Title
Dynamic Models of Appraisal Networks Explaining Collective Learning

Journal
IEEE Transactions on Automatic Control, 63(9)

ISSN
0018-9286

Authors
Mei, Wenjun
Friedkin, Noah E
Lewis, Kyle
et al.

Publication Date
2018-09-01

DOI
10.1109/tac.2017.2775963

Peer reviewed

Dynamic Models of Appraisal Networks Explaining Collective Learning

Wenjun Mei, Noah E. Friedkin, Kyle Lewis, and Francesco Bullo, *Fellow, IEEE*

***Abstract*—This paper proposes models of learning processes in teams of individuals who collectively execute a sequence of tasks and whose actions are determined by individual skill levels and networks of interpersonal appraisals and influence. The closely-related proposed models have increasing complexity, starting with a centralized manager-based assignment and learning model, and finishing with a social model of interpersonal appraisal, assignments, learning, and influences. We show how rational optimal behavior arises along the task sequence for each model, and discuss conditions of suboptimality. Our models are grounded in replicator dynamics from evolutionary games, influence networks from mathematical sociology, and transactive memory systems from organization science.**

***Index Terms*—Appraisal networks, collective learning, evolutionary games, influence networks, multiagent systems, replicator dynamics, transactive memory systems (TMS).**

I. Introduction

A. Transactive Memory System(TMS) in Applied Psychology

RESEARCHERS in sociology, psychology, and organization science have long studied the inner functioning and performance of teams with multiple individuals engaged in tasks. Extensive qualitative studies, conceptual model,s and empirical studies in the laboratory and field reveal some statistical features and various phenomena of teams [15], [17], [32], [33], but only a few quantitative and mathematical models are available [1], [20].

Manuscript received September 28, 2016; revised June 29, 2017 and October 31, 2017; accepted November 13, 2017. Date of publication November 20, 2017; date of current version August 28, 2018. This work was supported in part by the U.S. Army Research Laboratory and in part by the U.S. Army Research Office under Grant W911NF-15-1-0577 and Grant W911NF-16-1-0005. This paper was presented in part at the 55th Conference on Decision and Control, Las Vegas, NV, USA, December 12–14, 2016. Recommended by Associate Editor M. Cao.
(Corresponding author: Wenjun Mei.)

W. Mei and F. Bullo are with the Department of Mechanical Engineering and the Center for Control, Dynamical Systems, and Computation, University of California at Santa Barbara, Santa Barbara, CA 93106 USA (e-mail: meiwenjunbd@gmail.com; bullo@engineering.ucsb.edu).

N. E. Friedkin is with the Department of Sociology and the Center for Control, Dynamical Systems, and Computation, University of California at Santa Barbara, Santa Barbara, CA 93106 USA (e-mail: friedkin@soc.ucsb.edu).

K. Lewis is with the Technology Management Program, University of California at Santa Barbara, Santa Barbara, CA 93106 USA (e-mail: klewis@tmp.ucsb.edu).

Digital Object Identifier 10.1109/TAC.2017.2775963

A TMS is a conceptual model of team learning and performance well-established in organization science, see the seminal work by Wegner *et al.* [29] and other highly cited works [4], [15], [17], [30]. A TMS is a collective "memory" system that emerges in teams engaged in tasks, as the team members develop the collective knowledge on who possesses what expertise. TMS facilitates coordination and division of labor. Empirical research across a range of team types and settings [16], [17], [34], as well as some early simulation-based computational models [1], [25], [26], demonstrates a strong positive relationship between the development of a TMS and team performance. However, the mechanisms through which team members come to share an understanding of the distribution of expertise is typically treated as "black box" processes in TMS research. It remains an open problem how to mathematically characterize the TMS-related social and cognitive processes, such as the division of labor and the evolution of collective knowledge.

B. Problem Description

In this paper, we propose a class of multiagent dynamical systems as mathematical formalizations of some important aspects of the TMS theory. We consider a natural social process, in which a team of individuals, with unknown skill levels, is completing a sequence of tasks. Each task is completed by subdividing it into subtasks with different workloads and assigning one subtask to each team member. The team performance is maximized when the workload assignments are proportional to the individuals' underlying skill levels. We adopt the concept of *appraisal network*, or equivalently its corresponding row-stochastic appraisal matrix, to model the TMS of the team. The appraisal network represents how the team members evaluate each other's underlying skill level. The dynamics of the appraisal matrix is as follows: First, after completing the task, each individual receives a feedback signal equal to the deviation of her/his own performance from the weighted average performance of a subset of observed individuals. Second, based on the feedback signal, each individual adjusts her/his own appraisal and the appraisals of other team members. Third, the appraisal network may or may not be updated via an interpersonal influence process. Fourth, the workload division for the next tasks is computed as a function of the appraisal matrix. The evolution of the appraisal network corresponds to the development of a team's TMS. This paper aims to mathematically formalize this four-step process and investigate the conditions under which:

i) the team as an whole achieves asymptotically the optimal workload assignment;
ii) each individual learns asymptotically the true relative skill levels of all the team members; and
iii) the learning fails to occur.

We refer to property (ii) as *collective learning*.

C. Literature Review

To the best of our knowledge, this paper is the first attempt to model the development of TMS as a multiagent system and provide rigorous conditions for collective learning. To the best of our knowledge, the only related previous works are the computational models proposed by Palazzolo *et al.* [25], Ren *et al.* [26], and Anderson *et al.* [1]. The model in [1] is a two-dimension (2-D) ordinary differential equations system and treats the collective knowledge as a scalar variable, while the models in [25] and [26] are multiagent. Palazzolo *et al.* [25] consider time-varying skill levels. Ren *et al.* [26] consider multidimension skills and task requirements. Both models take into account numerous complicated and realistic individual/group actions, and the analysis of both models is based on simulation.

In our models, collective learning arises as the result of the coevolution of interpersonal appraisals and influence networks. Related previous work includes social comparison theory [7], averaging-based social learning [10], opinion dynamics [6], [9], [18], reflected appraisal mechanisms [8], [12], and the combined evolution of interpersonal appraisals and influence networks [11].

In the modeling and analysis of the evolution of appraisal and influence networks, we build an insightful connection between our model and the well-known replicator dynamics in evolutionary game theory; see the textbook [27], some control and optimization applications [2], [21], and the recent contributions [5], [19].

Our models are also marginally related to distributed optimization, e.g. [3], [23]. But, in this paper, we focus on modeling the natural social behavior of individuals. Moreover, the evolution of the decision variable, i.e., the workload assignment, is not directly modeled, but a byproduct of the dynamics for the appraisal network.

D. Contribution

First, based on a few natural assumptions, we propose three novel models with increasing complexity for the dynamics of teams: the manager dynamics, the assign/appraise dynamics, and the assign/appraise/influence dynamics. Without loosing mathematical tractability and intuitive insights, our work integrates several natural processes in a single model: the division of workload, the update of interpersonal appraisals via observation, and the opinion dynamics over the influence network. To the best of our knowledge, this is the first time that such an integration has been proposed and leads to rigorous and intuitive results. For the baseline manager dynamics, the workload assignment is adjusted in a centralized manner: the increase rate of workload assigned to an individual is equal to the deviation of his/her performance from the average. Under this intuitive assumption, the evolution of the workload assignment obeys the well-established replicator dynamics with novel fitness functions as the individual performances. The assign/appraise dynamics provides an insightful perspective on the connection between team performance and the appraisal network, by assuming that, instead of by the manager, the workload assignment is determined by the appraisal network in a social and distributed manner. The update of the appraisals is driven by the individuals' heterogeneous performance feedback. In the assign/appraise/influence dynamics model, we further incorporate the coevolution of appraisal and influence networks.

Second, we present comprehensive theoretical analysis on the dynamical properties of our models. For the assign/appraise dynamics and the assign/appraise/influence dynamics, we relate the models' asymptotic behavior with the connectivity property of the *observation network*, which defines the heterogeneous feedback signals each individual observes. Our theoretical results on the asymptotic behavior can be interpreted as the exploration of the most relaxed conditions for the emergence of asymptotic optimal workload assignment. Moreover, some theoretical results also reveal insightful interpretations that are consistent with the TMS theory studied in organization science. According to Lee *et al.* [14], in teams with well-developed TMS, members' agreements on the distribution of expertise facilitate high levels of coordination and division of labor, which a centralized manager might otherwise provide. In our paper, we prove that, along the assign/appraise dynamics and the assign/appraise/influence dynamics, the evolution of the workload assignment determined by the appraisal network does indeed satisfy the manager (a.k.a., replicator) dynamics in a generalized form. In addition, the assign/appraise/influence dynamics describes an emergence process by which team members' perception of "who knows what" become more similar over time, a fundamental feature of TMS [14], [24].

Third, besides the models in which the team eventually learns the individuals' true relative skill levels, we propose one variation in each of the three phases of the assign/appraise/influence dynamics: the assignment rule, the update of appraisal network based on feedback signal, and the opinion dynamics for the interpersonal appraisals. The variations reflect some sociological and psychological mechanisms known to prevent the team from learning. We investigate by simulation numerous possible causes of failure to learn.

E. Organization

The rest of this paper is organized as follows: Section II proposes our problem setup and centralized manager model; Section III introduces the assign/appraise dynamics. Section IV is the assign/appraise/influence model. Section V discusses some causes of failure to learn. Section VI provides some further discussions and conclusion. We put some preliminaries on evolutionary games and replicator dynamics in Appendix A.

II. Problem Setup and Manager Dynamics

In this section, we first mathematically formalize some concepts related to the social processes we aim to model, and

TABLE I
NOTATIONS FREQUENTLY USED IN THIS PAPER

$\succ$ ($\prec$ resp.)	entrywise greater than (less than resp.).
$\succeq$ ($\preceq$ resp.)	entrywise no less than (no greater than resp.).
$\mathbb{1}_n$ ($\mathbb{0}_n$ resp.)	n-dimension column vector with all entries equal to 1 (0 resp.)
$\boldsymbol{x}$	vector of individual skill levels, with $\boldsymbol{x} = (x_1, x_2, \dots, x_n)^\top \succ \mathbb{0}_n$ and $\boldsymbol{x}^\top \mathbb{1}_n = 1$.
$\boldsymbol{w}$	workload assignment. $\boldsymbol{w} \succ \mathbb{0}_n$ and $\boldsymbol{w}^\top \mathbb{1}_n = 1$
f	a concave, continuously differentiable and increasing function f: $[0, +\infty) \to [0, +\infty)$
$\boldsymbol{p}(\boldsymbol{w})$	vector of individual performances. $\boldsymbol{p}(\boldsymbol{w}) = \big(p_1(\boldsymbol{w}), \dots, p_n(\boldsymbol{w})\big)^\top$, where $p_i(\boldsymbol{w}) = f(w_i/x_i)$ is the performance of individual i.
A	appraisal matrix. $A = (a_{ij})_{n\times n}$, where a_{ij} is individual i's appraisal of j's skill level.
W	influence matrix. $W = (w_{ij})_{n\times n}$, where w_{ij} is the weight individual i assigns to j's opinion.
Δ_n	the n-simplex $\{\boldsymbol{y} \in \mathbb{R}^n \,\vert\, \boldsymbol{y}^\top \mathbb{1}_n = 1,\ \boldsymbol{y} \succeq \mathbb{0}_n\}$.
$\mathrm{int}(\Delta_n)$	the relative interior of Δ_n, i.e., $\mathrm{int}(\Delta_n) = \{\boldsymbol{y} \in \mathbb{R}^n \,\vert\, \boldsymbol{y}^\top \mathbb{1}_n = 1,\ \boldsymbol{y} \succ \mathbb{0}_n\}$.
$\boldsymbol{v}_{\text{left}}(A)$	the left dominant eigenvector of the non-negative and irreducible matrix A, i.e., the normalized entry-wise positive left eigenvector associated with the eigenvalue equal to A's spectral radius.
$G(B)$	the directed and weighted graph associated with the adjacency matrix $B \in \mathbb{R}^{n\times n}$.

illustrate them by a concrete example. Then, we introduce a baseline centralized model for team learning dynamics. Frequently used notations are listed in Table I.

A. Model Assumptions and Notations

1) Team, Tasks, and Assignments: The basic assumption on the individuals and the tasks are given below.

Assumption 1 (Team, Task Type, and Assignment): Consider a team of n individuals characterized by a fixed but unknown vector $\boldsymbol{x} = (x_1, \dots, x_n)^\top$ satisfying $\boldsymbol{x} \succ \mathbb{0}_n$ and $\boldsymbol{x}^\top \mathbb{1}_n = 1$, where each x_i denotes the *skill level* of individual i. The tasks being completed by the team are assumed to have the following properties:

1) the total workload of each task is characterized by a positive scalar and is fixed as 1 in this paper;
2) the task can be arbitrarily decomposed into n subtasks according to the *workload assignment* $\boldsymbol{w} = (w_1, \dots, w_n)^\top$, where each w_i is the subtask workload assigned to individual i. The workload assignment satisfies $\boldsymbol{w} \succ \mathbb{0}_n$ and $\boldsymbol{w}^\top \mathbb{1}_n = 1$. The subtasks are executed simultaneously.

The scalar skill levels can be interpreted in an abstract way as the individuals' overall abilities of contributing to the tasks, while the workload assignment corresponds to the individuals' relative responsibilities.

2) Individual Performance: The measure of individual performance is defined below.

Assumption 2 (Individual Performance): Given fixed skill levels, each individual i's performance, with the assignment $\boldsymbol{w}$, is measured by $p_i(\boldsymbol{w}) = f(x_i/w_i)$, where $f : [0, +\infty) \to [0, +\infty)$ is strictly concave, continuously differentiable and monotonically increasing.

The function f is assumed concave since it is widely adopted that the relation between the performance and individual ability obeys the power law, i.e., $f(x) \sim x^\gamma$, with $\gamma \in (0, 1)$ [1]. The specific form $f(\frac{x_i}{w_i})$ could be generalized by adopting different measures of x_i and w_i.

