag{2344}$$

The potential energy is related to the gravitational potential by $U = m\varphi_G$ where m is the mass of either superconducting plate. If we define D as the surface mass density and A as the area of a plate, then the mass of a plate is given by $m = DA$. We can therefore write the potential energy as

$$U = DA\varphi_G \tag{2345}$$

For the ground state energy of a gravitational field, we can use $U = \frac{1}{2}\hbar\omega$ from (2339). Substituting this into (2345) and solving for φ_G gives

$$\varphi_G = \frac{\hbar\omega}{2DA} \tag{2346}$$

Now we can insert (2346) into (2344) and consider ∇ to be on the order of $1/\lambda$ where λ is the wavelength of the gravitational oscillating field. This yields

$$\left(\frac{1}{\lambda}\right)^2 \left(\frac{\hbar\omega}{c^2 DA}\right) \approx \kappa\rho c^2 \tag{2347}$$

To simplify, we can use $\lambda = c/f$, $\hbar\omega = hf$, and $\kappa = 8\pi G/c^4$. Then solving for D gives

$$D \approx \left(\frac{h}{8\pi G c^2}\right)\left(\frac{f^3}{\mu A}\right) \tag{2348}$$

The mass density for niobium is $\rho \approx 8.6\ kg/m^3$. For microwave oscillations of the gravitational field, we can use $f \approx 10^{10} Hz$. A reasonable area for the plates is on the order of centimeters, $10^{-2} m$. Using these values to determine a result for the surface mass density (in SI units) gives

$$\boxed{D \approx 1.03 \times 10^{-10} kg/m^2} \tag{2349}$$

Thus we find that a very small surface mass density is required to have microwave gravitational oscillations in a quantum ground state between two superconducting niobium plates with an area on the order of centimeters. This indicates that reflection of gravitational fields does *not* require extreme mass densities which are present only in astrophysical systems. Rather, we find that incorporating quantum mechanics (namely, by introducing Planck's constant via a ground state energy) opens the possibility of probing the effects of gravitational radiation in a laboratory-scale setting. For details concerning the specific mechanism by which gravitational radiation can be reflected by superconductors, we refer the reader to the treatment found in [7]. Furthermore, if gravitational waves can be reflected, we anticipate that parametric amplification of these waves could also be achieved as described in [4].

23.8 Quantum versus classical sources of gravitons in a cavity

The Isaacson power flux formula for a gravitational wave is given in [43] as[258]

$$\mathscr{P}_{Isaacson} = \frac{c^3}{16\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle \quad (2350)$$

where $\mathscr{P}$ is the power per unit area (P/A) while h_+ and $h_\times$ are the plus and cross polarization strain fields. The particular polarization is not relevant to the analysis here so we can simply use h_0 for the amplitude and write a sinusoidal strain field as

$$h = h_0 \sin\left(\vec{k}\cdot\vec{x} - \omega t\right) \quad (2351)$$

Inserting this in the Isaacson power flux formula (2350) gives

$$\mathscr{P}_{Isaacson} = \frac{c^3}{16\pi G}\left\langle \omega^2 h_0^2 \sin^2\left(\vec{k}\cdot\vec{x} - \omega t\right)\right\rangle = \frac{c^3\omega^2 h_0^2}{32\pi G} \quad (2352)$$

We can also use the ground state energy of a single graviton $\left(E = \frac{1}{2}\hbar\omega\right)$, the area of the cavity (A), and the ring down time of the cavity ($\tau = Q/\omega$, where Q is the quality factor) to write the power flux for a single ground state graviton as

$$\mathscr{P}_{\substack{ground\\state}} = \frac{\frac{1}{2}\hbar\omega}{A\tau} = \frac{h\omega^2}{4\pi AQ} \quad (2353)$$

We can equate the power flux expressions in (2352) and (2353) and solve for the strain field amplitude, h_0. Using $Q = 10^9$ and an area with edges on the order of centimeters gives[259]

$$\boxed{h_0 = \sqrt{\frac{8Gh}{AQc^3}} \approx 1.45\times 10^{-37}} \quad (2354)$$

Gravitational strain field for the quantum ground state of a cavity

This is the strain field associated with a single graviton produced by the ground state quantum fluctuations of a cavity with quality factor Q and surface area A. It is interesting to note that the result is independent of the frequency. We can also consider what *classical* source would be required to produce the same strain. To do this, we can use the linearized Einstein equation for the transverse-traceless strain using the Helmholtz Decomposition formulation. From (361) we have

$$\Box h_{ij}^{\tau\tau} = -\frac{16\pi G}{c^4} T_{ij}^{\tau\tau} \quad (2355)$$

[258] Note that (2350) is typically derived in the transverse-traceless gauge where $h_{00} = h_{0i} = 0$ and $h_{ij} = h_{ij}^{TT}$. This leads to the gravitational wave field being expressed completely in terms of $h_{xx} = -h_{yy} = h_+$ for plus polarization, and $h_{xy} = h_{yx} = h_\times$ for cross polarization. However, the metric has the same form using the Helmholtz Decomposition formulation in the far field zone where $h_{00} = h_{0i} = 0$ and $h_{ij} = h_{ij}^{\tau\tau}$.

[259] Note that the quantity $A = Gh/c^3$ is the Planck area. Therefore we see that the quantity in (2354) is dimensionless as it should be for a strain field.

The retarded Green's function solution to this wave equation is

$$h_{ij}^{\tau\tau}(t,\vec{x}) \quad = \quad \frac{4G}{c^4}\int \frac{T_{ij}^{\tau\tau}(t_r,\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x' \tag{2356}$$

where $\vec{x}'$ is the spatial coordinate of each infinitesimal element of $T_{ij}^{\tau\tau}$ occupying a differential volume element d^3x. Also, $T_{ij}^{\tau\tau}(t_r,\vec{x}')$ is the stress-energy-momentum contribution at $\vec{x}'$ evaluated at a retarded time t_r and located at a distance $|\vec{x}-\vec{x}'|$ from the field point where $h_{ij}^{\tau\tau}$ is measured. We can therefore express the retarded time as $t_r = t - |\vec{x}-\vec{x}'|/c$. For simplicity, we can consider the case of relativistic dust (to order v^2/c^2) which we found in (431) gives

$$T_{ij}^{\tau\tau} \quad = \quad \rho\left(v_i v_j - \tfrac{1}{3}\delta_{ij}v^2\right) \tag{2357}$$

For plus polarization, we have $T_{\oplus} = \dfrac{2}{3}\rho v_x^2 = -\dfrac{2}{3}\rho v_y^2$. Likewise, for cross polarization, we have $T_{\otimes} = \rho v_x v_y$. Therefore, we can simply approximate $T \approx \rho v^2$ in (2356) and use h_{out} to represent the *outgoing* gravitational wave. This gives

$$h(t,\vec{x}) \quad = \quad \frac{4G}{c^4}\int \frac{\rho v^2}{|\vec{x}-\vec{x}'|}d^3x' \tag{2358}$$

where ρ and v^2 can be functions of $(t_r,\vec{x}')$. For small distance and time scales, we can neglect the dependence of ρ and v^2 on the retarded time and also consider them approximately uniform over a volume V. Let us also consider the distance from the source to the field point as remaining approximately a constant distance given by d. Then we have

$$h_0 \quad \approx \quad \frac{4G\rho v^2 V}{c^4 d} \tag{2359}$$

We can use $V = A\delta_G$ where $A \approx d^2$ is the surface area that the gravitational wave is incident upon and δ_G is gravitational penetration depth which characterizes the depth to which the gravitational wave field can induce transverse-traceless currents in the walls of the superconductor. Then solving for v gives

$$v \quad \approx \quad \sqrt{\frac{c^4 h_0}{4G\rho d\delta_G}} \tag{2360}$$

The classical result in (2360) is also independent of frequency just as the case for the previous calculation using the quantum ground state. However, (1334) shows that the gravitational penetration depth can be written in terms of the gravitational plasma frequency as $\delta_G = c/\omega_G$. We can consider a gravitational plasma frequency on the order of microwaves ($\omega_G \approx 10^{10}Hz$) and consider d to have a maximum value on the order of centimeters inside the cavity. We can also use the value of h_0 from (2354) and $\rho \approx 8.6\times 10^3 kg/m^3$ for Niobium. This gives

$$\boxed{v \approx \sqrt{\frac{c^4 h_0 \omega_G}{4G\rho dc}} \approx 1.7\times 10^6 m/s} \tag{2361}$$

This is a rough estimate of the velocities required for Niobium atoms to generate a single graviton at microwave frequencies.

We can also determine an estimate of the number of Niobium atoms that would be involved in producing this field. We can solve (2361) for ρ and multiply by a volume V to obtain the mass of Niobium atoms needed.

$$M \quad = \quad \rho V = \frac{c^4 h_0 V}{4Gv^2 d\delta_G} \tag{2362}$$

Once again, we can consider the volume containing the Niobium atoms to be described by the surface area of the cavity $\left(A \approx d^2\right)$ multiplied by a depth on the order of the gravitational penetration depth, δ_G. Then using (2362) and inserting the value found in (2361) gives

$$M \quad = \quad \frac{c^4 h_0 d}{4Gv^2} \approx 1.5 \times 10^{-8} kg \tag{2363}$$

Dividing by $m_{Niobium} \approx 1.5 \times 10^{-25} kg/atom$ for a single Niobium atom gives

$$\boxed{n \approx 10^{17} \text{ Niobium atoms}} \tag{2364}$$

This is the number of Niobium atoms within a gravitational penetration depth of the cavity wall which must move at the speed given in (2361) to produce a single microwave graviton.

Conclusion

In conclusion, the primary achievement of this dissertation was to develop a variety of methods for describing how gravity can be coupled to quantum mechanical and classical systems in ways that may yield experimentally testable results. Some examples include the interaction of gravity with superconductors and normal conductors, the possibility of scalar and vector gravitational Aharonov-Bohm effects, a gravitational Casimir effect between conducting parallel plates, and a gravitational wave transmitter-receiver system via superconducting ellipsoidal cavities. Some of the theoretical discoveries were found unexpectedly in the process of attempting to describe how gravitational waves interact with superconductors. Some examples are the formulation of a graviton mass in superconductors, and the discovery of a gravitationally induced dynamical Casimir effect in the phonon modes of the ionic lattice.

Perhaps one of the most important achievements of the dissertation was showing formally that a charge-separation effect occurs in a superconductor in response to a gravitational wave. This may provide a new way of detecting gravitational waves in a manner that was never previously attempted. It was also shown that this effect may play a role in a gravito-Meissner effect which would expel gravitational waves from superconductors in the DC limit, and hence lead to the possibility of reflection of incident gravitational waves.

However, a primary question that could not be conclusively answered is whether the generation and detection of gravitational waves in a laboratory setting is possible. Although a comprehensive mathematical framework was successfully developed to investigate this question, it remains an open question since there are still phenomenological parameters which require numerical values need to be known. In particular, the gravitational permeativity is the remaining undetermined quantity which will dictate whether lab scale superconductors can reflect gravitational waves. It still needs to be determined if this quantity can be theoretically estimated (similar to the London penetration depth) or whether it can only be experimentally measured. Therefore, further research of this topic is required. Such research may involve developing a “free stress tensor” (analogous to the free current in electromagnetism) and formulating gravitational wave auxiliary fields and constitutive equations in matter which could lead to a numerical estimate for the value of the gravitational permeativity of a superconductor.

On a related note, another important future research goal is to demonstrate conclusively how the gravitational penetration depth and London penetration depth are related. It was found that one can arrive at a result that implies that the gravitational penetration depth is half of the London penetration depth. However, the interpretation of this result is still open to debate. Further analysis and understanding of this topic is certainly needed.

Lastly, it would be a lofty but achievable goal to write a comprehensive textbook describing the way gravitation interacts with classical and quantum matter, similar to the comprehensive framework that already exists for electromagnetism. In fact, the work in this dissertation is ultimately intended to contribute to the broader goal of developing new areas of gravitational physics such as gravitational quantum optics and gravitational laser physics. This would undoubtedly lead to a new era of technology and advancements that have never before been imagined.

25 Appendices

Appendix A

Review of linearized General Relativity

The linearized Christoffel symbols and Riemann tensor

In the weak-field limit, the metric can be considered as a small perturbation about the flat Minkowski space-time metric.

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu} \qquad \text{where} \qquad \left|h_{\mu\nu}\right| << 1 \tag{2365}$$

We employ the usual process for evaluating the Einstein tensor by finding the linearized Christoffel symbols, Riemann tensor, Ricci tenor and Ricci scalar. The Christoffel symbols can be found from

$$\Gamma^{\mu}_{\nu\sigma} = \tfrac{1}{2} g^{\mu\rho} \left(\partial_\sigma g_{\rho\nu} + \partial_\nu g_{\sigma\rho} - \partial_\rho g_{\nu\sigma}\right) \tag{2366}$$

We can substitute (2365) into (2366) to find the Christoffel symbols. For the derivatives we can use $\partial_\gamma g_{\delta\beta} = \partial_\gamma \left(\eta_{\delta\beta} + h_{\delta\beta}\right) = \partial_\gamma h_{\delta\beta}$. Then for a lowest order approximation, we can use $g^{\mu\delta} \approx \eta^{\mu\delta}$ so that we only keep terms of order $\partial_\gamma h_{\delta\beta}$. Notice that using $g^{\mu\delta} \approx \eta^{\mu\delta} - h^{\mu\delta}$ would result in higher order terms of order $h^{\mu\rho}\left(\partial_\sigma h_{\rho\nu}\right)$. Therefore, we have

$$\boxed{\Gamma^{\mu}_{\nu\gamma} = \tfrac{1}{2}\eta^{\mu\rho}\left(\partial_\gamma h_{\rho\nu} + \partial_\nu h_{\gamma\rho} - \partial_\rho h_{\nu\gamma}\right) \qquad \textit{Linearized Christoffel symbols}} \tag{2367}$$

The Riemann tensor is given by

$$R^{\mu}_{\nu\gamma\delta} = \partial_\gamma \Gamma^{\mu}_{\nu\delta} - \partial_\delta \Gamma^{\mu}_{\nu\gamma} + \Gamma^{\mu}_{\rho\gamma}\Gamma^{\rho}_{\nu\delta} - \Gamma^{\mu}_{\rho\delta}\Gamma^{\rho}_{\nu\gamma} \tag{2368}$$

We can substitute (2367) into (2368) to find the *linearized* Riemann tensor. The first two terms will be of order $\partial_\gamma\partial_\nu h_{\delta\beta}$ while the second two terms will be of a higher order, $\left(\partial_\gamma h_{\delta\beta}\right)^2$. So we can neglect the second two terms in (2368) and insert (2367) to obtain

$$R^{\mu}{}_{\nu\gamma\delta} = \tfrac{1}{2}\eta^{\mu\rho}\partial_\gamma\left(\partial_\delta h_{\rho\nu} + \partial_\nu h_{\delta\rho} - \partial_\rho h_{\nu\delta}\right) - \tfrac{1}{2}\eta^{\mu\rho}\partial_\delta\left(\partial_\gamma h_{\rho\nu} + \partial_\nu h_{\gamma\rho} - \partial_\rho h_{\nu\gamma}\right) \tag{2369}$$

Canceling $\partial_\gamma\partial_\delta h_{\rho\nu}$ gives

$$\boxed{\begin{array}{c} R^{\mu}{}_{\nu\gamma\delta} = \tfrac{1}{2}\eta^{\mu\rho}\left(\partial_\gamma\partial_\nu h_{\delta\rho} - \partial_\gamma\partial_\rho h_{\nu\delta} - \partial_\delta\partial_\nu h_{\gamma\rho} + \partial_\delta\partial_\rho h_{\nu\gamma}\right) \\ \\ \textit{Linearized Riemann tensor} \end{array}} \tag{2370}$$

This is the weak-field, linearized Riemann tensor. The term “linearized” is common in the literature for this formulation, however, it may be somewhat misrepresentative. “Linear” typically refers to the first term after the constant term in a Taylor expansion or a polynomial expansion of a function. After that comes the first derivative (second-order term), the second derivative (third-order term, etc.) However, here the *lowest* order Riemann tensor has *second* derivatives of the metric perturbation. In other words, it is *third* order in the metric perturbation. Therefore, when speaking of *linearized* GR, it should be emphasized that this means that the highest order in the metric perturbation is technically *third* order.

For example, consider a component of the metric perturbation expressed as $h = Ae^{ikx}$ where A is a constant. In that case, notice that the term that was neglected from the Christoffel symbols is of the order $h^{\mu\rho}\left(\partial_\sigma h_{\rho\nu}\right) \sim A^2 k$. For example, if a gravitational wave has an amplitude $A \sim 10^{-21}$ and a frequency in the microwave range, $k = \omega/c \sim 10^{10}/10^{8} = 10^{2}$, then we are neglecting terms of order $A^2 k \sim 10^{-40}$ in the Christoffel symbols. We are only keeping terms of order $\partial_\gamma h_{\delta\beta} \sim Ak \sim 10^{-19}$. Likewise, in the Riemann tensor we are neglecting terms of order $\left(\partial_\gamma h_{\delta\beta}\right)^2 \sim A^2 k^2 \sim 10^{-38}$. However, we keep terms of order

$\partial_\gamma \partial_\nu h_{\delta\beta} \sim Ak^2 \sim 10^{-17}$. Hence, it is evident that when dealing with extremely weak gravitational fields (such as terrestrial gravitational waves), the approximations in linearized GR are completely appropriate.

In the example above, we dealt with a very small amplitude and a relatively high frequency. In that case, we find that higher order *products* of the metric perturbation (such as h^2, h^3, etc.) are much smaller terms which are neglected in the Riemann tensor. On the other hand, higher order *derivatives* (such as $\partial^3 h$, $\partial^4 h$, etc.) yield terms that continue to increase gradually. Therefore, in such a situation, we find that higher order *products* of the metric are discarded while higher order *derivatives* of the metric are permitted.

On the other hand, if we consider a situation with a much larger amplitude and a much smaller frequency, then the approximation scheme changes. For example, consider a field where $A \sim 10^{-9}$ (such as the gravitational field of the earth). We know that the potential falls off as $1/r$. Therefore, the variation goes as the derivative of $1/r$ which is $1/r^2$. Near the surface of the earth, we can use the radius of the earth which is approximately $6 \times 10^3 km$. Therefore $1/r^2$ gives approximately 10^{-8} which would play the effective role of k. Then we find that the term that was neglected from the Christoffel symbols is of the order $A^2 k \sim 10^{-26}$. We are only keeping terms of order $Ak \sim 10^{-17}$. Likewise, in the Riemann tensor we are neglecting terms of order $A^2 k^2 \sim 10^{-34}$. However, we are keeping terms that are of order $Ak^2 \sim 10^{-25}$. Now in this particular situation, we find that *both* the higher order products as well as the higher order derivatives must be neglected. For example, the third derivative of the metric will be of the order $Ak^3 \sim 10^{-33}$. This is comparable to the terms we neglected in the Riemann tensor which we stated are of the order $A^2 k^2 \sim 10^{-34}$. This means that we cannot justifiably keep the third derivative (or higher order derivatives) of the metric in this situation.

<u>The linearized Einstein tensor</u>

The Ricci tensor can be found by contracting the first and third index of the Riemann tensor. Setting $\mu = \gamma$ in (2370) gives $R_{\nu\delta} = R^{\gamma}{}_{\nu\gamma\delta}$. This becomes

$$R_{\nu\delta} = \tfrac{1}{2}\eta^{\gamma\rho}\left(\partial_\gamma \partial_\nu h_{\delta\rho} - \partial_\gamma \partial_\rho h_{\nu\delta} - \partial_\delta \partial_\nu h_{\gamma\rho} + \partial_\delta \partial_\rho h_{\nu\gamma}\right) \tag{2371}$$

$$= \tfrac{1}{2}\left(\partial^\rho \partial_\nu h_{\delta\rho} - \Box h_{\nu\delta} - \partial_\delta \partial_\nu h + \partial_\delta \partial^\gamma h_{\nu\gamma}\right) \tag{2372}$$

where h is the trace of $h_{\mu\nu}$ and $\Box = \partial_\mu \partial^\mu = \nabla^2 - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}$. The Ricci scalar is the trace of the Ricci tensor, $R = \eta^{\nu\delta} R_{\nu\delta} = R_\nu{}^\nu$. This gives

$$R = \tfrac{1}{2}\eta^{\nu\delta}\left(\partial^\rho \partial_\nu h_{\delta\rho} - \Box h_{\nu\delta} - \partial_\delta \partial_\nu h + \partial_\delta \partial^\gamma h_{\nu\gamma}\right) \tag{2373}$$

$$= \tfrac{1}{2}\left(\partial^\rho \partial^\delta h_{\delta\rho} - \Box h - \Box h + \partial^\nu \partial^\gamma h_{\nu\gamma}\right) \tag{2374}$$

$$= -\Box h + \partial^\nu \partial^\gamma h_{\nu\gamma} \tag{2375}$$

The Einstein tensor is just the trace-reversed Ricci tensor

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} R \eta_{\mu\nu} \tag{2376}$$

Substituting (2372) and (2375) into (2376) and using μ and ν for free indices gives

$$G_{\mu\nu} = \tfrac{1}{2}\left(\partial^\gamma \partial_\mu h_{\nu\gamma} - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h + \partial_\nu \partial^\gamma h_{\mu\gamma}\right) - \tfrac{1}{2}\left(-\Box h + \partial^\rho \partial^\gamma h_{\rho\gamma}\right)\eta_{\mu\nu} \tag{2377}$$

$$= \tfrac{1}{2}\left(\partial^\gamma \partial_\mu h_{\nu\gamma} - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h + \partial_\nu \partial^\gamma h_{\mu\gamma} + \eta_{\mu\nu}\Box h - \eta_{\mu\nu}\partial^\rho \partial^\gamma h_{\rho\gamma}\right) \tag{2378}$$

Rearranging terms gives the resulting linearized Einstein tensor.

$$\boxed{\begin{array}{c} G_{\mu\nu} = \frac{1}{2}\left(\partial^\gamma\partial_\mu h_{\gamma\nu} + \partial^\gamma\partial_\nu h_{\gamma\mu} + \eta_{\mu\nu}\Box h - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\partial^\rho\partial^\gamma h_{\rho\gamma}\right) \\ \\ \textit{Linearized Einstein tensor} \end{array}} \tag{2379}$$

From (2379) we can obtain each of the linearized Einstein tensor components, G_{00}, G_{0i}, and G_{ij}. First we sum over the indices in (2379) and expand the first box operator as $\Box = \partial_k\partial^k - \partial_t^2/c^2$.

$$\begin{aligned} G_{\mu\nu} &= \tfrac{1}{2}\Big[\Big(\partial^0\partial_\mu h_{0\nu} + \partial^k\partial_\mu h_{k\nu}\Big) + \Big(\partial^0\partial_\nu h_{0\mu} + \partial^k\partial_\nu h_{k\mu}\Big) + \eta_{\mu\nu}\partial_k\partial^k h - \eta_{\mu\nu}\ddot{h}/c^2 \\ &\quad -\Box h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{\mu\nu}\Big(\partial^0\partial^0 h_{00} + \partial^0\partial^k h_{0k} + \partial^k\partial^0 h_{k0} + \partial^k\partial^l h_{kl}\Big)\Big] \end{aligned} \tag{2380}$$

Writing the full trace of the metric perturbation as

$$h = \eta_{\mu\nu}h^{\mu\nu} = \eta_{00}h^{00} + \eta_{ii}h^{ii} = -h_{00} + H \tag{2381}$$

where $H = \delta^{ij}h_{ij} = h^i{}_i$ is the trace of the purely spatial part of the metric perturbation. Also, writing $\partial^0 = -\partial_t/c$ and using the dot "dot derivative" notation gives

$$\begin{aligned} G_{\mu\nu} &= \tfrac{1}{2}\Big[-\partial_\mu\dot{h}_{0\nu}/c + \partial^k\partial_\mu h_{k\nu} - \partial_\nu\dot{h}_{0\mu}/c + \partial^k\partial_\nu h_{k\mu} + \eta_{\mu\nu}\partial_k\partial^k\left(-h_{00} + H\right) \\ &\quad -\eta_{\mu\nu}\left(-\ddot{h}_{00} + \ddot{H}\right)/c^2 - \Box h_{\mu\nu} - \partial_\mu\partial_\nu\left(-h_{00} + H\right) \\ &\quad -\eta_{\mu\nu}\Big(\ddot{h}_{00}/c^2 - 2\partial^k\dot{h}_{0k}/c + \partial^k\partial^l h_{kl}\Big) \end{aligned} \tag{2382}$$

Canceling similar terms gives

$$\begin{aligned} G_{\mu\nu} &= \tfrac{1}{2}\Big[-\partial_\mu\dot{h}_{0\nu}/c + \partial^k\partial_\mu h_{k\nu} - \partial_\nu\dot{h}_{0\mu}/c + \partial^k\partial_\nu h_{k\mu} + \eta_{\mu\nu}\partial_k\partial^k\left(H - h_{00}\right) \\ &\quad -\eta_{\mu\nu}\ddot{H}/c^2 - \Box h_{\mu\nu} + \partial_\mu\partial_\nu\left(h_{00} - H\right) + \eta_{\mu\nu}\Big(2\partial^k\dot{h}_{0k}/c - \partial^k\partial^l h_{kl}\Big)\Big] \end{aligned} \tag{2383}$$

Evaluating G_{00} and expanding the box operator as $\Box = \partial_k\partial^k - \partial_t^2/c^2$ gives

$$\begin{aligned} G_{00} &= \tfrac{1}{2}\Big(-\ddot{h}_{00}/c^2 + \partial^k\dot{h}_{k0}/c - \ddot{h}_{00}/c^2 + \partial^k\dot{h}_{k0}/c - \partial_k\partial^k\left(H - h_{00}\right) \\ &\quad +\ddot{H}/c^2 - \Big(\partial_k\partial^k h_{00} - \ddot{h}_{00}/c^2\Big) + \left(\ddot{h}_{00} - \ddot{H}\right)/c^2 - 2\partial^k\dot{h}_{0k}/c + \partial^k\partial^l h_{kl}\Big) \end{aligned} \tag{2384}$$

Canceling similar terms, using $\partial_k\partial^k = \nabla^2$ and writing the last term as $\partial_i\partial_j h_{ij}$ gives

$$\boxed{G_{00} = \tfrac{1}{2}\Big(\partial_i\partial_j h_{ij} - \nabla^2 H\Big)} \tag{2385}$$

Returning to (2383) to evaluate G_{0i} and expanding the box operator as $\Box = \partial_k\partial^k - \partial_t^2/c^2$ gives

$$\begin{aligned} G_{0i} &= \tfrac{1}{2}\Big[-\partial_0\dot{h}_{0i}/c + \partial^k\partial_0 h_{ki} - \partial_\nu\dot{h}_{00}/c + \partial^k\partial_i h_{k0} + \eta_{0i}\partial_k\partial^k\left(H - h_{00}\right) \\ &\quad -\eta_{0i}\ddot{H}/c^2 - \Big(\partial_k\partial^k h_{0i} - \ddot{h}_{0i}/c^2\Big) + \partial_0\partial_i\left(h_{00} - H\right)\Big] \\ &\quad +\eta_{0i}\Big(2\partial^k\dot{h}_{0k}/c - \partial^k\partial^l h_{kl}\Big)\Big] \end{aligned} \tag{2386}$$

Canceling similar terms, using $\partial_k \partial^k = \nabla^2$, lowering spatial indices (which is valid to first order in the metric) and rearranging terms gives

$$\boxed{G_{0i} = \tfrac{1}{2}\left(\partial_k \dot{h}_{ki}/c - \partial_i \dot{H}/c + \partial_i \partial_k h_{k0} - \nabla^2 h_{0i}\right)} \tag{2387}$$

Lastly, returning to (2383) and evaluating G_{ij} gives

$$\begin{aligned} G_{ij} &= \tfrac{1}{2}\Big[-\partial_i \dot{h}_{0j}/c + \partial^k \partial_i h_{kj} - \partial_j \dot{h}_{0i}/c + \partial^k \partial_j h_{ki} + \eta_{ij} \partial_k \partial^k (H - h_{00}) \\ &\quad -\eta_{ij} \ddot{H}/c^2 - \Box h_{ij} + \partial_i \partial_j (h_{00} - H) + \eta_{ij}\left(2\partial^k \dot{h}_{0k}/c - \partial^k \partial^l h_{kl}\right)\Big] \end{aligned} \tag{2388}$$

Using $\partial_k \partial^k = \nabla^2$, writing $\eta_{ij} = \delta_{ij}$, lowering spatial indices and rearranging gives

$$\boxed{\begin{aligned} G_{ij} = \tfrac{1}{2}\Big[&-\Box h_{ij} - \left(\partial_i \dot{h}_{0j} + \partial_j \dot{h}_{0i}\right)/c + \partial_k \partial_i h_{kj} + \partial_k \partial_j h_{ki} + \partial_i \partial_j (h_{00} - H) \\ &+2\delta_{ij} \partial_k \dot{h}_{0k}/c - \delta_{ij} \partial_k \partial_l h_{kl} - \delta_{ij} \nabla^2 h_{00} + \delta_{ij} \nabla^2 H - \delta_{ij} \ddot{H}/c^2\Big] \end{aligned}} \tag{2389}$$

The linearized Einstein equation and conservation of the stress-energy-momentum tensor

Since the Einstein equation is given by $G_{\mu\nu} = \kappa T_{\mu\nu}$, then using (2379) we can write

$$2\kappa T_{\mu\nu} = \partial^\rho \partial_\mu h_{\rho\nu} + \partial^\rho \partial_\nu h_{\rho\mu} + \eta_{\mu\nu} \Box h - \Box h_{\mu\nu} - \eta_{\mu\nu} \partial^\sigma \partial^\rho h_{\sigma\rho} - \partial_\mu \partial_\nu h \tag{2390}$$

If we raise the indices (using $g^{\alpha\mu} g^{\beta\nu} \approx \eta^{\alpha\mu} \eta^{\beta\nu}$ to maintain only first order in the metric), then we have

$$\begin{aligned} 2\kappa T_{\mu\nu} &= \eta^{\beta\nu} \partial^\rho \partial^\alpha h_{\rho\nu} + \eta^{\alpha\mu} \partial^\rho \partial^\beta h_{\rho\mu} + \eta^{\alpha\beta} \Box h \\ &\quad -\Box h^{\alpha\beta} - \eta^{\alpha\beta} \partial^\sigma \partial^\rho h_{\sigma\rho} - \partial^\alpha \partial^\beta h \end{aligned} \tag{2391}$$

Now taking ∂_α of both sides gives

$$2\kappa \partial_\alpha T^{\alpha\beta} = \eta^{\beta\nu} \partial^\rho \Box h_{\rho\nu} + \partial^\mu \partial^\rho \partial^\beta h_{\rho\mu} + \partial^\beta \Box h - \partial_\alpha \Box h^{\alpha\beta} - \partial^\beta \partial^\sigma \partial^\rho h_{\sigma\rho} - \Box \partial^\beta h \tag{2392}$$

$$= \partial_\rho \Box h^{\beta\rho} + \partial^\mu \partial^\rho \partial^\beta h_{\rho\mu} + \partial^\beta \Box h - \partial_\alpha \Box h^{\alpha\beta} - \partial^\beta \partial^\sigma \partial^\rho h_{\sigma\rho} - \Box \partial^\beta h \tag{2393}$$

On the right side, we find that the first and fourth terms cancel, the second and fifth terms cancel, and the third and sixth terms cancel. Therefore $2\kappa \partial_\alpha T^{\alpha\beta} = 0$ and we can write the *linearized* conservation of mass-energy-momentum as

$$\boxed{\partial_\nu T^{\mu\nu} = 0 \qquad \textit{Linearized conversation of energy-momentum}} \tag{2394}$$

The full *non*-linear conservation of stress-energy-momentum is written in terms of covariant derivatives as $\nabla_\nu T^{\mu\nu} = \partial_\nu T^{\mu\nu} + \Gamma^\nu_{\nu\sigma} T^{\sigma\mu} + \Gamma^\mu_{\nu\sigma} T^{\nu\sigma}$. From (2367), we know that to first order in the metric, the Christoffel symbols involve terms with $\partial_\sigma h_{\mu\nu}$. Therefore, using $\partial_\nu T^{\mu\nu} = 0$ as the *linear* conservation of stress-energy-momentum involves neglecting terms of order $\left(\partial_\sigma h_{\mu\nu}\right) T^{\mu\nu}$.

We can also write (2394) in terms of the *covariant* stress tensor, $T_{\mu\nu}$. First we write (2394) as $\partial^\nu T^\mu{}_\nu = 0$. Using the inverse metric, we can also write this as

$$\partial^\nu T^\sigma{}_\nu = \partial^\nu \left(g^{\sigma\mu} T_{\mu\nu}\right) = 0 \tag{2395}$$

In (2415) we show that the inverse metric (to first order in the metric) is given by $g^{\sigma\mu} = \eta^{\sigma\mu} - h^{\sigma\mu}$ so the equation directly above becomes

$$\partial^{\nu}\left[\left(\eta^{\sigma\mu} - h^{\sigma\mu}\right)T_{\mu\nu}\right] = 0 \tag{2396}$$

As stated above, in this linearized approximation we must neglect terms involving $\left(\partial_{\sigma}h_{\mu\nu}\right)T^{\mu\nu}$. Therefore the equation above becomes

$$\partial^{\nu}\left(\eta^{\sigma\mu}T_{\mu\nu}\right) = 0 \tag{2397}$$

By the product rule we have

$$\left(\partial^{\nu}\eta^{\sigma\mu}\right)T_{\mu\nu} + \eta^{\sigma\mu}\left(\partial^{\nu}T_{\mu\nu}\right) = 0 \tag{2398}$$

The first term is obviously zero and therefore we are left with $\eta^{\sigma\mu}\left(\partial^{\nu}T_{\mu\nu}\right) = 0$. Applying $\eta_{\sigma\mu}$ to both sides gives

$$\boxed{\partial^{\nu}T_{\mu\nu} = 0} \tag{2399}$$

<u>Linearized gauge freedom</u>

In general, the Einstein tensor is invariant under a linear coordinate transformation (or diffeomorphism)

$$x'^{\mu} = x^{\mu} - \xi^{\mu} \tag{2400}$$

where ξ^{μ} is an arbitrary 4-displacement vector. We know the inverse metric transforms as

$$g'^{\mu\nu} = \frac{\partial x^{\mu'}}{\partial x^{\sigma}}\frac{\partial x^{\nu'}}{\partial x^{\rho}}g^{\sigma\rho} \tag{2401}$$

Using (2400) to evaluate one of the derivatives in (2401) gives

$$\frac{\partial x^{\mu'}}{\partial x^{\sigma}} = \frac{\partial}{\partial x^{\sigma}}\left(x^{\mu} - \xi^{\mu}\right) = \partial_{\sigma}x^{\mu} - \partial_{\sigma}\xi^{\mu} = \delta_{\sigma}{}^{\mu} - \partial_{\sigma}\xi^{\mu} \tag{2402}$$

Now substituting (2402) into the right side of (2401) gives

$$\begin{aligned} g'^{\mu\nu} &= \left(\delta_{\sigma}{}^{\mu} - \partial_{\sigma}\xi^{\mu}\right)\left(\delta_{\rho}{}^{\nu} - \partial_{\rho}\xi^{\nu}\right)g^{\sigma\rho} && (2403)\\ &= \left[\delta_{\sigma}{}^{\mu}\delta_{\rho}{}^{\nu} - \partial_{\sigma}\xi^{\mu}\delta_{\rho}{}^{\nu} - \delta_{\sigma}{}^{\mu}\partial_{\rho}\xi^{\nu} + \left(\partial_{\sigma}\xi^{\mu}\right)\left(\partial_{\rho}\xi^{\nu}\right)\right]g^{\sigma\rho} && (2404)\\ &= g^{\mu\nu} - \left(\partial_{\sigma}\xi^{\mu}\right)g^{\sigma\nu} - \left(\partial_{\rho}\xi^{\nu}\right)g^{\mu\rho} + \left(\partial_{\sigma}\xi^{\mu}\right)\left(\partial_{\rho}\xi^{\nu}\right)g^{\sigma\rho} && (2405) \end{aligned}$$

We can use the metric to raise the indices of the derivatives and obtain

$$\boxed{g'^{\mu\nu} = g^{\mu\nu} - \partial^{\nu}\xi^{\mu} - \partial^{\mu}\xi^{\nu} + \left(\partial_{\sigma}\xi^{\mu}\right)\left(\partial^{\sigma}\xi^{\nu}\right) \qquad \textit{Metric gauge freedom}} \tag{2406}$$

This is the full gauge freedom of the metric for a linear coordinate transformation. However, in (2365) we chose to consider the weak-field limit such that $g^{\mu\nu} \approx \delta^{\mu\nu} + h^{\mu\nu}$ with $|h^{\mu\nu}| << 1$. It is valid to consider that $g'^{\mu\nu}$ is also of the same order as $g^{\mu\nu}$ only if we require that $\partial^{\nu}\xi^{\mu}$ is of the same order as $h^{\mu\nu}$. In that case, if we only keep first order in $h^{\mu\nu}$, then we can only keep first order in $\partial^\nu\xi^\mu$ which means we must neglect the last term in (2406). This gives

$$g'^{\mu\nu} \approx g^{\mu\nu} - \partial^\nu\xi^\mu - \partial^\mu\xi^\nu \tag{2407}$$

To evaluate this in terms of the metric perturbation, we need an expression for the inverse metric. We can define the inverse metric as

$$g^{\mu\nu} = \eta^{\mu\nu} + f^{\mu\nu} \tag{2408}$$

The inverse metric must satisfy

$$g_{\mu\sigma}g^{\sigma\nu} = \delta_\mu^{\ \nu} \tag{2409}$$

Inserting (2365) and (2408) into (2409) gives

$$\left(\eta_{\mu\sigma} + h_{\mu\sigma}\right)\left(\eta^{\sigma\nu} + f^{\sigma\nu}\right) = \delta_\mu^{\ \nu} \tag{2410}$$

$$\eta_{\mu\sigma}\eta^{\sigma\nu} + \eta_{\mu\sigma}f^{\sigma\nu} + h_{\mu\sigma}\eta^{\sigma\nu} + h_{\mu\sigma}f^{\sigma\nu} = \delta_\mu^{\ \nu} \tag{2411}$$

To first order we can neglect $h_{\mu\sigma}f^{\sigma\nu}$. Also using $\eta_{\mu\sigma}\eta^{\sigma\nu} = \delta_\mu^{\ \nu}$ gives

$$\delta_\mu^{\ \nu} + \eta_{\mu\sigma}f^{\sigma\nu} + h_{\mu\sigma}\eta^{\sigma\nu} = \delta_\mu^{\ \nu} \tag{2412}$$

$$\eta_{\mu\sigma}f^{\sigma\nu} = -h_\mu^{\ \nu} \tag{2413}$$

We can apply $\eta^{\mu\rho}$ to both sides. On the left side we can use $\eta^{\mu\rho}\eta_{\mu\sigma} = \delta_\sigma^{\ \rho}$ and the right side we can use $\eta^{\mu\rho}$ to raise the index of $h_\mu^{\ \nu}$ to linear order. This gives

$$f^{\rho\nu} = -h^{\rho\nu} \tag{2414}$$

Then the inverse metric in (2408) can be written as

$$\boxed{g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \qquad \textit{Inverse metric (to first order)}} \tag{2415}$$

Substituting (2415) on both sides of (2407) gives

$$\eta'^{\mu\nu} - h'^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} - \partial^\nu\xi^\mu - \partial^\mu\xi^\nu \tag{2416}$$

$$h'^{\mu\nu} = h^{\mu\nu} + \partial^\mu\xi^\nu + \partial^\nu\xi^\mu \tag{2417}$$

To lower all the indices, we can apply the metric twice recognizing that to first order in the metric we have $g'_{\mu\sigma}g'_{\nu\rho} \approx \eta'_{\mu\sigma}\eta'_{\nu\rho}$ and $g_{\mu\sigma}g_{\nu\rho} \approx \eta_{\mu\sigma}\eta_{\nu\rho}$. Then we obtain $h'_{\sigma\rho} = h_{\sigma\rho} + \partial_\sigma\xi_\rho + \partial_\rho\xi_\sigma$. Changing the indices to μ and ν gives

$$\boxed{h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu \qquad \begin{array}{c}\textit{Linearized metric perturbation}\\ \textit{gauge freedom}\end{array}} \tag{2418}$$

This is the linearized gauge freedom of the metric to first order in $h_{\mu\nu}$ and $\partial_\mu\xi_\nu$ under a linear coordinate transformation $x'^\mu = x^\mu - \xi^\mu$. From (2402) we see that if $\partial_\mu\xi_\nu$ is small, then $\partial_\rho x'^\mu$ must also be correspondingly small. This implies that the coordinates x'^μ cannot have a large *rate of change* with respect to the coordinates x^μ. However, having $\partial_\mu\xi_\nu$ be small does not imply that we must keep ξ_μ itself small. In other words, we can transform between coordinates with an arbitrarily large 4-displacement between them,

$\xi^{\mu} = x^{\mu} - x'^{\mu}$, and still not violate the first order approximation of the metric. More concretely, this means that we can transform between coordinates that are separated by large spatial distances and large intervals of time without violating the first order approximation for the transformed metric. We simply cannot transform to coordinates where the spatial coordinates are greatly curved (or compressed/stretched), or the clocks run much faster/slower. In that case, if $\partial_{\mu}\xi_{\nu}$ is not small enough to be comparable to $h_{\mu\nu}$, then a first order approximation for the transformed metric may not be valid.

