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Publication Date

2008-09-25

To be submitted to Journal of
Mathematical Physics

UCRL-19793
Preprint

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DIFFRACTION RADIATION BY A LINE CHARGE MOVING PAST A COMB:
A MODEL OF RADIATION LOSSES IN AN ELECTRON RING ACCELERATOR^{*}

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April 23, 1970

ABSTRACT

A calculation is given of the radiated energy loss from a charged rod which moves at constant speed past an infinite set of parallel semi-infinite conducting plates of infinitesimal thickness, with the rod taken parallel to and at a fixed distance from the plate edges. The problem is analyzed using the Wiener-Hopf technique, and the resulting formulas are evaluated analytically in the limits of high rod speed and low rod speed, and compared with numerical evaluation over the full range of speeds.

I. INTRODUCTION

An electron ring accelerator accelerates heavy ions by trapping the ions in the potential well associated with a compact ring of relativistic electrons, and then accelerating the electrons by means of externally applied fields.¹ It is clear that the highly charged electron ring will, while being accelerated, radiate strongly because of its motion past the conducting surfaces of the acceleration column. Considerable theoretical effort has been devoted to determining the extent of this radiation; more than a dozen different calculations having been reported.²

The crucial point is the dependence of the ring radiation, at ultrarelativistic speeds, upon ring speed. If, for example, the radiation were to increase with increasing speed then the efficiency of an electron ring accelerator would decrease with increasing energy and there would result--in practice--an upper limit to the energy of the accelerator. Thus the very development of electron ring accelerators hinged upon demonstration that they would not be limited by radiation loss at high energies.

It is easy to estimate the radiation due to acceleration of the electron ring and to see that--at least in the relativistic limit--it is quite small. The radiation which is not small is the diffraction radiation due to the motion of the ring near conducting surfaces. Crudely speaking, one could say that image charges are being accelerated and hence there is radiation. It suffices to calculate the energy radiated by a ring moving at constant speed.

If the ring is approximated by a charge Q , then the net energy gain per unit length of the structure can be written in the form

$$\Delta U = A Q - B Q^2 .$$

For a charge moving at constant speed, A is proportional to the externally applied fields in the structure and is, clearly, the energy gain for an infinitesimal charge. The term BQ^2 is, by superposition, independent of the external fields on the structure. Thus it may be calculated for an unexcited structure. It is simply the radiated energy loss of a charge Q , moving at constant speed through the structure. The considerable theoretical effort, mentioned above, has been devoted to determining B which is, clearly, a function only of charge speed and the geometry of the accelerating structure.

The simplest model which has been considered is that of a charge passing through a closed cylindrical cavity. The radiation loss into the cavity was found to increase with increasing γ , where $\gamma = [1 - (v^2/c^2)]^{-1/2}$ and v is the charge speed and c the speed of light.³

It was suggested by Kolpakov and Kotov that a reasonable approximation to a cavity with entrance and exit ports will omit the radiation for modes with wavelengths less than the port dimensions. The radiation loss is then found to be γ -independent at large γ .⁴

A wave-diffraction model was employed by Lawson to study, more carefully, the short wavelength modes which were eliminated in the Kolpakov-Kotov approximation. Lawson found that they contributed energy loss which increased as $\gamma^{1/2}$ at large γ , and this result was obtained independently by Courant.^{5,6}

There remained the possibility that the radiation loss to an infinite periodic array was quite different from the loss to a single cavity. Voskresenskii and Bolotovskii had derived an expression for the

energy loss by a charged rod moving past a periodic array of semi-infinite planes,⁷ which they subsequently employed to show that asymptotically the radiation varied as $1/\gamma$ at large γ .⁸ A γ -independent asymptotic dependence was obtained by Kuznetsov and Rubin.⁹ Numerical evaluation of the Voskresenskii-Bolotovskii formula gave energy loss which fitted rather well--up to $\gamma \approx 300$ --a $\gamma^{-1/2}$ dependence.¹⁰

Thus it seemed likely that there wasn't a practical limit to the energy of an electron ring accelerator--at least up until exceedingly high energies--and development programs pressed ahead in four different laboratories. There remained, however, the question of reconciling the numerical results with the asymptotic evaluations, and this task is accomplished in this paper.

Also, clearly, the radiation loss had to be evaluated for structures which approximate actual acceleration columns. Keil has studied, numerically, a periodic array of cylindrical cavities connected by beam pipes.¹¹ His analysis--in contrast with the work on the planar problem--must be cut off at short wavelengths. He finds energy loss which is γ -independent, for large γ . The neglect of small wavelengths is supported by a numerical indication of convergences, and also by the results obtained in this paper. The difference between $\gamma^{-1/2}$ and γ^0 dependence, at large γ , is presumably a result of infinite transverse structure dimensions vs finite transverse dimensions. Energy balance arguments, presented in Appendix B, show that in a finite structure the energy loss can not decrease with increasing γ .

Still outstanding, at the present time, are results for periodic structures of finite length and for slightly imperfect structures. Efforts are, however, being put into these problems.¹² Rigorous analytic

results for periodic, finite transverse dimensional structures, would be most valuable, and hence worth the considerable effort they probably will demand.

Specifically, in this paper we compute the radiative energy loss from a charged rod which moves at constant speed past an infinite set of parallel semi-infinite conducting plates. The plates are uniformly spaced a distance $2\pi L$ apart, and the rod moves in the direction of their common normal at a distance x_0 below the plates' edges, as depicted in Fig. 1. We take the y direction as being perpendicular to the plane of the figure; note that all fields and currents may be assumed to be independent of y .

Radiation problems with this boundary configuration were apparently first considered by Carlson and Heins.¹³ However, the work of these authors, and some later studies by Heins,¹⁴ did not consider our particular form of radiating source, and did not have occasion to compute energy losses. A problem identical with ours was analyzed by Voskresenskii and Botolovskii.^{7,8} Despite this analysis, there are two reasons for reconsidering the problem here. First, as already stated, the work of Botolovskii and Voskresenskii is in conflict with the numerical evaluations.¹⁰ Secondly, it would be desirable to have expressions for the energy loss, valid in the limiting regimes of low and ultrarelativistic rod speed, in which the dependence upon rod speed and geometrical parameters is transparent, and from which numerical results readily may be obtained.

It is to this second task, i.e., the asymptotic evaluation of the energy loss, that our primary attention will be devoted. This is accomplished in Secs. 3 and 4. We first of all, however, derive in

Sec. 2 the formal solution to the boundary value problem, both in order to correct an error in Ref. 7 and for the sake of completeness. Finally, the modifications required to treat a slightly different situation, in which the charged rod is replaced by a moving current, are briefly considered in Appendix A.

The main results of our analysis are the formal expression of Eqs. (34) and (36), the asymptotic formula of Eqs. (64) - (67), and the low-speed formula of Eq. (71) with Eq. (86) (and Fig. 5) and Eq. (98). Comparison between the asymptotic formulas, which have a dominant $\gamma^{-1/2}$ dependence, and direct numerical evaluation of Eqs. (34) and (36) is presented in Fig. 4. The results for a current-carrying rod are given in (A.8) and (A.9).

2. SOLUTION TO THE BOUNDARY VALUE PROBLEM

It follows from Maxwell's equations that the electric field

$\underline{\mathcal{E}}$ and current density \underline{J} satisfy

$$\nabla \times (\nabla \times \underline{\mathcal{E}}) = -4\pi \frac{\partial \underline{J}}{\partial t} - \frac{\partial^2 \underline{\mathcal{E}}}{\partial t^2}. \quad (1)$$

(We use Gaussian units but set the light speed $c = 1$.) Here the left-hand side is

$$-\nabla^2 \underline{\mathcal{E}} + \nabla(\nabla \cdot \underline{\mathcal{E}}) = -\nabla^2 \underline{\mathcal{E}} + \nabla(4\pi\rho).$$

But the charge density ρ satisfies $\partial\rho/\partial t + \nabla \cdot \underline{J} = 0$, so that, by differentiating Eq. (1) with respect to time, we may obtain an equation in which ρ does not appear:

$$\frac{\partial}{\partial t} \left[-\nabla^2 \underline{\mathcal{E}} + \frac{\partial^2 \underline{\mathcal{E}}}{\partial t^2} \right] = 4\pi \left[\nabla(\nabla \cdot \underline{J}) - \frac{\partial^2 \underline{J}}{\partial t^2} \right]. \quad (2)$$

It suffices to solve the x component of this equation:

$$\frac{\partial}{\partial t} \left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2} \right] \mathcal{E}_x = 4\pi \left[\frac{\partial^2}{\partial x \partial z} J_z + \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) J_x \right]. \quad (3)$$

Here the unknowns are \mathcal{E}_x and the induced surface current J_x , since J_z is given in terms of the motion of the charged rod. In fact, if the rod has speed v and charge per unit length q ,

$$J_z = q v \delta(x + x_0) \delta(z - vt). \quad (4)$$

We have in addition the boundary condition that \mathcal{E}_x vanish on the surface of each (infinitely conducting) plate,

$$\mathcal{E}_x \Big|_{z=2\pi nL} = 0, \quad x > 0, \quad (5)$$

from which relation, together with the obvious fact that

$$J_x \Big|_{z=2\pi nL} = 0, \quad x < 0, \quad (6)$$

it is evident that our problem is amenable to the Wiener-Hopf technique.

