A simple method for resolving degeneracies in Delaunay triangulations

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Abstract

We characterize the conditions under which completing a Delaunay tessellation produces a configuration which is a nondegenerate Delaunay triangulation of an arbitrarily small perturbation of the original sites. One consequence of this result is a simple method for resolving degeneracies in Delaunay triangulations that does not require symbolic perturbation of the data.

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1 Introduction

A *data-induced degeneracy* (or simply *degeneracy*) in a geometric computation is a subset of the input that does not satisfy the "general position" assumptions appropriate for the computation. For example, a degeneracy in a line arrangement is a set of three or more concurrent lines. In the context of planar Delaunay triangulations, a degeneracy is either (1) a set of 4 or more cocircular generating sites such that the circle through the sites contains no other generating site in its interior, or (2) a set of three or more collinear generating sites on the boundary of the convex hull.

Handling degeneracies correctly is an important, and subtle, practical issue that arises in the implementation of geometric algorithms. It is generally desirable to resolve a data-induced degeneracy by computing a nondegenerate output that can be realized by an arbitrarily small perturbation of the input. General techniques, based on symbolic perturbation schemes, are developed in [9, 10, 21]. All of these techniques involve considerable computational overhead.

In this paper, we consider the special case of of two-dimensional Delaunay triangulations. It is well-known that not all possible triangulations are realizable as Delaunay triangulations [12, 6]. Indeed, only an exponentially small fraction of triangulations have such a realization [20]. Hence one would expect that some care is necessary when removing degeneracies from Delaunay triangulations and indeed, this turns out to be the case. However, the amount of care required turns out to be modest. In particular, we show that general symbolic-perturbation schemes are unnecessary, and that a much simpler method of resolving degeneracies suffices.

Our main result (Theorem 3.1) is a characterization of the conditions under which completing a degenerate Delaunay tessellation yields a configuration which is the nondegenerate triangulation of an arbitrarily small perturbation of the input. The proof of Theorem 3.1 is given in Section 5. The proof is based on a characterization of the conditions under which adding an edge to an inscribable polyhedron preserves inscribability (Theorem 5.1), which may be of independent interest. Practical consequences for Delaunay triangulation algorithms are discussed in Section 7.

Somewhat related to our result is a recent algorithm by Fortune for computing "approximate" Delaunay triangulations using fixed-precision arithmetic [11]. Fortune's algorithm uses $O(n^2)$ fixed-precision operations and produces a triangulation that satisfies an approximate Delaunay condition. However, there is no guarantee that the output of his algorithm will be the Delaunay triangulation of any input. Our theorem, like the general schemes of [9, 10, 21], is only applicable if exact arithmetic is being used (since otherwise it is impossible to correctly detect degeneracies.) If exact arithmetic is used, our theorem provides a means of ensuring that the output is a Delaunay triangulation of an arbitrarily small perturbation of the input.

2 Preliminaries

Except as noted, we use the graph-theoretical notation and definitions of [2]. In particular, $V(G)$ and $E(G)$ denote the set of vertices and edges of a graph G, respectively. A *triangulation* is a 2-connected plane graph in which all faces except possibly the outer face are bounded by triangles. A *maximal planar graph* is a planar graph in which all faces (including the outer face) are bounded by triangles. A graph G is *1-tough* [4] if for all nonempty $S \subseteq V(G)$, $c(G - S) \leq |S|$. (Here $|\cdot|$ denotes cardinality, and $c(\cdot)$ denotes the number of connected components.) G is *1-supertough* if, for all $S \subseteq V(G)$ with $|S| \geq 2$, $c(G - S) < |S|$.

The *Delaunay tessellation*, DT(S), of a planar set of points S is the unique graph with $V(G) = S$ such that the outer face is bounded by the convex hull of S, all vertices on the boundary of a common interior face are cocircular, the vertices of an interior face are exactly the points of S lying on the circumcircle of the face, and no points of S lie in the interior of a circumcircle of any interior face. DT(S) is said to be *nondegenerate* if it is a triangulation and all convex hull vertices of S are extreme points of S, *degenerate* otherwise. If DT(S) is nondegenerate, it is called the Delaunay triangulation. Elementary properties of the Delaunay tessellation/triangulation, and the more conventional definition as the dual of the Voronoi diagram, are developed in [1, 8, 15].