3) Optimal Assignment: It is reasonable to claim that, in a well-functioning team, individuals' relative responsibilities, characterized by the workload assignment, should be proportional to their true relative abilities. We thereby refer to $\boldsymbol{w}^* = \boldsymbol{x}$ as the *optimal assignment*. There are various team performance models for which $\boldsymbol{w}^*$ is the unique optimal solution in Δ_n. For example, define the measure of the mismatch between workload assignment and individual's true skill levels as $\mathcal{H}_1(\boldsymbol{w}) = \sum_{i=1}^n |\frac{w_i}{x_i} - 1|$. This mismatch is minimized at $\boldsymbol{w}^*$. Alternatively, if we define the team performance as the weighted average individual performance, i.e., $\mathcal{H}_2(\boldsymbol{w}) = \sum_{i=1}^n w_i f(\frac{x_i}{w_i})$, then the strict concavity of f implies that $\mathcal{H}_2(\boldsymbol{w})$ is maximized at $\boldsymbol{w}^* = \boldsymbol{x}$.

We introduce a simple and concrete example to illustrate the mathematical formalization introduced above.

Example (Intruder Detection Task): Consider a group of n individuals monitoring an environment. The environment is divided into numerous nonoverlapping regions with equal areas. Each region is monitored by a CCTV camera connected to its respective screen. The aim of the group is to detect the locations of randomly-appearing intruders via monitoring the screens. The appearance of the intruders is uniformly random in space and is a homogeneous Poisson process. An intruder is successfully detected if it is observed on a screen by one of the individuals within a certain time period since its appearance. The team performance over a given task period is the fraction of successfully detected intruders. The task is conducted in the follows way: each individual i monitors w_i number of screens and each screen is monitored by one and only one individual. Here, w_i is normalized such that $\sum_i w_i = 1$. Each individual i has an intrinsic but unknown normalized skill level x_i. Denote by $p_i(\boldsymbol{w})$ the probability that an intruder is successfully detected by individual i, given the division of cameras $\boldsymbol{w} \in \Delta_n$. This probability $p_i(\boldsymbol{w})$ increases with individual i's intrinsic skill level x_i and decreases with the number of screens monitored by i, i.e., w_i. A natural assumption is that $p_i(\boldsymbol{w}) = f(\frac{x_i}{w_i})$, where f is a concave and monotonically increasing function, with $f(0) = 0$ and $f(\infty) = 1$. One can check that the expected team performance is given by $\sum_i w_i f(\frac{x_i}{w_i})$, which is maximized at $\boldsymbol{w}^* = \boldsymbol{x}$.

B. Centralized Manager Dynamics

In this section, we introduce a continuous-time centralized model on the evolution of workload assignment, referred to as the *manager dynamics*. The diagram illustration is shown in Fig. 1(a). Suppose that, at each time t, a team is completing a task based on the assignment $\boldsymbol{w}(t)$. An outside manager observes the individuals' performance $\boldsymbol{p}\big(\boldsymbol{w}(t)\big)$. We adopt the intuitive assumption that the manager increases the workload assigned to individual i if her/his performance is above the weighted team average and vice versa. In addition, the sum of all the individuals' workloads remains 1. The manager is assumed to adjust

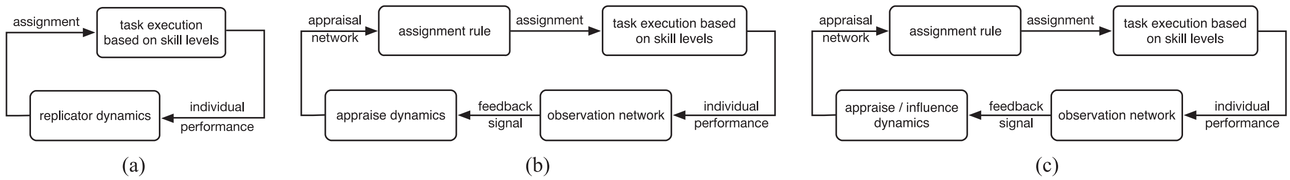


Fig. 1. Diagram illustrations of manager dynamics, assign/appraise dynamics, and assign/appraise/influence dynamics. (a) Manager. (b) Assign/appraise. (c) Assign/appraise/influence.

the workload assignment according to the replicator dynamics below, which is arguably the simplest model for the process described above

$$\dot{w}_i = w_i \left(p_i(\boldsymbol{w}) - \sum_{k=1}^{n} w_k p_k(\boldsymbol{w}) \right) \quad (1)$$

for any $i \in \{1, \dots, n\}$. Equation (1) takes the same form as the classic replicator dynamics from evolutionary game theory [5], [27], with the nonlinear fitness function $f(x_i/w_i)$. We refer to Appendix A for some preliminaries on evolutionary games and replicator dynamics.

Theorem 1 (Manager Dynamics): Consider (1) for the workload assignment as in Assumption 1 with performance as in Assumption 2. Then

1) the set $\text{int}(\Delta_n)$ is invariant;
2) for any $\boldsymbol{w}(0) \in \text{int}(\Delta_n)$, the manager's assignment $\boldsymbol{w}(t)$ converges to $\boldsymbol{w}^* = \boldsymbol{x}$, as $t \to \infty$.

The proof is given in Appendix B. We adopt the same Lyapunov function used for the asymptotic stability analysis of the replicator dynamics in [5] and [27]. The fitness function in the manager dynamics is novel.

III. Assign/Appraise Dynamics of the Appraisal Networks

Despite the desired property on the convergence of the workload assignment to optimality, the manager dynamics does not capture the evolution of the team's inner structures. In this section, we introduce a multiagent system, in which workload assignments are determined by the team members' interpersonal appraisals, rather than any outside authority, and the appraisal network is updated in a decentralized manner, driven by the team members' heterogeneous feedback signals.

A. Model Description and Problem Statement

1) Appraisal Network: Denote by a_{ij} the individual i's evaluation of j's skill levels and refer to $A = (a_{ij})_{n\times n}$ as the *appraisal matrix*. Since the evaluations are in the relative sense, we assume $A \succeq \mathbb{0}_{n\times n}$ and $A\mathbb{1}_n = \mathbb{1}_n$. The directed and weighted graph $G(A)$, referred to as the *appraisal network*, reflects the team's collective knowledge on the distribution of its members' abilities.

2) Assign/Appraise Dynamics: This multiagent model is illustrated by the diagram in Fig. 1(b). We model three phases: the workload assignment, the feedback signal, and the update of the appraisal network, specified by the following three assumptions, respectively.

Assumption 3 (Assignment Rule): At any time $t \geq 0$, the task is assigned according to the left dominant eigenvector of the appraisal matrix, i.e., $\boldsymbol{w}(t) = \boldsymbol{v}_{\text{left}}\big(A(t)\big)$.

Justification of Assumption 3 is given in Appendix C. For now, we assume $A(t)$ is row-stochastic and irreducible for all $t \geq 0$, so that $\boldsymbol{v}_{\text{left}}\big(A(t)\big)$ is always well defined. This will be proved later in this section.

Assumption 4 (Feedback Signal): After executing the workload assignment $\boldsymbol{w}$, each individual i observes, with no noise, the difference between her own performance and the quality of some part of the whole task, given by $\sum_k m_{ik} p_k(\boldsymbol{w})$, in which m_{ik} denotes the fraction of workload individual k contributes to the part of task observed by i. The matrix $M = (m_{ij})_{n\times n}$ defines a directed and weighted graph $G(M)$, referred to as the *observation network*, and satisfies $M \succeq \mathbb{0}_{n\times n}$ and $M\mathbb{1}_n = \mathbb{1}_n$ by construction.

The topology of the observation network defines the individuals' feedback signal structure. Notice that, the feedback signal for each individual i is only the deviation $p_i\big(\boldsymbol{w}(t)\big) - \sum_k m_{ik} p_k\big(\boldsymbol{w}(t)\big)$, while the matrix M is not necessarily known to the individuals.

Assumption 5 (Update of Interpersonal Appraisals): With the performance feedback signal defined as in Assumption 4, each individual i increases her self appraisal and decreases the appraisals of all the other individuals, if $p_i(\boldsymbol{w}) > \sum_k m_{ik} p_k(\boldsymbol{w})$, and vice versa. In addition, the appraisal matrix $A(t)$ remains row-stochastic.

The following dynamical system for the appraisal matrix, referred to as the *appraise dynamics*, is arguably the simplest model satisfying Assumptions 4 and 5:

$$\begin{cases} \dot{a}_{ii} = a_{ii}(1 - a_{ii}) \left(p_i(\boldsymbol{w}) - \sum_{k=1}^{n} m_{ik} p_k(\boldsymbol{w}) \right), \\ \dot{a}_{ij} = -a_{ii} a_{ij} \left(p_i(\boldsymbol{w}) - \sum_{k=1}^{n} m_{ik} p_k(\boldsymbol{w}) \right). \end{cases} \quad (2)$$

The matrix form of the appraise dynamics, together with the assignment rule as in Assumption 3, is given by

$$\begin{cases} \dot{A} = \text{diag}\big(\boldsymbol{p}(\boldsymbol{w}) - M\boldsymbol{p}(\boldsymbol{w})\big) A_{\text{d}}(I_n - A), \\ \boldsymbol{w} = \boldsymbol{v}_{\text{left}}(A), \end{cases} \quad (3)$$

and collectively referred to as the assign/appraise dynamics. Here, $A_{\text{d}} = \text{diag}(a_{11}, \dots, a_{nn})$.

3) Problem Statement: In Section III-B, we investigate the asymptotic behavior of dynamics (3), including:

1) convergence to the optimal assignment, which means that the team as an entirety eventually learns all its members' relative skill levels, i.e., $\lim_{t\to+\infty} \boldsymbol{w}(t) = \boldsymbol{x}$;
2) *appraisal consensus*, which means that the individuals asymptotically reach consensus on the appraisals of all the team members, i.e., $a_{ij}(t) - a_{kj}(t) \to 0$ as $t \to +\infty$, for any i, j, k.

Collective learning is the combination of the convergence to optimal assignment and appraisal consensus.

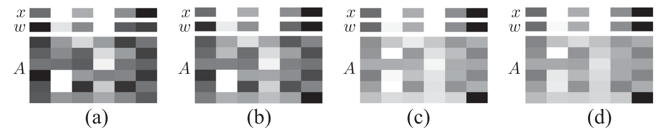


Fig. 2. Visualization of the evolution of $A(t)$ and $\boldsymbol{w}(t)$ obeying the assign/appraise dynamics with $n = 6$. The observation network is strongly connected. In these visualized matrices and vectors, the darker the entry, the higher value it has. (a) $t = 0$. (b) $t = 2$. (c) $t = 10$. (d) $t = 30$.

B. Dynamical Behavior of the Assign/Appraise Dynamics

We start by establishing that the appraisal matrix $A(t)$, as the solution to (3), is extensible to all $t \in [0, +\infty)$ and the assignment $\boldsymbol{w}(t)$ is well defined, in that $A(t)$ remains row-stochastic and irreducible. Moreover, some finite-time properties are investigated.

Theorem 2 (Finite-Time Properties of Assign/Appraise Dynamics): Consider the assign/appraise dynamics (3), based on Assumptions 3–5, describing a workload assignment as in Assumption 1, with performance as in Assumption 2. For any observation network $G(M)$, and any initial appraisal matrix $A(0)$ that is row-stochastic, irreducible and has strictly positive diagonal,

i) the appraisal matrix $A(t)$, as the solution to (3), is extensible to all $t \in [0, +\infty)$. Moreover, $A(t)$ remains row-stochastic, irreducible and has strictly positive diagonal for all $t \geq 0$;
ii) there exists a row-stochastic irreducible matrix $C \in \mathbb{R}^{n\times n}$ with zero diagonal such that
$$A(t) = \mathrm{diag}\big(\boldsymbol{a}(t)\big) + \big(I_n - \mathrm{diag}\big(\boldsymbol{a}(t)\big)\big)\, C \qquad (4)$$
for all $t \geq 0$, where $\boldsymbol{a}(t) = \big(a_1(t), \dots, a_n(t)\big)^\top$ and $a_i(t) = a_{ii}(t)$, for $i \in \{1, \dots, n\}$;
iii) define the *reduced assign/appraise dynamics* as
$$\begin{cases} \dot{a}_i = a_i(1-a_i)\left(p_i(\boldsymbol{w}) - \displaystyle\sum_{k=1}^{n} m_{ik} p_k(\boldsymbol{w})\right), \\ w_i = \dfrac{c_i}{(1-a_i)} \Big/ \displaystyle\sum_{k=1}^{n} \dfrac{c_k}{(1-a_k)}, \end{cases} \qquad (5)$$
where $\boldsymbol{c} = (c_1, \dots, c_n)^\top = \boldsymbol{v}_{\text{left}}(C)$. This dynamics is equivalent to system (3) in the following sense: The matrix $A(t)$'s each diagonal entry $a_{ii}(t)$ satisfies the dynamics (5) for $a_i(t)$, and, for any $t \geq 0$, $a_{ii}(t) = a_i(t)$ for any i, and $a_{ij}(t) = a_{ij}(0)\big(1 - a_i(t)\big)/\big(1 - a_i(0)\big)$ for any $i \neq j$;
iv) the set $\Omega = \big\{\boldsymbol{a} \in [0,1]^n \,\big|\, 0 \leq a_i \leq 1 - \zeta_i\big(\boldsymbol{a}(0)\big)\big\}$, where $\zeta_i\big(\boldsymbol{a}(0)\big) = \frac{c_i}{x_i} \min_k \frac{x_k}{c_k}\big(1 - a_k(0)\big)$, is a compact positively invariant set for the reduced assign/appraise dynamics (5);
v) the assignment $\boldsymbol{w}(t)$ satisfies the *generalized replicator dynamics* with time-varying fitness function $a_i(t)\Big(p_i\big(\boldsymbol{w}(t)\big) - \sum_l m_{il} p_l\big(\boldsymbol{w}(t)\big)\Big)$ for each i
$$\dot{w}_i = w_i \left(a_i\left(p_i(\boldsymbol{w}) - \sum_{l=1}^{n} m_{il} p_l(\boldsymbol{w})\right) - \sum_{k=1}^{n} w_k a_k \left(p_k(\boldsymbol{w}) - \sum_{l=1}^{n} m_{kl} p_l(\boldsymbol{w})\right)\right). \qquad (6)$$

The proof for Theorem 2 is presented in Appendix D. With the extensibility of $A(t)$ and the finite-time properties, we now present the main theorem of this section.

