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Constructions of constant scalar curvature Kähler metrics of Poincaré type

by

Yueqing Feng

A dissertation submitted in partial satisfaction of the

requirements for the degree of

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in

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in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Song Sun, Co-Chair
Professor Richard Bamler, Co-Chair
Professor Constantin Teleman
Professor Sung-Jin Oh

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To the readers

Abstract

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University of California, Berkeley

Professor Song Sun, Co-Chair

Professor Richard Bamler, Co-Chair

In this thesis, we study constant scalar curvature Kähler metrics of Poincaré type. We give constructions of new examples in the following setting.

- Consider a compact Kähler manifold admitting a constant scalar curvature Kähler metric and with no nontrivial holomorphic vector fields. After blowing up the manifold at finitely many points, we prove the existence of constant scalar curvature Kähler metrics on the complement of the exceptional divisors, with Poincaré-type singularities along them.
- Given a strictly unbounded toric symplectic 4-manifold, we explicitly construct complete toric scalar-flat Kähler metrics on the complement of a toric divisor. These symplectic 4-manifolds correspond to a specific class of non-compact Kähler surfaces. We also provide an alternative construction of toric scalar-flat Kähler metrics with conical singularity along the toric divisor, following the approach of Abreu and Sena-Dias.

We also make progress on the construction of a degenerate scalar-flat Kähler neck region in complex dimension two, motivated by degeneration problems in Calabi-Yau and negative Kähler-Einstein metrics. The neck region exhibits Poincaré-type singularities at both ends and may serve as a candidate local model for interpolating between appropriately chosen geometric ends.

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Chapter 1

Introduction

Let (X^n, ω) be an n -dimensional Kähler manifold, we define the scalar curvature $S(\omega)$ by

$$S(\omega) \wedge \omega^n = n \cdot \text{Ric}(\omega) \wedge \omega^{n-1}.$$

If $S(\omega)$ is constant on X , we say that (X, ω) is a constant scalar curvature Kähler (cscK) manifold. A particularly well-studied special case is when ω is Kähler-Einstein, i.e., $\text{Ric} \omega = \lambda \omega$, this implies that $S(\omega) = n\lambda$. More generally, Calabi introduced extremal Kähler metrics as critical points of the Calabi functional

$$Ca(\omega) = \int S(\omega)^2 \frac{\omega^n}{n!}.$$

The Euler–Lagrange equation for $Ca(\omega)$ is equivalent to requiring that the gradient vector field $\nabla_\omega S(\omega)$ be holomorphic. In the special case when $\nabla_\omega S(\omega) = 0$, the scalar curvature is constant and ω is cscK.

The existence of cscK metrics, particularly on compact manifolds, has been studied extensively in Kähler geometry. The Yau-Tian-Donaldson [74, 65, 28] conjecture relates the existence of a cscK metric in a polarization class with the algebro-geometric notion of stability of the polarized variety.

Conjecture 1.1. (*Yau-Tian-Donaldson conjecture*)

A compact polarized manifold (X, L) admits a constant scalar curvature Kähler metric in the first Chern class $c_1(L)$ if and only if (X, L) is K-stable.

In the special case of Fano Kähler-Einstein, the conjecture has been settled by Chen-Donaldson-Sun [21, 22, 23].

In this thesis, we study constant scalar curvature Kähler manifolds with Poincaré type singularities. Such metrics can be viewed as a generalization of the Poincaré cusp metric on $\mathbb{C}^2$ to higher dimension, exhibiting cuspidal singularity along a simple normal crossing divisor. The interest comes from several aspects:

Degeneration of smooth extremal Kähler metrics

In complex dimension one, Poincaré cusp singularities arise naturally from the degeneration of hyperbolic Riemann surfaces [72, 71, 52]. In the Deligne–Mumford compactification, a degenerating family of compact Riemann surfaces of genus at least two may converge to a stable nodal curve, locally modeled near a node by the plumbing family $xy = t$. As $t \rightarrow 0$, the degenerating hyperbolic metric near the node locally splits into complete finite-area hyperbolic metrics with cusp ends, each modeled on the Poincaré cusp metric $\omega_{\Delta^*} = \frac{\sqrt{-1}dz \wedge d\bar{z}}{(|z| \log |z|)^2}$ on the punctured unit disc Δ^* in $\mathbb{C}$.

In higher dimension, consider a ruled surface with a family of polarisations $\mathcal{L}_m$, Székelyhidi [64] constructs a sequence of extremal Kähler metrics whose degeneration yields complete extremal metrics of Poincaré type. More specifically, by minimizing the Calabi functional on the set of metrics constructed by the momentum constructions [40], it's shown that as m varies, these singular metrics appear along a continuity path between stable and unstable polarisations.

Connections with conical metrics

Poincaré type metrics are complete and are connected to the smooth metrics by a family of incomplete conical metrics. Intuitively, as the cone angle approaches 2π , we approach the smooth metrics; while when it approaches 0, we approach the Poincaré type cuspidal singularity. On the punctured unit disc Δ^* , the conical metrics $\omega_\beta := \frac{\beta^2 \sqrt{-1}dz \wedge d\bar{z}}{|z|^{2(1-\beta)}(1-|z|^{2\beta})^2}$ with cone angle $2\pi\beta$ has constant negative curvature, and converges pointwise to the Poincaré cusp metric ω_{Δ^*} .

In higher dimension, Guenancia [37] generalizes this observation and shows that for a compact Kähler manifold X and a divisor for which $K_X + D$ is ample, for a family of negative Kähler-Einstein metrics ω_β with cone angle $2\pi\beta$ along D , as β goes to zero, $\omega_\beta \rightarrow \omega_0$ a Poincaré type metric in $C_{loc}^\infty(X \setminus D)$. In particular, $(X \setminus D, \omega_\beta)$ converges to $(X \setminus D, \omega_0)$ in the pointwise Gromov-Hausdorff topology.

Yau-Tian-Donaldson type conjecture for Poincaré type metrics

First formulated by Székelyhidi, later refined by Apostolov-Auvray-Sektnan, the Yau-Tian-Donaldson type conjecture for extremal metrics of Poincaré type says the following.

Conjecture 1.2. (*Yau-Tian-Donaldson type conjecture for Poincaré type metrics [62], [8]*)
 (X, L, D) admits an extremal Kähler metric of Poincaré type on $X \setminus D$ if and only if

- (i) *Relative Donaldson–Futaki invariant of any test configuration for the triple (X, L, D) is non-negative, with equality holding if and only if the test configuration is product.*
- (ii) *Székelyhidi's numerical constraint holds.*

(iii) *The restriction of extremal vector field of X for $c_1(L)$, when restricted to D , is equal to the extremal vector field of D in $c_1(L)|_D$.*

The relationship between the relative K-stability (resp. K-stability) and the existence of extremal Kähler (resp. cscK) metric has been widely studied. Auvray [14, 16] proved the necessity of the Székelyhidi numerical constraint and the extremal vector field condition. For $n = 1$, given a log Riemann surface $(X, D = \sum_{i=1}^d p_i)$, [58] established a direct relationship between the complete cuspidal metric on $X \setminus D$ and the K-stability of the pair using quantization technique. Later, [7] shows if there is no non-trivial holomorphic vector field on D and the connected component of the identity in the holomorphic automorphism group preserving D is trivial, then K -semistability of the pair is a necessary condition. In the toric surface setting, Apostolov–Auvray–Sektnan [8] formulate a refined version of Conjecture 1.2 and verify it for all reduced torus-invariant divisors on the Hirzebruch surfaces $\mathbb{F}_m$, $\mathbb{CP}^1 \times \mathbb{CP}^1$, and $\mathbb{CP}^2$.

Main results

Known constructions of canonical Kähler metrics of Poincaré type include the negative Kähler-Einstein ones studied in Tian-Yau [66], Kobayashi [43] and Cheng-Yau [24]; (b) the toric extremal Kähler metrics of Poincaré type studied by Bryant [20] and Abreu [3]; (c) suitable blow-ups of (a) and (b) as discussed in [55]. In this thesis, we construct new examples of special metrics of Poincaré type.

In Chapter 3, we construct cscK metrics of Poincaré type with a gluing approach:

Theorem 1.1. [29] *Let X be a compact Kähler manifold of dimension n with no non-trivial holomorphic vector fields, meaning that there is no non-zero holomorphic vector field that vanishes somewhere on X . Suppose ω_X is a constant scalar curvature Kähler (cscK) metric on X . Fix finitely many points $p_1, \dots, p_k \in X$, let $Bl_{p_1, \dots, p_k} X$ be the blow-up of X at the chosen points, π be the blow-down map, and $E_i := \pi^{-1}(p_i)$ be the exceptional divisors. Then there exists a small constant $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, the manifold $Bl_{p_1, \dots, p_k} X \setminus \bigcup_{i=1}^k E_i$ admits a cscK metric with Poincaré type singularity along E_i within the given Kähler class*

$$\pi^*[\omega_X] - \epsilon^2 \sum_{i=1}^k [E_i]. \quad (1.1)$$

Here $[E_i]$ denotes the Poincaré dual of the E_i .

The assumptions can be relaxed, see Theorem 3.2. We also extend the constructions to extremal Kähler metrics of Poincaré type, see Theorem 3.3.

The gluing method has had widespread applications in the construction of special metrics in both Riemannian and Kähler geometry. Our construction builds on the cscK gluing

framework developed in [11, 12, 55, 63, 60, 61], and extends it to the Poincaré-type setting, where the singular region requires a delicate new analysis. The essential difficulty is to find the inverse of the linearization of the scalar curvature operator, which requires a detailed analysis for the fourth-order nonlinear differential operator near the singular region [Section 3.4]. This is done by identifying a scalar-flat model metric on the total space of line bundle $\mathcal{O}(-1)$ with the exceptional divisor removed [Section 3.1]. We construct the inverse by doing cut-off such that each function supports on a compact annulus, then we find its inverse on each annulus by carrying out a kernel and cokernel analysis for the closed, self-adjoint operator. The final step is to patch these local inverses together using a partition of unity. The total space carries a natural S^1 -action, and we decompose functions on $\text{Bl}_0 \mathbb{C}^n \setminus E$ into Fourier modes with respect to this action. For each Fourier component, we obtain uniform control of the inverse by adapting techniques from [17, 39].

In Chapter 4, we work within Donaldson's ansatz for toric scalar-flat Kähler metrics [26], which reduces the scalar-flat condition to an equation for axisymmetric harmonic functions. We explicitly construct suitable solutions to this equation and obtain complete toric scalar-flat Kähler metrics on the complement of a torus-invariant divisor in X .

Theorem 1.2. ([30], Theorem 4.2.1+Theorem 4.2.2) *Given X a strictly unbounded toric symplectic 4-manifold and $D = \sum_{i=1}^m D_i$ a divisor on X such that each D_i is an irreducible prime divisor fixed by the torus action and $D_i \cap D_j = \emptyset$ for any $1 \leq i \neq j \leq m$. On $X \setminus D$, we have*

- *a toric scalar-flat Kähler metric with Poincaré type singularity along D , and is asymptotically locally Euclidean(ALE) on the remaining end;*
- *a two-parameter family of toric scalar-flat Kähler metrics with Poincaré type singularity along D , and are asymptotic to either the generalized Taub-NUT metrics or the exceptional Taub-NUT metrics on the remaining end.*

This construction builds on the work of Abreu–Sena-Dias on complete toric scalar-flat Kähler metrics [4], and extends their method to the Poincaré-type singular setting [Section 4.2]. We also construct toric scalar-flat Kähler metrics with conical singularities [Theorem 4.1, Section 4.3], which clarify the relation between the smooth Abreu–Sena-Dias case and the Poincaré-type cusp case. Finally, under prescribed boundary behavior near the divisor, we prove a uniqueness result for the metrics constructed above [Theorem 4.3].

In Chapter 5, we construct a family of incomplete scalar-flat Kähler metrics with S^1 -symmetry on a singular S^1 -fibration over the cylindrical base $S^2 \times I$. Starting from the LeBrun ansatz, we allow the S^1 -action to have fixed points, the scalar-flat condition is reduced to an inhomogeneous equation with distributional terms. By solving this equation explicitly, we obtain a concrete description of the resulting geometry and show that the metrics admit a natural compactification across the singular locus.

The resulting metrics admit Poincaré type singularities near the ends, and may serve as a candidate region to interpolate geometries between appropriately chosen model ends.

Similar degeneration phenomenon has been demonstrated in the work of [59] for the Calabi-Yau setting and [50] for the negative Kähler-Einstein setting. The construction has not been included in any preprints before, and is of independent interest. It is included to record a self-contained argument and to highlight a local geometric model that may be useful in related problems.

Chapter 2

Preliminaries

2.1 Poincaré type Kähler metrics

The study of Poincaré type metrics stems from the standard Poincaré cusp metric

$$\omega_{\Delta^*} = \frac{\sqrt{-1}dz \wedge d\bar{z}}{(|z| \log |z|)^2} = -4\sqrt{-1}\partial\bar{\partial} \log(-\log |z|^2)$$

on the punctured unit disk $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$. To generalize this type of singularity to higher-dimensional manifold, we consider the following definition.

Definition 2.1.1. (*Poincaré type Kähler metric, [14, 13, 16]*) Let (X, ω_X) be a compact complex manifold, and let D be a smooth divisor in X with $\sigma \in H^0(X, \mathcal{O}(D))$ a holomorphic defining section. Fix a Hermitian metric $|\cdot|$ on $\mathcal{O}(D)$, let λ be a sufficiently large constant such that

$$\omega_h := \omega_X - \sqrt{-1}\partial\bar{\partial} \log(\lambda - \log(|\sigma|^2))$$

is a positive $(1, 1)$ -form on $X \setminus D$. We say a closed, smooth $(1, 1)$ -form

$$\omega_{PT} := \omega_X + \sqrt{-1}\partial\bar{\partial}\varphi$$

on $X \setminus D$ is a **Poincaré type Kähler metric** if

- ω_{PT} is quasi-isometric to ω_h , which means that there exists some $C > 0$, such that $\frac{1}{C}\omega_h \leq \omega_{PT} \leq C\omega_h$ and $\forall i \geq 1$, $\sup_{X \setminus D} |\nabla_{\omega_h}^i \omega_{PT}| < \infty$;
- for $h := \log(\lambda - \log(|\sigma|^2))$, φ is a smooth function on $X \setminus D$ with $\varphi = O(h)$, and $\forall i \geq 1$, $\sup_{X \setminus D} |\nabla_{\omega_h}^i \varphi| < \infty$.

Consider a point of D and choose local coordinates $(z_1, \dots, z_n)$ such that $U \cap D = \{z_1 = 0\}$, then near the divisor, the reference metric ω_h is mutually bounded with

$$\frac{\sqrt{-1}dz_1 \wedge d\bar{z}_1}{|z_1|^2 \log^2 |z_1|^2} + \sum_{i=2}^n \sqrt{-1}dz_i \wedge d\bar{z}_i.$$

Near every point on the divisor, the Poincaré type metric is then modeled on the product of the Poincaré metric on the punctured disc with a smooth metric on the divisor. In particular, metrics asymptotic to the product metric above near D are called asymptotically hyperbolic metrics. This is a special class of Poincaré type metrics. More generally, as discussed in [13, 16, 14], Poincaré type metrics are also defined in the case where D is a simple normal crossing divisor.

Comparison with Poincaré-Mok-Yau asymptotics

Mok-Yau [51] explicitly constructed a radially-symmetric complete metric ω with bounded Ricci curvature on $\mathbb{D}^n \setminus \{0\}$. It is piecing together the Fubini-Study metric with the Poincaré cusp metric, i.e.,

$$\omega_{MY} := \sqrt{-1}\partial\bar{\partial} (\log r^2 - \log(-\log r^2)).$$

Later, [31] extended the notion to a general compact Kähler manifold (X, ω_X) . Let S be a *higher-codimensional* subvariety of X , we say ω has the Poincaré-Mok-Yau asymptotic property on $X \setminus S$ if for some $a, b \in \mathbb{R}^+$,

$$\omega = \omega_X + \sqrt{-1}\partial\bar{\partial} \left(a \log r^2 - b \log(-\log r^2) + O\left(\frac{1}{\log r^2}\right) \right)$$

near S . The definition does *not* impose conditions on the asymptotics of higher derivatives and does *not* include the case where S is a divisor.

Comparison with complex hyperbolic cusp

Let $\mathbb{B}^n$ be the unit ball, and let $\Gamma \subset \text{Aut}(\mathbb{B}^n)$ be a torsion-free, discrete subgroup. The Bergman metric induces a Kähler-Einstein metric ω_{hyp} on $X^\circ = \Gamma/\mathbb{B}^n$. It is a quasi-projective manifold which can be compactified to X such that $X \setminus X^\circ$ is a divisor with simple normal crossing D . Near D , given a holomorphic chart $(z_1, \dots, z_n)$ on some open set U in X such that $D \cap U = (z_1 = 0)$, $\omega_{hyp}|_U$ is quasi-isometric to

$$\frac{\sqrt{-1}dz_1 \wedge d\bar{z}_1}{|z_1|^2 \log^2 |z_1|^2} + \frac{1}{-\log |z_1|} \sum_{i=2}^n \sqrt{-1}dz_i \wedge d\bar{z}_i.$$

Up to a finite cover, any such cusp can be realized as the punctured unit disc bundle of a negative line bundle over an abelian variety [18, 32]. Let (L, h) be a Hermitian line bundle over D . Under the Calabi ansatz, the Kähler-Einstein metric on L , which is interpreted as the normal bundle of D , takes the following form

$$\omega_{hyp} = -(n+1)\sqrt{-1}\partial\bar{\partial}\log(-\log h).$$

Biquard and Guenancia [18] show that assuming that there is a conical family of Kähler-Einstein metrics ω_β with cone angle $2\pi\beta$ along D , they converge to the complex hyperbolic metric ω_{hyp} in $C_{loc}^\infty(X \setminus D)$. This is in contrast to the case where $K_X + D$ is ample, the limit is the Poincaré type cusp metric.

Non-Poincaré-type singularity

Despite the continuous focus on Poincaré type singularity, as pointed out by Székelyhidi [64, 62], Apostolov-Auvray-Sektnan [8], there are complete conical metrics that are not of Poincaré type near the asymptotic end. Addison [5] provided a generalization of the cusp metrics by taking into account the automorphism of the ends. Given a trivial holomorphic disk bundle over a compact Kähler manifold, by integrating the flow of a real holomorphy potential in the direction of a selected holomorphic vector field on X , Addison defined the gnarled metric. These metrics naturally arise in the study of the LeBrun-Simanca openness problem of the cscK Poincaré type Kähler metric.

2.2 LeBrun ansatz: past and present

One method for constructing scalar-flat Kähler surfaces comes from the LeBrun ansatz. Under an additional S^1 symmetry assumption, LeBrun gives the following explicit expression of such metrics.