Let S be a set of generating sites, DT(S) its Delaunay tessellation. Define a *completion* of DT(S) to be a triangulation obtained by

1. Adding diagonals to non-triangular interior faces of DT(S);
2. Declaring each non-extreme site on the convex hull of S to be either "extreme" or "non-extreme," and
3. Adding new edges so that the sites incident on the outer face are exactly the sites that either are extreme sites or were declared "extreme" in step 2.

If DT(S) is nondegenerate, it has only one completion, namely itself. If DT(S) is degenerate, there is more than one way to complete it. In this case, we call *any* completion of DT(S) a *degenerate Delaunay triangulation.*

3 The main result

We say a Delaunay triangulation is *valid* if it can be realized as a nondegenerate Delaunay triangulation of an arbitrarily small perturbation of its input. Obviously, any nondegenerate Delaunay triangulation is valid. Theorem 3.1, below, characterizes the completions of Delaunay tessellations that produce valid Delaunay triangulations.

The *augmented Delaunay tessellation* of S, ADT(S), is the graph obtained from DT(S) by adding a new vertex ∞, representing the point at infinity, and connecting all extreme points of S to ∞. If ADT(S) is a bipartite graph, assume that its vertices are two-colored red and blue. In this case, a *weakly symmetric* completion of DT(S) is a

completion obtained by first adding edges to ADT(S) to obtain a maximal planar graph, in such a way that at least one red-red edge and at lease one blue-blue edge are added, and then removing the vertex ∞ (and all attached edges).

Theorem 3.1 *Let S be a set of planar points.*

(a) *If the augmented Delaunay tessellation* ADT(S) *is not bipartite, then any completion of $DT(S)$ is valid.*

(b) *If* ADT(S) *is bipartite, then a completion of* DT(S) *is valid if and only if it is weakly symmetric. If the completion is not weakly symmetric, then it is not the Delaunay triangulation of* any *input.*

The proof of this theorem is given in Section 5.

Theorem 3.1(a) says that except in highly degenerate cases, *any* completion of a Delaunay tessellation yields a valid Delaunay triangulation. In particular, if ADT(S) is bipartite, every face of DT(S) has even valence (so DT(S) has no triangular faces!) and no two extreme points of S appear consecutively on the convex hull of S. Such behavior, while pathological, can occur: we give an example in Section 6. Since testing ADT(S) for bipartiteness (and ensuring weak symmetry of the completion, if necessary) is straightforward, Theorem 3.1 provides a simple method for postprocessing a degenerate Delaunay triangulation to obtain a triangulation that is a nondegenerate Delaunay triangulation of an arbitrarily small perturbation of the input. We discuss this in more detail in Section 7.

4 Some facts about inscribable graphs

The proof of Theorem 3.1 uses some theoretical results about inscribable graphs, which we summarize here. A graph G is *polyhedral* if it can be realized as the edges and vertices of a noncoplanar set of points in 3-space. A famous theorem of Steinitz (see [12]) asserts that a graph is polyhedral if and only if it is 3-connected and planar. A graph G is *inscribable* if it can be realized as the edges and vertices of the convex hull of a noncoplanar set of points on the surface of a sphere in 3-space. An *inscription* of G is an assignment of coordinates to the vertices of G achieving this realization. A *cutset* in a graph is a minimal set of edges whose removal increases the number of components. A cutset is *noncoterminous* if its edges do not all have a common endpoint.