Theorem 3 (Asymptotic Behavior of Assign/Appraise Dynamics): Consider the dynamics (3), based on Assumptions 3–5, with the workload assignment as in Assumption 1 and the performance as in Assumption 2. Assume the observation network $G(M)$ is strongly connected. For any initial appraisal matrix $A(0)$ that is row-stochastic, irreducible and has positive diagonal,

1) the solution $A(t)$ converges, i.e., there exists $A^* \in \mathbb{R}^{n\times n}$ such that $\lim_{t\to\infty} A(t) = A^*$;
2) the limit appraisal matrix A^* is row-stochastic and irreducible. Moreover, the workload assignment satisfies $\lim_{t\to\infty} \boldsymbol{w}(t) = \boldsymbol{v}_{\text{left}}(A^*) = \boldsymbol{x}$.

The proof is presented in Appendix E. Theorem 3 indicates that, the teams obeying the assign/appraise dynamics asymptotically achieves the optimal workload assignment, but do not necessarily reach appraisal consensus. Fig. 2 gives a visualized illustration of the asymptotic behavior of the assign/appraise dynamics.

Remark 4: From the proof for Theorem 3, we know that, the teams obeying the following dynamics
$$\begin{cases} \dot{a}_{ii} = \gamma_i(t) a_{ii}(1-a_{ii})\big(p_i(\boldsymbol{w}) - \sum_k m_{ik} p_k(\boldsymbol{w})\big), \\ \dot{a}_{ij} = -\gamma_i(t) a_{ii} a_{ij}\big(p_i(\boldsymbol{w}) - \sum_k m_{ik} p_k(\boldsymbol{w})\big), \end{cases}$$
also asymptotically achieve the optimal assignment, if each $\gamma_i(t)$ remains strictly bounded from 0. This result indicates that our model can be generalized to the case of heterogeneous sensitivities to performance feedback.

IV. Assign/Appraise/Influence Dynamics of the Appraisal Networks

In this section, we further elaborate the assign/appraise dynamics by assuming that the appraisal network is updated via not only the performance feedback, but also the influence process inside the team.

A. Model Description

The new model, named the *assign/appraise/influence dynamics*, is defined by three components: the assignment rule as in Assumption 3, the appraise dynamics based on Assumptions 4 and 5, and the *influence dynamics*, which is the opinion exchanges among individuals on the interpersonal appraisals. Denote by w_{ij} the weight individual i assigns to j (including self weight w_{ii}) in the opinion exchange. The matrix $W = (w_{ij})_{n\times n}$ defines a directed and weighted graph, referred to as the *influence network*, is row-stochastic and possibly time-varying.

The diagram illustration of assign/appraise/influence dynamics is presented in Fig. 1(c), and the general form is given as follows:

$$\begin{cases} \dot{A} = \frac{1}{\tau_{\text{ave}}} F_{\text{ave}}(A, W) + \frac{1}{\tau_{\text{app}}} F_{\text{app}}(A, \boldsymbol{w}), \\ \boldsymbol{w} = \boldsymbol{v}_{\text{left}}(A). \end{cases} \quad (7)$$

The time index t is omitted for simplicity. The term $F_{\text{app}}(A, \boldsymbol{w})$ corresponds to the appraise dynamics given by the right-hand side of the first equation in (3), while the term $F_{\text{ave}}(A, W)$ corresponds to the influence dynamics specified by the assumption. Parameters τ_{ave} and τ_{app} are positive, and relate to the time scales of influence dynamics and appraise dynamics, respectively.

Assumption 6 (Influence Dynamics): For the assign/appraise/influence dynamics, assume that, at each time $t \geq 0$, the influence network is identical to the appraisal network, i.e., $W(t) = A(t)$. Moreover, assume that the individuals obey the classic DeGroot opinion dynamics [6] for the interpersonal appraisals, i.e., $F_{\text{ave}}(W, A) = -(I_n - W)A$.

Based on (7) and Assumptions 3–6, the assign/appraise/influence dynamics is written as

$$\begin{cases} \dot{A} = & \frac{1}{\tau_{\text{ave}}}(A^2 - A) \\ & + \frac{1}{\tau_{\text{app}}} \operatorname{diag}\big(\boldsymbol{p}(\boldsymbol{w}) - M\boldsymbol{p}(\boldsymbol{w})\big) A_{\text{d}}(I_n - A), . \\ \boldsymbol{w} = & \boldsymbol{v}_{\text{left}}(A), \end{cases} \quad (8)$$

In the following section, we relate the topology of the observation network $G(M)$ to the asymptotic behavior of the assign/appraise/influence dynamics, i.e., the convergence to optimal assignment and the appraisal consensus.

B. Dynamical Behavior of the Assign/Appraise/Influence Dynamics

The following lemma shows that, for the assign/appraise/influence dynamics, we only need to consider the all-to-all initial appraisal network.

Lemma 5 (Entry-Wise Positive for Initial Appraisal): Consider the assign/appraise/influence dynamics (8) based on Assumptions 3–6, with the workload assignment and performance as in Assumptions 1 and 2, respectively. For any initial appraisal matrix $A(0)$ that is primitive and row-stochastic, there exists $\Delta t > 0$ such that $A(t) \succ \mathbb{0}_{n\times n}$ for any $t \in (0, \Delta t]$.

The proof is given in Appendix F. Before discussing the asymptotic behavior, we state a technical assumption.

Conjecture 6 (Strict Lower Bound of the Interpersonal Appraisals): Consider the assign/appraise/influence dynamics (8) based on Assumptions 3–6, with the workload assignment and performance as in Assumptions 1 and 2, respectively. For any $A(0)$ that is entry-wise positive and row-stochastic, there exists $a_{\min} > 0$, depending on $A(0)$, such that $A(t) \succ a_{\min}\mathbb{1}_n\mathbb{1}_n^\top$ for any time $t \geq 0$, as long as $A(\tau)$ and $\boldsymbol{w}(\tau)$ are well defined for all $\tau \in [0, t]$.

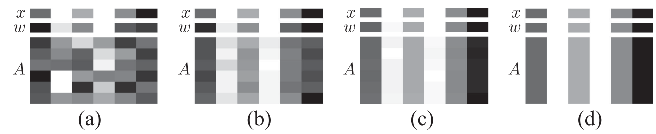


Fig. 3. Visualization of the evolution of $A(t)$ and $\boldsymbol{w}(t)$ obeying the assign/appraise/influence dynamics with $n = 6$. The observation network contains a globally reachable node. In these visualized matrices and vectors, the darker the entry, the higher value it has. (a) $t = 0$. (b) $t = 2$. (c) $t = 10$. (d) $t = 30$.

Monte Carlo validation and a sufficient condition for Conjecture 6 are presented in Appendix G. Now we state the main results of this section.

Theorem 7 (Assign/Appraise/Influence Dynamical Behavior): Consider the assign/appraise/influence dynamics (8) based on Assumptions 3–6, with the task assignment and performance as in Assumptions 1 and Assumption 2, respectively. Suppose that Conjecture 6 holds. Assume that the observation network $G(M)$ contains a globally reachable node. For any initial appraisal matrix $A(0)$ that is entrywise positive and row-stochastic, and any time scales $\tau_{\text{ave}} > 0$ and $\tau_{\text{app}} > 0$ in (8):

i) the solution $A(t)$ exists and $\boldsymbol{w}(t) = \boldsymbol{v}_{\text{left}}\big(A(t)\big)$ is well defined for all $t \in [0, +\infty)$. Moreover, $A(t) \succ \mathbb{0}_{n\times n}$ and $A(t)\mathbb{1}_n = \mathbb{1}_n$ for any $t \geq 0$;

ii) the assignment $\boldsymbol{w}(t)$ obeys the generalized replicator dynamics (6), and $\xi_0\mathbb{1}_n \preceq \boldsymbol{w}(t) \preceq \big(1 - (n-1)\xi_0\big)\mathbb{1}_n$, where

$$\xi_0 = \left(1 + (n-1)\frac{\max_k x_k}{\min_l x_l}\gamma_0\right)^{-1} \quad \text{and}$$

$$\gamma_0 = \frac{\max_k x_k / w_k(0)}{\min_l x_l / w_l(0)}$$

iii) as $t \to +\infty$, $A(t)$ converges to $\mathbb{1}_n\boldsymbol{x}^\top$ and thereby $\boldsymbol{w}(t)$ converges to $\boldsymbol{x}$.

The proof is given in Appendix H. As Theorem 7 indicates, the team obeying the assign/appraise/influence dynamics achieves collective learning. A visualized illustration of the dynamics is shown in Fig. 3.

Theorem 7 indicates that the asymptotic behavior of the assign/appraise/influence dynamics is independent of the time scales τ_{ave} and τ_{app}. The following argument adds some intuition to this observation. The assign/appraise/influence dynamics can be regarded a combination of the assign/appraise * dynamics (3) and the influence dynamics $\dot{A} = A^2$. As shown in Section III, for an appraisal matrix $A(t)$ obeying the assign/appraise dynamics (3), the left dominant eigenvector $\boldsymbol{v}_{\text{left}}(A(t))$ converges to the optimal assignment $\boldsymbol{x}$. Moreover, along the dynamics $\dot{A} = A^2$, the eigenvector $\boldsymbol{v}_{\text{left}}(A(t))$ remains unchanged. Theorem 7 states that the introduction of the influence dynamics does not affect the convergence of the left dominant eigenvector of $A(t)$ to $\boldsymbol{x}$.

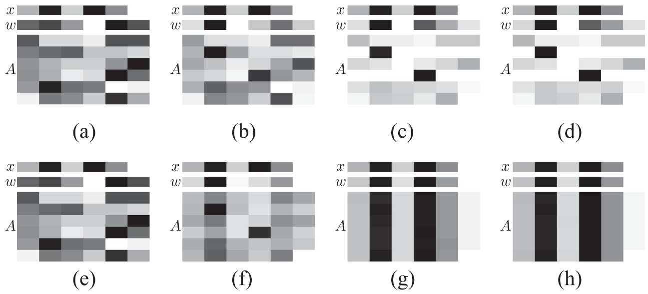


Fig. 4. Examples of the assign/appraise (first row) and the assign/appraise/influence (second row) dynamics in which the assignment is based on the individuals' in-degree centrality. The assign/appraise dynamics does not achieve the collective learning, while the assign/appraise/influence dynamics does. (a) no influence dynamics, $t = 0$. (b) no influence dynamics, $t = 2$. (c) no influence dynamics, $t = 30$. (d) no influence dynamics, $t = 50$. (e) with influence dynamics, $t = 0$. (f) with influence dynamics, $t = 2$. (g) with influence dynamics, $t = 30$. (h) with influence dynamics, $t = 50$.

V. Model Variations: Causes of Failure to Learn

The assign/appraise/influence dynamics (8) consists of three phases: the assignment rule, the appraise dynamics, and the influence dynamics. In this section, we propose one variation in each of the three phases, based on some sociopsychological mechanisms that may cause a failure in team learning. We investigate the behavior of each model variation by numerical simulation.

A. Variation in the Assignment Rule: Workload Assignment Based on Degree Centrality

In Assumption 3, the workload assignment is based on the individuals' eigenvector centrality in the appraisal network. If we assume instead that the assignment is based on the individuals' normalized in-degree centrality in the appraisal network, i.e., $\boldsymbol{w}(t) = A^\top(t)\mathbb{1}_n/\mathbb{1}_n^\top A(t)\mathbb{1}_n$, then the numerical simulation, see Fig. 4, shows the following results: the team obeying the assign/appraise dynamics does not necessarily achieve collective learning, while the team obeying the assign/appraise/influence dynamics still achieves collective learning.

B. Variation in the Appraise Dynamics: Partial Observation of Performance Feedback

According to Assumption 4, the observation network $G(M)$ determines the feedback signals received by each individual. If the observation network does not have the desired connectivity property, the individuals do not have sufficient information to achieve collective learning. Simulation results in Fig. 5 shows that, if $G(M)$ is not strongly connected for the assign/appraise dynamics, or if $G(M)$ does not contain a globally reachable node for the assign/appraise/influence dynamics, the team does not necessarily achieve collective learning.

C. Variation in the Influence Dynamics: Prejudice Model

In Assumption 6, we assume that the individuals obey the DeGroot opinion dynamics. If we instead adopt the Friedkin–Johnsen opinion dynamics, given by

$$F_{\text{ave}}(A, W) = -\Lambda(I_n - W)A + (I_n - \Lambda)(A(0) - A)$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ and each λ_i characterizes individual i's attachment to her initial appraisals. Numerical simulation, see Fig. 6, shows that the team does not necessarily achieve collective learning. The Friedkin–Johnsen model captures the social-psychological mechanism, in which individuals show an attachment to their initial opinions, which causes the failure to learn.

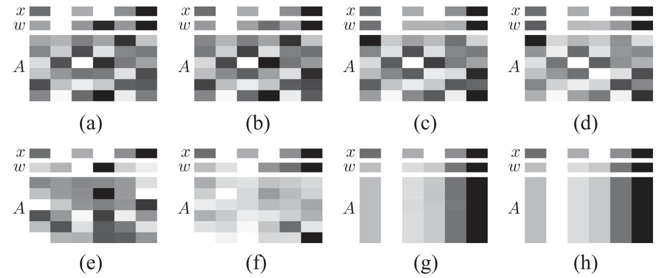


Fig. 5. Examples of failure to learn with partial observation for a six-individual team. The figures in the first row correspond to the assign/appraise dynamics, in which the observation network is not strongly connected but contains a globally reachable node. The figures in the second row correspond to the assign/appraise/influence dynamics, in which the observation network does not contain a globally reachable node. In both cases, $A(t)$ converges but $\lim_{t\to+\infty} \boldsymbol{w}(t) \neq \boldsymbol{x}$. (a) No influence dynamics, $t = 0$. (b) No influence dynamics, $t = 1$. (c) No influence dynamics, $t = 5$. (d) No influence dynamics, $t = 10$. (e) With influence dynamics, $t = 0$. (f) With influence dynamics, $t = 5$. (g) With influence dynamics, $t = 50$. (h) With influence dynamics, $t = 60$.

