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Second-order Pseudo-Gaussian Shaper

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The purpose of this document is to provide a calculus spreadsheet for the design of second-order pseudo-gaussian shapers. A very interesting reference is given by C.H. Mosher "Pseudo-Gaussian Transfer Functions with Superlative Recovery", IEEE TNS Volume 23, p. 226-228 (1976). Fred Goulding and Don Landis have studied the structure of those filters and their implementation and this document will outline the calculation leading to the relation between the coefficients of the filter.

The general equation of the second order pseudo-gaussian filter is:

$$f(t) = P_0 \cdot e^{-3kt} \cdot \sin^2(kt) \qquad (1)$$

The parameter k is a normalization factor.

1. Filter Transfer Function

A possible implementation of the filter is based on the following schematic (modified Bridge T):

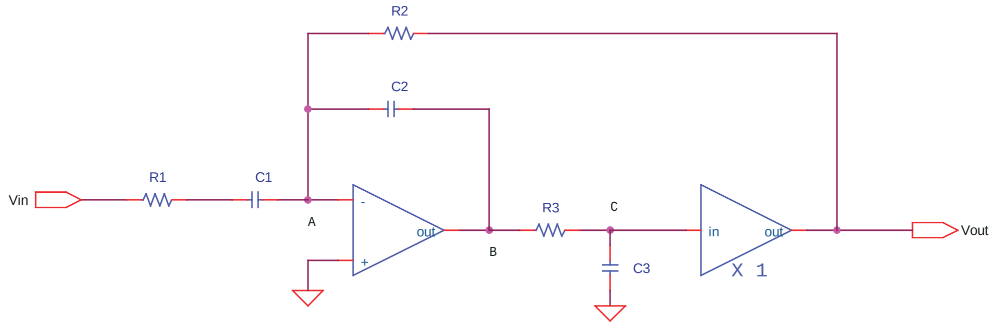

Figure 1: Schematic of the second-order pseudo gaussian shaper.

The analysis conducted hereafter assumes that all the components of the schematic of Figure 1 are ideal.

1.1. Notations

The following notations are adopted for the calculations:

- $Z_1 = R_1 // C_1 = \dfrac{1 + sR_1C_1}{sC_1}$
- $Z_{C2} = \dfrac{1}{sC_2}$
- $a_i = \dfrac{1}{R_iC_i}$
- The voltage potentials on nodes A, B or C are noted V_A, V_B, or V_C.

1.2. Transfer function calculation

The analysis of the circuit leads to the following system of equations:

- Summing all the currents at node A gives:

$$\frac{V_{in}}{Z_1} + \frac{V_B}{Z_{C2}} + \frac{V_{out}}{R_2} = 0 \tag{2}$$

- The potential V_C is equal to V_{out} and can be expressed as a function of V_B:

$$V_{out} = \frac{1}{1 + sR_3C_3} V_B \tag{3}$$

Equation (3) can be rewritten as:

$$V_B = (1 + sR_3C_3) V_{out} \tag{4}$$

Replacing V_B in Equation (2) with the expression of Equation (4) leads to the filter transfer function:

$$\boxed{H(s) = \frac{V_{out}}{V_{in}} = \frac{-sR_2C_1}{(1 + sR_1C_1)[1 + sR_2C_2(1 + sR_3C_3)]}} \tag{5}$$

Introducing the a_i coefficients, Equation (5) can be rewritten as:

$$\boxed{H(s) = \frac{V_{out}}{V_{in}} = \frac{-s\, a_1 a_2 a_3 R_2 C_1}{(s + a_1)(s^2 + a_3\, s + a_2 a_3)}} \tag{6}$$

2. Step Response

In the Laplace space, the unity step response of the filter is given by the following equation:

$$F(s) = H(s) \times \frac{1}{s} = \frac{-a_1 a_2 a_3 R_2 C_1}{(s + a_1)(s^2 + a_3\, s + a_2 a_3)} \tag{7}$$

The pole separation is the next step of the analysis, and Equation (7) must be expressed as:

$$F(s) = \frac{\alpha}{s + a_1} + \frac{\beta + \gamma\, s}{s^2 + a_3\, s + a_2 a_3}$$

The coefficients α,β,γ can be determined by identification with Equation (7), leading to the following set of equations:

$$\begin{cases} \alpha + \gamma = 0 \\ a_3 \cdot \alpha + \beta + a_1 \cdot \gamma = 0 \\ a_2 a_3 \cdot \alpha + a_1 \cdot \beta = -a_1 a_2 a_3 R_2 C_1 \end{cases}$$