More specifically, the situation here differs from the usual Wiener-Hopf problem only in the periodicity of the mixed boundary conditions (5) and (6). This difference is conveniently dealt with by noting the symmetry

$$\mathcal{E}_x(z, t) = \mathcal{E}_x(z + 2\pi nL, t + \frac{2\pi nL}{v}), \quad (7)$$

which suggests for \mathcal{E}_x the appropriately modified Fourier representation

$$\mathcal{E}_x = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega \exp[i\omega(\frac{z}{v} - t)] e^{inz/L} \mathcal{E}_n(x, \omega). \quad (8)$$

Here and below, ω should be assumed to have a small positive imaginary part, so that only outgoing waves are obtained. The currents, which possess the same symmetry, (7), as \mathcal{E}_x , may be similarly expressed:

$$\begin{aligned} J_x &= 2\pi L j(x, z - vt) \sum_{n=-\infty}^{\infty} \delta(z - 2\pi nL) \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega j(x, \omega) \exp[i\omega(\frac{z}{v} - t)] e^{inz/L}, \end{aligned} \quad (9)$$

$$J_z = \frac{1}{2\pi} \sum_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega q \delta_{n0} \delta(x + x_0) \exp[i\omega(\frac{z}{v} - t)] e^{inz/L} . \quad (10)$$

Upon substituting these representations into Eq. (3) we find that \mathcal{E}_n and j must satisfy

$$\left[\frac{\partial^2}{\partial x^2} + \omega^2 - \left(\frac{\omega}{v} + \frac{n}{L} \right)^2 \right] \mathcal{E}_n(x, \omega) = \frac{4\pi}{i\omega} \left(\omega^2 + \frac{\partial^2}{\partial x^2} \right) j(x, \omega) + \frac{4\pi q}{v} \delta_{n0} \delta'(x + x_0). \quad (11)$$

The boundary conditions (5) and (6) now take the form

$$\sum_{-\infty}^{\infty} \mathcal{E}_n(x, \omega) = 0, \quad x > 0, \quad (12)$$

$$j(x, \omega) = 0, \quad x < 0. \quad (13)$$

The system (11) - (13) may be solved by--essentially--the conventional Wiener-Hopf technique.¹⁵ We first Fourier-transform in x , according to the convention

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx f(x) e^{-ikx}$$

and note that Eqs. (12) and (13) imply analyticity properties for the transformed functions. Thus Eq. (11) becomes

$$(\alpha_n^2 + k^2) \tilde{\mathcal{E}}_n(k, \omega) = -\frac{4\pi}{i\omega} (\omega^2 - k^2) \tilde{j}_-(k, \omega) - \frac{4\pi q}{v} \delta_{n0} ik e^{ikx_0}, \quad (14)$$

where

$$\alpha_n \equiv \left[\left(\frac{\omega}{v} + \frac{n}{L} \right)^2 - \omega^2 \right]^{1/2} \quad (15)$$

is defined to have a positive real part. The subscript on \tilde{j}_- serves to remind us that this function must, by Eq. (13), be analytic (in k) in the half-plane $\text{Im}(k) < 0$. Similarly Eq. (12) implies that the function

$$E_+(k) \equiv \sum_{-\infty}^{\infty} \tilde{\mathcal{E}}_n(k, \omega)$$

is analytic for $\text{Im}(k) > 0$. But, from Eq. (14),

$$E_+(k) = -\frac{4\pi i q}{v} \frac{ke^{ikx_0}}{k^2 + \alpha_0^2} - \frac{4\pi i}{\omega} \tilde{j}_-(k, \omega) (k^2 - \omega^2) V(k), \quad (16)$$

where

$$V(k) \equiv \sum_{-\infty}^{\infty} \frac{1}{k^2 + \alpha_n^2}, \quad (17)$$

so that both the unknown functions \tilde{j}_- and E_+ may be determined from their analyticity properties, as follows. We suppose there exist

functions $V_+(k)$ and $V_-(k)$ such that

- (i) $V_+(k)[V_-(k)]$ is analytic and nonzero in the half-plane $\text{Im}(k) \geq 0$ [$\text{Im}(k) \leq 0$].
- (ii) Both the $V_{\pm}(k)$ have at most polynomial growth for large k .
- (iii) $V_+(k) V_-(k) = V(k)$.

We will compute the $V_{\pm}(k)$ explicitly below; for the present it suffices to note that they have the asymptotic behavior

$$V_{\pm}(k) \sim k^{-1/2} \quad \text{for} \quad k \rightarrow \infty, \quad \text{Im}(k) \geq 0, \quad (18)$$

and that they evidently allow us to rewrite Eq. (16) in the form

$$\frac{E_+(k)}{V_+(k)} (k - i\alpha_0) + \frac{4\pi i q}{v} \frac{ke^{ikx_0}}{(k + i\alpha_0)} \frac{1}{V_+(k)} = - \frac{4\pi i}{\omega} (k^2 - \omega^2)(k - i\alpha_0) \tilde{j}_-(k) V_-(k). \quad (19)$$

Since the left- (right)-hand side of this equation is analytic in the upper- (lower)-half k -plane, it defines, by analytic continuation, an entire function. That the entire function must be a polynomial--of degree one at most--follows from Eq. (18) and the fact that, for physically acceptable fields and currents, E_+ must vanish for large k .

Thus we have, in particular,

$$\tilde{j}_-(k) = \frac{i\omega}{4\pi} \frac{A_0 + A_1 k}{(k^2 - \omega^2)(k - i\alpha_0) V_-(k)}. \quad (20)$$

The constants A_0 and A_1 are easily determined. We recall that ω has a small positive imaginary part, so that Eq. (20) is consistent with the analyticity property of \tilde{j}_- only if $A_0 + A_1 k = B(k + \omega)$; and the left-hand side of Eq. (19) may be evaluated at $k = i\alpha_0$ to yield

$$B = \frac{2\pi i q}{v} \frac{e^{-\alpha_0 x_0}}{(\omega + i\alpha_0)} \frac{1}{V_+(i\alpha_0)}.$$

Finally then, using Eq. (14),

$$\begin{aligned} \tilde{C}_n^{\alpha}(k, \omega) = & - \frac{4\pi i q}{v} \delta_{n0} \frac{ke^{ikx_0}}{k^2 + \alpha_0^2} \\ & + \frac{2\pi i q}{v} \frac{e^{-\alpha_0 x_0} (k + \omega)}{(\omega + i\alpha_0) V_+(i\alpha_0) (k^2 + \alpha_n^2) (k - i\alpha_0) V_-(k)}. \end{aligned} \quad (21)$$

We recognize the first term here as the infinite space solution. Hence the x component of that field, $\mathcal{E}_{\sim s}$, which arises purely from the surface currents in the plates, is given by Eq. (8) and

$$\mathcal{E}_{\text{sn}}(x, \omega) = + \frac{iq}{v} \int_{-\infty}^{\infty} dk \frac{e^{-\alpha_0 x_0} (k + \omega) e^{ikx}}{(\omega + i\alpha_0) V_+(i\alpha_0) (k^2 + \alpha_n^2) (k - i\alpha_0) V_-(k)} . \quad (22)$$

Note that, for $x < 0$, the integral over k is entirely trivial. In particular

$$\mathcal{E}_{\text{sn}}(-x_0, \omega) = - \frac{\pi q}{v} \frac{e^{-(\alpha_0 + \alpha_n)x_0} (\omega - i\alpha_n)}{(\omega + i\alpha_0) V_+(i\alpha_0) \alpha_n (\alpha_0 + \alpha_n) V_-(-i\alpha_n)} . \quad (23)$$

We have now solved Eqs. (3) - (6), except for the determination of V_+ and V_- . We will not examine the field structure here, but will restrict our attention to computing the rate of energy loss, \dot{W} , to the plates. This quantity must equal the power needed to move the charged rod through the field due to the plates:

$$\begin{aligned} \dot{W} &= - \int_{\mathcal{J}_z} \mathcal{E}_{\text{sz}}(x, z, t) dx dz \\ &= - qv \mathcal{E}_{\text{sz}}(-x_0, vt, t) . \end{aligned} \quad (24)$$

Note that at any point not on the plates $\nabla \cdot \mathcal{E}_{\sim s} = 0$, so that for such points the Fourier components [in the representation of the form (8)] of \mathcal{E}_{sz} may easily be related to those of \mathcal{E}_{sx} . In this way Eq. (24) becomes

$$\dot{W} = \frac{qv}{2\pi i} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega e^{i\omega t/L} \frac{\alpha_n}{\left(\frac{\omega}{v} + \frac{n}{L}\right)} \mathcal{E}_{sn}(-x_0, t) . \quad (25)$$

Of primary interest is the time-averaged energy loss, to which only the $n = 0$ term in Eq. (25) contributes. We denote the average energy loss per plate by $q^2 U$, that is,

$$U \equiv \frac{2\pi L}{q^2 v} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \dot{W} dt .$$

Equations (23) and (25) give

$$\begin{aligned} U &= -\frac{iLv}{q} \int_{-\infty}^{\infty} d\omega \frac{\alpha_0}{\omega} \mathcal{E}_{s0}(-x_0, \omega) \\ &= \frac{\pi iL}{2} \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \frac{e^{-2\alpha_0 x_0}}{\alpha_0} \frac{(\omega - i\alpha_0)}{(\omega + i\alpha_0)} \frac{1}{V_+(i\alpha_0)V_-(-i\alpha_0)} . \end{aligned} \quad (26)$$

We now turn our attention to the explicit Wiener-Hopf factorization of $V(k)$. This may be accomplished by a conventional procedure.^{13,14} We first of all decompose each term of the right-hand side of Eq. (17) into partial fractions $(k \pm i\alpha_n)^{-1}$. Using then the identity $\pi \cotn \pi a = \sum (n+a)^{-1}$, we find

$$V(k) = \frac{\pi L}{2(\omega^2 - k^2)^{1/2}} \frac{\sin 2\pi L(\omega^2 - k^2)^{1/2}}{\sin \pi L \left[\frac{\omega}{v} - (\omega^2 - k^2)^{1/2} \right] \sin \pi L \left[\frac{\omega}{v} + (\omega^2 - k^2)^{1/2} \right]} . \quad (27)$$