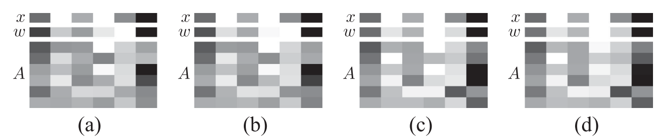


Fig. 6. Example of the evolution of $A(t)$ and $\boldsymbol{w}(t)$ in the prejudice model with $n = 6$. The darker the entry, the higher value it has. The simulation result shows that $A(t)$ converges but $\boldsymbol{w}(t) = \boldsymbol{v}_{\text{left}}\big(A(t)\big)$ does not necessarily converges to $\boldsymbol{x}$. (a) $t = 0$. (b) $t = 1$. (c) $t = 5$. (d) $t = 10$.

VI. Further Discussion and Conclusion

A. Connections With TMS Theory

1) TMS Structure: As discussed in the introduction, one important aspect of TMS is the members' shared understanding about who possess what expertise. For the case of 1-D skill, TMS structure is approximately characterized by the appraisal matrix, and thus, the development of TMS corresponds to the collective learning on individuals' true skill levels. Simulation results in Fig. 7 compare the evolution of some features among the teams obeying the assign/appraise/influence model, the assign/appraise model, and the team that randomly assigns the subtasks, respectively. Fig. 7(a) shows that, for both the assign/appraise/influence dynamics and the assign/appraise dynamics, the team performance measure $\mathcal{H}_1(\boldsymbol{w})$, defined by the mismatch between workload assignment and individual skill

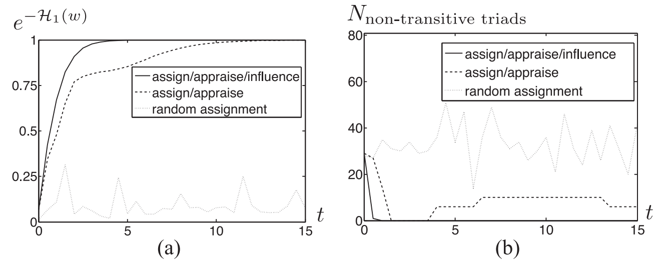


Fig. 7. Evolution of the measure of mismatch between assignment and individual skill levels, and the number of nontransitive triads in the comparative appraisal graph. The solid curves represent the team obeying the assign/appraise/influence dynamics. The dash curves represent the team obeying the assign/appraise dynamics. The dotted curves represent the team that randomly assign subtask workloads. (a) $e^{-\mathcal{H}_1(\boldsymbol{w},\boldsymbol{x})}$. (b) Number of non-transitive triads.

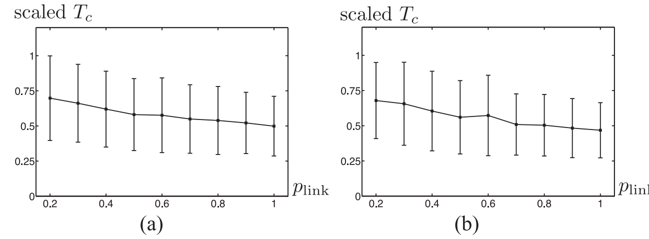


Fig. 8. Error bar plots for the mean convergence time of 100 random realizations, as a function of the link probability of the Erdős–Rényi observation network. The errors are set to be the standard deviation of the convergence time. Fig. 8(a) depicts the realizations for the assign/appraise dynamics, while Fig. 8(b) depicts the realizations for the assign/appraise/influence dynamics. (a) Assign/appraise. (b) Assign/appraise/influence.

levels, converges to 0, which exhibits the advantage of a developing TMS.

2) Transitive triads: As Palazzolo [24] points out, transitive triads are indicative of a well-organized TMS. The underlying logic is that inconsistency of interpersonal appraisals lowers the efficiency of locating the expertise and allocating the incoming information. In order to reveal the evolution of triad transitivity in our models, we define an unweighted and directed graph, referred to as the *comparative appraisal graph* $\widetilde{G}(A) = (V, E)$, with $V = \{1, \dots, n\}$, as follows: for any $i, j \in V$, $(i, j) \in E$ if $a_{ij} \geq a_{ii}$, i.e., if individual i thinks j has no lower skill level than i herself. We adopt the standard notion of triad transitivity and use the number of nontransitive triads as the indicator of inconsistency in a team. Fig. 7(b) shows that, the nontransitive triads vanish along the assign/appraise/influence dynamics, but persist along the assign/appraise dynamics or the random assignments.

B. Observation Network Structure and Learning Speed

Simulation results illustrate how the structure of the observation network affects the convergence speeds of our models, characterized by the *convergence time* $T_c = \min\left\{t \geq 0 \,\middle|\, e^{-\mathcal{H}_1(\boldsymbol{w}(t))} \geq 0.99\right\}$. T_c is a function of the skill level $\boldsymbol{x}$, the initial condition $A(0)$, and the observation network. We run 100 independent realizations of the assign/appraise dynamics for a team with 7 individuals. In each realization, we first randomly generate $\boldsymbol{x}$ and $A(0)$, and then randomly generate 9 strongly connected observation networks, $G_1, \dots, G_9$, where each G_i is an Erdős–Rényi graph with the link probability $p_{\text{link},i} = 0.2 + 0.1(i-1)$ and the individuals' out-degrees normalized to 1. With the same $\boldsymbol{x}$ and $A(0)$, we run the assign/appraise dynamics with the observation networks $G_1, \dots, G_9$, respectively, and denote by $T_{c,i}$ the convergence time with respect to the observation network G_i. In each realization, $T_{c,1}, \dots, T_{c,9}$ are scaled by dividing them by $\max_i T_{c,i}$. For the 100 realizations, we compute the mean value of each $T_{c,i}$ and plot it as a function of $p_{\text{link},i}$, see Fig. 8(a). The same simulation study has also been done for the assign/appraise/influence dynamics, see Fig. 8(b). Simulation results clearly indicate that, for both the assign/appraise and the assign/appraise/influence dynamics with Erdős–Rényi observation network, the convergence speed increases with the link probability.

C. Conclusion

This paper proposes a class of models closely connected with the TMS theory in organization science. We generalize from qualitative TMS theory the following two arguments, as the staring point of the mathematical modeling: first, team performance depends on whether the team members' relative responsibilities are proportional to their relative abilities in the team; second, the team members' relative responsibilities are determined by how they evaluate each other's relative ability. Theoretical analysis of the assign/appraise dynamics and the assign/appraise influence dynamics can be interpreted as the exploration of the most relaxed condition for the convergence to optimal workload assignment, concluded as follows:

i) each individual only needs to know, as feedback, the difference between her own performance and the average performance of some subgroup of individuals, but do not need to know exactly whom she is compared with;
ii) the individuals can have heterogeneous but strictly positive sensitivities to the performance feedback;
iii) with opinion exchange, the observation network with one globally reachable node is sufficient for the convergence to optimal assignment;
iv) without opinion exchange, strongly connected observation network is sufficient for the convergence to optimal assignment.

The theoretical results in this paper can be broadly interpreted as follows. First, we note that the connectivity requirement on the observation network for asymptotic optimal assignment is more relaxed in the assign/appraise/influence model than in the assign/appraise model. Therefore, our models lend credence to the argument that opinion exchanges inside the group can compensate for the lack of sufficiently-rich observation of performance feedback. Second, the numerical comparison between the assignment simply by the average appraisals, i.e., $\boldsymbol{w}(t) = \mathbb{1}_n^\top A(t)/n$, and the assignment by the appraisal centrality, i.e., $\boldsymbol{w}(t) = \boldsymbol{v}_{\text{left}}(A(t))$, shows that the former does not always leads to asymptotic optimal assignment in the assign/appraise dynamics, while the latter does. The main difference between these two assignment rules is that, for the assignment by the appraisal centrality, the opinions of the "highly-appraised individuals" on how the workload should be assigned are more important than those of the "lowly appraised," whereas, for the assignment by the average appraisals, all the individuals' opinions are equally important. The interpretation of this observation is that, in a well-functioning team, the individuals with higher appraisal should have higher weights in decision-making processes. Third, as illustrated by the numerical study of the causes of failure to learn, our models indicate that individuals' persistent attachment to their initial appraisals, i.e., prejudice, generally impedes collective learning and thus should be avoided in team tasks.

Future research directions might include more realistic models considering noisy observation and finite individual memory.

Appendix A
Preliminaries

Evolutionary games apply game theory to evolving populations adopting different strategies. Consider a game with n pure strategies, denoted by the unit vectors $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$, respectively. A mixed strategy $\boldsymbol{w}$ is thereby a vector in the n-dimension simplex denoted by Δ_n. Denote by $\pi(\boldsymbol{v}, \boldsymbol{w})$ the expected payoff for any mixed strategy $\boldsymbol{v}$ against mixed strategy $\boldsymbol{w}$. A strategy $\boldsymbol{w}^*$ is a locally *evolutionarily stable strategy* (ESS) if there exists a deleted neighborhood $\breve{U}(\boldsymbol{w}^*)$ in $\text{int}(\Delta_n)$ such that $\pi(\boldsymbol{w}^*, \boldsymbol{w}) > \pi(\boldsymbol{w}, \boldsymbol{w})$ for any $\boldsymbol{w} \in \breve{U}(\boldsymbol{w}^*)$, which implies that, in a population adopting strategy $\boldsymbol{w}$, a sufficiently small mutated subpopulation adopting strategy $\boldsymbol{w}^*$ gets more payoff than the majority population.

Replicator dynamics models the evolution of subpopulations adopting different strategies. The total population is divided into n subpopulations. Individuals in each subpopulation i adopt the pure strategy $\boldsymbol{e}_i$. Denote by $w_i(t)$ the fraction of subpopulation i in the total population at time t. The fitness of subpopulation i, denoted by $\pi_i\big(\boldsymbol{w}(t)\big)$, depends on the subpopulation distribution $\boldsymbol{w}(t) = \big(w_1(t), \dots, w_n(t)\big)^\top$ and is defined as the expected payoff $\pi\big(\boldsymbol{e}_i, \boldsymbol{w}(t)\big)$. The growth rate of subpopulation i is equal to the deviation of its fitness from the population average. The replicator dynamics is given by

$$\dot{w}_i = w_i\Big(\pi_i(\boldsymbol{w}) - \sum_{k=1}^n w_k \pi_k(\boldsymbol{w})\Big). \tag{9}$$

There is a simple connection between the locally ESS and the replicator dynamics [5]: generally, a locally ESS in $\text{int}(\Delta_n)$ is a locally asymptotic equilibrium of the replicator dynamics; specifically, if there exists a matrix A such that $\pi(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v}^\top A \boldsymbol{w}$ for any $\boldsymbol{v}, \boldsymbol{w} \in \Delta_n$, then, a locally ESS in $\text{int}(\Delta_n)$ is a globally asymptotic stable equilibrium of the replicator dynamics. In addition, the replicator dynamics is also a mean-field approximation of some stochastic population process, which is out of the scope of this paper.

Appendix B
Proof for Theorem 1

The vector form of (1) is written as

$$\dot{\boldsymbol{w}} = \text{diag}(\boldsymbol{w})\left(\boldsymbol{p}(\boldsymbol{w}) - \boldsymbol{w}^\top \boldsymbol{p}(\boldsymbol{w}) \mathbb{1}_n\right). \tag{10}$$

Left multiply both sides by $\mathbb{1}_n^\top$. We get $d(\mathbb{1}_n^\top \boldsymbol{w})/dt = 0$. Moreover, since $\dot{w}_i = 0$ whenever $w_i = 0$, the n-dimension simplex Δ_n is an positively invariant set.

Since the function f is continuously differentiable, the right-hand side of (10) is continuously differentiable and locally Lipschitz in $\text{int}(\Delta_n)$. Define

$$V(\boldsymbol{w}) = -\sum_{i=1}^n x_i \log \frac{w_i}{x_i}.$$

Due to the strict concavity of log function and $\mathbb{1}_n^\top \boldsymbol{w} = 1$, we have that $V(\boldsymbol{w}) \geq 0$ for any $\boldsymbol{w} \in \Delta_n$ and $V(\boldsymbol{w}) = 0$ if and only if $\boldsymbol{w} = \boldsymbol{x}$. Moreover, since $V(\boldsymbol{w})$ is continuously differentiable in $\boldsymbol{w}$, the level set $\{\boldsymbol{w} \in \text{int}(\Delta_n) | V(\boldsymbol{w}) = \xi\}$ is a compact subset of $\text{int}(\Delta_n)$. Since the function f is monotonically increasing, along the trajectory

$$\begin{aligned}\frac{dV(\boldsymbol{w})}{dt} = &- \sum_{i \in \theta_1(\boldsymbol{w})} (x_i - w_i) f(x_i/w_i) \\ &- \sum_{i \in \theta_2(\boldsymbol{w})} (x_i - w_i) f(x_i/w_i) < 0\end{aligned}$$

where $\theta_1(\boldsymbol{w}) = \{i | x_i \geq w_i\}$ and $\theta_2(\boldsymbol{w}) = \{i | x_i < w_i\}$. This concludes the proof for the invariant set and the asymptotic stability of $\boldsymbol{w}^* = \boldsymbol{x}$, and one can infer, from the inequality above, that $\boldsymbol{w}^* = \boldsymbol{x}$ is the ESS for the evolutionary game with the payoff function $\pi_i(\boldsymbol{w}) = f(x_i/w_i)$. Moreover, since $V(\boldsymbol{w}) \to +\infty$ as $\boldsymbol{w}$ tends to the boundary of Δ_n, the region of attraction is $\text{int}(\Delta_n)$.

Appendix C
Justifications of Assumption 3

We provide some justification of Assumption 3 on the workload assignment rule $\boldsymbol{w} = \boldsymbol{v}_{\text{left}}(A)$. First, the entries of $\boldsymbol{v}_{\text{left}}(A)$ correspond to the individuals' eigenvector centrality in the appraisal network and thus reflect how much each individual is appraised by the team. Second, each row i of $A(t)$ can be considered as individual i's opinion on how to divide the workload for the task at time t. Suppose the group of individuals exchange their opinions over the influence network defined by $W = A(t)$ and eventually reach consensus on the workload assignment. We have that the consensus workload assigned to any individual j, denoted by $w_j(t)$, satisfies $w_j(t) = \lim_{k\to\infty} W^k \boldsymbol{A}_j(t) = \mathbb{1}_n \boldsymbol{v}_{\text{left}}(A(t))^\top \boldsymbol{A}_j(t)$, where $\boldsymbol{A}_j(t)$ denotes the jth column of $A(t)$. Therefore, $\boldsymbol{w}^\top(t) = \boldsymbol{v}_{\text{left}}(A(t))^\top A(t)$, which leads to $\boldsymbol{w}(t) = \boldsymbol{v}_{\text{left}}(A(t))$. Third, our eigenvector assignment rule is consistent with the following natural property: in a team without performance feedback, due to the lack of information inflow, the team's task assignment does not change. These arguments justify Assumption 3; recall also Section V-A with a numerical evaluation of a different assignment rule.