Transformation of derivatives of the metric to first order in the metric

In general, we can show that a linear coordinate transformation, $x'^{\mu} = x^{\mu} - \xi^{\mu}$, does not affect differentiation. That is $\partial_{\mu'} \approx \partial_{\mu}$ to first order in $h_{\mu\nu}$ and $\partial^{\mu}\xi^{\nu}$. To demonstrate this we may apply the chain rule to write

$$\partial_{\mu} = \frac{\partial x'^{\nu}}{\partial x^{\mu}}\frac{\partial}{\partial x'^{\nu}} = \frac{\partial x'^{\nu}}{\partial x^{\mu}}\partial_{\nu'} \tag{2419}$$

To evaluate $\frac{\partial x'^{\nu}}{\partial x^{\mu}}$, we can substitute $x'^{\mu} = x^{\mu} - \xi^{\mu}$ and apply the chain rule.

$$\frac{\partial x'^{\nu}}{\partial x^{\mu}} = \frac{\partial\left(x^{\nu} - \xi^{\nu}\right)}{\partial x^{\mu}} = \delta^{\nu}_{\mu} - \partial_{\mu}\xi^{\nu} \tag{2420}$$

Inserting this into (2419) gives

$$\partial_{\mu} = \left(\delta^{\nu}_{\mu} - \partial_{\mu}\xi^{\nu}\right)\partial_{\nu'} = \partial_{\mu'} - \partial_{\mu}\xi^{\nu}\partial_{\nu'} \tag{2421}$$

Applying this to the metric perturbation gives

$$\partial_{\mu}h^{\rho\sigma} = \partial_{\mu'}h^{\rho\sigma} - \partial_{\mu}\xi^{\nu}\partial_{\nu'}h^{\rho\sigma} \tag{2422}$$

Since we are neglecting terms or order $\partial_{\mu}\xi^{\nu}\partial_{\nu'}h^{\rho\sigma}$, then we simply have

$$\partial_{\mu}h^{\rho\sigma} = \partial_{\mu'}h^{\rho\sigma} \tag{2423}$$

Therefore, we find that to first order in $h_{\mu\nu}$ and $\partial^{\mu}\xi^{\nu}$ we have

$$\boxed{\partial_{\mu'} \approx \partial_{\mu} \qquad \textit{Gauge invariance of the four-derivative}} \tag{2424}$$

We can also show that this result is consistent with the linearized gauge freedom for $h'_{\mu\nu}$ as given in (2418). Taking the derivative of $h'_{\mu\nu}$ in (2418) gives

$$\partial'_{\sigma}h'_{\mu\nu} = \frac{\partial h'_{\mu\nu}}{\partial x'^{\sigma}} = \frac{\partial\left(h_{\mu\nu} + \partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu}\right)}{\partial\left(x^{\sigma} - \xi^{\sigma}\right)} \tag{2425}$$

Dividing top and bottom of the right side by ∂x^{ρ} gives

$$\partial'_{\sigma}h'_{\mu\nu} = \frac{\partial\left(h_{\mu\nu} + \partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu}\right)/\partial x^{\rho}}{\partial\left(x^{\sigma} - \xi^{\sigma}\right)/\partial x^{\rho}} = \frac{\partial_{\rho}\left(h_{\mu\nu} + \partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu}\right)}{\delta_{\rho}{}^{\sigma} - \partial_{\rho}\xi^{\sigma}} \tag{2426}$$

Since we only keep $\partial_\rho \xi^\sigma$ to first order, then we can approximate $\left(\delta_\rho{}^\sigma - \partial_\rho \xi^\sigma\right)^{-1} \approx \delta_\rho{}^\sigma + \partial_\rho \xi^\sigma$. This gives

$$\partial'_\sigma h'_{\mu\nu} = \partial_\rho \left(h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu\right) \left(\delta_\rho{}^\sigma + \partial_\rho \xi^\sigma\right) \tag{2427}$$

Once again, to keep only fist order in $h_{\mu\nu}$ and $\partial_\mu \xi_\nu$, we must neglect squared terms in $\partial_\sigma \xi^\mu$ and products of $h_{\mu\nu}$ and $\partial_\mu \xi_\nu$. Therefore we must neglect $\partial_\rho \xi^\sigma$ in the last parentheses of (2427) which gives

$$\partial'_\sigma h'_{\mu\nu} = \partial_\sigma \left(h_{\mu\nu} + \partial_\sigma \partial_\mu \xi_\nu + \partial_\sigma \partial_\nu \xi_\mu\right) \tag{2428}$$

$$\partial'_\sigma h'_{\mu\nu} = \partial_\sigma h'_{\mu\nu} \tag{2429}$$

This means that in our linearized approximations, again we obtain $\partial'_\mu \approx \partial_\mu$. Therefore, throughout our treatment we consider that taking derivatives does not break gauge-invariance in the linearized approximation. The derivative of any gauge-invariant quantity is still a gauge-invariant quantity, provided we require that any linear coordinate transformation given by $x'^\mu = x^\mu - \xi^\mu$ satisfies $\left|\partial_\nu \xi^\mu\right| << 1$.

However, to maintain consistency with the linearization of the Riemann tensor at the beginning of this appendix, we still require that higher order derivatives do not yield terms which are larger than the terms that were neglected in the Riemann tensor. Otherwise, we must return to the calculation of the Riemann tensor and preserve higher order terms. This would significantly altar all of the formulation used throughout this entire dissertation.

Appendix B

Linearized General Relativity in the harmonic gauge

The trace-reversed metric perturbation

First we define the *trace-reversed* metric perturbation as

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}h \tag{2430}$$

where h is the trace. Since we know from (2381) that the trace can be written as $h = -h_{00} + H$, then the separate components of (2430) can be written as

$$\bar{h}_{00} = \tfrac{1}{2}\left(h_{00} + H\right), \qquad \bar{h}_{0i} = h_{0i}, \qquad \bar{h}_{ij} = h_{ij} + \tfrac{1}{2}\delta_{ij}\left(h_{00} - H\right) \tag{2431}$$

Note that if we take the trace of (2430) using $g^{\mu\nu} \approx \eta^{\mu\nu}$ to first order in the metric, then we obtain

$$\eta^{\mu\nu}\bar{h}_{\mu\nu} = \eta^{\mu\nu}h_{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\eta_{\mu\nu}h \tag{2432}$$

$$\bar{h}^{\nu}{}_{\nu} = h^{\nu}{}_{\nu} - \frac{1}{2}4h \tag{2433}$$

$$\bar{h} = -h \tag{2434}$$

Hence the trace of $\bar{h}_{\mu\nu}$ is the negative trace of $h_{\mu\nu}$, which is the reason for the name "trace-reversed" perturbation. Substituting $\bar{h} = -h$ into (2430) gives $\bar{h}_{\mu\nu} = h_{\mu\nu} + \dfrac{1}{2}\eta_{\mu\nu}\bar{h}$. Solving for $h_{\mu\nu}$ gives

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} \tag{2435}$$

Notice this has the same form as (2430) but with $h_{\mu\nu}$ and h exchanged with $\bar{h}_{\mu\nu}$ and $\bar{h}$, respectively.

The trace-reversed harmonic gauge[260]

Next we consider the divergence of $h'_{\mu\nu}$ from (2418) which gives

$$\partial^{\nu}h'_{\mu\nu} = \partial^{\nu}h_{\mu\nu} + \partial^{\nu}\partial_{\mu}\xi_{\nu} + \Box\xi_{\mu} \tag{2436}$$

This can be expressed in terms of the trace-reversed metric perturbation by inserting (2435).

$$\partial^{\nu}\left(\bar{h}'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}'\right) = \partial^{\nu}\left(\bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}\right) + \partial^{\nu}\partial_{\mu}\xi_{\nu} + \Box\xi_{\mu} \tag{2437}$$

$$\partial^{\nu}\bar{h}'_{\mu\nu} - \frac{1}{2}\partial_{\mu}\bar{h}' = \partial^{\nu}\bar{h}_{\mu\nu} - \frac{1}{2}\partial_{\mu}\bar{h} + \partial^{\nu}\partial_{\mu}\xi_{\nu} + \Box\xi_{\mu} \tag{2438}$$

[260]Typically, only the term "harmonic gauge" is used however, we use "trace-reversed harmonic gauge" to emphasize that the harmonic gauge is applied to the *trace-reversed* metric perturbation. As we show near the end of this appendix, applying the harmonic gauge to the *non*-trace-reversed metric perturbation does not yield the desired final result.

We can relate $\overline{h}'$ to $\overline{h}$ by taking the trace of (2418) which gives

$$\eta^{\mu\nu}h'_{\mu\nu} = \eta^{\mu\nu}h_{\mu\nu} + \eta^{\mu\nu}\partial_\mu\xi_\nu + \eta^{\mu\nu}\partial_\nu\xi_\mu \qquad (2439)$$

$$h' = h + \partial^\nu\xi_\nu + \partial^\mu\xi_\mu \qquad (2440)$$

Using $\overline{h} = -h$ from (2434) and combining the last two terms with a single repeating index gives

$$\overline{h}' = \overline{h} - 2\partial^\rho\xi_\rho \qquad (2441)$$

Substituting this into (2438) gives

$$\partial^\nu\overline{h}'_{\mu\nu} - \frac{1}{2}\partial_\mu\left(\overline{h} - 2\partial^\rho\xi_\rho\right) = \partial^\nu\overline{h}_{\mu\nu} - \frac{1}{2}\partial_\mu\overline{h} + \partial^\nu\partial_\mu\xi_\nu + \Box\xi_\mu \qquad (2442)$$

$$\partial^\nu\overline{h}'_{\mu\nu} = \partial^\nu\overline{h}_{\mu\nu} + \Box\xi_\mu \qquad (2443)$$

We can select ξ_μ so that $\partial^\nu\overline{h}_{\mu\nu} = \Box\xi_\mu$ which means $\partial^\nu\overline{h}'_{\mu\nu} = 0$. Dropping the prime gives

$$\boxed{\partial^\nu\overline{h}_{\mu\nu} = 0 \qquad \textit{Trace-reversed harmonic gauge}} \qquad (2444)$$

This is effectively a gauge choice[261] which is often referred to as the harmonic gauge[262]. It involves four constraint equations (one for each value of the index μ) and therefore removes four gauge degrees of freedom. Since the metric is a symmetric 4x4 tensor consisting of ten non-redundant degrees of freedom (four components on the diagonal and six off-diagonal), then this gauge choice leaves only six independent degrees of freedom remain. This is precisely the number of physical degrees of freedom predicted in GR since the four gauge degrees of freedom are explicitly determined by the four components of the gauge vector ξ^μ that generate linear transformations. The harmonic gauge is the basis for the gravito-electromagnetic formulation in Part I which is one of the most popular forms of gravito-electromagnetism used by various authors in the literature. (ref.)

Lastly, we also mention that the harmonic gauge is the usual starting point for developing the transverse-traceless (TT) gauge which is typically used for describing gravitational waves as shown in the following two appendices. We say it is the *usual* starting point because it is certainly not the *only* starting point. It is simply convenient because it already removes four gauge degrees of freedom which leaves only another four gauge degrees of freedom for the case of vacuum solutions which yields the TT gauge. In other words, the linearized gauge transformation is reduced from (2418) to (2443). In the TT gauge, this is reduced further with the choice of $\Box\xi_\mu = 0$. However, it is certainly possible to begin with *all* the degrees of freedom in the metric and eliminate eight gauge degrees of freedom through a single process in order to arrive at the TT gauge in vacuum. Both approaches are shown in Appendix F.

The Einstein tensor components in terms of the trace-reversed metric perturbation

We can express the Einstein tensor in terms of the trace-reversed metric perturbation by substituting

[261] This is directly analgous to the case in EM where the gauge freedom is $A'^\mu = A^\mu + \partial^\mu\chi$ so the divergence gives $\partial_\mu A'^\mu = \partial_\mu A^\mu + \Box\chi$. For the Lorenz gauge we choose $\partial_\mu A^\mu = -\Box\chi$ so that $\partial_\mu A'^\mu = 0$.

[262] This gauge has also been referred to as the *transverse or Einstein* or *Hilbert* or *de Donder gauge*, or the *harmonic coordinate system.*

(2435) and (2434) into (2379). This gives

$$\begin{aligned} G_{\mu\nu} &= \frac{1}{2}\left[\partial^{\gamma}\partial_{\mu}\left(\overline{h}_{\gamma\nu}-\frac{1}{2}\eta_{\gamma\nu}\overline{h}\right)+\partial^{\gamma}\partial_{\nu}\left(\overline{h}_{\gamma\mu}-\frac{1}{2}\eta_{\gamma\mu}\overline{h}\right)-\eta_{\mu\nu}\Box\overline{h}\right. \\ &\left.-\Box\left(\overline{h}_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}\overline{h}\right)+\partial_{\mu}\partial_{\nu}\overline{h}-\eta_{\mu\nu}\partial^{\rho}\partial^{\gamma}\left(\overline{h}_{\rho\gamma}-\frac{1}{2}\eta_{\rho\gamma}\overline{h}\right)\right] \end{aligned} \tag{2445}$$

Distributing and contracting indices gives

$$\begin{aligned} G_{\mu\nu} &= \frac{1}{2}\left(\partial^{\gamma}\partial_{\mu}\overline{h}_{\gamma\nu}-\frac{1}{2}\partial_{\nu}\partial_{\mu}\overline{h}+\partial^{\gamma}\partial_{\nu}\overline{h}_{\gamma\mu}-\frac{1}{2}\partial_{\mu}\partial_{\nu}\overline{h}-\eta_{\mu\nu}\Box\overline{h}\right. \\ &\left.-\Box\overline{h}_{\mu\nu}+\frac{1}{2}\eta_{\mu\nu}\Box\overline{h}+\partial_{\mu}\partial_{\nu}\overline{h}-\eta_{\mu\nu}\partial^{\rho}\partial^{\gamma}\overline{h}_{\rho\gamma}+\frac{1}{2}\eta_{\mu\nu}\Box\overline{h}\right) \end{aligned} \tag{2446}$$

Canceling and combining terms as well as rearranging gives

$$\boxed{\begin{gathered} G_{\mu\nu}=\tfrac{1}{2}\left(\partial^{\gamma}\partial_{\mu}\overline{h}_{\gamma\nu}+\partial^{\gamma}\partial_{\nu}\overline{h}_{\gamma\mu}-\eta_{\mu\nu}\partial^{\rho}\partial^{\gamma}\overline{h}_{\rho\gamma}-\Box\overline{h}_{\mu\nu}\right) \\ \textit{Linearized Einstein tensor in terms of the trace-reversed metric perturbation} \end{gathered}} \tag{2447}$$

Expanding the summations gives

$$\begin{aligned} G_{\mu\nu} &= \tfrac{1}{2}\left[\left(\partial^{0}\partial_{\mu}\overline{h}_{0\nu}+\partial^{k}\partial_{\mu}\overline{h}_{k\nu}\right)+\left(\partial^{0}\partial_{\nu}\overline{h}_{0\mu}+\partial^{k}\partial_{\nu}\overline{h}_{k\mu}\right)\right. \\ &\left.-\eta_{\mu\nu}\left(\partial^{0}\partial^{0}\overline{h}_{00}+2\partial^{0}\partial^{k}\overline{h}_{0k}+\partial^{k}\partial^{l}\overline{h}_{kl}\right)-\Box\overline{h}_{\mu\nu}\right] \end{aligned} \tag{2448}$$

Next we write an expression for each of the components of the Einstein tensor: G_{00}, G_{0i}, and G_{ij}. Evaluating G_{00} and expanding the box operator as $\Box=\partial_0\partial^0+\partial_k\partial^k$ gives

$$\begin{aligned} G_{00} &= \tfrac{1}{2}\left[\left(\partial^{0}\partial_{0}\overline{h}_{00}+\partial^{k}\partial_{0}\overline{h}_{k0}\right)+\left(\partial^{0}\partial_{0}\overline{h}_{00}+\partial^{k}\partial_{0}\overline{h}_{k0}\right)\right. \\ &\left.+\left(\partial^{0}\partial^{0}\overline{h}_{00}+2\partial^{0}\partial^{k}\overline{h}_{0k}+\partial^{k}\partial^{l}\overline{h}_{kl}\right)-\left(\partial_{0}\partial^{0}+\partial_{k}\partial^{k}\right)\overline{h}_{00}\right] \end{aligned} \tag{2449}$$

Since $\partial_0=-\partial^0$, then we can cancel terms to obtain

$$\boxed{G_{00}=\tfrac{1}{2}\left(\partial^{k}\partial^{l}\overline{h}_{kl}-\partial_{k}\partial^{k}\overline{h}_{00}\right)} \tag{2450}$$

Returning to (2448) to evaluate G_{0i} and expanding the box operator as $\Box=\partial_0\partial^0+\partial_k\partial^k$ gives

$$\begin{aligned} G_{0i} &= \tfrac{1}{2}\left[\left(\partial^{0}\partial_{0}\overline{h}_{0i}+\partial^{k}\partial_{0}\overline{h}_{ki}\right)+\left(\partial^{0}\partial_{i}\overline{h}_{00}+\partial^{k}\partial_{i}\overline{h}_{k0}\right)\right. \\ &\left.-\eta_{0i}\left(\partial^{0}\partial^{0}\overline{h}_{00}+2\partial^{0}\partial^{k}\overline{h}_{0k}+\partial^{k}\partial^{l}\overline{h}_{kl}\right)-\left(\partial_{0}\partial^{0}+\partial_{k}\partial^{k}\right)\overline{h}_{0i}\right] \end{aligned} \tag{2451}$$

Canceling terms and using $\eta_{0i}=0$ gives

$$\boxed{G_{0i}=\tfrac{1}{2}\left(\partial^{k}\partial_{0}\overline{h}_{ki}+\partial^{0}\partial_{i}\overline{h}_{00}+\partial^{k}\partial_{i}\overline{h}_{k0}-\partial_{k}\partial^{k}\overline{h}_{0i}\right)} \tag{2452}$$

Lastly, returning to (2448) to evaluate G_{ij} gives

$$\boxed{\begin{aligned} G_{ij} = \tfrac{1}{2}\big[&\partial^0\partial_i\bar{h}_{0j} + \partial^k\partial_i\bar{h}_{kj} + \partial^0\partial_j\bar{h}_{0i} + \partial^k\partial_j\bar{h}_{ki} \\ &-\delta_{ij}\left(\partial^0\partial^0\bar{h}_{00} + 2\partial^0\partial^k\bar{h}_{0k} + \partial^k\partial^l\bar{h}_{kl}\right) - \Box\bar{h}_{ij}\big] \end{aligned}} \tag{2453}$$

The Einstein tensor in the harmonic gauge

We now apply the harmonic gauge from (2444) to the Einstein tensor in (2447). This immediately eliminates three terms and only leaves

$$\boxed{G_{\mu\nu} = -\tfrac{1}{2}\Box\bar{h}_{\mu\nu} \qquad \begin{array}{c}\textit{Linearized Einstein tensor}\\ \textit{in the trace-reversed harmonic gauge}\end{array}} \tag{2454}$$

A remarkable property of this gauge choice (in linearized GR) is that the components of the Einstein tensor $(G_{00},\, G_{0i},\, G_{ij})$ are each expressed only in terms of the corresponding component of the (trace-reversed) metric perturbation $\left(\bar{h}_{00},\, \bar{h}_{0i},\, \bar{h}_{ij}\right)$. There is no mixing of the metric perturbation components between the Einstein tensor components. This is not immediately apparent when looking at the separate Einstein tensor components found in (2450), (2452) and (2453) since they each contain a mix of $\bar{h}_{00}$, $\bar{h}_{0i}$, and $\bar{h}_{ij}$. However, it can be shown that by choosing the harmonic gauge given in (2444) as $\partial^\nu\bar{h}_{\mu\nu} = 0$, then the Einstein tensor components each reduce to $G_{\mu\nu} = -\frac{1}{2}\Box\bar{h}_{\mu\nu}$. To see this, note that summing over indices in the harmonic gauge gives $\partial^0\bar{h}_{\mu 0} + \partial^k\bar{h}_{\mu k} = 0$. This leads to two equations: for $\mu = 0$ and for $\mu = i$ we have, respectively,

$$\partial^0\bar{h}_{00} + \partial^k h_{0k} = 0 \qquad \text{and} \qquad \partial^0\bar{h}_{i0} + \partial^k\bar{h}_{ik} = 0 \tag{2455}$$

Taking a time derivative (∂_0) of the first equation and a spatial derivative (∂_i) of the second equation gives

$$\partial_0\partial^0\bar{h}_{00} + \partial_0\partial^k\bar{h}_{0k} = 0 \qquad \text{and} \qquad \partial_i\partial^0\bar{h}_{i0} + \partial_i\partial^k\bar{h}_{ik} = 0 \tag{2456}$$

In the second expression, we can freely raise the spatial indices (since we are working to first order in the metric). We can also use $\partial^0 = -\partial_0$ and change i to k in the first term (since it is a repeated index in both terms). Then the expression becomes

$$-\partial^k\partial_0\bar{h}_{k0} + \partial_i\partial^k\bar{h}_{ik} = 0 \tag{2457}$$

Then adding this to the first expression in (2456) will cancel terms with $\partial_0\partial^k\bar{h}_{0k}$ and give

$$\partial_0\partial^0\bar{h}_{00} + \partial^l\partial^k\bar{h}_{ik} = 0 \tag{2458}$$

This means $\partial^l\partial^k\bar{h}_{ik} = -\partial_0\partial^0\bar{h}_{00}$ which substituted into (2450) gives

$$G_{00} \;=\; \tfrac{1}{2}\left(-\partial_0\partial^0\bar{h}_{00} - \partial^k\partial_k\bar{h}_{00}\right) = -\tfrac{1}{2}\Box\bar{h}_{00} \tag{2459}$$

For G_{0i} in (2452), we see that the two middle terms can be written together as $\partial_i\left(\partial^0\bar{h}_{00} + \partial^k\bar{h}_{k0}\right)$ which vanishes according to the first equation in (2455). This leaves

$$G_{0i} = \tfrac{1}{2}\left(\partial^k\partial_0\bar{h}_{ki} - \partial^k\partial_k\bar{h}_{0i}\right) \tag{2460}$$

The first term can be replaced using the first equation in (2456) which gives

$$G_{0i} \;=\; \tfrac{1}{2}\left(-\partial_0\partial^0\bar{h}_{00} - \partial^k\partial_k\bar{h}_{0i}\right) = -\tfrac{1}{2}\Box\bar{h}_{0i} \tag{2461}$$

For G_{ij} in (2453), we can write $2\partial^0\partial^k\overline{h}_{k0}$ as $\partial^0\left(\partial^k\overline{h}_{k0}\right)+\partial^k\left(\partial^0\overline{h}_{k0}\right)$. Then the three terms in the parentheses multiplying δ_{ij} in (2453) become

$$\partial^0\partial^0\overline{h}_{00}+2\partial^0\partial^k\overline{h}_{0k}+\partial^k\partial^l\overline{h}_{kl}=\partial\left({}^0\partial^0\overline{h}_{00}+\partial^k\overline{h}_{k0}\right)+\partial^k\left(\partial^0\overline{h}_{k0}+\partial^l\overline{h}_{kl}\right)\tag{2462}$$

Using the two expressions in (2455), we see that both sets of parentheses above become zero. Then (2453) reduces to

$$G_{ij}=\frac{1}{2}\left(\partial^0\partial_i\overline{h}_{0j}+\partial^k\partial_i\overline{h}_{kj}+\partial^0\partial_j\overline{h}_{0i}+\partial^k\partial_j\overline{h}_{ki}-\Box\overline{h}_{ij}\right)\tag{2463}$$

Regrouping gives

$$G_{ij}=\frac{1}{2}\left[\partial^0\left(\partial_i\overline{h}_{0j}+\partial_j\overline{h}_{0i}\right)+\partial^k\left(\partial_i\overline{h}_{kj}+\partial_j\overline{h}_{ki}\right)-\Box\overline{h}_{ij}\right]\tag{2464}$$

Note that taking ∂_j of the second expression in (2455) gives

$$\partial_j\partial^0\overline{h}_{i0}+\partial_j\partial^k\overline{h}_{ik}=0\tag{2465}$$

Also, expressing the second expression in (2455) using an index j (instead of i) and taking ∂_i gives

$$\partial_i\partial^0\overline{h}_{i0}+\partial_i\partial^k\overline{h}_{jk}=0\tag{2466}$$

Using (2465) and (2466) in (2464) leaves $G_{ij}=-\frac{1}{2}\Box\overline{h}_{ij}$. Therefore, we conclude that in the harmonic gauge, each of the Einstein tensor components satisfies $G_{\mu\nu}=-\frac{1}{2}\Box\overline{h}_{\mu\nu}$.

Lastly, we point out that if we did not use the *trace-reversed* metric perturbation and simply used the harmonic gauge of the non-trace-reversed metric perturbation, $\partial^\nu h_{\mu\nu}=0$, then the linearized Einstein tensor in (2379) would become

$$G_{\mu\nu}=\frac{1}{2}\left(\eta_{\mu\nu}\Box h-\Box h_{\mu\nu}-\partial_\mu\partial_\nu h\right)\tag{2467}$$

This is obviously not the same as (2454). Hence we find that it is only the *combination* of expressing the Einstein tensor in terms of $\overline{h}_{\mu\nu}$ and imposing $\partial^\nu\overline{h}_{\mu\nu}=0$ that we arrive at the simple result in (2454).

The Einstein field equations in the harmonic gauge

Since the Einstein field equations are given by $G_{\mu\nu}=-2\kappa T_{\mu\nu}$, then using the Einstein tensor in the harmonic gauge from (2454) gives

$$\Box\overline{h}_{\mu\nu}=-2\kappa T_{\mu\nu}\tag{2468}$$

where $\kappa=8\pi G/c^4$. So we can also write this as

$$\Box\overline{h}_{\mu\nu}=-\frac{16\pi G}{c^4}T_{\mu\nu}\tag{2469}$$

The retarded Green's function solution to the D'Alembert (wave) operator is $\dfrac{-\delta\left(t-\left|\vec{x}-\vec{x}'\right|/c\right)}{4\pi\left|\vec{x}-\vec{x}'\right|}$. Therefore, the Green's function solution to the wave equation above is

$$\overline{h}_{\mu\nu}\left(t,\vec{x}\right)=-\frac{1}{4\pi}\left(-\frac{16\pi G}{c^4}\right)\int\frac{T_{\mu\nu}\left(t_r,\vec{x}'\right)}{\left|\vec{x}-\vec{x}'\right|}d^3x'\tag{2470}$$

$$=\frac{4G}{c^4}\int\frac{T_{\mu\nu}\left(t_r,\vec{x}'\right)}{\left|\vec{x}-\vec{x}'\right|}d^3x'\tag{2471}$$

where $\vec{x}'$ is the spatial coordinate of each infinitesimal element of $T_{\mu\nu}$ occupying a differential volume element d^3x. Also, $T_{\mu\nu}\left(t_r,\vec{x}'\right)$ is the stress-energy-momentum contribution at $\vec{x}'$ evaluated at a retarded time t_r and located at a distance $\left|\vec{x}-\vec{x}'\right|$ from the field point where $\overline{h}_{\mu\nu}$ is measured. We can therefore express the retarded time as $t_r=t-\left|\vec{x}-\vec{x}'\right|/c$. From this expression we find that each component of $T_{\mu\nu}$ is directly related to the corresponding component of $h_{\mu\nu}$.

Appendix C

The Bianchi identity applied to the linearized Riemann tensor

Components of the linearized Riemann tensor

First, the components of the linearized Riemann tensor are evaluated. In (2370) of Appendix A, the linearized Riemann tensor was found to be

$$R^{\mu}{}_{\nu\gamma\delta} \quad = \quad \tfrac{1}{2}\eta^{\mu\rho}\left(\partial_\gamma\partial_\nu h_{\delta\rho} - \partial_\gamma\partial_\rho h_{\nu\delta} - \partial_\delta\partial_\nu h_{\gamma\rho} + \partial_\delta\partial_\rho h_{\nu\gamma}\right) \tag{2472}$$

This can be written with all lowered indices (to lowest order in the metric) using $R_{\beta\rho\gamma\sigma} = \eta_{\beta\mu}R^{\mu}{}_{\rho\gamma\sigma}$. If $\eta_{\beta\mu}$ is applied to (2472) while remaining to first order in the metric, then $\eta_{\beta\mu}\eta^{\mu\beta} \approx \delta_{\alpha\beta}$ since $R_{\beta\rho\gamma\sigma}$ is already first order in the metric. This gives

$$R_{\beta\rho\gamma\sigma} \quad = \quad \tfrac{1}{2}\left(\partial_\gamma\partial_\rho h_{\sigma\beta} - \partial_\gamma\partial_\beta h_{\rho\sigma} - \partial_\sigma\partial_\rho h_{\gamma\beta} + \partial_\sigma\partial_\beta h_{\rho\gamma}\right) \tag{2473}$$

It is immediately evident that $R_{0000} = 0$ since all the terms would be identical and would therefore cancel. To find the other independent components, the properties of the Riemann tensor can be applied, namely, symmetry under exchange of the first and second pairs of indices, and anti-symmetry under exchange of the first two indices with each other, or the last two indices with each other. This leads to the following relations.

$$\begin{aligned} &R_{i000} = R_{00i0} = -R_{0i00} = -R_{000i}, \qquad && R_{0i0j} = -R_{i00j} = -R_{0ij0} \\ &R_{ij00} = R_{00ij}, && R_{ijk0} = R_{koij} = -R_{0kij} = -R_{ij0k} \end{aligned} \tag{2474}$$

Notice that the set of components in the top-left contains only one temporal index, the set in the bottom-left contains two temporal indices appearing as the first pair of indices or the last pair of indices. The set in the top-right also contains two temporal index values, however, they are separated between the first pair of indices and the second pair of indices. Finally, the set in the bottom-right contains only a single temporal index.

Using (2473) to evaluate only the first Riemann tensor component in each set above gives

$$R_{i000} \quad = \quad \tfrac{1}{2}\left(\partial_0^2 h_{0i} - \partial_0\partial_i h_{00} - \partial_0^2 h_{0i} + \partial_0\partial_i h_{00}\right) = 0 \tag{2475}$$

$$R_{ij00} \quad = \quad \tfrac{1}{2}\left(\partial_0\partial_j h_{0i} - \partial_0\partial_i h_{0j} - \partial_0\partial_j h_{0i} + \partial_0\partial_i h_{0j}\right) = 0 \tag{2476}$$

$$R_{0i0j} \quad = \quad \tfrac{1}{2}\left(\partial_0\partial_i h_{0j} - \partial_0\partial_0 h_{ij} - \partial_i\partial_j h_{00} + \partial_j\partial_0 h_{0i}\right) \tag{2477}$$

$$R_{ijk0} \quad = \quad \tfrac{1}{2}\left(\partial_k\partial_i h_{0i} - \partial_k\partial_i h_{0j} - \partial_0\partial_j h_{ki} + \partial_0\partial_i h_{jk}\right) \tag{2478}$$

Lastly, there is one more independent Riemann tensor component given by all spatial indices.

$$R_{ijkl} \quad = \quad \tfrac{1}{2}\left(\partial_k\partial_i h_{li} - \partial_k\partial_i h_{jl} - \partial_l\partial_j h_{ki} + \partial_l\partial_i h_{jk}\right) \tag{2479}$$

In summary, the only non-zero Riemann tensor components and their corresponding symmetry properties are in the following table. Using the notation $\partial_0 h_{\mu\nu} = \frac{1}{c}\dot{h}_{\mu\nu}$ gives

$$R_{0i0j} = \tfrac{1}{2}\left(-\partial_i\partial_j h_{00} + \frac{1}{c}\partial_i\dot{h}_{0j} + \frac{1}{c}\partial_j\dot{h}_{0i} - \frac{1}{c^2}\ddot{h}_{ij}\right) \quad \text{where} \quad R_{0i0j} = -R_{i00j} = -R_{0ij0}$$

$$R_{ijk0} = \tfrac{1}{2}\left(\partial_k\partial_i h_{0i} - \partial_k\partial_i h_{0j} - \frac{1}{c}\partial_j\dot{h}_{ki} - \frac{1}{c}\partial_i h_{jk}\right) \quad \text{where} \quad R_{ijk0} = R_{k0ij} = -R_{0kij} = -R_{ij0k}$$

$$R_{ijkl} = \tfrac{1}{2}\left(\partial_k\partial_i h_{li} - \partial_k\partial_i h_{jl} - \partial_l\partial_j h_{ki} + \partial_l\partial_i h_{jk}\right)$$

The non-zero components of the linearized Riemann tensor

(2480)

The linearized Bianchi identity

The Bianchi identity for the Riemann tensor is

$$\nabla_\lambda R_{\beta\rho\gamma\sigma} + \nabla_\sigma R_{\beta\rho\lambda\gamma} + \nabla_\lambda R_{\beta\rho\sigma\lambda} = 0 \tag{2481}$$

where ∇_μ is the covariant derivative defined as

$$\nabla_\mu V_\nu = \partial_\mu V_\nu + \Gamma^\sigma_{\mu\nu} V_\sigma \tag{2482}$$

with V_ν being an arbitrary vector. Applying this to the Riemann tensor and remaining to first order in the metric will reduce the covariant derivative to a partial derivative. This leads to

$$\partial_\lambda R_{\beta\rho\gamma\sigma} + \partial_\sigma R_{\beta\rho\lambda\gamma} + \partial_\lambda R_{\beta\rho\sigma\lambda} = 0 \tag{2483}$$

If the Riemann tensor component in the first term above is $R_{\beta\rho\gamma\sigma} = R_{0i0j}$, then the identity will become

$$\partial_\lambda R_{0i0j} + \partial_j R_{0i\lambda 0} + \partial_0 R_{0ij\lambda} = 0 \tag{2484}$$

Since λ is a free index, then there are two cases: $\lambda = 0$ and $\lambda = k$. These lead to

$$\partial_0 R_{0i0j} + \partial_j R_{0i00} + \partial_0 R_{0ij0} = 0 \qquad \text{and} \qquad \partial_k R_{0i0j} + \partial_j R_{0ik0} + \partial_0 R_{0ijk} = 0 \tag{2485}$$

It has already been shown that $R_{0i00} = 0$. Also, using $R_{0i0j} = -R_{0ij0}$ means that the first identity in (2485) vanishes. Returning to (2483) and using $R_{\beta\rho\gamma\sigma} = R_{ijkl}$ gives

$$\partial_\lambda R_{ijk0} + \partial_0 R_{ij\lambda k} + \partial_k R_{ij0\lambda} = 0 \tag{2486}$$

Again, there are two cases: $\lambda = 0$ and $\lambda = l$. These lead to

$$\partial_0 R_{ijk0} + \partial_0 R_{ij0k} + \partial_k R_{ij00} = 0 \qquad \text{and} \qquad \partial_l R_{ijk0} + \partial_0 R_{ijlk} + \partial_k R_{ij0l} = 0 \tag{2487}$$

It has already been shown that $R_{ij00} = 0$. Also, using $R_{ijk0} = -R_{ij0k}$ means that the first identity in (2487) vanishes. Lastly, returning to (2483) and using $R_{\beta\rho\gamma\sigma} = R_{ijkl}$ gives

$$\partial_\lambda R_{ijkl} + \partial_l R_{ij\lambda k} + \partial_k R_{ijl\lambda} \quad = \quad 0 \tag{2488}$$

Again, there are two cases: $\lambda = 0$ and $\lambda = m$. These lead to

$$\partial_0 R_{ijkl} + \partial_l R_{ij0k} + \partial_k R_{ijl0} \quad = \quad 0 \qquad \text{and} \qquad \partial_m R_{ijkl} + \partial_l R_{ijmk} + \partial_k R_{ijlm} = 0 \tag{2489}$$

In the first identity above, permuting indices on all three terms leads to $\partial_0 R_{ijlk} + \partial_l R_{ijkl} + \partial_k R_{ij0l} = 0$ which is identical to the second identity in (2487). The second identity in (2489) also vanishes when inserting the third Riemann tensor component from (2480). Therefore, the only unique, non-vanishing Bianchi identities for the linearized Riemann tensor are given by the second identity in each of (2485) and (2487). The results are summarized below, with the terms rearranged and using $\partial_0 = \frac{1}{c}\partial_t$.

left side and a time-derivative on the right side.

$$\boxed{\begin{gathered} \partial_k R_{0i0j} - \partial_j R_{0i0k} = -\frac{1}{c}\partial_t R_{kji0} \\[1em] \partial_k R_{ijl0} - \partial_l R_{ijk0} = -\frac{1}{c}\partial_t R_{ijkl} \\[1em] \textit{The unique, non-vanishing Bianchi identities} \\ \textit{of the linearized Riemann tensor} \end{gathered}} \tag{2490}$$

Writing the non-vanishing Bianchi identities in this form is suggestive of the Faraday-like relationships that they predict. Specifically, notice that there is an anti-symmetric spatial derivative on the left side and a time-derivative on the right side.