Lemma 4.1 ([13, 16, 17, 19]) *A graph is inscribable if and only if it is polyhedral and weights w can be assigned to its edges such that:*

(W1) *For each edge e, $0 < w(e) < 1/2$.*

(W2) *For each vertex v, the total weight of all edges incident on v is equal to 1.*

(W3) *For each noncoterminous cutset $C \subseteq E(G)$, the total weight of all edges in C is strictly greater than 1.*

The weights in Theorem 4.1 correspond to certain normalized hyperbolic angles in a realization of the polyhedron as a convex polyhedron in hyperbolic 3-space with vertices on the ideal sphere. It can be shown [18] that these angles uniquely determine an inscription, up to homothetic transformations. Furthermore, there is a continuity relation between the weights and the inscription, which is precisely formulated in the following lemma.

Lemma 4.2 (**[18]**) *Let G be an inscribable graph, w a weighting, I an inscription of G that realizes w, $\epsilon > 0$.*

(a) *There exists a real number $\delta = \delta(G, w, \epsilon)$ such that if w' is any other weighting of G satisfying conditions (W1)–(W3) and for which $|w(e) - w'(e)| < \delta$ for all $e \in E(G)$, then there is an inscription I' of G realizing w' with $d(I(v), I'(v)) < \epsilon$ for all $v \in V(G)$.*

(b) *Let H be a planar graph obtained from G by adding edges $e_1, \ldots, e_k$. There exists a real number $\delta = \delta(G, w, \epsilon)$ such that if w' is any weighting of H satisfying conditions (W1)–(W3) and the additional conditions that $|w(e) - w'(e)| < \delta$ for all $e \in E(G)$ and $0 < w'(e_i) < \delta$ for $1 \leq i \leq k$, then there is an inscription I' of H realizing w' with $d(I(v), I'(v)) < \epsilon$ for all $v \in V(G)$.* ■

The following lemma describes the connection between Delaunay tessellations and inscribable graphs. It is a different formulation from that in [3]. The operation of *stellating* a face f in a plane graph G consists of adding a vertex inside the face f and then connecting all vertices incident on f to the new vertex.

Lemma 4.3 *A plane graph G is realizable as* DT(S) *for some set S, with f as the unbounded face, if and only if the graph G' obtained from G by stellating f is inscribable. In this case, G' is realizable as* ADT(S).

Proof The lemma follows immediately from the fact that stereographic projection of a plane (together with the point at ∞) onto a sphere maps lines onto circles passing through the pole, and circles onto circles not passing through the pole. See [5] for a discussion of stereographic projection. ■

5 Proof of Theorem 3.1

In this section, we prove Theorem 3.1. We first establish the circumstances under which adding an edge to an inscribable graph preserves inscribability.

Theorem 5.1 *Let G be an inscribable graph. Suppose that H is obtained from G by performing any of the following transformations in such a way that H remains planar.*

(T1) *If G is nonbipartite, adding an edge to G.*

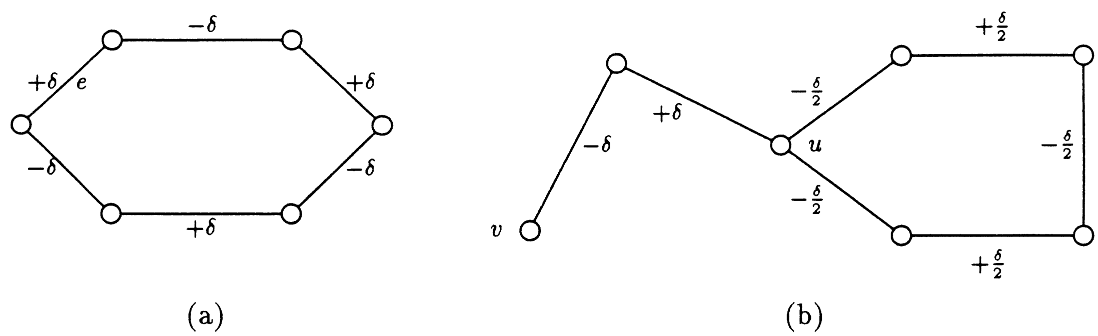

Figure 1: (a) Adding an edge e when at least one face of H incident on e has even valence. (b) Adding an edge e when some face of G has odd valence.