Proposition 2.2.1. *(LeBrun [46], 1991) Given an open set $U \subset \mathbb{R}^3$, consider $w : U \subset \mathbb{R}^3 \rightarrow \mathbb{R}^+$ satisfying*

$$\partial_x^2 w + \partial_y^2 w + \partial_z^2 (we^u) = 0. \quad (2.1)$$

and $u : U \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ satisfying

$$\partial_x^2 u + \partial_y^2 u + \partial_z^2 (e^u) = 0. \quad (2.2)$$

Assume the following closed 2-form

$$\frac{1}{2\pi}\alpha = \frac{1}{2\pi}(w_x dy \wedge dz + w_y dz \wedge dx + (we^u)_z dx \wedge dy)$$

is integral. Consider an S^1 -bundle $M \rightarrow U$ whose first Chern class is given by $[\frac{1}{2\pi}\alpha]$, and a 1-form on M whose curvature is α . Then M admits a scalar-flat Kähler metric given by

$$g = e^u w(dx^2 + dy^2) + w dz^2 + w^{-1}(dt + \theta)^2.$$

Conversely, given any scalar-flat Kähler surface with S^1 symmetry, it locally arises from the above construction.

Thus, one approach to constructing scalar-flat Kähler surfaces is to find solutions (u, w) to Equation (2.2) and Equation (2.1). One special case is when $w(x, y, z) = w(z)$, then Equation 2.1 reduces to the following

$$\partial_z^2(e^u w) = 0.$$

This suggests that the metric is a warped product of D and I . There are three possible cases. Let x, y be the coordinates on D and let z be that on I .

1. D is $\mathbb{CP}^1$. By solving Equation 2.2 and Equation 2.1 we obtain

$$w = \frac{az + b}{z^2 + nz + k}, \quad u = \log \frac{z^2 + nz + k}{\left(1 + \frac{x^2 + y^2}{4}\right)^2}.$$

Completeness of the metric at $z = 0$ requires $b \neq 0, k = 0$. We see Burns metric on $\mathcal{O}(-1)$, under the action chosen by Hwang-Singer¹ can be written in this ansatz as follows

$$w = \frac{1}{2z} = \frac{\frac{z}{2} + 1}{z^2}, \quad u = \log \frac{z^2 + nz}{\left(1 + \frac{x^2 + y^2}{4}\right)^2}.$$

Similarly, the Eguchi-Hanson metric on $\mathcal{O}(-2)$, and LeBrun metrics on $\mathcal{O}(-k)$, with $k \geq 3$, under the action chosen by Hwang-Singer, can be written in this ansatz

$$w = \frac{1 + \frac{kz}{2}}{z^2 + 2z}, \quad u = \log \frac{z^2 + 2z}{\left(1 + \frac{x^2 + y^2}{4}\right)^2}.$$

On $D \times \mathcal{H}$, we write the scalar-flat Kähler metrics as

$$w = \frac{1}{z^2}, \quad u = \log \frac{z^2}{\left(1 + \frac{x^2 + y^2}{4}\right)^2}.$$

Besides all the above, we derive a complete singular metric given by

$$w = \frac{1 + \frac{kz}{2}}{z^2}, \quad u = \log \frac{z^2}{\left(1 + \frac{x^2 + y^2}{4}\right)^2}.$$

¹this action is different from the one chosen by LeBrun

We call this metric the Hwang-Singer metric, which we show in Section 3.1 that it exhibits Poincaré type singularity along the exceptional divisor. We also explicitly write down a one parameter family of conical metrics given by

$$w = \frac{1 + \frac{kz}{2}}{z(z + 2\beta)}, \quad u = \log \frac{z(z + 2\beta)}{\left(1 + \frac{x^2 + y^2}{4}\right)^2},$$

with $\beta \in (0, 1)$. It interpolates between the Hwang-Singer cuspidal metric and the Burns' metric.

2. D is $\mathbb{T}^2$. By solving Equation 2.2 and Equation 2.1 we obtain

$$w = \frac{az + b}{nz + k}, \quad u = \log(nz + k).$$

As pointed out in [70], we see the exceptional half-plane metric on $\mathbb{C}^2$ can be written in this ansatz as

$$w = 1 + \frac{1}{2z}, \quad u = \log 2z.$$

In fact, as pointed out in [50], all complete scalar-flat Kähler metrics on $\mathcal{O}_{\mathbb{C}}$ (whose total space is $\mathbb{C}^2$) that arise from the Hwang-Singer construction correspond to the case where $w = -\frac{\beta}{2} + \frac{1}{2z}$. Here the degree $\beta < 0$. Further, if

$$h_{\beta} := \exp(\beta(\text{Im}z)^2)$$

defined on $\mathbb{C}^2$ is invariant under translation, then we get a complete scalar-flat Kähler metric on $\mathbb{C}/\mathbb{Z} \times \mathbb{C}$ and on the total space of line bundle over elliptic curve.

When $w = 2z, u = 1$, this gives us the Calabi model, resulting in a Ricci flat Kähler metric on a positive degree line bundle.

3. D is higher-genus complex curve with metric given by $g_D := g^2(x, y)(dx^2 + dy^2)$. By solving 2.2 and 2.1 we obtain

$$w = \frac{az + b}{-z^2 + nz + k}, \quad u = \log \frac{-z^2 + nz + k}{g^2(x, y)}.$$

This case, as pointed out in [40], does not result in complete constant scalar curvature metrics. Instead, complete extremal Kähler metrics occur.

There are many examples constructed in the general case where $w(x, y, z)$ depends on all x, y, z variables. The Gibbons-Hawking ansatz corresponds to the case where $u = 0$. By further letting $w = w(x, y, z) = 1 + \frac{1}{r_{x,y,z}}$, we obtain the Taub-NUT metric. Besides, LeBrun constructs scalar-flat Kähler metrics on ruled surfaces [47] using the ansatz. More concretely, starting from a higher genus curve Σ_g , take V on $\mathcal{U} = \{(x, y, z) : y > 0, z \in (-1, 1)\}$ defined by $V := 1 + \sum G_j$. Here $\Delta G_j = 2\pi\delta_j$ and $\lim_{t \rightarrow \pm 1} G_j = 0$, then $\Delta V = 2\pi \sum \delta_j$ and $\lim_{t \rightarrow \pm 1} V = 1$. Let $w := \frac{V}{1-z^2}$ and $u := \log \frac{1-z^2}{y^2}$, we obtain a S^1 -bundle $\pi_0 : M_0 \rightarrow \Sigma_g \times (-1, 1) - \{q_1, \dots, q_m\}$ with metric

$$g = (1 - z^2)(Vh + V^{-1}\theta^2).$$

To compactify M_0 , we further assume $[\frac{1}{2\pi} \star dV] \in H^2(\Sigma_g \times (-1, 1) - \{q_1, \dots, q_m\})$. This is equivalent to say that $\frac{1}{2} \sum (1 + z(q_j)) \in \mathbb{Z}$. A particular case is when the points are in pairs, meaning that $q_j = (a, b, c)$ and $q'_j = (a, b, -c)$. Under this condition, we add two copies of Σ_g at $t = \pm 1$ and m isolated points corresponding to q_j 's, and this results in a S^1 -fibration $\pi : M \rightarrow \Sigma_g \times [-1, 1]$.

Chapter 3

Gluing Construction

There is another perspective for Theorem 1.1. For fixed point $p \in X$, let $d_p(z)$ be the distance function from z to p . Given λ sufficiently large, we define the $(1, 1)$ -form

$$\omega_h := \omega_X + \sqrt{-1} \partial \bar{\partial} (\log d_p^2(z) - \log(\lambda - \log d_p^2(z))). \quad (3.1)$$

We say $\omega_{PT} := \omega_X + \sqrt{-1} \partial \bar{\partial} \varphi$ is a metric has Poincaré-Mok-Yau type singularity if it satisfies the two conditions in Definition 2.1.1 with the sup of norms taken over $X \setminus \{p\}$. This definition can be easily extended to a compact manifold with finitely many points $p_1, \dots, p_k$ removed, simply by choosing sufficiently large constants $\lambda_1, \dots, \lambda_k$ such that

$$\omega_h := \omega_X + \sqrt{-1} \partial \bar{\partial} \sum_{i=1}^k (\log d_{p_i}^2(z) - \log(\lambda_i - \log d_{p_i}^2(z)))$$

is a positive $(1, 1)$ -form on $X \setminus \{p_1, \dots, p_k\}$. This type of singularity is discussed in [51, 31]. In that paper, the Poincaré-Mok-Yau asymptotic is defined in a more general setting in the higher-codimensional complements of a manifold, and the definition requires only the quasi-isometric property stated in Definition 2.1.1.

Theorem 1.1 thus implies the existence of constant scalar curvature Kähler metrics on the punctured manifold $X \setminus \{p_1, \dots, p_k\}$ with complete Poincaré-Mok-Yau type singularity at each p_i . From this perspective, we compare Theorem 1.1 with the following proposition.

Theorem 3.1. (*Gao-Yau-Zhou, [33]*) *Given X a compact Kähler manifold and $S \subset X$ a subvariety with codimension higher than or equal to 2, then there are no complete Kähler-Einstein metrics on $X \setminus S$.*

Theorem 1.1 implies that it is **not** true in the cscK setting.

The scalar curvature of the resulting cscK metric in Theorem 1.1 is not the same as that of ω_X , see Section 3.7, but in a special situation, the Theorem 1.1 can be used to construct scalar-flat Kähler metrics of Poincaré type:

Corollary 3.1.1. *Given a compact Kähler manifold X of dimension n with no non-trivial holomorphic vector fields, assume it is endowed with a scalar-flat Kähler(SFK) metric ω_X . Assume $c_1(X) \neq 0$, given finitely many points $p_1, \dots, p_k \in X$, then the punctured manifold $X \setminus \{p_1, \dots, p_k\}$ admits a **scalar-flat Kähler(SFK) metric** with Poincaré type singularity at each p_i .*

With additional assumptions, we can extend our result to those X admits non-trivial holomorphic vector fields.

Theorem 3.2. *Given a compact Kähler manifold X and a maximal compact subgroup K of $G = \text{Aut}(X)$, let $\mathfrak{k}$ be its Lie algebra. Assume X admits a cscK metric ω_X invariant under K . Let $\mu : X \rightarrow \mathfrak{k}^*$ be a normalized moment map with respect to ω_X . Choose points $p_1, \dots, p_k$ and $a_1, \dots, a_k > 0$ such that the genericity condition $\mu(p_1), \dots, \mu(p_k)$ span $\mathfrak{k}$ and the balancing condition $\sum_{i=1}^k a_i \mu(p_i) = 0$ are satisfied. Assume also that the extremal vector field $\mathcal{X}_\epsilon$ determined by the Kähler class*

$$\pi^*[\omega_X] - \epsilon^2 \sum_{i=1}^k a_i [E_i] \quad (3.2)$$

and the subgroup K satisfies

$$\mathcal{X}_\epsilon|_{E_i} = 0, i = 1, \dots, k. \quad (3.3)$$

Then $Bl_{p_1, \dots, p_k} X \setminus \bigcup_{i=1}^k E_i$ admits a cscK metric with Poincaré type singularity along E_i within the Kähler class (3.2).

The **extremal vector field** for the Poincaré type Kähler metric setting is defined using the Futaki character introduced in [15], and we will discuss in more detail in Section 3.7. Here, we just mention that it is a holomorphic vector field depending only on the Kähler class and maximal compact subgroup K . From [13] Theorem 4, the condition (3.3) is a necessary condition to the existence of cscK metric of Poincaré type in the given Kähler class (3.2).

The above results are all concerned with cscK metrics; in fact, the arguments can be extended to Poincaré type extremal Kähler metrics as well.

Theorem 3.3. *Given a compact Kähler manifold X endowed with an extremal Kähler(extK) metric ω_X . Assume ω_X is invariant under the action of the maximal compact subgroup K of $G = \text{Aut}(X)$ containing its extremal vector field. Let $T \subset K$ be a compact torus and H be the centralizer of T in K . Use $\mathfrak{t}, \mathfrak{k}, \mathfrak{h}$ to denote the corresponding lie algebras and μ to denote a normalized moment map. Choose points $p_1, \dots, p_k$ on X and $a_1, \dots, a_k > 0$ such that the extremal vector field vanishes at each point, and further, the balancing condition $\sum_{i=1}^k a_i^{n-1} \mu(p_i) \in \mathfrak{t}$ and the genericity condition $\mathfrak{t} + \langle \mu(p_1), \dots, \mu(p_k) \rangle = \mathfrak{h}$ are satisfied. Assume also the extremal vector field X_ϵ satisfies*

$$\mathcal{X}_\epsilon|_{E_i} = 0, i = 1, \dots, k.$$

Then $Bl_{p_1, \dots, p_k} X \setminus \bigcup_{i=1}^k E_i$ admits an extremal Kähler(*extK*) metric with Poincaré type singularity along E_i in the Kähler class (3.2).

The conditions on the choice of points are the same as those required in the blowing up situation in [10]. The vector field condition, again seen from [13] Theorem 4, is a necessary condition for the existence of extremal Kähler metric of Poincaré type in the given Kähler class (3.2).

We also discuss the conical family of metrics connecting the smooth and the cuspidal metric. In Section 3.1, we will show that there exists a family of conical metrics with cone angle $\beta \in (0, 1)$ along the zero section connecting the classical Burns-Simanca metric with the cuspidal metric ω_{HS} . If we use this family to do the gluing construction, then we expect to obtain a family of conical metrics on $Bl_{p_1, \dots, p_k} X \setminus \bigcup_{i=1}^k E_i$ connecting the smooth metric constructed in [11] by Arezzo-Pacard and our Poincaré type metric in Theorem 1.1.

Notations of weight near Poincaré type singularity The Definition 3.19 of weight for Hölder spaces near Poincaré type cusp may differ by a sign in different contexts. To avoid confusion, we mention here that the sign we choose aligns with the one chosen by Auvray in [14], [15], [13], [16] and by Lockhart-McOwen in [49]. It differs by a negative sign from the one chosen by Sektnan in [56], [55], by Pacard in [53] and by Xu-Zheng in [73].

3.1 Momentum construction of scalar-flat Kähler metric

In this section, we discuss how the momentum construction in [40] gives rise to the cuspidal SFK metric on the punctured line bundle $\mathcal{O}(-1)^*$ appeared in the introduction. Let $p : (L, h) \rightarrow (M^m, \omega_M)$ be an Hermitian holomorphic line bundle with curvature form $\gamma = -\sqrt{-1}\partial\bar{\partial} \log h$, where M^m is a Kähler manifold of dimension m . For $r = \log |\cdot|_h$ the log of the fiberwise norm function defined by h , consider the Calabi ansatz

$$\omega = p^* \omega_M + 2\sqrt{-1}\partial\bar{\partial} f(r). \quad (3.4)$$

Then consider the natural S^1 -action on L , and we use ξ to denote its generator. Let τ be the corresponding moment map determined by $\iota_\xi \omega = -d\tau$, then $|\xi|_h^2$ is constant on the level sets of τ , which implies the existence of a function $w : I \rightarrow (0, \infty)$ satisfying $w = |\xi|_h^2$. The inverse of w , denoted by φ , is called the *momentum profile* of ω_M . Then we rewrite the metric 3.4 as

$$\omega_\varphi = p^* \omega_M + 2\sqrt{-1}\partial\bar{\partial} \varphi(\tau).$$

We choose the interval I such that the closed $(1, 1)$ -form

$$\omega_M(\tau) := \omega_M - \tau\gamma$$

is positive for all $\tau \in I$. Let $g_M(\tau)$ and $Scal(g_M(\tau))$ denote the associated metric and the scalar curvature, respectively. From [40], we know the existence of complete cscK metrics on (punctured) line/disc bundles:

Theorem 3.1.1. ([40], Theorem C) *Starting from the above data, the holomorphic line bundle $p : (L, h) \rightarrow (M^m, \omega_M)$ and interval I . Assume $\gamma \leq 0$, $\gamma \neq 0$ and g_M is complete. Then there exists a real number $c_0 \in \mathbb{R}$ such that for any $c < c_0$ and finitely many $c > c_0$, the punctured disc bundle, denoted by $\Delta^*(L)$, carries a complete metric g_c with constant scalar curvature c . If $c = c_0$, the metric extends to a complete metric on the punctured line bundle L^* .*

From the proof, we know the completeness of the metric near $\tau = 0$ implies that φ vanishes to order two, i.e., $\varphi(0) = \varphi'(0) = 0$. Define $Q : I \times M^m \rightarrow \mathbb{R}$ to be

$$Q(\tau) := \frac{\omega_M(\tau)^m}{\omega_M^m}.$$

In our case, take

$$M = \mathbb{CP}^1, \quad L_0 := \mathcal{O}(-1), \quad L := kL_0 = \mathcal{O}(-k), \quad \omega_M = 2\omega_{FS},$$

then

$$I = (0, \infty), \quad \gamma = -\frac{k}{2}\omega_{FS}, \quad Q(\tau) = 1 + \frac{k\tau}{2}, \quad Scal(g_M(\tau)) = \frac{1}{1 + \frac{k\tau}{2}}.$$

Here on $\mathbb{CP}^1$, ω_{FS} satisfies $Ric \omega_{FS} = 2\omega_{FS}$ and in homogeneous coordinate Z can be written as $\sqrt{-1}\partial\bar{\partial} \log |Z|^2$. To obtain the explicit expression of momentum profile, first, we recall from [40] the expression of $Scal(g_M(\tau))$:

Theorem 3.1.2. ([40], Theorem A) *Starting from the above data the holomorphic line bundle $p : (L, h) \rightarrow (M^m, \omega_M)$ and interval I , let g_φ be the metric associated to ω_φ , then*

$$Scal(g_\varphi) = Scal(g_M(\tau)) - \frac{1}{2Q} \frac{\partial^2}{\partial \tau^2} (Q\varphi)(\tau). \quad (3.5)$$

Then Equation (3.5) becomes

$$\frac{\partial^2}{\partial \tau^2} \left(\left(1 + \frac{k\tau}{2} \right) \varphi \right) = 2.$$

Set $\varphi(0) = \varphi'(0) = 0$, we obtain the solution

$$\varphi(\tau) = \frac{\tau^2}{1 + \frac{k}{2}\tau}.$$

It's clear from the expression that $\varphi(\tau) > 0$ on $I = (0, \infty)$. From the proof of Theorem 3.1.1, we know for our case, $c_0 > 0$ and Theorem 3.1.1 ensures the existence of a complete SFK metric g_0 on the punctured disc bundle. In fact, it can be further extended to the punctured line bundle in our special situation.

Lemma 3.1.2.1. *The metric g_0 on $\Delta^*(L)$ can be extended to a complete SFK metric on L^* , and near $\tau = 0$, it's a finite-area cusp.*

Proof. Given the momentum profile φ , note the log of the fiberwise norm function has the expression $r = \int_{\tau_0}^{\tau} \frac{dx}{\varphi(x)}$. We define $s(r) := \int_{\tau_0}^{\tau} \frac{dx}{\sqrt{\varphi(x)}}$. Then as $x \rightarrow \infty$, $\varphi(x)$ grows linearly, suggesting that the r integral is divergent, hence the metric lives on the punctured line bundle L^* . The s integral diverges, which means the distance to the level set $\{r = r_\infty\}$ is unbounded; hence the metric is complete. Here $r_\infty = \lim_{\tau \rightarrow \infty} \int_{\tau_0}^{\tau} \frac{dx}{\varphi(x)}$.