Appendix D

Properties of plane-fronted gravitational waves

The trace-reversed metric perturbation for a plane-fronted gravitational wave can be written as

$$\bar{h}_{\mu\nu} = \bar{A}_{\mu\nu} e^{ik_\sigma x^\sigma} \tag{2491}$$

where $\bar{A}_{\mu\nu}$ is a constant amplitude and k_σ is a constant wave four-vector. We will demonstrate that gravitational plane-fronted waves have the following properties.

1. They are covariantly transverse $\left(\bar{A}_{\mu\nu} k^\nu = 0\right)$ in the harmonic gauge.
2. They are spatially transverse $\left(A_{ij} k^j = 0\right)$ in the TT gauge.
3. They have a null wave four-vector $\left(k^\sigma k_\sigma = 0\right)$ in vacuum.
4. They have a dispersion equation in matter (assuming a particular constitutive equation).

1. Plane waves are covariantly transverse $\left(\bar{A}_{\mu\nu} k^\nu = 0\right)$ in the harmonic gauge.

In (2444) of Appendix B, we found the trace-reversed harmonic gauge as $\partial_\nu \bar{h}^{\mu\nu} = 0$. Here we show that plane wave solutions in this gauge are necessarily transverse. Applying the gauge condition to the plane wave solution in (2491) gives

$$\partial^\nu \bar{h}_{\mu\nu} \quad = \quad \partial^\nu \left(\bar{A}_{\mu\nu} e^{ik_\sigma x^\sigma}\right) = 0 \tag{2492}$$

Since $\bar{A}_{\mu\nu}$ is a constant amplitude, once again we can bring it out of the four-derivative and apply the chain rule to $e^{ik_\sigma x^\sigma}$

$$e^{ik_\sigma x^\sigma} \bar{A}_{\mu\nu} \partial^\nu \left(k_\sigma x^\sigma\right) \quad = \quad 0 \tag{2493}$$

From (2502) we know that $\partial^\nu \left(k_\sigma x^\sigma\right) = k^\nu$. So we have

$$e^{ik_\sigma x^\sigma} \bar{A}_{\mu\nu} k^\nu \quad = \quad 0 \tag{2494}$$

This result requires either the trivial solution $e^{ik^\alpha x_\alpha} = 0$ (which means there is no wave) or

$$\boxed{k^\nu \bar{A}_{\mu\nu} = 0 \qquad \textit{covariantly transverse plane waves in the harmonic gauge}} \tag{2495}$$

2. Plane waves are spatially transverse $\left(A_{ij} k^j = 0\right)$ in the TT gauge.

Summing over ν in (2495) gives

$$k^0 \bar{A}_{\mu 0} + k^j \bar{A}_{\mu j} = 0 \tag{2496}$$

In Appendix F, it is shown that specializing to the transverse-traceless (TT) gauge requires[263]

$$h_{0\mu} = 0 \qquad \text{and} \qquad h = 0 \tag{2497}$$

Having $h_{0\mu} = 0$ means that $A_{0\mu} = 0$ so the first term in (2496) vanishes and the second term is non-zero only for $\mu = i$. So we have

$$\boxed{A_{ij}k^j = 0 \qquad \textit{spatially transverse plane waves in the TT gauge}} \tag{2498}$$

3. Plane waves have a null wave four-vector $(k^\sigma k_\sigma = 0)$ in vacuum.

From (2468) we find that the linearized Einstein equation in the harmonic gauge for the case of vacuum $\left(T_{\mu\nu} = 0\right)$ is simply

$$\Box \bar{h}_{\mu\nu} = 0 \tag{2499}$$

Substituting (2491) into (2499) gives

$$\Box \bar{h}_{\mu\nu} = \partial^\rho \partial_\rho \left(\bar{A}_{\mu\nu} e^{ik_\sigma x^\sigma}\right) = 0 \tag{2500}$$

Since $\bar{A}_{\mu\nu}$ is constant, then we can bring it out of the four-derivatives and divide it from both sides. Now applying the four-derivative ∂_ρ on $e^{ik_\sigma x^\sigma}$ gives

$$\partial^\rho \left[e^{ik_\sigma x^\sigma} \partial_\rho \left(k_\sigma x^\sigma\right)\right] = 0 \tag{2501}$$

Since k_σ is constant, then we can pull it out of the derivative and $\partial_\rho \left(k_\sigma x^\sigma\right)$ becomes

$$\partial_\rho \left(k_\sigma x^\sigma\right) = k_\sigma \partial_\rho x^\sigma = k_\sigma \delta_\rho{}^\sigma = k_\rho \tag{2502}$$

Substituting this result into (2501) gives

$$\partial^\rho \left(e^{ik_\sigma x^\sigma} k_\rho\right) = e^{ik_\sigma x^\sigma} k_\rho \partial^\rho \left(k_\sigma x^\sigma\right) \tag{2503}$$

$$= e^{ik_\sigma x^\sigma} k_\rho k_\sigma g^{\rho\lambda} \partial_\lambda x^\sigma \tag{2504}$$

$$= e^{ik_\sigma x^\sigma} k^\lambda k_\sigma \delta_\lambda{}^\sigma \tag{2505}$$

$$= e^{ik_\sigma x^\sigma} k^\sigma k_\sigma \tag{2506}$$

From (2501) we know this quantity much vanish.

$$e^{ik_\sigma x^\sigma} k^\sigma k_\sigma = 0 \tag{2507}$$

This requires either the trivial solution $e^{ik_\sigma x^\sigma} = 0$ (which means no wave) or

$$\boxed{k^\sigma k_\sigma = 0 \qquad \textit{null wave four-vector for a plane wave in vacuum}} \tag{2508}$$

[263] Since the trace is zero in the TT gauge, then there is no longer a distinction between $h_{\mu\nu}$ and $\bar{h}_{\mu\nu}$ (which is trace-reversed) and we can drop the "bar" notation if desired.

Since $k^\sigma = \left(k^0, k^i\right)$ where $k^0 = \omega/c$, then

$$k^\sigma k_\sigma = -\omega^2/c^2 + k^2 \tag{2509}$$

Using (2508) immediately leads to

$$k = \omega/c \tag{2510}$$

Therefore, we find that gravitational waves in vacuum propagate at the speed of light. Quantum mechanically, this would imply that gravitational waves correspond to massless gravitons.

4. Plane waves have a dispersion equation in matter.

The linearized Riemann tensor is found in (2370) as

$$R^\mu_{\ \nu\gamma\delta} = \tfrac{1}{2}\eta^{\mu\rho}\left(\partial_\gamma\partial_\nu h_{\delta\rho} - \partial_\gamma\partial_\rho h_{\nu\delta} - \partial_\delta\partial_\nu h_{\gamma\rho} + \partial_\delta\partial_\rho h_{\nu\gamma}\right) \tag{2511}$$

We can express $h_{\mu\nu}$ in terms of $\overline{h}_{\mu\nu}$ by using (2435) which gives

$$h_{\mu\nu} = \overline{h}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\overline{h} \tag{2512}$$

We will need the trace of $\overline{h}_{\mu\nu} = \overline{A}_{\mu\nu}e^{ik_\sigma x^\sigma}$ from (2491) which is

$$\overline{h} = \eta^{\mu\nu}\overline{h}_{\mu\nu} = \eta^{\mu\nu}\overline{A}_{\mu\nu}e^{ik_\sigma x^\sigma} = \overline{A}^\nu_{\ \nu}e^{ik_\sigma x^\sigma} \tag{2513}$$

Substituting (2491) and (2513) into (2512) gives

$$h_{\mu\nu} = \overline{A}_{\mu\nu}e^{ik_\sigma x^\sigma} - \tfrac{1}{2}\eta_{\mu\nu}\overline{A}^\nu_{\ \nu}e^{ik_\sigma x^\sigma} \tag{2514}$$

$$= \left(\overline{A}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\overline{A}\right)e^{ik_\sigma x^\sigma} \tag{2515}$$

where $\overline{A} = \overline{A}^\nu_{\ \nu}$ is the trace of $\overline{A}_{\mu\nu}$. Then we can simply identify the *non*-trace-reversed amplitude tensor as

$$A_{\mu\nu} = \overline{A}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\overline{A} \tag{2516}$$

and the *non*-trace-reversed metric perturbation for a plane wave is just

$$h_{\mu\nu} = A_{\mu\nu}e^{ik_\sigma x^\sigma} \tag{2517}$$

Now we can substitute (2517) into (2511) to evaluate the Riemann tensor in the linearized approximation.

$$\begin{aligned} R^\mu_{\ \nu\gamma\delta} = {} & \tfrac{1}{2}\eta^{\mu\rho}\Big[\partial_\gamma\partial_\nu\left(A_{\delta\rho}e^{ik_\sigma x^\sigma}\right) - \partial_\gamma\partial_\rho\left(A_{\nu\delta}e^{ik_\sigma x^\sigma}\right) \\ & - \partial_\delta\partial_\nu\left(A_{\gamma\rho}e^{ik_\sigma x^\sigma}\right) + \partial_\delta\partial_\rho\left(A_{\nu\gamma}e^{ik_\sigma x^\sigma}\right)\Big] \end{aligned} \tag{2518}$$

Since each term in the bracket is similar, we can consider just the first term.

$$\partial_\gamma\partial_\nu\left(A_{\delta\rho}e^{ik_\sigma x^\sigma}\right) = \partial_\gamma\left(A_{\delta\rho}e^{ik_\sigma x^\sigma}ik_\sigma\delta_\nu^{\ \sigma}\right) \tag{2519}$$

$$= ik_\nu A_{\delta\rho}\partial_\gamma e^{ik_\sigma x^\sigma} \tag{2520}$$

$$= i^2 k_\nu A_{\delta\rho}e^{ik_\sigma x^\sigma}k_\sigma\delta_\gamma^{\ \sigma} \tag{2521}$$

$$= -k_\nu k_\gamma A_{\delta\rho}e^{ik_\sigma x^\sigma} \tag{2522}$$

We can apply the same procedure to the other three terms in (2518) and factor out $e^{ik_\sigma x^\sigma}$ which we will write as e^{ikx}. Then rearranging terms gives

$$R^\mu_{\ \nu\gamma\delta} = \tfrac{1}{2}e^{ikx}\eta^{\mu\rho}\left(k_\gamma k_\rho A_{\nu\delta} + k_\nu k_\delta A_{\gamma\rho} - k_\nu k_\gamma A_{\delta\rho} - k_\delta k_\rho A_{\nu\gamma}\right) \tag{2523}$$

The Ricci tensor can be found by contracting the first and third index of the Riemann tensor: $R_{\nu\delta} = R^\mu_{\ \nu\gamma\delta}$. Setting $\mu = \gamma$ in (2523) and distributing the inverse metric $\eta^{\gamma\rho}$ gives

$$R_{\nu\delta} = \tfrac{1}{2}e^{ikx}\left(\eta^{\gamma\rho}k_\gamma k_\rho A_{\nu\delta} + \eta^{\gamma\rho}k_\nu k_\delta A_{\gamma\rho} - \eta^{\gamma\rho}k_\delta k_\rho A_{\nu\gamma} - \eta^{\gamma\rho}k_\nu k_\gamma A_{\delta\rho}\right) \tag{2524}$$

To first order, we have $g^{\gamma\rho} \approx \eta^{\gamma\rho}$ so we can contract indices using $\eta^{\gamma\rho}$. This gives

$$R_{\nu\delta} = \tfrac{1}{2}e^{ikx}\left(k_\gamma k^\gamma A_{\nu\delta} + k_\nu k_\delta A - k_\delta k^\gamma A_{\nu\gamma} - k_\nu k^\rho A_{\delta\rho}\right) \tag{2525}$$

where $A = A^\rho_{\ \rho}$. From (2512) we know $A_{\mu\nu} = \bar{A}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{A}$. Taking the trace of this leads to $A = -\bar{A}$. Substituting these relations into (2525) gives

$$R_{\nu\delta} = \tfrac{1}{2}e^{ikx}\left[-k_\gamma k^\gamma \bar{A}_{\nu\delta} - k_\nu k_\delta \bar{A} - k_\delta k^\gamma\left(\bar{A}_{\nu\gamma} - \tfrac{1}{2}\eta_{\nu\gamma}\bar{A}\right) - k_\nu k^\rho\left(\bar{A}_{\delta\rho} - \tfrac{1}{2}\eta_{\delta\rho}\bar{A}\right)\right] \tag{2526}$$

Distributing and lowering indices with $\eta_{\nu\gamma}$ and $\eta_{\delta\rho}$ gives

$$R_{\nu\delta} = \tfrac{1}{2}e^{ikx}\left(-k_\gamma k^\gamma \bar{A}_{\nu\delta} - k_\nu k_\delta \bar{A} - k_\delta k^\gamma \bar{A}_{\nu\gamma} + \tfrac{1}{2}k_\delta k_\nu \bar{A} - k_\nu k^\rho \bar{A}_{\delta\rho} + \tfrac{1}{2}k_\nu k_\delta \bar{A}\right) \tag{2527}$$

The second, fourth, and sixth terms cancel with one another. Using γ for all repeated indices gives

$$R_{\nu\delta} = \tfrac{1}{2}e^{ikx}\left(-k_\gamma k^\gamma \bar{A}_{\nu\delta} - k_\delta k^\gamma \bar{A}_{\nu\gamma} - k_\nu k^\gamma \bar{A}_{\delta\gamma}\right) \tag{2528}$$

The Ricci scalar is found by contracting the Ricci tensor: $R = \eta^{\nu\delta}R_{\nu\delta} = R^\delta_{\ \delta}$. This gives

$$R = \tfrac{1}{2}e^{ikx}\eta^{\nu\delta}\left(-k_\gamma k^\gamma \bar{A}_{\nu\delta} - k_\delta k^\gamma \bar{A}_{\nu\gamma} - k_\nu k^\gamma \bar{A}_{\delta\gamma}\right) \tag{2529}$$

Contracting indices makes the last two terms identical so they can be combined. Then using γ and ρ for all repeated indices and writing $\bar{A}^\delta_{\ \delta}$ as simply $\bar{A}$ gives

$$R = \tfrac{1}{2}e^{ikx}\left(-k_\gamma k^\gamma \bar{A} - 2k^\rho k^\gamma \bar{A}_{\rho\gamma}\right) \tag{2530}$$

The linearized Einstein tensor written as the trace-reversed Ricci tensor is $G_{\mu\nu} \approx R_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}R$. Using (2528) and (2530) with μ and ν for free indices, and factoring out e^{ikx} gives

$$G_{\mu\nu} = \tfrac{1}{2}e^{ikx}\left[\left(-k_\gamma k^\gamma \bar{A}_{\mu\nu} - k_\nu k^\gamma \bar{A}_{\mu\gamma} - k_\mu k^\gamma \bar{A}_{\nu\gamma}\right) + \tfrac{1}{2}\eta_{\mu\nu}\left(-k_\gamma k^\gamma \bar{A} - 2k^\rho k^\gamma \bar{A}_{\rho\gamma}\right)\right] \tag{2531}$$

Rearranging to gather common terms gives

$$G_{\mu\nu} = \tfrac{1}{2}e^{ikx}\left[-k_\gamma k^\gamma\left(\bar{A}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\bar{A}\right) - k^\gamma k_\mu \bar{A}_{\nu\gamma} - k^\gamma k_\nu \bar{A}_{\mu\gamma} - \eta_{\mu\nu}k^\rho k^\gamma \bar{A}_{\rho\gamma}\right] \tag{2532}$$

We can fix the gauge by choosing the trace-reversed harmonic gauge. Then (2495) gives $k^{\nu}\bar{A}_{\mu\nu} = 0$ and therefore the last three terms in (2532) vanish. Also, since (2516) gives $A_{\mu\nu} = \bar{A}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{A}$, then (2532) becomes

$$\boxed{\begin{array}{c} G_{\mu\nu} = -\frac{1}{2}k_{\gamma}k^{\gamma}A_{\mu\nu}e^{ikx} \\ \\ \textit{Linearized Einstein tensor for a plane-fronted gravitational wave} \\ \textit{in the trace-reversed harmonic gauge} \end{array}} \tag{2533}$$

This result is consistent with inserting a wave, $\bar{h}_{\mu\nu} = A_{\mu\nu}e^{ik_{\sigma}x^{\sigma}}$, into the linearized Einstein tensor in the trace-reversed harmonic gauge found in (2454) as $G_{\mu\nu} = -\frac{1}{2}\Box\bar{h}_{\mu\nu}$. We can now use the Einstein field equation, $G_{\mu\nu} = \kappa T_{\mu\nu}$, to obtain

$$-\frac{1}{2}e^{ikx}k_{\gamma}k^{\gamma}A_{\mu\nu} = \kappa T_{\mu\nu} \tag{2534}$$

Using $\kappa = 8\pi G/c^4$ gives

$$e^{ikx}k_{\gamma}k^{\gamma}A_{\mu\nu} \quad = \quad -\frac{16\pi G}{c^4}T_{\mu\nu} \tag{2535}$$

If we consider a constitutive equation given by $T_{\mu\nu} = -\mu_{G(SC)}e^{ikx}A_{\mu\nu}$, where $\mu_{G(SC)}$ is a positive constant with dimensions of energy density, then we obtain a dispersion relation given by

$$k_{\gamma}k^{\gamma} - \frac{16\pi G\mu}{c^2} = 0 \tag{2536}$$

Using $k^{\sigma}k_{\sigma} = -\omega^2/c^2 + k^2$ gives

$$k^2 - \frac{\omega^2}{c^2} - \frac{16\pi G\mu}{c^2} \quad = \quad 0 \tag{2537}$$

Note that this dispersion relation is not unique to any particular component of the metric (h_{00}, h_{0i}, h_{ij}) or the corresponding stress tensor components (T_{00}, T_{0i}, T_{ij}). However, as shown by the field equations in the Helmholtz Decomposition treatment (353), only the transverse-traceless components of the metric perturbation ($h^{\tau\tau}_{ij}$) are radiating degrees of freedom. Therefore we expect that a gravitational wave dispersion relation such as (2537) would only apply to $h^{\tau\tau}_{ij}$.

Lastly, we point out that for a wave with a null wave four-vector $\left(k_{\gamma}k^{\gamma} = 0\right)$, we find from (2533) that $G_{\mu\nu}$ vanishes. Since $G_{\mu\nu} = \kappa T_{\mu\nu}$, then this requires $T_{\mu\nu} = 0$ which means the wave is a vacuum solution. As expected, this implies a plane-fronted gravitational wave will have no dispersion in vacuum.

Appendix E

Relating gauge freedom and conservation laws in GR and EM

In Appendix A, we found that there are four degrees of gauge freedom due to the four components of the gauge four-vector ξ^{μ}. In Appendix B, we applied the harmonic gauge to remove the four degrees of gauge freedom from the metric. We also showed in Appendix D that a plane wave solution satisfies the linearized Einstein equation in vacuum and that in the harmonic gauge, plane waves must be transverse. Here we will show that in vacuum, the harmonic gauge does not completely fix the gauge. There are four *additional* degrees of gauge freedom. We will show that this is a result of the absence of energy-momentum conservation in vacuum since there are no sources to conserve.

Relating gauge freedom and conservation laws for gravitation

In (2443) of Appendix B, we found that the divergence of the gauge transformation for the metric perturbation is

$$\partial^{\nu}\overline{h}'_{\mu\nu} = \partial^{\nu}\overline{h}_{\mu\nu} + \Box\xi_{\mu} \tag{2538}$$

In the trace-reversed harmonic gauge[264], we choose

$$\partial^{\nu}\overline{h}_{\mu\nu} = -\Box\xi_{\mu} \tag{2539}$$

so that $\partial^{\nu}\overline{h}'_{\mu\nu} = 0$. This leads to the Einstein equation found in (2468) as

$$\Box\overline{h}_{\mu\nu} = -2\kappa T_{\mu\nu} \tag{2540}$$

If we define a four-vector, $\Upsilon_{\mu} \equiv \Box\xi_{\mu}$, and apply a box operator on (2538), then we have

$$\partial^{\nu}\Box\overline{h}'_{\mu\nu} = \partial^{\nu}\Box\overline{h}_{\mu\nu} + \Box\Upsilon_{\mu} \tag{2541}$$

Using (2540) gives

$$-2\kappa\partial^{\nu}T'_{\mu\nu} = -2\kappa\partial^{\nu}T_{\mu\nu} + \Box\Upsilon_{\mu} \tag{2542}$$

Now we have the gauge freedom related to the stress tensor rather than the metric perturbation. Conservation of energy-momentum (in linearized GR) requires $\partial^{\nu}T_{\mu\nu} = 0$, so (2542) becomes

$$\Box\Upsilon_{\mu} = 0 \tag{2543}$$

We can also factor out a derivative from (2542) to obtain

$$\partial^{\nu}\left(2\kappa T'_{\mu\nu} - 2\kappa T_{\mu\nu} + \partial_{\nu}\Upsilon_{\mu}\right) = 0 \tag{2544}$$

In curved space-time, the divergence of $T^{\mu\nu}$ is actually given by[265]

$$\nabla_{\nu}T^{\mu\nu} = \partial_{\nu}T^{\mu\nu} + \Gamma^{\rho}_{\rho\nu}T^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_{\nu}\left(\sqrt{-g}T^{\mu\nu}\right) \tag{2545}$$

Also, if we apply a time-like Killing vector $k^{\mu} = \delta^{\mu}_{0}$ such that

$$\partial^{\nu}\left(k^{\mu}T_{\mu\nu}\right) = \partial^{\nu}J_{\nu} = 0 \qquad \text{and} \qquad \partial^{\nu}\left(k^{\mu}\partial_{\nu}\Upsilon_{\mu}\right) = \partial^{\nu}L_{\nu} = 0 \tag{2546}$$

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[264]For brevity, we will simply refer to this as the “harmonic gauge” rather than the “trace-reversed harmonic gauge” throughout the rest of this appendix.

[265]Here we are following the formulation found in [104].

then we can write (2544) as

$$\frac{1}{\sqrt{-g}}\partial^{\nu}\left[\sqrt{-g}\left(2\kappa J'_{\nu}-2\kappa J_{\nu}+L_{\nu}\right)\right] = 0 \tag{2547}$$

Taking a volume integral over the proper 4-volume $dV=\sqrt{-g}d^4x$ gives

$$\int \partial^{\nu}\left[\sqrt{-g}\left(2\kappa J'_{\nu}-2\kappa J_{\nu}+L_{\nu}\right)\right]d^4x = 0 \tag{2548}$$

Then applying Gauss's law in curved space-time gives

$$\oint\left(2\kappa J'_{\nu}-2\kappa J_{\nu}+L_{\nu}\right)\hat{n}_{\nu}\sqrt{-g}d^3x=0 \tag{2549}$$

where $\hat{n}_{\mu}$ is the outward unit normal to the hypersurface d^3x. Assuming $J'^{\mu}, J^{\mu}, \partial^{\mu}\Psi\to 0$ as $r\to\infty$ (for a finite stress tensor distribution), then if we evaluate the hypersurface integral at infinity, the integrand must vanish. Denoting the longitudinal components as $J_{\nu\,\|}$ gives

$$2\kappa J'_{\nu\,\|}-2\kappa J_{\nu\,\|}+L_{\nu\,\|}=0 \tag{2550}$$

Using (2546) gives

$$k^{\mu}\left(2\kappa T'_{\mu\nu\,\|}-2\kappa T_{\mu\nu\,\|}+\partial_{\nu}\Upsilon_{\mu\,\|}\right)=0 \tag{2551}$$

Therefore, this is the unique solution to (2544). Since $\Upsilon_{\mu}\equiv\Box\xi_{\mu}$, then we have

$$2\kappa T'_{\mu\nu\,\|}-2\kappa T_{\mu\nu\,\|}+\partial_{\nu}\Box\xi_{\mu\,\|}=0 \tag{2552}$$

Therefore we find that only the longitudinal component of $T_{\mu\nu}$ is associated with the longitudinal component of the gauge freedom described by ξ_{μ}. Assuming that the stress tensor is gauge-invariant leads to

$$\boxed{\partial_{\nu}\Box\xi_{\mu\,\|}=\textit{constant (in matter)}} \tag{2553}$$

Hence we find that conservation of energy-momentum inside matter imposes a restriction on the gauge freedom in (2538). In contrast to this, we can consider the case for fields in vacuum where we find there are four additional degrees of gauge freedom. The Einstein equation (2468) in vacuum becomes $\Box\bar{h}_{\mu\nu}=0$. Using this in (2541) immediately leads to $\partial_{\nu}\Box\xi_{\mu}=0$. This means that we have

$$\boxed{\Box\xi_{\mu}=\textit{constant (in vacuum)}} \tag{2554}$$

Comparing (2553) for matter with (2554) for vacuum shows that in vacuum, the gauge freedom is relaxed by one derivative. It is no surprise that we find additional gauge freedom in vacuum that is not present in matter. The presence of matter introduces additional conservation laws which *reduce* the symmetry of the gauge. There are effectively four conservation laws in $\partial_{\nu}T^{\mu\nu}=0$ (one for each value of μ) and hence there are four less degrees of gauge freedom in matter. We may consider this to be a consequence of Noether's theorem which shows that conservation laws are always associated with symmetries. Since the vacuum has no energy-momentum conservation law and has a higher symmetry than matter,[266] then we expect that there would be more degrees of gauge freedom in vacuum.

[266]The vacuum has higher symmetry than matter because any (finite) matter distribution must have a boundary and a center of mass. These features necessarily break translational symmetry and therefore reduce the symmetry of matter compared to vacuum.

Comparing the gauge freedom in matter and in vacuum

To demonstrate more explicitly the difference in the gauge freedom in matter versus vacuum, we can compare the gauge freedom solutions for each case. For the case of fields in matter, we can write the solution to (2543) as a superposition of plane waves

$$\Upsilon_\mu(\vec{x}) = \mathrm{Re}\int \frac{d^3\vec{k}}{(2\pi)^3} B_\mu\left(\vec{k}\right) e^{ik_\sigma x^\sigma} \tag{2555}$$

where $k^\sigma = \left(\omega, k^i\right)$ and $k^\sigma k_\sigma = -\omega^2 + k^2 = 0$. Since we defined $\Upsilon_\mu = \Box\xi_\mu$, then we also have a Green's function solution for ξ_μ given by

$$\xi_\mu(t,\vec{x}) = -\frac{1}{4\pi}\int \frac{\Upsilon_\mu(t_r,\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' \tag{2556}$$

where t_r is the retarded time expressed as $t_r = t - |\vec{x}-\vec{x}'|/c$. Since (2556) gives ξ_μ in terms of Υ_μ, and (2555) gives Υ_μ in terms of B_μ, then we find that B_μ ultimately determines the four degrees of gauge freedom that are possible in order to satisfy conservation of energy-momentum in matter.

Now in the harmonic gauge given by (2539), we have already removed four gauge degrees of freedom from the metric. Therefore, it would seem from (2555) that there are another four degrees of gauge freedom given by B_μ, which would imply that there are a total of *eight* degrees of gauge freedom in matter. However, B_μ cannot be chosen arbitrarily. Rather, we find that B_μ is already fixed due to the fact that we must satisfy $\Upsilon_\mu = \Box\xi_\mu$ where $\Box\xi_\mu = -\partial^\nu \bar{h}_{\mu\nu}$ in the harmonic gauge (2539). Therefore, we find that using the harmonic gauge and also enforcing conservation of energy-momentum does *not* leave any remaining degrees of gauge freedom.

In a sense, conservation of energy-momentum removes four degrees of gauge freedom which leaves only four degrees of gauge freedom in matter. Then the harmonic gauge completely fixes the gauge in matter by removing the remaining four degrees of gauge freedom. This is done by imposing four constraint equations, $\partial^\nu \bar{h}_{\mu\nu} = -\Box\xi_\mu$, where each value of μ gives one constraint.

Now we consider the case of fields in vacuum. Setting the constant in (2554) to zero gives $\Box\xi_\mu = 0$. Notice that $\Box\xi_\mu = 0$ still satisfies the harmonic gauge condition in (2539) which simply becomes $\partial^\nu \bar{h}_{\mu\nu} = 0$. The solution to $\Box\xi_\mu = 0$ can be written as a superposition of plane waves

$$\xi_\mu(\vec{x}) \quad = \quad \mathrm{Re}\int \frac{d^3\vec{k}}{(2\pi)^3} \varepsilon_\mu\left(\vec{k}\right) e^{ik_\sigma x^\sigma} \tag{2557}$$

Notice that (2557) simply expresses the four degrees of freedom from ξ_μ in terms of the four components of ε_μ. Therefore, choosing the constant in (2554) to be zero and obtaining (2557) did not fix any gauge degrees of freedom.

We now have four functions given by ε_μ in (2557) which can be chosen arbitrarily. This implies that in vacuum, there are four additional gauge degrees of freedom beyond the four that the harmonic gauge already fixed. In vacuum, there are no constraints imposed by energy-momentum conservation that would determine ε_μ. This is in contrast to the case in matter where the constraint $\Box\Upsilon_\mu = 0$ (due to energy-momentum conservation) combined with the harmonic gauge condition, $\Upsilon_\mu = \Box\xi_\mu = -\partial^\nu \bar{h}_{\mu\nu}$, does not permit the functions B_μ in (2555) to be chosen arbitrarily. In the next appendix, we will show that if we set $\Box\xi_\mu = 0$ in vacuum, then the transverse-traceless (TT) gauge will specify conditions on the four components of ε_μ in (2557). The conditions are found in (2627) in terms of the wave vector k and the components of the wave amplitude tensor $A_{\mu\nu}$.

The effect of imposing the TT gauge inside matter

It may seem that for the case of fields in matter, we can also set $\Box\xi_\mu = 0$ since this still satisfies the harmonic gauge condition. However, since $\Upsilon_\mu = \Box\xi_\mu$, then this would mean $\Upsilon_\mu = 0$ and (2542) would give $T'_{\mu\nu} = T_{\mu\nu}$. This implies that the stress tensor would be invariant under linear coordinate transformations. Such a result is certainly not consistent with the linear coordinate transformation of the stress tensor which can be shown to be[267]

$$T'_{\mu\nu} = T_{\mu\nu} - \left(\partial^\rho \xi_\mu\right) T_{\rho\nu} - \left(\partial^\rho \xi_\nu\right) T_{\mu\rho} \tag{2558}$$

For example, consider the simple case of a coordinate shift that is purely spatial, $\xi_\mu = (0, \xi_i)$, and is uniform, $\partial_k \xi_i = 0$. Using (2558) to find T'_{0i} gives

$$T'_{0i} = T_{0i} - \left(\partial^\rho \xi_i\right) T_{0\rho} - \left(\partial^\rho \xi_i\right) T_{0\rho} \tag{2559}$$

$$= T_{0i} - \left(\partial^0 \xi_i\right) T_{00} \tag{2560}$$

If we consider a stress tensor describing dust (a pressureless ideal fluid), then $T_{\mu\nu} = \rho u_\mu u_\nu$. To lowest order in the metric, we can use $\partial^0 \approx -\frac{1}{c}\partial_t$ and $u_\mu \approx (-c, u_i)$. This gives

$$-\rho c u'_i = -\rho c u_i + \frac{1}{c}\dot{\xi}_i \rho c^2 \tag{2561}$$

Defining $U_i \equiv \dot{\xi}_i$ and simplifying gives

$$u'_i = u_i - U_i \tag{2562}$$

This transformation implies that an observer can always boost into the rest frame of a dust particle by choosing $U_i = -u_i$ so that $u'_i = 0$. However, if we insist that $T'_{\mu\nu} = T_{\mu\nu}$, then we must have $u'_i = u_i$ regardless of the value of U_i. This means that boosting into a different frame has no effect on the observed velocity of the particle. This clearly violates the basic tenants of relativity. Therefore, we see that choosing the transverse-traceless (TT) gauge in matter (which requires setting $\Box\xi_\mu = 0$ and leads to $T'_{\mu\nu} = T_{\mu\nu}$) is an unphysical condition to impose on the stress tensor.

Relating gauge freedom and conservation laws for electromagnetism (EM)

Similar to the treatment shown above, in EM we find that *charge* conservation reduces the gauge freedom of the four-potential by one degree of freedom. The gauge freedom is given by

$$A'^\mu = A^\mu + \partial^\mu \chi \tag{2563}$$

Taking a divergence gives

$$\partial_\mu A'^\mu = \partial_\mu A^\mu + \Box\chi \tag{2564}$$

Similar to the harmonic gauge, in the Lorenz gauge we choose

$$\partial_\mu A^\mu = -\Box\chi \tag{2565}$$

[267] We should note that (2542) cannot be derived from (2558) since (2542) was a result of using the Einstein equation in conjunction with the metric transformation (2538). Therefore, (2542) and (2558) cannot be checked for consistency regardless of the discussion here about gauge freedom. Nevertheless, (2558) still demonstrates that the stress tensor should not be invariant under coordinate transformations as (2542) would imply if we set $\Upsilon_\mu = 0$.

so that $\partial_\mu A'^\mu = 0$. If we define $\Psi \equiv \Box\chi$ and apply a box operator on (2564), then we have

$$\partial_\mu \Box A'^\mu = \partial_\mu \Box A^\mu + \Box\Psi \tag{2566}$$

Since the Maxwell equations in the Lorenz gauge give $\Box A^\mu = -\mu_0 J^\mu$, then in matter we have

$$-\mu_0 \partial_\mu J'^\mu = -\mu_0 \partial_\mu J^\mu + \Box\Psi \tag{2567}$$

Since charge conservation requires $\partial_\mu J^\mu = 0$, then

$$\boxed{\Box\Psi = 0 \textit{ (in charged matter)}} \tag{2568}$$

Hence we find that conservation of charge imposes a restriction on the gauge freedom in (2564). We can compare this result to the case of fields in vacuum. Starting from (2563) and applying a box operator gives

$$\Box A'^\mu \quad = \quad \Box A^\mu + \partial^\mu \Box\chi \tag{2569}$$

In vacuum we have $\Box A^\mu = 0$ which means $\partial^\mu \Box\chi = 0$. Again using $\Psi \equiv \Box\chi$ gives

$$\boxed{\partial^\mu \Psi = \textit{constant (in vacuum)}} \tag{2570}$$

Comparing (2568) for matter with (2570) for vacuum shows that in vacuum, the gauge freedom is relaxed by one derivative. It is no surprise that we find this additional gauge freedom in vacuum since the presence of charge introduces an additional conservation law which *reduces* the symmetry of the gauge. Conservation of charge, $\partial_\mu J^\mu = 0$, is a single constraint equation and therefore removes a single degree of freedom in matter.

For fields in charged matter, we can write the solution to (2568) as

$$\Psi(\vec{x}) \quad = \quad \mathrm{Re} \int \frac{d^3\vec{k}}{(2\pi)^3} D\left(\vec{k}\right) e^{ik_\sigma x^\sigma} \tag{2571}$$

Since we defined $\Psi \equiv \Box\chi$, then we also have a Green's function solution for χ given by

$$\chi(t,\vec{x}) \quad = \quad -\frac{1}{4\pi} \int \frac{\Psi(t_r,\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' \tag{2572}$$

Notice that (2572) gives χ in terms of Ψ, and (2571) gives Ψ in terms of D, therefore we find that D gives the single degree of gauge freedom that remains after satisfying conservation of charge in matter.

Now in the Lorenz gauge given by (2565), we have already removed a single gauge degree of freedom from the four-potential by imposing a single constraint equation. Therefore, it would seem from (2555) that we still have another degree of gauge freedom which would imply a total of two degrees of gauge freedom in matter. However, D cannot be chosen arbitrarily. We must satisfy $\Psi = \Box\chi$ where $\Box\chi$ must satisfy $\Box\chi = -\partial_\mu A^\mu$ in the Lorenz gauge (2565). Therefore, we find that using the Lorenz gauge and also enforcing the conservation of charge does *not* leave any remaining gauge degrees of freedom. In a sense, conservation of charge removes one degree of gauge freedom in charged matter and therefore the Lorenz gauge completely fixes the gauge in charged matter by imposing a single constraint equation that removes the last remaining degree of gauge freedom.

Now we consider the case of fields in vacuum. Setting the constant in (2570) to zero and using $\Psi = \Box\chi$ gives $\Box\chi = 0$. The solution for χ can be written as a superposition of plane waves

$$\chi(\vec{x}) \quad = \quad \mathrm{Re}\int \frac{d^3\vec{k}}{(2\pi)^3} F\left(\vec{k}\right) e^{ik_\sigma x^\sigma} \tag{2573}$$

Notice that $\Box\chi = 0$ still satisfies the harmonic gauge condition in (2565) which becomes $\partial_\mu A^\mu = 0$. However, we now have an additional degree of freedom given by F in (2573) which can be chosen arbitrarily. This implies that in vacuum, there is another gauge degree of freedom in addition to the gauge degree of freedom already removed by the Lorenz gauge. In vacuum, there are no constraints imposed by charge conservation that would determine F. This is in contrast to the case in charged matter where the constraint $\Box\Psi = 0$ (due to charge conservation) and the Lorenz gauge condition given as $\Psi = \Box\chi = -\partial_\mu A^\mu$, does not permit the function F in (2573) to be chosen arbitrarily.