APPENDIX D
PROOF FOR THEOREM 2

Before the proof, we state a useful lemma summarized from Pages 62–67 of [31].

Lemma 8 (Continuity of eigenvalue and eigenvector): Suppose $A, B \in \mathbb{R}^{n\times n}$ satisfy $|a_{ij}| < 1$ and $|b_{ij}| < 1$ for any $i, j \in \{1, \dots, n\}$. For sufficiently small $\epsilon > 0$:

1) the eigenvalues λ and λ' of A and $(A + \epsilon B)$, respectively, can be put in one-to-one correspondence so that $|\lambda' - \lambda| < 2(n+1)^2(n^2\epsilon)^{\frac{1}{n}}$;
2) if λ is a simple eigenvalue of A, then the corresponding eigenvalue $\lambda(\epsilon)$ of $A + \epsilon B$ satisfies $|\lambda(\epsilon) - \lambda| = O(\epsilon)$;
3) if $\boldsymbol{v}$ is an eigenvector of A associated with a simple eigenvalue λ, then the eigenvector $\boldsymbol{v}(\epsilon)$ of $A + \epsilon B$ associated with the corresponding eigenvalue $\lambda(\epsilon)$ satisfies $|v_i(\epsilon) - v_i| = O(\epsilon)$ for any $i \in \{1, \dots, n\}$.

Proof of Theorem 2: In this proof, we extend the definition of $\boldsymbol{v}_{\text{left}}(A)$ to the normalized entry-wise positive left eigenvector, associated with the eigenvalue of A with the largest magnitude, if such an eigenvector exists and is unique. According to Perron–Frobenius theorem and Lemma 8, vector $\boldsymbol{v}_{\text{left}}(A)$, as long as well defined, depends continuously on the entries of A. Therefore, for system (3), there exists a sufficiently small $\tau > 0$ such that $A(t)$ and $\boldsymbol{w}(t)$ are well defined and continuously differentiable at any $t \in [0, \tau]$, and, moreover, $p_i\big(\boldsymbol{w}(t)\big) - \sum_k m_{ik} p_k\big(\boldsymbol{w}(t)\big)$ remains finite. Therefore, for any $t \in [0, \tau]$ and $i, j \in \{1, \dots, n\}$, $a_{ij}(t) > 0$ if $a_{ij}(0) > 0$; $a_{ij}(t) = 0$ if $a_{ij}(0) = 0$, and thus, $A(t)$ is row-stochastic and primitive for any $t \in [0, \tau]$.

For any $i \in \{1, \dots, n\}$, there exists $k \neq i$ such that $a_{ik}(0) > 0$. According to (2)

$$\frac{da_{ij}(t)}{da_{ik}(t)} = \frac{a_{ij}(t)}{a_{ik}(t)},\ \forall t \in [0, \tau],\ \forall j \in \{1, \dots, n\} \setminus \{i, k\}$$

which leads to $a_{ij}(t)/a_{ik}(t) = a_{ij}(0)/a_{ik}(0)$. Let C be an $n \times n$ matrix with the entries c_{ij} defined as:

1) $c_{ii} = 0$ for any $i \in \{1, \dots, n\}$;
2) $c_{ij} = a_{ij}(0)/\big(1 - a_{ii}(0)\big)$ for any $j \neq i$.

One can check that C is row-stochastic and $A(t)$ is given by (4), for any $t \in [0, \tau]$, where $\boldsymbol{a}(t) = \big(a_1(t), \dots, a_n(t)\big)^\top$ with $a_i(t) = a_{ii}(t)$. Since the digraph, with C as the adjacency matrix, has the same topology with the digraph associated with $A(0)$, matrix C is irreducible and $\boldsymbol{c} = \boldsymbol{v}_{\text{left}}(C)$ is well defined.

Since the matrix $A(t)$ has the structure given by (4), according to Lemma 2.2 in [12], for any $t \in [0, \tau]$

$$w_i(t) = \frac{c_i}{1 - a_i(t)} \Big/ \sum_k \frac{c_k}{1 - a_k(t)}.$$

Therefore, for any $t \in [0, \tau]$

$$p_i\big(\boldsymbol{w}(t)\big) = f\left(\frac{x_i}{c_i}\big(1 - a_i(t)\big) \sum_k w_k(t) \frac{c_k}{1 - a_k(t)}\right).$$

According to (2), $\dot{a}_j(t) \leq 0$ for any $j \in \operatorname{argmin}_k \frac{x_k}{c_k}\big(1 - a_k(t)\big)$. Therefore, $\operatorname{argmin}_k \frac{x_k}{c_k}\big(1 - a_k(t)\big)$ is increasing, and similarly, $\operatorname{argmax}_k \frac{x_k}{c_k}\big(1 - a_k(t)\big)$ is decreasing with t, which implies that, the set

$$\Omega_A\big(A(0)\big) = \Big\{A \in \mathbb{R}^{n\times n}\ \Big|\ A = \operatorname{diag}(\boldsymbol{a}) + (I - \operatorname{diag}(\boldsymbol{a}))C,\quad 0 \leq a_i \leq 1 - \frac{c_i}{x_i} \min_k \frac{x_k}{c_k}\big(1 - a_{kk}(0)\big), \forall i\Big\}$$

is a compact positive invariant set for system (3), as long as $A(0)$ is row-stochastic, irreducible, and has strictly positive diagonal. Moreover, one can check that, for any $A \in \Omega_A\big(A(0)\big)$, $\boldsymbol{w} = \boldsymbol{v}_{\text{left}}(A)$ is well defined and strictly lower (upper resp.) bounded from 0 (1 resp.). Therefore, the solution $A(t)$ is extensible to all $t \in [0, +\infty)$ and (4) and (5) hold for any $t \in [0, +\infty)$. Moreover, since $p_i\big(\boldsymbol{w}(t)\big) - \sum_k m_{ik} p_k\big(\boldsymbol{w}(t)\big)$ remains bounded, we have $a_{ij} > 0$ if $a_{ij}(0) > 0$ and $a_{ij}(t) = 0$ if $a_{ij}(0) = 0$. This concludes the proof for (i)–(iv).

For statement (v), differentiate both sides of the equation $\boldsymbol{w}^\top(t)A(t) = \boldsymbol{w}^\top(t)$ and substitute (3) into the differentiated equation. We obtain

$$\left(\boldsymbol{w}^\top \operatorname{diag}(\boldsymbol{p}(\boldsymbol{w}) - M\boldsymbol{p}(\boldsymbol{w}))A_d - \frac{d\boldsymbol{w}^\top}{dt}\right)(I_n - A) = \mathbb{0}_n^\top$$

where time index t is omitted for simplicity. Equation (6) in statement (v) is obtained due to $\boldsymbol{w}^\top(t)\mathbb{1}_n = 1$.

APPENDIX E
PROOF FOR THEOREM 3

We prove the theorem by analyzing the generalized replicator dynamics (6) for $\boldsymbol{w}(t)$, and the reduced assign/appraise dynamics (5) for $\boldsymbol{a}(t)$, given any constant, normalized and entrywise positive vector $\boldsymbol{c}$. According to (5), the assignment $\boldsymbol{w} = \boldsymbol{v}_{\text{left}}(A)$ can be considered as a function of the self appraisal vector $\boldsymbol{a}$, that is, $\boldsymbol{w}(t) = \boldsymbol{w}\big(\boldsymbol{a}(t)\big)$ for any $t \geq 0$. In this proof, let $\boldsymbol{\phi}(\boldsymbol{a}) = \boldsymbol{p}\big(\boldsymbol{w}(\boldsymbol{a})\big) - M\boldsymbol{p}\big(\boldsymbol{w}(\boldsymbol{a})\big)$ and denote by $\mathcal{D} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ the distance induced by the two-norm in $\mathbb{R}^n$. For any $\boldsymbol{x} \in \mathbb{R}^n$ and subset S of $\mathbb{R}^n$, defined $\mathcal{D}(\boldsymbol{x}, S) = \inf_{\boldsymbol{y}\in S} \mathcal{D}(\boldsymbol{x}, \boldsymbol{y})$.

First of all, for any given $\boldsymbol{a}(0) \in (0,1)^n$, we know that the set Ω, as defined in Theorem 2(iv), is a compact positively invariant set for dynamics (5), and $\boldsymbol{w}(t)$ is well defined and entrywise strictly lower (upper resp.) bounded from $\mathbb{0}_n$ ($\mathbb{1}_n$ resp.), for all $t \in [0, +\infty)$.

Second, for any $\boldsymbol{a} \in \Omega$, define a scalar function $V : \mathbb{R}^n \to \mathbb{R}$ as

$$V(\boldsymbol{a}) = \log \frac{\max_k x_k/w_k(\boldsymbol{a})}{\min_k x_k/w_k(\boldsymbol{a})}$$

and the following index sets

$$\overline{\theta}(\boldsymbol{a}) = \Big\{i\ \Big|\ \exists t_i > 0 \text{ s.t. } \frac{x_i}{w_i\big(\boldsymbol{a}(t)\big)} = \max_k \frac{x_k}{w_k\big(\boldsymbol{a}(t)\big)}\ \text{for any } t \in [0, t_i],\ \text{with } \boldsymbol{a}(0) = \boldsymbol{a}\Big\},\ \text{and}$$

$$\underline{\theta}(\boldsymbol{a}) = \Big\{ j \,\Big|\, \exists t_j > 0 \text{ s.t. } \frac{x_j}{w_j\big(\boldsymbol{a}(t)\big)} = \min_k \frac{x_k}{w_k\big(\boldsymbol{a}(t)\big)} \text{ for any } t \in [0, t_j], \text{ with } \boldsymbol{a}(0) = \boldsymbol{a} \Big\}.$$

Then, the right time derivative of $V\big(\boldsymbol{a}(t)\big)$, along the solution $\boldsymbol{a}(t)$, is given by

$$\frac{d^+ V\big(\boldsymbol{a}(t)\big)}{dt} = a_j(t)\phi_j\big(\boldsymbol{a}(t)\big) - a_i(t)\phi_i\big(\boldsymbol{a}(t)\big)$$

for any $i \in \overline{\theta}\big(\boldsymbol{a}(t)\big)$ and $j \in \underline{\theta}\big(\boldsymbol{a}(t)\big)$. Define

$$\begin{aligned} E &= \big\{\boldsymbol{a} \in \Omega \,\big|\, a_j\phi_j(\boldsymbol{a}) - a_i\phi_i(\boldsymbol{a}) = 0 \\ &\qquad \text{for any } i \in \overline{\theta}(\boldsymbol{a}), j \in \underline{\theta}(\boldsymbol{a})\big\} \\ E_1 &= \big\{\boldsymbol{a} \in E \,\big|\, \boldsymbol{\phi}(\boldsymbol{a}) = \mathbb{0}_n\big\} \\ E_2 &= \big\{\boldsymbol{a} \in E \,\big|\, \boldsymbol{\phi}(\boldsymbol{a}) \neq \mathbb{0}_n\big\}. \end{aligned}$$

One can check that E and E_1 are compact subsets of Ω, $E = E_1 \cup E_2$, and $E_1 \cap E_2$ is empty. Denote by $\hat{E}$ the largest invariant subset of E. Applying the LaSalle Invariance Principle, see Theorem 3 in [13], we have $\mathcal{D}\big(\boldsymbol{a}(t), \hat{E}\big) \to 0$ as $t \to +\infty$. Note that, $\lim_{t\to+\infty} \mathcal{D}\big(\boldsymbol{a}(t), \hat{E}\big) = 0$ does not necessarily leads to $\lim_{t\to+\infty} \boldsymbol{w}(t) = \boldsymbol{x}$. We need to further refine the result.

For set E_1, it is straightforward to see that $E_1 \subset \hat{E}$ and $\boldsymbol{w}(\boldsymbol{a}) = \boldsymbol{x}$ for any $\boldsymbol{a} \in E_1$. Now we prove by contradiction that, if $E_2 \cap \hat{E}$ is not empty, then, for any $\boldsymbol{a} \in E_2 \cap \hat{E}$, there exists $i \in \overline{\theta}(\boldsymbol{a})$ such that $a_i = 0$. Suppose $a_i > 0$ for any $i \in \overline{\theta}(\boldsymbol{a})$. Since the observation network $G(M)$ is strongly connected, there exists a directed path $i, k_1, \dots, k_q, j$ on $G(M)$, where $i \in \overline{\theta}(\boldsymbol{a})$ and $j \in \underline{\theta}(\boldsymbol{a})$. We have $k_1 \in \overline{\theta}(\boldsymbol{a})$, otherwise, starting with $\tilde{\boldsymbol{a}}(0) = \boldsymbol{a}$, there exists sufficiently small $\Delta t > 0$ such that $\phi_i\big(\tilde{\boldsymbol{a}}(t)\big) > 0$ and $\tilde{a}_i(t) > 0$, which contradicts the fact that $\boldsymbol{a}$ is in the largest invariant set of E. Repeating this argument, we have $j \in \overline{\theta}(\boldsymbol{a})$, which contradicts $\boldsymbol{\phi}(\boldsymbol{a}) \neq \mathbb{0}_n$. Similarly, we have that, for any $\boldsymbol{a} \in E_2 \cap \hat{E}$, there exists $j \in \underline{\theta}(\boldsymbol{a})$ with $a_j = 0$.

If the fixed vectors $\boldsymbol{c}$ and $\boldsymbol{x}$ satisfy $\boldsymbol{c} = \boldsymbol{x}$, then, there can not exist $\boldsymbol{a} \in E_2 \cap \hat{E}$ satisfying all the following three properties:

i) there exists $i \in \overline{\theta}(\boldsymbol{a})$ such that $a_i = 0$;
ii) there exists $j \in \underline{\theta}(\boldsymbol{a})$ such that $a_j = 0$;
iii) $\boldsymbol{\phi}(\boldsymbol{a}) \neq \mathbb{0}_n$. In this case, $E_2 \cap \hat{E}$ is an empty set, which implies that $\boldsymbol{a}(t) \to \hat{E} = E_1$ and thus $\boldsymbol{w}(t) \to \boldsymbol{x}$ as $t \to +\infty$.