Near $\tau = 0$, $\varphi(\tau) \leq \tau^2$, the distance to the level set $\{\tau = 0\}$ is infinite, with the nearby metric being a finite area cusp. $\square$

Note that locally, we can rewrite the S^1 -invariant scalar-flat Kähler metric ω_{HS} under the LeBrun ansatz in [46]. On a neighborhood $U \subset \mathbb{R}^3$ with coordinates x, y, τ , define $w : U \rightarrow \mathbb{R}$ to be the inverse of $\varphi(\tau)$, and we define a function $u : U \rightarrow \mathbb{R}$ by

$$u := \log \frac{\tau^2}{\left(1 + \frac{x^2 + y^2}{2}\right)^2}.$$

Then u is a solution to the $SU(\infty)$ Toda-Lattice equation

$$\partial_x^2 u + \partial_y^2 u + \partial_\tau^2 (e^u) = 0, \quad (3.6)$$

and u, w satisfies

$$\partial_x^2 w + \partial_y^2 w + \partial_\tau^2 (we^u) = 0. \quad (3.7)$$

Hence locally, for some connection 1-form θ , we can write

$$g_{HS} = e^u w (dx^2 + dy^2) + w d\tau^2 + w^{-1} \theta^2. \quad (3.8)$$

As discussed in [40], when $\varphi(0) = 0, \varphi'(0) = 2$, we obtain the classical Burns-Simanca metric. In fact, assuming $\varphi(0) = 0, \varphi'(0) = 2\beta$ for $\beta \in (0, 1)$, we get

$$\varphi(\tau) = \frac{\tau(\tau + 2\beta)}{1 + \frac{k}{2}\tau},$$

and hence, obtain a family of conical scalar-flat Kähler metrics with cone angle β connecting the classical Burns-Simanca and cuspidal metrics. Locally, for $\beta \in [0, 1]$, we can write down this family of S^1 -invariant metrics in the LeBrun ansatz as in 3.8, with

$$w = \frac{1 + \frac{k}{2}\tau}{\tau(\tau + 2\beta)}, \quad u = \log \frac{\tau(\tau + 2\beta)}{\left(1 + \frac{x^2 + y^2}{4}\right)^2}$$

satisfying Equations 3.6 and 3.7.

3.2 Further study on the cuspidal scalar-flat Kähler metric

In this section, we will study the behavior of the metric near both the asymptotically Euclidean(AE) end and the cusp end. We start with the simplest setting, where the metric lives on the total space of the punctured line bundle $\mathcal{O}(-1)^*$ over $\mathbb{CP}^1$, then extend the arguments to the general setting of total space of $\mathcal{O}(-k)^*$ over $\mathbb{CP}^{n-1}$. Recall on $\mathcal{O}(-1)^*$,

$$\varphi(\tau) = \frac{\tau^2}{1 + \frac{\tau}{2}},$$

from the relations given by the momentum construction, $\frac{dr}{d\tau} = \varphi(\tau)^{-1}$, $\frac{df(r)}{d\tau} = \frac{\tau}{\varphi(\tau)}$, we get

$$r = \int \frac{1 + \frac{x}{2}}{x^2} dx = \frac{1}{2} \log \tau - \frac{1}{\tau}, \quad (3.9)$$

$$f(r) = \int \frac{1 + \frac{x}{2}}{x} dx = \log \tau + \frac{\tau}{2}.$$

Then from (3.9), we can express τ in terms of $|z|_h^2$

$$\tau = \frac{2}{W\left(\frac{2}{|z|_h^2}\right)},$$

where W is the product log function. A key property of the product log function we are going to use is that as $x \rightarrow \infty$,

$$W(x) = \log x - \log \log x + o(1).$$

Behavior near E

As $|z|_h \rightarrow 0$, the dominating term for $W(x)$ is $\log x$, more precisely, for a given $\delta > 0$ sufficiently small, there exists ϵ , such that when $0 < |z|_h < \epsilon$, we have

$$W\left(\frac{2}{|z|_h^2}\right) = (1 - \delta) \log \left(\frac{2}{|z|_h^2}\right) + o(1),$$

and $f(t) = \log \frac{2}{1-\delta} - \log(\log 2 - \log |z|_h^2) + \frac{1}{(1-\delta)(\log 2 - \log |z|_h^2)}$. Then

$$\omega_{HS} = 2p^*\omega_{FS} - 2\sqrt{-1}\partial\bar{\partial} \left(\log(\log 2 - \log |z|_h^2) + \frac{1}{(1-\delta)(\log 2 - \log |z|_h^2)} \right).$$

Furthermore, we can control the growth of the norm of the derivatives of ω_{HS} . Note

$$W'(x) = \frac{W(x)}{x(W(x) + 1)},$$

when x is sufficiently large, we get $\frac{W(x)}{W(x)+1} = 1 + O\left(\frac{1}{\log x}\right)$, thus we write in this case

$$W'(x) = \frac{1}{x} \left(1 + O\left(\frac{1}{\log x}\right) \right).$$

In general, the n th derivatives of the product log function is given by

$$W^{(n)}(x) = \frac{W^{n-1}(x)}{z^n (1 + W(x))^{2n-1}} \sum_{k=1}^n a_{k,n} W^k(x).$$

When x is sufficiently large, the dominating term for $W^{(n)}(x)$ is $a_{n,n} \frac{W^{2n-1}(x)}{z^n (1+W(x))^{2n-1}}$. Note the term $a_{n,n}$ satisfies the recurrence relation $a_{n,n} = -(n-1)a_{n-1,n-1}$ with $a_{1,1} = 1$. Thus

$$a_{n,n} = (-1)^{n-1} (n-1)!.$$

Hence, when x is sufficiently large, we can write

$$W^{(n)}(x) = \frac{(-1)^{n-1} (n-1)!}{x^n} \left(1 + O\left(\frac{1}{\log x}\right) \right).$$

At the same time, we know

$$\frac{d^n \log x}{dx^n} = \frac{(-1)^{n-1} (n-1)!}{x^n},$$

then for index I with $|I| = k$, we know

$$\frac{\partial^k \tau}{\partial z^I} = \frac{d^k \left(\frac{2}{\log \frac{2}{|z|_h^2}} \right)}{dz^I} \left(1 + O\left(\frac{1}{\log |z|_h^2}\right) \right).$$

The goal is to show that the Hwang-Singer metric is of Poincaré type. For this, we consider the reference metric $\omega_h := 2p^*\omega_{FS} - \sqrt{-1}\partial\bar{\partial} \log(\log 2 - \log |z|_h^2)$, and let $f_h := -\log(\log 2 - \log |z|_h^2)$. From [16], we know that if we take a polydisc U around E and write $E \cap U = \{z_1 = 0\}$, $U \setminus E = \Delta^* \times \Delta$, then near E , ω_h is mutually bounded by a product of

Poincaré metric on punctured disc with a smooth metric $\omega_U = \frac{\sqrt{-1}dz_1 \wedge d\bar{z}_1}{|z_1|^2 \log^2 |z_1|^2} + \omega_h|_E$, which has bounded derivatives at any order with respect to an orthonormal frame to be defined below. Write $\tau_h := \frac{2}{\log \frac{2}{|z|_h^2}}$, then $\tau = \frac{1}{1-\delta}\tau_h$, and for $k \geq 2$,

$$\frac{\partial^k f}{\partial^k \tau} = \frac{\partial^{k-1} \left(\frac{1}{\tau} + \frac{1}{2} \right)}{\partial^{k-1} \tau} = \frac{\partial^{k-2} \left(\frac{1}{-\tau^2} \right)}{\partial^{k-2} \tau}, \quad \frac{\partial^k f_h}{\partial^k \tau_h} = \frac{\partial^{k-1} \left(\frac{1}{\tau_h} \right)}{\partial^{k-1} \tau_h} = \frac{\partial^{k-2} \left(\frac{1}{-\tau_h^2} \right)}{\partial^{k-2} \tau_h}.$$

It follows that $\frac{\partial^k f}{\partial^k \tau} = \frac{1}{(1-\delta)^k} \frac{\partial^k f_h}{\partial^k \tau_h}$.

We fix a coframe $e^1 = dz^1, e^2 = \frac{1}{|z|_h \log |z|_h} dz^2$, then for $k \geq 2$,

$$\begin{aligned} \frac{\partial^k f}{\partial e^I} &= \frac{\partial^k f}{\partial^k \tau} \frac{\partial^k \tau}{\partial z^I} \frac{\partial z^I}{\partial e^I} de^I \\ &= \frac{1}{(1-\delta)^k} \frac{\partial^k f_h}{\partial^k \tau_h} \frac{\partial \tau^k}{\partial z^I} \frac{\partial z^I}{\partial e^I} de^I \\ &= \frac{1 + O\left(\frac{1}{\log |z|_h^2}\right)}{(1-\delta)^k} \frac{\partial^k f_h}{\partial^k \tau_h} \frac{d^k \tau_h}{dz^I} \frac{\partial z^I}{\partial e^I} de^I \\ &= \frac{1 + O\left(\frac{1}{\log |z|_h^2}\right)}{(1-\delta)^k} \frac{\partial^k f_h}{\partial e^I}, \end{aligned}$$

hence for $j \geq 2$, given δ and $|z|_h$ sufficiently small,

$$\nabla_{\omega_U}^j f = \nabla_{\omega_U}^j f_h \cdot \left(1 + O\left(\frac{1}{\log |z|_h^2}\right) \right).$$

Or equivalently, $\nabla_{\omega_h}^j f = \nabla_{\omega_h}^j f_h \cdot \left(1 + O\left(\frac{1}{\log |z|_h^2}\right) \right)$ for $j \geq 2$. Similar arguments show that $\frac{1}{2}\omega_h \leq \omega_{HS} \leq 2\omega_h$ and $\nabla_{\omega_h} f = \nabla_{\omega_h} f_h \cdot \left(1 + O\left(\frac{1}{\log |z|_h^2}\right) \right)$. From these discussions, we conclude that ω_{HS} is a Poincaré type Kähler metric.

Remark 3.2.1. *In fact, from our computation, we further see the Hwang-Singer metric is actually asymptotically hyperbolic, which is a special type of Poincaré type Kähler metric. We also mention that unlike here, where we used explicit computation to show the metric is asymptotically hyperbolic, Szekelyhidi provided a different proof; for details, see [62] Theorem 5.1.3.*

Asymptotic behavior away from E

As $|z|_h \rightarrow \infty$, $\log \tau = 2 \log |z|_h + O(\tau^{-1})$, thus

$$f = \log |z|_h^2 + \frac{1}{2}|z|_h^2 + 1 + O(|z|_h^{-2}),$$

and $\omega_{HS} = \sqrt{-1}\partial\bar{\partial}(|z|_h^2 + 4\log|z|_h^2 + O(|z|_h^{-2}))$. We rescale ω_{HS} by a factor of $\frac{1}{2}$ so that it lives in the Kähler class $[\omega_{FS}]$ and perform change of variables $z \mapsto \frac{z}{\sqrt{2}}$, then near the AE end,

$$\omega_{HS} = \sqrt{-1}\partial\bar{\partial}(|z|_h^2 + 2\log|z|_h + \varphi_2(z)), \quad (3.10)$$

for some $\varphi_2 = O(|z|_h^{-2})$; near E ,

$$\omega_{HS} = \sqrt{-1}\partial\bar{\partial}\left(\log|z|_h^2 - \log(\log 2 - \log|z|_h^2) + O\left(\frac{1}{\log|z|_h^2}\right)\right).$$

$\mathcal{O}(-k)$ case

We have studied the cuspidal metric on $\mathcal{O}(-1)^*$; in fact, similar behavior of the metric can be seen in $\mathcal{O}(-k)^*$ for $k > 1$. Note

$$r = \int \frac{1 + \frac{kx}{2}}{x^2} dx = \frac{k}{2} \log \tau - \frac{1}{\tau}, \quad (3.11)$$

$$f(r) = \int \frac{1 + \frac{kx}{2}}{x} dx = \log \tau + \frac{k\tau}{2}.$$

Near $\tau = 0$, we solve for τ , and get

$$\tau = \frac{\frac{2}{k}}{W\left(\frac{2}{k} \left(\frac{1}{|z|_h^2}\right)^{1/k}\right)}.$$

A similar analysis shows that it is still of Poincaré type as $\tau \rightarrow 0$. For asymptotic behavior away from E , let $|z|_h^2 \rightarrow \infty$, we have $\log \tau = \frac{2}{k} \log|z|_h + O(\tau^{-1})$. Consider

$$q : \mathcal{O}(-r) \rightarrow \mathbb{C}^n / \Gamma_k,$$

precomposing with it we obtain $f(r \circ q^{-1}) = (\log \tau + k\tau/2) \circ q^{-1}$, and the expression for the potential is written as

$$\frac{k}{2} \sum |z_i|^2 + \log \sum |z_i|^2.$$

Similar to above, after suitable rescaling of the metric and change of variables, we can normalize ω_{HS} such that near the AE end

$$\omega_{HS,k} = \sqrt{-1}\partial\bar{\partial}(|z|_h^2 + c \log|z|_h + \varphi_{2,k}(z)),$$

for some $\varphi_{2,k} = O(|z|_h^{-2})$ and some constant c .

Higher dimension case

The previous discussions are restricted to where the base manifold is $\mathbb{CP}^1$. The next step is to discuss the situation of higher dimension where the base manifold is $\mathbb{CP}^{n-1}$ for $n \geq 3$. On $\mathcal{O}(-k)$ and $\beta = -\frac{k}{2}$, take $\omega_M = \omega_{FS}$ where $\text{Ric}\omega_{FS} = n\omega_{FS}$, then solving the SFK equation we obtain

$$\varphi(\tau) = \frac{2}{n\beta^2} \left(1 - \beta\tau + \frac{n-1}{(1-\beta\tau)^{n-1}} - \frac{n}{(1-\beta\tau)^{n-2}} \right). \quad (3.12)$$

Let $\beta = -1$. Then as $\tau \rightarrow \infty$, we have

$$\begin{aligned} r &= \frac{1}{2} \int \frac{1}{1+x} \left(1 + \frac{1+nx}{(1+x)^n} + O(x^{2-2n}) \right) dx \\ &= \frac{1}{2} \log(1+\tau) + d_1(1+\tau)^{1-n} + d_2(1+\tau)^{-n} + O(\tau^{2-2n}), \end{aligned}$$

$$\begin{aligned} f(r) &= \frac{1}{2} \int \left(1 - \frac{1}{1+x} \right) \left(1 + \frac{1+nx}{(1+x)^n} + O(x^{2-2n}) \right) dx \\ &= \frac{1}{2} \tau - \frac{1}{2} \log(1+\tau) + d_3(1+\tau)^{1-n} + d_4(1+\tau)^{-n} + O(\tau^{3-2n}), \end{aligned}$$

here $d_1, \dots, d_4$ are all constants. It follows that

$$f = \frac{1}{2} |z|_h^2 - \frac{1}{2} \log |z|_h + d_3 |z|_h^{4-2n} + |z|^{2-2n} + O(|z|_h^{6-4n}),$$

and the metric is written as

$$\omega_{HS} = \sqrt{-1} \partial \bar{\partial} (|z|_h^2 + \varphi_2(z)), \quad (3.13)$$

for some $\varphi_2 = d_3 |z|_h^{4-2n} + O(|z|_h^{2-2n})$.

For small τ , as $\tau \rightarrow 0$, write $r = a_{-1} \frac{1}{\tau} + a_0 \log \tau + O(\tau)$, then $\log |z|_h^2 = \frac{a_{-1}}{\tau} + a_0 \log \tau + O(\tau)$, giving us

$$\tau = \frac{-a_{-1}}{a_0 W \left(\frac{-a_{-1}}{a_0} \left(\frac{1}{|z|_h^2} \right)^{2/a_0} \right)},$$

showing that they are still of Poincaré type. Tracing the coefficients more carefully, we obtain the following expansion

$$\omega_{HS} = \sqrt{-1} \partial \bar{\partial} \left(\log |z|_h^2 - \frac{2}{n(n-1)} \log(a - \log |z|_h^2) + O \left(\frac{1}{\log |z|_h^2} \right) \right),$$

for some constant a . This expansion is also discussed in [31]. For other $k \neq 2$, the situation for $\mathcal{O}(-k)$ is similar, as $|z|_h^2 \rightarrow \infty$,

$$\omega_{HS,k} = \sqrt{-1} \partial \bar{\partial} (|z|_h^2 + \varphi_{2,k}(z)),$$

for some $\varphi_{2,k} = O(|z|_h^{4-2n})$. As $|z|_h^2 \rightarrow 0$, it is again of Poincaré type.

3.3 Set up of the construction

With the scalar-flat Kähler Poincaré type metric ω_{HS} , we are ready to discuss the cscK equation we aim to solve for the gluing construction. Given $p \in X$, consider a small neighborhood of $p \in X$, with z being the complex normal coordinate, WLOG we assume $|z| < 1$. Now we rewrite ω_X as

$$\omega_X = \sqrt{-1} \partial \bar{\partial} (|z|^2 + \varphi_1(z)),$$

for some $\varphi_1 = O(|z|^4)$. For a small constant $\epsilon > 0$, we consider the radius $0 < r_\epsilon < 1$ of the form

$$r_\epsilon = \epsilon^{\frac{2n-1}{2n+1}}. \quad (3.14)$$

This particular exponent is chosen for later estimates to be discussed in Section 3.6. We aim at gluing the metric

$$\omega_{HS} = \sqrt{-1} \partial \bar{\partial} (|z|^2 + \log |z| + \varphi_2(z)), \quad \text{for } n = 2, \text{ with } \varphi_2 = O(|z|^{-2}),$$

$$\omega_{HS} = \sqrt{-1} \partial \bar{\partial} (|z|^2 + \varphi_2(z)), \quad \text{for } n \geq 3, \text{ with } \varphi_2 = O(|z|^{-2n+4})$$

on $Bl_0 \mathbb{C}^n \setminus E$ to ω_X on an annulus region $A_{r_\epsilon} := B_{2r_\epsilon} \setminus B_{r_\epsilon}$. For this, we perform coordinate transform of rescaling by ϵ^{-1} and write

$$\epsilon^2 \omega_{HS} = \sqrt{-1} \partial \bar{\partial} (|z|^2 + \epsilon^2 \log |\epsilon^{-1} z| + \epsilon^2 \varphi_2(\epsilon^{-1} z)).$$

Consider a cut-off function $\gamma : \mathbb{R} \rightarrow [0, 1]$ satisfying

$$\gamma(x) = \begin{cases} 0 & x \leq 1, \\ 1 & x \geq 2 \end{cases}$$

we rescale it to $\gamma_1(z) = \gamma\left(\frac{|z|}{r_\epsilon}\right)$, and let $\gamma_2 = 1 - \gamma_1$. Note although both γ_1 and γ_2 depend on ϵ , we will omit them in the notation for simplification. Now, we define an approximate solution of the cscK equation as our candidate for the actual solution,

$$\omega_\epsilon := \begin{cases} \omega_X & \text{on } X \setminus B_{2r_\epsilon}, \\ \sqrt{-1}\partial\bar{\partial}(|z|^2 + \gamma_1(z)\varphi_1(z) + \epsilon^2\gamma_2(z)\varphi_2(\epsilon^{-1}z)) & \text{on } A_{r_\epsilon} \text{ for } n \geq 3, \\ \sqrt{-1}\partial\bar{\partial}(|z|^2 + \gamma_1(z)\varphi_1(z) + \epsilon^2\gamma_2(z)\varphi_2(\epsilon^{-1}z + \log|\epsilon^{-1}z|)) & \text{on } A_{r_\epsilon} \text{ for } n = 2, \\ \epsilon^2\omega_{HS} & \text{on } B_{r_\epsilon} \setminus p. \end{cases} \quad (3.15)$$

First note given ϵ sufficiently small, the metric ω_ϵ is positive definite everywhere. It's because under direct calculation, we see on A_{r_ϵ} , $\gamma_1(z)\varphi_1(z) + \epsilon^2\gamma_2(z)(\varphi_2(\epsilon^{-1}z) + \log|\epsilon^{-1}z|) = O(|z|^4)$ for $n = 2$ and $\gamma_1(z)\varphi_1(z) + \epsilon^2\gamma_2(z)(\varphi_2(\epsilon^{-1}z)) = O(|z|^4)$ for $n \geq 3$. Then by considering the preimage of B_{r_ϵ} under the blow-down map π , we can view ω_ϵ as living on $Bl_p X \setminus E$.