Next we can obtain a relationship between J'^μ and J^μ to consider the role of the gauge function with regard to the four-current. Factoring out a derivative from (2567) gives

$$\partial_\mu\left(-\mu_0 J'^\mu + \mu_0 J^\mu - \partial^\mu\Psi\right) = 0 \tag{2574}$$

Taking a volume integral over the proper 4-volume $dV = \gamma d^4x$ gives

$$\int \partial_\mu\left(\mu_0 J'^\mu - \mu_0 J^\mu + \partial^\mu\Psi\right)\gamma d^4x = 0 \tag{2575}$$

Applying Gauss's law in covariant form gives

$$\oint\left(\mu_0 J'^\mu - \mu_0 J^\mu + \partial^\mu\Psi\right)\hat{n}_\mu \gamma d^3x = 0 \tag{2576}$$

where $\hat{n}_\mu$ is the outward unit normal to the hypersurface d^3x. Assuming $J'^\mu, J^\mu, \partial^\mu\Psi \to 0$ as $r \to \infty$ (for a finite four-current distribution), then if we evaluate the hypersurface integral at infinity, the integrand must vanish. Denoting the longitudinal components as $J_\parallel^\mu$ gives

$$\mu_0 J_\parallel^{'\mu} - \mu_0 J_\parallel^\mu + \left(\partial^\mu\Psi\right)_\parallel = 0 \tag{2577}$$

Therefore, this is the unique solution to (2574). Using $\Psi = \Box\chi$ gives

$$\mu_0 J'^\mu = \mu_0 J^\mu - \left(\partial^\mu\Box\chi\right)_\parallel \tag{2578}$$

Therefore we find that only the longitudinal component of J^μ is associated with the longitudinal component of the gauge freedom described by $\partial^\mu\chi$. This result may appear to imply that the four-current is a gauge-dependent quantity which can be changed by a choice of χ. However, since the gauge freedom χ is not observable and the four-current J^μ *is* observable, then we know this cannot be the case. Earlier we applied conservation of charge by setting the divergence of (2578) to zero. This led to the condition in (2568) which was enough to demonstrate that charge conservation reduces the gauge degrees of freedom in charged matter. However, here we see that to insure J^μ cannot be changed by an unobservable gauge function, we must require the stricter condition that $\left(\partial^\mu\Box\chi\right)_\parallel = 0$. In terms of $\Psi = \Box\chi$, this means

$$\boxed{\left(\partial^\mu\Psi\right)_\parallel = 0 \textit{ (in charged matter)}} \tag{2579}$$

Note that the condition for vacuum obtained in (2570) will not change since the stricter condition obtained here is only relevant to charged matter where J^μ must remain a gauge-invariant quantity. Note that in the case of gravitation, we have the opposite situation. There was no need to consider a stricter condition in matter as we have done here because the "gauge" function in (2542) involves a coordinate transformation, not an unobservable gauge function. Although we expect that the stress tensor should transform according to the coordinate transformation in (2558) rather than the relationship in (2542), still we cannot simply set $\partial_\nu \Upsilon_\mu = 0$ in (2542) otherwise this would imply that $T'_{\mu\nu} = T_{\mu\nu}$ which we have demonstrated leads to violations of basic tenants of relativity.

Therefore, we observe a fundamental difference between the source in electromagnetism, J^μ, and the source in gravitation, $T_{\mu\nu}$. The key difference is that J'^μ and J^μ in (2578) are not related by a *coordinate* transformation. They are only related by a *gauge* transformation where the gauge freedom is given in (2563) as $A'^\mu = A^\mu + \partial^\mu \chi$. This means that gauge choices involving χ can never conflict with transformations of J^μ which involve boosts, rotations, translations, etc. This is in contrast to the stress tensor where $T'_{\mu\nu}$ and $T_{\mu\nu}$ in (2542) are related by a gauge function $\partial_\nu \Upsilon_\mu$ (which is related to the gauge freedom of the metric) and they are also related by a coordinate transformation given in (2558). As a result, there are stricter conditions that must be satisfied for the stress tensor than for the four-current.

<u>Comparing the Coulomb gauge in EM to the transverse-traceless (TT) gauge in GR</u>

Next we consider the implications of using the Coulomb gauge in electromagnetism and compare the result with using the TT gauge in gravitation. The Coulomb gauge is imposed by setting

$$\partial_i A^i = 0 \tag{2580}$$

It may seem that for the case of fields inside charged matter, we can also set $\Box\chi = 0$ since this still satisfies the Lorenz gauge condition. However, since $\Psi = \Box\chi$, then this would mean $\Psi = 0$. To consider the implications of this, first we can use

$$\partial_\mu \left(\mu_0 J'^\mu\right) = \partial_\mu \left(\mu_0 J^\mu + \partial^\mu \Psi\right) \tag{2581}$$

This implies that the stress tensor would be invariant under linear coordinate transformations. Such a result is certainly not consistent with the linear coordinate transformation of the stress tensor which is

$$T'_{\mu\nu} = T_{\mu\nu} - \left(\partial^\rho \xi_\mu\right) T_{\rho\nu} - \left(\partial^\rho \xi_\nu\right) T_{\mu\rho} \tag{2582}$$

Therefore, we conclude that charge conservation is a conservation law which effectively reduces the symmetry of the gauge. Note that charge conservation is a *single* conservation law (as opposed to $\partial_\nu T^{\mu\nu} = 0$ which is effectively *four* conservation laws), therefore charge conservation removes only a single degree of gauge freedom from (2564). Since we already used the Lorenz gauge $\left(\partial_\mu A^\mu = 0\right)$ in (2565), then typically the Coulomb gauge is *also* imposed so that $\partial_i A^i = 0$. This is being used as an *additional* gauge condition which removes another degree of freedom from A^μ. The Coulomb gauge is the common gauge for treating electromagnetic radiation (hence it is sometimes referred to as the radiation gauge). It is a gauge that makes A^i a purely transverse field. As we will show in the next appendix, it is directly analogous to the TT gauge for gravitational radiation where we arrive at $\partial^j h_{ij}^{TT} = 0$ so that h_{ij}^{TT} is also a purely transverse field. However, because h_{ij}^{TT} is a tensor, we have the additional property that it is also *traceless*, $\delta^{ij} h_{ij}^{TT}$, hence the name "transverse-traceless" (TT) gauge.

Appendix F

Linearized General Relativity in the transverse-traceless (TT) gauge

In the previous appendix, we showed that in vacuum, the harmonic gauge does not completely fix the gauge. There are four *additional* degrees of gauge freedom as a result of the absence of energy-momentum conservation in vacuum. Here we will specialize to the transverse-traceless (TT) gauge by eliminating the additional degrees of gauge freedom in a manner that leaves only two physical degrees of freedom which describe the polarization states for gravitational waves. The validity of this gauge will be demonstrated in two ways: first by a physical argument based on choosing a particular frame of reference, and secondly based on a more mathematically formal treatment by applying an appropriate gauge transformation to the metric.

The TT gauge derived by a *physical* argument using a particular coordinate frame

When considering gravitational waves, the most common treatment is to consider plane-fronted transverse waves in vacuum. In Appendix D, we show that a plane wave of the form

$$h_{\mu\nu} = A_{\mu\nu} e^{ik_\sigma x^\sigma} \tag{2583}$$

is transverse $\left(k^\nu A_{\mu\nu} = 0\right)$ in the harmonic gauge $\left(\partial^\nu h_{\mu\nu} = 0\right)$. If the wave has a null wave four-vector $(k_\sigma x^\sigma = 0)$ then it is also a solution to the linearized vacuum field equation (2499). We will assume these properties in the following discussion.

Now applying the harmonic gauge removes four of the ten degrees of freedom leaving six independent degrees of freedom for the metric perturbation. To see this, consider a wave traveling in the z-direction, which would have a wave four-vector given by $k^\sigma = (\omega/c, 0, 0, k)$. Since a plane wave has a null wave four-vector $(k^\sigma k_\sigma = 0)$, then $\omega/c = k$. In that case, the wave four-vector can be written as

$$k^\sigma = (k, 0, 0, k) \tag{2584}$$

We also showed in (2495) that a plane wave in the harmonic gauge is transverse so that

$$k^\nu A_{\mu\nu} = 0 \tag{2585}$$

Using (2584) in (2585) means we have

$$k^0 A_{\mu 0} + k^3 A_{\mu 3} = 0 \tag{2586}$$

Since $k^0 = k^3$, then $A_{0\mu} = -A_{\mu 3}$ which we can write as

$$A_{00} = -A_{03}, \qquad A_{01} = -A_{13}, \qquad A_{02} = -A_{23}, \qquad A_{30} = -A_{33} \tag{2587}$$

This removes four degrees of freedom. Therefore, the remaining six degrees of freedom are

$$A_{00}, \quad A_{01}, \quad A_{02}, \quad A_{11}, \quad A_{12}, \quad A_{22} \tag{2588}$$

Note that the first and last relation in (2587) are also equal since $A_{03} = A_{30}$ by symmetry of the metric tensor. This means that we also know $A_{33} = A_{00}$. So the amplitude tensor can be written as

$$A_{\mu\nu} \quad = \quad \begin{pmatrix} A_{00} & A_{01} & A_{02} & -A_{00} \\ A_{01} & A_{11} & A_{12} & -A_{01} \\ A_{02} & A_{12} & A_{22} & -A_{02} \\ -A_{00} & -A_{01} & -A_{02} & A_{00} \end{pmatrix} \tag{2589}$$

We can show that there is still some gauge freedom remaining by choosing a frame of reference moving with four-velocity $u^\mu = \left(\gamma c, \gamma v^i\right)$ that is transverse to the wave. Then we have

$$u^\mu h_{\mu\nu} = 0 \tag{2590}$$

Expanding the summation gives

$$\gamma c h_{0\nu} + \gamma v^i h_{i\nu} = 0 \tag{2591}$$

If we choose to observe the wave from a frame of reference such that $v^i = 0$, then we must have

$$h_{0\nu} = 0 \tag{2592}$$

This implies that these components are not *physical* degrees of freedom of the wave since they can be removed by a simple choice of velocity for the observer.[268] Therefore (2592) removes another three degrees of freedom in (2589) by setting $A_{00} = A_{01} = A_{02} = 0$. Then (2589) becomes

$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & A_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2593}$$

The harmonic gauge, $\partial^\mu h_{\nu\mu} = 0$, also gives

$$\partial^0 h_{\nu 0} + \partial^i h_{\nu i} = 0 \tag{2594}$$

For $\nu = 0$ this becomes

$$\partial^0 h_{00} + \partial^i h_{0i} = 0 \tag{2595}$$

Since $h_{0\nu} = 0$ according to (2592), then the second term above is zero and we have $\partial^0 h_{00} = 0$. This confirms that h_{00} is constant in time and is therefore a static gravitational potential that does not contribute to radiation fields and can be eliminated from the wave metric. Using $\nu = i$ for the free index and j for the repeating index in (2594) gives

$$\partial^0 h_{i0} + \partial^j h_{ij} = 0 \tag{2596}$$

Since $h_{0\nu} = 0$ according to (2592), then the first term above is zero and we are left with $\partial^i h_{ij} = 0$. This means the divergence of the spatial part of the amplitude tensor is zero. Using $h_{\mu\nu} = A_{\mu\nu} e^{ik_\sigma x^\sigma}$ from (2583) gives

$$\partial^j \left(A_{ij} e^{ik_\sigma x^\sigma}\right) = 0 \tag{2597}$$

Since A_{ij} and k_σ are constants, then we can bring them out of the derivative. Also, since $k_0 = -k$ and $k_3 = k$ from (2584), and because the wave is propagating in the $x_3 = z$ direction, then $k_\sigma x^\sigma = k_0 x^0 + k_l x^l = k(-ct + x_3)$. So (2597) becomes

$$A_{ij} e^{ik_\sigma x^\sigma} ik \partial^j (-ct + x_3) = A_{ij} e^{ik_\sigma x^\sigma} ik \delta_{j3} = ik A_{i3} e^{ik_\sigma x^\sigma} = 0 \tag{2598}$$

This result requires either the trivial solution $e^{ik^\alpha x_\alpha} = 0$ (which means no wave) or $A_{i3} = 0$. These components have in fact been eliminated from the metric as shown in (2593). Therefore, we confirm that (2593) also satisfies all the conditions of the harmonic gauge.

Lastly, we found in (333) that the only radiating field satisfying a wave equation is $h_{ij}^{\tau\tau}$ which is both transverse and *traceless*. For the wave amplitude in (2593) to be traceless, we must have $A_{11} = -A_{22}$. This means that the metric for a transverse plane wave in vacuum has only *two* independent degrees of freedom.

[268] Note that because the Einstein field equation in the trace-reversed harmonic gauge is $\Box \bar{h}_{\mu\nu} = -2\kappa T_{\mu\nu}$, then setting $h_{00} = 0$ in the TT gauge means we have $T_{00} = 0$ which can only be true in vacuum.

We can let $A_{11} = -A_{22} = A_{\oplus}$ for a *plus*-polarization gravitational wave and $A_{12} = A_{21} = A_{\otimes}$ for a *cross*-polarization gravitational wave. Then the metric perturbation in the *transverse-traceless* (TT) gauge is

$$\boxed{h^{TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{\oplus} & A_{\otimes} & 0 \\ 0 & A_{\otimes} & -A_{\oplus} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ikx} \qquad \textit{The TT gauge metric perturbation}} \tag{2599}$$

In order to determine the TT gauge components above, we used the harmonic gauge and some physical arguments concerning frames of reference when observing a transverse plane wave in vacuum. This allowed us to identify which components of the metric are purely gauge artifacts and therefore can be removed.

The TT gauge derived by a *mathematical* argument using a gauge transformation

Here we use a more formal mathematical treatment showing how the non-zero components of the metric are determined in the TT gauge. Once again, we begin with the harmonic gauge for the purpose of removing the four degrees of gauge freedom that we know exist in the metric in both matter and vacuum. We can then consider a plane wave in vacuum and identify the additional gauge freedom that results. Specifically, we can observe from (2400) that we can make any choice we wish for ξ^{μ} as long as we remain in the harmonic gauge (and the first-order approximation). Notice (2443) indicates that we can let

$$\Box \xi^{\mu} = 0 \tag{2600}$$

while still respecting (2444). This means that we are still in the harmonic gauge since this choice simply means that *both* $\partial_{\nu} h^{\mu\nu}$ *and* $\Box \xi^{\mu}$ are zero in (2443).[269] Solutions to the wave equation in (2600) are[270]

$$\xi^{\mu} = i\varepsilon^{\mu} e^{ik^{\sigma}x_{\sigma}} \tag{2601}$$

where ε^{μ} is a constant. Inserting (2601) into (2400) gives

$$x'^{\mu} = x^{\mu} + i\varepsilon^{\mu} e^{ik^{\sigma}x_{\sigma}} \tag{2602}$$

which implies that there are "waves" in the coordinates. Therefore, although we are free to choose ξ^{μ} in whatever way will satisfy (2600), it would be ideal to choose ε^{μ} so that it removes these nonphysical degrees of freedom from the metric. To do this, we can apply a coordinate transformation to the metric given in (2418) as

$$h'^{\mu\nu} = h^{\mu\nu} + \partial^{\mu}\xi^{\nu} + \partial^{\nu}\xi^{\mu} \tag{2603}$$

Using $h^{\mu\nu} = \bar{h}^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\bar{h}$ from (2435) to express this in terms of the trace-reversed metric perturbation gives

$$\bar{h}'^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu}\bar{h}' \quad = \quad \bar{h}^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu}\bar{h} + \partial^{\mu}\xi^{\nu} + \partial^{\nu}\xi^{\mu} \tag{2604}$$

Using the trace-reversed metric perturbation given in (2583) as a plane wave, $\bar{h}^{\mu\nu} = A^{\mu\nu}e^{ik_{\sigma}x^{\sigma}}$, and the trace from (2441) expressed as $\bar{h}' = \bar{h} - 2\partial_{\rho}\xi^{\rho}$, gives

$$A'^{\mu\nu}e^{ik_{\sigma}x^{\sigma}} - \tfrac{1}{2}\eta^{\mu\nu}\left(\bar{h} - 2\partial_{\rho}\xi^{\rho}\right) = A^{\mu\nu}e^{ik_{\sigma}x^{\sigma}} - \tfrac{1}{2}\eta^{\mu\nu}\bar{h} + \partial^{\mu}\xi^{\nu} + \partial^{\nu}\xi^{\mu} \tag{2605}$$

[269] This is directly analogous to being in *both* the Lorenz gauge and the Coulomb gauge in electromagnetism, which is the common approach for dealing with electromagnetic radiation.

[270] We insert an "i" only for the purpose of obtaining simpler equations when we take the derivative of ξ^{μ} later in the appendix.

$$A'^{\mu\nu}e^{ik_\sigma x^\sigma} = A^{\mu\nu}e^{ik_\sigma x^\sigma} + \partial^\mu \xi^\nu + \partial^\nu \xi^\mu - \eta^{\mu\nu}\partial_\rho \xi^\rho \tag{2606}$$

Inserting $\xi^\mu = i\varepsilon^\mu e^{ik^\sigma x_\sigma}$ from (2601) gives

$$A'^{\mu\nu}e^{ik_\sigma x^\sigma} = A^{\mu\nu}e^{ik_\sigma x^\sigma} + \partial^\mu i\varepsilon^\nu e^{ik^\sigma x_\sigma} + \partial^\nu i\varepsilon^\mu e^{ik^\sigma x_\sigma} - \eta^{\mu\nu}\partial_\rho i\varepsilon^\rho e^{ik^\sigma x_\sigma} \tag{2607}$$

Since ε_ρ is a constant, then the derivative can move past it. Also, from (2502) we know that $\partial^\mu (k_\sigma x^\sigma) = k^\mu$ and similarly, $\partial_\mu (k_\sigma x^\sigma) = k_\mu$. Then we have

$$A'^{\mu\nu}e^{ik_\sigma x^\sigma} = A^{\mu\nu}e^{ik_\sigma x^\sigma} - \varepsilon^\nu k^\mu e^{ik^\sigma x_\sigma} - \varepsilon^\mu k^\nu e^{ik^\sigma x_\sigma} + \eta^{\mu\nu}\varepsilon^\rho k_\rho e^{ik^\sigma x_\sigma} \tag{2608}$$

$$A'^{\mu\nu} = A^{\mu\nu} - \varepsilon^\mu k^\nu - \varepsilon^\nu k^\mu + \eta^{\mu\nu}\varepsilon^\rho k_\rho \tag{2609}$$

For a plane wave propagating in the z-direction, (2584) gives $k^\rho = (k,0,0,k)$ and $k_\mu = (-k,0,0,k)$. Then the expression above can be written as

$$A'^{\mu\nu} = A^{\mu\nu} - \varepsilon^\mu k^\nu - \varepsilon^\nu k^\mu + \eta^{\mu\nu}k\left(\varepsilon^3 - \varepsilon^0\right) \tag{2610}$$

The ten components of the transformed metric (omitting redundant components due to symmetry of the metric) will give the following transformation relations.

$$\begin{aligned}
&A'^{00} = A^{00} - k\left(\varepsilon^0 + \varepsilon^3\right), && A'^{01} = A^{01} - k\varepsilon^1, && A'^{13} = A^{13} - \varepsilon^1 k \\
&A'^{11} = A^{11} + k\left(\varepsilon^3 - \varepsilon^0\right), && A'^{02} = A^{02} - k\varepsilon^2, && A'^{12} = A^{12} \\
&A'^{22} = A^{22} + k\left(\varepsilon^3 - \varepsilon^0\right), && A'^{03} = A^{03} - k\left(\varepsilon^0 + \varepsilon^3\right), && A'^{23} = A^{23} - \varepsilon^2 k \\
&A'^{33} = A^{33} - k\left(\varepsilon^0 + \varepsilon^3\right)
\end{aligned} \tag{2611}$$

We would like to choose the constants ε^μ so as to eliminate as many components as possible in order to remove all gauge artifacts from the metric. If we set $A'^{00} = 0$, then we have

$$A^{00} = k\left(\varepsilon^3 + \varepsilon^0\right) \tag{2612}$$

Setting $A'^{01} = 0$ and $A'^{02} = 0$ will give, respectively,

$$\varepsilon^1 = A^{01}/k \qquad \text{and} \qquad \varepsilon^2 = A^{02}/k \tag{2613}$$

We also found in (2587) that in the harmonic gauge, a plane wave propagating in the z-direction requires that

$$A_{00} = -A_{03}, \qquad A_{01} = -A_{13}, \qquad A_{02} = -A_{23}, \qquad A_{03} = -A_{33} \tag{2614}$$

Since symmetry of the metric requires $A_{30} = A_{03}$, then we can combine the first and last relations into a single equality. We can also write all the relations with upper indices since we have (to first order in the metric) $A_{00} = A^{00}$, $A_{i0} = -A^{i0}$ and $A_{ij} = A^{ij}$. Therefore, the relations above become

$$A^{00} = A^{03} = A^{33}, \qquad A^{01} = A^{31}, \qquad A^{02} = A^{32} \tag{2615}$$

Since $A^{00}=A^{03}=A^{33}$, then using (2612) we have $A^{03}=A^{33}=k\left(\varepsilon^3+\varepsilon^0\right)$. Substituting these into A'^{03} and A'^{33} in (2611) gives $A'^{03}=A'^{33}=0$. Also, using $A^{01}=A^{31}$ from (2615) as well as $A^{01}=k\varepsilon^1$ from (2613) requires that $A^{31}=k\varepsilon^1$. Substituting this into A'^{13} in (2611) means $A'^{13}=0$. Similarly, using $A^{02}=A^{32}$ from (2615) and $A^{02}=k\varepsilon^2$ from (2613) requires that $A'^{32}=A'^{23}=0$. With all these conditions, the expressions in (2611) reduce to just

$$A'^{11}=A^{11}+k\left(\varepsilon^3-\varepsilon^0\right),\qquad A'^{22}=A^{22}+k\left(\varepsilon^3-\varepsilon^0\right),\qquad A'^{12}=A^{12} \tag{2616}$$

Solving (2612) for ε^0 gives $\varepsilon^0=A^{00}/k-\varepsilon^3$. Substituting this into (2616) gives

$$A'^{11}=A^{11}-A^{00},\qquad A'^{22}=A^{22}-A^{00},\qquad A'^{12}=A^{12} \tag{2617}$$

We have now used all the expressions which relate the metric perturbation components to the gauge freedom represented by ε^μ. Although it appears that there are *three* final independent degrees of freedom in (2617), we can make one additional gauge choice. By adding A'^{11} and A'^{22} and setting the sum to zero we have

$$A'^{11}+A'^{22}=A^{11}+A^{22}-2A^{00}=0 \tag{2618}$$

This means that

$$A'^{11}=-A'^{22} \tag{2619}$$

The choice in (2618) can be justified by choosing ε^3 and ε^0 in (2616) so that the sum of A'^{11} and A'^{22} is zero. This means the sum of A'^{11} and A'^{22} can be written as

$$A'^{11}+A'^{22}=A^{11}+A^{22}+2k\left(\varepsilon^3-\varepsilon^0\right)=0 \tag{2620}$$

Solving for $\varepsilon^3-\varepsilon^0$ gives

$$\varepsilon^3-\varepsilon^0=-\frac{A^{11}+A^{22}}{2k} \tag{2621}$$

Now we simply require consistency between the relation for $\left(\varepsilon^3-\varepsilon^0\right)$ in (2621) and the relation for $\left(\varepsilon^3+\varepsilon^0\right)$ from (2612). We can write (2612) as

$$\varepsilon^3+\varepsilon^0=A^{00}/k \tag{2622}$$

Adding (2621) and (2622) gives

$$2\varepsilon^3 = -\frac{A^{11}+A^{22}}{2k}+\frac{A^{00}}{k} \tag{2623}$$

$$\varepsilon^3 = \frac{2A^{00}-A^{11}-A^{22}}{4k} \tag{2624}$$

Also, subtracting (2621) and (2622) gives

$$-2\varepsilon^0 = -\frac{A^{11}+A^{22}}{2k}-\frac{A^{00}}{k} \tag{2625}$$

$$\varepsilon^0 = \frac{2A^{00}+A^{11}+A^{22}}{4k} \tag{2626}$$

Therefore, we summarize our choices for ε^μ as follows

$$\begin{aligned} &\varepsilon^0=\left(2A^{00}+A^{11}+A^{22}\right)/4k, &&\varepsilon^1=A^{01}/k \\ &\varepsilon^2=A^{02}/k, &&\varepsilon^3=\left(2A^{00}-A^{11}-A^{22}\right)/4k \end{aligned} \tag{2627}$$

We also conclude from the last relation in (2617) and from (2619) that the only two remaining independent *physical* degrees of freedom are

$$A^{12} = A^{21} \qquad \text{and} \qquad A^{11} = -A^{22} \tag{2628}$$

From this we see that for a wave traveling in the z-direction, the amplitude of the wave is only in the x and y directions. In other words, the wave is *transverse*. Also, applying the second condition in (2628) means that the trace of the amplitude tensor is zero so that it is *traceless*. Hence we arrive at the *transverse-traceless* (TT) gauge. Once again, we can let $A^{11} = -A^{22} = A_{\oplus}$ for the "plus" polarization state and let $A^{12} = A^{21} = A_{\otimes}$ for the "cross" polarization state of the gravitational wave. Similar to the result found in (2599), we can write the metric perturbation tensor in the TT gauge as

$$h_{\mu\nu}^{TT} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{\oplus} & A_{\otimes} & 0 \\ 0 & A_{\otimes} & -A_{\oplus} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ikx} \tag{2629}$$

The full metric in the TT gauge is

$$g_{\mu\nu}^{TT} = \eta_{\mu\nu} + h_{\mu\nu}^{TT} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1+h_{\oplus} & h_{\otimes} & 0 \\ 0 & h_{\otimes} & 1-h_{\oplus} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2630}$$

Then the relativistic invariant interval can be written as

$$ds^2 = -cdt^2 + (1+h_{\oplus})\,dx^2 + (1-h_{\oplus})\,dy^2 + 2h_{\otimes}dxdy + dz^2 \tag{2631}$$

For *plus* polarization, $h_{\oplus} \neq 0$ and $h_{\otimes} = 0$. Then the invariant is

$$ds^2 = -cdt^2 + (1+A_{\oplus})\,dx^2 + (1-A_{\oplus})\,dy^2 + dz^2 \tag{2632}$$

For *cross* polarization, $h_{\oplus} = 0$ and $h_{\otimes} \neq 0$. Then the invariant is

$$ds^2 = -cdt^2 + dx^2 + dy^2 + dz^2 + 2h_{\otimes}dxdy \tag{2633}$$

Independence of polarization fields

Although $h_{\oplus}$ and $h_{\otimes}$ are independent degrees of freedom, they are not completely unrelated. Since $\partial_i h_{ij}^{TT} = 0$, then we must have

$$\partial_x h_{\oplus} + \partial_y h_{\otimes} = 0 \qquad \text{and} \qquad \partial_x h_{\otimes} - \partial_y h_{\oplus} = 0 \tag{2634}$$

However, we can show that these differential equations can be decoupled. Taking ∂_x of both equations in (2634) gives

$$\partial_x^2 h_{\oplus} + \partial_x \partial_y h_{\otimes} = 0 \qquad \text{and} \qquad \partial_x^2 h_{\otimes} - \partial_x \partial_y h_{\oplus} = 0 \tag{2635}$$

Also taking ∂_y of both equations in (2634) gives

$$\partial_y \partial_x h_{\oplus} + \partial_y^2 h_{\otimes} = 0 \qquad \text{and} \qquad \partial_x \partial_y h_{\otimes} - \partial_y^2 h_{\oplus} = 0 \tag{2636}$$

Subtracting the first equation in (2635) from the second equation in (2636) gives

$$\partial_x^2 h_\oplus + \partial_y^2 h_\oplus = 0 \tag{2637}$$

Likewise, adding the second equation in (2635) to the first equation in (2636) gives

$$\partial_x^2 h_\otimes + \partial_y^2 h_\otimes = 0 \tag{2638}$$

Therefore, we find that $h_\oplus$ and $h_\otimes$ each satisfy their own independent, second-order differential equation.

Appendix G

The determinant of the metric to second order

Since $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, then writing the metric as an explicit matrix gives

$$g_{\mu\nu} = \begin{pmatrix} -1+h_{00} & h_{01} & h_{02} & h_{03} \\ h_{10} & 1+h_{11} & h_{12} & h_{13} \\ h_{20} & h_{21} & 1+h_{22} & h_{23} \\ h_{30} & h_{31} & h_{32} & 1+h_{33} \end{pmatrix} \tag{2639}$$

Then the determinant of $g_{\mu\nu}$ is

$$\begin{aligned} g &= (-1+h_{00}) \begin{vmatrix} 1+h_{11} & h_{12} & h_{13} \\ h_{21} & 1+h_{22} & h_{23} \\ h_{31} & h_{32} & 1+h_{33} \end{vmatrix} - h_{01} \begin{vmatrix} h_{10} & h_{12} & h_{13} \\ h_{20} & 1+h_{22} & h_{23} \\ h_{30} & h_{32} & 1+h_{33} \end{vmatrix} \\ &+ h_{02} \begin{vmatrix} h_{10} & 1+h_{11} & h_{13} \\ h_{20} & h_{21} & h_{23} \\ h_{30} & h_{31} & 1+h_{33} \end{vmatrix} - h_{03} \begin{vmatrix} h_{10} & 1+h_{11} & h_{12} \\ h_{20} & h_{21} & 1+h_{22} \\ h_{30} & h_{31} & h_{32} \end{vmatrix} \end{aligned} \tag{2640}$$

For convenience, this can be written as

$$g = A - B + C - D \tag{2641}$$

Evaluating A gives

$$\begin{aligned} A &= (-1+h_{00}) \Bigg((1+h_{11}) \begin{vmatrix} 1+h_{22} & h_{23} \\ h_{32} & 1+h_{33} \end{vmatrix} \\ &- h_{12} \begin{vmatrix} h_{21} & h_{23} \\ h_{31} & 1+h_{33} \end{vmatrix} + h_{13} \begin{vmatrix} h_{21} & 1+h_{22} \\ h_{31} & h_{32} \end{vmatrix} \Bigg) \end{aligned} \tag{2642}$$

$$\begin{aligned} &= (-1+h_{00}) \{ (1+h_{11}) [(1+h_{22})(1+h_{33}) - h_{23}h_{32}] \\ &- h_{12} [h_{21}(1+h_{33}) - h_{23}h_{13}] + h_{13} [h_{21}h_{32} - (1+h_{22}) h_{31}] \} \end{aligned} \tag{2643}$$

Multiplying out terms gives

$$\begin{aligned} A &= (-1+h_{00}) [(1+h_{11})(1+h_{22}+h_{33}+h_{22}h_{33}-h_{23}h_{32}) \\ &- h_{12}(h_{21}+h_{21}h_{33}-h_{23}h_{13}) + h_{13}(h_{21}h_{32}-h_{31}-h_{22}h_{31})] \end{aligned} \tag{2644}$$

Distributing and eliminating terms that are higher than second order gives

$$\begin{aligned} A &= \left(-1+h_{00}\right)\left(1+h_{22}+h_{33}+h_{22}h_{33}-h_{23}h_{32}+h_{11}\right. \\ &\qquad \left. +h_{11}h_{22}+h_{11}h_{33}-h_{12}h_{21}-h_{13}h_{31}\right) \end{aligned} \tag{2645}$$

Multiplying out terms and eliminating more terms that are higher than second order gives

$$\begin{aligned} A &= -1-h_{22}-h_{33}-h_{22}h_{33}+\left(h_{23}\right)^2-h_{11}-h_{11}h_{22}-h_{11}h_{33} \\ &\quad +\left(h_{12}\right)^2+\left(h_{13}\right)^2+h_{00}\left(1+h_{22}+h_{33}+h_{11}\right) \end{aligned} \tag{2646}$$

The trace of h_{ij} can be written as $H=\delta^{ij}h_{ij}=h_{11}+h_{22}+h_{33}$, then the result above becomes

$$A = -1-H+h_{00}\left(1+H\right)-h_{11}h_{22}-h_{22}h_{33}-h_{11}h_{33}+\left(h_{12}\right)^2+\left(h_{13}\right)^2+\left(h_{23}\right)^2 \tag{2647}$$

Now evaluating B gives

$$B = h_{01}\left(h_{10}\begin{vmatrix} 1+h_{22} & h_{23} \\ h_{32} & 1+h_{33} \end{vmatrix} - h_{12}\begin{vmatrix} h_{20} & h_{23} \\ h_{30} & 1+h_{33} \end{vmatrix} + h_{13}\begin{vmatrix} h_{20} & 1+h_{22} \\ h_{30} & h_{32} \end{vmatrix}\right) \tag{2648}$$

$$\begin{aligned} &= h_{01}\left\{h_{10}\left[\left(1+h_{22}\right)\left(1+h_{33}\right)-h_{23}h_{32}\right]-h_{12}\left[h_{20}\left(1+h_{33}\right)-h_{23}h_{30}\right]\right. \\ &\quad \left. +h_{13}\left[h_{20}h_{32}-\left(1+h_{22}\right)h_{30}\right]\right\} \end{aligned} \tag{2649}$$

Multiplying out terms and eliminating terms that are higher than second order gives

$$B = \left(h_{01}\right)^2 \tag{2650}$$

Now evaluating C gives

$$C = h_{02}\left(h_{10}\begin{vmatrix} h_{21} & h_{23} \\ h_{31} & 1+h_{33} \end{vmatrix} - \left(1+h_{11}\right)\begin{vmatrix} h_{20} & h_{23} \\ h_{30} & 1+h_{33} \end{vmatrix} + h_{13}\begin{vmatrix} h_{20} & h_{21} \\ h_{30} & h_{31} \end{vmatrix}\right) \tag{2651}$$

$$\begin{aligned} &= h_{02}\left\{h_{10}\left[h_{21}\left(1+h_{33}\right)-h_{23}h_{31}\right]-\left(1+h_{11}\right)\left[h_{20}\left(1+h_{33}\right)-h_{23}h_{30}\right]\right. \\ &\quad \left. +h_{13}\left[h_{20}h_{31}-h_{21}h_{30}\right]\right\} \end{aligned} \tag{2652}$$

Multiplying out terms and eliminating terms that are higher than second order gives

$$C = -\left(h_{02}\right)^2 \tag{2653}$$

Now evaluating D gives

$$D = h_{03}\left(h_{10}\begin{vmatrix} h_{21} & 1+h_{22} \\ h_{31} & h_{32} \end{vmatrix} - (1+h_{11})\begin{vmatrix} h_{20} & 1+h_{22} \\ h_{30} & h_{32} \end{vmatrix} + h_{12}\begin{vmatrix} h_{20} & h_{21} \\ h_{30} & h_{31} \end{vmatrix}\right) \quad (2654)$$

$$= h_{03}\{h_{10}[h_{21}h_{32} - (1+h_{22})h_{31}] - (1+h_{11})[h_{20}h_{32}(1+h_{22})h_{30}]$$

$$+h_{12}[(h_{20}h_{31} - h_{21}h_{30})]\} \quad (2655)$$

Multiplying out terms and eliminating terms that are higher than second order gives

$$D = (h_{03})^2 \quad (2656)$$

Inserting (2647), (2650), (2653), and (2656) into (2641) gives

$$\boxed{\begin{aligned} g = -1 - H + h_{00}(1+H) - h_{11}h_{22} - h_{22}h_{33} - h_{11}h_{33} \\ +(h_{12})^2 + (h_{13})^2 + (h_{23})^2 - (h_{01})^2 - (h_{02})^2 - (h_{03})^2 \end{aligned}} \quad (2657)$$

Metric determinant to second order in the metric

For the transverse-traceless metric perturbation in (177), the metric becomes $g^{\tau\tau}_{\mu\nu} = \eta_{\mu\nu} + h^{\tau\tau}_{\mu\nu}$. The matrix is

$$g^{\tau\tau}_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1+h_{11} & h_{12} & h_{13} \\ 0 & h_{21} & 1+h_{22} & h_{23} \\ 0 & h_{31} & h_{32} & 1+h_{33} \end{pmatrix} \quad (2658)$$

Since $h^{\tau\tau}_{ij}$ is spatially traceless, then $H = 0$. Also, since $h^{\tau\tau}_{0\mu} = 0$, then the determinant of the metric in (2657) reduces to

$$g^{\tau\tau} = -1 - h_{11}h_{22} - h_{22}h_{33} - h_{11}h_{33} + (h_{12})^2 + (h_{13})^2 + (h_{23})^2 \quad (2659)$$

For a gravitational wave propagating in the z-direction, $h^{\tau\tau}_{i3} = 0$ (since $h^{\tau\tau}_{ij}$ is transverse). Also $h^{\tau\tau}_{11} = -h^{\tau\tau}_{22} = h_{\oplus}$ for plus-polarization, and $h^{\tau\tau}_{12} = h^{\tau\tau}_{21} = h_{\otimes}$ for cross-polarization. Then the metric reduces further to

$$g^{\tau\tau}_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1+h_{\oplus} & h_{\otimes} & 0 \\ 0 & h_{\otimes} & 1-h_{\oplus} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2660)$$

Correspondingly, the determinant of the metric in (2659) reduces further to

$$\boxed{g^{\tau\tau} = -1 + h_{\oplus}^2 + h_{\otimes}^2 \qquad \textit{Determinant for the transverse-traceless metric}} \quad (2661)$$

Note that this result could have been obtained directly from the metric in (2660) by taking the determinant. This leads to $g^{\tau\tau} = -[(1+h_{\oplus})(1-h_{\oplus})] - h_{\otimes}^2$ which reduces to (2661). Hence, it is found that the determinant of the transverse-traceless metric does not contain any linear terms. The lowest order terms are second order. It also does not contains any *higher* order terms than second order.

Appendix H

The linearized transverse-traceless Landau-Lifshitz pseudotensor

In (2468) of Appendix B, we find the linearized Einstein field equation in the trace-reversed harmonic gauge is

$$\Box\bar{h}_{\mu\nu} = -2\kappa T_{\mu\nu} \tag{2662}$$

Here we consider the case where $T_{\mu\nu}$ is a stress tensor induced by some incoming gravitational wave $h^{(in)}_{\mu\nu}$ which is incident on some matter distribution. The resulting back-action could produce an outgoing gravitational wave $h^{(out)}_{\mu\nu}$. Since gravitation is a self-coupling field, then the incoming gravitational wave is also essentially a "source" of gravitation in addition to the stress tensor of the matter. To account for this, a common approach is to use $t_{\mu\nu}$ as a stress-energy pseudotensor for the gravitational field. As shown in MTW [11] (p. 465), the total stress tensor now becomes

$$T^{\mu\nu}_{\text{eff}} = T_{\mu\nu} + t^{\text{L-L}}_{\mu\nu} \tag{2663}$$

Then the resulting Einstein field equation in (2662) becomes

$$\Box\bar{h}_{\mu\nu} = -2\kappa\left(T_{\mu\nu} + t_{\mu\nu}\right) \tag{2664}$$

As shown in MTW [11] (p. 466), we can use the Landau-Lifshitz pseudotensor, $t^{\text{L-L}}_{\mu\nu}$. Then the Einstein equation can be written as[271]

$$\partial_\alpha\partial_\beta H^{\mu\alpha\nu\beta}_{\text{L-L}} = 2\kappa(-g)\left(T^{\mu\nu} + t^{\mu\nu}_{\text{L-L}}\right) \qquad \text{where} \qquad H^{\mu\alpha\nu\beta}_{\text{L-L}} = \mathfrak{g}^{\mu\nu}\mathfrak{g}^{\alpha\beta} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\mu\beta} \tag{2665}$$

Here we have $\mathfrak{g}^{\mu\nu} \equiv (-g)^{1/2} g^{\mu\nu}$ and $g \equiv \det g_{\mu\nu}$. Combining the equations above and using the Einstein equation, $G^{\mu\nu} = \kappa T^{\mu\nu}$, yields

$$\partial_\alpha\partial_\beta\left[(-g)\left(g^{\mu\nu}g^{\alpha\beta} - g^{\alpha\nu}g^{\mu\beta}\right)\right] = 2\kappa(-g)\left(\frac{1}{\kappa}G^{\mu\nu} + t^{\mu\nu}_{\text{L-L}}\right) \tag{2666}$$

Solving for $t^{\mu\nu}_{L-L}$ gives

$$\boxed{\begin{gathered} t^{\mu\nu}_{\text{L-L}} = -\frac{1}{\kappa}G^{\mu\nu} + \frac{1}{2\kappa(-g)}\partial_\alpha\partial_\beta\left[(-g)\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta}\right)\right] \\ \textit{Landau-Lifshitz pseudotensor} \end{gathered}} \tag{2667}$$

For gravitational waves, only the transverse-traceless part of the metric perturbation, $h^{\tau\tau}_{ij}$, is the propagating field.[272] In that case, (2664) becomes

$$\Box h^{\tau\tau\ (out)}_{ij} = -2\kappa\left(T^{\tau\tau}_{ij} + t^{\tau\tau}_{ij\ \text{L-L}}\right) \tag{2668}$$

[271]Note that MTW [11] set $G = c = 1$ so that $\kappa = 8\pi$. However, here we keep κ explicit in the calculation.