Leading to the solution:

$$\begin{cases} \alpha = \dfrac{-a_1 a_2 a_3 R_2 C_1}{a_1^{\ 2} - a_1 a_3 + a_2 a_3} \\ \beta = (a_1 - a_3) \cdot \alpha \\ \gamma = -\alpha \end{cases}$$

Finally Equation (7) can be written as:

$$\boxed{F(s) = \frac{-a_1 a_2 a_3 R_2 C_1}{a_1^{\ 2} - a_1 a_3 + a_2 a_3} \left\{ \frac{1}{s + a_1} - \frac{s + (a_3 - a_1)}{s^2 + a_3\, s + a_2 a_3} \right\}} \tag{8}$$

2.1. Pseudo-Gaussian Function

Equation (8) represents the general Laplace response of the two-pole filter to a step function. In order to obtain a pseudo-gaussian response similar to Equation (1), the coefficients a_i must follow some criteria. The first term of the parenthesis corresponds to a delay function in the time domain, which can be associated to the delay component of the pseudo-gaussian function (hypothesis #1). The second term has either complex poles or real poles depending on the coefficient values. In the case of complex poles, the time-domain equivalent function will contain a sinusoidal component (hypothesis #2).

The denominator of the second term of the parenthesis is given by the following equation:

$$s^2 + a_3 s + a_2 a_3 = \left(s + \frac{a_3}{2}\right)^2 + a_2 a_3 - \left(\frac{a_3}{2}\right)^2 \tag{9}$$

- ***Hypothesis #1:*** $a_3 = 2 a_1$

  Equation (9) can be rewritten as:

$$s^2 + a_3 s + a_2 a_3 = (s + a_1)^2 + 2a_1 a_2 - a_1^2 \tag{10}$$

- ***Hypothesis #2:*** $2a_1 a_2 - a_1^2 > 0 \Leftrightarrow 2a_2 > a_1$ ***(since*** $a_1 > 0$***)***

  In this case, the roots of the polynomial expression represented by Equation (10) are complex, insuring that the time-domain response is sinusoidal.

$$s^2 + a_3 s + a_2 a_3 = (s + a_1)^2 + b^2 \tag{11}$$

With $$\boxed{b^2 = 2a_1 a_2 - a_1^2} \tag{12}$$

With those conditions on the coefficients, the constant term of Equation (8) can be simplified:

$$\frac{-a_1 a_2 a_3 R_2 C_1}{a_1^2 - a_1 a_3 + a_2 a_3} = \frac{-2a_2}{2a_2 - a_1} \times \frac{R_2}{R_1}$$

The step response of the filter can be expressed as:

$$\boxed{F(s) = \frac{-2a_2}{2a_2 - a_1} \times \frac{R_2}{R_1} \left\{ \frac{1}{s + a_1} - \frac{s + a_1}{(s + a_1)^2 + b^2} \right\}} \tag{13}$$

Finally the time domain response of the filter is:

$$\boxed{f(t) = \frac{-2a_2}{2a_2 - a_1} \times \frac{R_2}{R_1} \times \exp(-a_1 \cdot t) \times [1 - \cos(b \cdot t)]} \qquad (14)$$

The fourth term of the expression can be expressed as a sine function the following way:

$$1 - \cos(b \cdot t) = 2\sin^2\left(\frac{b}{2} \cdot t\right)$$

Leading to the expression:

$$\boxed{f(t) = \frac{-4a_2}{2a_2 - a_1} \times \frac{R_2}{R_1} \times \exp(-a_1 \cdot t) \times \sin^2\left(\frac{b}{2} \cdot t\right)} \qquad (15)$$

This expression is only valid under the conditions of the hypotheses #1 and #2.

2.2. Peaking Time

When designing any shaper, the peaking time is one of the main criteria. The peaking time corresponds to the maximum of the function *f* described by Equation (15). The peaking time is therefore one of the roots of the function's derivative.

$$f'(t) = \frac{-4a_2}{2a_2 - a_1} \times \frac{R_2}{R_1} \times \exp(-a_1 \cdot t) \times \sin\left(\frac{b}{2} \cdot t\right) \times \cos\left(\frac{b}{2} \cdot t\right) \times \left[b - a_1 \cdot \tan\left(\frac{b}{2} \cdot t\right)\right]$$

The solutions of the equation $f'(t) = 0$ are numerous and are:

$$\begin{cases} t = 0 \\ t = \dfrac{2}{b} Arctg\left(\dfrac{b}{a_1}\right) + n \cdot \pi, \quad \text{n} = 0,1,2... \end{cases}$$

The first solution (*t = 0)* shows that the response starts with a horizontal tangent. For the other solutions, it can be shown that n=0 corresponds to the maximum of the function. The others roots (n>0) lead to ripples on the relaxation of the pulse and it can be shown that the amplitude of those is negligible.