Remark 3.3.1. *For a geometric meaning of ϵ , ϵ^{2n-2} can be viewed as the volume of E for our gluing construction.*

We decompose the operator into linear and non-linear parts to approach the metric defined by this non-linear PDE of the constant scalar curvature equation. Let

$$L_\omega(\varphi) := \left. \frac{d}{dt} \right|_{t=0} S(\omega + t\sqrt{-1}\partial\bar{\partial}\varphi) = -\mathcal{D}_\omega^* \mathcal{D}_\omega \varphi + g^{j\bar{k}} \partial_j S(\omega) \partial_{\bar{k}} \varphi$$

be the linearization of the scalar curvature operator. Here $\mathcal{D}_\omega \varphi = \bar{\partial} \circ \nabla_\omega^{1,0} \varphi$, and

$$\mathcal{D}_\omega^* \mathcal{D}_\omega \varphi = \frac{1}{2} \Delta_\omega^2 \varphi + \langle i\partial\bar{\partial}\varphi, \rho \rangle + \frac{1}{2} \langle d\varphi, dS(\omega) \rangle,$$

is known as the Lichnerowicz operator. ρ denotes the Ricci form for ω . When ω is cscK, the linearization $L_\omega(\varphi)$ coincides with the negative of the Lichnerowicz operator.

The idea is to follow Arezzo-Pacard's blowing-up construction, which we recall as follows. We decompose $Bl_p X \setminus E$ into two pieces. Set

$$R_\epsilon := \frac{r_\epsilon}{\epsilon}, \quad X_{r_\epsilon} := X \setminus B_{r_\epsilon},$$

then

$$Bl_p X \setminus E = X_{r_\epsilon} \coprod Bl_p B_{R_\epsilon} \setminus E.$$

We want to study the Lichnerowicz operator and construct cscK metrics on both pieces separately, then match them over the gluing annulus with appropriately chosen boundary data. A major difference in our case is that, due to the complexity introduced by the behavior of ω_{HS} , we need to first consider solving a modified cscK equation on $Bl_0 \mathbb{C}^n \setminus E$, to be introduced in Section 3.6.

For later purposes, we will modify ω_ϵ a bit on the gluing annulus to obtain a better approximation. In the annulus region, the metric is close to the Euclidean metric, and thus,

the Lichnerowicz operator is close to the standard bilaplacian operator. We want to find biharmonic functions on each piece with given boundary data.

Lemma 3.3.0.1. ([12], Proposition 4.3, 4.4) *Given $h \in C^{4,\alpha}(\partial B_1), k \in C^{2,\alpha}(\partial B_1)$, there exists $H_{h,k}^1 \in C^{4,\alpha}(\bar{B}_1)$ satisfying*

$$\Delta_{Euc}^2 H_{h,k}^1 = 0 \text{ in } B_1; \quad H_{h,k}^1 = h, \quad \Delta_{Euc} H_{h,k}^1 = k \text{ on } \partial B_1,$$

with the following control of bounds

$$\|H_{h,k}^1\|_{C^{4,\alpha}(\bar{B}_1)} \lesssim \|h\|_{C^{4,\alpha}(\partial B_1)} + \|k\|_{C^{2,\alpha}(\partial B_1)}.$$

Given $h \in C^{4,\alpha}(\partial B_1), k \in C^{2,\alpha}(\partial B_1)$, and further $\int_{\partial B_1} k = 0$, there exists $H_{h,k}^0 \in C_{4-2n}^{4,\alpha}(\mathbb{C}^n \setminus B_1)$ satisfying

$$\Delta_{Euc}^2 H_{h,k}^0 = 0 \text{ in } \mathbb{C}^n \setminus \bar{B}_1; \quad H_{h,k}^0 = h, \quad \Delta_{Euc} H_{h,k}^0 = k \text{ on } \partial B_1,$$

with the following control of bounds

$$\|H_{h,k}^0\|_{C_{3-2n}^{4,\alpha}(\mathbb{C}^n \setminus B_1)} \lesssim \|h\|_{C^{4,\alpha}(\partial B_1)} + \|k\|_{C^{2,\alpha}(\partial B_1)}.$$

Here $H_{h,k}^0$ lives in the weighted Hölder space and we postpone its definition to the next section. With this Lemma, for radius R_ϵ , we construct perturbations on each of the two pieces $Bl_0 B_{R_\epsilon} \setminus E$ and X_{r_ϵ} . On $Bl_0 B_{R_\epsilon} \setminus E$, given boundary data $\tilde{h}, \tilde{k}$, for cut-off function χ^1 which vanishes on $Bl_0 B_1 \setminus E$ and equal to 1 on $Bl_0 \mathbb{C}^n \setminus Bl_0 B_2$, we define on $Bl_0 B_{R_\epsilon} \setminus E$ the perturbed function

$$H_{\epsilon, \tilde{h}, \tilde{k}}^1(z) := \chi^1(z) \left(H_{\tilde{h}^\perp, \tilde{k}}^1 \left(\frac{z}{R_\epsilon} \right) - H_{\tilde{h}, \tilde{k}}^1(0) \right) + H_{\tilde{h}, \tilde{k}}^1(0). \quad (3.16)$$

On X_{r_ϵ} , given boundary data h, k , for a cut-off function χ^0 which vanishes on $X \setminus B_2$ and equals to 1 on $B_1 \setminus p$, we define on X_{r_ϵ}

$$H_{\epsilon, h, k}^0(z) := \chi^0(z) H_{h, k}^0 \left(\frac{z}{r_\epsilon} \right).$$

On X_{r_ϵ} , we follow [12] and perturb the metric with smooth function $\varphi_{\epsilon, h, k}$ such that

$$\omega_{\epsilon, h, k} := \omega_X + \sqrt{-1} \partial \bar{\partial} (H_{\epsilon, h, k}^0 + \varphi_{\epsilon, h, k}) \quad (3.17)$$

satisfying cscK equation, for a more precise statement, see Theorem 3.6.1.

On $Bl_0 B_{R_\epsilon} \setminus E$, taking the perturbed biharmonic extension into account, we want to find a smooth potential function $u_{\epsilon, \tilde{h}, \tilde{k}}$ such that the metric

$$\omega_{\epsilon, \tilde{h}, \tilde{k}, HS} := \omega_{HS} + \sqrt{-1} \partial \bar{\partial} (H_{\epsilon, \tilde{h}, \tilde{k}}^1 + u_{\epsilon, \tilde{h}, \tilde{k}}) \quad (3.18)$$

satisfies a modified cscK equation, which we will introduce in Section 3.6. To solve the modified cscK equation, we will first aim at understanding the linearized operator $L_{\omega_{HS}}$ in later sections.

Note that our discussions here and in the upcoming sections are restricted to the situation of blowing up one point, but there are no essential differences when extending the arguments to the situation of blowing up finitely many points $\{p_1, \dots, p_k\}$. With sufficiently small ϵ , let

$$X_{r_\epsilon}^* := X - \{B_{r_\epsilon}(p_1), \dots, B_{r_\epsilon}(p_k)\}, \quad B_{R_\epsilon}^i := B_{R_\epsilon}(p_i)$$

for $i = 1, \dots, k$. On $X_{r_\epsilon}^*$, standard arguments from [11] show the existence of cscK metric on it. On each $Bl_0 B_{R_\epsilon}^i \setminus E_i$, we solve the modified cscK equation as in the case of blowing up one point. The matching argument for finitely many gluing annuli is essentially the same for matching over one annulus. Hence, understanding the situation of blowing up one point of p suffices.

3.4 Weighted Hölder space

In this section, we will introduce the appropriate notion of doubly weighted Hölder space for functions on $Bl_0 \mathbb{C}^n \setminus E$ and we will work in these spaces to study the linearized operator $L_{\omega_{HS}}$. We point out it is modified from the ones studied by Auvray and Sektnan in [56], [16].

To define Hölder spaces on $Y \setminus D$, consider

$$\Phi_\xi : \left(\frac{3}{4}\Delta\right)^k \times \Delta^{n-k} \rightarrow (\Delta^*)^k \times \Delta^{n-k},$$

given by $(w_1, \dots, w_k, z_{k+1}, \dots, z_n) \mapsto (\varphi_{\xi_1}(w_1), \dots, \varphi_{\xi_k}(w_k), z_{k+1}, \dots, z_n)$. Let U be a polydisc neighborhood of D sufficiently small such that $U \cap D = \{z_1 = 0\}$, we define for function on $U \setminus D$ the Hölder norm

$$\|f\|_{C^{k,\alpha}(U \setminus D)} := \sup_{\xi \in (0,1)^k} \|\Phi_\xi^* f\|_{C^{k,\alpha}\left(\left(\frac{3}{4}\Delta\right)^k \times \Delta^{n-k}\right)}.$$

Let $\{U_i\}_{i=1}^k$ be a finite set of charts such that $\left(\bigcup_{i=1}^k U_i\right) \cap D$ covers D and let U_0 be an open set satisfying $Y = \bigcup_{i=0}^k U_i$. We define for function on $Y \setminus D$

$$\|f\|_{C^{k,\alpha}(Y \setminus D)} := \|f\|_{C^{k,\alpha}(U_0)} + \max_{i=1}^k \|f\|_{C^{k,\alpha}(U_i \setminus D)}.$$

The next goal is to add *weight* to this Hölder space. Note for Poincaré metric ω_{Δ^*} , we can change to the *logarithmic polar coordinates* $(t, \theta) \in \mathbb{R} \times \mathbb{S}^1$, where θ is the usual angular coordinate, and

$$t := \log(-\log(|z|^2)).$$

For ω_{PT} on $Y \setminus D$, on a neighborhood of D , we can similarly define a function t which is invariant under the S^1 -action and differs from $h = \log(\lambda - \log |\sigma|^2)$ up to $O(e^{-h})$ term. Then, the weighted Hölder norm is defined as

$$\|f\|_{C_\eta^{k,\alpha}(Y \setminus D)} := \|e^{t\eta} f\|_{C^{k,\alpha}(Y \setminus D)}. \quad (3.19)$$

Remark 3.4.1. As pointed out in [14] Section 3 and [56] Chapter 6, if we consider a neighborhood of D in the holomorphic normal bundle, it can be identified with a tubular neighborhood $N(D)$ of D in Y via the exp map defined by ω_Y . Note $N(D)$ admits an S^1 -action coming from that in the holomorphic normal bundle, and it gives us an S^1 -invariant projection $\pi : N(D) \setminus D \rightarrow D$. Write $N(D)_a := \{x \in N(D) \setminus D : t(x) \geq a\}$, we get an S^1 -fibration

$$\Pi = (\pi, t) : N(D)_a \setminus D \rightarrow D \times [a, \infty).$$

Further, we can define a connection 1-form Θ as follows. Let ξ be the infinitesimal generator of the S^1 -action with φ_s being the associated flow. Then we define

$$\hat{\Theta}_x := \int_0^{2\pi} \varphi_s^* \left(\frac{g_x(\cdot, \xi)}{g_x(\xi, \xi)} \right) ds, \quad \Theta_x := 2\pi \frac{\hat{\Theta}_x}{\int_{S^1} \hat{\Theta}_x}.$$

Note for $\theta := \sqrt{-1}(\log z - \frac{1}{2}e^t)$, we have $\|\nabla^j(\Theta - d\theta)\| = O(1)$, we can write

$$g = dt^2 + e^{-2t}\Theta^2 + \pi^*g_D + O(e^{-t}). \quad (3.20)$$

For $f \in C_\eta^{k,\alpha}(Y \setminus D)$, we can write an orthogonal decomposition of f as $f = f^0 + f^{\neq 0}$, where f^0 is an S^1 -invariant function on $N(D)$ and $f^{\neq 0}$ has zero mean on each fiber of Π . We see f^0 can be viewed as a function $[a, \infty) \times D \rightarrow \mathbb{R}$ and our norm defined near the Poincaré type cusp is **equivalent to that for the cylinder**. More explicitly, the $C^{k,\alpha}$ norm on the cylinder $[a, \infty) \times D$ defined as $\sup_{s \geq a+1} \|f^{cyl}\|_{C^{k,\alpha}([s-1, s+1] \times D, dt^2 + g_D)}$ is equivalent to the Poincaré type norm for S^1 -invariant function. For details, see [56], Lemma 6.7. We immediately see that the weighted norms for the cylinder can be defined similarly and are equivalent to those for Poincaré type cusp.

Let $Y = Bl_0 B_1 \setminus E$, then the above discussion gives us the weighted norm on Y . For a function $f : \mathbb{R}^{2n} - \{0\} \rightarrow \mathbb{R}$, consider $f_r : B_2 \setminus B_1 \rightarrow \mathbb{R}$ defined as

$$f_r(z) := r^{-\delta} f(rz), \quad (3.21)$$

then its weighted norm is defined as

$$\|f\|_{C_\delta^{k,\alpha}(\mathbb{R}^{2n} - \{0\})} := \sup_{r>0} \|f_r\|_{C^{k,\alpha}(B_2 \setminus B_1)}. \quad (3.22)$$

To define weighted Hölder space on $Bl_0\mathbb{C}^n \setminus E$, we use **double index** η, δ which are independent of each other to monitor the growth rate of f near the divisor and on the AE end separately.

Definition 3.4.1. For a function f defined on $Bl_0\mathbb{C}^n \setminus E$, the Hölder norm with weight η, δ is defined as

$$\|f\|_{C_{\eta,\delta}^{k,\alpha}(Bl_0\mathbb{C}^n \setminus E)} := \|f\|_{C_{\eta}^{k,\alpha}(Bl_0B_2 \setminus E)} + \|\gamma f\|_{C_{\delta}^{k,\alpha}(Bl_0\mathbb{C}^n \setminus B_1)}, \quad (3.23)$$

here γ is a cut-off function which equals 1 outside $Bl_0B_2 \setminus E$ and equals 0 in $Bl_0B_1 \setminus E$, then γf can be viewed as a function on $\mathbb{R}^{2n} - \{0\}$, whose norm is defined as in Equation 3.22.

We also introduce the weighted Sobolev space on $Bl_0\mathbb{C}^n \setminus E$ for later use.

Definition 3.4.2. For a function f defined on $Bl_0\mathbb{C}^n \setminus E$, the Sobolev norm with weights η, δ is defined as

$$\|f\|_{W_{\eta,\delta}^{2,k}(Bl_0\mathbb{C}^n \setminus E)} := \sum_{i=0}^k \|\nabla^i f\|_{L_{\eta,\delta}^2(Bl_0\mathbb{C}^n \setminus E)}, \text{ here } \|f\|_{L_{\eta,\delta}^2(Bl_0\mathbb{C}^n \setminus E)} := \|\Gamma_{\eta,\delta} f\|_{L^2(Bl_0\mathbb{C}^n \setminus E)}. \quad (3.24)$$

$\Gamma_{\eta,\delta}$ is a smooth positive function which coincides with $e^{\eta t}$ near the divisor and coincide with $r^{-\delta}$ near the AE end.

An inclusion relation between Sobolev and Hölder weighted spaces is given as follows:

Lemma 3.4.0.1. Given $\eta, \eta', \delta \in \mathbb{R}$,

$$C_{\eta,\delta}^{k,\alpha}(Bl_0\mathbb{C}^n \setminus E) \subset W_{\eta',\delta}^{2,k}(Bl_0\mathbb{C}^n \setminus E) \iff \eta' < \eta + \frac{1}{2}.$$

Proof. Note the weight for the two spaces on the AE end are the same, then the result follows immediately from Lemma 2.3 in [55]. $\square$

3.5 Linear theory

In this section, we will study the properties of the linearized operator $L_{\omega_{HS}}$ on $Bl_0\mathbb{C}^n \setminus E$. The main goal is to construct a surjective linear map. The complexity of $L_{\omega_{HS}}$ on $Bl_0\mathbb{C}^n \setminus E$ is introduced by the existence of the divisor E and the holomorphy potential on it. As a first step, we introduce the Schauder estimate result for $L_{\omega_{HS}}$.