[272]Since $h^{\tau\tau}_{ij}$ is the only metric component that satisfies a wave equation, then all the other components of the metric will vanish in the far-field approximation.

where $h_{ij}^{\tau\tau\ (out)}$ represents the outgoing gravitational wave.[273] Then the *transverse-traceless* Landau-Lifshitz pseudotensor can be written using (2667) as

$$t_{ij\ \text{L-L}}^{\tau\tau} = -\frac{1}{\kappa}\left\{G_{ij} - \frac{1}{2g}\partial^\alpha\partial^\beta\left[g\left(g_{ij}g_{\alpha\beta} - g_{i\alpha}g_{j\beta}\right)\right]\right\}^{\tau\tau} \tag{2669}$$

In (268), it is found that the transverse-traceless part of G_{ij} is $G_{ij}^{\tau\tau} = -\frac{1}{2}\Box h_{ij}^{\tau\tau}$. Inserting this into (2669) gives

$$t_{ij\ \text{L-L}}^{\tau\tau} = \frac{1}{2\kappa}\left\{\Box h_{ij}^{\tau\tau} + \frac{1}{g^{\tau\tau}}\partial^\alpha\partial^\beta\left[g^{\tau\tau}\left(g_{ij}^{\tau\tau}g_{\alpha\beta}^{\tau\tau} - g_{i\alpha}^{\tau\tau}g_{j\beta}^{\tau\tau}\right)\right]\right\} \tag{2670}$$

For the transverse-traceless metric perturbation in (177), the metric becomes $g_{\mu\nu}^{\tau\tau} = \eta_{\mu\nu} + h_{\mu\nu}^{\tau\tau}$. The explicit matrix is

$$g_{\mu\nu}^{\tau\tau} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1+h_{11} & h_{12} & h_{13} \\ 0 & h_{21} & 1+h_{22} & h_{23} \\ 0 & h_{31} & h_{32} & 1+h_{33} \end{pmatrix} \tag{2671}$$

Using the result in (2659), it is evident that eliminating second order terms of the determinant leaves $g_{\mu\nu}^{\tau\tau} = 1$. Then (2670) simply becomes

$$t_{ij\ \text{L-L}}^{\tau\tau} = \frac{1}{2\kappa}\left\{\Box h_{ij}^{\tau\tau} + \partial^\alpha\partial^\beta\left(g_{ij}^{\tau\tau}g_{\alpha\beta}^{\tau\tau} - g_{i\alpha}^{\tau\tau}g_{j\beta}^{\tau\tau}\right)\right\} \tag{2672}$$

Summing over α gives

$$t_{ij\ \text{L-L}}^{\tau\tau} = \frac{1}{2\kappa}\left\{\Box h_{ij}^{\tau\tau} + \partial^\beta\left[\partial^0\left(g_{ij}^{\tau\tau}g_{0\beta}^{\tau\tau} - g_{i0}^{\tau\tau}g_{j\beta}^{\tau\tau}\right) + \partial^k\left(g_{ij}^{\tau\tau}g_{k\beta}^{\tau\tau} - g_{ik}^{\tau\tau}g_{j\beta}^{\tau\tau}\right)\right]\right\} \tag{2673}$$

From (2671) it is evident that $g_{i0}^{\tau\tau} = 0$, so the terms involving $g_{i0}^{\tau\tau}g_{j\beta}^{\tau\tau}$ can be eliminated. Then summing over β gives

$$\begin{aligned} t_{ij\ \text{L-L}}^{\tau\tau} = \frac{1}{2\kappa}\Big\{&\Box h_{ij}^{\tau\tau} + \partial^0\left[\partial^0\left(g_{ij}^{\tau\tau}g_{00}^{\tau\tau}\right) + \partial^k\left(g_{ij}^{\tau\tau}g_{k0}^{\tau\tau} - g_{ik}^{\tau\tau}g_{j0}^{\tau\tau}\right)\right] \\ &+\partial^l\left[\partial^0\left(g_{ij}^{\tau\tau}g_{0l}^{\tau\tau}\right) + \partial^k\left(g_{ij}^{\tau\tau}g_{kl}^{\tau\tau} - g_{ik}^{\tau\tau}g_{jl}^{\tau\tau}\right)\right]\Big\} \end{aligned} \tag{2674}$$

Again, using $g_{k0}^{\tau\tau} = 0$ and $g_{00}^{\tau\tau} = -1$ gives

$$t_{ij\ \text{L-L}}^{\tau\tau} = \frac{1}{2\kappa}\left\{\Box h_{ij}^{\tau\tau} - \partial^0\partial^0 g_{ij}^{\tau\tau} + \partial^l\partial^k\left(g_{ij}^{\tau\tau}g_{kl}^{\tau\tau} - g_{ik}^{\tau\tau}g_{jl}^{\tau\tau}\right)\right\} \tag{2675}$$

Using $g_{ij}^{\tau\tau} = \eta_{ij} + h_{ij}^{\tau\tau}$ and $\partial^0 \approx -\frac{1}{c}\partial_t$ (to first order in the metric perturbation) gives

$$\begin{aligned} t_{ij\ \text{L-L}}^{\tau\tau} = \frac{1}{2\kappa}\Big\{&\Box h_{ij}^{\tau\tau} - \frac{1}{c^2}\ddot{h}_{ij}^{\tau\tau} \\ &+\partial^l\partial^k\left[\left(\eta_{ij} + h_{ij}^{\tau\tau}\right)\left(\eta_{kl} + h_{kl}^{\tau\tau}\right) - \left(\eta_{ik} + h_{ik}^{\tau\tau}\right)\left(\eta_{jl} + h_{jl}^{\tau\tau}\right)\right]\Big\} \end{aligned} \tag{2676}$$

[273]The Landau-Lifshitz pseudotensor in (2667) is in terms of the *incoming* wave, $h_{ij}^{\tau\tau\ (in)}$. However, the superscript “in” is dropped for brevity and $h_{ij}^{\tau\tau}$ (with no superscript) is used to refer to the *incoming* wave.

Keeping only first order terms in the metric and recognizing that derivatives of η_{ij} are zero gives

$$t^{\tau\tau}_{ij\,\text{L-L}} = \frac{1}{2\kappa}\left\{\Box h^{\tau\tau}_{ij} - \frac{1}{c^2}\ddot{h}^{\tau\tau}_{ij} + \partial^l\partial^k\left[\left(\eta_{kl}h^{\tau\tau}_{ij} + \eta_{ij}h^{\tau\tau}_{kl}\right) - \left(\eta_{jl}h^{\tau\tau}_{ik} + \eta_{ik}h^{\tau\tau}_{jl}\right)\right]\right\} \tag{2677}$$

Distributing the derivatives and using $\eta_{ij} = \delta_{ij}$ gives

$$t^{\tau\tau}_{ij\,\text{L-L}} = \frac{1}{2\kappa}\left(\Box h^{\tau\tau}_{ij} - \frac{1}{c^2}\ddot{h}^{\tau\tau}_{ij} + \nabla^2 h^{\tau\tau}_{ij} + \partial^l\partial^k h^{\tau\tau}_{kl}\eta_{ij} - \partial_j\partial^k h^{\tau\tau}_{ik} - \partial_i\partial^l h^{\tau\tau}_{jl}\right) \tag{2678}$$

Since $h^{\tau\tau}_{kl}$ is transverse, then $\partial^k h^{\tau\tau}_{kl} = 0$, so the last three terms vanish. Using $\Box = -\frac{1}{c^2}\partial_t^2 + \nabla^2$ to combine the other terms gives[274]

$$\boxed{t^{\tau\tau}_{ij\,\text{L-L}} = \frac{1}{\kappa}\Box h^{\tau\tau}_{ij} \qquad \textit{The transverse-traceless Landau-Lifshitz pseudotensor to first order in the metric}} \tag{2679}$$

Lastly, substituting this back into (2668) gives

$$\Box h^{\tau\tau\,(out)}_{ij} = -2\kappa T^{\tau\tau}_{ij} - 2\Box h^{\tau\tau}_{ij} \tag{2680}$$

where $h^{\tau\tau\,(out)}_{ij}$ is the outgoing gravitational wave and $h^{\tau\tau}_{ij}$ is understood to be the incoming wave. Note that both fields must be transverse-traceless for consistency, while $T^{\tau\tau}_{ij}$ is the transverse-traceless part of the full stress tensor, $T_{\mu\nu}$.

[274] Alternatively, we could have seen from (2672) that because $g^{\tau\tau}_{i\alpha}$ is transverse, then $\partial^\alpha\partial^\beta g^{\tau\tau}_{i\alpha}g^{\tau\tau}_{j\beta} = 0$ so the last term in (2672) vanishes. Also, using the metric to lower the index of a derivative gives $\partial^\alpha\partial^\beta g^{\tau\tau}_{ij}g^{\tau\tau}_{\alpha\beta} = \partial^\alpha\partial_\alpha g^{\tau\tau}_{ij} = \Box g^{\tau\tau}_{ij}$. Therefore, the first two terms in (2672) combine and we immediately have $t^{\tau\tau}_{ij\,\text{L-L}} = \frac{1}{\kappa}h^{\tau\tau}_{ij}$ as found in (2679). This avoids the need for summing over indices explicitly as was done above.

Appendix I

The linearized ideal fluid stress tensor

The covariant components of the ideal fluid stress tensor

The ideal fluid stress tensor is given by

$$T_{\mu\nu}^{(ideal\ fluid)} = \left(\rho + P/c^2\right) u_\mu u_\nu + P g_{\mu\nu} \tag{2681}$$

where u^μ is the four-velocity and P is the pressure. Substituting $u_\mu u_\nu = g_{\mu\sigma} g_{\nu\rho} u^\sigma u^\rho$ and expressing the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ gives[275]

$$T_{\mu\nu}^{(ideal\ fluid)} = \left(\rho + P/c^2\right)\left(\eta_{\mu\sigma} + h_{\mu\sigma}\right)\left(\eta_{\nu\rho} + h_{\nu\rho}\right) u^\sigma u^\rho + P\left(\eta_{\mu\nu} + h_{\mu\nu}\right) \tag{2682}$$

As noted in Appendix A, using $\partial^\nu T_{\mu\nu} = 0$ as the *linear* conservation of stress-energy-momentum involves neglecting terms of order $\left(\partial_\sigma h_{\mu\nu}\right) T^{\mu\nu}$. This does *not* require neglecting terms of order $h_{\mu\nu} T^{\mu\nu}$. Therefore, to first order in $h_{\mu\nu}$, we only neglect terms involving $h_{\mu\sigma} h_{\nu\rho}$.

$$T_{\mu\nu}^{(ideal\ fluid)} = \left(\rho + P/c^2\right)\left(\eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\sigma} h_{\nu\rho} + h_{\mu\sigma}\eta_{\nu\rho}\right) u^\sigma u^\rho + P\left(\eta_{\mu\nu} + h_{\mu\nu}\right) \tag{2683}$$

We can distribute terms and use $u^\mu = \gamma v^\mu$, where $v^\mu = \left(c, v^i\right)$ and γ is given by (2738) as

$$\gamma = \left(1 - h_{00} - 2h_{0i}\frac{v^i}{c} - \frac{v^2}{c^2} - h_{ij}\frac{v^i v^j}{c^2}\right)^{-1/2} \tag{2684}$$

Then we have

$$T_{\mu\nu}^{(ideal\ fluid)} = \left(\rho + P/c^2\right)\left(\eta_{\mu\sigma}\eta_{\nu\rho} v^\sigma v^\rho + \eta_{\mu\sigma} h_{\nu\rho} v^\sigma v^\rho + h_{\mu\sigma}\eta_{\nu\rho} v^\sigma v^\rho\right)\gamma^2 + P\left(\eta_{\mu\nu} + h_{\mu\nu}\right) \tag{2685}$$

We now evaluate each of the stress tensor components. For T_{00} we have

$$T_{00}^{(ideal\ fluid)} = \left(\rho + P/c^2\right)\left(\eta_{0\sigma}\eta_{0\rho} v^\sigma v^\rho + \eta_{0\sigma} h_{0\rho} v^\sigma v^\rho + h_{0\sigma}\eta_{0\rho} v^\sigma v^\rho\right)\gamma^2 + P\left(-1 + h_{00}\right) \tag{2686}$$

Summing over σ and ρ gives

$$\boxed{T_{00}^{(ideal\ fluid)} = \left(\rho + P/c^2\right)\gamma^2\left(c^2 - 2c^2 h_{00} - 2c h_{0k} v^k\right) - P + P h_{00}} \tag{2687}$$

For T_{0i}, (2685) gives

$$T_{0i}^{(ideal\ fluid)} = \left(\rho + P/c^2\right)\left(\eta_{0\sigma}\eta_{i\rho} v^\sigma v^\rho + \eta_{0\sigma} h_{i\rho} v^\sigma v^\rho + h_{0\sigma}\eta_{i\rho} v^\sigma v^\rho\right)\gamma^2 + P h_{0i} \tag{2688}$$

[275] As noted in Appendix A, using $\partial^\nu T_{\mu\nu} = 0$ as the *linear* conservation of stress-energy-momentum involves neglecting terms of order $\left(\partial_\sigma h_{\mu\nu}\right) T^{\mu\nu}$. This does *not* require neglecting terms of order $h_{\mu\nu} T^{\mu\nu}$.

Summing over σ and ρ gives

$$\boxed{T_{0i}^{(ideal\ fluid)} = \left(\rho + P/c^2\right)\gamma^2\left(-cv_i - c^2 h_{i0} - ch_{ik}v^k + ch_{00}v_i + h_{0k}v^k v_i\right) + Ph_{0i}} \tag{2689}$$

For T_{ij}, (2685) gives

$$T_{ij}^{(ideal\ fluid)} = \left(\rho + P/c^2\right)\left(\eta_{i\sigma}\eta_{j\rho}v^\sigma v^\rho + \eta_{i\sigma}h_{j\rho}v^\sigma v^\rho + h_{i\sigma}\eta_{j\rho}v^\sigma v^\rho\right)\gamma^2 + P\left(\eta_{ij} + h_{ij}\right) \tag{2690}$$

Summing over σ and ρ gives

$$\boxed{T_{ij}^{(ideal\ fluid)} = \left(\rho + P/c^2\right)\gamma^2\left(v_i v_j + ch_{i0}v_j + ch_{j0}v_i + h_{ik}v^k v_j + h_{jk}v^k v_i\right) + P\left(\eta_{ij} + h_{ij}\right)} \tag{2691}$$

Neglecting self-coupling and terms higher order than v^2/c^2

If we neglect any self-coupling of the gravitational field to the stress tensor sources, then we can neglect terms involving $h_{\mu\nu}T_{\rho\sigma}$ and γ^2 reduces to the usual Lorentz factor in Special Relativity, $\gamma^2 = \left(1 - v^2/c^2\right)^{-1}$. In that case, (2687), (2689) and (2691) become, respectively,

$$T_{00}^{(ideal\ fluid)} \approx \left(\rho c^2 + P\right)\gamma^2 - P \tag{2692}$$

$$T_{0i}^{(ideal\ fluid)} \approx -\left(\rho c + P/c\right)v_i\gamma^2 \tag{2693}$$

$$T_{ij}^{(ideal\ fluid)} \approx \left(\rho + P/c^2\right)v_i v_j\gamma^2 + P\eta_{ij} \tag{2694}$$

Furthermore, if we are only interested in keeping source terms to order v^2/c^2, then we can use a binomial approximation to write $\gamma^2 = \left(1 - v^2/c^2\right)^{-1} \approx 1 + v^2/c^2$. Substituting this into the expressions above and keeping only v^2/c^2 to highest order gives

$$T_{00}^{(ideal\ fluid)} \approx \left(\rho c^2 + P\right)\left(1 + v^2/c^2\right) - P \tag{2695}$$

$$\boxed{T_{00}^{(ideal\ fluid)} \approx \rho c^2\left(1 + v^2/c^2\right) + Pv^2/c^2} \tag{2696}$$

$$T_{0i}^{(ideal\ fluid)} \approx -\left(\rho c + P/c\right)\left(1 + v^2/c^2\right)v_i \tag{2697}$$

$$\boxed{T_{0i}^{(ideal\ fluid)} \approx -\left(\rho c + P/c\right)v_i} \tag{2698}$$

$$T_{ij}^{(ideal\ fluid)} \approx \left(\rho v_i v_j + P v_i v_j/c^2\right)\left(1+v^2/c^2\right) + P\eta_{ij} \tag{2699}$$

$$\boxed{T_{ij}^{(ideal\ fluid)} \approx \rho v_i v_j + P v_i v_j/c^2 + P\eta_{ij}} \tag{2700}$$

Linearized conservation of the stress-energy-momentum tensor

The linearized conservation law for the stress-energy-momentum tensor is given in (2394) of Appendix A as

$$\partial^\nu T_{\mu\nu} = 0 \tag{2701}$$

Summing over ν gives

$$\partial^0 T_{\mu 0} + \partial^i T_{\mu i} = 0 \tag{2702}$$

For $\mu = 0$, we have the following mass-momentum continuity equation.

$$\partial^0 T_{00} + \partial^i T_{0i} = 0 \tag{2703}$$

Since we are neglecting terms involving $h_{\mu\nu}T_{\rho\sigma}$, then we can lower indices using $g_{\mu\nu} \approx \eta_{\mu\nu}$. Substituting (2692) and (2693) into the expression above gives

$$\partial^0\left[\left(\rho c^2 + P\right)\gamma^2 - P\right] - \partial^i\left[\left(\rho c + P/c\right)v_i\gamma^2\right] \quad = \quad 0 \tag{2704}$$

Applying the product rule for the spatial derivative gives

$$-\tfrac{1}{c}\left[\left(\dot\rho c^2 + \dot P\right)\gamma^2 + \left(\rho c^2 + P\right)2\gamma\dot\gamma + \dot P\right] - \left(\partial^i \rho c + \partial^i P/c\right)v_i\gamma^2$$

$$-\left(\rho c + P/c\right)\left(\partial^i v_i\right)\gamma^2 - \left(\rho c + P/c\right)v_i 2\gamma\partial^i\gamma \quad = \quad 0 \tag{2705}$$

For a fluid with incompressible flow, we have $\partial_i v_i = 0$. Also, if the mass density remains uniform, then $\partial_i \rho = 0$. Lastly, if there are no pressure gradients, then $\partial_i P = 0$. In that case, (2705) simply becomes

$$-\left(\dot\rho c^2 + \dot P\right)\gamma^2 - \left(\rho c^2 + P\right)2\gamma\dot\gamma - \dot P \quad = \quad 0 \tag{2706}$$

$$2c^2\gamma\dot\gamma\rho + \gamma^2 c^2\dot\rho + 2\gamma\dot\gamma P + \left(\gamma^2+1\right)\dot P \quad = \quad 0 \tag{2707}$$

For the case of a relativistic dust (no pressures), we have

$$2\dot\gamma\rho + \gamma\dot\rho = 0 \tag{2708}$$

Since $\gamma^2 = \left(1 - v^2/c^2\right)^{-1}$, then

$$\dot\gamma \quad = \quad -\left(1-v^2/c^2\right)^{-2}\partial_t\left(1-v^2/c^2\right) \tag{2709}$$

$$= \quad \left(1-v^2/c^2\right)^{-2}\left(2va/c^2\right) \tag{2710}$$

$$= \quad 2va\gamma^4/c^2 \tag{2711}$$

Then (2708) becomes

$$4\gamma^3 \rho v a/c^2 + \dot{\rho} \quad = \quad 0 \tag{2712}$$

To order v^2/c^2, we have $\gamma = \left(1 - v^2/c^2\right)^{-1/2} \approx 1 + \dfrac{v^2}{2c^2}$. Then writing the expression above to order v^2/c^2 gives

$$4\rho v a/c^2 + \dot{\rho} \quad = \quad 0 \tag{2713}$$

This is effectively an equation of motion for relativistic dust (to order v^2/c^2). If we take the non-relativistic limit, then $va/c^2 \approx 0$ and we simply have

$$\dot{\rho} = 0 \tag{2714}$$

This means that the mass density distribution of a non-relativistic, pressureless, ideal fluid cannot change in time. If we return to (2394) and let $\mu = i$, then we have the following momentum-stress conservation equation.

$$\partial^0 T_{i0} + \partial^j T_{ij} \quad = \quad 0 \tag{2715}$$

Inserting (2693) and (2694) gives

$$\partial^0 \left[-\left(\rho c + P/c\right) v_i \gamma^2\right] + \partial^j \left[\left(\rho + P/c^2\right) v_i v_j \gamma^2 + P\eta_{ij}\right] \quad = \quad 0 \tag{2716}$$

Once again, we can set $\partial_i v_i = \partial_i \rho = \partial_i P = 0$. Then we have

$$\left(\dot{\rho} c + \dot{P}/c\right) v_i \gamma^2 + \left(\rho c + P/c\right) a_i \gamma^2 + \left(\rho c + P/c\right) v_i 2\gamma\dot{\gamma} \quad = \quad 0 \tag{2717}$$

Using (2711) gives

$$\left(\dot{\rho} c + \dot{P}/c\right) v_i + \left(\rho c + P/c\right)\left(a_i + 4va\gamma^3 v_i/c^2\right) \quad = \quad 0 \tag{2718}$$

For the case of a relativistic dust (no pressures), we have

$$\dot{\rho} v_i + \rho a_i + 4\rho v a \gamma^3 v_i/c^2 \quad = \quad 0 \tag{2719}$$

To order v^2/c^2, we have $\gamma = \left(1 - v^2/c^2\right)^{-1/2} \approx 1 + \dfrac{v^2}{2c^2}$. Then writing the expression above to order v^2/c^2 gives

$$\dot{\rho} v_i + \rho a_i + 4\rho a v v_i/c^2 \quad = \quad 0 \tag{2720}$$

If we take the non-relativistic limit, then $va/c^2 \approx 0$ and we simply have

$$\dot{\rho} v_i + \rho a_i = 0 \tag{2721}$$

This is effectively an equation of motion for a mass element with mass density ρ, velocity v_i and acceleration a_i. If the mass element doesn't accelerate, then $a_i = 0$ and we simply have $\dot{\rho} v_i = 0$ which indicates that the mass element is moving with a constant velocity and is changing in mass density.

Appendix J

The linearized geodesic equation of motion

Reparameterizing the geodesic equation of motion in terms of coordinate time

The geodesic equation of motion is given by

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0 \tag{2722}$$

We can reparameterize the equation in terms of t instead of τ by using a chain rule

$$\frac{dx^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dx^\mu}{dt}=\gamma v^\mu \tag{2723}$$

where $\gamma = dt/d\tau$ and $v^\mu = \left(c,\dot{x}^i\right) = \left(c,v^i\right)$. We can obtain $\dfrac{d^2x^\mu}{d\tau^2}$ by applying $\dfrac{d}{d\tau}$ again on (2723). Using the chain rule again gives

$$\frac{d^2x^\mu}{d\tau^2} = \frac{d}{d\tau}(\gamma v^\mu) \tag{2724}$$

$$= \frac{dt}{d\tau}\frac{d}{dt}(\gamma v^\mu) \tag{2725}$$

$$= \gamma\frac{d}{dt}(\gamma v^\mu) \tag{2726}$$

Now applying the product rule gives

$$\frac{d^2x^\mu}{d\tau^2} = \gamma(\gamma a^\mu+\dot{\gamma}v^\mu) \tag{2727}$$

$$= \gamma^2a^\mu+\gamma\dot{\gamma}v^\mu \tag{2728}$$

where $a^\mu = \left(0,\dot{v}^i\right)$. Substituting this result into (2722) and using $\dfrac{dx^\mu}{d\tau}=\gamma v^\mu$ for the terms contracted with the Christoffel symbols gives

$$\boxed{\gamma^2a^\mu+\gamma\dot{\gamma}v^\mu+\gamma^2\Gamma^\mu_{\rho\sigma}v^\rho v^\sigma = 0 \qquad \textit{Geodesic equation of motion in terms of coordinate time}} \tag{2729}$$

The "Lorentz factor" in curved space-time

To find an expression for $\gamma = dt/d\tau$, recall that the proper time $d\tau$ is defined in terms of the invariant interval ds by

$$ds^2 = -c^2 d\tau^2 \tag{2730}$$

We can also express the invariant interval in terms of the metric as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2731}$$

This gives

$$ds^2 = c^2 g_{00} dt^2 + 2c g_{0i} dt dx^i + g_{ij} dx^i dx^j \tag{2732}$$

$$= c^2 dt^2 \left(g_{00} + \frac{2}{c} g_{0i} \frac{dx^i}{dt} + \frac{1}{c^2} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \right) \tag{2733}$$

$$= c^2 dt^2 \left(g_{00} + \frac{2}{c} g_{0i} v^i + \frac{1}{c^2} g_{ij} v^i v^j \right) \tag{2734}$$

Equating (2730) and (2734) gives

$$\frac{d\tau^2}{dt^2} = g_{00} + 2 g_{0i} \frac{v^i}{c} + g_{ij} \frac{v^i v^j}{c^2} \tag{2735}$$

Writing the metric as a perturbation added to flat Minkowski space-time, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, gives

$$\frac{d\tau^2}{dt^2} = -(\eta_{00} + h_{00}) - 2(\eta_{0i} + h_{0i}) \frac{v^i}{c} - (\eta_{ij} + h_{ij}) \frac{v^i v^j}{c^2} \tag{2736}$$

Since $\eta_{00} = -1$, $\eta_{0i} = 0$, and $\eta_{ij} = \delta_{ij}$, then we have

$$\frac{d\tau^2}{dt^2} = 1 - h_{00} - 2h_{0i} \frac{v^i}{c} - \frac{v^2}{c^2} - h_{ij} \frac{v^i v^j}{c^2} \tag{2737}$$

Since the Lorentz factor in Special Relativity is defined as $\gamma \equiv dt/d\tau$, then we can define a "Lorentz factor" in curved space-time by solving (2737) for $dt/d\tau$. This gives

$$\boxed{\gamma = \left(1 - h_{00} - 2h_{0i} \frac{v^i}{c} - \frac{v^2}{c^2} - h_{ij} \frac{v^i v^j}{c^2} \right)^{-1/2} \qquad \textit{Lorentz factor in GR}} \tag{2738}$$

Note that for a flat Minkowski space-time, we have $h_{\mu\nu} = 0$, and the Lorentz factor reduces to the familiar form in Special Relativity: $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$.

Conservation of energy momentum

Recall that in Special Relativity, the energy and momentum are not conserved independently. Rather they are conserved together according to the relation $E^2 - c^2 p^2 = m^2 c^4$. Taking the derivative of this expression with respect to proper time (for a particle with constant mass) gives

$$E\frac{dE}{d\tau} - c^2\frac{dp}{d\tau} \quad = \quad 0 \tag{2739}$$

This expression can be related to the geodesic equation of motion by writing (2722) in terms of $p^\mu = m\dfrac{dx^\mu}{d\tau}$. This gives

$$\frac{dp^\mu}{d\tau} + \frac{1}{m}\Gamma^\mu_{\rho\sigma}p^\rho p^\sigma \quad = \quad 0 \tag{2740}$$

Since $p^\mu = \left(E/c,\ p^i\right)$, then for $\mu = 0$ and $\mu = i$ we have, respectively,

$$\frac{dE}{d\tau} + \frac{c}{m}\Gamma^0_{\rho\sigma}p^\rho p^\sigma \quad = \quad 0 \qquad \text{and} \qquad \frac{dp^i}{d\tau} + \frac{1}{m}\Gamma^i_{\rho\sigma}p^\rho p^\sigma = 0 \tag{2741}$$

The first expression represents conservation of energy while the second represents conservation of momentum. As shown in (2739), these quantities are not actually conserved independently, but rather their *difference* is conserved. To relate this to the acceleration, we can evaluate (2729) for $\mu = 0$ and for $\mu = i$, and use $a^\mu = \left(0, \dot{v}^i\right)$ and $v^\mu = \left(c, v^i\right)$. This gives

$$c\gamma\dot{\gamma} + \gamma^2\Gamma^0_{\rho\sigma}v^\rho v^\sigma \quad = \quad 0 \qquad \text{and} \qquad \gamma^2 a^i + \gamma\dot{\gamma}v^i + \gamma^2\Gamma^i_{\rho\sigma}v^\rho v^\sigma = 0 \tag{2742}$$

Multiplying the first equation by v^i/c and subtracting it from the second gives

$$\boxed{a^i = -\Gamma^i_{\rho\sigma}v^\rho v^\sigma + \Gamma^0_{\rho\sigma}v^\rho v^\sigma v^i/c \qquad \begin{matrix}\textit{Geodesic equation of motion}\\ \textit{in terms of coordinate time}\end{matrix}} \tag{2743}$$

This is the geodesic equation of motion in terms of the metric perturbation and Christoffel symbols with absolutely no approximations. Notice that the terms with $\gamma\dot{\gamma}$ by subtraction from one another, and γ^2 vanishes when divided from each term. Therefore, (2738) does not need to be inserted. Matching this result to (2741), we find that the first term (involving $\Gamma^i_{\rho\sigma}$) corresponds to accelerations due to changes in *momentum*, while the second term (involving $\Gamma^0_{\rho\sigma}$) corresponds to accelerations due to changes in *energy*. Summing over repeated indices and rearranging leads to

$$a^i \quad = \quad -\left(c^2\Gamma^i_{00} + 2c\Gamma^i_{0j}v^j + \Gamma^i_{jk}v^j v^k\right) + \left(\Gamma^0_{00}c + 2\Gamma^0_{j0}v^j + \Gamma^0_{jk}v^j v^k/c\right)v^i \tag{2744}$$

This result matches [28], equation (3.2). In order to evaluate this expression to linear order in $h_{\mu\nu}$, we can use the linearized Christoffel symbols. These must now be determined.

The Christoffel symbols to first order in the metric

To obtain an expression completely in terms of the metric, we can evaluate the Christoffel symbols using (2367) which gives

$$\Gamma^{\alpha}_{\beta\gamma} = \tfrac{1}{2}\eta^{\alpha\rho}\left(\partial_\gamma h_{\rho\beta} + \partial_\beta h_{\gamma\rho} - \partial_\rho h_{\beta\gamma}\right) \tag{2745}$$

For $\alpha = 0$, the only non-vanishing Christoffel symbols have $\rho = 0$ so that $\eta^{\alpha\rho} = -1$. This gives

$$\Gamma^0_{\beta\gamma} = -\tfrac{1}{2}\left(\partial_\gamma h_{0\beta} + \partial_\beta h_{\gamma 0} - \partial_0 h_{\beta\gamma}\right) \tag{2746}$$

Setting $\beta = 0$ gives

$$\Gamma^0_{0\gamma} = -\tfrac{1}{2}\left(\partial_\gamma h_{00} + \partial_0 h_{\gamma 0} - \partial_0 h_{0\gamma}\right) = -\tfrac{1}{2}\partial_\gamma h_{00} \tag{2747}$$

For $\gamma = 0$, (2747) becomes

$$\boxed{\Gamma^0_{00} = -\tfrac{1}{2}\partial_0 h_{00}} \tag{2748}$$

For $\gamma = i$, (2747) becomes

$$\boxed{\Gamma^0_{0i} = -\tfrac{1}{2}\partial_i h_{00}} \tag{2749}$$

Returning to (2746) and setting $\beta = i$ gives

$$\Gamma^0_{i\gamma} = -\tfrac{1}{2}\left(\partial_\gamma h_{0i} + \partial_i h_{\gamma 0} - \partial_0 h_{i\gamma}\right) \tag{2750}$$

For $\gamma = 0$, (2750) becomes

$$\Gamma^0_{i0} = -\tfrac{1}{2}\left(\partial_0 h_{0i} + \partial_i h_{00} - \partial_0 h_{i0}\right) = -\tfrac{1}{2}\partial_i h_{00} \tag{2751}$$

This matches (2749) as expected since Christoffel symbols are symmetric in the lower two indices. For $\gamma = j$, (2750) becomes

$$\boxed{\Gamma^0_{ij} = -\tfrac{1}{2}\left(\partial_j h_{0i} + \partial_i h_{j0} - \partial_0 h_{ij}\right)} \tag{2752}$$

Now we return to the general expression for the Christoffel symbols in (2745) and choose $\alpha = i$.

$$\Gamma^{i}_{\beta\gamma} = \tfrac{1}{2}\eta^{i\rho}\left(\partial_\gamma h_{\rho\beta} + \partial_\beta h_{\gamma\rho} - \partial^i h_{\beta\gamma}\right) \tag{2753}$$

$$\Gamma^{i}_{\beta\gamma} = \tfrac{1}{2}\left(\partial_\gamma h^i{}_\beta + \partial_\beta h^i{}_\gamma - \partial^i h_{\beta\gamma}\right) \tag{2754}$$

Setting $\beta = 0$ gives

$$\Gamma^i_{0\gamma} = \tfrac{1}{2}\left(\partial_\gamma h^i{}_0 + \partial_0 h^i{}_\gamma - \partial^i h_{0\gamma}\right) \tag{2755}$$

For $\gamma = 0$, (2755) becomes

$$\Gamma^i_{00} = \tfrac{1}{2}\left(\partial_0 h^i{}_0 + \partial_0 h^i{}_0 - \partial^i h_{00}\right) \tag{2756}$$

$$\boxed{\Gamma^i_{00} = \partial_0 h^i{}_0 - \tfrac{1}{2}\partial^i h_{00}} \tag{2757}$$

For $\gamma = j$, (2755) becomes

$$\boxed{\Gamma^i_{0j} = \tfrac{1}{2}\left(\partial_j h^i{}_0 + \partial_0 h^i{}_j - \partial^i h_{0j}\right)} \tag{2758}$$

Returning to (2754) and setting $\beta = j$ gives

$$\Gamma^i_{j\gamma} \quad = \quad \tfrac{1}{2}\left(\partial_\gamma h^i{}_j + \partial_j h^i{}_\gamma - \partial^i h_{j\gamma}\right) \tag{2759}$$

For $\gamma = 0$, (2759) becomes

$$\Gamma^i_{j0} \quad = \quad \tfrac{1}{2}\left(\partial_0 h^i{}_j + \partial_j h^i{}_0 - \partial^i h_{0j}\right) \tag{2760}$$

This matches (2758) as expected again since Christoffel symbols are symmetric in the lower two indices. For $\gamma = k$, (2759) becomes

$$\boxed{\Gamma^i_{jk} = \tfrac{1}{2}\left(\partial_k h^i{}_j + \partial_j h^i{}_k - \partial^i h_{jk}\right)} \tag{2761}$$

In summary, the following is the set of all non-vanishing Christoffel symbols.[276]

$$\boxed{\begin{gathered} \Gamma^0_{00} = -\tfrac{1}{2c}\dot{h}_{00} \\ \Gamma^0_{0i} = \Gamma^0_{i0} = -\tfrac{1}{2}\partial_i h_{00} \\ \Gamma^0_{ij} = -\tfrac{1}{2}\left(\partial_j h_{0i} + \partial_i h_{j0} - \tfrac{1}{c}\dot{h}_{ij}\right) \\ \Gamma^i_{00} = \tfrac{1}{c}\dot{h}_{0i} - \tfrac{1}{2}\partial_i h_{00} \\ \Gamma^i_{0j} = \Gamma^i_{j0} = \tfrac{1}{2}\left(\partial_j h_{0i} + \tfrac{1}{c}\dot{h}_{ij} - \partial_i h_{0j}\right) \\ \Gamma^i_{jk} = \tfrac{1}{2}\left(\partial_k h_{ij} + \partial_j h_{ik} - \partial_i h_{jk}\right) \end{gathered}} \tag{2762}$$

<u>The geodesic equation of motion to first order in $h_{\mu\nu}$</u>

We now make use of the Christoffel symbols found above to expand the geodesic equation in (2744).

$$\begin{aligned} a^i \quad = \quad & -c\dot{h}_{0i} + \frac{c^2}{2}\partial_i h_{00} - \left(c\partial_j h_{0i} - c\partial_i h_{0j} + \dot{h}_{ij}\right)v^j - \tfrac{1}{2}\left(\partial_k h_{ij} + \partial_j h_{ik} - \partial_i h_{jk}\right)v^j v^k \\ & - \left[\tfrac{1}{2}\dot{h}_{00} + v^j \partial_j h_{00} + \tfrac{1}{2c}\left(\partial_k h_{0j} + \partial_j h_{k0} - \tfrac{1}{c}\dot{h}_{jk}\right)v^j v^k\right]v^i \end{aligned} \tag{2763}$$

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[276]We express $\partial_0 h_{\mu\nu}$ as $\frac{1}{c}\dot{h}_{\mu\nu}$. We also apply the metric to terms such as $h^i{}_0$ to write them with all lower indces. Note that the metric applied to $h_{\mu\nu}$ will simply be $g_{\mu\nu} \approx \eta_{\mu\nu}$ to maintain linear order in $h_{\mu\nu}$. Therefore spatial indices can be freely raised and lowered while temporal indices bring in a negative sign.