But¹⁵

$$\begin{aligned}
 \frac{\sin 2\pi L(\omega^2 - k^2)^{1/2}}{2\pi L(\omega^2 - k^2)^{1/2}} &= \prod_{n=1}^{\infty} \left[1 - \frac{4L^2(\omega^2 - k^2)}{n^2} \right] \\
 &= \prod_{n=1}^{\infty} \left(-i \frac{2L}{n} \right) \left\{ k + i \left[\left(\frac{n}{2L} \right)^2 - \omega^2 \right]^{1/2} \right\} e^{i2kL/n} \\
 &\times \prod_{n=1}^{\infty} \left(i \frac{2L}{n} \right) \left\{ k - i \left[\left(\frac{n}{2L} \right)^2 - \omega^2 \right]^{1/2} \right\} e^{-i2kL/n}, \tag{28}
 \end{aligned}$$

where the $\pm i e^{\mp i2kL/n}$ factors have been inserted so as to make each infinite product converge separately. In a similar manner we obtain

$$\begin{aligned}
 2 \sin \pi L \left[\frac{\omega}{v} - (\omega^2 - k^2)^{1/2} \right] \sin \pi L \left[\frac{\omega}{v} + (\omega^2 - k^2)^{1/2} \right] \\
 = 2 \pi^2 L^2 (k + i\alpha_0) \prod_{n=1}^{\infty} (-1) \left(\frac{L}{n} \right)^2 (k + i\alpha_n)(k + i\alpha_{-n}) e^{i2kL/n} \\
 \times (k - i\alpha_0) \prod_{n=1}^{\infty} (-1) \left(\frac{L}{n} \right)^2 (k - i\alpha_n)(k - i\alpha_{-n}) e^{-i2kL/n}. \tag{29}
 \end{aligned}$$

Upon substituting the representations (28) and (29) into Eq. (27), one may by inspection obtain a factorization

$$V(k) = \hat{V}_+(k) \hat{V}_-(k)$$

in which the factors

$$\hat{V}_+(k) = \frac{1}{(k + i\alpha_0)} \prod_{n=1}^{\infty} \left(\frac{2ni}{L} \right) \frac{\left\{ k + i \left[\left(\frac{n}{2L} \right)^2 - \omega^2 \right]^{1/2} \right\}}{(k + i\alpha_n)(k + i\alpha_{-n})} \quad (30)$$

and

$$\hat{V}_-(k) = \frac{1}{(k - i\alpha_0)} \prod_{n=1}^{\infty} \left(-\frac{2ni}{L} \right) \frac{\left\{ k - i \left[\left(\frac{n}{2L} \right)^2 - \omega^2 \right]^{1/2} \right\}}{(k - i\alpha_n)(k - i\alpha_{-n})} \quad (31)$$

clearly have the desired analyticity properties. These factors are, however, unsatisfactory because they are not polynomial-bounded for large k .¹⁶ In fact

$$\hat{V}_{\pm}(k) \sim 2^{\pm 2ikL} k^{-1/2}, \quad k \rightarrow \infty, \quad \text{Im}(k) \geq 0. \quad (32)$$

[The relation (32) can be seen to follow from Eqs. (30) and (31) by noting the identity

$$\prod_n [1 + (z/n)] e^{-z/n} = \left[\Gamma(z) z e^{cz} \right]^{-1}$$

and recalling that $\Gamma(z) \sim e^{-z} z^{z-1/2}$ for large z .¹⁷] A proper choice for the $V_{\pm}(k)$ is obviously

$$V_{\pm}(k) = 2^{\mp 2ikL} \hat{V}_{\pm}(k), \quad (33)$$

which functions clearly have the asymptotic behavior we anticipated in Eq. (18).

We substitute the results (30) - (33) into Eq. (25), and obtain for the average energy loss per plate (per unit charge-squared) the expression

$$U = \frac{2\pi L}{v \gamma} i \int_{-\infty}^{\infty} d\omega \frac{|\omega|}{\omega} \exp\left[-\frac{2|\omega|x_0}{v \gamma}\right] \frac{(\omega - i \alpha_0)}{(\omega + i \alpha_0)}$$

$$\times \left[\frac{2^{-2|\omega|L/v\gamma} \prod_{n=1}^{\infty} \frac{(\alpha_0 + \alpha_n)(\alpha_0 + \alpha_{-n})}{\left\{ \alpha_0 + \left[\left(\frac{n}{2L}\right)^2 - \omega^2 \right]^{1/2} \right\}} \frac{L}{2n}} \right]^2.$$

Here γ is the relativistic factor $(1 - v^2)^{-1/2}$ and we have noted, from Eq. (15), that $\alpha_0 = |\omega|/v\gamma$. It is convenient to replace the integration variable by $\lambda \equiv \omega L/v$, and to introduce the abbreviation

$$P(\lambda, \gamma) \equiv \exp\left[-2 \frac{|\lambda|}{\gamma} \ln 2\right]$$

$$\times \prod_{n=1}^{\infty} \frac{\left\{ \frac{|\lambda|}{\gamma} + \left[(n-\lambda)^2 - \lambda^2 v^2 \right]^{1/2} \right\} \left\{ \frac{|\lambda|}{\gamma} + \left[(n+\lambda)^2 - \lambda^2 v^2 \right]^{1/2} \right\}}{n \left[\frac{2|\lambda|}{\gamma} + \left(n^2 - 4 \lambda^2 v^2 \right)^{1/2} \right]}.$$

(34)

Then

$$U = \frac{2\pi i}{\gamma} \int_{-\infty}^{\infty} d\lambda \frac{|\lambda|}{\lambda} \exp\left[-2|\lambda| \frac{x_0}{L\gamma}\right] \left(\frac{\gamma v \lambda - i |\lambda|}{\gamma v \lambda + i |\lambda|} \right) P^2(\lambda, \gamma).$$

(35)

Recalling now that ω --and hence λ --is to be considered as having a small positive imaginary part, we find that $P(-\lambda, \gamma) = P^*(\lambda, \gamma)$.

This guarantees that U is real and positive, and allows us to rewrite Eq. (35) in the more convenient form

$$U = - \operatorname{Im} \frac{4\pi}{\gamma} \left(\frac{1 - i/v\gamma}{1 + i/v\gamma} \right) \int_0^{\infty} d\lambda \exp \left[-2\lambda \frac{x_0}{L\gamma} \right] F^2(\lambda, \gamma) . \quad (36)$$

The integral in Eq. (36) has been evaluated numerically with the help of Esther Schroeder.¹⁰ The remainder of this report will be devoted to an analytic evaluation of U , for each of the two limiting situations $\gamma \gg 1$ and $\gamma \approx 1$. Specifically, in the case of large γ we will derive a closed form asymptotic expression which is correct to $\mathcal{O}(\gamma^{-3/2})$. In the opposite limiting case our expression for U will involve a very easily (but nonanalytically) evaluated integral, and will be correct to $\mathcal{O}(v^2)$. In both cases a systematic means of obtaining more accurate expressions will be clear, but the labor seems unjustified, especially since our results compare well with the numerical evaluation.

3. ENERGY LOSS FROM AN ULTRARELATIVISTIC ROD

Considering first the case of large γ , we begin with the observation that

$$P(\lambda, \infty) = 1, \tag{37}$$

as is evident from the definition, Eq. (34). Thus our procedure will be to let

$$z(\lambda, \gamma) \equiv 2 \ln P(\lambda, \gamma), \tag{38}$$

and expand

$$P^2 = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \tag{39}$$

Our first task, then, is to derive a sufficiently accurate expression for z . The definition, Eq. (38), may be substantially simplified if we use Eq. (37) to write

$$P(\lambda, \gamma) = P(\lambda, \gamma)/P(\lambda, \infty). \tag{40}$$

One term in the logarithm of Eq. (38) is

$$\begin{aligned} \ln \left\{ \frac{1}{\gamma} + \left[\left(\frac{n}{\lambda}\right)^2 - 2\left(\frac{n}{\lambda}\right) + \frac{1}{\gamma^2} \right]^{1/2} \right\} - \ln \left[\left(\frac{n}{\lambda}\right)^2 - 2\left(\frac{n}{\lambda}\right) \right]^{1/2} \\ = \sinh^{-1} \frac{1}{\gamma \left[\left(\frac{n}{\lambda}\right)^2 - 2\left(\frac{n}{\lambda}\right) \right]^{1/2}}, \end{aligned}$$

and the other terms may be treated similarly. We find

$$z(\lambda) = -4 \frac{\lambda}{\gamma} \ln 2 + 2 \sum_{n=1}^{\infty} F\left(\frac{n}{\lambda}\right), \tag{41}$$

where

$$\begin{aligned}
 F(\phi) \equiv & \sinh^{-1} \frac{1}{\gamma(\phi^2 - 2\phi)^{1/2}} + \sinh^{-1} \frac{1}{\gamma(\phi^2 + 2\phi)^{1/2}} \\
 & - \sinh^{-1} \frac{2}{\gamma(\phi^2 - 4)^{1/2}} .
 \end{aligned} \tag{42}$$

Because of Eq. (36) we may restrict our attention to $\text{Re}(\lambda) > 0$.

Recalling our convention $\text{Im}(\lambda) > 0$, we see that the relevant singularities of $F(\phi)$ occur in the upper-half $\phi = (n/\lambda)$ -plane, at

$$\phi = 1 \pm (1 - 1/\gamma^2)^{1/2} + i0 ,$$

$$\phi = 2(1 - 1/\gamma^2)^{1/2} + i0 ,$$

$$\phi = i0 ,$$

$$\phi = 2 + i0 .$$

These are all branch-points; we choose the the cuts of $F(\phi)$ for $\text{Re}(\phi) > 0$ to extend upwards as indicated in Fig. 2. It is easily verified that

$$\int_0^\infty dn F\left(\frac{n}{\lambda}\right) = \frac{2\lambda}{\gamma} \ln 2 . \tag{43}$$

Using also the relation

$$2 \sum_{n=1}^{\infty} F\left(\frac{n}{\lambda}\right) = \frac{\lambda}{i} \int_C d\phi F(\phi) \text{ctn } \pi\lambda\phi ,$$

in which the path $C = C_1 + C_2$ remains below the branch-points of F as in Fig. 2, we find

$$z(\lambda) = 2 \left[\sum_{n=1}^{\infty} - \int_0^{\infty} dn \right] F\left(\frac{n}{\lambda}\right) \quad (44)$$

$$= \frac{\lambda}{i} \int_{C_1} d\phi F(\phi) (\operatorname{ctn} \pi\lambda\phi + i) + \frac{\lambda}{i} \int_{C_2} d\phi F(\phi) (\operatorname{ctn} \pi\lambda\phi - i). \quad (45)$$

We now observe that there will be contributions to Eq. (45) from the region $\phi \lesssim 1/\gamma^2$ only if $\lambda/\gamma^2 = n$ for some integer n . Since such contributions will be weighted by $e^{-\gamma}$ in our formula (36) for U , they may clearly be neglected in the present large- γ analysis.