Theorem 3.5.1. Given $f \in L_{\eta+\frac{1}{2},\delta}^2(Bl_0\mathbb{C}^n \setminus E)$ and suppose $L_{\omega_{HS}} f \in C_{\eta,\delta-4}^{0,\alpha}(Bl_0\mathbb{C}^n \setminus E)$, then $f \in C_{\eta,\delta}^{4,\alpha}(Bl_0\mathbb{C}^n \setminus E)$ and moreover,

$$\|f\|_{C_{\eta,\delta}^{4,\alpha}(Bl_0\mathbb{C}^n \setminus E)} \lesssim \|f\|_{L_{\eta+\frac{1}{2},\delta}^2(Bl_0\mathbb{C}^n \setminus E)} + \|L_{\omega_{HS}} f\|_{C_{\eta,\delta-4}^{0,\alpha}(Bl_0\mathbb{C}^n \setminus E)}.$$

Proof. For any compact set K in $\text{Bl}_0 \mathbb{C}^n \setminus E$, the η, δ -norms are equivalent to the unweighted norm and we get

$$\|f\|_{C_{\eta,\delta}^{4,\alpha}(K)} \lesssim \|f\|_{L_{\eta+\frac{1}{2},\delta}^2(\text{Bl}_0 \mathbb{C}^n \setminus E)} + \|L_{\omega_{HS}} f\|_{C_{\eta,\delta-4}^{0,\alpha}(\text{Bl}_0 \mathbb{C}^n \setminus E)}.$$

Then it remains to deal with the situation near the divisor and the AE end. For the control near the divisor, decompose f into f^0 and f^{cyl} as in Remark 3.4.1, we identify the S^1 -invariant function f^0 with a function on $[\lambda, +\infty) \times E$, then for $s > \lambda + 2$,

$$\|e^{\eta t} f^0\|_{C^{4,\alpha}([s-1, s+1] \times E)} \lesssim \|f^0\|_{L_{\eta+\frac{1}{2},\delta}^2(\text{Bl}_0 \mathbb{C}^n \setminus E)} + \|e^{\eta t} L_{\omega_{HS}} f^0\|_{C^0([s-2, s+2] \times E)},$$

and for f^{cyl} we have

$$\|f^{cyl}\|_{C_{\eta,\delta}^{4,\alpha}} \lesssim \|f^{cyl}\|_{L_{\eta+\frac{1}{2},\delta}^2(\text{Bl}_0 \mathbb{C}^n \setminus E)} + \|L_{\omega_{HS}} f^{cyl}\|_{C_{\eta,\delta-4}^{0,\alpha}(\text{Bl}_0 \mathbb{C}^n \setminus E)}.$$

Combining these two estimates gives the desired control near the divisor. For details, we refer to [55] Proposition 3.2. For the control near the AE end, f is viewed as a function on $[\mu, \infty) \times S^{n-1}$, then for $s > \mu + 2$, with similar arguments, we get

$$\|r^{-\delta} f\|_{C^{4,\alpha}([s-1, s+1] \times E)} \lesssim \|f\|_{L_{\eta+\frac{1}{2},\delta}^2(\text{Bl}_0 \mathbb{C}^n \setminus E)} + \|r^{-\delta} L_{\omega_{HS}} f\|_{C^0([s-2, s+2] \times E)},$$

for details, we refer to Lemma 12.1.1 in [53]. Combining estimates on each region, we get the desired estimates. $\square$

Theorem 3.5.2. *For $\eta \in (0, \kappa)$ and $\delta \in (0, 1)$, the operator*

$$G_{\omega_{HS}} : C_{\eta,\delta}^{4,\alpha}(\text{Bl}_0 \mathbb{C}^n \setminus E) \oplus \ker \mathcal{D}_E^* \mathcal{D}_E \oplus \mathbb{R} \rightarrow \tilde{C}_{\eta,\delta-4}^{0,\alpha}(\text{Bl}_0 \mathbb{C}^n \setminus E)$$

given by

$$(\phi, f, \lambda) \mapsto \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(\phi + \psi_f + t\lambda)$$

has a bounded right inverse.

Proof. Consider a cut-off function γ_1 supported on a tubular neighborhood of radius R near E , denoted by $T_R \setminus E$, and let $\nabla \gamma_1$ supported on $T_R \setminus T_{\frac{R}{2}}$ with $\|\gamma_1\|_{C^{4,\alpha}} \leq c$. Define $\gamma_2 := 1 - \gamma_1$ satisfying $\|\gamma_2\|_{C^{4,\alpha}} \leq c$. Consider $\varphi \in \tilde{C}_{\eta,\delta-4}^{0,\alpha}$, then $\gamma_1 \varphi$ is viewed as a function on $T_R \setminus E$ and $\gamma_2 \varphi$ is viewed as a function on $\text{Bl}_0 \mathbb{C}^n \setminus \text{Bl}_0 B_R$.

Given $\gamma_2 \varphi$ supported on a AE manifold $\text{Bl}_0 \mathbb{C}^n \setminus \text{Bl}_0 B_R$, under the weight $\delta \in (0, 1)$, in order to show the map $\mathcal{D}^* \mathcal{D} : C_{\delta}^{4,\alpha} \rightarrow C_{\delta-4}^{0,\alpha}$ is surjective, it suffices to show its adjoint operator $\mathcal{D}^* \mathcal{D} : C_{4-2n-\delta}^{4,\alpha} \rightarrow C_{-2n-\delta}^{0,\alpha}$ is injective. To find the kernel $\mathcal{D}^* \mathcal{D} f = 0$, we first improve the decay rate of the solution at infinity. Note $4 - 2n - \delta < 0$, we use standard biharmonic analysis for Euclidean space and obtain $f \in C_{4-2n}^{4,\alpha}$ for $n \geq 3$, thus integration

by parts is justified, which implies $\mathcal{D}f = 0$. This gives rise to a holomorphic vector field and thus holomorphic function on the complement of $\mathbb{C}^n$ of a ball. Then Hartogs's extension theorem implies the holomorphic function is extended to all of $\mathbb{C}^n$, and hence is a constant. Note again $4 - 2n - \delta < 0$, we know the constant is zero, thus we conclude the triviality of the kernel. Then there exists $P_2(\gamma_2\varphi)$ such that $\mathcal{D}^*\mathcal{D}(P_2(\gamma_2\varphi)) = \gamma_2\varphi$ and satisfies

$$\|P_2(\gamma_2\varphi)\|_{C_\delta^{4,\alpha}} \lesssim \|\gamma_2\varphi\|_{C_{\delta-4}^{0,\alpha}} \lesssim \|\varphi\|_{\tilde{C}_{\eta,\delta-4}^{0,\alpha}}.$$

Given $\gamma_1\varphi$ supported on $T_R \setminus E$, satisfying $\gamma_1\varphi|_{\partial T_R} = 0$. For a function $f \in C^{k,\alpha}(T_R \setminus E)$, consider the orthogonal decomposition $f = f^0 + f^{\neq 0}$, where f^0 is invariant under the holomorphic S^1 action on $(Bl_0\mathbb{C}^n \setminus E, \omega_{HS})$, and $f^{\neq 0}$ has zero mean on each fiber. Note ω_{HS} is S^1 -invariant, and thus the associated Licherowicz operator maps S^1 -invariant part to S^1 -invariant and zero mean part to zero mean part. To solve $L_{\omega_{HS}}\phi = \gamma_1\varphi$ on $T_R \setminus E$, we solve these two parts separately.

For $(\gamma_1\varphi)^0$, it is viewed as a function on $T_R/S^1 = [0, \infty) \times E$ with a cylinder metric $dt^2 + \omega_{FS} + O(e^{-t})$ with volume form $e^{-t}dzdvol_E$, here $t = \log(-\log|z|_h^2) + O(e^{-t})$. The Lichnerowicz operator has the expression

$$L_{\omega_{HS}} = \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}} = \frac{1}{2}(\partial_z - \partial_z^2)^2 + \mathcal{D}_E^* \mathcal{D}_E + \Delta_{FS}(\partial_z - \partial_z^2) + (\partial_z - \partial_z^2) + O(e^{-t}).$$

Consider

$$C_{\eta;0}^{k,\alpha}([0, \infty) \times E) := \{f \in C_\eta^{k,\alpha}([0, \infty) \times E), f(0, \cdot) = \partial_z f(0, \cdot) \equiv 0\}.$$

Given $\{f_i\}$ a basis of $\ker \mathcal{D}_E^* \mathcal{D}_E$, and a cut-off function $\chi(t)$, define $\psi_{f_i} := \chi(t)\pi^* f_i$. From [16] Lemma 3.10, for $\eta \in (0, \kappa)$, we obtain an isomorphism

$$L_{\omega_{HS}} : C_{\eta;0}^{4,\alpha}([0, \infty) \times E) \oplus \text{Span}(\psi_{f_i}) \rightarrow C_\eta^{0,\alpha}([0, \infty) \times E). \quad (3.25)$$

Define the following **modified Hölder space**

$$\tilde{C}_\eta^{0,\alpha} := C_\eta^{0,\alpha} \oplus \text{Span}(\chi(t)\pi^* 1) \subset C_0^{0,\alpha}.$$

Lemma 3.5.2.1. *The following operator*

$$C_{\eta;0}^{4,\alpha}([0, \infty) \times E) \oplus \text{Span}(\psi_{f_i}) \oplus \mathbb{R} \rightarrow \tilde{C}_\eta^{0,\alpha}([0, \infty) \times E)$$

given by

$$(\phi, f, \lambda) \mapsto \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(\phi + \psi_f + t\lambda)$$

has a bounded right inverse.

Proof. Given $\psi \in \tilde{C}_\eta^{0,\alpha}([0, \infty) \times E)$, note

$$\mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(t\chi\pi^*1) = \chi\pi^*1 + O(e^{-t}),$$

hence there exists $\lambda \in \mathbb{R}$ such that $\psi - \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(t\chi\pi^*\lambda) \in C_\eta^{0,\alpha}$. Then, we conclude the surjectivity from (3.25). $\square$

It remains to deal with the function $(\gamma_1\varphi)^{\neq 0}$. Consider the weighted Sobolev space which vanishes on ∂T_R

$$W_{\eta;0}^{4,2} := \{f : f \in W_\eta^{4,2}, f|_{\partial T_R} = 0\}.$$

We first prove the following lemma:

Lemma 3.5.2.2. *Given $f \in W_{0;0}^{4,2}$, $\eta \in (0, \kappa)$, we have the following inequality*

$$\|\mathcal{D}^* \mathcal{D}f\|_{L_2^2} \geq C\|f\|_{L^2}. \quad (3.26)$$

Proof. Recall

$$\omega_{HS} = (1 + \frac{\tau}{2})p^*\omega_{FS} + d\tau \wedge \Theta,$$

where $\Theta = \varphi^{-1}d^c\tau$. First we show that Θ satisfies

$$\Theta(\partial_\tau) = 0.$$

It is equivalent to show $d^c\tau(\partial_\tau) = d\tau(J\partial_\tau) = 0$. For $r = \log|z|_h$, from the momentum construction, we know $\varphi dr = d\tau$, then $d\tau(J\partial_\tau) = \varphi dr(\varphi^{-1}J\partial_\tau) = 0$. Locally, when we pick a chart on D , we can write $\Theta = dz + \theta$ where θ is a connection 1-form depending on D .

Write

$$df = d_E^\theta f + \partial_\tau f d\tau + \partial_z f \Theta,$$

here d_E^θ is the modified operator in the D direction. Then

$$d^c f = d_E^{\theta,c} f + \partial_\tau f \varphi \Theta - \partial_z f \varphi^{-1} d\tau,$$

further we get

$$\begin{aligned} dd^c f &= d_E^\theta d_E^{\theta,c} f - d_E^\theta(\partial_z f \varphi^{-1}) \wedge d\tau + d\tau \wedge \partial_\tau(d_E^{\theta,c} f) + \Theta \wedge (\partial_z d_E^{\theta,c} f) + \varphi^{-1} \partial_z^2 f d\tau \wedge \Theta + \partial_\tau f \varphi d\Theta \\ &\quad + \partial_\tau(\partial_z f \varphi^{-1}) d\tau \wedge \Theta + d_E^\theta(\partial_\tau f \varphi) \wedge \Theta. \end{aligned}$$

Consider the Fourier decomposition of the functions on $Bl_0\mathbb{C}^n \setminus E$ under the infinitesimal action of S^1 action ∂_z . We write

$$(\gamma_1 \varphi)^{\neq 0} = \sum_{k \neq 0} f_k$$

with f_k satisfying $\mathcal{L}_{\partial_z} f_k = ik f_k$. We want to further decompose f_k 's. To achieve this, we apply a similar technique as in [39] Section 4.1. Note that the function f_k corresponds to a section $\hat{f}_k$ of $\mathcal{O}(k)$ over $\mathbb{CP}^{n-1}$. Consider the Fourier decomposition of the Dolbeault Laplacian operator Δ_k for the eigensections of $\mathcal{O}(k)$, we write

$$\hat{f}_k = \sum_{\lambda_0} \hat{f}_{k,\lambda_0}, \text{ for } \Delta_k \hat{f}_{k,\lambda_0} = \lambda_0 \hat{f}_{k,\lambda_0}.$$

Consider on $Bl_0 \mathbb{C}^n \setminus E$ the operator

$$\Delta_{FS}^\theta f_{k,\lambda} := (n-1) \frac{d_E^\theta d_E^{\theta,c} \wedge p^* \omega_D^{n-2}}{p^* \omega_D^{n-1}} f_{k,\lambda}.$$

Pick a local chart, with coordinate x_i, y_i on D and coordinate ξ on fiber. Let $e_0(z)$ be a holomorphic trivialization, we write

$$f(x_i, y_i, \xi) = f(x_i, y_i) \xi^k, \quad \hat{f}(x_i, y_i) = f(x_i, y_i) (e_0(z)^*)^k.$$

Then $\Delta_{FS}^\theta f_{k,\lambda}$ becomes

$$\frac{1}{4} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} \right) f_{k,\lambda}.$$

Applying the Bochner-Kodaira-Nakano formula to Δ_k , write Δ_k^∂ as the ∂ -laplacian, we know $\Delta_k = \Delta_k^\partial + k(n-1)$ and thus $\lambda_0 \geq k(n-1)$. Note from $\Delta_k \hat{f}_{k,\lambda_0} = \lambda_0 \hat{f}_{k,\lambda_0}$ we deduce that

$$\frac{1}{4} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} \right) f_{k,\lambda} = \left(-\lambda_0 + \frac{k(n-1)}{2} \right) f_{k,\lambda}.$$

We denote $\lambda := -\lambda_0 + \frac{k(n-1)}{2}$. Note

$$d\Theta = d(d^c \tau \cdot \varphi^{-1}) = \varphi^{-1} \cdot dd^c \tau + \partial_\tau(\varphi^{-1}) d\tau \wedge d^c \tau.$$

We recall from the momentum construction that given a function $\psi(\tau)$ on the total space of line bundle, we have

$$dd^c \psi(\tau) = (\varphi \psi')(\tau) \frac{1}{2} \pi^* \omega_{FS} + \frac{1}{\varphi} (\varphi \psi')'(\tau) d\tau \wedge d^c \tau.$$

Thus

$$dd^c \tau = \frac{1}{2} \varphi \pi^* \omega_{FS} + \varphi^{-1} \varphi_\tau d\tau \wedge d^c \tau.$$

In particular, we note $d\Theta$ does not have contribution of mixed term of fiber and base. Write $\theta = \theta^{x_i} dx_i + \theta^{y_i} dy_i$, we have

$$\partial_\tau(\theta^{x_i}) = \partial_\tau(\theta^{y_i}) = 0.$$

From this we know θ and thus $d_E^\theta, d_E^{\theta,c}$ are invariant under τ . From now on we abbreviate $f_{k,\lambda}$ as f for simplification. Then

$$\Delta_{\omega_{HS}} f = n \frac{dd^c f \wedge \omega_{HS}^{n-1}}{\omega_{HS}^n} = \frac{\lambda}{1 + \frac{\tau}{2}} f - \varphi^{-1} k^2 f + \partial_\tau(\varphi \partial_\tau f) + \frac{n-1}{2(1 + \frac{\tau}{2})} \varphi \partial_\tau f.$$

To compute the Licherowicz operator

$$\mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}} f = \frac{1}{2} \Delta_{\omega_{HS}}^2 f + \langle \rho_{\omega_{HS}}, dd^c f \rangle,$$

we need to compute the Ricci curvature of ω_{HS} . From momentum construction, the expression for the Ricci curvature is

$$\rho = \pi^* \omega_{FS} + \frac{1}{2Q} (\varphi Q)'(\tau) \pi^* \gamma - \frac{1}{2} \left[\frac{1}{Q} (\varphi Q)' \right]'(\tau) d\tau \wedge \Theta,$$

here $\gamma = -\sqrt{-1} \partial \bar{\partial} \log h$ for hermitian metric h on the line bundle. In our case, plugging in $Q(\tau) = (1 + \frac{\tau}{2})^{n-1}$ and $\varphi(\tau) = \frac{8}{n} \left(1 + \frac{\tau}{2} - \frac{1 + \frac{n}{2} \tau}{(1 + \frac{\tau}{2})^{n-1}} \right)$, we obtain

$$\rho_{\omega_{HS}} = \frac{1}{(1 + \frac{\tau}{2})^{n-1}} \pi^* \omega_D - \frac{1}{(1 + \frac{\tau}{2})^n} d\tau \wedge \Theta,$$

then

$$\langle \rho_{\omega_{HS}}, dd^c f \rangle = \frac{1}{(1 + \frac{\tau}{2})^{n+1}} (\lambda f + \frac{1}{2} \varphi f_\tau) + \frac{1}{(1 + \frac{\tau}{2})^n} (\varphi^{-1} k^2 f - \partial_\tau(\varphi f_\tau)).$$

Note given $f \in W_{0,0}^{4,2}$, we integrate over $T_R \setminus E$, applying integration by parts, we get

$$\begin{aligned}
 \int 2\mathcal{D}^*\mathcal{D}f \cdot f \, d\text{vol} &= \int (\Delta_{\omega_{HS}}f)^2 \, d\text{vol} + \int 2\langle \rho_{\omega_{HS}}, dd^c f \rangle f \, d\text{vol} \\
 &= \int \left(-\varphi^{-1}k^2f + \varphi_\tau f_\tau + \frac{n-1}{2(1+\frac{\tau}{2})}\varphi f_\tau + \varphi f_{\tau\tau} \right)^2 + 2f \cdot f_\tau \cdot \varphi_\tau \left(\frac{\lambda}{1+\frac{\tau}{2}} - \frac{1}{(1+\frac{\tau}{2})^n} \right) \\
 &\quad + \int f \cdot f_\tau \cdot \varphi \left(\frac{(n-1)\lambda}{(1+\frac{\tau}{2})^2} + \frac{1}{(1+\frac{\tau}{2})^{n+1}} \right) + \int 2f \cdot f_{\tau\tau} \cdot \varphi \left(\frac{\lambda}{1+\frac{\tau}{2}} - \frac{1}{(1+\frac{\tau}{2})^n} \right) \\
 &\quad + \left(2\varphi^{-1}k^2 \left(\frac{1}{(1+\frac{\tau}{2})^n} - \frac{\lambda}{1+\frac{\tau}{2}} \right) + \frac{2\lambda}{(1+\frac{\tau}{2})^{n+1}} + \frac{\lambda^2}{(1+\frac{\tau}{2})^2} \right) f^2 \\
 &= \int \left(-\varphi^{-1}k^2f + \varphi_\tau f_\tau + \frac{n-1}{2(1+\frac{\tau}{2})}\varphi f_\tau + \varphi f_{\tau\tau} \right)^2 \\
 &\quad + \int \left(2\varphi^{-1}k^2 \left(\frac{1}{(1+\frac{\tau}{2})^n} - \frac{\lambda}{1+\frac{\tau}{2}} \right) + \frac{2\lambda}{(1+\frac{\tau}{2})^{n+1}} + \frac{\lambda^2}{(1+\frac{\tau}{2})^2} \right) f^2 \\
 &\quad + f \cdot f_\tau \cdot \varphi \left(\frac{n\lambda}{(1+\frac{\tau}{2})^2} - \frac{n-1}{(1+\frac{\tau}{2})^{n+1}} \right) + \left(\frac{2\varphi}{(1+\frac{\tau}{2})^n} - \frac{2\lambda\varphi}{1+\frac{\tau}{2}} \right) f_\tau^2 \\
 &\geq \int C\tau^{-2}f^2 \geq \int C'e^{2t}f^2 \, d\text{vol}
 \end{aligned}$$

Applying Cauchy-Schwartz inequality to the above, we obtain the desired inequality for $f_{k,\lambda}$. Using the orthogonality of the eigenfunctions corresponding to different eigenvalues, we obtain the inequality for f . $\square$

We construct the inverse of $(\gamma_1\varphi)^{\neq 0}$ under the operator $\mathcal{D}^*\mathcal{D}$ by first doing cut-off so that it supports in a compact annulus, then we find its inverse under $\mathcal{D}^*\mathcal{D}$ on each annulus, and finally we glue them together using partition of unity. This strategy is first explored by Biquard in [17]. The inequality (3.26) above is used to improve the regularity of the inverse.