In the last term of the first line, we have $\partial_k h_{ij}$ and $\partial_j h_{ik}$ both being summed identically over $v^j v^k$. Therefore we can combine them. Similarly, $\partial_k h_{0j}$ and $\partial_j h_{k0}$ in the second line can be combined. Then after some rearranging, we have

$$\boxed{\begin{aligned} a^i = \frac{c^2}{2}\partial_i h_{00} - c\dot{h}_{0i} - c\left(\partial_i h_{0j} - \partial_j h_{0i}\right)v^j - \frac{1}{2}\dot{h}_{00}v^i - \dot{h}_{ij}v^j \\ -\left(\partial_j h_{00}\right)v^j v^i + \left(\frac{1}{2}\partial_i h_{jk} - \partial_k h_{ij}\right)v^j v^k + \left(\frac{1}{c}\partial_i h_{j0} - \frac{1}{2c^2}\dot{h}_{ij}\right)v^i v^j v^k \\ \textit{Geodesic equation of motion to first order in the metric perturbation} \end{aligned}} \tag{2764}$$

This expression gives the acceleration of test particles to first order in the metric perturbation and with no approximations applied to the particle velocity. The first three terms on the right can be viewed as a gravitational "Lorentz force." analogous to $\frac{m}{q}a^i = -\partial_i\varphi - A^i + v^j\left(\partial_i A_j - \partial_j A_i\right)$ in electromagnetism. The next two terms on the right side of (2764) contain temporal derivatives of the metric perturbation: $\dot{h}_{00}v^i$ and $\dot{h}_{ij}v^j$. These do not have analogous quantities in the electromagnetic Lorentz force. The remaining terms in the second line involve the velocity to second and third order, which also does not occur in the electromagnetic Lorentz force.

The terms containing temporal and spatial derivatives of h_{ij} describe how a gravitational *strain* field couple to matter. This includes how gravitational waves affect test masses. In the far-field zone, we have $h_{00} = h_{0i} \approx 0$ and $h_{ij} = h_{ij}^{\tau\tau}$. This gives

$$a^i = -v^j\dot{h}_{ij}^{\tau\tau} + \left(\frac{1}{2}\partial_i h_{jk}^{\tau\tau} - \partial_j h_{ik}^{\tau\tau}\right)v^j v^k - \frac{1}{2c^2}\dot{h}_{jk}^{\tau\tau}v^j v^k v^i \tag{2765}$$

This the geodesic equation of motion which describes the motion of test particles in the presence of a gravitational wave (to first order in the metric perturbation).

The equation of motion for one-dimensional motion

In general, the acceleration on the left side of (2764) cannot be isolated to obtain an expression for a^i appearing alone on one side. The reason is because a^i is summed over other quantities and therefore its components are distributed over multiple terms. They cannot be factored out to form a single vector for a^i multiplying the rest of the quantities in the parentheses. This fact is a consequence of Special Relativity, not General Relativity. It arises due to the fact that finding the acceleration in (2727) involves a term with $\dot{\gamma}$ where γ involves v^2. Taking the time derivative of v^2 will lead to a term with $v_i a_i$ which involves a_i being summed with v_i. Therefore, the only way to avoid a_i appearing in multiple terms is to go to lower order than $(v/c)^2$ (which simply recovers the Newtonian, non-relativistic acceleration) or to consider the case of one-dimensional motion. In that case a_i can be completely factored out. For example, if there is only motion in the x_1 direction, then we have

$$\begin{aligned} a_1 + v_1 a_j\left(v_j + h_{00}v_j + ch_{0j} + v_k h_{jk}\right)/c^2 \quad = \quad & \frac{c^2}{2}\partial_1 h_{00}\left(1 - 2\frac{v^2}{c^2}\right) - c\dot{h}_{01}\left(1 - 2\frac{v^2}{c^2}\right) \\ & + cv^j\left(\partial_1 h_{0j} - \partial_j h_{01}\right) - \frac{1}{2}v_1\dot{h}_{00} - \dot{h}_{0j}v^j v_1/c \\ & - v^j\dot{h}_{1j} + v^j v^k\left(\tfrac{1}{2}\partial_1 h_{jk} - \partial_k h_{1j}\right) \end{aligned} \tag{2766}$$

Since we only have $j = k = 1$, then we can sum on the left side and factor out a_1. Also, simplifying the right side gives

$$\begin{aligned} a_1\left[1+\frac{v^2}{c^2}+\frac{v^2}{c^2}(h_{00}+h_{11})+\frac{v_1}{c}h_{01}\right] \\ = \frac{c^2}{2}\partial_1 h_{00}\left(1-2\frac{v^2}{c^2}\right) - c\dot{h}_{01}\left(1-2\frac{v^2}{c^2}\right) \\ -\frac{v_1}{2}\dot{h}_{00} - \frac{v^2}{c}\dot{h}_{01} - v_1\dot{h}_{11} + \frac{v^2}{2}\partial_1 h_{11} \end{aligned} \tag{2767}$$

Dividing by the bracket on the left side means we will have $\left[1+\frac{v^2}{c^2}+\frac{v^2}{c^2}(h_{00}+h_{11})+\frac{v_1}{c}h_{01}\right]^{-1}$ multiplying the right side. We can use a weak-field approximation in the form

$$\left[1+\frac{v^2}{c^2}+\frac{v^2}{c^2}(h_{00}+h_{11})+\frac{v_1}{c}h_{01}\right]^{-1} \approx 1+\frac{v^2}{c^2}-\frac{v^2}{c^2}(h_{00}+h_{11})-\frac{v_1}{c}h_{01} \tag{2768}$$

Then (2767) becomes

$$\begin{aligned} a_1 &= \left[\frac{c^2}{2}\partial_1 h_{00}\left(1-2\frac{v^2}{c^2}\right) - c\dot{h}_{01}\left(1-3\frac{v^2}{c^2}\right) - \frac{v_1}{2}\dot{h}_{00} - v_1\dot{h}_{11} + \frac{v^2}{2}\partial_1 h_{11}\right] \\ &\quad \cdot\left[1+\frac{v^2}{c^2}-\frac{v^2}{c^2}(h_{00}+h_{11})-\frac{v_1}{c}h_{01}\right] \end{aligned} \tag{2769}$$

We can immediately eliminate the last two terms in the last line since they will yield results that are second order in $h_{\mu\nu}$. Distributing each of the other two terms in the last bracket and eliminating any terms that will yield results that are higher order than v^2/c^2 gives

$$\begin{aligned} a_1 &= \left[\frac{c^2}{2}\partial_1 h_{00}\left(1-2\frac{v^2}{c^2}\right) - c\dot{h}_{01}\left(1-3\frac{v^2}{c^2}\right) - \frac{v_1}{2}\dot{h}_{00} - v_1\dot{h}_{11} + \frac{v^2}{2}\partial_1 h_{11}\right] \\ &\quad +\frac{v^2}{c^2}\left(\frac{c^2}{2}\partial_1 h_{00} - c\dot{h}_{01}\right) \end{aligned} \tag{2770}$$

Finally, combining common terms gives

$$\boxed{a_1 = \frac{c^2}{2}\partial_1 h_{00}\left(1-\frac{v^2}{c^2}\right) - \dot{h}_{00}\frac{v_1}{2} - c\dot{h}_{01}\left(1-2\frac{v^2}{c^2}\right) - \dot{h}_{11}v_1 + \frac{v^2}{2}\partial_1 h_{11}} \tag{2771}$$

Geodesic equation of motion in the weak-field limit with v^2/c^2 test masses for motion in 1-D

This expression gives the acceleration of test particles in the weak-field limit for test mass velocities to order v^2/c^2 and motion in only the x_1 direction.

Analysis of the approximation to order v^2/c^2

As discussed at the end of Section 26, it is important to recognize that this equation of motion technically goes beyond the first-order approximation of linearized GR. This is due to the fact that to first order in the metric, the Einstein equation predicts no coupling between the Christoffel symbols and the stress tensor. Therefore, the equation of motion in (2764) is valid as long as it is understood to only describe the lowest order response of matter to an external gravitational field. It *cannot* be used to predict the *net* field that is a result of *both* the external gravitational field *and* the gravitational field generated by the moving matter.

It is also important to consider the approximations that were used to obtain (2764) which required that we restrict the accelerations permitted for the test masses. Specifically, we eliminated terms of order $a\,(v/c)^n$ with n being integers from 1 to 4. For $n=1$, the lowest order acceleration term neglected is $a\,(v/c)$. If we consider a velocity given by $v = v_0 e^{i\omega t}$, then the acceleration is $a \sim v\omega_{particle}$ where $\omega_{particle}$ characterizes how rapidly the velocity of the particle varies. Then the lowest order acceleration term neglected is $a\,(v/c) \sim v^2\omega_{particle}/c$.

We can consider the case of a gravitational wave given by $h = Ae^{i\omega t}$. From (2765) we see that the equation of motion for test particles involves the time derivative of the wave, $\dot{h} \sim A\omega_{wave}$. The equation of motion also involves the spatial derivative of the wave which can be approximated using the wave vector, $k = \omega_{wave}/c$. Then $\nabla h \sim Ak \sim \omega_{wave}/c$.

With these quantities, we may now consider the conditions associated with our approximations. Specifically, we must insure that the lowest order terms neglected are much less than the terms retained in (2765). Looking back at (2764), we see that if v^i is distributed through the last line, then the last term in the expression is, $\sim hv^2a/c^2$, which involves the coupling of the acceleration to gravitational waves. Since this term was neglected, then we must insure that it is much smaller than the terms in (2765). Using the first term in (2765) requires

$$hv^2a/c^2 << \dot{h}v \tag{2772}$$

On left we can use $a \sim v\omega_{particle}$ and $h \sim A$ while on the right we can use $\dot{h} \sim A\omega_{wave}$. We can also assume that the frequency of the motion of the test particle is the same as the wave. Then we have

$$Av^3\omega_{particle}/c^2 << A\omega_{wave}v \qquad \Longrightarrow \qquad v^2/c^2 << 1 \tag{2773}$$

This result is consistent with retaining velocities to order $(v/c)^2$ in our approximation. We can also compare the same term from (2764) to any of the other terms in (2765). Then we have

$$hv^2a/c^2 << \nabla hv^2 \tag{2774}$$

Again we use $a \sim v\omega_{particle}$ and $h \sim A$ as well as $\nabla h \sim Ak \sim A\omega_{wave}/c$. We also assume that the frequency of particle motion is the same as the wave. Then we have

$$Av^3\omega_{particle}/c^2 << A\omega_{wave}v^2/c \qquad \Longrightarrow \qquad v/c << 1 \tag{2775}$$

Once again, this result is consistent with our approximation. It is a stricter constraint than we obtained in (2773), however, it does not violate our approximation.

The insight we may draw from (2773) and (2775) is that the first term in (2765) which involves $\dot{h}_{ij}v_j$, will have a greater effect on the test particle (by a factor of v/c) when compared with the effect of the other terms involving ∇hv^2. This is naturally expected since the time variation is much more substantial than the spatial variation for a gravitational wave in the microwave range.

However, for a particle with a velocity approaching c, we find that all the terms in (2765) will contribute equally. However, in that case, we no longer have the right to neglect higher order velocity terms. In fact, as we demonstrated above in (2773) and (2775), neglecting the lowest order terms involving the acceleration in (2764) requires that the speed is limited to $v/c << 1$ or at least $v^2/c^2 << 1$. If we abandon this low-velocity limit, then we cannot justifiably eliminate other terms from (2764). In that case, they must all appear. However, if we still work with gravitational wave in the far-field, then (2764) becomes

$$a^i = -\dot{h}_{ij}v^j - \tfrac{1}{2}v^j v^k \left(\partial_k h_{ij} + \partial_j h_{ki} - \partial_i h_{jk}\right) - \frac{1}{2}v^i\left(1 - \frac{v^2}{c^2}\right)\left(\frac{1}{c^2}\dot{h}_{jk}v^j v^k + \frac{2}{c^2}v_j a^j + \frac{2}{c^2}h_{jk}v_j a_k\right) \tag{2776}$$

This is the appropriate equation of motion for particles moving with $v \sim c$ in the presence of gravitational waves (in the TT gauge). The only approximation is the weak-field approximation, but there are no approximations concerning the velocities or accelerations.

Notice that for velocities approaching c, the entire last half of the equation is suppressed due to the factor of $\left(1 - v^2/c^2\right)$. If we take the extreme case of $v \approx c$ and use $\dot{h} \sim A\omega_{wave}$ as well as $\nabla h \sim A\omega_{wave}/c$, then from (2776) we obtain $a \sim Ac\omega$. If we consider the case of a gravitational wave with an amplitude $A \sim 10^{-20}$ and a frequency in the microwave range, $\omega_{wave} \sim 10^{10}$, then this gives an acceleration of $a \sim 10^{-2}$ m/s^2. Therefore, we find that for the case of gravitational waves, we are completely justified in neglecting higher order acceleration terms since the lowest order acceleration is extremely small even for the case of high frequency waves and an extremely high particle velocity. The reason is obviously due to how small the field is $\left(A \sim 10^{-20}\right)$.

Appendix K

The equation of motion of a test mass orbiting a Schwarzschild metric

The Schwarzschild metric is

$$ds^2 = (1 - R_S/r)^{-1} dr^2 + r^2 d\Omega^2 - (1 - R_S/r) c^2 dt^2 \tag{2777}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and the Schwarzschild radius is $R_s = 2GM/c^2$. The non-zero Christoffel symbols for the Schwarzschild metric are the following:

$$\begin{aligned}
\Gamma^r_{rr} &= \frac{R_S}{2r(R_S - r)}, & \Gamma^\theta_{r\theta} &= \Gamma^\theta_{\theta r} = 1/r, \\
\Gamma^r_{\phi\phi} &= (R_S - r)\sin^2\theta, & \Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta, \\
\Gamma^r_{\theta\theta} &= R_S - r, & \Gamma^\phi_{r\phi} &= \Gamma^\phi_{\phi r} = 1/r, \\
\Gamma^r_{tt} &= \frac{R_S(r - R_S)}{2r^3} & \Gamma^\phi_{\phi\theta} &= \Gamma^\phi_{\theta\phi} = \cot\theta \\
& & \Gamma^t_{rt} = \Gamma^t_{tr} &= \frac{R_S}{2(r^2 - R_S r)}
\end{aligned} \tag{2778}$$

Now we use the geodesic equation of motion[277]

$$\ddot{x}^\mu + \Gamma^\mu_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma = 0 \tag{2779}$$

For $x^\mu = r$, the geodesic equation gives

$$\ddot{r} + \Gamma^r_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma = 0 \tag{2780}$$

Using the Christoffel symbols of the form $\Gamma^r_{\rho\sigma}$ in (2778) gives

$$\ddot{r} + \Gamma^r_{rr}\dot{r}\dot{r} + \Gamma^r_{\phi\phi}\dot{\phi}\dot{\phi} + \Gamma^r_{\theta\theta}\dot{\theta}\dot{\theta} + \Gamma^r_{tt}(c\dot{t})(c\dot{t}) = 0 \tag{2781}$$

$$\ddot{r} + \frac{R_S}{2r(R_S - r)}\dot{r}^2 + (R_S - r)\sin^2\theta\dot{\phi}^2 + (R_S - r)\dot{\theta}^2 + \frac{R_S(r - R_S)}{2r^3}c^2\dot{t}^2 = 0 \tag{2782}$$

To determine the velocity of the test mass, $\vec{u} = d\vec{r}/d\tau$ we must take the derivative of the displacement vector $\vec{r}$ with respect to proper time. By the chain rule we have

$$\vec{u} = \frac{d\vec{r}}{d\tau} = \frac{d\vec{r}}{dt}\frac{dt}{d\tau} = \frac{d\vec{r}}{dt}\dot{t} \tag{2783}$$

Since $\vec{r} = r\hat{r}$, then by the product rule $\frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r}) = \hat{r}\frac{dr}{dt} + r\frac{d\hat{r}}{dt}$ so (2783) becomes

$$\vec{u} = \left(\hat{r}\frac{dr}{dt} + r\frac{d\hat{r}}{dt}\right)\dot{t} \tag{2784}$$

[277] Note that $\dot{x}^\mu$ is a derivative of x^μ with respect to *proper* time τ, not coordinate time t.

Since we have reserved the dot-derivative for a derivative with respect to *proper* time, then we will use the *prime* to represent a derivative with respect to *coordinate* time t. So (2784) becomes

$$\vec{u} = \dot{t}\left(\hat{r}r' + r\hat{r}'\right) \tag{2785}$$

The time derivative of the unit vector $\hat{r}$ can be found from any classical mechanics reference to be

$$\hat{r}' = \theta'\hat{\theta} + \phi'\sin\theta\hat{\phi} \tag{2786}$$

Substituting this into (2785) gives

$$\vec{u} = \dot{t}\left(\hat{r}\frac{dr}{dt} + r\frac{d\theta}{dt}\hat{\theta} + r\frac{d\phi}{dt}\sin\theta\hat{\phi}\right) \tag{2787}$$

So (2787) becomes

$$\vec{u} \quad = \quad \dot{t}\left(r'\hat{r} + r\theta'\hat{\theta} + r\phi'\sin\theta\hat{\phi}\right) \tag{2788}$$

Notice that this is just $\vec{u} = \dot{t}\vec{v}$ where $\vec{v}$ is $d\vec{r}/dt$. In other words, by the chain rule we can recognize that taking the derivative of $\vec{r}$ with respect to τ is the same as taking the derivative of $\vec{r}$ with respect to t and then multiplying by $\dot{t}$. Therefore, if we want the *proper* acceleration, $d^2\vec{r}/d\tau^2$ we can simply multiply the coordinate acceleration $\vec{a}$ by $\dot{t}^2$. We can use the acceleration in spherical coordinates from any classical mechanics reference to write the proper acceleration as

$$\begin{aligned}\frac{d^2\vec{r}}{d\tau^2} \quad &= \quad \dot{t}^2\vec{a} = \dot{t}^2\left(r'' - r\theta'^2 - r\sin^2\theta\phi'^2\right)\hat{r} + \dot{t}^2\left(r\theta'' + 2r'\theta' - r\sin\theta\cos\theta\phi'^2\right)\hat{\theta} \\ &\quad + \dot{t}^2\left(r\sin\theta\phi'' + 2r\cos\theta\theta'\phi' + 2r'\phi'\sin\theta\right)\hat{\phi}\end{aligned} \tag{2789}$$

Since the gravitational force of a central mass will point only in the radial direction, then we can use only the radial component of (2789). Also, working in the equatorial plane where $\theta = \pi/2$ and $\dot{\theta} = 0$, then we have

$$\frac{d^2\vec{r}}{d\tau^2} \quad = \quad \dot{t}^2\left(r'' - r\phi'^2\right)\hat{r} \tag{2790}$$

$$= \quad \left(\ddot{r} - r\dot{\phi}^2\right)\hat{r} \tag{2791}$$

Now we look at the geodesic equation of motion in (2782). If we confine the motion to the equatorial plane so that $\theta = \pi/2$ and $\dot{\theta} = 0$ then we have

$$\ddot{r} + \frac{R_S}{2r\left(R_S - r\right)}\dot{r}^2 + \left(R_S - r\right)\dot{\phi}^2 + \frac{R_S\left(r - R_S\right)}{2r^3}c^2\dot{t}^2 \quad = \quad 0 \tag{2792}$$

Since $d^2\mathbf{r}/d\tau^2 = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{r}$ from (2791), then we can rearrange the terms in (2792) to obtain $\ddot{r} - r\dot{\phi}^2$ on one side.

$$\ddot{r} - r\dot{\phi}^2 \quad = \quad -\frac{R_S}{2r\left(R_S - r\right)}\dot{r}^2 - R_S\dot{\phi}^2 - \frac{R_S\left(r - R_S\right)}{2r^3}c^2\dot{t}^2 \tag{2793}$$

We can recall that the components of the four-momentum were determined from the Schwarzschild metric as follows.

$$p_0 = g_{00}p^0 = -\left(1 - R_S/r\right)m\dot{x}^0 = -\frac{E}{c} \tag{2794}$$

where $\dot{x}^0 = c\dot{t}$. So from this we can see that

$$\dot{t} \quad = \quad \frac{E}{(1-R_S/r)\,mc^2} \tag{2795}$$

Likewise, we also found that

$$p_3 \quad = \quad g_{33}p^3 = mr^2 \sin\theta\dot{\phi} = L \tag{2796}$$

Since $\theta = \pi/2$ in the equatorial plane, then solving for $\dot{\phi}$ gives

$$\dot{\phi} = \frac{L}{mr^2} = \frac{l}{r^2} \tag{2797}$$

where $l = L/m$. We also found the equation of motion to be

$$\dot{r}^2 + c^2\left(1-\frac{R_S}{r}\right)\left(1+\frac{(l/c)^2}{r^2}\right) \quad = \quad \left(\frac{E}{mc}\right)^2 \tag{2798}$$

$$\dot{r}^2 \quad = \quad \left(\frac{E}{mc}\right)^2 - c^2\left(1-\frac{R_S}{r}\right)\left(1+\frac{(l/c)^2}{r^2}\right) \tag{2799}$$

Substituting (2795), (2797), and (2799) into (2793) gives

$$\begin{aligned}\ddot{r} - r\dot{\phi}^2 \quad = \quad & -\frac{R_S}{2r\,(R_S-r)}\left[\left(\frac{E}{mc}\right)^2 - c^2\left(1-\frac{R_S}{r}\right)\left(1+\frac{(l/c)^2}{r^2}\right)\right] \\ & -\frac{R_S l^2}{r^4} - \frac{R_S\,(r-R_S)}{2r^3}c^2\left(\frac{E}{(1-R_S/r)\,mc^2}\right)^2\end{aligned} \tag{2800}$$

Now we work to algebraically simplify the expression.

$$\begin{aligned}\ddot{r} - r\dot{\phi}^2 \quad = \quad & -\frac{R_S}{2r\,(R_S-r)}\left[\left(\frac{E}{mc}\right)^2 - c^2\left(1+\frac{(l/c)^2}{r^2} - \frac{R_S}{r} - \frac{R_S\,(l/c)^2}{r^3}\right)\right] \\ & -\frac{R_S l^2}{r^4} - \frac{R_S\,(r-R_S)\,c^2}{2r^3}\left(\frac{Er}{(r-R_S)\,mc^2}\right)^2\end{aligned} \tag{2801}$$

Distributing gives

$$\begin{aligned}\ddot{r} - r\dot{\phi}^2 \quad = \quad & -\frac{R_S E^2}{2r\,(R_S-r)\,m^2c^2} + \frac{R_S c^2}{2r\,(R_S-r)}\left(1+\frac{(l/c)^2}{r^2} - \frac{R_S}{r} - \frac{R_S\,(l/c)^2}{r^3}\right) \\ & -\frac{R_S l^2}{r^4} - \frac{R_S E^2}{2r\,(r-R_S)\,m^2c^2}\end{aligned} \tag{2802}$$

We can cancel the first and last terms and also distribute in the second term.

$$\ddot{r} - r\dot{\phi}^2 = \frac{R_S c^2}{2r(R_S - r)}\left(1 + \frac{(l/c)^2}{r^2} - \frac{R_S}{r} - \frac{R_S(l/c)^2}{r^3}\right) - \frac{R_S l^2}{r^4} \tag{2803}$$

$$= \frac{R_S c^2}{2r(R_S - r)} + \frac{R_S l^2}{2r^3(R_S - r)} - \frac{R_S^2 c^2}{2r^2(R_S - r)} - \frac{R_S^2 l^2}{2r^4(R_S - r)} - \frac{R_S l^2}{r^4} \tag{2804}$$

Factoring out a common $R_S/2$ and getting a common denominator gives

$$\ddot{r} - r\dot{\phi}^2 = \frac{R_S}{2}\left(\frac{c^2r^3 + l^2r - R_S c^2 r^2 - R_S l^2 - 2l^2(R_S - r)}{r^4(R_S - r)}\right) \tag{2805}$$

Combining similar terms gives

$$\ddot{r} - r\dot{\phi}^2 = \frac{R_S}{2}\left(\frac{c^2r^3 + 3l^2r - R_S c^2 r^2 - 3R_S l^2}{r^4(R_S - r)}\right) \tag{2806}$$

Rearranging terms and factoring out $(R_S - r)$ gives

$$\ddot{r} - r\dot{\phi}^2 = \frac{R_S}{2}\left(\frac{c^2r^2(r - R_S) + 3l^2(r - R_S)}{r^4(R_S - r)}\right) \tag{2807}$$

$$= -\frac{R_S}{2}\left(\frac{c^2r^2 + 3l^2}{r^4}\right) \tag{2808}$$

Substituting $R_S = 2GM/c^2$ gives

$$\ddot{r} - r\dot{\phi}^2 = -\frac{GM}{c^2}\left(\frac{c^2}{r^2} + \frac{3l^2}{r^4}\right) \tag{2809}$$

$$= -\frac{GM}{r^2}\left(1 + \frac{3(l/c)^2}{r^4}\right) \tag{2810}$$

From (2791) we know that the left side is just $\vec{a} = d^2\vec{r}/d\tau^2$ so we have

$$\vec{a} = -\frac{GM}{r^2}\left[1 + 3\frac{(l/c)^2}{r^2}\right]\hat{r} \tag{2811}$$

Here we have effectively recovered the Newtonian gravitational force with an additional relativistic correction.

Appendix L

The geodesic deviation equation

Here we follow an approach similar to [105] to derive the geodesic deviation equation. We can consider a particle with worldline $x^{\mu}(\tau)$ following a geodesic

$$Du^{\mu} = \frac{du^{\mu}}{d\tau} + \Gamma^{\mu}_{\sigma\rho}(x)\, u^{\sigma}u^{\rho} = 0 \tag{2812}$$

and a second particle with worldline $\tilde{x}^{\mu}(t)$ following a neighboring geodesic

$$D\tilde{u}^{\mu} = \frac{d\tilde{u}^{\mu}}{d\tau} + \Gamma^{\mu}_{\sigma\rho}(\tilde{x})\, \tilde{u}^{\sigma}\tilde{u}^{\rho} = 0 \tag{2813}$$

The two particles are separated by a coordinate distance $L^{\mu} = \tilde{x}^{\mu} - x^{\mu}$. From this relation we can find the proper acceleration[278] of L^{μ} which is the *relative* acceleration between x^{μ} and $\tilde{x}^{\mu}$.

$$\frac{d^2}{d\tau^2}L^{\mu} = \frac{d}{d\tau}\tilde{u}^{\mu} - \frac{d}{d\tau}u^{\mu} \tag{2814}$$

$$L''^{\mu} = \tilde{u}'^{\mu} - u'^{\mu} \tag{2815}$$

$$= -\Gamma^{\mu}_{\sigma\rho}(\tilde{x})\, \tilde{u}^{\sigma}\tilde{u}^{\rho} - \Gamma^{\mu}_{\sigma\rho}(x)\, u^{\sigma}u^{\rho} \tag{2816}$$

$$= \Gamma^{\mu}_{\sigma\rho}(\tilde{x})\left(u^{\sigma} + L'^{\sigma}\right)\left(u^{\rho} + L'^{\rho}\right) - \Gamma^{\mu}_{\sigma\rho}(x)\, u^{\sigma}u^{\rho} \tag{2817}$$

Multiplying out terms and staying to first order in L'^{σ} gives

$$L''^{\mu} = -\Gamma^{\mu}_{\sigma\rho}(\tilde{x})\left(u^{\sigma}u^{\rho} + L'^{\sigma}u^{\rho} + u^{\sigma}L'^{\rho}\right) + \Gamma^{\mu}_{\sigma\rho}(x)\, u^{\sigma}u^{\rho} \tag{2818}$$

Expanding $\Gamma^{\mu}_{\sigma\rho}(\tilde{x})$ to first order about x^{μ} gives

$$\Gamma^{\mu}_{\sigma\rho}(\tilde{x}) \approx \Gamma^{\mu}_{\sigma\rho}(x) + \left(\partial_{\gamma}\Gamma^{\mu}_{\sigma\rho}(x)\right)\left(\tilde{x}^{\gamma} - x^{\gamma}\right) \tag{2819}$$

$$\approx \Gamma^{\mu}_{\sigma\rho}(x) + \left(\partial_{\gamma}\Gamma^{\mu}_{\sigma\rho}(x)\right) L^{\gamma} \tag{2820}$$

We can insert this into (2818) and drop the function notation since all quantities are now functions of x.

$$L''^{\mu} = -\left[\Gamma^{\mu}_{\sigma\rho} + \left(\partial_{\gamma}\Gamma^{\mu}_{\sigma\rho}\right) L^{\gamma}\right]\left(u^{\sigma}u^{\rho} + L'^{\sigma}u^{\rho} + u^{\sigma}L'^{\rho}\right) + \Gamma^{\mu}_{\sigma\rho}u^{\sigma}u^{\rho} \tag{2821}$$

Canceling common terms and eliminating the higher order terms containing $L^{\gamma}L'^{\rho}$ gives

$$L''^{\mu} = -\Gamma^{\mu}_{\sigma\rho}L'^{\sigma}u^{\rho} - \Gamma^{\mu}_{\sigma\rho}u^{\sigma}L'^{\rho} - L^{\gamma}\left(\partial_{\gamma}\Gamma^{\mu}_{\sigma\rho}\right)u^{\sigma}u^{\rho} \tag{2822}$$

[278] In this section we use the notation $x'^{\mu} = dx^{\mu}/d\tau$ as the derivative with respect to *proper* time in order distinguish from $\dot{x}^{\mu} = dx^{\mu}/dt$ which is the derivative with respect to *coordinate* time.

Since $\Gamma^{\mu}_{\sigma\rho}$ is symmetric in σ and ρ, then we can combine the first two terms to express the geodesic deviation equation as

$$\boxed{L''^{\mu} = -2\Gamma^{\mu}_{\sigma\rho} L'^{\sigma} u^{\rho} - \left(\partial_{\gamma}\Gamma^{\mu}_{\sigma\rho}\right) u^{\sigma} u^{\rho} L^{\gamma} \qquad \begin{array}{c}\textit{Coordinate dependent}\\ \textit{geodesic deviation equation}\end{array}} \tag{2823}$$

If we wish to express the geodesic deviation equation in terms of the Riemann tensor, we can start with the covariant derivative acting on L^{μ}.

$$DL^{\mu} = L'^{\mu} + \Gamma^{\mu}_{\gamma\rho} L^{\gamma} u^{\rho} \tag{2824}$$

Then applying the covariant derivative to L^{μ} twice gives

$$D^2 L^{\mu} = \frac{d}{d\tau}\left(L'^{\mu} + \Gamma^{\mu}_{\gamma\rho} L^{\gamma} u^{\rho}\right) + \Gamma^{\mu}_{\alpha\beta}\left(L'^{\alpha} + \Gamma^{\alpha}_{\gamma\rho} L^{\gamma} u^{\rho}\right) u^{\beta} \tag{2825}$$

Using the product rule to evaluate the derivative and distributing gives

$$D^2 L^{\mu} = L''^{\mu} + \Gamma'^{\mu}_{\gamma\rho} L^{\gamma} u^{\rho} + \Gamma^{\mu}_{\gamma\rho} L'^{\gamma} u^{\rho} + \Gamma^{\mu}_{\gamma\rho} L^{\gamma} u'^{\rho} + \Gamma^{\mu}_{\alpha\beta} L'^{\alpha} u^{\beta} + \Gamma^{\mu}_{\alpha\beta}\Gamma^{\alpha}_{\gamma\rho} L^{\gamma} u^{\rho} u^{\beta} \tag{2826}$$

Using the chain rule, we can write $\Gamma'^{\mu}_{\gamma\rho}\left(x^{\mu}\right)$ as

$$\Gamma'^{\mu}_{\gamma\rho}\left(x^{\mu}\right) = \frac{d}{d\tau}\Gamma^{\mu}_{\gamma\rho}\left(x^{\mu}\right) = \frac{dx^{\sigma}}{d\tau}\frac{d}{dx^{\sigma}}\Gamma^{\mu}_{\gamma\rho} = u^{\sigma}\left(\partial_{\sigma}\Gamma^{\mu}_{\gamma\rho}\right) \tag{2827}$$

We can substitute this into the second term of (2826). We can also substitute (2823) in the first term and (2812) in the fourth term to obtain

$$\begin{aligned} D^2 L^{\mu} = {} & -2\Gamma^{\mu}_{\rho\sigma} L'^{\sigma} u^{\rho} - \left(\partial_{\gamma}\Gamma^{\mu}_{\sigma\rho}\right) u^{\sigma} u^{\rho} L^{\gamma} + \partial_{\sigma}\Gamma^{\mu}_{\gamma\rho} L^{\gamma} u^{\rho} u^{\sigma} \\ & + \Gamma^{\mu}_{\gamma\rho} L'^{\gamma} u^{\rho} + \Gamma^{\mu}_{\gamma\rho} L^{\gamma}\left(-\Gamma^{\rho}_{\sigma\nu} u^{\sigma} u^{\nu}\right) + \Gamma^{\mu}_{\alpha\beta} L'^{\alpha} u^{\beta} + \Gamma^{\mu}_{\alpha\beta}\Gamma^{\alpha}_{\gamma\rho} L^{\gamma} u^{\rho} u^{\beta} \end{aligned} \tag{2828}$$

The three terms involving $\Gamma^{\mu}_{\sigma\rho} L'^{\sigma} u^{\rho}$ cancel to give

$$D^2 L^{\mu} = -\left(\partial_{\gamma}\Gamma^{\mu}_{\sigma\rho}\right) L^{\gamma} u^{\sigma} u^{\rho} + \partial_{\sigma}\Gamma^{\mu}_{\gamma\rho} L^{\gamma} u^{\rho} u^{\sigma} - \Gamma^{\mu}_{\gamma\rho}\Gamma^{\rho}_{\sigma\nu} L^{\gamma} u^{\sigma} u^{\nu} + \Gamma^{\mu}_{\alpha\beta}\Gamma^{\alpha}_{\gamma\rho} L^{\gamma} u^{\rho} u^{\beta} \tag{2829}$$

We can change repeated indices so they match in each term and then factor out $L^{\gamma} u^{\sigma} u^{\rho}$.

$$D^2 L^{\mu} = -\left(\partial_{\gamma}\Gamma^{\mu}_{\sigma\rho}\right) u^{\sigma} u^{\rho} L^{\gamma} + \partial_{\sigma}\Gamma^{\mu}_{\gamma\rho} L^{\gamma} u^{\rho} u^{\sigma} - \Gamma^{\mu}_{\gamma\alpha}\Gamma^{\alpha}_{\sigma\rho} L^{\gamma} u^{\sigma} u^{\rho} + \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\alpha}_{\gamma\rho} L^{\gamma} u^{\rho} u^{\sigma} \tag{2830}$$

$$= \left(-\partial_{\gamma}\Gamma^{\mu}_{\sigma\rho} + \partial_{\sigma}\Gamma^{\mu}_{\gamma\rho} - \Gamma^{\mu}_{\gamma\alpha}\Gamma^{\alpha}_{\sigma\rho} + \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\alpha}_{\gamma\rho}\right) L^{\gamma} u^{\rho} u^{\sigma} \tag{2831}$$

Lastly, since the Riemann tensor given in (2368) can be written as

$$R^{\mu}{}_{\rho\gamma\sigma} = \partial_{\gamma}\Gamma^{\mu}_{\sigma\rho} - \partial_{\sigma}\Gamma^{\mu}_{\gamma\rho} + \Gamma^{\mu}_{\gamma\alpha}\Gamma^{\alpha}_{\rho\sigma} - \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\alpha}_{\rho\gamma} \tag{2832}$$

Recognizing that the term in parentheses in (2831) is $-R^{\mu}{}_{\rho\gamma\sigma}$ gives

$$\boxed{D^2L^{\mu} = -R^{\mu}{}_{\rho\gamma\sigma}L^{\gamma}u^{\rho}u^{\sigma} \qquad \textit{Coordinate-free geodesic deviation equation}} \tag{2833}$$

We can choose to consider the local Lorentz frame of x^{μ} so that $\Gamma^{\mu}_{\sigma\rho}\left(x^{\mu}\right) = 0$. Then (2826) gives $D^2L^{\mu} = L''^{\mu}$. If we are also in the *proper* (rest) frame of x^{μ}, then $L''^{\mu} = \ddot{L}^{\mu}$. Then (2833) becomes

$$\boxed{\ddot{L}^{\mu} = -R^{\mu}{}_{\rho\gamma\sigma}L^{\gamma}u^{\rho}u^{\sigma} \qquad \begin{array}{c}\textit{Geodesic deviation equation}\\ \textit{in the proper Lorentz frame}\end{array}} \tag{2834}$$

We can also write this result in terms of the metric perturbation. In (2473) of Appendix C, we found the linearized Riemann tensor to be

$$R^{\mu}{}_{\rho\gamma\sigma} \quad = \quad \tfrac{1}{2}\eta^{\mu\alpha}\left(\partial_{\gamma}\partial_{\rho}h_{\sigma\alpha} - \partial_{\gamma}\partial_{\alpha}h_{\rho\sigma} - \partial_{\sigma}\partial_{\rho}h_{\gamma\alpha} + \partial_{\sigma}\partial_{\alpha}h_{\rho\gamma}\right) \tag{2835}$$

In that case, the geodesic deviation equation in (2834) becomes

$$\boxed{\ddot{L}^{\mu} = -R_{\mu\rho\gamma\sigma}L^{\gamma}u^{\rho}u^{\sigma} \qquad \begin{array}{c}\textit{Linearized geodesic deviation equation}\\ \textit{in the proper Lorentz frame}\end{array}} \tag{2836}$$

Appendix M

Lagrangians for relativistic charged particles in curved space-time

The Lagrangian in flat space-time

First we consider the action for particle on a given path written in terms of the Lagrangian L as

$$S = \int_{t_1}^{t_2} L dt \tag{2837}$$

where we are choosing to parameterize the path by t. Minimizing the action $(\delta S = 0)$ leads to the Euler-Lagrange equations of motion. For a relativistic particle, the action should only involve relativistic invariants so we can use $dt = \gamma d\tau$ to express the *coordinate* time, dt, in terms of the invariant *proper* time, $d\tau$.

$$S = \int_{\tau_1}^{\tau_2} \gamma L d\tau \tag{2838}$$

We also know that in the rest frame of the particle, the Lagrangian is just the rest energy of the particle, $E = mc^2$. Therefore, it is also possible to minimize the action with

$$S = \int_{\tau_1}^{\tau_2} mc^2 d\tau \tag{2839}$$

Equating the action expressions in (2838) and (2839) gives $\gamma L = mc^2$. In flat space-time, we have $\gamma^{-1} = \sqrt{1-(v/c)^2}$, where v is the speed of the particle[279] in this case, then we have

$$L = mc^2\sqrt{1-(v/c)^2} \tag{2840}$$

As an alternative method, we could also obtain the same results using the invariance of the relativistic length interval:

$$ds^2 = dx^\mu dx_\mu = \eta_{\mu\nu} dx^\mu dx^\nu \tag{2841}$$

Using $x^\mu = \left(ct, x^i\right)$ gives

$$ds^2 = -c^2dt^2 + \left(dx^i\right)^2 = \left[-1 + \frac{\left(dx^i\right)^2}{c^2dt^2}\right] c^2dt^2 = \left[-1 + (v/c)^2\right] c^2dt^2 \tag{2842}$$

In that case, we would minimize the action given by the path length:

$$S = mc\int \sqrt{ds^2} = mc\int_{t_1}^{t_2} \sqrt{\left[-1+(v/c)^2\right] c^2dt^2} = mc^2\int_{t_1}^{t_2} \sqrt{(v/c)^2 - 1} dt \tag{2843}$$

Hence we find the Lagrangian is

$$L = mc^2\sqrt{(v/c)^2 - 1} \tag{2844}$$

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[279] Formally, the speed v found in $\gamma^{-1} = \sqrt{1-(v/c)^2}$ is the relative speed between two moving inertial frames. However, since we are choosing one frame to be the rest frame of the particle (with time τ), and the other frame to be moving relative to the rest frame with velocity $\vec{v}$ (and with time t) then the velocity of the particle would be observed in the frame with time t to be moving with velocity $-\vec{v}$. When the expression for the Lagrangian only contains v^2 then the negative is irrelevant. In fact, multiplying the lagrangian with an overall negative sign does not change the Euler-Lagrange equation of motion, $\frac{d}{d\tau}\frac{\partial L}{\partial \vec{v}} = \frac{\partial L}{\partial \vec{x}}$. Therefore, this lagrangian is often found written with a negative sign.