Hence we allow C_1 and C_2 to coalesce onto the real axis up to some point δ , where we choose $\delta \gtrsim 1/\gamma^2$. Next, we take advantage of the fact that $(\operatorname{ctn} \pi\lambda\phi \pm i)$ becomes exponentially small in the limit $\phi \rightarrow \pm i\infty$, by deforming the rest of the contour C_2 into the lower-half plane, and by "wrapping" the contour C_1 around the branch cuts in the usual way. The result of these mutilations is to leave Eq. (45) in the form

$$z(\lambda) \cong \frac{\lambda}{i} \int_{I+II} (\operatorname{ctn} \pi\lambda\phi + i) F(\phi) d\phi + \frac{\lambda}{i} \int_{III} (\operatorname{ctn} \pi\lambda\phi - i) F(\phi) d\phi \\ + 2\lambda \int_{IV} F(\phi) d\phi,$$

which is conveniently decomposed as follows:

$$z(\lambda) = z_1(\lambda) + z_2(\lambda) + z_3(\lambda), \quad (46)$$

$$z_1(\lambda) \equiv \frac{\lambda}{i} \int_I (\operatorname{ctn} \pi\lambda\phi + i) F(\phi) d\phi, \quad (47)$$

$$\begin{aligned} z_2(\lambda) \equiv & -\frac{\lambda}{i} \int_{II} (\operatorname{ctn} \pi\lambda\phi + i) \sinh^{-1} \frac{2}{\gamma(\phi^2 - 4)^{1/2}} d\phi \\ & - \frac{\lambda}{i} \int_{III} (\operatorname{ctn} \pi\lambda\phi - i) \sinh^{-1} \frac{2}{\gamma(\phi^2 - 4)^{1/2}} d\phi \\ & - 2\lambda \int_{IV} \sinh^{-1} \frac{2}{\gamma(\phi^2 - 4)^{1/2}} d\phi, \end{aligned} \quad (48)$$

$$\begin{aligned} z_3(\lambda) \equiv & \frac{\lambda}{i} \int_{II} (\operatorname{ctn} \pi\lambda\phi + i) f(\phi) d\phi + \frac{\lambda}{i} \int_{III} (\operatorname{ctn} \pi\lambda\phi - i) f(\phi) d\phi \\ & + 2\lambda \int_{IV} f(\phi) d\phi, \end{aligned} \quad (49)$$

where

$$f(\phi) \equiv \sinh^{-1} \frac{1}{\gamma(\phi^2 - 2\phi)^{1/2}} + \sinh^{-1} \frac{1}{\gamma(\phi^2 + 2\phi)^{1/2}}. \quad (50)$$

The contours I - IV are depicted in Fig. 3.

We re-emphasize that the error in Eq. (46) corresponds to an error in U which is exponentially small; up to this point, no serious approximations have been made. Now, however, we commit ourselves to

keeping only $\mathcal{O}(\gamma^{-3/2})$ terms in Eq. (36), which then becomes

$$U = - \operatorname{Im} \frac{4\pi}{\gamma} \left(1 - \frac{2i}{v\gamma}\right) \int_0^{\infty} d\lambda \exp\left[-2 \frac{x_0}{L\gamma} \lambda\right] \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!}\right) + \mathcal{O}(\gamma^{-2}). \quad (51)$$

[We anticipate here that $z = \mathcal{O}(\gamma^{-1/2})$.] The contribution from z_1 to Eq. (51) is easily disposed of. Consider

$$U_1 \equiv - \operatorname{Im} \frac{4\pi}{\gamma} \left(1 - \frac{2i}{\gamma}\right) \int_0^{\infty} d\lambda \exp\left[-2 \frac{x_0}{L\gamma} \lambda\right] z_1$$

$$\propto \frac{1}{\gamma^2} \int_I F(\phi) \sum_{m=1}^{\infty} \frac{d\phi}{\left(\frac{2x_0}{L\gamma} - 2\pi i m \phi\right)^2}, \quad (52)$$

where we have used the identity

$$\operatorname{ctn} \frac{x}{2} = \frac{2}{i} \sum_{n=1}^{\infty} e^{inx} - i - \frac{2\pi}{i} \sum_{-\infty}^{\infty} \delta(x - 2\pi n), \quad (53)$$

in which the δ -function terms are here irrelevant, to explicitly perform the integral over λ . Since $|\phi|$ is not small on the path I , it is clear that

$$U_1 = \mathcal{O}(\gamma^{-2}).$$

The higher-order terms in z_1 may be similarly treated, and we conclude that, in our approximation, the term z_1 may be omitted from Eq. (46).

With regard to z_2 , we observe first of all that the definition (48) is equivalent to

$$z_2 = -\frac{\lambda}{i} \int_{i\infty}^{-i\infty} \text{ctn } \pi\lambda\phi \sinh^{-1} \frac{2}{\gamma(\phi^2 - 4)^{1/2}} d\phi, \quad (54)$$

where the path lies to the left of all poles of $\text{ctn } \pi\lambda\phi$ except the one at $\phi = 0$. Hence we may evaluate the integral in Eq. (54) in terms of principal value and pole contributions in the usual way; since the integrand is odd in ϕ , the principal value vanishes and we are left with the semi-residue

$$\begin{aligned} z_2 &= \sinh^{-1} \frac{i}{\gamma} \\ &= \frac{i}{\gamma} + \mathcal{O}(\gamma^{-3}). \end{aligned} \quad (55)$$

The calculation of z_3 [Eq. (49)] is more complicated. It is helpful to observe first, from Eq. (51), that a term in z_3 of the form $\lambda^\alpha \gamma^\beta$ can contribute to U at most a term of order $\gamma^{\alpha+\beta}$ (its contribution will be smaller if the term is purely real). Thus we may drop such terms whenever $\alpha + \beta < -3/2$. With this in mind we expand Eq. (50):

$$f(\phi) = f^0(\phi) + f^1(\phi) + \mathcal{O}(\phi^{3/2}), \quad (56)$$

where

$$f^0(\phi) \equiv \sinh^{-1} \frac{1}{\gamma(2\phi)^{1/2}} + \sinh^{-1} \frac{1}{\gamma(-2\phi)^{1/2}}, \quad (57)$$

$$f^1(\phi) \equiv \frac{(i-1)}{4\sqrt{2}} \frac{\phi^{1/2}}{\gamma}. \quad (58)$$

The error in Eq. (56) would contribute to z_3 a term of order $\lambda^{-3/2} \gamma^{-1}$ and, by our remarks above, may be ignored. To compute the contribution to z_3 from f^0 , it is simplest to revert from Eq. (49) to our earlier formulation, Eq. (44):

$$\begin{aligned}
 z_3^0 &= 2 \left[\sum_{n=1}^{\infty} - \int_0^{\infty} dn \right] f^0\left(\frac{n}{\lambda}\right) \\
 &= 2 \left[\sum_{n=1}^{\infty} - \int_0^{\infty} \right] \left[\left(\frac{\lambda}{2n}\right)^{1/2} \frac{(1+i)}{\gamma} \right] \\
 &\quad - \frac{2^{-3/2}}{3} \frac{\lambda^{3/2}}{\gamma^3} (1-i) \sum_{n=1}^{\infty} n^{-3/2} \\
 &\quad + 2 \int_0^{\infty} dn \left[\sinh^{-1} \frac{1}{\gamma(2n/\lambda)^{1/2}} - \frac{1}{\gamma(2n/\lambda)^{1/2}} + \sinh^{-1} \frac{i}{\gamma(2n/\lambda)^{1/2}} \right. \\
 &\quad \left. - \frac{i}{\gamma(2n/\lambda)^{1/2}} \right]. \quad (59)
 \end{aligned}$$

Here we have expanded $\sinh^{-1} x = x - (x^3/6) + \mathcal{O}(x^5)$. The error term in Eq. (59) is again irrelevant and the last integral vanishes identically, as may be verified by integrating by parts twice. There remains

$$z_3^0 = \sqrt{2} (1+i) \zeta\left(\frac{1}{2}\right) \frac{(\lambda)^{1/2}}{\gamma} - \frac{(1-i)}{6\sqrt{2}} \zeta\left(\frac{3}{2}\right) \frac{\lambda^{3/2}}{\gamma^3}, \quad (60)$$

where ζ is the Riemann zeta function.¹⁸

Finally we must compute the contribution to z_3 of f^1 . According to Eqs. (49) and (58), this is

$$z_3^1 = \frac{(1-i)}{4\sqrt{2}} i \frac{\lambda}{\gamma} \left\{ \int_{II} (\operatorname{ctn} \pi\lambda\phi + i) \phi^{1/2} d\phi + \int_{III} (\operatorname{ctn} \pi\lambda\phi - i) \phi^{1/2} d\phi + 2i \int_{IV} \phi^{1/2} d\phi \right\}. \quad (61)$$

By the nature of the path IV the last term here is $\mathcal{O}(\lambda/\gamma^3)$ and may be neglected; the other terms may be integrated explicitly by means of Eq. (53) (in which again the δ functions do not contribute). We have then

$$\begin{aligned} z_3^1 &= \frac{(1-i)\lambda}{2\sqrt{2}\gamma} \left\{ \int_{i\infty}^0 \sum_{n=1}^{\infty} e^{2\pi i\lambda n\phi} \phi^{1/2} d\phi - \int_0^{-i\infty} \sum_{n=1}^{\infty} e^{-2\pi i\lambda n\phi} \phi^{1/2} d\phi \right\} \\ &= \frac{(1-i)\lambda}{2\gamma} \int_0^{\infty} \sum_{n=1}^{\infty} e^{-2\pi\lambda ny} y^{1/2} dy \\ &= \frac{(1-i)}{8\sqrt{2}\pi} \frac{\lambda^{-1/2}}{\gamma} \zeta(3/2). \end{aligned} \quad (62)$$