Let $(\xi_A)_{A \in \mathbb{Z}_+}$ be a partition of unity to the intervals $(A-1, A+1)$ in coordinate t . Let U_A be the annulus whose coordinate t lives in $(A-1, A+1)$. From (3.26), we know the map we know the map

$$\mathcal{D}^*\mathcal{D} : W_0^{4,2}(U_A) \subset L^2(U_A) \rightarrow L^2(U_A)$$

is injective. Here, $W_0^{4,2}(U_A)$ is the closure of $C_c^\infty(U_A)$ in $W^{4,2}(U_A)$. Note the operator above is densely defined, closed, and self-adjoint, then from [19] Theorem 2.19, we know $\text{Im } \mathcal{D}^*\mathcal{D}$ being closed is equivalent to $\text{Im } \mathcal{D}^*\mathcal{D} = (\ker \mathcal{D}^*\mathcal{D})^\perp$. Note we have the following elliptic a priori estimates

$$\|f\|_{W^{4,2}(U_A)} \lesssim \|\mathcal{D}^*\mathcal{D}f\|_{L^2(U_A)} + \|f\|_{L^2(U_A)}.$$

Then from standard arguments we know $\text{Im } \mathcal{D}^* \mathcal{D}$ is closed, see [73] Lemma 4.3, [53] Theorem 9.2.1. Hence the map is surjective, for any $B \in \mathbb{Z}_+$, we use $(\mathcal{D}^* \mathcal{D})^{-1}(\zeta_B g)$ to denote the inverse. On the annulus U_A , let 1_A denote the function on it which is constantly equal to 1. We consider the following weighted estimate

$$\begin{aligned}
\int_{U_A} |\mathcal{D}^* \mathcal{D} f|^2 e^{(2\eta-3)t} &\geq \min\{e^{(2\eta-3)(A-1)}, e^{(2\eta-3)(A+1)}\} \int_{U_A} |\mathcal{D}^* \mathcal{D} f|^2 \\
&= e^{-2|2\eta-3|} \max\{e^{(2\eta-3)(A-1)}, e^{(2\eta-3)(A+1)}\} \int_{U_A} |\mathcal{D}^* \mathcal{D} f|^2 \\
&\geq e^{-2|2\eta-3|} \max\{e^{(2\eta-3)(A-1)}, e^{(2\eta-3)(A+1)}\} \int_{U_A} |f|^2 e^{4t} \\
&\geq e^{-2|2\eta-3|} \int_{U_A} |f|^2 e^{(2\eta+1)t}.
\end{aligned}$$

Note that $e^{-2|2\eta-3|}$ is a constant independent of A , then any $A, B \in \mathbb{Z}_+$, applying the above inequality, we get

$$\begin{aligned}
\|\zeta_A (\mathcal{D}^* \mathcal{D})^{-1}(\zeta_B g)\|_{L^2_{\eta+\frac{1}{2}}} &\leq e^{(\eta-\eta_1)A} \|\zeta_A (\mathcal{D}^* \mathcal{D})^{-1}(\zeta_B g)\|_{L^2_{\eta+\frac{1}{2}}} \\
&\lesssim e^{(\eta-\eta_1)A} \|1_A \cdot (\mathcal{D}^* \mathcal{D})^{-1}(\zeta_B g)\|_{L^2_{\eta+\frac{1}{2}}} \\
&\lesssim e^{(\eta-\eta_1)A} \|\zeta_B g\|_{L^2_{\eta_1-\frac{3}{2}}} \\
&\lesssim e^{(\eta-\eta_1)(A-B)} \|\zeta_B g\|_{L^2_{\eta_1-\frac{3}{2}}}
\end{aligned}$$

For each A , choose η_1 such that $\eta - \eta_1 = \epsilon \cdot \text{sgn}(A - B)$, then the above expression becomes

$$\|\zeta_A (\mathcal{D}^* \mathcal{D})^{-1}(\zeta_B g)\|_{L^2_{\eta+\frac{1}{2}}} \lesssim e^{\epsilon|A-B|} \|\zeta_B g\|_{L^2_{\eta_1-\frac{3}{2}}}$$

Combining the above, we have the following estimate

$$\begin{aligned}
\left\| \sum_A \sum_B \zeta_A (\mathcal{D}^* \mathcal{D})^{-1}(\zeta_B g) \right\|_{L^2_{\eta+\frac{1}{2}}} &\leq \left(\sum_A \left\| \sum_B \zeta_A (\mathcal{D}^* \mathcal{D})^{-1}(\zeta_B g) \right\|_{L^2_{\eta+\frac{1}{2}}}^2 \right)^{\frac{1}{2}} \\
&\leq \left(\sum_A \left(\sum_B \|\zeta_A (\mathcal{D}^* \mathcal{D})^{-1}(\zeta_B g)\|_{L^2_{\eta+\frac{1}{2}}} \right)^2 \right)^{\frac{1}{2}} \\
&\lesssim \left(\sum_A \left(\sum_B e^{-\epsilon|A-B|} \|\zeta_B g\|_{L^2_{\eta_1-\frac{3}{2}}} \right)^2 \right)^{\frac{1}{2}} \\
&\lesssim \left(\sum_B \|\zeta_B g\|_{L^2_{\eta_1-\frac{3}{2}}}^2 \right)^{\frac{1}{2}} \leq \|g\|_{L^2_{\eta-\frac{3}{2}}} \lesssim \|g\|_{C_\eta^{0,\alpha}}
\end{aligned}$$

From elliptic regularity Theorem 3.5.1, we conclude $\sum_A \sum_B \zeta_A (\mathcal{D}^* \mathcal{D})^{-1} (\zeta_B g) \in C_\eta^{4,\alpha}$. Combining this construction with Lemma 3.5.2.1, we obtain $P_1(\gamma_1 \varphi)$ satisfying $G_{\omega_{HS}}(P_1(\gamma_1 \varphi)) = \gamma_1 \varphi$ and

$$\|P_1(\gamma_1 \varphi)\|_{C_\eta^{4,\alpha} \times \ker \mathcal{D}_E^* \mathcal{D}_E \times \mathbb{R}} \lesssim \|\varphi\|_{\tilde{C}_{\eta, \delta-4}^{0,\alpha}}$$

Consider for $i = 1, 2$ the cutoff functions β_i with $\beta_i = 1$ on the support of γ_i , and $\nabla \beta_1$ supports on $T_{2R} \setminus T_R$, $\nabla \beta_2$ supports on $T_{\frac{R}{2}} \setminus T_{\frac{R}{4}}$, further, $\|\nabla \beta_1\|_{C^{3,\alpha}} \leq \frac{C}{\log R}$ and $\|\nabla \beta_2\|_{C^{3,\alpha}} \leq \frac{C}{\log R}$. This can be achieved by defining $\beta_1 := \beta\left(\frac{|z|}{\log R}\right)$ where $\beta(r) = 1$ for $r < \frac{R}{\log R}$ and $\beta(r) = 0$ for $r > \frac{2R}{\log R}$; $\beta_2 := \tilde{\beta}\left(\frac{|z|}{\log R}\right)$ where $\tilde{\beta}(r) = 1$ for $r > \frac{R}{2 \log R}$ and $\tilde{\beta} = 0$ for $r < \frac{R}{4 \log R}$. Define

$$P(\varphi) = \beta_1 P_1(\gamma_1 \varphi) + \beta_2 P_2(\gamma_2 \varphi)$$

First we show that $\varphi \mapsto P(\varphi)$ gives an approximate inverse, i.e.,

$$\|G_{\omega_{HS}}(P(\varphi)) - \varphi\|_{\tilde{C}_{\eta, \delta-4}^{0,\alpha}} \leq \frac{1}{2} \|\varphi\|_{\tilde{C}_{\eta, \delta-4}^{0,\alpha}}. \quad (3.27)$$

It reduces to show

$$\|G_{\omega_{HS}}(\beta_i P_i(\gamma_i \varphi)) - \gamma_i \varphi\| \leq \frac{1}{4} \|\varphi\|.$$

Write

$$G_{\omega_{HS}}(\beta_i P_i(\gamma_i \varphi)) - \gamma_i \varphi = \beta_i \gamma_i \varphi - \gamma_i \varphi + D^3(\nabla \beta_i \star P_i(\gamma_i \varphi)),$$

here D^3 denotes a third-order differential operator, and $\star$ a bilinear operator. Note

$$\|D^3(\nabla \beta_1 \star P_1(\gamma_1 \varphi))\|_{C_\eta^{0,\alpha}} \lesssim \|\nabla \beta_1\|_{C^{3,\alpha}} \|P_1(\gamma_1 \varphi)\|_{C_\eta^{3,\alpha}} = o(1) \|\varphi\|_{C_\eta^{0,\alpha}},$$

$$\|D^3(\nabla \beta_2 \star P_2(\gamma_2 \varphi))\|_{C_{\delta-4}^{0,\alpha}} \lesssim \|\nabla \beta_2\|_{C_{-1}^{3,\alpha}} \|P_2(\gamma_2 \varphi)\|_{C_\delta^{3,\alpha}} = o(1) \|\varphi\|_{C_{\delta-4}^{0,\alpha}}.$$

Here $\nabla \beta_2$ is supported on $T_{\frac{R}{2}} \setminus T_{\frac{R}{4}}$, and $\|\nabla \beta_2\|_{C_{-1}^{3,\alpha}} \leq \frac{C}{\log R}$, thus proving (3.27). From (3.27), we see there is a uniformly bounded inverse $(G_{\omega_{HS}} \circ P)^{-1}$, thus $G_{\omega_{HS}}$ has a right inverse $P \circ (G_{\omega_{HS}} \circ P)^{-1}$. □

3.6 Existence of solution

To obtain a cscK metric on $Bl_p X \setminus E$, we want to solve a boundary value problem for the cscK equation on the two pieces X_{r_ϵ} and $Bl_0 B_{R_\epsilon} \setminus E$. The arguments for the piece X_{r_ϵ} are the same as Arezzo-Pacard's case. We can directly take from [11].

Theorem 3.6.1. ([11], Proposition 6.1) *Fix a constant $\Lambda > 0$ large enough, there exist $c_\Lambda > 0$, $\epsilon_\Lambda > 0$ such that $\forall \epsilon \in (0, \epsilon_\Lambda)$, and all boundary data $h \in C^{4,\alpha}(\partial B_1)$, $k \in C^{2,\alpha}(\partial B_1)$ satisfying*

$$\|h\|_{C^{4,\alpha}(\partial B_1)} \leq \Lambda r_\epsilon^4, \quad \|k\|_{C^{2,\alpha}(\partial B_1)} \leq \Lambda r_\epsilon^4, \quad (3.28)$$

there exists $\varphi_{\epsilon,h,k}$ such that the Kähler form $\omega_{\epsilon,h,k}$ defined by Equation 3.17 on X_{r_ϵ} has constant scalar curvature $s(\omega_X) + \nu_{\epsilon,h,k}$. If we use $\hat{\varphi}$ to denote the component in φ which is defined on X_p , then

$$\|\hat{\varphi}_{\epsilon,h,k}|_{B_{2r_\epsilon} \setminus B_{r_\epsilon}}(r_\epsilon \cdot) - H_{h,k}^0\|_{C^{4,\alpha}(B_2 \setminus B_1)} \leq c_\Lambda r_\epsilon^{2n+\delta}, \quad |\nu_{\epsilon,h,k}| \leq c_\Lambda r_\epsilon^{2n}.$$

Moreover,

$$\|\hat{\varphi}_{\epsilon,h,k} - \hat{\varphi}_{\epsilon,h',k'}|_{B_{2r_\epsilon} \setminus B_{r_\epsilon}}(r_\epsilon \cdot) - H_{h,k}^0\|_{C^{4,\alpha}(B_2 \setminus B_1)} \leq c_\Lambda r_\epsilon^{2n-4+\delta} \|h - h', k - k'\|, \quad (3.29)$$

$$|\nu_{\epsilon,h,k} - \nu_{\epsilon,h',k'}| \leq c_\Lambda r_\epsilon^{2n-4} \|h - h', k - k'\|. \quad (3.30)$$

The main difference lies in the piece of $Bl_0 \mathbb{C}^n \setminus E$, which we discuss as follows.

Theorem 3.6.2. *Fix a constant $\Lambda > 0$ large enough, there exist $c > 0$ independent of Λ and $\epsilon_\Lambda > 0$ such that $\forall \epsilon \in (0, \epsilon_\Lambda)$, and for all $\nu \in \mathbb{R}$ and boundary data $\tilde{h} \in C^{4,\alpha}(\partial B_1)$, $\tilde{k} \in C^{2,\alpha}(\partial B_1)$ satisfying*

$$|\nu| \leq |s(\omega_X)| + 1, \quad \|\tilde{h}\|_{C^{4,\alpha}(\partial B_1)} \leq \Lambda R_\epsilon^{3-2n}, \quad \|\tilde{k}\|_{C^{2,\alpha}(\partial B_1)} \leq \Lambda R_\epsilon^{3-2n}, \quad (3.31)$$

there exist $\phi_\epsilon \in C_{\eta,\delta}^{4,\alpha}$, constant $\lambda_\epsilon \in \mathbb{R}$ and $f_\epsilon \in \ker \mathcal{D}_E^ \mathcal{D}_E$ such that the scalar curvature of the Kähler form*

$$\omega_{\epsilon,\tilde{h},\tilde{k},HS} = \omega_{HS} + \sqrt{-1} \partial \bar{\partial} (H_{\epsilon,\tilde{h},\tilde{k}}^1 + \phi_\epsilon + \lambda_\epsilon t)$$

is

$$\epsilon^2 \nu + \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(\psi_{f_\epsilon}) + \frac{1}{2} \langle \nabla \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(\psi_{f_\epsilon}), \nabla (H_{\epsilon,\tilde{h},\tilde{k}}^1 + \phi_\epsilon + \lambda_\epsilon t) \rangle.$$

Further,

$$\|\phi_\epsilon(R_\epsilon \cdot)\|_{C^{4,\alpha}(B_2 \setminus B_1)} \leq c R_\epsilon^{3-2n}, \quad |\lambda_\epsilon| + \|f_\epsilon\| \leq c R_\epsilon^{3-2n-\delta}.$$

If $\phi_\epsilon, \lambda_\epsilon$ are determined by boundary data $\tilde{h}, \tilde{k}$ and $\phi'_\epsilon, \lambda'_\epsilon$ are determined by boundary data $\tilde{h}', \tilde{k}'$, then

$$\|f_\epsilon - f'_\epsilon\| + |\lambda_\epsilon - \lambda'_\epsilon| + \|\phi_\epsilon(R_\epsilon \cdot) - \phi'_\epsilon(R_\epsilon \cdot)\| \leq c_\Lambda (R_\epsilon^{\delta-1} \|(\tilde{h} - \tilde{h}', \tilde{k} - \tilde{k}')\| + R_\epsilon^{3-2n} |\nu - \nu'|); \quad (3.32)$$

Proof. In the proof, we omit the subscript ϵ for $\phi_\epsilon, \lambda_\epsilon, f_\epsilon$ for simplicity concern. We want to solve the equation

$$s(\omega_{HS} + \sqrt{-1}\partial\bar{\partial}(H_{\epsilon, \tilde{h}, \tilde{k}}^1 + \phi + \lambda t)) = \epsilon^2\nu + \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(\psi_f) + \frac{1}{2} \langle \nabla \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(\psi_f), \nabla(H_{\epsilon, \tilde{h}, \tilde{k}}^1 + \phi + \lambda t) \rangle, \quad (3.33)$$

for a function ϕ , a constant λ and a holomorphy potential f on E . If we separate the equation into linear and non-linear terms, we get

$$\begin{aligned} \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(\phi + \psi_f + t\lambda) &= Q_{\omega_{HS}}(H_{\epsilon, \tilde{h}, \tilde{k}}^1 + \phi + \lambda t) - \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(H_{\epsilon, \tilde{h}, \tilde{k}}^1) - \epsilon^2\nu \\ &\quad - \frac{1}{2} \langle \nabla \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(\psi_f), \nabla(H_{\epsilon, \tilde{h}, \tilde{k}}^1 + \phi + \lambda t) \rangle, \end{aligned}$$

and we denote the RHS by Q_ϵ . From Theorem 3.5.2, we know the operator

$$(\phi, f, \lambda) \mapsto \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(\phi + \psi_f + t\lambda)$$

has a bounded right inverse, given our choice of weights in Theorem 3.5.2. We consider an extension operator following [12] Section 5.2. For given $R \geq 1$, consider $\mathcal{E}_R : C_{\eta, \delta}^{0, \alpha}(Bl_0 B_R \setminus E) \rightarrow C_{\eta, \delta}^{0, \alpha}(Bl_0 \mathbb{C}^n \setminus E)$ defined by

$$\mathcal{E}_R(f(z)) = \begin{cases} f(z) & \text{in } Bl_0 \bar{B}_R \setminus E \\ \bar{\chi}\left(\frac{|z|}{R}\right) f\left(R \frac{z}{|z|}\right) & \text{in } Bl_0 B_{2R} \setminus Bl_0 B_R \\ 0 & \text{in } Bl_0 \mathbb{C}^n \setminus Bl_0 B_{2R} \end{cases} \quad (3.34)$$

where $t \mapsto \bar{\chi}(t)$ is a smooth cut-off function which equals to 1 for $t < \frac{5}{4}$ and vanishes for $t > \frac{7}{4}$. Let $\mathcal{E}_\epsilon$ be the extension operator for R_ϵ , from [12] we know it is a bounded operator whose norm is independent of ϵ . With this operator, the Equation 3.33 can be rewritten as a fixed point problem

$$(\phi, \lambda, f) = N_\epsilon(\phi, \lambda, f)$$

here we define

$$\mathcal{N}_\epsilon := G_\epsilon^{-1} \circ \mathcal{E}_\epsilon \circ Q_\epsilon : C_{\eta, \delta}^{4, \alpha}(Bl_0 \mathbb{C}^n \setminus E) \times \ker \mathcal{D}_E^* \mathcal{D}_E \times \mathbb{R} \rightarrow C_{\eta, \delta}^{4, \alpha}(Bl_0 \mathbb{C}^n \setminus E) \times \ker \mathcal{D}_E^* \mathcal{D}_E \times \mathbb{R}. \quad (3.35)$$