The Lagrangian in (2844) is the same as (2840) except for a minus sign inside the root which does not affect the equation of motion.

The Lagrangian in *curved* space-time

To generalize the Lagrangian in (2840) to *curved* space-time, we can first use the Lagrangian in (2840) to write the action in flat space-time as

$$S=\int_{\tau_1}^{\tau_2} mc^2\sqrt{1-(v/c)^2}d\tau=\int_{\tau_1}^{\tau_2} mc\sqrt{c^2-v^2}d\tau=\int_{\tau_1}^{\tau_2} mc\sqrt{\eta_{\mu\nu}u^\mu u^\nu}d\tau \tag{2845}$$

Then in curved space-time, we could just replace $\eta_{\mu\nu}$ with $g_{\mu\nu}$.

$$S=\int_{\tau_1}^{\tau_2} mc\sqrt{g_{\mu\nu}u^\mu u^\nu}d\tau=\int_{\tau_1}^{\tau_2} mc\sqrt{g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau \tag{2846}$$

From here we can immediately identify the Lagrangian as $L=mc\sqrt{g_{\mu\nu}u^\mu u^\nu}$. If we are using a metric with signature $diag(-1,1,1,1)$, then we need to include a negative inside the root to keep the Lagrangian real-valued. So we use $L=mc\sqrt{-g_{\mu\nu}u^\mu u^\nu}$. It is apparent that this Lagrangian is associated with the invariance of the four-velocity:

$$\sqrt{-g_{\mu\nu}u^\mu u^\nu}=c \tag{2847}$$

For an electron-pair that is also coupled to an electromagnetic field, A_μ, then minimal coupling requires that we have[280]

$$\boxed{L=-mc\sqrt{-g_{\mu\nu}u^\mu u^\nu}+eA_\mu u^\mu \qquad \textit{Four-velocity invariant Lagrangian in covariant form}} \tag{2848}$$

Note that we intentionally neglect the electron spin in this Lagrangian since we are ultimately interested in describing electron-*pairs* such as the Cooper pairs of a superconductor. Consequently, a single Cooper pair may be thought of as a composite boson particle with no spin since there must be one spin-up electron and one spin-down electron paired in a single state as required by the Pauli Exclusion Principle. Therefore, the Lagrangians throughout this paper are specifically describing these spinless bosons as particles with a mass that is twice the mass of an electron, $m=2m_e$, and a charge that is twice the charge of an electron, $q=2e$. Formally speaking, the Lagrangian in (2848) would be written $L_1=2m_ec\sqrt{-g_{\mu\nu}u^\mu u^\nu}+2eA_\mu u^\mu$. However, we will drop the subscript on the mass and drop the common factor of 2 (which doesn't affect the equation of motion) and simply write the Lagrangian as shown in (2848).

Alternatively, to generalize the results from (2841) to *curved* space-time, we could replace $\eta_{\mu\nu}$ with $g_{\mu\nu}$ in (2841) so it becomes

$$ds^2=dx^\mu dx_\mu=g_{\mu\nu}dx^\mu dx^\nu=g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}d\tau^2 \tag{2849}$$

Again we can write the action in terms of the path length as[281]

$$S=mc\int\sqrt{-ds^2}=mc\int_{\tau_1}^{\tau_2}\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau \tag{2850}$$

[280]Notice that we have written a negative sign in front of the kinetic term. In the case of the free particle, this was not necessary since it would just be an overall negative to the entire lagrangian. However, in the case of the coupled particle, we must make sure the kinetic term and the coupled term have opposite signs in order to obtain the correct equaton of motion. This Lagrangian is also found in Jackson [40] (eq. 12.31).

[281]We must use a minus sign inside the square root in order to keep it from becoming complex since $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu=-c^2$ for a metric with signature $\text{diag}(-1,1,1,1)$.

Again we can immediately identify the Lagrangian as $L = mc\sqrt{-g_{\mu\nu}u^\mu u^\nu}$ and if the electron-pair is coupled to an electromagnetic field, A_μ, then we can make the kinetic term negative and the Lagrangian would be the same as the one found in (2848).

$$L = -mc\sqrt{-g_{\mu\nu}u^\mu u^\nu} + eA_\mu u^\mu \tag{2851}$$

Note that this is the same Lagrangian used by DeWitt in equation (1) of [42].

Yet another alternative is to begin with a form of the Lagrangian which is similar to the form in non-relativistic mechanics. For a free particle, the Lagrangian is just the kinetic energy given by

$$L = \frac{\vec{p}^2}{2m} \tag{2852}$$

where $\vec{p}$ is the momentum 3-vector. We may anticipate that the generalization to Special Relativity would involve the momentum four-vector instead.[282]

$$L = \frac{p^\mu p_\mu}{2m} \tag{2853}$$

Since $p^\mu = (E/c, \vec{p})$, then $p^\mu p_\mu = -E^2 + p^2c^2$. Also, since $p^\mu = mu^\mu$, then $p^\mu p_\mu = m^2u^\mu u_\mu = -m^2c^2$. Equating these two results leads to $E^2 = m^2c^4 + p^2c^2$ which is the standard formulation of relativistic energy ($E = \gamma mc^2$) in terms of the rest-mass energy (mc^2) and the kinetic energy (pc) of a free particle. To generalize the Lagrangian to *curved* space-time, we could write

$$L = \frac{g_{\mu\nu}p^\mu p^\nu}{2m} \tag{2854}$$

The action associated with this Lagrangian could be written without the factor of 2 as

$$S = \int \frac{g_{\mu\nu}p^\mu p^\nu}{m} d\tau \tag{2855}$$

It is apparent that this Lagrangian is associated with the invariance of the four-momentum:

$$g_{\mu\nu}p^\mu p^\nu = -m^2c^2 \tag{2856}$$

For this reason, we refer to this Lagrangian as the "*four-momentum invariant Lagrangian*." If the electron-pair is also coupled to an electromagnetic field, A_μ, then minimal coupling gives

$$\boxed{L_2 = \frac{1}{2m}g_{\mu\nu}p^\mu p^\nu + eA_\mu u^\mu \qquad \textit{Four-momentum invariant Lagrangian in covariant form}} \tag{2857}$$

Throughout this paper we will consistently use L_1 to refer to the "four-*velocity* invariant Lagrangian" and L_2 to refer to the "four-*momentum* invariant Lagrangian."

Reparameterizing the Lagrangian in terms of coordinate time

Notice that the Lagrangian is parameterized in the action by proper time, τ. We can reparameterize the Lagrangian so that rather than being a function of four coordinates (x^μ) parameterized by proper time τ, instead it can be a function of three coordinates $\left(x^i\right)$ parameterized by coordinate time t. To do this, we use the chain rule to write

$$\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} \tag{2858}$$

[282] The factor of 1/2 does not have any significance since it does not affect the equation of motion.

We can define the "dot derivative" as a derivative with respect to t, the coordinate time.[283] So we can write (2858) as

$$\frac{dx^\mu}{d\tau} = \dot{x}^\mu \frac{dt}{d\tau} \tag{2859}$$

Then the action for the Lagrangian in (2848) can be written as

$$S_1 = \int_{\tau_1}^{\tau_2} \left(mc\sqrt{-g_{\mu\nu}u^\mu u^\nu} + eA_\mu u^\mu \right) d\tau \tag{2860}$$

$$= \int_{\tau_1}^{\tau_2} \left(mc\sqrt{-g_{\mu\nu}\left(\dot{x}^\mu \frac{dt}{d\tau}\right)\left(\dot{x}^\nu \frac{dt}{d\tau}\right)} + eA_\mu \dot{x}^\mu \frac{dt}{d\tau} \right) d\tau \tag{2861}$$

$$= \int_{t_1}^{t_2} \left(mc\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} + eA_\mu \dot{x}^\mu \right) dt \tag{2862}$$

Hence we can write the reparameterized "space + time" Lagrangian as

$$\boxed{\tilde{L}_1 = mc\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} + eA_\mu \dot{x}^\mu \qquad \textit{Four-velocity invariant Lagrangian in "space+time" form}} \tag{2863}$$

The Lagrangian used by DeWitt follows similarly from (2851) as

$$L = -mc\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} + eA_\mu \dot{x}^\mu \tag{2864}$$

Likewise, the four-momentum invariant Lagrangian in (2857) becomes

$$\boxed{\tilde{L}_2 = \dot{x}^\mu p_\mu + eA^\mu \dot{x}_\mu \qquad \textit{Four-momentum invariant Lagrangian in "space+time" form}} \tag{2865}$$

where we have dropped the prefactor of $1/2$ in the kinetic term.[284]

In the process of reparameterizing each Lagrangian, the action went from an integral of proper time $(d\tau)$ to an integral of coordinate time (dt). In a sense, this corresponds to deparameterizing a four-dimensional configuration space $\{x^\mu(\tau)\}$ parameterized in τ, to a three-dimensional configuration space $\left\{x^i\left(x^0\right)\right\}$ parameterized in x^0. It is for this reason that the new Lagrangian must depend on x^i and $v^i = dx^i/dx^0$ rather than x^μ and $u^\mu = dx^\mu/d\tau$. (This is the same convention used by DeWitt [42] when writing the Lagrangian in terms of $\dot{x}^\mu$ with the "dot" being a derivative with respect to x^0.) This formulation is valid as long as the corresponding Euler-Lagrange equation is written in terms of *coordinate* time as

$$\frac{dL}{dt} - \frac{\partial L}{\partial\left(dx^\mu/dx^0\right)} = 0 \tag{2866}$$

rather than in terms of *proper* time as

$$\frac{dL}{d\tau} - \frac{\partial L}{\partial\left(dx^\mu/d\tau\right)} = 0 \tag{2867}$$

[283] DeWitt [42] does the same except that he defines it as a derivative with respect to x^0 with $c = 1$. If we do not set $c = 1$, then it is actually a derivative with respect to $t = x^0/c$.

[284] As shown by the action in (2855), the prefactor of $1/2$ in the kinetic term is not necessary. It was included in (2857) only to demonstrate in Section 43 that the Lagrangian and Hamiltonian for a free particle are found to be the same, and the Hamiltonian for a particle coupled to an electromagnetic field simply becomes $H_2 = \frac{1}{2m}\left(p^\mu - qA^\mu\right)^2$.

Appendix N

The spatial inverse metric

We can define the "spatial inverse metric" as $\tilde{g}^{ik}$ where

$$\tilde{g}^{ik} g_{jk} = \delta^{i}{}_{j} \tag{2868}$$

To find an expression for $\tilde{g}^{ik}$, we can develop relations between the metric and inverse metric components. Since $g^{\mu\nu} g_{\lambda\nu} = \delta^{\mu}_{\lambda}$, then summing over ν gives

$$g^{\mu 0} g_{\lambda 0} + g^{\mu k} g_{\lambda k} = \delta^{\mu}_{\lambda} \tag{2869}$$

We can consider the various combinations of choosing space and time components for μ and λ. Using (2869) and recognizing that $\delta^{i}_{i} = \delta^{0}_{0} = 1$ and $\delta^{i}_{0} = 0$, gives the following.

$$\text{For } \mu = i,\ \lambda = j \implies g^{i0} g_{j0} + g^{ik} g_{jk} = \delta^{i}{}_{j} \tag{2870}$$

$$\text{For } \mu = 0,\ \lambda = j \implies g^{00} g_{j0} + g^{0k} g_{jk} = \delta^{0}{}_{j} \implies g^{0k} g_{jk} = -g^{00} g_{j0} \tag{2871}$$

$$\text{For } \mu = j,\ \lambda = 0 \implies g^{j0} g_{00} + g^{jk} g_{0k} = \delta^{j}{}_{0} \implies g^{jk} g_{0k} = -g^{j0} g_{00} \tag{2872}$$

$$\text{For } \mu = 0,\ \lambda = 0 \implies g^{00} g_{00} + g^{0k} g_{0k} = \delta^{0}{}_{0} \implies g^{0k} g_{0k} = 1 - g^{00} g_{00} \tag{2873}$$

Inserting (2870) into (2868) and dividing by g_{jk} gives

$$\tilde{g}^{ik} = \frac{g^{i0} g_{j0} + g^{ik} g_{jk}}{g_{jk}} = \frac{g^{i0} g_{j0}}{g_{jk}} + g^{ik} \tag{2874}$$

From (2871) we also have $g_{jk} = -\dfrac{g^{00} g_{j0}}{g^{0k}}$. Inserting this in the first term of (2874) gives

$$\boxed{\tilde{g}^{jk} = g^{ik} - \frac{g^{0i} g^{0k}}{g^{00}} \qquad \textit{Spatial inverse metric}} \tag{2875}$$

To first order in the metric, we have $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$. Also using $\eta^{0i} = 0$ and $\eta^{00} = -1$ gives

$$\tilde{g}^{jk} = g^{ik} - \frac{h^{0i} h^{0k}}{-1 - h^{00}} = g^{ik} + \frac{h^{0i} h^{0k}}{1 + h^{00}} \tag{2876}$$

Approximating the denominator to first order gives $\left(1 + h^{00}\right)^{-1} \approx 1 - h^{00}$. This means we have

$$\tilde{g}^{jk} \approx g^{ik} + h^{0i} h^{0k} \left(1 - h^{00}\right) \tag{2877}$$

Therefore, if we remain to first order in the metric, we simply have $\tilde{g}^{jk} \approx g^{ik}$. However, (2875) is valid to *all* orders in the metric. There are two quantities that appear in the Hamiltonian in (677) that we can evaluate here. They are $\tilde{g}^{ik}g_{0k}$ and $\left(\tilde{g}^{jk}g_{0j}g_{0k}-g_{00}\right)$. Writing (2871) with $\mu=0,\ \lambda=k$ and using j for the repeated index gives

$$g^{00}g_{k0}+g^{0j}g_{kj}=\delta^{0}_{\ k} \qquad \Longrightarrow \qquad g_{jk}=-\frac{g^{00}g_{k0}}{g^{0j}} \tag{2878}$$

Inserting this into (2868) gives

$$\tilde{g}^{ik}\left(-\frac{g^{00}g_{k0}}{g^{0j}}\right)=\delta^{i}_{\ j} \tag{2879}$$

$$\boxed{\tilde{g}^{ik}g_{0k}=-\frac{g^{0i}}{g^{00}}} \tag{2880}$$

The other quantity, $\left(\tilde{g}^{jk}g_{0j}g_{0k}-g_{00}\right)$, can be evaluated by inserting (2875) into it. This gives

$$\tilde{g}^{jk}g_{0j}g_{0k}-g_{00} \quad = \quad \left(g^{jk}-\frac{g^{0j}g^{0k}}{g^{00}}\right)g_{0j}g_{0k}-g_{00} \tag{2881}$$

$$= \quad g^{jk}g_{0j}g_{0k}-\frac{g^{0j}g^{0k}}{g^{00}}g_{0j}g_{0k}-g_{00} \tag{2882}$$

Inserting (2872) in the first term and factoring out $g^{0j}g_{0j}$ from two terms gives

$$\tilde{g}^{jk}g_{0j}g_{0k}-g_{00} \quad = \quad \left(-g^{j0}g_{00}\right)g_{0j}-\frac{g^{0j}g^{0k}}{g^{00}}g_{0j}g_{0k}-g_{00} \tag{2883}$$

$$= \quad g^{0j}g_{0j}\left[-g_{00}-\frac{g^{0k}g_{0k}}{g^{00}}\right]-g_{00} \tag{2884}$$

Inserting (2873) twice and simplifying gives

$$\tilde{g}^{jk}g_{0j}g_{0k}-g_{00} \quad = \quad \left(1-g^{00}g_{00}\right)\left[-g_{00}-\frac{\left(1-g^{00}g_{00}\right)}{g^{00}}\right]-g_{00} \tag{2885}$$

$$\boxed{\tilde{g}^{jk}g_{0j}g_{0k}-g_{00}=-\frac{1}{g^{00}}} \tag{2886}$$

Appendix O

The quantum phase and local gauge-invariance of the wavefunction

In the electromagnetic case, we know that a magnetic field, $\vec{B}_G = \nabla \times \vec{A}$, leads to a coupling rule for the canonical momentum given by

$$\vec{p} \implies \vec{p} - q\vec{A} \tag{2887}$$

Since the kinetic momentum is $\vec{\pi} = \vec{p} - q\vec{A}$, then the (non-relativistic) Hamiltonian becomes

$$H = \frac{\vec{\pi}^2}{2m} = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 \tag{2888}$$

Quantizing the Hamiltonian and canonical momentum, and acting on a wavefunction gives the following Schrödinger equation.

$$i\hbar\partial_t \Psi(\vec{r},t) = \left[\frac{1}{2m}\left(i\hbar\nabla - q\vec{A}\right)^2\right]\Psi(\vec{r},t) \tag{2889}$$

If we $\Psi(\vec{r},t)$ is a solution to (2889), then we can consider a different wavefunction with a phase factor, $e^{i\tilde{\phi}}$, where $\tilde{\phi}$ is the phase.[285]

$$\Psi = e^{i\tilde{\phi}}\tilde{\Psi} \tag{2890}$$

For a quantum particle in the vicinity of a mass solenoid such as the one depicted in Section 9, the phase is related to $\vec{A}$ by

$$\tilde{\phi} = \frac{q}{\hbar}\int_0^r \vec{A}(\vec{r}')\cdot d\vec{r}' \tag{2891}$$

To evaluate the Schrödinger equation in (2889), we must first find $\nabla\Psi$. Applying the product rule to (2890) gives

$$\nabla\Psi = e^{i\tilde{\phi}}\left(\nabla i\tilde{\phi}\right)\tilde{\Psi} + e^{i\tilde{\phi}}\nabla\tilde{\Psi} \tag{2892}$$

Since $\tilde{\phi} = \frac{q}{\hbar}\int_0^r \vec{A}(\vec{r}')\cdot d\vec{r}'$, then $\nabla\tilde{\phi} = \frac{q}{\hbar}\vec{A}$ (by the fundamental theorem of calculus). So we have

$$\nabla\Psi = e^{i\tilde{\phi}}\left(i\frac{q}{\hbar}\vec{A}\right)\tilde{\Psi} + e^{i\tilde{\phi}}\nabla\tilde{\Psi} \tag{2893}$$

Next we substitute $\Psi = e^{i\tilde{\phi}}\tilde{\Psi}$ for the first term on the right, combine terms with Ψ, and multiply through by $-i\hbar$.

$$\left(-i\hbar\nabla - q\vec{A}\right)\Psi = -i\hbar e^{i\tilde{\phi}}\nabla\tilde{\Psi} \tag{2894}$$

Now we substitute $\Psi = e^{i\tilde{\phi}}\tilde{\Psi}$ on the left and cancel $e^{i\tilde{\phi}}$ on both sides.

$$\left(-i\hbar\nabla - q\vec{A}\right)\tilde{\Psi} = -i\hbar\nabla\tilde{\Psi} \tag{2895}$$

We can consider both sides of (2895) as being operators acting on $\tilde{\Psi}$. Applying the same operators on both sides again gives

$$\left(-i\hbar\nabla - q\vec{A}\right)^2\tilde{\Psi} = (-i\hbar\nabla)^2\tilde{\Psi} \tag{2896}$$

$$\left(i\hbar\nabla + q\vec{A}\right)^2\tilde{\Psi} = -\hbar\nabla^2\tilde{\Psi} \tag{2897}$$

[285]We will use the notation $\tilde{\phi}$ to distinguish the phase from the scalar potential found in $h_{00} = -2\phi/c^2$.

Substituting $\tilde{\Psi} = \Psi / e^{i\tilde{\phi}}$ on the left and multiplying by $e^{i\tilde{\phi}}$ gives

$$\left(i\hbar\nabla + q\vec{A}\right)^2 \Psi = -\hbar e^{i\tilde{\phi}} \nabla^2 \tilde{\Psi} \tag{2898}$$

Next we substitute this back into the right side of Schrödinger's equation (2889).

$$i\hbar\partial_t \Psi(\vec{r},t) = -\frac{\hbar}{2m} e^{i\tilde{\phi}} \nabla^2 \tilde{\Psi}(\vec{r},t) \tag{2899}$$

Then substituting $\Psi = e^{i\tilde{\phi}}\tilde{\Psi}$ on the left and canceling $e^{i\tilde{\phi}}$ on both sides gives

$$i\hbar\partial_t \tilde{\Psi}(\vec{r},t) \quad = \quad -\frac{\hbar}{2m} \nabla^2 \tilde{\Psi}(\vec{r},t) \tag{2900}$$

This result is the Schrödinger equation for a *free* particle with a wavefunction $\tilde{\Psi}(\vec{r},t)$. In other words, we find that $\tilde{\Psi}(\vec{r},t)$ is also a solution of Schrödinger's equation but with the absence of $\vec{A}$. This means that the presence of $\vec{A}$ just causes the wavefunction to acquire a phase factor. So we conclude that multiplying the wavefunction by a phase factor, $e^{i\tilde{\phi}}$, so that $\Psi = e^{i\tilde{\phi}}\tilde{\Psi}$ where $\tilde{\phi} = \frac{q}{\hbar}\int_0^r \vec{A}(\vec{r}') \cdot d\vec{r}$, causes the Schrödinger equation given in (2889) for a *coupled* particle with *no phase*, to become the Schrödinger equation given in (2900) for a *free* particle which *has a phase*.

Here the analysis has been done using the coupling rule in (2887) which involves $\vec{A}$. However, this analysis could be carried out for *any* vector field, $\vec{V}$, which transforms the canonical momentum according to $\vec{p} \implies \vec{p} - c\vec{V}$ where c is the particular "charge" quantity of the particle which couples to the arbitrary vector field $\vec{V}$. Then the corresponding phase would obviously be expressed as $\tilde{\phi} = \frac{c}{\hbar}\int_0^r \vec{V}(\vec{r}') \cdot d\vec{r}$.

Appendix P

Quantum phase interference due to a Newtonian potential

Here we examine a basic example of a quantum wavefunction phase shift due to a common Newtonian gravitational field, such as the gravitational field of the earth. We consider the case of a nearly monoenergetic beam of particles that is split into two parts and then brought together as shown in the following diagram from Sakurai, *Modern Quantum Mechanics*, [71] (pp. 136-139).

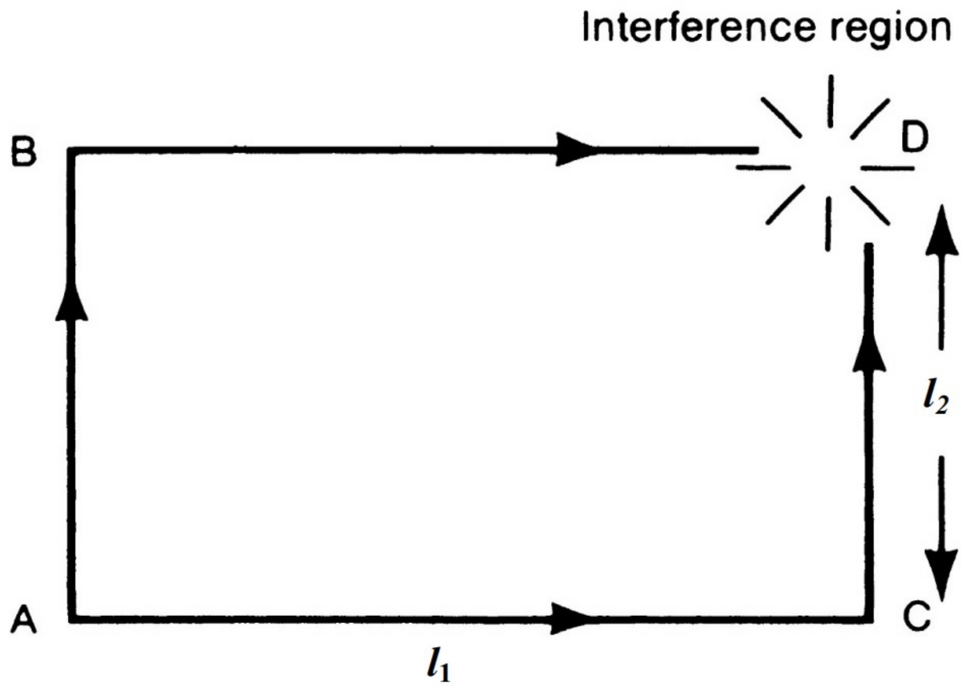

Figure 31: The quantum wave function of a beam of particles is split into paths at point A and then brought together again point D. One path is ABD while the other path is ACD. A quantum phase interference will occur due to the Newtonian potential of Earth's gravitational field.

Assuming that the size of the wave packet is much smaller than the macroscopic dimension of the loop formed by the two alternate paths, then we can apply the concept of a classical trajectory. Let us describe the path ABC as Path 1 and the path ACD as Path 2. If we consider the wave function on each path to be a plane wave, then the wave functions are

$$\Psi_1(\vec{r},t) = Ae^{i\left(\vec{k}_1\cdot\vec{r}_1-\omega_1 t_1\right)} \qquad \text{and} \qquad \Psi_2(\vec{r},t) = Ae^{i\left(\vec{k}_2\cdot\vec{r}_2-\omega_2 t_2\right)} \tag{2901}$$

If the loop is rotated by an angle δ about the segment AC, then the path of each wave will rise in height as the wave goes from A to D. The energy of the particle is $E = \frac{p^2}{2m} + mgz$, so a rise in height will increase the gravitational potential energy and therefore decrease the kinetic energy. Decreasing kinetic energy means decreasing momentum and therefore increasing the wavelength, $\lambda = h/p$. The sum of these changes in wavelength over the length of the path will cause an effective phase change at the interference region. So the wave function will have a phase factor $e^{i\phi}$. By gauge invariance of the wave function, we can multiply the wave function by $e^{i\phi}$ and it will still satisfy the Schrödinger wave equation. Then the two wave functions given above we will be

$$\Psi_1(\vec{r},t) = Ae^{i\phi_1}e^{i(\vec{p}_1\cdot\vec{r}_1/\hbar-r_1/v_1)} \qquad \text{and} \qquad \Psi_2(\vec{r},t) = Ae^{i\phi_2}e^{i(\vec{p}_2\cdot\vec{r}_2/\hbar-r_2/v_2)} \tag{2902}$$

We can write the phase for each wave as

$$\phi_1 = \int_{Path\ 1} \vec{k}_1 \cdot d\vec{l} \qquad \text{and} \qquad \phi_2 = \int_{Path\ 2} \vec{k}_2 \cdot d\vec{l} \tag{2903}$$

The total path length is the same for the two waves. However, the segment BD along Path 1 is at a greater height than the segment AC along Path 2. Therefore, Ψ_1 has more of its path at a higher gravitational potential than Ψ_2 and consequently $\lambda_1 = h/p_1$ will be larger than $\lambda_2 = h/p_2$ when comparing segments BD and CD. As a result there will be a phase difference between the two waves. This difference is

$$\Delta\phi = \phi_1 - \phi_2 = \int_{Path\ 1} \vec{k}_1 \cdot d\vec{l} - \int_{Path\ 2} \vec{k}_2 \cdot d\vec{l} \tag{2904}$$

Note that the segment AB on Path 1 and the segment CD on path 2 will not contribute to the phase difference between the wave functions because the heights of the wave functions increase identically on these segments. Therefore, we can consider only the segment BD on Path 1 and AC on Path 2. Then we have

$$\Delta\phi = k_1 l_1 - k_2 l_1 \tag{2905}$$

The energy, $E = \frac{p^2}{2m} + mgz$, for either path must be the same since it is a single particle. So we have

$$\frac{p_1^2}{2m} + mgz_1 = \frac{p_2^2}{2m} + mgz_2 \tag{2906}$$

$$p_1^2 = p_2^2 + 2m^2 g\,(z_2 - z_1) \tag{2907}$$

$$\hbar^2 k_1^2 = \hbar^2 k_2^2 + 2m^2 g\,(z_2 - z_1) \tag{2908}$$

$$k_1^2 = k_2^2 - \frac{2m^2 g}{\hbar^2}\,(z_1 - z_2) \tag{2909}$$

We are considering the case when the loop is rotated by an angle δ that makes the segment BD (from path 1) higher than the segment AC (from path 2). So the difference in heights is given by $z_1 - z_2 = l_2 \sin\delta$. Then we have

$$k_1^2 = k_2^2 - \frac{2m^2 g l_2 \sin\delta}{\hbar^2} \tag{2910}$$

We can now combine equations (2905) and (2910) to solve for $\Delta\phi$. Rearranging the terms in (2905) and squaring both sides gives

$$(\Delta\phi + k_2 l_1)^2 = k_1^2 l_1^2 \tag{2911}$$

Using (2910) to substitute for k_1^2 gives

$$(\Delta\phi + k_2 l_1)^2 = \left(k_2^2 - \frac{2m^2 g l_2 \sin\delta}{\hbar^2}\right) l_1^2 \tag{2912}$$

$$\Delta\phi^2 + 2k_2 l_1 \Delta\phi + k_2^2 l_1^2 = k_2^2 l_1^2 - \frac{2m^2 g l_1^2 l_2 \sin\delta}{\hbar^2} \tag{2913}$$

$$\Delta\phi^2 + 2k_2 l_1 \Delta\phi + 2m^2 g l_2 l_1^2 \sin\delta/\hbar^2 = 0 \tag{2914}$$

We can use the quadratic formula with $a = 1,\ b = 2k_2 l_1$, and $c = 2m^2 g l_2 l_1^2 \sin\delta/\hbar^2$.

$$\Delta\phi = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2915}$$

$$= \frac{-2k_2 l_1 \pm \sqrt{4k_2^2 l_1^2 - 8m^2 g l_2 l_1^2 \sin\delta/\hbar^2}}{2} \tag{2916}$$

$$= -k_2 l_1 \pm \sqrt{k_2^2 l_1^2 - 2m^2 g l_2 l_1^2 \sin\delta/\hbar^2} \tag{2917}$$

This is an exact answer. If we would like to simplify this, we must make some approximations. First we can factor out $k_2^2 l_1^2$ from the square root.

$$\Delta\phi = -k_2 l_1 \pm k_2 l_1 \sqrt{1 + \frac{2m^2 g l_2 \sin\delta}{k_2^2 \hbar^2}} \tag{2918}$$

Notice that the fraction inside the square root can be written as $\frac{mgl_2 \sin\delta}{p^2/2m}$ which is the ratio of the gravitational potential energy to kinetic energy. For a very small mass (such as neutrons which were used to experimentally verify this effect), the kinetic energy is much greater than the gravitational potential energy so $\frac{mgl_2}{p^2/2m} << 1$. This means that we can apply the binomial expansion: $\sqrt{1-x} \approx 1 - \frac{1}{2}x$ for $x << 1$. So we obtain

$$\Delta\phi \approx -k_2 l_1 \pm k_2 l_1 \left(1 - \frac{m^2 g l_2 \sin\delta}{k_2^2 \hbar^2}\right) \tag{2919}$$

$$\approx -k_2 l_1 \pm \left(k_2 l_1 - \frac{m^2 g l_1 l_2 \sin\delta}{k_2 \hbar^2}\right) \tag{2920}$$

Here we have the option of using the positive or negative sign from the "$\pm$." The negative sign will result in a factor of $-2k_2 l_1$ contributing to the phase difference. Since we are only interested in the effect of the gravitational potential on the phase difference, then we can apply the positive sign in order for the $k_2 l_1$ terms to cancel. This gives

$$\Delta\phi \approx -\frac{m^2 g l_1 l_2 \sin\delta}{k_2 \hbar^2} \tag{2921}$$

Therefore we see that even a common Newtonian gravitational field such as the earth's can produce a phase shift in the wavefunction.

Appendix Q

The London equations and penetration depth in electromagnetism

The London Equations

The brothers F. and H. London developed two equations which lead to an expression for the penetration depth for EM fields incident on a superconductor [72]. By modeling the electrons in a superconductor as flowing with no resistance, the electrons will accelerate uniformly according to the Lorentz force law

$$m\vec{a} = q\vec{E} \tag{2922}$$

Using $\vec{a} = \partial_t \vec{v}$ and $q = -e$ gives

$$\partial_t \vec{v} = -\frac{e}{m_e}\vec{E} \tag{2923}$$

The supercurrent can be written as

$$\vec{J}_s = n_s q\vec{v} = -n_s e\vec{v} \tag{2924}$$

where $\vec{J}_s$ is the charge current and n_s is the number density of superconducting carriers (electrons). Then (2923) becomes what is often referred to as the *first* London equation. Since it involves the electric field, we will refer to it as the *electric* London equation.

$$\boxed{\partial_t \vec{J}_s = \frac{n_s e^2}{m_e}\vec{E} \qquad \textit{Electric London equation}} \tag{2925}$$

Note that inserting $\vec{E} = -\nabla\varphi - \partial_t \vec{A}$ into the expression above and rearranging gives

$$\partial_t \left(\vec{J}_s + \frac{n_s e^2}{m_e}\vec{A}\right) = -\frac{n_s e^2}{m_e}\nabla\varphi \tag{2926}$$

In (2931) it is found that $\vec{J}_s = -\frac{n_s e^2}{m_e}\vec{A}$ which means that the entire term in parentheses above must be zero. Then the equation above requires that $\nabla\varphi = 0$ inside a superconductor.[286]

Next, we can take the curl of (2925) and apply Faraday's law, $\nabla \times \vec{E} = -\partial_t \vec{B}$, to obtain

$$\partial_t \nabla \times \vec{J}_s = -\frac{n_s e^2}{m_e}\partial_t \vec{B} \tag{2927}$$

Integrating both sides with respect to time and setting the integration constant to zero gives what is often referred to as the *second* London equation. Since it involves the magnetic field, we will refer to it as the *magnetic* London equation.[287]

$$\boxed{\nabla \times \vec{J}_s = -\frac{n_s e^2}{m_e}\vec{B} \qquad \textit{Magnetic London equation}} \tag{2928}$$

[286] This means the electric field does not have a contribution due to the scalar potential. It must be written as: $E_i = -\partial_t A_i$. Taking a divergence and applying the London gauge, $\partial_i A_i = 0$, also means $\partial_i E_i = 0$. By Gauss's law this means $\rho/\varepsilon = 0$. Therefore, either $\rho_c = 0$ or $\varepsilon \to \infty$ inside a superconductor.

[287] Note that (2925) and (2928) are *constituent* equations, not field equations. These equations describe the current density produced in the superconductor due to external electric and magnetic fields. These equations do not describe the fields produced by the current density in the superconductor. The field equations are still the standard Maxwell field equations.

This using $\vec{B} = \nabla \times \vec{A}$ gives

$$\nabla \times \left(\vec{J}_s + \frac{n_s e^2}{m_e} \vec{A} \right) = 0 \tag{2929}$$

The solution to this differential equation is

$$\vec{J}_s + \frac{n_s e^2}{m_e} \vec{A} + \nabla f(r,t) \quad = \quad 0 \tag{2930}$$

For a steady state current, we have $\dot{\rho} = 0$ and therefore $\nabla \cdot \vec{J}_s = 0$ by the continuity equation. Also, in the Coulomb (or London) gauge, we have $\nabla \cdot \vec{A} = 0$. Therefore, taking the divergence of (2930) requires that $\nabla^2 f(r,t) = 0$. Assuming $\vec{J}_s$ and $\vec{A}$ go to zero as $r \to \infty$, then we must also have that $\nabla f(r,t) \to 0$ as $r \to \infty$ which means that $f(r,t) \to \textit{constant}$ as $r \to \infty$. Then the unique solution to $\nabla^2 f(r,t) = 0$ is $f(r,t) = \textit{constant}$ everywhere and therefore $\nabla f(r,t) = 0$ everywhere. Therefore (2930) simply reduces to

$$\boxed{\vec{J}_s = -\frac{n_s e^2}{m_e} \vec{A} \qquad \textit{London constitutive equation}} \tag{2931}$$

The equation in (2931) can be thought of as a constitutive equation[288] which describes how the supercurrent in a superconductor responds to an external field $\vec{A}$. The equation in (2931) can also be thought of as a *single* London equation which combines the previous two London equations into one relationship. We can observe this by taking a time derivative of (2931) and using $\vec{E} = -\partial_t \vec{A}$ to obtain the electric London equation in (2925). Also, taking the curl of (2931) and using $\vec{B} = \nabla \times \vec{A}$ gives the magnetic London equation in (2928).

We may also derive (2931) by use of the minimal coupling rule for a charged particle coupled to an EM field. The kinetic momentum, $m_e \vec{v}$, is expressed in terms of the canonical momentum, $\vec{p}_{can}$, according to (842) as

$$\vec{p}_{can} = m_e \vec{v} - e\vec{A} \tag{2932}$$

Since the superconducting state of a system is a zero-momentum state, then the canonical momentum becomes zero and we have $\vec{v} = e\vec{A}/m_e$. Substituting this into (2924) gives

$$\vec{J}_s = -\frac{n_s e^2}{m_e} \vec{A} \tag{2933}$$

This matches the result found in (2931).

<u>The London penetration depth for the vector potential and the magnetic field</u>

Ampere's law for a steady-state supercurrent is $\nabla \times \vec{B}_s = \mu \vec{J}_s$. Taking the curl gives

$$\nabla \times \nabla \times \vec{B}_s = \mu \nabla \times \vec{J}_s \tag{2934}$$

Since $\nabla \cdot \vec{B}_s = 0$, then the vector calculus identity $\nabla \times \nabla \times \vec{B}_s = \nabla \left(\nabla \cdot \vec{B}_s \right) - \nabla^2 \vec{B}_s$ will become $\nabla \times \nabla \times \vec{B}_s = -\nabla^2 \vec{B}_s$. Then the equation above becomes

$$-\nabla^2 \vec{B}_s = \mu \nabla \times \vec{J}_s \tag{2935}$$

[288] This is similar to Ohm's law, $\vec{J} = \sigma_c \vec{E}$, where σ_c is the conductivity that characterizes the response of a conductor to an electric field.