Before discussing the case when $\boldsymbol{c} \neq \boldsymbol{x}$, we present some properties of the individual performance measure:

P1: For any $k, l \in \{1, \dots, n\}$, $\frac{x_k}{c_k}(1 - a_k) \leq \frac{x_l}{c_l}(1 - a_l)$ leads to $p_k(\boldsymbol{a}) \leq p_l(\boldsymbol{a})$, and $\frac{x_k}{c_k}(1 - a_k) > \frac{x_l}{c_l}(1 - a_l)$ leads to $p_k(\boldsymbol{a}) > p_l(\boldsymbol{a})$;

P2: If there exists $\tau \geq 0$ such that $i \in \overline{\theta}\big(\boldsymbol{a}(\tau)\big)$ and $a_i(\tau) = 0$, then, $i \in \overline{\theta}\big(\boldsymbol{a}(t)\big)$ for all $t \geq \tau$;

P3: $\boldsymbol{p}\big(\boldsymbol{a}(t)\big)$ is finite and strictly bounded from 0, satisfying $f\big(\frac{x_i}{c_i}(1 - \zeta_i(\boldsymbol{a}(0)))\big) \leq p_i(\boldsymbol{a}(t)) \leq f\big(\frac{x_i}{c_i}\sum_k \frac{c_k}{\zeta_k(\boldsymbol{a}(0))}\big)$, with $\zeta_i(\boldsymbol{a})$ defined in Theorem 2(iv).

For the case when $\boldsymbol{c} \neq \boldsymbol{x}$, consider the partition $\varphi_1, \dots, \varphi_m$ of the index set $\{1, \dots, n\}$, with $m \leq n$, satisfying the following two properties:

1) $x_k/c_k = x_l/c_l$ for any k, l in the same subset φ_r;
2) $x_k/c_k > x_l/c_l$ for any $k \in \varphi_r$, $l \in \varphi_s$, with $r < s$.

For any $\boldsymbol{a} \in E_2 \cap \hat{E}$, since there exists $j \in \underline{\theta}(\boldsymbol{a})$ with $a_j = 0$, we have $\varphi_m \subset \underline{\theta}(\boldsymbol{a})$. For any $i \in \cup_{r=1}^{m-1}\varphi_r$, let

$$\begin{aligned} E_{2,i} = \Big\{\boldsymbol{a} \in \Omega \,\Big|\, &a_i = 0,\ a_j = 0 \text{ for any } j \in \varphi_m, \\ &1 - \frac{x_i}{c_i}\frac{c_k}{x_k} \leq a_k \leq 1 - \min_{l\in\{1,\dots,n\}} \frac{x_l}{c_l}\frac{c_k}{x_k}, \\ &\text{for any } k \in \varphi_1 \cup \dots \cup \varphi_{m-1} \setminus \{i\}\Big\}. \end{aligned}$$

With properties P1 and P2 of $\boldsymbol{p}(\boldsymbol{a})$, for any $\boldsymbol{a} \in E_{2,i}$, we have $i \in \overline{\theta}(\boldsymbol{a})$ and $a_i = 0$. Moreover

1) $E_{2,i} \subset \mathbb{R}^n$ is compact for any $i \in \varphi_1 \cup \dots \cup \varphi_{m-1}$;
2) $\cup_{i\in\varphi_1} E_{2,i}, \dots, \cup_{i\in\varphi_{m-1}} E_{2,i}$ are disjoint and compact subsets of $\mathbb{R}^n$;
3) $E_2 \cap \hat{E} \subset \bigcup_{i\in\varphi_1\cup\dots\cup\varphi_{m-1}} E_{2,i}$.

For any $\boldsymbol{a} \in E_2 \cap \hat{E}$, since there exists $i \in \overline{\theta}(\boldsymbol{a})$ and $j \in \underline{\theta}(\boldsymbol{a})$ such that $a_i = a_j = 0$, on the observation network $G(M)$, there must exists a path $i, k_1, \dots, k_q$ satisfying:

i) $i \in \overline{\theta}(\boldsymbol{a})$ and $a_i = 0$;
ii) $a_{k_q} = 0$ and $x_{k_q}/c_{k_q} < x_i/c_i$;
iii) $a_{k_l} > 0$ for any $l \in \{1, \dots, q-1\}$.

Consider the trajectory $\tilde{\boldsymbol{a}}(t)$ with $\tilde{\boldsymbol{a}}(0) = \boldsymbol{a}$, we have

$$\begin{aligned} \dot{\tilde{a}}_{k_{q-1}} &\geq \tilde{a}_{k_{q-1}}(1 - \tilde{a}_{k_{q-1}}) \\ &\cdot \left(f\left(\frac{x_{k_{q-1}}}{c_{k_{q-1}}}(1 - \tilde{a}_{k_{q-1}}) \sum_{l=1}^n \frac{c_l}{1 - \tilde{a}_l}\right) \right. \\ &\left. - f\left(\left(m_{k_{q-1}k_q}\frac{x_{k_q}}{c_{k_q}} + (1 - m_{k_{q-1}k_q})\frac{x_i}{c_i}\right) \sum_{l=1}^n \frac{c_l}{1 - \tilde{a}_l}\right)\right). \end{aligned}$$

The inequality is due to properties P1–P3 of $p_i(\boldsymbol{a})$ for $i \in \overline{\theta}(\boldsymbol{a})$ with $a_i = 0$, and the concavity of the function f. Moreover, since $\tilde{a}_{k_{q-1}}$ is strictly bounded from 1 and $\sum_l c_l/(1 - \tilde{a}_l)$ is strictly lower bounded from 0, there exists $T_{k_{q-1}}(M, \boldsymbol{a}(0), \boldsymbol{a}) > 0$ such that

$$p_{k_{q-1}}\big(\tilde{\boldsymbol{a}}(t)\big) < \frac{2 - m_{k_{q-1}k_q}}{2} p_i\big(\tilde{\boldsymbol{a}}(t)\big) + \frac{m_{k_{q-1}k_q}}{2} p_{k_q}\big(\tilde{\boldsymbol{a}}(t)\big).$$

Applying the same argument to $k_{q-2}, \dots, k_1$, we have that, there exists $T_{k_1}(M, \boldsymbol{a}(0), \boldsymbol{a}) > 0$ and $\eta_{ik_1\dots k_q}(M) \in (0, 1)$ such that, for the solution $\tilde{\boldsymbol{a}}(t)$ with $\tilde{\boldsymbol{a}}(0) = \boldsymbol{a}$

$$\begin{aligned} p_{k_1}\big(\tilde{\boldsymbol{a}}(t)\big) &< \big(1 - \eta_{ik_1\dots k_q}(M)\big) p_i\big(\tilde{\boldsymbol{a}}(t)\big) \\ &\quad + \eta_{ik_1\dots k_q}(M) p_{k_q}\big(\tilde{\boldsymbol{a}}(t)\big) \end{aligned}$$

for all $t \geq T_{k_1}(M, \boldsymbol{a}(0), \boldsymbol{a})$. This inequality implies that

$$\phi_i(\tilde{\boldsymbol{a}}(t)) \geq m_{ik_1}\eta_{ik_1\dots k_q}(M)\Big(p_i(\tilde{\boldsymbol{a}}(t)) - p_{k_q}(\tilde{\boldsymbol{a}}(t))\Big)$$
$$\geq m_{ik_1}\eta_{ik_1\dots k_q}(M) f'\left(\frac{x_i}{c_i}\right)$$
$$\cdot \sum_{l=1}^{n} \frac{c_l}{1-\zeta_l(\boldsymbol{a}(0))}\left(\frac{x_i}{c_i} - \frac{x_{k_q}}{c_{k_q}}\right) > 0.$$

Since the choices of i and the paths $i, k_1, \dots, k_q$ are finite, there exists a constant $\eta > 0$ such that, for any $\boldsymbol{a} \in E_2 \cap \hat{E}$, there exists $T(\boldsymbol{a}(0), \boldsymbol{a}) > 0$ such that, for any $t \geq T(\boldsymbol{a}(0), \boldsymbol{a}) > 0$, the solution $\tilde{\boldsymbol{a}}(t)$, with $\tilde{\boldsymbol{a}}(0) = \boldsymbol{a}$, satisfies $i \in \overline{\theta}(\tilde{\boldsymbol{a}}(t))$ and $\phi_i(\tilde{\boldsymbol{a}}(t)) \geq \eta > 0$.

For any $i \in \varphi_1 \cup \dots \cup \varphi_{m-1}$, define

$$\hat{E}_{2,i} = \left\{\boldsymbol{a} \in E_{2,i} \,\middle|\, p_i(\boldsymbol{a}) - \sum_{k=1}^{n} m_{ik} p_k(\boldsymbol{a}) \geq \eta\right\}.$$

We have: 1) each $\hat{E}_{2,i}$ is a compact subset of $\mathbb{R}^n$; 2) $\cup_{i\in\varphi_1}\hat{E}_{2,i}, \dots, \cup_{i\in\varphi_{m-1}}\hat{E}_{2,i}$ are disjoint and compact subsets of $\mathbb{R}^n$. Let $\hat{E}_2 = \cup_{r=1}^{m-1}\big(\cup_{r\in\varphi_r}\hat{E}_{2,i}\big)$. For dynamics (5), due to the continuous dependency on the initial condition, for any $\boldsymbol{a} \in (E_2 \cap \hat{E}) \setminus (\hat{E}_2 \cap \hat{E})$, there exists $\delta > 0$ such that, for any $\tilde{\boldsymbol{a}}(0) \in \mathcal{U}(\boldsymbol{a}, \delta) \cap (E_2 \cap \hat{E})$, where $\mathcal{U}(\boldsymbol{a}, \delta) = \{\boldsymbol{b} \in \Omega \,|\, \mathcal{D}(\boldsymbol{b}, \boldsymbol{a}) \leq \delta\}$, $\tilde{\boldsymbol{a}}(t) \in \hat{E}_2 \cap \hat{E}$ for sufficiently large t. Therefore, $\boldsymbol{a}$ can not be an ω-limit point of $\boldsymbol{a}(0)$. We thus obtain that, the ω-limit set of $\boldsymbol{a}(0)$ is in the set $E_1 \cup (\hat{E}_2 \cap \hat{E})$. Moreover, since $E_1, \cup_{i\in\varphi_1}\hat{E}_{2,i}, \dots, \cup_{i\in\varphi_{m-1}}\hat{E}_{2,i}$ are disjoints compact subsets of $\mathbb{R}^n$, and the ω-limit set of $\boldsymbol{a}(0)$ is connected and compact, $\boldsymbol{a}(t)$ can only converge to one of the sets $E_1, \cup_{i\in\varphi_1}\hat{E}_{2,i}, \dots, \cup_{i\in\varphi_{m-1}}\hat{E}_{2,i}$.

Now, we prove $\lim_{t\to+\infty}\mathcal{D}(\boldsymbol{a}(t), E_1) = 0$ by contradiction. Suppose $\omega(\boldsymbol{a}(0)) \in \cup_{i\in\varphi_r}\hat{E}_{2,i}$ for some $r \in \{1, \dots, m-1\}$. Since each $\hat{E}_{2,i}$ is a compact set, there exists $\epsilon > 0$ and $\eta(\epsilon) > 0$ such that $\phi_i(\boldsymbol{a}) \geq \eta(\epsilon) > 0$ for any $\boldsymbol{a} \in \mathcal{U}(\hat{E}_{2,i}, \epsilon)$. For this given $\epsilon > 0$, since $\omega(\boldsymbol{a}(0)) \in \cup_{i\in\varphi_r}\hat{E}_{2,i}$ leads to $\mathcal{D}(\boldsymbol{a}(t), \cup_{i\in\varphi_r}\hat{E}_{2,i}) \to 0$ as $t \to +\infty$, we conclude that, there exists $T > 0$ such that, for any $t \geq T$, $\boldsymbol{a}(t) \in \cup_{i\in\varphi_r}\mathcal{U}(\hat{E}_{2,i}, \epsilon)$. Define $V_r(\boldsymbol{a}) = \min_{i\in\varphi_r} a_i$, for any $\boldsymbol{a} \in \cup_{i\in\varphi_r}\mathcal{U}(\hat{E}_{2,i}, \epsilon)$. The function $V_r(\boldsymbol{a})$ satisfies that, $V_r(\boldsymbol{a}) \geq 0$ for any $\boldsymbol{a} \in \cup_{i\in\varphi_r}\mathcal{U}(\hat{E}_{2,i}, \epsilon)$ and $V_r(\boldsymbol{a}) = 0$ if and only if $\boldsymbol{a} \in \cup_{i\in\varphi_r}\hat{E}_{2,i}$. Therefore, $\mathcal{D}(\boldsymbol{a}(t), \cup_{i\in\varphi_r}\hat{E}_{2,i}) \to 0$ leads to $V_r(\boldsymbol{a}(t)) \to 0$ as $t \to +\infty$. Moreover since $\boldsymbol{a} \in \mathcal{U}(\hat{E}_{2,i}, \epsilon)$ for any $i \in \operatorname{argmin}_{k\in\varphi_r} a_k$, we have

$$\frac{d^+ V_r(\boldsymbol{a}(t))}{dt} = \min_{i\in\operatorname{argmin}_{k\in\varphi_r} a_k(t)} \dot{a}_i(t) \geq \delta a_i(t)(1 - a_i(t)).$$

According to Theorem 2(i), for any given $\boldsymbol{a}(0) \in (0,1)^n$, $\boldsymbol{a}(t) \in (0,1)^n$ for all $t \geq 0$. Therefore, $d^+V_r(\boldsymbol{a}(t))/dt > 0$ for all $t \geq T$, which contradicts $\lim_{t\to+\infty} V_r(\boldsymbol{a}(t)) = 0$. Therefore, we have $\lim_{t\to+\infty}\mathcal{D}(\boldsymbol{a}(t), E_1) = 0$ and $\lim_{t\to+\infty}\boldsymbol{w}(t) = \boldsymbol{x}$.