Lemma 3.6.2.1. *Given $\|\phi\|_{C_{0,2}^{4, \alpha}}, |\lambda| \leq c$, let*

$$\omega'_{HS} := \omega_{HS} + \sqrt{-1}\partial\bar{\partial}H_{\epsilon, \tilde{h}, \tilde{k}}^1, \quad \omega_\lambda := \omega'_{HS} + \sqrt{-1}\partial\bar{\partial}\lambda t, \quad \omega_{\phi, \lambda} := \omega'_{HS} + \sqrt{-1}\partial\bar{\partial}(\phi + \lambda t).$$

Then there exists $C > 0$, such that

$$\|L_{\omega_{\phi, \lambda}} - L_{\omega'_{HS}}\|_{C_{\eta, \delta}^{4, \alpha} \times \mathbb{R} \rightarrow \tilde{C}_{\eta, \delta-4}^{0, \alpha}} \leq C \|(\phi, \lambda)\|_{C_{0,2}^{4, \alpha} \times \mathbb{R}}.$$

Proof. First we show there exists $C_1 > 0$ such that $\|g'_{HS}\|_{C^{2,\alpha}(Bl_0\mathbb{C}\setminus E)} \leq C_1$. For this, we divide the regions into three parts, pulling back the metric components of g'_{HS} to each region. Note that it is different from pulling back the metric tensors here, and no additional rescaling factor will be introduced. Near E , the metric is uniformly equivalent to the Poincaré hyperbolic metric; on the neck annulus, the metric is uniformly equivalent to the Euclidean flat metric; on $X \setminus B_1$, it's the cscK metric we started with. In each case, we have a uniform upper bound on the metric. Similarly, we get $\|g'_{HS}\|_{C^{2,\alpha}(Bl_0\mathbb{C}^n \setminus E)}^{-1} \leq C_2$ for some $C_2 > 0$. Then we have

$$\begin{aligned} \|L_{\omega_\lambda}(\varphi, \mu) - L_{\omega'_{HS}}(\varphi, \mu)\|_{\tilde{C}_{\eta, \delta-4}^{0,\alpha}} &= \left\| \left(g_\lambda^{j\bar{k}} g_\lambda^{p\bar{q}} - g'_{HS}{}^{j\bar{k}} g'_{HS}{}^{p\bar{q}} \right) \nabla_p \nabla_j \nabla_{\bar{k}} \nabla_{\bar{q}} (\varphi + \mu t) \right\|_{\tilde{C}_{\eta, \delta-4}^{0,\alpha}} \\ &\leq \left\| g_\lambda^{j\bar{k}} \left(g_\lambda^{p\bar{q}} - g'_{HS}{}^{p\bar{q}} \right) - \left(g_\lambda^{j\bar{k}} - g'_{HS}{}^{j\bar{k}} \right) g'_{HS}{}^{p\bar{q}} \right\|_{C^{2,\alpha}} \|(\varphi, \mu)\|_{C_{\eta, \delta}^{4,\alpha} \times \mathbb{R}}. \end{aligned} \quad (3.36)$$

$$\begin{aligned} \|L_{\omega_{\phi, \lambda}}(\varphi, \mu) - L_{\omega_\lambda}(\varphi, \mu)\|_{\tilde{C}_{\eta, \delta-4}^{0,\alpha}} &= \left\| \left(g_{\phi, \lambda}^{j\bar{k}} g_{\phi, \lambda}^{p\bar{q}} - g_\lambda^{j\bar{k}} g_\lambda^{p\bar{q}} \right) \nabla_p \nabla_j \nabla_{\bar{k}} \nabla_{\bar{q}} (\varphi + \mu t) \right\|_{\tilde{C}_{\eta, \delta-4}^{0,\alpha}} \\ &\leq \left\| g_{\phi, \lambda}^{j\bar{k}} \left(g_{\phi, \lambda}^{p\bar{q}} - g_\lambda^{p\bar{q}} \right) - \left(g_{\phi, \lambda}^{j\bar{k}} - g_\lambda^{j\bar{k}} \right) g_\lambda^{p\bar{q}} \right\|_{C^{2,\alpha}} \|(\varphi, \mu)\|_{C_{\eta, \delta}^{4,\alpha} \times \mathbb{R}} \end{aligned} \quad (3.37)$$

We observe that with this inequality, now it suffices to give bounds on $g_{\phi, \lambda}^{-1} - g_\lambda^{-1}$ and $g_\lambda^{-1} - g'_{HS}{}^{-1}$. Note

$$g_{\phi, \lambda}^{-1} - g_\lambda^{-1} = g_{\phi, \lambda}^{-1} (g_\lambda - g_{\phi, \lambda}) g_\lambda^{-1}, \quad g_\lambda^{-1} - g'_{HS}{}^{-1} = g_\lambda^{-1} (g'_{HS} - g_\lambda) g'_{HS}{}^{-1},$$

then

$$\|g_{\phi, \lambda}^{-1} - g_\lambda^{-1}\|_{C^{2,\alpha}} \leq \|g_{\phi, \lambda}^{-1}\|_{C^{2,\alpha}} \|\sqrt{-1} \partial \bar{\partial} \phi\|_{C^{2,\alpha}} \|g_\lambda^{-1}\|_{C^{2,\alpha}} \leq C_3 \|\phi\|_{C_{0,2}^{4,\alpha}}. \quad (3.38)$$

Note that g'_{HS} is cscK metric near the divisor, we know from [13] Theorem 3.1 (which we quote as Theorem 3.7.1 in Section 3.7) that

$$(g'_{HS})_{i\bar{j}} = \begin{pmatrix} \frac{1}{|z_1|^2 \log^2 |z_1|^2} & 0 \\ 0 & g_{FS, i\bar{j}} \end{pmatrix} + O(e^{-\delta t}).$$

Let $A := \begin{pmatrix} \frac{1}{|z_1|^2 \log^2 |z_1|^2} & 0 \\ 0 & g_{FS, i\bar{j}} \end{pmatrix}$, then $A^{-1} = \begin{pmatrix} |z_1|^2 \log^2 |z_1|^2 & 0 \\ 0 & g_{FS}^{i\bar{j}} \end{pmatrix}$ which has bounded norm. Then for $(X)_{i\bar{j}} := (g)_{i\bar{j}} - (A)_{i\bar{j}}$, we know $(A + X)^{-1} = A^{-1} - A^{-1} X A^{-1} + O(\|X\|^2)$. Thus

$$(g'_{HS})^{i\bar{j}} = A^{-1} + O(e^{-\delta t}).$$

From $(g'_{HS} - g_\lambda)_{i\bar{j}} = \begin{pmatrix} \frac{\lambda}{|z_1|^2 \log^2 |z_1|^2} & 0 \\ 0 & 0 \end{pmatrix}$, we see $\|(g'_{HS} - g_\lambda) g'_{HS}{}^{-1}\|_{C^{2,\alpha}} \leq C_4 |\lambda|$, and thus

$$\|g_\lambda^{-1} - g'_{HS}^{-1}\|_{C^{2,\alpha}} \leq \|g_\lambda^{-1}\|_{C^{2,\alpha}} \|(g'_{HS} - g_\lambda)g'_{HS}^{-1}\|_{C^{2,\alpha}} \leq C_5|\lambda|. \quad (3.39)$$

Substituting Equation 3.38 back to Equation 3.36 and Equation 3.39 back to Equation 3.37, we get the desired result. $\square$

Lemma 3.6.2.2. *There is a constant $c > 0$ independent of Λ and $c_\Lambda > 0$, $\epsilon_\Lambda > 0$ such that for $\epsilon \in (0, \epsilon_\Lambda)$,*

$$\|\mathcal{N}_\epsilon(0, 0, 0)\|_{C_{\eta,\delta}^{4,\alpha} \times \ker \mathcal{D}_E^* \mathcal{D}_E \times \mathbb{R}} \leq cR_\epsilon^{3-2n-\delta}. \quad (3.40)$$

If

$$\|(\phi_1, f_1, \lambda_1)\|_{C_{\eta,\delta}^{4,\alpha} \times \ker \mathcal{D}_E^* \mathcal{D}_E \times \mathbb{R}}, \quad \|(\phi_2, f_2, \lambda_2)\|_{C_{\eta,\delta}^{4,\alpha} \times \ker \mathcal{D}_E^* \mathcal{D}_E \times \mathbb{R}} \leq 2cR_\epsilon^{3-2n-\delta},$$

then

$$\|\mathcal{N}_\epsilon(\phi_1, f_1, \lambda_1) - \mathcal{N}_\epsilon(\phi_2, f_2, \lambda_2)\| \leq c_\Lambda R_\epsilon^{3-2n-\delta} \|(\phi_1, f_1, \lambda_1) - (\phi_2, f_2, \lambda_2)\|, \quad (3.41)$$

and let $\tilde{\mathcal{N}}_\epsilon$ be the map taking into account the set of ν and the boundary data $\tilde{h}', \tilde{k}'$, then

$$\|\tilde{\mathcal{N}}_\epsilon(\phi, f, \lambda, \nu, \tilde{h}, \tilde{k}) - \tilde{\mathcal{N}}_\epsilon(\phi, f, \lambda, \nu', \tilde{h}', \tilde{k}')\| \leq c_\Lambda (R_\epsilon^{-1} \|(\tilde{h} - \tilde{h}', \tilde{k} - \tilde{k}')\|_{C^{4,\alpha} \times C^{2,\alpha}} + R_\epsilon^{3-2n-\delta} |\nu - \nu'|). \quad (3.42)$$

Proof. For Equation 3.40, note from 3.16, $H_{\epsilon,\tilde{h},\tilde{k}}^1$ is supported away from the cusp end, we can argue similarly as in [12] Lemma 5.3. More precisely, we have

$$\|\nabla^2 H_{\epsilon,\tilde{h},\tilde{k}}^1\|_{C^{0,2}(Bl_0 B_2 \setminus E)} \leq c_\Lambda R_\epsilon^{2-2n},$$

then note $L_{\omega_{HS}} H_{h,k}^1 = (L_{\omega_{HS}} - \Delta_{Euc}^2) H_{h,k}^1$ on the annulus $Bl_0 B_{R_\epsilon} \setminus Bl_0 B_2$, we see from the expansions 3.10, 3.13, the coefficients of ∇^{2+i} lives in $C_{i-2n}^{0,\alpha}$ for $i = 0, 1, 2$ and thus

$$\|L_{\omega_{HS}} H_{\epsilon,\tilde{h},\tilde{k}}^1\|_{\tilde{C}_{\eta,\delta-4}^{0,\alpha}} = \|L_{\omega_{HS}} H_{\epsilon,\tilde{h},\tilde{k}}^1\|_{C_{\delta-4}^{0,\alpha}} \leq c_\Lambda R_\epsilon^{2-2n} R_\epsilon^{-2n} R_\epsilon^{4-\delta} = c_\Lambda R_\epsilon^{2-2n},$$

and

$$\|\mathcal{E}(Q_{\omega_{HS}}(\nabla^2 H_{\epsilon,\tilde{h},\tilde{k}}^1))\|_{\tilde{C}_{\eta,\delta-4}^{0,\alpha}} = \|\mathcal{E}(Q_{\omega_{HS}}(\nabla^2 H_{\epsilon,\tilde{h},\tilde{k}}^1))\|_{C_{\delta-4}^{0,\alpha}} \leq c_\Lambda R_\epsilon^{4-4n} R_\epsilon^{-4n} R_\epsilon^{4-\delta} = R_\epsilon^{4-4n}.$$

Also

$$\|\epsilon^2 \nu\|_{\tilde{C}_{\eta,\delta-4}^{0,\alpha}} = \|\epsilon^2 \nu\|_{C_{\delta-4}^{0,\alpha}} \leq c R_\epsilon^{4-\delta} \epsilon^2 \leq R_\epsilon^{3-2n-\delta}. \quad (3.43)$$

Combining these inequalities with the bound for G_ϵ^{-1} , we obtain Equation 3.40.

For Equation 3.41, note for our choice of weights, on $Bl_0 B_{R_\epsilon} \setminus E$,

$$\|\phi\|_{C_{0,2}^{4,\alpha}} \leq c\|\phi\|_{C_{\eta,\delta}^{4,\alpha}},$$

then as in [60] Lemma 8.18 and Lemma 8.21, it follows from Lemma 3.6.2.1 by the mean value theorem that the Q term is bounded. It remains to estimate the following

$$\frac{1}{2} \|\langle \nabla \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(\psi_{f_2}), \nabla(H_{\epsilon,\tilde{h},\tilde{k}}^1 + \phi_2 + \lambda_2 t) \rangle - \langle \nabla \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(\psi_{f_1}), \nabla(H_{\epsilon,\tilde{h},\tilde{k}}^1 + \phi_1 + \lambda_1 t) \rangle\|_{\tilde{C}_{\eta,\delta-4}^{0,\alpha}}. \quad (3.44)$$

Note $\|\nabla H_{\epsilon,\tilde{h},\tilde{k}}^1\|_{C^{0,2}} \leq c_\Lambda R_\epsilon^{3-2n}$, $\|\nabla \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(\psi_{f_2} - \psi_{f_1})\|_{C^{0,\alpha}} \leq c\|f_1 - f_2\|$, then

$$\|\langle \nabla \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(\psi_{f_2} - \psi_{f_1}), \nabla H_{\epsilon,\tilde{h},\tilde{k}}^1 \rangle\| \leq c_\Lambda R_\epsilon^{3-2n-\delta} \|f_1 - f_2\|,$$

and for the remaining part of 3.44, we have

$$\begin{aligned} & \|\langle \nabla \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}\psi_{f_2}, \nabla(\phi_2 + \lambda_2 t) \rangle - \langle \nabla \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}\psi_{f_1}, \nabla(\phi_1 + \lambda_1 t) \rangle\|_{\tilde{C}_{\eta,\delta-4}^{0,\alpha}} \\ & \leq \|\langle \nabla \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}\psi_{f_2}, \nabla(\phi_2 + \lambda_2 t) - \nabla(\phi_1 + \lambda_1 t) \rangle\|_{\tilde{C}_{\eta,\delta-4}^{0,\alpha}} + \|\langle \nabla \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(\psi_{f_2} - \psi_{f_1}), \nabla(\phi_1 + \lambda_1 t) \rangle\|_{\tilde{C}_{\eta,\delta-4}^{0,\alpha}} \\ & \leq \|\nabla \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}\psi_{f_2}\|_{C_{0,-3}^{0,\alpha}} \|(\phi_2, \lambda_2) - (\phi_1, \lambda_1)\| + \|(\phi_1, \lambda_1)\| \|\nabla \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(\psi_{f_2} - \psi_{f_1})\|_{C_{0,-3}^{0,\alpha}} \\ & \leq c(\|f_2\| \cdot \|(\phi_2, \lambda_2) - (\phi_1, \lambda_1)\| + \|(\phi_1, \lambda_1)\| \cdot \|f_2 - f_1\|) \end{aligned}$$

These inequalities combined with the bounds of G_ϵ^{-1} and $\mathcal{E}_\epsilon$ give the desired estimate.

For Equation 3.42, again since $H_{\epsilon,\tilde{h},\tilde{k}}^1$ is supported away from the cusp end, we have from [12] Lemma 5.3 the following estimates

$$\|L_{\omega_{HS}}(H_{\epsilon,\tilde{h},\tilde{k}}^1 - H_{\epsilon,\tilde{h}',\tilde{k}'}^1)\|_{C_{\delta-4}^{0,\alpha}} \leq c_\Lambda R_\epsilon^{-1} \|(\tilde{h} - \tilde{h}', \tilde{k} - \tilde{k}')\|,$$

and

$$\|\mathcal{E}(Q_{\omega_{HS}}(\nabla^2 H_{\epsilon,\tilde{h},\tilde{k}}^1 - \nabla^2 H_{\epsilon,\tilde{h}',\tilde{k}'}^1))\|_{C_{\delta-4}^{0,\alpha}} \leq c_\Lambda R_\epsilon^{-1} \|(\tilde{h} - \tilde{h}', \tilde{k} - \tilde{k}')\|.$$

The estimate for the variance in ν, ν' follows from equation 3.43. □

Note by reducing c_Λ if necessary, we have $c_\Lambda R_\epsilon^{3-2n-\delta} \leq \frac{1}{2}$, then with the above estimates, we know the map $\mathcal{N}_\epsilon$ is a contraction on the set

$$\mathcal{U}_\epsilon : \{(\phi, f, \lambda) \in C_{\eta,\delta}^{4,\alpha} \times \ker \mathcal{D}_E^* \mathcal{D}_E \times \mathbb{R} : \|(\phi, f, \lambda)\|_{C_{\eta,\delta}^{4,\alpha} \times \ker \mathcal{D}_E^* \mathcal{D}_E \times \mathbb{R}} \leq c R_\epsilon^{3-2n-\delta}\},$$

for ϵ sufficiently small, then $\mathcal{N}_\epsilon$ is a contraction on $\mathcal{U}$ and $\mathcal{N}_\epsilon(\mathcal{U}) \subset \mathcal{U}$. In particular, $\mathcal{N}_\epsilon$ has a fixed point, finishing the proof of Theorem 3.6.2. □

With these metrics on each piece, upon applying Theorem 3.6.2 to

$$\nu := s(\omega_X) + \nu_{\epsilon, h, k},$$

we define on $Bl_0 B_1 \setminus Bl_0 B_{\frac{1}{2}}$

$$\psi^1 := \epsilon^2(\varphi_{HS} + H_{\epsilon, \tilde{h}, \tilde{k}}^1 + \phi_\epsilon).$$

From Theorem 3.6.1, we define on $B_2 \setminus B_1$

$$\psi^0 := \varphi_X + H_{\epsilon, h, k}^0 + \varphi_{\epsilon, h, k}.$$

To match up these two metrics, we note for our choice of r_ϵ in 3.14, $\|\psi^0 - H_{\epsilon, h, k}^0\|_{C^{4, \alpha}} \leq cr_\epsilon^4$ and $\|\psi^1 - \epsilon^2 H_{\epsilon, \tilde{h}, \tilde{k}}^1\|_{C^{4, \alpha}} \leq cR_\epsilon^{3-2n}$. Combined with the estimates 3.29, 3.30 and 3.32, we have the following.

Theorem 3.6.3. ([12], Section 5.3) *Consider functions ψ^0 and ψ^1 defined as above. There exists boundary data h, k satisfying 3.28 and $\tilde{h}, \tilde{k}$ satisfying 3.31, such that on ∂B_1 ,*

$$\psi^0 = \psi^1, \quad \partial_r \psi^0 = \partial_r \psi^1, \quad \Delta \psi^0 = \Delta \psi^1, \quad \partial_r \Delta \psi^0 = \partial_r \Delta \psi^1,$$

here $r = |x|$, with x being the coordinates in B_2 . Then we can glue along ∂B_1 the functions ψ^0 and ψ^1 and obtain a function on $Bl_p X \setminus E$, which results in a metric ω_{sol} on $Bl_p X \setminus E$.