Equations (60) and (62) provide a sufficiently accurate expression for $z_3 = z_3^0 + z_3^1$. This result, together with Eq. (55), is now of

course to be substituted into Eq. (51). Thus, after performing the elementary integration over λ , we find that the average energy loss per plate is

$$\begin{aligned}
 q^2 U = & -q^2 \frac{\zeta(1/2)}{2\sqrt{2}} \left(\frac{2\pi L}{x_0} \right)^{3/2} \gamma^{-1/2} + q^2 \left[1 - \frac{\zeta^2(1/2)}{2\pi} \left(\frac{2\pi L}{x_0} \right) \right] \left(\frac{2\pi L}{x_0} \right) \gamma^{-1} \\
 & + q^2 \left\{ \frac{\zeta(3/2)}{4\sqrt{2}} + \frac{\zeta(1/2)}{2\sqrt{2}} \left(\frac{2\pi L}{x_0} \right) \right. \\
 & \quad \left. - \frac{1}{8\sqrt{2}\pi} \left[\zeta^3(1/2) + \frac{\zeta(3/2)}{8} \right] \left(\frac{2\pi L}{x_0} \right)^2 \right\} \left(\frac{2\pi L}{x_0} \right)^{1/2} \gamma^{-3/2} \\
 & + \mathcal{O}(\gamma^{-2}) . \tag{63}
 \end{aligned}$$

Recall that $2\pi L$ is the actual distance between plates. We introduce the abbreviation

$$\rho \equiv 2\pi L/x_0 ,$$

and evaluate¹⁹ the numerical coefficients in Eq. (63), which then becomes

$$U = a \gamma^{-1/2} + b \gamma^{-1} + c \gamma^{-3/2} + \mathcal{O}(\gamma^{-2}) , \tag{64}$$

with

$$a \approx 0.516 \rho^{3/2} , \tag{65}$$

$$b \approx \rho(1 - 0.339\rho) , \tag{66}$$

$$c \approx \rho^{1/2} (0.462 - 0.516\rho + 0.0784\rho^2) . \tag{67}$$

In Fig. 4 evaluation of Eq. (64)--for three values of ρ --is presented and compared with numerical evaluation of Eq. (36). The accuracy of the asymptotic formulas--even to γ as low as 2.0--was, of course, unexpected.

4. ENERGY LOSS FROM A SLOW ROD

We now turn our attention to the case in which γ is close to 1. It is convenient to begin again with our exact formula (36), but here we will always drop terms of order v^3 . The small velocity limit is analytically somewhat awkward because of the form of $\lim \operatorname{Re}(P)$. Equation (34) gives

$$P(\lambda, 1) = e^{-2\lambda \ln 2} \prod_{n=1}^{[\lambda]} \frac{2\lambda - n}{n}, \quad (68)$$

where $[\lambda]$ is the largest integer less than λ . On the other hand we will find that

$$\operatorname{Im}[P^2(\lambda, \gamma)] = \mathcal{O}(v^2), \quad v \approx 0, \quad (69)$$

from which it follows in particular that we may approximate

$$\operatorname{Im}(P^2) \cong 2 P^2(\lambda, 1) \operatorname{Im}(\ln P). \quad (70)$$

Similarly expanding

$$-\frac{1}{\gamma} \frac{1 - i/v\gamma}{1 + i/v\gamma} = 1 + 2iv - \frac{5}{2} v^2 + \mathcal{O}(v^3),$$

where the v^2 term is irrelevant in view of Eq. (69), we find that

Eq. (36) may be written in the form

$$U = 8\pi(v U_0 + U_1) + \mathcal{O}(v^3), \quad (71)$$

$$U_0 \equiv \int_0^\infty d\lambda e^{-\kappa\lambda} P^2(\lambda, 1), \quad (72)$$

$$U_1 \equiv \int_0^\infty d\lambda e^{-\kappa\lambda} P^2(\lambda, 1) \operatorname{Im}(\ln P). \quad (73)$$

Here of course $\kappa = 2x_0/L$, and we need evaluate $\text{Im}(\ln P)$ only to lowest order.

We consider first U_0 . The definition (72) may be rewritten as

$$U_0 = \sum_{m=0}^{\infty} \int_0^1 dx e^{-\kappa(m+x)} P^2(m+x, 1). \quad (74)$$

From Eq. (68),

$$P(m+x, 1) = e^{-2(m+x)\ln 2} \binom{2m + 2x - 1}{m}, \quad (75)$$

where the second factor is the usual binomial coefficient. Introducing the quantity

$$z = \frac{1}{4} e^{-\kappa/2},$$

we have

$$U_0 = \int_0^1 dx z^{2x} \sum_{m=0}^{\infty} \left[z^m \binom{2m + 2x - 1}{m} \right]^2. \quad (76)$$

The sum of squares may be rewritten as the square of a sum by means of the artifice

$$\sum_{m=0}^{\infty} \left[z^m \binom{2m + 2x - 1}{m} \right]^2 = \frac{1}{2\pi} \int_0^{2\pi} d\theta \left| \sum_{m=0}^{\infty} (e^{i\theta} z)^m \binom{2m + 2x - 1}{m} \right|^2. \quad (77)$$

We denote the sum on the right-hand side by $S(x, \theta)$, so that

$$U_0 = \int_0^1 dx z^{2x} \frac{1}{2\pi} \int_0^{2\pi} d\theta |S(x, \theta)|^2. \quad (78)$$

By the binomial theorem and Cauchy's theorem,

$$S(x, \theta) = \frac{1}{2\pi i} \sum_{m=0}^{\infty} (z e^{i\theta})^{2m} \oint \frac{dw}{w^{m+1}} \left(1 + \frac{w}{z e^{i\theta}}\right)^{2m+2x-1}, \quad (79)$$

where the integration contour must enclose the origin of the w plane in such a way as to include only the pole at $w = 0$. We choose it to be a circle with radius only slightly less than $1/4$. The series is now geometric and easily summed:

$$S(x, \theta) = \frac{1}{2\pi i} \oint \frac{dw}{w} \left(1 + \frac{w}{z e^{i\theta}}\right)^{2x-1} \frac{w}{w - (w + z e^{i\theta})^2}. \quad (80)$$

The integrand here has poles at

$$w = \frac{1}{2} - z e^{i\theta} \pm \frac{1}{2} (1 - 4z e^{i\theta})^{1/2}$$

and by our contour choice only the smaller of these is enclosed. Since the integrand is otherwise analytic inside the contour,

$$S(x, \theta) = (z e^{i\theta})^{1-2x} \frac{\left[\frac{1}{2} - \frac{1}{2}(1 - 4z e^{i\theta})^{1/2}\right]^{2x-1}}{(1 - 4z e^{i\theta})^{1/2}}. \quad (81)$$

It is now convenient to replace z by

$$u \equiv 4z = e^{-\kappa/2}$$

so that upon substituting Eq. (81) into Eq. (78) we have

$$U_0 = \frac{1}{4} \int_0^1 dx u^{2x} \frac{1}{2\pi} \int_0^{2\pi} d\theta u^{2-4x} \times \frac{\left[1 - (1 - ue^{i\theta})^{1/2}\right]^{2x-1} \left[1 - (1 - ue^{-i\theta})^{1/2}\right]^{2x-1}}{(1 - ue^{i\theta})^{1/2} (1 - ue^{-i\theta})^{1/2}} \quad (82)$$

Notice that x appears now only in exponents, so that the x -integration could readily be performed. However, we defer this step in order first to simplify the integral over θ . To this end we replace the integration variable by $y = e^{i\theta}$, whence

$$U_0 = \frac{1}{4} \int_0^1 dx u^{2(1-x)} \frac{1}{2\pi i} \oint \frac{dy}{y} \times \frac{\left\{ \left[1 - (1 - uy)^{1/2}\right] \left[1 - (1 - u/y)^{1/2}\right] \right\}^{2x-1}}{(1 - uy)^{1/2} (1 - u/y)^{1/2}} \quad (83)$$

Here the contour is the unit circle. The integrand has branch points at $y = 0, u, 1/u, \infty$. Since it is evident from Eq. (82) that the integrand is continuous throughout the domain of integration, no branch cut can cross the contour. Thus the branch-points at 0 and u must be connected by a cut, and we shrink the contour to surround this cut in the obvious way. On the new contour y is real and $0 \leq y \leq u$ so that $(1 - u/y)^{1/2}$ is always imaginary. We thus make the further substitution

$$(1 - u/y)^{1/2} = i \tan \theta ,$$

and obtain

$$\begin{aligned} U_0 &= \frac{1}{4\pi} \int_0^1 dx u^{2(1-x)} \int_{-\pi/2}^{\pi/2} d\theta \\ &\times \frac{\left\{ \left[1 - (1 - u^2 \cos^2 \theta)^{1/2} \right] (1 - i \tan \theta) \right\}^{2x-1}}{(1 - u^2 \cos^2 \theta)^{1/2}} \\ &= \frac{1}{4\pi} \int_{-\pi/2}^{\pi/2} d\theta \frac{u^2 e^{i\theta} \cos \theta}{(1 - u^2 \cos^2 \theta)^{1/2} \left[1 - (1 - u^2 \cos^2 \theta)^{1/2} \right]} \\ &\times \int_0^1 dx \left[\frac{1 - (1 - u^2 \cos^2 \theta)^{1/2}}{e^{i\theta} u \cos \theta} \right]^{2x} . \end{aligned} \quad (84)$$

Performing the x integral we find, after some simple manipulations

$$U_0 = \frac{1}{4\pi} \int_{-\pi/2}^{\pi/2} d\theta \frac{(1 - u^2 \cos^2 \theta)^{1/2} - i \tan \theta}{(1 - u^2 \cos^2 \theta)^{1/2}} \frac{1}{\cosh^{-1} \frac{1}{u \cos \theta} - i\theta} . \quad (85)$$

This form may be further simplified if we observe that

$$\frac{-i \tan \theta}{(1 - u^2 \cos^2 \theta)^{1/2}} = -i \frac{\partial}{\partial \theta} \left(\cosh^{-1} \frac{1}{u \cos \theta} - i\theta \right) + 1 .$$

Hence the $\tan \theta$ term in Eq. (85) may be integrated by parts; we thus find its contribution is equal to that of the other term. We have then, finally,

$$\begin{aligned}
 U_0 &= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{d\theta}{\cosh^{-1} \frac{1}{u \cos \theta} - i \theta} \\
 &= \frac{1}{\pi} \int_0^{\pi/2} d\theta \frac{\cosh^{-1} \frac{1}{u \cos \theta}}{\left(\cosh^{-1} \frac{1}{u \cos \theta}\right)^2 + \theta^2} . \tag{86}
 \end{aligned}$$

This integral is easily evaluated numerically; a plot of $U_0(u)$ is given in Fig. 5. It has been checked--approximately--by extrapolations of numerical evaluations of Eq. (36).