Since $\dot{A}(t) \to \mathbb{0}_{n\times n}$ as $\boldsymbol{\phi}(\boldsymbol{a}(t)) \to \mathbb{0}_n$, there exists an entry-wise nonnegative and irreducible matrix A^*, depending on $A(0)$ and satisfying $\boldsymbol{v}_{\text{left}}(A^*) = \boldsymbol{x}$, such that $A(t) \to A^*$ as $t \to +\infty$.

Appendix F
Proof for Lemma 5

Since $A(0)$ is primitive and row-stochastic, following the same argument in the proof for Theorem 2(i), we have that, there exists $\Delta\tilde{t} > 0$ such that, for any $t \in [0, \Delta\tilde{t}]$:

1) $\boldsymbol{w}(t)$ is well defined and $\boldsymbol{w}(t) \succ \mathbb{0}_n$;
2) $A(t)$ is bounded, continuously differentiable to t, and satisfies $A(t)\mathbb{1}_n = \mathbb{1}_n$;
3) $\boldsymbol{p}(\boldsymbol{w}(t)) - M\boldsymbol{p}(\boldsymbol{w}(t))$ is bounded.

Therefore, for any $t \geq 0$, there exists μ, depending on t and $A(0)$, such that $\dot{A}(t) \succeq \frac{1}{\tau_{\text{ave}}}A^2(t) - (\frac{1}{\tau_{\text{ave}}} + \mu)A(t)$.

Consider the equation $\dot{B}(t) = \frac{1}{\tau_{\text{ave}}}B^2(t) - (\frac{1}{\tau_{\text{ave}}} + \mu)B(t)$, with $B(0) = A(0)$. According to the comparison theorem, $A(t) \succeq B(t)$ for any $t \geq 0$. Let $\boldsymbol{b}_i(t)$ be the ith column of $B(t)$ and let $\boldsymbol{y}_k(t) = e^{(\frac{1}{\tau_{\text{ave}}}+\mu)t}\boldsymbol{b}_k(t)$. We obtain $\dot{\boldsymbol{y}}_k(t) = \frac{1}{\tau_{\text{ave}}}B(t)\boldsymbol{y}_k(t)$.

Denote by $\Phi(t, 0)$ the state transition function for the equation $\dot{\boldsymbol{y}}_k(t) = \frac{1}{\tau_{\text{ave}}}B(t)\boldsymbol{y}_k(t)$, which is written as $\Phi(t,0) = I_n + \sum_{k=1}^{\infty}\Phi_k(t)$, where $\Phi_1(t) = \int_0^t B(\tau_1)d\tau_1$ and $\Phi_l(t) = \int_0^t B(\tau_1)\int_0^{\tau_1}\dots B(\tau_{l-1})\int_0^{\tau_{l-1}} B(\tau_l)d\tau_l$ for $l \geq 2$. By computing the MacLaurin expansion for each $\Phi_k(t)$ and summing them together, we obtain that

$$\Phi(t,0) = I_n + h_1(t)B(0) + h_2(t)B^2(0) + \dots$$
$$+ h_{n-1}(t)B^{n-1}(0) + O(t^n)$$

where $h_k(t)$ is a polynomial with the form $h_k(t) = \eta_{k,k}t^k + \eta_{k,k+1}t^{k+1} + \dots$, and, moreover, $\eta_{k,k} > 0$ for any $k \in \mathbb{N}$. Therefore, for t sufficiently small, we have $h_k(t) > 0$ for any $k \in \{1, \dots, n-1\}$. Moreover, since $B^k(0) \succeq \mathbb{0}_{n\times n}$ for any $k \in \mathbb{N}$ and $B(0) + \dots + B^{n-1}(0) \succ \mathbb{0}_{n\times n}$, there exists $\Delta t \leq \Delta\tilde{t}$ such that $\Phi(t,0) \succ \mathbb{0}_{n\times n}$ for any $t \in [0, \Delta t]$.

Appendix G
Discussion on Conjecture 6

The Monte Carlo method [28] is adopted to estimate the probability that Conjecture 6 holds. For any randomly generated $A(0) \in \text{int}(\Delta_n)$, define the random variable $Z : \text{int}(\Delta_n) \to \{0,1\}$ as

1) $Z(A(0)) = 1$ if there exists $a_{\min} > 0$ such that $A(t) \succeq a_{\min}\mathbb{1}_n\mathbb{1}_n^\top$ for all $t \in [0, 1000]$;
2) $Z(A(0)) = 0$ otherwise.

Let $p = \mathbb{P}[Z(A(0)) = 1]$. For N independent random samples $Z_1, \dots, Z_N$, in each of which $A(0)$ is randomly generated in $\text{int}(\Delta_n)$, define $\hat{p}_N = \sum_{i=1}^N Z_i/N$. For any accuracy $1 - \epsilon \in (0,1)$ and confidence level $1 - \xi \in (0,1)$, $|\hat{p}_N - p| < \epsilon$ with probability greater than $1 - \xi$ if

$$N \geq \frac{1}{2\epsilon^2}\log\frac{2}{\xi}. \tag{11}$$

For $\epsilon = \xi = 0.01$, the Chernoff bound (11) is satisfied by $N = 27\,000$. We run 27 000 independent MATLAB simulations of the assign.appraise/influence dynamics with $n = 7$ and find that $\hat{p}_N = 1$. Therefore, for any $A(0) \in \text{int}(\Delta_n)$, with 99%

confidence level, there is at least 0.99 probability that $A(t)$ is entrywise strictly lower bounded from $\mathbb{0}_{n\times n}$ for all $t \in [0, 10\,000]$.

Moreover, we present in the following lemma a sufficient condition for Conjecture 6 on the initial appraisal matrix $A(0)$ and the parameters τ_{ave}, τ_{app}.

Lemma 9 (Strictly Positive Lower Bound of Appraisals): Consider the assign/appraise/influence dynamics (8), based on Assumptions 3–6, with the assignment $\boldsymbol{w}(t)$ and performance $\boldsymbol{p}(\boldsymbol{w})$ as in Assumptions 1 and 2, respectively. For any initial appraisal matrix $A(0)$ that is entrywise positive and row-stochastic, as long as

$$\frac{\tau_{\text{app}}}{\tau_{\text{ave}}} \geq \frac{1-\xi_0}{\xi_0}\left(f\left(\frac{x_{\max}}{\xi_0}\right) - f\left(\frac{x_{\min}}{1-(n-1)\xi_0}\right)\right)$$

where the constant ξ_0 is defined as in Theorem 7(ii), then there exists $a_{\min} > 0$ such that $A(t) \succeq a_{\min}\mathbb{1}_n\mathbb{1}_n^\top$.

Proof: First of all, by definition we have $w_s(t) = \sum_k w_k(t)a_{ks}(t)$. The right-hand side of this equation is a convex combination of $\{a_{1s}(t), \dots, a_{ns}(t)\}$. Therefore, $\max_k a_{ks}(t) \geq w_s(t) \geq \xi_0$ for all $t \in [0, +\infty)$.

At any time $t \geq 0$, for any pair (i, j) such that $a_{ij}(t) = \min_{k,l} a_{kl}(t)$, the dynamics for $a_{ij}(t)$ is

$$\dot{a}_{ij}(t) = \frac{1}{\tau_{\text{ave}}}\left(\sum_k a_{ik}(t)a_{kj}(t) - a_{ij}(t)\right) - \frac{1}{\tau_{\text{app}}}a_{ii}(t)a_{ij}(t)\Big(p_i\big(\boldsymbol{w}(t)\big) - \sum_{k=1}^n m_{ik}p_k\big(\boldsymbol{w}(t)\big)\Big).$$

For simplicity, in this proof, denote $\phi_i = p_i\big(\boldsymbol{w}(t)\big) - \sum_{k=1}^n m_{ik}p_k\big(\boldsymbol{w}(t)\big)$. Suppose $a_{mj}(t) = \max_k a_{kj}(t)$. We have

$$\dot{a}_{ij}(t) \geq \frac{1}{\tau_{\text{ave}}}a_{ij}(t)a_{mj}(t) - \frac{1}{\tau_{\text{ave}}}a_{ij}^2(t) - \frac{1}{\tau_{\text{app}}}a_{ii}(t)a_{ij}(t)\phi_i.$$

Therefore

$$\frac{\dot{a}_{ij}}{a_{ij}} \geq \frac{1}{\tau_{\text{ave}}}\xi_0 - \frac{1}{\tau_{\text{app}}}(1-\xi_0)\left(f\left(\frac{x_{\max}}{\xi_0}\right) - f\left(\frac{x_{\min}}{1-(n-1)\xi_0}\right)\right).$$

The condition on $\frac{1}{\tau_{\text{ave}}}/\frac{1}{\tau_{\text{app}}}$ in Lemma 9 guarantees that $\dot{a}_{ij}(t)/a_{ij}(t)$ is positive if $a_{ij}(t) = \min_{k,l} a_{kl}(t)$. ■

Appendix H
Proof for Theorem 7

Statement (i) is proved following the same argument in the proof for Theorem 2(i). For any given $A(0)$ that is row-stochastic and entrywise positive, the closed and bounded invariant set Ω for $A(t)$ is given by $\Omega = \big\{A \in \mathbb{R}^{n\times n} \,\big|\, A \succ a_{\min}\mathbb{1}_n\mathbb{1}_n^\top, A\mathbb{1}_n = \mathbb{1}_n\big\}$, where $a_{\min} > 0$ is given by Conjecture 6.

Since $\boldsymbol{w}^\top(t)\big(A^2(t) - A(t)\big) = \mathbb{0}_n^\top$ for all $t \geq 0$, we conclude that, $\boldsymbol{w}(t)$ in the assign/appraise/influence dynamics also obeys the generalized replicator dynamics (6). Consider $\boldsymbol{w}(t)$ as a function of $A(t)$. Define $\boldsymbol{\phi}(A) = \boldsymbol{p}\big(\boldsymbol{w}(A)\big) - M\boldsymbol{p}\big(\boldsymbol{w}(A)\big)$ and

$$V(A) = \log\frac{\max_k x_k/w_k(A)}{\min_k x_k/w_k(A)}.$$

For any $t \in [0, +\infty)$, there exists $i \in \operatorname{argmax}_k x_k/w_k\big(A(t)\big)$ and $j \in \operatorname{argmin}_k x_k/w_k\big(A(t)\big)$ such that $V\big(A(t)\big) = \log\Big(x_iw_j\big(A(t)\big)/x_jw_i\big(A(t)\big)\Big)$, and $\frac{d^+V(A)}{dt} = a_{jj}\phi_j(A) - a_{ii}\phi_i(A) \leq 0$. Therefore, $V\big(A(t)\big)$ is nonincreasing with t, which in turn implies

$$\frac{x_i}{x_j}\frac{w_j(t)}{w_i(t)} \leq \frac{\max_k x_k/w_k(0)}{\min_k x_k/w_k(0)} = \gamma_0$$

for any $i, j \in \{1, \dots, n\}$. This inequality, combined with the fact that $\sum_k w_k(t) = 1$ for any $t \geq 0$, leads to the inequalities in statement (ii).

Similar to the proof for Theorem 3, define

$$\overline{\theta}(A) = \left\{i \,\middle|\, \exists\, t_i > 0 \text{ s.t. } \frac{x_i}{w_i\big(A(t)\big)} = \max_k \frac{x_k}{w_k\big(A(t)\big)} \text{ for any } t \in [0, t_i] \text{ with } A(0) = A\right\}$$

$$\underline{\theta}(A) = \left\{j \,\middle|\, \exists\, t_j > 0 \text{ s.t. } \frac{x_j}{w_j\big(A(t)\big)} = \min_k \frac{x_k}{w_k\big(A(t)\big)} \text{ for any } t \in [0, t_j] \text{ with } A(0) = A\right\}$$

and let $E = \big\{A \in \Omega \,\big|\, d^+V(A)/dt = 0\big\}$. For any $A \in E$, since $A \succeq a_{\min}\mathbb{1}_n\mathbb{1}_n^\top$, we have $\phi_i(A) = \phi_j(A) = 0$ for any $i \in \overline{\theta}(A)$ and $j \in \underline{\theta}(A)$. Suppose individual s is a globally reachable node in the observation network. There exists a directed path $i, k_1, \dots, k_q, s$. Without loss of generality, suppose $q \geq 1$. For any A in the largest invariant subset of E, we have $k_1 \in \overline{\theta}(A)$ and therefore $\phi_{k_1}(A) = 0$. This iteration of argument leads to $s \in \overline{\theta}(A)$. Following the same line of argument, we have $s \in \underline{\theta}(A)$. Therefore, for any given $A(0) \succ \mathbb{0}_{n\times n}$ that is row-stochastic, the solution $A(t)$ converges to $\hat{E} = \{A \in \Omega | \boldsymbol{\phi}(A) = \mathbb{0}_n\} = \{A \in \Omega | \boldsymbol{v}_{\text{left}}(A) = \boldsymbol{x}\}$.

Let $\tilde{A} = \max_j\big(\max_k a_{kj} - \min_k a_{kj}\big)$. One can check that $d^+\tilde{V}(A)/dt$ along the dynamics (8) is a continuous function of A for any $A \in \Omega$. Define $\hat{E}_{\epsilon/2} = \big\{A \in \hat{E} \,\big|\, \|A - \mathbb{1}_n\boldsymbol{x}^\top\|_2 \geq \epsilon/2\big\}$. Since $\hat{E}$ is compact, $\hat{E}_{\epsilon/2}$ is also a compact set. For any $A \in \hat{E}_{\epsilon/2}$, since $d^+\tilde{V}(A)/dt$ is strictly negative and depends continuously on A, there exists a neighborhood $\mathcal{U}(A, r_A) = \{\tilde{A} \in \Omega | \|\tilde{A} - A\|_2 \leq r_A\}$ such that $d^+\tilde{V}(\tilde{A})/dt < 0$ for any $\tilde{A} \in \mathcal{U}(A, r_A)$. Due to the compactness of $\hat{E}_{\epsilon/2}$ and according to the Heine–Borel finite cover theorem, there exists $K \in \mathbb{N}$ and $\{A_k, r_k\}_{k\in\{1,\dots,K\}}$, where $A_k \in \hat{E}_{\epsilon/2}$ and $r_k > 0$ for any $k \in \{1, \dots, K\}$, such that $\hat{E}_{\epsilon/2} \subset \cup_{k=1}^K \mathcal{U}(A_k, r_k)$.