The peaking time of the second-order pseudo-gaussian shaper is given by

$$\boxed{\tau_p = \frac{2}{b} Arctg\left(\frac{b}{a_1}\right)} \tag{16}$$

2.3. Peak Value

The expression of the peak value is obtained by replacing *t* by τ_p in Equation (15).

$$Peak\_Value = f(\tau_p) = \frac{-4a_2}{2a_2 - a_1} \times \frac{R_2}{R_1} \times \exp(-a_1 \cdot \tau_p) \times \sin^2\left(Arctg\left(\frac{b}{a_1}\right)\right)$$

Taking into account that:

$$\sin^2(x) = \frac{\tan^2(x)}{1 + \tan^2(x)}$$

Then: $\sin^2[Arctg(y)] = \dfrac{y^2}{y^2 + 1}$

Yielding to,

$$\boxed{Peak\_Value = f(\tau_p) = \frac{-4a_2}{2a_2 - a_1} \times \frac{R_2}{R_1} \times \exp(-a_1 \cdot \tau_p) \times \frac{b^2}{b^2 + a_1^2}} \tag{17}$$

3. Filter Synthesis

The analysis conducted so far did not include the specific expression of Equation (1) in which the relation between a_1 and b is fixed.

$$f(t) = P_0 \cdot e^{-3kt} \cdot \sin^2(kt) \qquad (1)$$

The results from the previous sections will be applied to this particular example. After determining the expression of coefficients a_i, a few examples will be given.

3.1. Expression of the Filter coefficients as a function of k

This expression has to be compared to Equation (15), which leads to:

$$a_1 = 3k$$
$$b = 2k$$

From those parameters it is possible to calculate a_2 from the expression of b (Equation 12) and a_3 from hypothesis #1:

$$a_2 = \frac{13}{6}k$$
$$a_3 = 6k$$

Hypothesis #2 is always verified for any value of k.

With these coefficients, the expression of the coefficient P_0 is the following:

$$P_0 = -\frac{13}{2} \times \frac{R_2}{R_1}$$

The filter step response can then be written as:

$$f(t) = -\frac{13}{2} \times \frac{R_2}{R_1} \cdot e^{-3kt} \cdot \sin^2(kt)$$

The peaking time expression is then:

$$\boxed{\tau_p = \frac{1}{k} Arctg\left(\frac{2}{3}\right)} \qquad (18)$$

Finally, using Equation (17), it is also possible to calculate the peak value:

$$\boxed{Peak\_Value = -2 \times \frac{R_2}{R_1} \times \exp\left[-3 \cdot Arctg\left(\frac{2}{3}\right)\right]} \tag{19}$$

The peak value defines the ratio of R_2 over R_1 and can be made independent of the values of the peaking time and of the parameter k.

Since the filter synthesis always starts with the choice of the peaking time, it is important to calculate the relations of the coefficients to the peaking time for this particular function.

3.2. Expression of the Filter coefficients as a function of the peaking time

The peaking time expression given by (18) yields:

$$\boxed{k = \frac{1}{\tau_p} Arctg\left(\frac{2}{3}\right)} \tag{20}$$

It is therefore possible to express all the coefficients as a function of the peaking time.

Peaking time	τ_p
a_1	$a_1 = \frac{3}{\tau_p} Arctg\left(\frac{2}{3}\right)$
a_2	$a_2 = \frac{13}{6 \cdot \tau_p} Arctg\left(\frac{2}{3}\right)$
a_3	$a_3 = \frac{6}{\tau_p} Arctg\left(\frac{2}{3}\right)$

Table 1: Filter Coefficients vs. Peaking Time

3.3. Example

- Coefficients' Calculation

Peaking Time		**1.00E-07**
K		5.88E+06
A1=3K		1.76E+07
A2=13K/6		1.27E+07
A3=2A1=6K		3.53E+07
A1=1/(R1xC1)	C1	6.80E-11
	R1	8.34E+02
A2=1/(R2xC2)	C2	6.80E-11
	R2	1.15E+03
A3=1/(R3xC3)	C3	6.80E-11
	R3	4.17E+02
Peak Value		4.75E-01