The asymptotic forms of U_0 for large and small $\kappa = 2x_0/L$ are easily determined. Considering first the case of large κ , we note that $\cosh^{-1} \frac{1}{u \cos \theta} \approx -\ln 2u \cos \theta$, for $u \approx 0$. Hence in this limit

$$U_0 \approx -\frac{1}{2} \frac{1}{\ln u} = \frac{L}{2x_0}, \quad x_0 \gg L. \tag{87}$$

On the other hand, for κ small and $\theta \lesssim \pi/4$,

$$\cosh^{-1} \frac{1}{u \cos \theta} \approx (\kappa + \theta^2)^{1/2} \quad \text{for} \quad \kappa \approx 0, \quad \theta \lesssim \pi/4,$$

so that the main contribution to the integral in Eq. (86) comes from the lower end-point. It is in fact clear that if we choose δ to be proportional to, but larger than, $\kappa^{1/2}$, the small κ form of U_0 will follow the small δ form of

$$\frac{1}{\pi} \int_{\delta}^{\pi/2} \frac{\cosh^{-1} \frac{1}{u \cos \theta}}{\left(\cosh^{-1} \frac{1}{u \cos \theta}\right)^2 + \theta^2} d\theta$$

$$\approx \frac{1}{\pi} \int_{\delta}^{\pi/2} \left[\frac{\ln \frac{1 + \sin \theta}{\cos \theta}}{\left(\ln \frac{1 + \sin \theta}{\cos \theta}\right)^2 + \theta^2} - \frac{1}{2\theta} \right] d\theta + \frac{1}{2\pi} \int_{\delta}^{\pi/2} \frac{d\theta}{\theta} .$$

(88)

Here we have set $u = 1$ in the integrand (since $\theta^2 \geq \delta^2 > \kappa$) and isolated the singular part of U_0 ($u=1$). Now letting $\delta \propto \kappa^{1/2}$ become arbitrarily small we find

$$U_0 \approx -\frac{1}{2\pi} \ln \delta , \quad \delta \approx 0 ,$$

$$\approx -\frac{1}{4\pi} \ln \frac{x_0}{L} , \quad x_0 \ll L , \quad (89)$$

since the first integral in Eq. (88) remains finite as $\delta \rightarrow 0$.

In order to evaluate U_1 [cf. Eq. (73)], we first note from Eq. (34) that for small v

$$\text{Im}(\ln P) = \text{Im} \sum_{n=1}^{\infty} \ln \left\{ \frac{\lambda}{\gamma} + \left[(n - \lambda)^2 - \lambda^2 v^2 \right]^{1/2} \right\} \quad (90)$$

$$\approx \text{Im} \sum_{n=1}^{\infty} \ln \left\{ 1 + \left[\left(\frac{n}{\lambda} - 1\right)^2 - v^2 \right]^{1/2} \right\} \quad (91)$$

since, as we have remarked, it suffices to compute U_1 to lowest order. We now set $\lambda = m + x$, but here, unlike the case of Eq. (74), we choose $|x| \leq \frac{1}{2}$. Then

$$\text{Im}(\ln P) \approx \sum_{n=1}^{\infty} \text{Im} \ln \left\{ 1 + i \left[v^2 - \left(\frac{n}{m+x} - 1 \right)^2 \right]^{1/2} \right\} \quad (92)$$

$$\approx \theta \left(\frac{mv}{1-v} - |x| \right) \left[v^2 - \left(\frac{x}{m+x} \right)^2 \right]^{1/2}, \quad (93)$$

where θ is the usual Heaviside function: it is equal to one (zero) when its argument is positive (negative). Equation (73) has become

$$U_1 = \sum_{m=1}^{\infty} \int_{-\eta}^{\eta} dx e^{-\kappa(m+x)} P^2(m+x,1) \left[v^2 - \left(\frac{x}{m+x} \right)^2 \right]^{1/2}, \quad (94)$$

where $\eta \equiv mv/(1-v)$. Because of the exponential factor in the integrand, we may assume η is small. It follows that to lowest order

$$U_1 = \sum_{m=1}^{\infty} e^{-\kappa m} P^2(m,1) \int_{-\eta}^{\eta} \left[v^2 - \left(\frac{x}{m+x} \right)^2 \right]^{1/2} dx. \quad (95)$$

The integral is $(\pi/2)mv^2 + \mathcal{O}(v^3)$, whence

$$U_1 = -\frac{\pi}{2} v^2 \frac{\partial}{\partial \kappa} \sum_{m=0}^{\infty} e^{-\kappa m} P^2(m,1). \quad (96)$$

Here we have included the $m=0$ term--which clearly does not contribute --so that the sum may be recognized from Eq. (79):

$$\sum_{m=0}^{\infty} e^{-\kappa m} P^2(m,1) = \frac{1}{2\pi} \int_0^{2\pi} d\theta |S(0,\theta)|^2.$$

We have thus merely to repeat the procedure of Eqs. (74)-(84) above, setting everywhere $x = 0$. The result is

$$\begin{aligned}
 U_1 &= \frac{-v^2}{8} \frac{\partial}{\partial \kappa} \int_{-\pi/2}^{\pi/2} \frac{d\theta}{(1 - u^2 \cos^2 \theta)^{1/2}} \\
 &= \frac{-v^2}{4} \frac{\partial}{\partial \kappa} K(e^{-\kappa/2}), \tag{97}
 \end{aligned}$$

where K is the complete elliptic integral of the first kind. The identity¹⁹

$$\frac{dK}{dk} = \frac{1}{k} \left[\frac{E(k)}{1 - k^2} - K(k) \right]$$

in which E is the complete elliptic integral of the second kind, finally gives

$$U_1 = \frac{v^2}{8} \left\{ \frac{E(e^{-\kappa/2})}{1 - e^{-\kappa}} - K(e^{-\kappa/2}) \right\}. \tag{98}$$

Using known¹⁹ properties of the functions E and K , we find that U_1 vanishes exponentially for large κ , while

$$U_1 \approx \frac{v^2}{8} \frac{1}{\kappa} \quad \text{for} \quad \kappa \approx 0. \tag{99}$$

Combining now Eqs. (71), (86), and (98), we conclude

$$\begin{aligned}
 U &= 8v \int_0^{\pi/2} d\theta \frac{\cosh^{-1} \left(\frac{e^{\kappa/2}}{\cos \theta} \right)}{\left[\cosh^{-1} \left(\frac{e^{\kappa/2}}{\cos \theta} \right) \right]^2 + \theta^2} + \pi v^2 \left[\frac{E(e^{-\kappa/2})}{1 - e^{-\kappa}} - K(e^{-\kappa/2}) \right] \\
 &\quad + O(v^3). \tag{100}
 \end{aligned}$$

APPENDIX A: THE CASE OF A MOVING CURRENT

Our analysis is very easily carried over to the solution of a slightly different problem, namely, that in which the moving rod has no net charge, but carries a current in the y direction. In this Appendix we briefly outline the necessary modifications to the arguments and results presented above.

The essential difference between the two problems is that in the moving current case, the relevant component of the field is \mathcal{E}_y , and this function satisfies an equation significantly simpler than that [Eq. (3)] for \mathcal{E}_x in the moving charge case. In fact we have now to solve

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} \right) \mathcal{E}_y = 4\pi \frac{\partial J_y}{\partial t}, \quad (\text{A.1})$$

where \mathcal{E}_y satisfies the usual boundary condition (5) and

$$J_y = q v' \delta(x + x_0) \delta(z - vt) + J_{sy}. \quad (\text{A.2})$$

Here x_0 and v are as in Fig. 1, and v' is the y-directed velocity, associated with the given current, of the charge per unit length q .

[The q in Eq. (A.2) has the same numerical value as that in Eq. (4), in so far as the positive charge carried by a ring in an electron ring accelerator is small compared to the negative charge.] The unknown current in the plates is J_{sy} ; we note that it may be represented in the same form [Eq. (9)] as J_x in the previous problem, and that its transform also satisfies Eq. (13).

Thus Eq. (A.1) differs in structure from Eq. (3) only in having somewhat fewer derivatives, and the Fourier transform procedure of Eqs. (8) - (17) may be carried through with only minor changes. In this way we obtain the analogues of Eq. (14):

$$(k^2 + \alpha_n^2) \mathcal{E}_n(k, \omega) = 4\pi i \omega \tilde{j}_- + 4\pi i \omega \frac{qv'}{v} \delta_{n0} e^{ikx_0}, \quad (A.3)$$

and of Eq. (19):

$$\frac{E_+}{V_+} (k - i\alpha_0) - \frac{4\pi i \omega qv' e^{ikx_0}}{v V_+ (k + i\alpha_0)} = 4\pi i \omega \tilde{j}_- V_-(k - i\alpha_0). \quad (A.4)$$

Here of course the unknowns E_+ and \tilde{j}_- have different physical meanings from the moving charge case: they refer to y components rather than x components of the field and current. More significantly, the absence in Eq. (A.4) of the factor $(k^2 - \omega^2)$ on the right-hand side allows us to conclude, by the usual Wiener-Hopf argument, that both sides must equal a constant [rather than, as in the case of Eq. (19), a first-degree polynomial]. The constant may be evaluated as usual by setting $k = i\alpha_0$; thus \tilde{j}_- is determined, and Eq. (A.3) yields

$$\mathcal{E}_{sn}(x, \omega) = - \frac{qv' \omega e^{-\alpha_0 x_0}}{v \alpha_0 V_+(i\alpha_0)} \int_{-\infty}^{\omega} dk \frac{e^{ikx}}{(k - i\alpha_0) V_-(k) (k^2 + \alpha_n^2)} \quad (A.5)$$

where the subscript s again indicates that we have omitted the infinite space solution.