(T2) *If G is bipartite, adding a red-blue edge to G.*

(T3) *If G is bipartite, adding a red-red edge and a blue-blue edge to G.*

Then H is inscribable, and can be realized through an arbitrarily small perturbation of the vertices of G.

Proof Since G is polyhedral, so is H. Let $\epsilon > 0$ be given. Let w be a weighting of the edges of G satisfying (W1)–(W3) of Lemma 4.1, and extend this weighting to H by (temporarily) assigning the new edge(s) a weight of 0. Let α be less than the minimum slack in all the strict inequalities in (W1) and (W3) in the weighting of G, and let n be the number of vertices in G. Finally, let $\delta = \min(\alpha/n, \epsilon)/2$. We show that the temporary weights assigned to H can be modified so that (1) the modified weights provide a weighting of H satisfying (W1)–(W3), and (2) no edge weight is modified by more than 2δ. The theorem then follows from Theorems 4.1 and 4.2.

Suppose first that a single edge $e = vx$ is added (Transformation (T1) or (T2)). Since H is planar and G is 3-connected, v and x are nonconsecutive vertices incident on a common face f of G. Adding e splits f into two new faces, f_1 and f_2. If either of these two faces has even valence, alternately modify the weights of edges about this face by $\pm\delta$, with e being modified by $+\delta$, as shown in Figure 1(a). Notice that when (T2) is applied, this case must occur.

When (T1) is applied, if both f_1 and f_2 have odd valence, then f must have even valence. Since G is nonbipartite, there must be a face of G (and hence of H) with odd valence. Let Z be the cycle bounding this face. Choose a path Φ in G from v to a vertex u of Z. Modify the weights of the edges along Φ by $\pm\delta$, alternating signs, so that the edge incident on v is modified by $-\delta$. Modify the weights along Z by $\pm\delta/2$, alternating signs, so that the two edges of Z incident with u have the opposite sign from the edge of Φ incident on u. (See Figure 1(b)). Process vertex x in a similar fashion. Assign edge e a weight of δ. It is easy to see that this modification satisfies the necessary properties.

Finally, consider Transformation (T3), and let e_1 and e_2 be the two new edges. Choose any cycle Z in H passing through e_1 and e_2. Increment the weights along Z by

$\pm\delta$, alternating signs, so that $w(e_1)$ is incremented by $+\delta$. Since G is bipartite, e_1 is red-red, and e_2 is blue-blue, it follows that Z has even length and $w(e_2)$ is also incremented by $+\delta$. Hence all (W1) constraints are preserved. ■

Proof of Theorem 3.1 Suppose first that the hypotheses of Theorem 3.1(a) or (b) hold. By Lemma 4.3, ADT(S) is inscribable. It then follows from repeated applications of Theorem 5.1 that the completion of ADT(S) is inscribable, and the inscription may be achieved through an arbitrarily small perturbation of the inscription of ADT(S). Hence (a), and the "if" part of (b), follow from Lemma 4.3 and the fact that stereographic projection is a bicontinuous mapping between the sphere and the extended plane.

Now suppose that ADT(S) is bipartite, but that the completion is not weakly symmetric. Let K' be the maximal planar graph obtained by adding edges to ADT(S) and assume, without loss of generality, that all the edges are blue-blue. Let K be the graph obtained by deleting ∞ from K'. Let b and r denote, respectively, the number of blue and red vertices of ADT(S). Since all inscribable graphs are 1-tough ([6, Theorem 3.2]), $r = b$. Hence removing the b blue vertices from K' decomposes it into b components, each consisting of a single red vertex. Since K' is maximal planar and not 1-supertough, K' cannot be inscribable ([7, Theorem 2.2]). It follows from Lemma 4.3 that K is not realizable as a Delaunay triangulation of any input. ■

6 An example of a non-realizable completion

Theorem 3.1 provides a characterization of those completions of degenerate Delaunay tessellation that can be realized as Delaunay triangulations of arbitrarily small perturbations of the input. An example of a completion that fails the weak symmetry test of Theorem 3.1(b) is shown in Figure 2. This example was originally described by Kantabutra in a somewhat different context [14]. The set S of generating sites consists of the three vertices of an equilateral triangle, the midpoints of the edges, and the centroid. DT(S) is shown on the left of the figure. The two triangulations obtained by completing DT(S) in a non-weakly-symmetric fashion are shown on the right of the figure. Neither of these triangulations is realizable as a Delaunay triangulation, as they fail to satisfy the necessary conditions of Theorems 3.1 and 3.2 of [6], respectively. By Theorem 3.1, any completion of the tessellation in Figure 2, other than the two shown, can be realized by an arbitrarily small perturbation of the vertices of S.