Table 2: Calculation Example

- Implementation

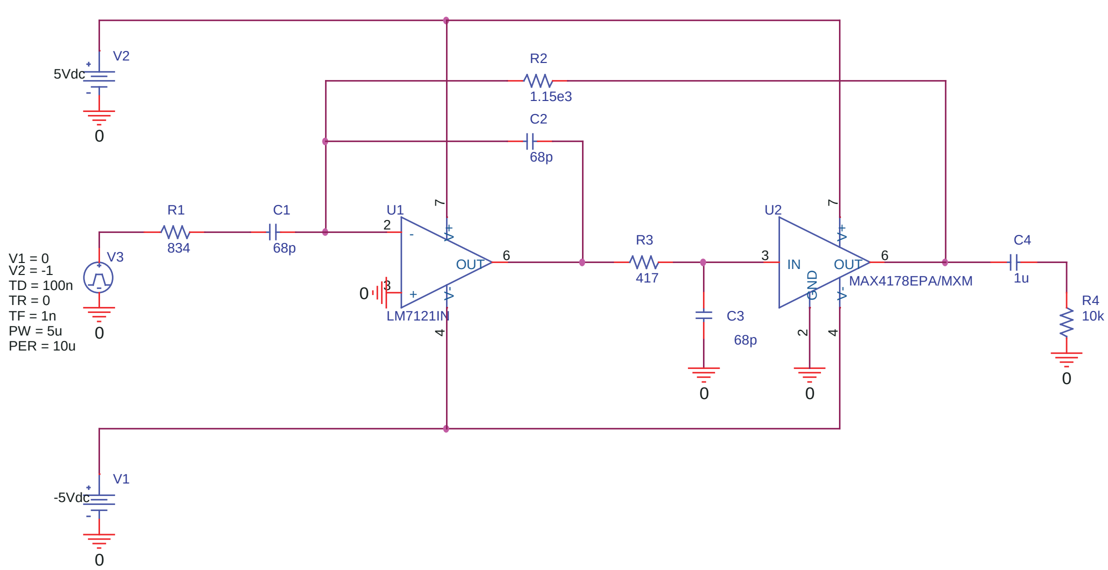

Figure 2: Filter Implementation

- Simulation Results

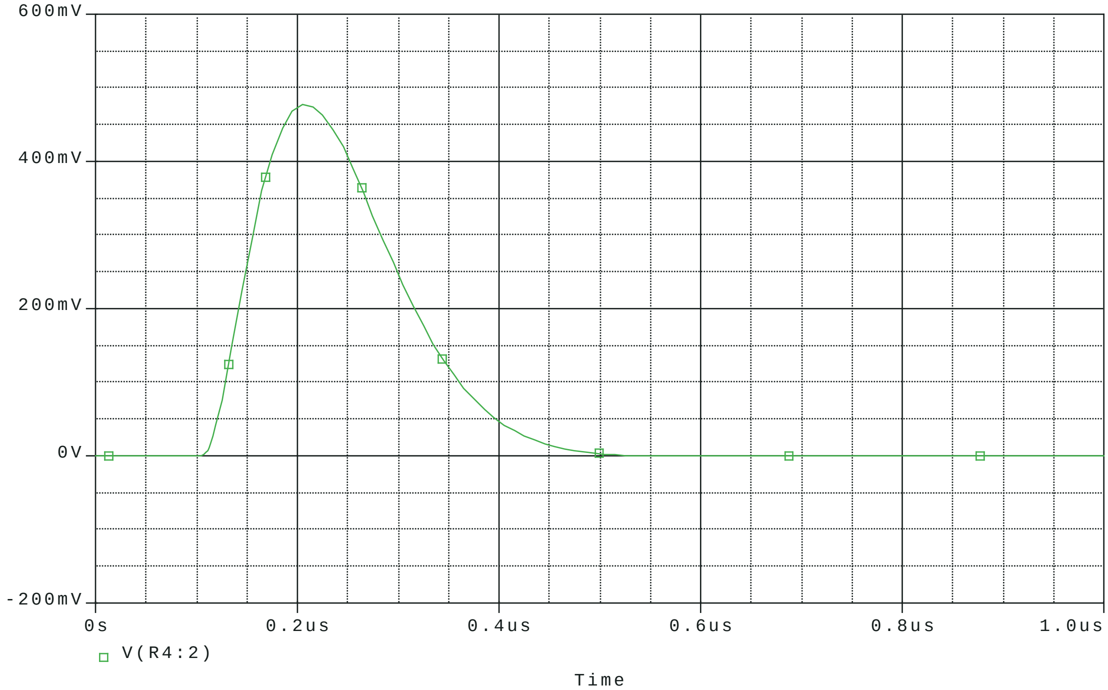

Figure 3: Response of the filter to a voltage step (Amplitude –1V).

The peaking time of the pulse is 100ns and its amplitude is 475mV, as the calculation predicted.