Finally we compute the time-averaged energy loss per plate:

$$U' \equiv - \frac{2\pi L}{q^2 v} \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T}^T dt \int dx dz \underline{J}_0 \cdot \underline{E}'_s, \quad (A.6)$$

where \underline{J}_0 is of course the given current, and we use a prime to distinguish the energy loss in the moving current case. Equation (A.5) yields, in the usual way,

$$U' = -4\pi v'^2 \gamma \operatorname{Im} \int_0^\infty d\lambda \exp\left[-\frac{2x_0}{L\gamma} \lambda\right] P^2(\lambda, \gamma), \quad (A.7)$$

where $P(\lambda, \gamma)$ is defined by Eq. (34).

Comparing the exact Eqs. (36) and (A.7), we observe that the relation between U' and U is analytically very simple; in particular, both quantities involve the same integral. It follows that our asymptotic evaluations of U need be only trivially modified.

Specifically, in the ultrarelativistic case we find
[remember $\gamma = (1 - v_z^2)^{1/2}$]

$$U' = v'^2 \gamma^2 [a' \gamma^{-1/2} + b' \gamma^{-1} + c' \gamma^{-3/2} + \mathcal{O}(\gamma^{-2})], \quad (A.8)$$

where

$$\left. \begin{aligned} a' &= a = 0.516 \rho^{3/2}, \\ b' &= -\rho(1 + 0.339\rho), \\ c' &= \rho^{1/2} (0.462 + 0.516\rho + 0.0784\rho^2), \end{aligned} \right\} \quad (A.9)$$

and we recall $\rho = 2\pi L/x_0$. Notice the surprisingly similarity between b', c' and b, c [cf. Eqs. (65) - (67)].

Note that if we accelerate a rod in the z direction, then the y momentum ($v'\gamma$) is invariant. Hence if the rod before acceleration

($\gamma = 1$) is relativistic, then $v'\gamma = 1$, and we may expect the radiation due to the y current to equal that from the charge in the limit $\gamma \gg 1$.

In the case of small v , Eq. (A.7) implies

$$U' = 8\pi v'^2 U_1 + \mathcal{O}(v^3), \quad (\text{A.10})$$

where U_1 is given by Eq. (98).

APPENDIX B: ENERGY BALANCE ARGUMENTS

In this appendix we present arguments which yield a lower limit to the amount of diffraction radiation produced by a charge, Q , passing at constant speed, v , through an accelerating structure of finite dimensions. The discussion is only barely novel; related arguments have been made by Eberhard Keil, John Lawson, and others.

The net energy gain in traversing the structure, ΔU , may, for an electromagnetically linear device, be written in the form:

$$\Delta U = AQ - BQ^2. \quad (\text{B.1})$$

The coefficient A is proportional to the applied field:

$$A = k_E \mathcal{E}, \quad (\text{B.2})$$

where \mathcal{E} is the field, in the absence of the charge Q , measured at some reference position. The coefficient B is the quantity we wish to bound.

The accelerating structure has a total stored energy, W , prior to the introduction of the charge Q , which is proportional to \mathcal{E}^2 :

$$W = k_W \mathcal{E}^2. \quad (\text{B.3})$$

Clearly by energy conservation

$$\Delta U \leq W, \quad (\text{B.4})$$

which implies, by Eqs. (B.1), (B.2), and (B.3):

$$B > \frac{A}{Q^2} \left(Q - \frac{k_W}{k_E} A \right). \quad (\text{B.5})$$

Taking the maximum of the right-hand side of Eq. (B.5), yields the limit:

$$B \geq \frac{k_E^2}{4k_W} . \quad (\text{B.6})$$

Physically, it is clear that k_W is finite, and it is also clear that there exist accelerating structures for which k_E is nonzero. In particular, even for extreme relativistic particles an efficient acceleration column can be designed; i.e., k_E need not decrease with increasing γ , where

$$\gamma = [1 - (v^2/c^2)]^{-1/2} .$$

For these structures--which are just the structures of physical interest--it follows, from Eq. (B.6), that B can not decrease without limit with increasing γ . The restriction to electromagnetically linear structures is not a severe restriction; one can, for example, imagine disconnecting a structure from the--generally nonlinear--power supplies after it has been excited.

ACKNOWLEDGMENTS

We are very grateful to Edward P. Lee for correcting numerous mistakes in the original manuscript and to Esther Schroeder for performing the numerical evaluations.

FOOTNOTES AND REFERENCES

- * Supported in part by the U. S. Atomic Energy Commission.
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FIGURE CAPTIONS

- Fig. 1. Rod, plates, and coordinate system. The origin is at the edge of one of the plates.
- Fig. 2. Branch cuts and integration contours in the complex ϕ plane.
- Fig. 3. Transformed integration contours.
- Fig. 4. Energy loss per plate, per square of the unit charge per unit length on the rod, as a function of rod speed, v , expressed in terms of the relativistic factor $\gamma = (1 - v^2)^{-1/2}$. The solid lines are the asymptotic evaluation of Eq. (64) and the dashed lines are the numerical evaluation of Eq. (36). Curves are presented--as indicated on the figure--for three values of $\rho = 2\pi L/x_0$ which span the range of practical interest.
- Fig. 5. The integral U_0 , defined in Eq. (36), as a function of $u = e^{-x_0/L}$. To lowest order in rod speed, v , the energy loss per plate, per square of the unit charge per unit length on the rod, is given by $U = 8\pi v U_0$.

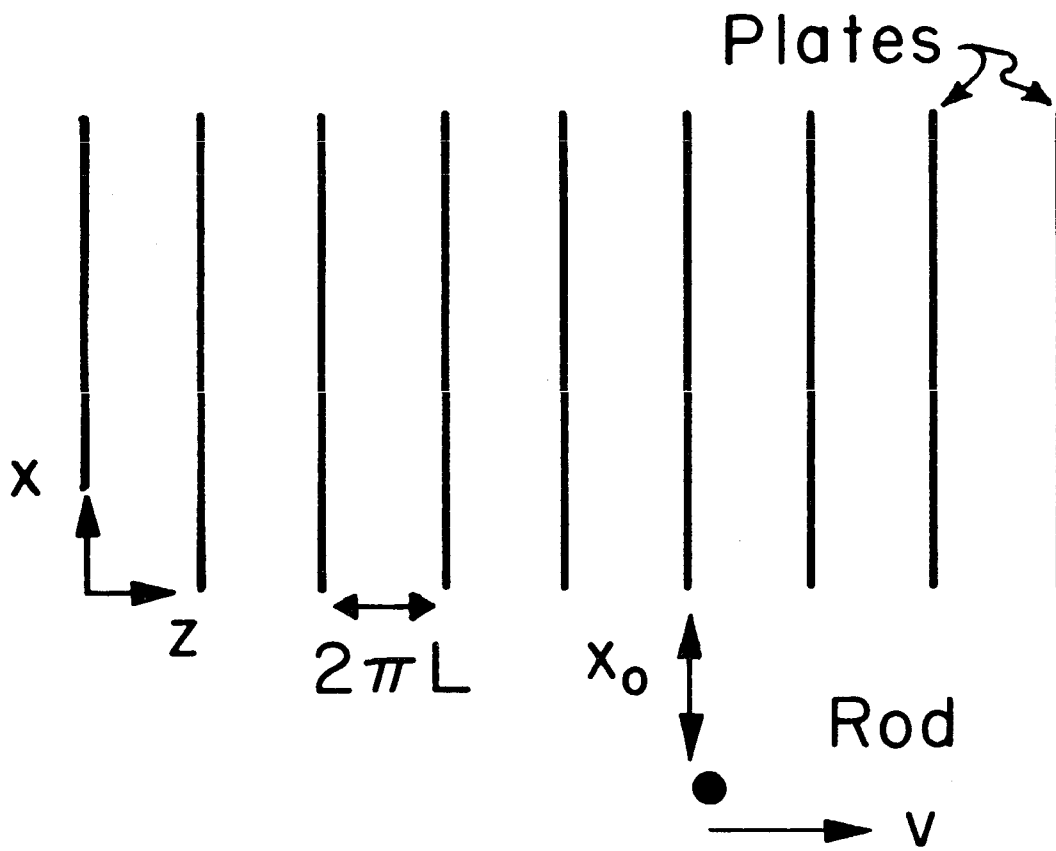


Fig. 1

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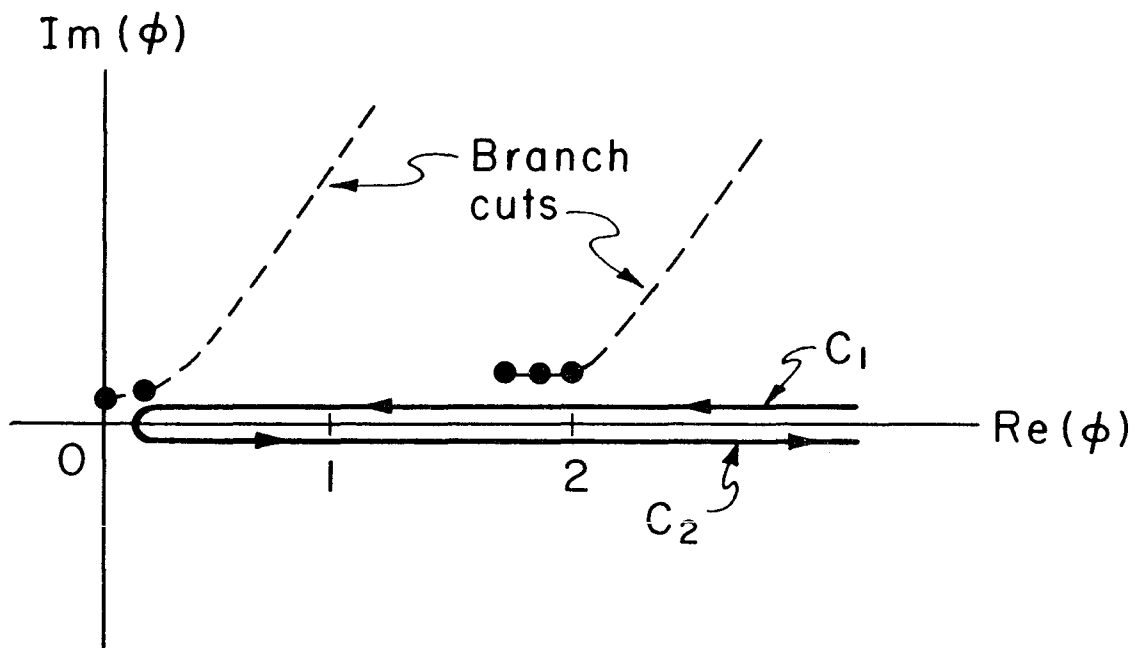