Define the distance $\mathcal{D} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ as in the proof for Theorem 3. Let $\delta = \min\{r_1, \dots, r_k, \epsilon/2\}$ and

$$B_1 = \left\{A \in \Omega \,\middle|\, \mathcal{D}(A, \hat{E}) \leq \delta, \mathcal{D}(A, \hat{E}_{\epsilon/2}) > \delta\right\}$$
$$B_2 = \left\{A \in \Omega \,\middle|\, \mathcal{D}(A, \hat{E}) \leq \delta, \mathcal{D}(A, \hat{E}_{\epsilon/2}) \leq \delta\right\}.$$

We have $B_1 \cap B_2$ is empty. For any $A \in B_1$, since $\mathcal{D}(A, \hat{E}) \leq \delta$, $\mathcal{D}(A, \hat{E}_{\epsilon/2}) > \delta$, there exists $\tilde{A} \in \hat{E}_{\epsilon/2}$ such that $\mathcal{D}(A, \tilde{A}) \leq \delta$. Since $\mathcal{D}(\tilde{A}, \mathbb{1}_n \boldsymbol{x}^\top) < \epsilon/2$, we have $\mathcal{D}(A, \mathbb{1}_n \boldsymbol{x}^\top) \leq \mathcal{D}(A, \tilde{A}) + \mathcal{D}(\tilde{A}, \mathbb{1}_n \boldsymbol{x}^\top) < \epsilon$. Therefore, $B_1 \subset \mathcal{U}(\mathbb{1}_n \boldsymbol{x}^\top, \epsilon)$. Moreover, since B_2 is compact, $\tilde{V}(A)$ is lower bounded and $d^+ \tilde{V}(A)/dt$ is strictly upper bounded from 0 in B_2. Since $\lim_{t\to+\infty} \mathcal{D}(A(t), \hat{E}) = 0$, there exists $t_0 > 0$ such that $A(t) \in B_1 \cup B_2$ for any $t \geq 0$. Therefore, for any $t \geq t_0$, there exists $t_1 \geq t$ such that $A(t_1) \in B_1$. This argument is valid for any $\epsilon > 0$, which implies that $\mathbb{1}_n \boldsymbol{x}^\top$ is an ω-limit point for any given $A(0)$.

Since $\hat{E}$ is compact, $\mathcal{D}(A, \tilde{E})$ is strictly positive. Since $\lim_{t\to+\infty} \mathcal{D}\big(A(t), \hat{E}\big) = 0$, any $A \in \Omega \setminus \hat{E}$ can not be an ω-limit point of $A(0)$. For any $A \in \hat{E} \setminus \{\mathbb{1}_n \boldsymbol{x}^\top\}$, since the solution passing through A asymptotically converges to $\mathbb{1}_n \boldsymbol{x}^\top$, $A \in \hat{E} \setminus \{\mathbb{1}_n \boldsymbol{x}^\top\}$ can not be an ω-limit point of $A(0)$ either. Therefore, the ω-limit set of $A(0)$ is $\{\mathbb{1}_n \boldsymbol{x}^\top\}$.

Acknowledgment

The content of the information does not necessarily reflect the position or the policy of the Government, and no official endorsement should be inferred. A preliminary version [22] has been accepted by the 53rd IEEE Conference on Decision and Control. Compared with [22], this paper proposes more generalized models, discusses numerous model variations, and provides all the mathematical proofs absent in [22].

References

[1] E. G. Anderson, Jr and K. Lewis, "A dynamic model of individual and collective learning amid disruption," *Org. Sci.*, vol. 25, no. 2, pp. 356–376, 2013, doi:10.1287/orsc.2013.0854.

[2] J. Barreiro-Gomez, N. Quijano, and C. Ocampo-Martinez, "Constrained distributed optimization: A population dynamics approach," *Autom.*, vol. 69, pp. 101–116, 2016, doi:10.1016/j.automatica.2016.02.004.

[3] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, "Randomized gossip algorithms," *IEEE Trans. Inf. Theory*, vol. 52, no. 6, pp. 2508–2530, Jun. 2006, doi:10.1109/TIT.2006.874516.

[4] S. Y. Choi, H. Lee, and Y. Yoo, "The impact of information technology and transactive memory systems on knowledge sharing, application, and team performance: A field study," *Manage. Inf. Syst. Quart.*, vol. 34, no. 4, pp. 855–870, 2010. [Online]. Available: http://www.jstor.org/stable/25750708

[5] R. Cressman and Y. Tao, "The replicator equation and other game dynamics," *Proc. Nat. Acad. Sci.*, vol. 111, pp. 10810–10817, 2014, doi:10.1073/pnas.1400823111.

[6] M. H. DeGroot, "Reaching a consensus," *J. Amer. Stat. Assoc.*, vol. 69, no. 345, pp. 118–121, 1974, doi:10.1080/01621459.1974.10480137.

[7] L. Festinger, "A theory of social comparison processes," *Human Relations*, vol. 7, no. 2, pp. 117–140, 1954, doi:10.1177/001872675400700202.

[8] N. E. Friedkin, "A formal theory of reflected appraisals in the evolution of power," *Adm. Sci. Quart.*, vol. 56, no. 4, pp. 501–529, 2011, doi:10.1177/0001839212441349.

[9] N. E. Friedkin and C. E. Johnsen, "Attitude change, affect control, and expectation states in the formation of influence networks," *Adv. Group Process.*, vol. 20, pp. 1–29, 2003, doi:10.1016/S0882–6145(03)20001-1.

[10] A. Jadbabaie, A. Sandroni, and A. Tahbaz-Salehi, "Non-Bayesian social learning," *Games Econ. Behav.*, vol. 76, no. 1, pp. 210–225, 2012, doi:10.1016/j.geb.2012.06.001.

[11] P. Jia, N. E. Friedkin, and F. Bullo, "The coevolution of appraisal and influence networks leads to structural balance," *IEEE Trans. Netw. Sci. Eng.*, vol. 3, no. 4, pp. 286–298, Oct.–Dec. 2016, doi:10.1109/TNSE.2016.2600058.

[12] P. Jia, A. MirTabatabaei, N. E. Friedkin, and F. Bullo, "Opinion dynamics and the evolution of social power in influence networks," *SIAM Rev.*, vol. 57, no. 3, pp. 367–397, 2015, doi:10.1137/130913250.

[13] J. P. LaSalle, "Stability theory for ordinary differential equations," *J. Differ. Equ.*, vol. 4, pp. 57–65, 1968, doi:10.1016/0022–0396(68)90048-X.

[14] J. Y. Lee, D. G. Bachrach, and K. Lewis, "Social network ties, transactive memory, and performance in groups," *Org. Sci.*, vol. 25, no. 3, pp. 951–967, 2014, doi:10.1287/orsc.2013.0884.

[15] K. Lewis, "Measuring transactive memory systems in the field: Scale development and validation," *J. Appl. Psychol.*, vol. 88, no. 4, pp. 587–604, 2003, doi:10.1037/0021–9010.88.4.587.

[16] K. Lewis, "Knowledge and performance in knowledge-worker teams: A longitudinal study of transactive memory systems," *Manage. Sci.*, vol. 50, pp. 1519–1533, 2004, doi:10.1287/mnsc.1040.0257.

[17] D. W. Liang, R. Moreland, and L. Argote, "Group versus individual training and group performance: The mediating role of transactive memory," *Personality Social Psychol. Bull.*, vol. 21, pp. 384–393, 1995, doi:10.1177/0146167295214009.

[18] J. Lorenz and D. A. Lorenz, "On conditions for convergence to consensus," *IEEE Trans. Autom. Control*, vol. 55, pp. 1651–1656, Jul. 2010, doi:10.1109/TAC.2010.2046086.

[19] D. Madeo and C. Mocenni, "Game interactions and dynamics on networked populations," *IEEE Trans. Autom. Control*, vol. 60, pp. 1801–1810, Jul. 2015, doi:10.1109/TAC.2014.2384755.

[20] J. G. March, "Exploration and exploitation in organizational learning," *Org. Sci.*, vol. 2, no. 1, pp. 71–87, 1991, doi:10.1287/orsc.2.1.71.

[21] J. R. Marden, H. P. Young, G. Arslan, and J. S. Shamma, "Payoff-based dynamics for multiplayer weakly acyclic games," *SIAM J. Control Optim.*, vol. 48, no. 1, pp. 373–396, 2009, doi:10.1137/070680199.

[22] W. Mei, N. E. Friedkin, K. Lewis, and F. Bullo, "Dynamic models of appraisal networks explaining collective learning," in *Proc. IEEE Conf. Decis. Control*, Dec. 2016, pp. 3554–3559, doi:10.1109/CDC.2016.7798803.

[23] A. Nedić and A. Ozdaglar, "Distributed subgradient methods for multiagent optimization," *IEEE Trans. Autom. Control*, vol. 54, no. 1, pp. 48–61, Jan. 2009, doi:10.1109/TAC.2008.2009515.

[24] E. T. Palazzolo, "Organizing for information retrieval in transactive memory systems," *Commun. Res.*, vol. 32, no. 6, pp. 726–761, 2005, doi:10.1177/0093650205281056.

[25] E. T. Palazzolo, D. A. Serb, Y. C. She, C. K. Su, and N. S. Contractor, "Coevolution of communication and knowledge networks in transactive memory systems: Using computational models for theoretical development," *Commun. Theory*, vol. 16, pp. 223–250, 2006, doi:10.1111/j.1468–2885.2006.00269.x.

[26] Y. Ren, K. M. Carley, and L. Argote, "The contingent effects of transactive memory: When is it more beneficial to know what others know?," *Manage. Sci.*, vol. 52, pp. 671–682, 2006 doi:10.1287/mnsc.1050.0496.

[27] W. H. Sandholm. *Population Games and Evolutionary Dynamics*. Cambridge, MA, USA: MIT Press, 2010.

[28] R. Tempo, G. Calafiore, and F. Dabbene, *Randomized Algorithms for Analysis and Control of Uncertain Systems*. New York, NY, USA: Springer, 2005.

[29] D. M. Wegner, "Transactive memory: A contemporary analysis of the group mind," in *Theories of Group Behavior*, B. Mullen and G. R. Goethals, Eds. New York, NY, USA: Springer, 1987, pp. 185–208, doi:10.1007/978–1-4612-4634-3_9.

[30] D. M. Wegner, R. Erber, and P. Raymond, "Transactive memory in close relationships," *J. Personality Social Psychol.*, vol. 61, pp. 923–929, 1991, doi:10.1037/0022–3514.61.6.923.

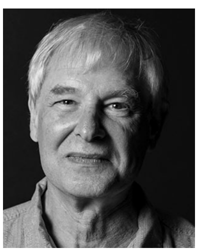
[31] J. H. Wilkinson, *The Algebraic Eigenvalue Problem*. London, U.K.: Oxford Univ. Press, 1965.

[32] A. W. Woolley, C. F. Chabris, A. Pentland, N. Hasnmi, and T. W. Malone, "Evidence for a collective intelligence factor in the performance of human groups," *Science*, vol. 330, pp. 686–688, 2010, doi:10.1126/science.1193147.

[33] S. Wuchty, B. F. Jones, and B. Uzzi, "The increasing dominance of teams in production of knowledge," *Science*, vol. 316, pp. 1036–1039, 2007, doi:10.1126/science.1136099!.

[34] Y. C. Yuan, I. Carboni, and K. Ehrlich, "The impact of awareness and accessibility on expertise retrieval: A multilevel network perspective," *J. Amer. Soc. Inf. Sci. Technol.*, vol. 61, no. 4, pp. 700–714, 2010, doi:10.1002/asi.21287.

Wenjun Mei received the Bachelor of Science degree in theoretical and applied mechanics from Peking University, Beijing, China, in 2011. He is currently working toward the Ph.D. degree at the Department of Mechanical Engineering, University of California at Santa Barbara, Santa Barbara, CA, USA.



Noah E. Friedkin received the B.A. and Ph.D. degrees in sociology from the University of Chicago, Chicago, IL, USA, in 1969 and 1977, respectively.

He is currently a Professor of sociology and a faculty member of the Center for Control, Dynamical-Systems, and Computation at the University of California at Santa Barbara, Santa Barbara, CA, USA. He was the Chair of the Mathematical Sociology Section of the American Sociological Association twice, chair of the Sociology Department, and is currently the Editor-in-Chief of the Journal of Mathematical Sociology.

Dr. Friedkin received the two Harrison White Outstanding Book Awards from the Mathematical Sociology Section of the American Sociological Association and election to the Sociological Research Association.

Kyle Lewis received the B.A. degrees in both computer science and mathematics at Duke University, Durham, NC, USA, in 1983, the M.S. degree in industrial administration from Carnegie-Mellon University, Pittsburgh, PA, USA, in 1990, and the Ph.D. degree in organizational behavior and human resources management from the University of Maryland, College Park, MD, USA, in 1999.

She is currently a Professor of Technology Management at University of California at Santa Barbara, Santa Barbara, CA, USA. She has published articles in Management Science, Academy of Management Review, Academy of Management Journal, Organization Science, Journal of Management, Organizational Behavior and Human Decision Processes, and Group Dynamics. He was a Division Chair in the Academy of Management and a Senior Editor for Organization Science.

Francesco Bullo (F'10) received the Laurea degree "summa cun laude" in electrical engineering from the University of Padova, in 1994, and the Ph.D. degree in control and dynamical systems from the California Institute of Technology, in 1999.

He is a Professor with Department of the Mechanical Engineering and the Center for Control, Dynamical Systems and Computation, University of California at Santa Barbara, Santa Barbara, CA, USA. He was previously associated with the University of Padova, the California Institute of Technology, and the University of Illinois. His research interests include network systems and distributed control with application to robotic coordination, power grids, and social networks. He is the coauthor of Geometric Control of Mechanical Systems (Springer, 2004) and Distributed Control of Robotic Networks (Princeton, 2009); his forthcoming Lectures on Network Systems is available on his website.

Mr. Bullo received the Best Paper Awards for his work in IEEE CONTROL SYSTEMS, *Automatica*, *SIAM Journal on Control and Optimization*, IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS, and IEEE TRANSACTIONS ON CONTROL OF NETWORK SYSTEMS. He is a Fellow of IFAC. He was with the editorial boards of the IEEE, SIAM, and ESAIM journals, and will serve as IEEE CSS President in 2018.