7 Consequences for Delaunay Triangulation algorithms

Suppose that a Delaunay triangulation algorithm is correct in the sense that it produces the Delaunay triangulation if its input is nondegenerate and produces *some* degenerate Delaunay triangulation if its input is degenerate. We can ensure that the algorithm always produces a valid Delaunay triangulation by postprocessing its output as follows. Let S be an input to such an algorithm, G its output.

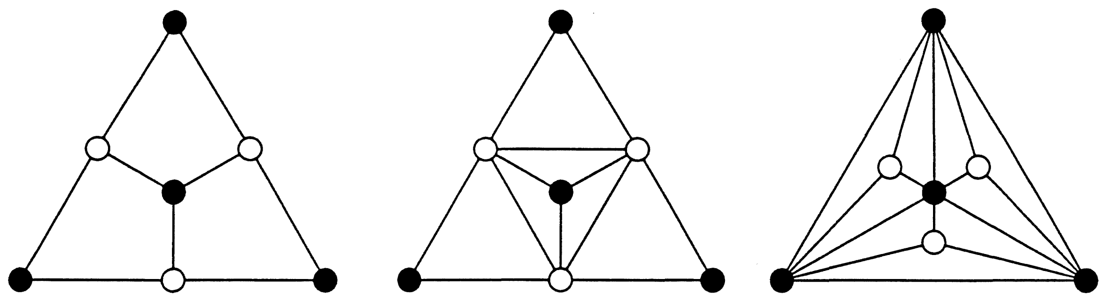

Figure 2: A degenerate Delaunay tessellation, DT(S) such that ADT(S) is bipartite (shown on the left), and the two triangulations resulting from non-weakly-symmetric completions of $ADT(S)$.

1. Add a new vertex to G, and label it ∞. Connect every vertex on the outer face of G to ∞. Call the augmented graph, which is maximal planar, G'.
2. Label each edge $e = uv$ of G' as either "real" or "artificial" as follows:
   (a) If either u or v is ∞, (assume $u = \infty$), label e as "real" if v is an extreme point of S, "artificial" otherwise. Notice that this can be done in constant time by determining whether angle $xvw = 180°$, where x and w are the two neighbors of v adjacent to u about v.
   (b) If e is an edge of the outer face of G, let x be the (unique) vertex such that $x \neq \infty$ and uxv bounds a face of G'. Label e as "real" if angle $uxv < 180°$, "artificial" if angle $uxv = 180°$.
   (c) If neither of the preceding two cases apply, e is an interior edge of G. Let w and x be the two vertices of G such that uwv and uwv bound faces of G. Label e "artificial" if u, v, w, and x are cocircular, "real" otherwise.
3. Determine whether the subgraph of G' induced by the "real" edges is bipartite. If so, 2-color the vertices of this graph red and blue, and proceed to Step 4. Otherwise, exit.
4. If there is at least one red-red "artificial" edge and at least one blue-blue "artificial" edge, exit.
5. Choose any "artificial" edge of G', say uv and delete it, creating a 4-valent face $uxvw$. Add the "opposite diagonal" xw to G'. Make the corresponding changes in G (i.e., delete uv if $u \neq \infty$ and $v \neq \infty$, add xw if $x \neq \infty$ and $w \neq \infty$).

It follows from Theorem 3.1 that the preceding postprocessing sequence produces a graph G that is a valid Delaunay triangulation of S. In the worst case, $O(|S|)$ operations are required. It is never necessary to perform more than one edge deletion and one edge addition.

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