Fig. 2

XBL 705-2901

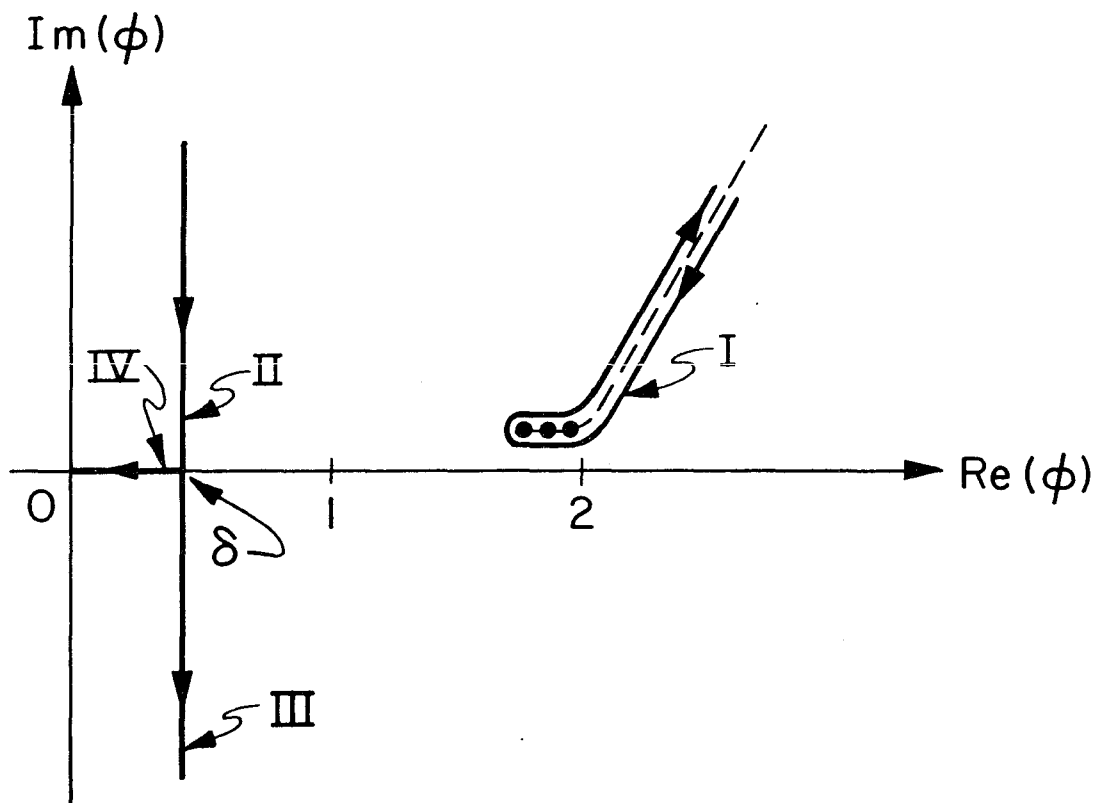


Fig. 3

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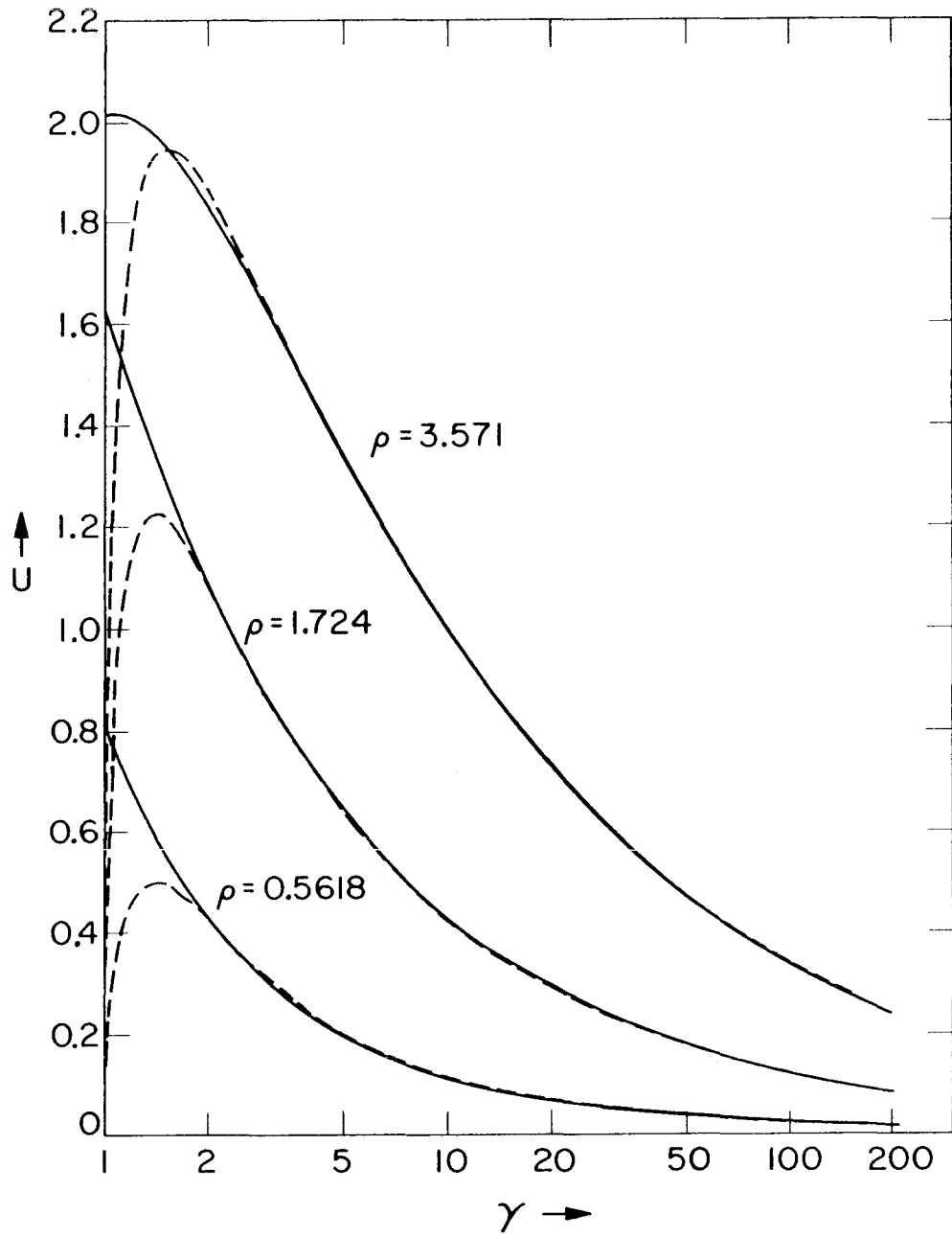


Fig. 4

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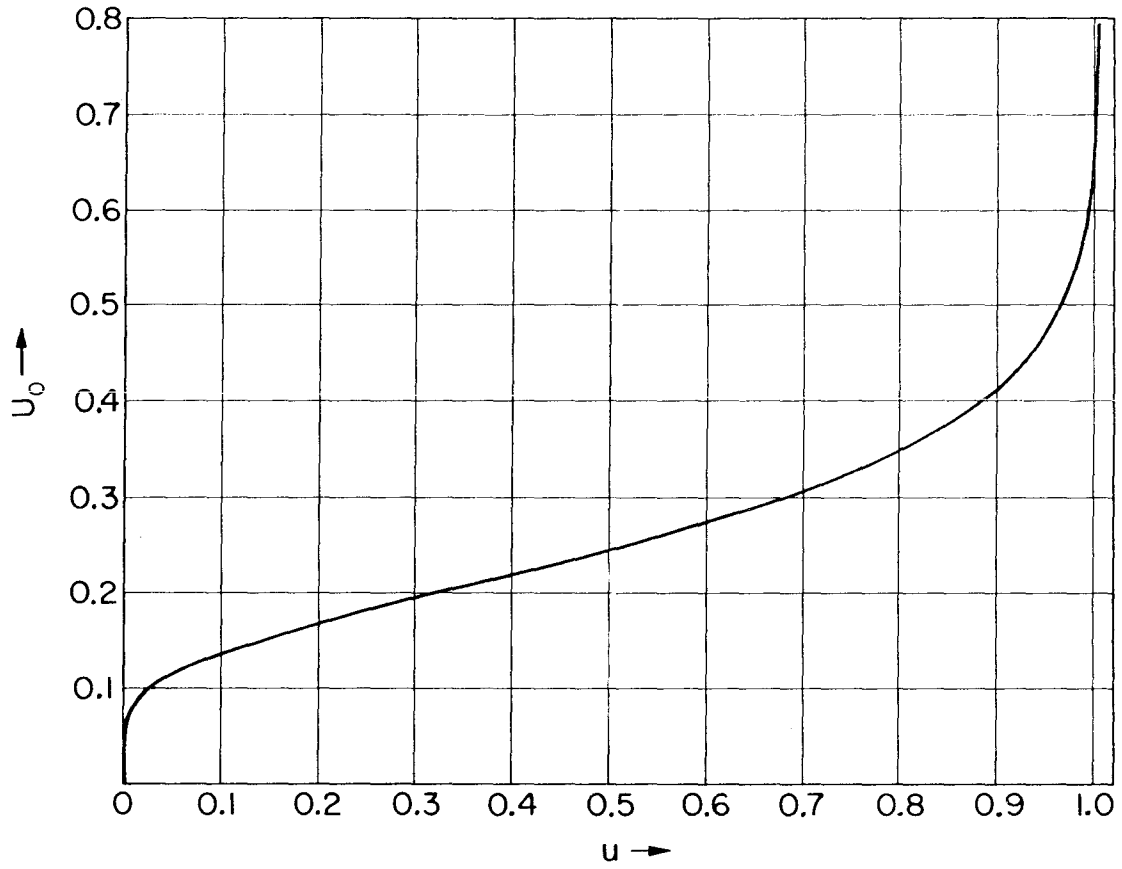


Fig. 5

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