Inserting (2928) into the right side of (2935) gives[289]

$$\nabla^2\vec{B} = \frac{\mu n_s e^2}{m_e}\vec{B} \tag{2936}$$

We can write the prefactor on the right side as $\frac{\mu n_s e^2}{m_e} = \frac{1}{\lambda_L^2}$. Then we have

$$\boxed{\nabla^2\vec{B} - \frac{1}{\lambda_L^2}\vec{B} = 0} \tag{2937}$$

The solution to this equation is $B(x) = B_0 e^{-x/\lambda_L}$ where the London penetration depth is

$$\boxed{\lambda_L = \sqrt{\frac{m_e}{\mu n_s e^2}} \qquad \textit{London penetration depth}} \tag{2938}$$

This determines the characteristic depth to which an external magnetic field can penetrate into the interior of a superconductor. This can also be expressed as a plasma frequency $\omega_p = \frac{c}{\lambda_L}$. This gives[290]

$$\omega_p = c\sqrt{\frac{\mu n_s e^2}{m_e}} \tag{2939}$$

We can return again to Ampere's law for a steady-state supercurrent, $\nabla \times \vec{B} = \mu \vec{J}_s$. This time, we substitute $\vec{B} = \nabla \times \vec{A}$ on the left and insert (2931) on the right.

$$\nabla \times \nabla \times \vec{A} = -\frac{\mu n_s e^2}{m_e}\vec{A} \tag{2940}$$

Using the Coulomb (or London) gauge, $\nabla \cdot \vec{A} = 0$, makes the vector calculus identity given by $\nabla \times \nabla \times \vec{A} = \nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2\vec{A}$ become $\nabla \times \nabla \times \vec{A} = -\nabla^2\vec{A}$. Then the equation above becomes

$$\nabla^2\vec{A} \quad = \quad \frac{\mu n_s e^2}{m_e}\vec{A} \tag{2941}$$

Writing this in terms of the London penetration depth from (2938) gives

$$\boxed{\nabla^2\vec{A} - \frac{1}{\lambda_L^2}\vec{A} = 0} \tag{2942}$$

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[289] Formally speaking $\vec{B}$ and $\vec{B}_s$ are not the same since $\vec{B}_s$ is the magnetic field *produced by* the superconductor while $\vec{B}$ is an *external* magnetic field *acting on* the superconductor. In fact, the *external* $\vec{B}$ is related to an *external* $\vec{E}$ (by Faraday's law used earlier) which provided an *external* Lorentz force according to (2923). Therefore, technically the differential equaton constains two separate $\vec{B}$ fields that cannot be treated as the same field. However, this is often overlooked by simply viewing $\vec{B}$ as taking into account *all* the fields, including those produced by the material of the superconductor as well as those introduced externally.

[290] Since $c = 1/\sqrt{\varepsilon_0\mu_0}$, then (2938) could also be written as $\lambda_L = c\sqrt{\frac{\varepsilon_0 m_e}{n_s e^2}}$ and (2939) could also be written as $\omega_p = \sqrt{\frac{n_s e^2}{\varepsilon_0 m_e}}$.

Therefore, the vector potential is also expected to vanish within the London penetration depth.

The London penetration depth for the supercurrent and electric field

Taking the curl of the magnetic London equation in (2928) gives

$$\nabla\times\nabla\times\vec{J}_s = -\frac{n_s e^2}{m_e}\nabla\times\vec{B} \tag{2943}$$

Since $\dot{\rho}=0$ for a steady state current, then by the continuity equation we have $\nabla\cdot\vec{J}_s=0$. In that case, the vector calculus identity $\nabla\times\nabla\times\vec{J}_s=\nabla\left(\nabla\cdot\vec{J}_s\right)-\nabla^2\vec{J}_s$ becomes $\nabla\times\nabla\times\vec{J}_s=-\nabla^2\vec{J}_s$. Then the equation above can be written as

$$\nabla^2\vec{J}_s = \frac{n_s e^2}{m_e}\nabla\times\vec{B} \tag{2944}$$

Substituting Ampere's law for a steady-state supercurrent, $\nabla\times\vec{B}=\mu\vec{J}_s$, gives

$$\nabla^2\vec{J}_s = \frac{\mu n_s e^2}{m_e}\vec{J}_s \tag{2945}$$

Writing this in terms of the London penetration depth from (2938) gives

$$\boxed{\nabla^2\vec{J}_s-\frac{1}{\lambda_L^2}\vec{J}_s=0} \tag{2946}$$

Therefore, the supercurrent is also expected to vanish within the London penetration depth. This is consistent with the fact the magnetic field (which would drive the supercurrent) also penetrates into the superconductor only up to the London penetration depth. Taking the time derivative of (2946) and using the electric London equation from (2925) gives

$$\boxed{\nabla^2\vec{E}-\frac{1}{\lambda_L^2}\vec{E}=0} \tag{2947}$$

Therefore, the electric field is also expected to vanish within the London penetration depth.

An alternative approach to the London equations

We begin by expressing the Lorentz force for a differential volume element containing a charge density $\rho_c=dq/dV$ and mass density $\rho_m=dm/dt$.

$$\rho_m\vec{a} = \rho_c\left(\vec{E}+\vec{v}\times\vec{B}\right) \tag{2948}$$

Using $\vec{a}=\partial_t\vec{v}$ and $\vec{J}_s=\rho_c\vec{v}$ for the supercurrent density gives

$$\rho_m\partial_t\vec{J}_s = \rho_c^2\left(\vec{E}+\vec{v}\times\vec{B}\right) \tag{2949}$$

For a supercurrent of electrons, we have $\rho_c/\rho_m = -e/m_e$ which gives

$$\partial_t \vec{J}_s \quad = \quad -\frac{e\rho_c}{m_e}\left(\vec{E}+\vec{v}\times\vec{B}\right) \tag{2950}$$

This is effectively the same as the electric London equation in (2928) except we have retained the contribution of the magnetic force. We can take the curl of this equation and apply Faraday's law, $\nabla\times\vec{E} = -\partial_t\vec{B}$. This gives

$$\partial_t \nabla\times\vec{J}_s \quad = \quad -\frac{e\rho_c}{m_e}\left[-\partial_t\vec{B}+\nabla\times\left(\vec{v}\times\vec{B}\right)\right] \tag{2951}$$

We can apply the identity $\nabla\times\left(\vec{v}\times\vec{B}\right) = \vec{v}\left(\nabla\cdot\vec{B}\right) - (\nabla\cdot\vec{v})\vec{B}$ and use $\nabla\cdot\vec{B} = 0$. Also, for an incompressible flow, we have $\nabla\cdot\vec{v} = 0$. Therefore, the equation above becomes

$$\partial_t \nabla\times\vec{J}_s \quad = \quad \frac{e\rho_c}{m_e}\partial_t\vec{B} \tag{2952}$$

Integrating both sides with respect to time gives

$$\nabla\times\vec{J}_s = \frac{e\rho_c}{m_e}\vec{B} + f(\vec{x}) \tag{2953}$$

where $f(\vec{x})$ is an integrating function of position but not time. Setting $f(\vec{x}) = 0$ and using $\rho_c = n_s e$ gives

$$\nabla\times\vec{J}_s \quad = \quad \frac{n_s e^2}{m_e}\vec{B} \tag{2954}$$

This is the same as the magnetic London equation found in (2928). Using $\vec{B} = \nabla\times\vec{A}$ gives

$$\nabla\times\left(\vec{J}_s + \frac{e\rho_c}{m_e}\vec{A}\right) \quad = \quad 0 \tag{2955}$$

Once again, we can argue that the unique solution to this differential equation is

$$\vec{J}_s \quad = \quad -\frac{e\rho_c}{m_e}\vec{A} \tag{2956}$$

which is the same as the single London equation in (2931) except in terms of ρ_c rather than $n_s e$.

<u>The negligibility of the displacement current</u>

We now express the full Ampere's law in terms of the supercurrent *without* neglecting the displacement current. For an oscillating field, we can use $\partial_t\vec{E} = \omega\vec{E}$ so that we have

$$\nabla\times\vec{B} = \mu\vec{J}_s + \frac{\omega}{c^2}\vec{E} \tag{2957}$$

For an oscillating current, we can also use $\partial_t\vec{J} = \omega\vec{J}$. Inserting this into the electric London equation from (2925) and gives

$$\omega\vec{J}_s = \frac{n_s e^2}{m_e}\vec{E} \tag{2958}$$

Solving this for $\vec{E}$ and substituting it into (2957) gives

$$\nabla \times \vec{B} = \mu \vec{J}_s + \frac{\omega^2 m_e}{c^2 e^2 n_s} \vec{J}_s \tag{2959}$$

This can be written in terms of the London penetration depth from (2938).

$$\nabla \times \vec{B} = \mu \vec{J}_s + \frac{\mu \lambda_L^2 \omega^2}{c^2} \vec{J}_s \tag{2960}$$

From this expression we can easily compare the strength of the charge current (determined by μ) to the strength of the displacement current, determined by the prefactor $\mu\lambda_L^2\omega^2/c^2$. We can let $\mu \approx \mu_0 \approx 1.3 \times 10^{-6}$ (SI units). We can also consider the case of microwave frequencies, $f \sim 10^{10} Hz$, and a Niobium superconductor, $\lambda_L \approx 10^{-9}$. Then the prefactor for the displacement current becomes $\mu\lambda_L^2\omega^2/c^2 \approx 5.7 \times 10^{-20}$ (SI units). Therefore, the contribution by the displacement current is weaker than the contribution by the charge current by a factor on the order of 10^{-14}. Therefore the displacement current can be neglected in the London equations even for high frequencies.

The "London prescription"

The process used for determining the London penetration depth can be considered as essentially a "prescription" which is summarized as follows.

1. Express the velocity in terms of the current density: $v = -\dfrac{J}{n_s e}$.
2. Substitute this into the Lorentz force to obtain the electric London equation in (2925). This will determine the proportionality constant that ultimately appears in (2931).
3. Use Faraday's law to express the electric London equation in terms of $\vec{B}$. This gives the magnetic London equation. Express $\vec{B}$ as $\vec{B} = \nabla \times \vec{A}$ to obtain the single London equation given by (2931).
4. Use Ampere's law to formulate a differential equation in terms of $\vec{B}$ or $\vec{A}$ which can then be used to determine the penetration depth of the field based on an exponential decay function.

Appendix R

Electrodynamics within a superconductor

First we use Maxwell equations in the Lorenz gauge to obtain dispersion relations and penetration depths for the vector potential, $\vec{A}$, for two cases: 1) Ohm's law and 2) London's constitutive equation. Then we use the Maxwell equations to derive non-homogenous wave equations for $\vec{E}$ and $\vec{B}$ in matter. The source terms are evaluated for four cases: 1) Ohm's law; 2) London's constitutive equation; 3) the two fluid model; and 4) the full Lorentz force density. These are shown to lead to different dispersion relations and penetration depths for $\vec{E}$ and $\vec{B}$.

A dispersion relation, plasma frequency, and penetration depth for $\vec{A}$ and $\vec{J}$

The wave equation for the vector potential describing electromagnetic waves in the Lorenz gauge is given by $\Box A^i = -\mu_0 J^i$. The London constitutive equation is $J^i = -\Lambda_L A^i$, where we found in (2931) that $\Lambda_L = n_s e^2/m_e$. Inserting this into the wave equation and expanding the box operator gives

$$-\tfrac{1}{c^2}\partial_t^2 A^i + \nabla^2 A^i = \mu_0 \Lambda_L A^i \tag{2961}$$

To obtain a dispersion relation, we can use a monochromatic, plane-fronted wave propagating in the z-direction given by

$$A(z,t) = A_0 e^{i(kz-\omega t)} \tag{2962}$$

where A_0 is a constant amplitude. Then the wave equation above gives[291]

$$-\frac{(-i\omega)^2}{c^2}A^i + (ik)^2 A^i = \mu_0 \Lambda_L A^i \tag{2963}$$

$$\left(k^2 - \frac{\omega^2}{c^2} + \mu_0\Lambda_L\right)A^i = 0 \tag{2964}$$

For a non-trivial solution $\left(A^i \neq 0\right)$, we have

$$k^2 = \frac{\omega^2}{c^2} - \mu_0\Lambda_L \tag{2965}$$

We can factor out ω^2/c^2 to obtain the following dispersion relation for electromagnetic waves in a superconductor.

$$\boxed{k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\mu_0 c^2 \Lambda_L}{\omega^2}\right) \qquad \textit{Dispersion relation for electromagnetic waves in a superconductor}} \tag{2966}$$

We can define an electromagnetic plasma frequency as[292]

$$\boxed{\omega_{EM}^2 \equiv \mu_0 c^2 \Lambda_L \qquad \textit{Electromagnetic plasma frequency}} \tag{2967}$$

[291] Note that that the frequency of the incoming electromagnetic wave (on the right) is assumed to have the *same* frequency as the outgoing electromagnetic wave (on the left).

[292] Since $\Lambda_L = \mu/\lambda_L^2$ where λ_L is the London penetration depth, then $\omega_{EM} = c/\lambda_L$. For Niobium $(\lambda_L \approx 39nm)$, this gives $\omega_{EM} \approx 8 \times 10^{15} Hz$.

Then (2966) can also be written as

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_{EM}^2}{\omega^2}\right) \tag{2968}$$

It is clear from (2968) that reflection or absorption will occur when $\omega \leq \omega_{EM}$ which means k becomes imaginary. We can also write (2968) as

$$k^2 = \frac{\omega^2}{c^2} n^2(\omega) \tag{2969}$$

where we are using an index of refraction defined as

$$\boxed{n^2(\omega) \equiv \left(1 - \frac{\omega_{EM}^2}{\omega^2}\right) \qquad \textit{Electromagnetic index of refraction}} \tag{2970}$$

This characterizes the reflection and refraction of electromagnetic waves in matter. Next we can evaluate an electromagnetic wave penetration depth. We can define a complex wave number as

$$k = K + i\kappa_0 \tag{2971}$$

where K and κ_0 are real quantities. Inserting this into the plane wave of (2962) and separating the real and imaginary parts of the phase gives

$$A(z,t) = A_0 e^{i[(K+i\kappa)z-\omega t]} = A_0 e^{-\kappa z} e^{i(Kz-\omega t)} \tag{2972}$$

Here we clearly see that the wave falls off exponentially with distance where κ as the exponential decay factor. The square of the wave number in (2971) is

$$k^2 = K^2 - \kappa_0^2 + 2iK\kappa_0 \tag{2973}$$

Since k^2 in (2965) is only real, then we must have either $K = 0$ or $\kappa_0 = 0$ to eliminate the last cross term above. Setting $\kappa_0 = 0$ and using (2965) gives

$$K^2 = \frac{\omega^2}{c^2} - \mu_0 \Lambda_L \tag{2974}$$

Note that this condition is only valid for $\omega^2/c^2 \geq \mu\Lambda_L$ since K is real. Using the electromagnetic plasma frequency in (2967), we can write this condition as $\omega \geq \omega_{EM}$. Then the plane wave of (2972) becomes

$$\boxed{\begin{array}{c} A(z,t) = A_0 e^{i(Kz-\omega t)} \qquad \text{where} \qquad K = \pm\sqrt{\frac{\omega^2}{c^2} - \mu_0 \Lambda_L} \\ \\ \textit{Propagating solution for an electromagnetic wave} \\ \textit{in a superconductor (for } \omega \geq \omega_{EM}\textit{)} \end{array}} \tag{2975}$$

This result corresponds to a wave with a frequency *above* the plasma frequency which therefore cannot interact with the material and hence propagates with no attenuation. On the other hand, setting $K = 0$ and using (2965) and (2973) gives

$$\kappa_0^2 = \mu_0 \Lambda_L - \frac{\omega^2}{c^2} \tag{2976}$$

Note that this condition is only valid for $\omega^2/c^2 \le \mu_0\Lambda_L$ since κ_0 is real. Using the electromagnetic plasma frequency in (2967), we can write this condition as $\omega \le \omega_G$. Then the plane wave of (2972) becomes[293]

$$\boxed{\begin{array}{c} A(z,t) = A_0 e^{-\kappa_0 z} e^{-i\omega t} \qquad \text{where} \qquad \kappa_0 = \sqrt{\mu_0\Lambda_L - \frac{\omega^2}{c^2}} \\ \\ \textit{Exponentially decaying solution for an electromagnetic wave} \\ \textit{in a superconductor for } (\omega \le \omega_{EM}) \end{array}} \tag{2977}$$

This result corresponds to a wave with a frequency *below* the plasma frequency. Such a wave will interact with the material and is therefore attenuated as it propagates in the material. Therefore, we can identify a characteristic frequency-dependent penetration depth as $\delta_{EM} = 1/\kappa_0$. Using (2976) gives

$$\delta_{EM}^2 = \frac{c}{c^2\mu_0\Lambda_L - \omega^2} \tag{2978}$$

Using $\Lambda_L = n_s e^2/m_e$, we can write this as

$$\boxed{\delta_{EM}^2 = \frac{c}{c^2\mu_0 n_s e^2/m_e - \omega^2} \qquad \textit{Electromagnetic wave penetration depth}} \tag{2979}$$

We can also write this in terms of the electromagnetic plasma frequency in (2967) as

$$\delta_{EM}^2 = \frac{c}{\omega_{EM}^2 - \omega^2} \tag{2980}$$

In this form, we see that as ω approaches ω_{EM}, we have δ_{EM} approaching infinity. This means that as the frequency approaches the plasma frequency, the wave is no longer attenuated with depth. On the other hand, for $\omega >> \omega_{EM}$, we find that $\delta_{EM} \approx c/\omega_{EM}$. In fact, for the "DC" limit ($\omega = 0$), the penetration depth is no longer frequency-dependent and using (2967) gives[294]

$$\boxed{\delta_{EM} = \frac{c}{\omega_{EM}} = \frac{c^2}{\sqrt{\mu_0 c^2 \Lambda_L}} \qquad \textit{Electromagnetic penetration depth for } \omega = 0} \tag{2981}$$

If we consider the static limit of (2961), then we have a Yukawa-like equation given as

$$\nabla^2 A^i - \mu_0\Lambda_L A^i = 0 \tag{2982}$$

Since $\Lambda_L = n_s e^2/m_e$, then we can write the prefactor as $\dfrac{1}{\lambda_L^2} = \mu n_s e^2/m_e$ so the solution to (2982) is $A(z) = A_0 e^{-z/\lambda_L}$ where

$$\lambda_L \quad = \quad \sqrt{\frac{m_e}{\mu_0 n_s e^2}} \tag{2983}$$

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[293]Formally, κ_0 should have a positive and negative root. However, if we consider the case of $z < 0$ representing the vacuum and $z > 0$ representing the superconductor, then we can write κ_0 with only the *positive* root in order to obtain an exponential *decay* solution and avoid a diverging exponential growth solution.

[294]Another way of arguing this is to consider that the wave can only penetrate the skin of the superconductor to a depth on the order of a wavelength. Then using $k = c/\omega_{EM}$ and $\omega_{EM}^2 = \mu c^2 \Lambda_L$ leads to (2981).

This is the electromagnetic London penetration depth for a static vector potential field expelled from a superconductor as found in (2938). From (2967), the electromagnetic plasma frequency can be written as $\omega_{EM}^2 = \mu_0 c^2 \Lambda_L = \mu_0 c^2 n_s e^2/m_e$. Therefore, we find $\lambda_L = c/\omega_{EM}$ as expected. This is equivalent to the electromagnetic penetration depth found in (2981) in the static or "DC" limit.

Lastly, we point out that because the London constitutive equation, $J^i = -\Lambda_L A^i$, is just a proportionality between J^i and A^i, then we can replace A^i with J^i all throughout the analysis above and obtain the same exact results for the dispersion relation and penetration depth. Therefore, the solutions in (2975) and (2977) can also be used to describe the current density when the London constitutive equation applies.[295] This means that for $\omega \geq \omega_{EM}$ we have

$$J(z,t) = J_0 e^{i(Kz-\omega t)} \qquad \text{where} \qquad K = \pm\sqrt{\tfrac{\omega^2}{c^2} - \mu_0 \Lambda_L} \tag{2984}$$

and for $\omega \leq \omega_{EM}$ we have

$$J(z,t) = J_0 e^{-\kappa_0 z} e^{-i\omega t} \qquad \text{where} \qquad \kappa_0 = \sqrt{\mu_0 \Lambda_L - \tfrac{\omega^2}{c^2}} \tag{2985}$$

Also, the penetration depth for this exponentially decaying current density in (2985) is given by (2979).

Inhomogenous wave equations for $\vec{E}$ and $\vec{B}$

The four Maxwell field equations in matter are

$$\begin{aligned} &\nabla \cdot \vec{E} = \rho/\varepsilon && \nabla \cdot \vec{B} = 0 \\ &\nabla \times \vec{E} = -\dot{\vec{B}} && \nabla \times B = \mu \vec{J} + \mu\varepsilon \dot{\vec{E}} \end{aligned} \tag{2986}$$

Taking the curl of Faraday's law and applying the identity $\nabla \times \left(\nabla \times \vec{E}\right) = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E}$ gives

$$\nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E} = -\nabla \times \dot{\vec{B}} \tag{2987}$$

Using Gauss's law on the left side and Ampere's law on the right side gives

$$\varepsilon \nabla \rho - \nabla^2 \vec{E} = -\mu \dot{J} - \mu\varepsilon \ddot{\vec{E}} \tag{2988}$$

Rearranging and using $\mu\varepsilon = 1/c^2$ gives

$$\boxed{\nabla^2 \vec{E} - \frac{1}{c^2}\ddot{\vec{E}} = \varepsilon \nabla \rho + \mu \dot{\vec{J}} \quad \textit{Inhomogeneous wave equation for } \vec{E}} \tag{2989}$$

Next we take the curl of Ampere's law and again apply $\nabla \times \left(\nabla \times \vec{B}\right) = \nabla\left(\nabla \cdot \vec{B}\right) - \nabla^2 \vec{B}$ to obtain

$$\nabla\left(\nabla \cdot \vec{B}\right) - \nabla^2 \vec{B} = \mu\left(\nabla \times \vec{J}\right) + \left(\mu\varepsilon \nabla \times \dot{\vec{E}}\right) \tag{2990}$$

Using Gauss's law for $\vec{B}$ on the left side and Faraday's law on the right side gives

$$-\nabla^2 \vec{B} = \mu\left(\nabla \times \vec{J}\right) - \mu\varepsilon \ddot{\vec{B}} \tag{2991}$$

[295] This is similar to Ohm's constituent equation, $J^i = \sigma \partial_t A^i$, however, Ohm's equation required the additional step of taking a time derivative of the wave equation to write it in terms of the current.

Rearranging and using $\mu_0\varepsilon_0 = 1/c^2$ gives

$$\boxed{\nabla^2\vec{B} - \frac{1}{c^2}\ddot{\vec{B}} = -\mu\nabla\times\vec{J} \quad \textit{Inhomogeneous wave equation for } \vec{B}} \tag{2992}$$

<u>Dispersion relations for $\vec{E}$ and $\vec{B}$ using Ohm's law</u>

Inserting Ohm's law, $\vec{J} = \sigma\vec{E}$ into the electric field wave equation (2989) gives

$$\nabla^2\vec{E} - \frac{1}{c^2}\ddot{\vec{E}} \quad = \quad \varepsilon\nabla\rho + \mu\sigma\dot{\vec{E}} \tag{2993}$$

Using complex sinusoid functions for the field, $\vec{E} = \vec{E}_0 e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$ and charge density, $\rho = \rho_0 e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$ gives

$$-k^2E_0 + \frac{\omega^2}{c^2}E_0 \quad = \quad \varepsilon ik\rho_0 + \mu\sigma\omega E_0 \tag{2994}$$

Using the continuity equation, Gauss's law, and Ohm's law, we found in (1359) that the charge density is given by $\rho = \rho_0 e^{-t/\tau}$ where $\tau = \varepsilon/\sigma_c$ is the characteristic time scale that describes how rapidly and net charge in the interior of the conductor will move to the surface. For a conductor like copper, this time scale is on the order of $10^{-19}s$ therefore we can drop the term involving ρ.... This matches the results found in Griffiths [29] (p. 394)

<u>Wave equations for $\vec{E}$ and $\vec{B}$ in terms of the Lorentz force density</u>

The Lorentz force, $m\vec{a} = q\vec{E} + q\vec{v}\times\vec{B}$, can be written in terms of mass density, ρ_m, and charge density, ρ_c, as

$$\rho_m\vec{a} = \rho_c\vec{E} + \rho_c\vec{v}\times\vec{B} \tag{2995}$$

Using $\vec{a} = \dot{\vec{v}}$, the current density, $\vec{J} = \rho_c\vec{v}$, and the conversion, $\rho_m = \dfrac{q}{m}\rho_c$, gives

$$\boxed{\begin{array}{c} \dfrac{q}{m}\dot{\vec{J}} = \rho_c\vec{E} + \vec{J}\times\vec{B} \\[2ex] \textit{Lorentz force density in terms of current density} \end{array}} \tag{2996}$$

Inserting this into the electric field wave equation (2989) gives

$$\nabla^2\vec{E} - \frac{1}{c^2}\ddot{\vec{E}} \quad = \quad \varepsilon\nabla\rho + \frac{m\mu}{q}\left(\rho_c\vec{E} + \vec{J}\times\vec{B}\right) \tag{2997}$$

Using complex sinusoid functions for the fields and sources gives

$$\nabla^2\vec{E} - \frac{1}{c^2}\ddot{\vec{E}} \quad = \quad \varepsilon\nabla\rho + \frac{m\mu}{q}\left(\rho_c\vec{E} + \vec{J}\times\vec{B}\right) \tag{2998}$$

Appendix S

The frequency of gravitational waves from a single mass oscillator

Here we show that a single mass oscillator with frequency ω will produce gravitational waves with double the frequency, 2ω. Consider a particle of mass m oscillating along the z-axis according to $z = z_o cos\omega t$. The gravitational radiation power due to the oscillator is given by

$$P = -\frac{G}{5c^9}\left(\dddot{Q}^{ij}\dddot{Q}_{ij}\right) \tag{2999}$$

where Q^{ij} is the mass quadrupole moment given by

$$Q^{ij} = \int x^i x^j T^{00} d^3x \tag{3000}$$

Since the mass is oscillating along the z-axis according to $z = z_o cos\omega t$, then we can describe T^{00} with delta functions.

$$T^{00} = mc^2\delta(x)\,\delta(y)\,\delta(z - z_0\cos\omega t) \tag{3001}$$

Then (3017) gives

$$Q^{ij} = \int x^i x^j mc^2\delta(x)\,\delta(y)\,\delta(z - z_0\cos\omega t)\,d^3x \tag{3002}$$

Note that the delta functions in x and y are zero for all x and y except when $x = y = 0$. However, when x^i or x^j are zero, then the integral will become zero. Thus when i, j, or both have values of 1 or 2 then the integral will become zero. So we are left with only Q^{33}. Taking the integral of the delta functions will simply give the integrand.

$$Q^{33} = z^2mc^2 \tag{3003}$$

$$= z_0^2\cos^2(\omega t)\,mc^2 \tag{3004}$$

Using a half-angle formula gives

$$Q^{33} = z_0^2mc^2\left[\frac{1}{2} + \frac{1}{2}\cos(2\omega t)\right] \tag{3005}$$

Taking the first, second, and third time derivatives gives

$$\dot{Q}^{33} = z_0^2mc^2\left[-\omega\sin(2\omega t)\right] \tag{3006}$$

$$\ddot{Q}^{33} = -2z_0^2mc^2\omega^2\cos(2\omega t) \tag{3007}$$

$$\dddot{Q}^{33} = 4z_0^2mc^2\omega^3\sin(2\omega t) \tag{3008}$$

Substituting (3008) into (2999) gives

$$P = -\frac{G}{5c^9}\left[4z_0^2mc^2\omega^3\sin(2\omega t)\right]^2 \tag{3009}$$

$$\boxed{P = -\frac{16G}{5c^5}z_0^4m^2\omega^6\sin^2(2\omega t)} \tag{3010}$$

As expected, we find that the power of the gravitational waves has a frequency that is double the frequency of oscillation of the source given by $z = z_o cos\omega t$. If we take a time-average over several periods then $\left\langle\sin^2(2\omega t)\right\rangle = 1/2$ so we have

$$\langle P\rangle = -\frac{8G}{5c^5}z_0^4m^2\omega^6 \tag{3011}$$

Appendix T

Charge/mass dipole and quadrupole moments of a Lorentz oscillator

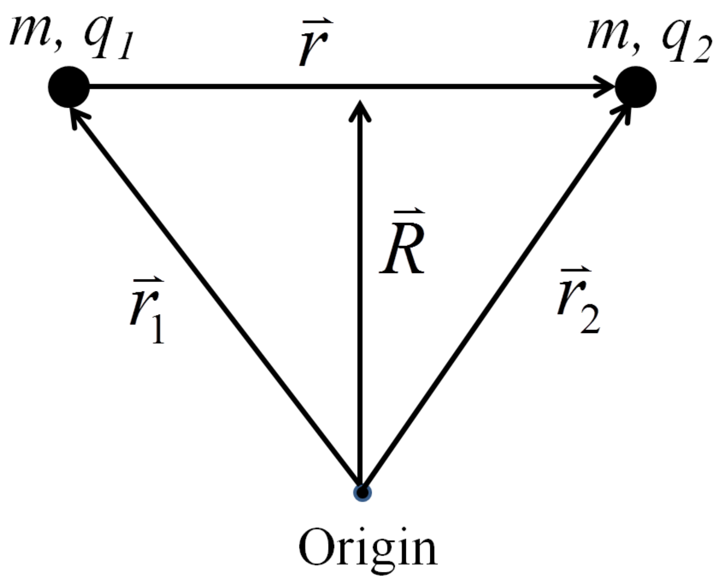

Figure 32: Two spherical charged masses are shown with a displacement vector $\vec{r}$ between them. The particle at position $\vec{r}_1$ has mass m and charge q_1, while the particle at position $\vec{r}_2$ has mass m and charge q_2. The positions $\vec{r}_1$ and $\vec{r}_2$ are equidistant from an arbitrary origin. The vector $\vec{R}$ gives the position of the center of mass of the system as measured from the origin.

The charge dipole moment for the arrangement in the figure above can be found as

$$\vec{p} = \sum_{i=1}^{2} q_i \vec{r}_i = q_1 \vec{r}_1 + q_2 \vec{r}_2 \tag{3012}$$

Note that $\vec{r}_1 = \vec{R} - \frac{1}{2}\vec{r}$ and $\vec{r}_2 = \vec{R} + \frac{1}{2}\vec{r}$. Then the dipole moment becomes

$$\vec{p} = \left[q_1 \left(\vec{R} - \frac{1}{2}\vec{r} \right) + q_2 \left(\vec{R} + \frac{1}{2}\vec{r} \right) \right] \tag{3013}$$

If the charges are identical, then we can write $q_1 = q_2 = q$, and we have

$$\vec{p}_{(q_1=q_2)} = 2q\vec{R} \tag{3014}$$

If the charges equal and opposite, then $q_1 = -q_2 = -q$ and (3013) becomes

$$\vec{p}_{(q_1=-q_2)} = -q \left[\left(\vec{R} - \frac{1}{2}\vec{r} \right) + \left(\vec{R} + \frac{1}{2}\vec{r} \right) \right] \tag{3015}$$

$$= 2q\vec{r} \tag{3016}$$

The charge quadrupole moment can also be found as[296]

$$Q = 2\sum_{i=1}^{2} q_i \vec{r}_i^{\,2} = 2\left(q_1\vec{r}_1^{\,2} + q_2\vec{r}_2^{\,2}\right) \tag{3017}$$

Using $\vec{r}_1 = \vec{R} - \frac{1}{2}\vec{r}$ and $\vec{r}_2 = \vec{R} + \frac{1}{2}\vec{r}$ gives

$$Q = 2\left[q_1\left(\vec{R} - \frac{1}{2}\vec{r}\right)^2 + q_2\left(\vec{R} + \frac{1}{2}\vec{r}\right)^2\right] \tag{3018}$$

$$= 2\left[q_1\left(\vec{R}^2 - \vec{R}\cdot\vec{r} + \frac{1}{4}\vec{r}^2\right) + q_2\left(\vec{R}^2 + \vec{R}\cdot\vec{r} + \frac{1}{4}\vec{r}^2\right)\right] \tag{3019}$$

If the charges are identical, then we can write $q_1 = q_2 = q$ and we have

$$Q_{(q_1=q_2)} = q\left(4\vec{R}^2 + \vec{r}^2\right) \tag{3020}$$

If the charges equal and opposite, then $q_1 = -q_2 = q$ and (3019) becomes

$$Q_{(q_1=-q_2)} = 4q\vec{R}\cdot\vec{r} \tag{3021}$$

We can also perform this same analysis for a *mass* dipole moment, $\vec{p}_m$, and a *mass* quadrupole moment, Q_m, given respectively as

$$\vec{p}_m = \sum_{i=1}^{2} m_i\vec{r}_i \qquad \text{and} \qquad Q_m = 2\sum_{i=1}^{2} m_i\vec{r}_i^{\,2} \tag{3022}$$

Since the mass is the same for both particles, then the mass dipole and mass quadrupole will have the same form as (3014) and (3020), respectively.

$$\vec{p}_m = 2m\vec{R} \qquad \text{and} \qquad Q_m = m\left(4\vec{R}^2 + \vec{r}^2\right) \tag{3023}$$

Now we consider the particles accelerating in anti-symmetric directions along the line between them and evaluate the time-derivative of the moments. Since the center of charge/mass does not move relative to the origin, then $\dot{\vec{R}} = 0$. The time-derivative of the charge dipole moment in (3014) will therefore be zero. Likewise, the time-derivative of the mass dipole moment in (3023) will also be zero (as required by conservation of linear momentum). Notice that $Q_{(q_1=-q_2)}$ in (3021) depends on $\vec{R}$ which is defined using an arbitrary coordinate system. This means it is possible to choose a coordinate system where $\vec{R} = 0$ (that is, choosing the origin to be at the center-of-charge of the system). Therefore, $Q_{(q_1=-q_2)}$ is not a coordinate free quantity and can be omitted. Then the time-derivatives of the remaining charge and mass moments in (3016), (3020) and (3023) will be, respectively,

$$\dot{\vec{p}}_{(q_1=-q_2)} = 2q\dot{\vec{r}}, \qquad \dot{Q}_{(q_1=q_2)} = 2q\vec{r}\cdot\dot{\vec{r}}, \qquad \dot{Q}_m = 2m\vec{r}\cdot\dot{\vec{r}} \tag{3024}$$

[296]In Jackson [40] (equ. 4.9), we find that formally the electric quadrupole moment is a *tensor* quantity given by $Q_{ij} = \int\rho(\vec{r})\left(3r_ir_j - r^2\delta_{ij}\right)d^3r$. For discreet charges, this becomes $Q_{ij} = \sum_k q_k\left(3r_{ik}r_{jk} - r_k^2\delta_{ij}\right)$. For charges on a single axis we have $i = j$ and therefore Q_{ij} can be written as the *scalar* quantity shown in (3017).

We can also consider the *second* time-derivatives of the charge and mass moments above.

$$\ddot{\vec{p}}_{(q_1=-q_2)} \quad = \quad 2q\ddot{\vec{r}}, \qquad \ddot{Q}_{(q_1=q_2)} = 2q\left(\dot{\vec{r}}^2 + \vec{r}\cdot\ddot{\vec{r}}\right), \qquad \ddot{Q}_m = 2m\left(\dot{\vec{r}}^2 + \vec{r}\cdot\ddot{\vec{r}}\right) \tag{3025}$$

Since quadrupole radiation involves the third time-derivative of the quadrupole moment, then we can take another time-derivative of the quadruples to obtain

$$\dddot{Q}_{(q_1=q_2)} \quad = \quad 2q\left(3\dot{\vec{r}}\ddot{\vec{r}} + \vec{r}\cdot\dddot{\vec{r}}\right), \qquad \dddot{Q}_m = 2m\left(3\dot{\vec{r}}\ddot{\vec{r}} + \vec{r}\cdot\dddot{\vec{r}}\right) \tag{3026}$$

Therefore, we find that for a Lorentz oscillator, the electromagnetic radiation can be dipolar to lowest order (if the charges are oppositely signed) or quadrupolar to lowest order (if the charges have the same sign). We also find that the gravitational radiation is quadrupolar to lowest order. These results can be summarized by writing the following charge and mass moments which are the only moments that are relevant for radiation by a Lorentz oscillator.

$$\boxed{\ddot{\vec{p}}_{(q_1=-q_2)} = 2q\ddot{\vec{r}}, \qquad \dddot{Q}_{(q_1=q_2)} = 2q\left(3\dot{\vec{r}}\ddot{\vec{r}} + \vec{r}\cdot\dddot{\vec{r}}\right), \qquad \dddot{Q}_m = 2m\left(3\dot{\vec{r}}\ddot{\vec{r}} + \vec{r}\cdot\dddot{\vec{r}}\right)} \tag{3027}$$

Appendix U

Standing wave from the superposition of opposite traveling waves

In (2280) and (2281) we combine the waves propagating in both z-directions to obtain

$$\vec{E} = \vec{E}_+ + \vec{E}_- = E_0 \left[\cos(kz - \omega t) + \cos(kz + \omega t)\right]\hat{y} \tag{4}$$

$$\vec{B} = \vec{B}_+ + \vec{B}_- = \frac{E_0}{c}\left[\cos(kz - \omega t) - \cos(kz + \omega t)\right]\hat{x} \tag{5}$$

For (2280) we can use the trigonometric identity $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$. This gives

$$\begin{aligned} \cos(kz - \omega t) + \cos(kz + \omega t) &= \left[\cos(kz)\cos(\omega t) + \sin(kz)\sin(\omega t)\right] \\ &\quad + \left[\cos(kz)\cos(\omega t) - \sin(kz)\sin(\omega t)\right] && (3028) \\ &= 2\cos(kz)\cos(\omega t) && (3029) \end{aligned}$$

Therefore we can write the standing electric field wave as

$$\boxed{\vec{E} = 2E_0 \cos(kz)\cos(\omega t)\,\hat{y}} \tag{3030}$$

For (2281) again we use the trigonometric identity $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$. This gives

$$\begin{aligned} \cos(kz - \omega t) - \cos(kz + \omega t) &= \left[\cos(kz)\cos(\omega t) + \sin(kz)\sin(\omega t)\right] \\ &\quad - \left[\cos(kz)\cos(\omega t) - \sin(kz)\sin(\omega t)\right] && (3031) \\ &= 2\sin(kz)\sin(\omega t) && (3032) \end{aligned}$$

Therefore we can write the standing magnetic field wave as

$$\boxed{\vec{B} = \frac{2E_0}{c}\sin(kz)\sin(\omega t)\,\hat{x}} \tag{3033}$$

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