Proof of Theorem 1.1. Noticing Equation 3.32, the existence of such boundary data can be obtained with the same arguments as in [12], Section 5.3. Near E , we know $\phi + \lambda t = O(t)$ from the definition, which is indeed of Poincaré type.

We want to show that ω_{sol} is indeed cscK by arguing that the component $\mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(\psi_f)$ is zero. Note the metric g_{HS} near E has expansion

$$g_{HS} = \frac{1}{n(n-1)}(dt^2 + e^{-2t}\Theta^2) + p^* \omega_{FS} + O(e^{-t}),$$

then near E , ω_{sol} has expansion

$$g_{sol} = \epsilon^2 \left(\frac{1}{n(n-1)} - \frac{1}{2} \lambda_\epsilon \right) (dt^2 + e^{-2t}\Theta^2) + p^* \omega_{FS} + O(e^{-\eta' t}) \quad (3.45)$$

for some $\eta' > 0$. Hence $\nabla^{1,0} s(\omega_{sol})$, when restricted to E , becomes the extremal vector field $\nabla^{1,0} s(\omega_{FS})$ on E , which equals to zero. On the other hand, recall that

$$s(\omega_{sol}) = \epsilon^2 \nu + \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(\psi_{f_\epsilon}) + \frac{1}{2} \langle \nabla \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(\psi_{f_\epsilon}), \nabla(H_{\epsilon, \tilde{h}, \tilde{k}}^1 + \phi_\epsilon + \lambda_\epsilon t) \rangle,$$

then restricting the associated vector field to E , the parts involving $\mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(\psi_{f_\epsilon})$ component gives nonzero term, and $\nabla^{1,0}(\epsilon^2 \nu)$ gives zero. Hence, $\mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(\psi_{f_\epsilon})$ is forced to

vanish, and the metric is indeed cscK. The smoothness of ϕ follows from standard elliptic bootstrap arguments. $\square$

3.7 Applications and Discussions

Proof of Corollary

Note near E , we have an expansion of g_{sol} as in Equation 3.45. In fact, we have a more precise relation of the coefficient before t in the expression of potential with the average scalar curvature. Recall from [13] the following:

Theorem 3.7.1. (*Auvray, [13], Theorem 3.1*)

Given D a smooth divisor and ω_φ a cscK Poincaré type Kähler metric on $Y \setminus D$ in the class $[\omega_Y]$, we have the asymptotic expression of metric g_φ

$$g_\varphi = a(dt^2 + e^{-2t}\Theta^2) + \pi^*g_D + O(e^{\eta_0 t}),$$

for some constants $\eta_0 < 0$ and $a > 0$, more precisely, let

$$\bar{s}(\omega_Y) := n \frac{-c_1(K_Y[D]) \cup [\omega_Y]^{n-1}}{[\omega_Y]^n}, \quad \bar{s}(\omega_Y|_D) := (n-1) \frac{-c_1(K_Y[D]|_D) \cup [\omega_Y|_D]^{n-2}}{[\omega_Y|_D]^{n-1}},$$

we have an explicit expression of a , given by

$$a = \frac{1}{\bar{s}(\omega_Y|_D) - \bar{s}(\omega_Y)}.$$

Applying this theorem to $Y = Bl_p X$, $D = E$, we have

$$\frac{1}{\epsilon^2 \left(\frac{1}{n(n-1)} - \frac{\lambda_\epsilon}{2} \right)} = \bar{s}(\omega_\epsilon|_E) - \bar{s}(\omega_{sol}).$$

We are interested in how fast the RHS grows; for this purpose, we calculate the scalar curvature of the new cscK metric. Note ω_ϵ and our solution ω_{sol} to the cscK equation are in the same Kähler class, which is $\pi^*[\omega_X] - \epsilon^2[E]$, at the same time, we have

$$c_1(Bl_p X) = \pi^*c_1(X) - (n-1)[E].$$

Recall that $[E]$ denotes the Poincaré dual of E and below by $c_1([E])$ we mean $c_1(\mathcal{O}(E))$, we have

$$c_1(Bl_p X) - c_1([E]) = \pi^*c_1(X) - (n-1)[E] - [E] = \pi^*c_1(X) - n[E].$$

Recall from [36] that $[E]^n = (-1)^{n-1}$, we get

$$(c_1(Bl_p X) - c_1([E])) \cup [\omega_\epsilon]^{n-1} = c_1(X) \cup [\omega_X]^{n-1} - n\epsilon^{2n-2}, \quad [\omega_\epsilon]^n = [\omega_X]^n - \epsilon^{2n},$$

thus

$$\bar{s}(\omega_{sol}) = \frac{n(c_1(X) \cup [\omega_X]^{n-1} - n\epsilon^{2n-2})}{[\omega_X]^n - \epsilon^{2n}}, \quad (3.46)$$

which is a decreasing function for small ϵ and converges to $s(\omega_X)$ as $\epsilon \rightarrow 0$.

We also compute the average scalar curvature of metric on E in the class $[\omega_\epsilon|_E]$,

$$c_1(K_{Bl_p X}[E]) \cup [\omega_\epsilon|_E]^{n-2} = (\pi^* c_1(X) - n[E]) \cdot (\pi^* [\omega_X] - \epsilon^2[E])^{n-2} \cdot [E] = \epsilon^{2n-4}n,$$

$$[\omega_\epsilon|_E]^{n-1} = [\omega_\epsilon]^{n-1} \cdot c_1([E]) = \epsilon^{2n-2},$$

thus

$$\bar{s}(\omega_\epsilon|_E) = \frac{n(n-1)}{\epsilon^2}.$$

Proof of Corollary 3.1.1. As in the proof of [11] Corollary 8.1, we can apply LeBrun-Simanca and obtain a family of cscK metrics $\omega(y)$ such that $\omega(0) = \omega_X$ and $s(\omega(y)) < 0$ for $y \in [-y_0, 0)$ and $s(\omega(y)) > 0$ for $y \in (0, y_0]$. Apply Theorem 1.1 to each $\omega(y)$, we obtain a family $\omega_\epsilon(y)$ of cscK metrics with a uniform bound for ϵ as y varies in $[-y_0, y_0]$, denoted by ϵ_0 . Arguing similarly as in [11], we know $\omega_\epsilon(y)$ depends continuously on y . From the discussion at the beginning of this section, if ϵ_0 is small enough, we know $s(\omega_\epsilon(y)) < 0$ if $y < 0$ and $s(\omega_\epsilon(y)) > 0$ if $y > 0$. Now that $s(\omega_\epsilon(y))$ depends continuously on y , there exists y' such that $s(\omega_\epsilon(y')) = 0$. $\square$

Extension of results to more general situations

In this subsection, we extend the arguments to a cscK manifold X which admits non-trivial holomorphic vector fields. We will show that if we choose the blow-up points carefully, the gluing construction we have still work under some modifications. Now to construct cscK metric on X_{r_ϵ} piece, we define function

$$G(z) := \begin{cases} -\log |z| & \text{when } n = 2 \\ |z|^{4-2n} & \text{when } n \geq 3 \end{cases}$$

which are homogeneous solutions to $L_{\omega_{Euc}}$. On the gluing annulus, these can be perturbed to $\tilde{G}$, which are homogeneous solutions to L_{ω_X} on $B_{r_0}^*$ for sufficiently small r_0 . We define the deficiency space as in [12] Section 4.1

$$D_\delta(L_{\omega_X}) := \text{Span} \{ \chi_i \tilde{G}_i, i = 1, \dots, k \}.$$

With this, given the genericity condition on the points, we obtain a surjective operator $C_\delta^{4,\alpha}(X_p) \times D_\delta(L_{\omega_X}) \rightarrow C_{\delta-4}^{0,\alpha}(X_p)$ given by $(\phi, \beta) \mapsto L_{\omega_X}(\phi + \beta)$. For the first perturbation on the gluing annulus, we require an additional term derived from the deficiency space to deal with the complexity introduced by $\mathfrak{k}$ as in [12]. The next step is to find the cscK metric on both pieces. Choose r_ϵ and R_ϵ as before, on X_{r_ϵ} , the existence of cscK metric follows from [12] Proposition 5.1. On $Bl_0 B_{R_\epsilon} \setminus E$, we perturb $a^2 \omega_{HS}$ for a given constant $a > 0$ and we require $a \in [a_{\min}, a_{\max}]$ for some fixed constants $0 < a_{\min} < a_{\max}$. Then, with essentially similar estimates as in Section 3.6, we solve a modified cscK equation.

Theorem 3.7.2. *For a constant $\Lambda > 0$ large enough, there exists $c > 0$ independent of Λ and $\epsilon_\Lambda > 0$ such that $\forall \epsilon \in (0, \epsilon_\Lambda)$, and for all $\nu \in \mathbb{R}$ and boundary data $\tilde{h} \in C^{4,\alpha}(\partial B_1)$, $\tilde{k} \in C^{2,\alpha}(\partial B_1)$ satisfying 3.31, and $a \in [a_{\min}, a_{\max}]$, $\nu \in [\nu_{\min}, \nu_{\max}]$, then there exists $\phi_\epsilon \in C_{\eta,\delta}^{4,\alpha}$, constant $\lambda_\epsilon \in \mathbb{R}$ and $f_\epsilon \in \ker \mathcal{D}_E^* \mathcal{D}_E$ such that the scalar curvature of the Kähler form*

$$\omega_{\epsilon, \tilde{h}, \tilde{k}, HS} = a^2 \omega_{HS} + \sqrt{-1} \partial \bar{\partial} (H_{\epsilon, \tilde{h}, \tilde{k}}^1 + a^{2n-2} R_\epsilon^{4-2n} G + \phi_\epsilon + \lambda_\epsilon t)$$

defined on $Bl_0 B_{R_\epsilon} \setminus E$ is

$$\epsilon^2 \nu + \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(\psi_{f_\epsilon}) + \frac{1}{2} \langle \nabla \mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(\psi_{f_\epsilon}), \nabla (H_{\epsilon, \tilde{h}, \tilde{k}}^1 + \phi_\epsilon + \lambda_\epsilon t) \rangle.$$

Further,

$$\|\phi_\epsilon(R_\epsilon \cdot / a)\|_{C^{4,\alpha}(B_2 \setminus B_1)} \leq c R_\epsilon^{3-2n}, \quad |\lambda_\epsilon| + \|f_\epsilon\| \leq c R_\epsilon^{3-2n-\delta}.$$

If $\phi_\epsilon, \lambda_\epsilon$ are determined by boundary data $\tilde{h}, \tilde{k}$ and $\phi'_\epsilon, \lambda'_\epsilon$ are determined by boundary data $\tilde{h}', \tilde{k}'$, then

$$\|f_\epsilon - f'_\epsilon\| + |\lambda_\epsilon - \lambda'_\epsilon| + \|\phi(R_\epsilon \cdot / a) - \phi'(R_\epsilon \cdot / a')\| \leq c_\Lambda (R_\epsilon^{\delta-1} \|(\tilde{h} - \tilde{h}', \tilde{k} - \tilde{k}')\| + R_\epsilon^{3-2n} (|\nu - \nu'| + |a - a'|)). \quad (3.47)$$

The matching of the metrics constructed on the two pieces is independent of the Poincaré type behavior near the exceptional divisor, and the matching arguments in [12] Section 5.3 still hold in our case. Finally, because of the condition (3.3), the metric on $Bl_p X \setminus E$ is cscK. We postpone the details of this argument to the more complicated situation of extremal Kähler metrics at the end of the following subsection.

Extremal Kähler metric

We discuss the situation of extremal Kähler(extK) metrics in Theorem 3.3. To solve the extK equation, we need to choose T -invariant coordinates when defining weighted Hölder spaces,

and we want to solve for T -invariant potential functions. Comparing the expansion of ω_{HS} in 3.10 and 3.13 with those of Burns-Simanca metrics, we see the arguments of perturbations for ω_X to an extK metric that on [10] Proposition 6.1.1. Again, the difference lies in the piece of $Bl_0 B_{R_\epsilon} \setminus E$, where we solve a modified extK equation.

Theorem 3.7.3. *Given $\nu \in \mathbb{R}$, $X \in \mathfrak{t}$ satisfying*

$$|\epsilon^2 \nu| + \|X\| \leq c,$$

there exists $\theta > 0, c > 0$ and for all $\Lambda > 0$, there exists $\epsilon_\Lambda > 0$ such that for all $\epsilon \in (0, \epsilon_\Lambda)$, $a \in [a_{\min}, a_{\max}]$, and T -invariant boundary data satisfying

$$\|\tilde{h}\|_{C^{4,\alpha}(\partial B_1)} + \|\tilde{k}\|_{C^{2,\alpha}(\partial B_1)} \leq \Lambda R_\epsilon^{3-2n}, \quad \int_{\partial B_1} (4n\tilde{h} - \tilde{k}) dvol = 0,$$

there exists a T -invariant function ϕ_ϵ , a constant λ_ϵ and $f_\epsilon \in \ker \mathcal{D}_E^ \mathcal{D}_E$ such that the scalar curvature of the Kähler form*

$$\omega_{\epsilon, \tilde{h}, \tilde{k}, HS} = a^2 \omega_{HS} + \sqrt{-1} \partial \bar{\partial} (H_{\epsilon, \tilde{h}, \tilde{k}}^1 + \phi_\epsilon + \lambda_\epsilon t)$$

defined on $Bl_0 B_{R_\epsilon} \setminus E$ is

$$Y_\epsilon + \frac{1}{2} \langle \nabla Y_\epsilon, \nabla (H_{\epsilon, \tilde{h}, \tilde{k}}^1 + \phi_\epsilon + \lambda_\epsilon t) \rangle,$$

where $Y_\epsilon = \epsilon^4 Y + a^{-4} \mathcal{D}_{\omega_{HS}}^ \mathcal{D}_{\omega_{HS}}(\psi_{f_\epsilon})$, with Y being the potential function for X , and*

$$\frac{1}{|\partial B_1|} \int_{\partial B_1} s(\omega_{\epsilon, \tilde{h}, \tilde{k}, HS}(R_\epsilon \cdot / a)) dvol = \epsilon^2 \nu.$$

Further,

$$\|\phi_\epsilon(R_\epsilon \cdot / a)\|_{C^{4,\alpha}(B_2 \setminus B_1)} \leq c R_\epsilon^{3-2n}, \quad |\lambda_\epsilon| + \|f_\epsilon\| \leq c R_\epsilon^{3-2n-\delta}.$$

If $\phi_\epsilon, \lambda_\epsilon$ are determined by boundary data $\tilde{h}, \tilde{k}$ and $\phi'_\epsilon, \lambda'_\epsilon$ are determined by boundary data $\tilde{h}', \tilde{k}'$, then

$$\begin{aligned} \|f_\epsilon - f'_\epsilon\| + |\lambda_\epsilon - \lambda'_\epsilon| + \|\phi(R_\epsilon \cdot / a) - \phi'(R_\epsilon \cdot / a')\| &\leq c_\Lambda (R_\epsilon^{\delta-1} \|(\tilde{h} - \tilde{h}', \tilde{k} - \tilde{k}')\| \\ &\quad + R_\epsilon^{3-2n} (|\nu - \nu'| + |a - a'|) + R_\epsilon^{4-4n} \|Y - Y'\|). \end{aligned}$$

Assume we already have this theorem, we know from standard arguments in [10] that on X_{r_ϵ} , there exists extK metric $\omega_{\epsilon, h, k}$ with extremal vector field $X_{\epsilon, h, k, a}$, then apply Theorem 3.7.3 to

$$\nu = \frac{1}{|\partial B_1|} \int_{\partial B_1} s(\omega_{\epsilon, h, k}) dvol, \quad X = X_{\epsilon, h, k, a}.$$

With these data, we match the metric on $Bl_0 B_{R_\epsilon} \setminus E$ with the extK metric on X_{r_ϵ} . To prove this theorem, the modified extK equation we want to solve is

$$s(a^2 \omega_{HS} + \sqrt{-1} \partial \bar{\partial} (H_{\epsilon, \tilde{h}, \tilde{k}}^1 + \phi_\epsilon + \lambda_\epsilon t)) = Y_\epsilon + \frac{1}{2} \langle \nabla Y_\epsilon, \nabla (H_{\epsilon, \tilde{h}, \tilde{k}}^1 + \phi_\epsilon + \lambda_\epsilon t) \rangle. \quad (3.48)$$

Expanding the scalar curvature operator into linear $\mathcal{D}^* \mathcal{D}$ term and nonlinear Q term, we rewrite the equation as

$$a^{-4} \mathcal{D}^* \mathcal{D}(\phi_\epsilon + \lambda_\epsilon t + \psi_{f_\epsilon}) = Q(H_{\epsilon, \tilde{h}, \tilde{k}}^1 + \phi_\epsilon + \lambda_\epsilon t) - \mathcal{D}^* \mathcal{D}(H_{\epsilon, \tilde{h}, \tilde{k}}^1) - \frac{1}{2} \langle \nabla(\epsilon^4 Y + a^{-4} \mathcal{D}^* \mathcal{D} \psi_{f_\epsilon}), \nabla(H_{\epsilon, \tilde{h}, \tilde{k}}^1 + \phi_\epsilon + \lambda_\epsilon t) \rangle,$$

and we denote the RHS by Q_ϵ . Again, we have a bounded inverse of the LHS and an extension operator whose norm is independent of ϵ . Thus it suffices to solve the fixed point problem for $N_\epsilon = G_\epsilon^{-1} \circ \mathcal{E}_\epsilon \circ Q_\epsilon$.

Lemma 3.7.3.1. *There is a constant $c > 0$ independent of Λ and $c_\Lambda > 0$, $\epsilon_\Lambda > 0$ such that for $\epsilon \in (0, \epsilon_\Lambda)$,*

$$\|\mathcal{N}_\epsilon(0, 0, 0)\|_{C_{\eta, \delta}^{4, \alpha} \times \ker \mathcal{D}_E^* \mathcal{D}_E \times \mathbb{R}} \leq c R_\epsilon^{3-2n-\delta}. \quad (3.49)$$

If

$$\|(\phi_1, f_1, \lambda_1)\|_{C_{\eta, \delta}^{4, \alpha} \times \ker \mathcal{D}_E^* \mathcal{D}_E \times \mathbb{R}}, \quad \|(\phi_2, f_2, \lambda_2)\|_{C_{\eta, \delta}^{4, \alpha} \times \ker \mathcal{D}_E^* \mathcal{D}_E \times \mathbb{R}} \leq 2c R_\epsilon^{3-2n-\delta},$$

then

$$\|\mathcal{N}_\epsilon(\phi_1, f_1, \lambda_1) - \mathcal{N}_\epsilon(\phi_2, f_2, \lambda_2)\| \leq c_\Lambda (R_\epsilon^{3-2n-\delta} + \epsilon^4 R_\epsilon^4) \|(\phi_1, f_1, \lambda_1) - (\phi_2, f_2, \lambda_2)\|, \quad (3.50)$$

and let $\tilde{\mathcal{N}}_\epsilon$ be the map taking into account the divisor volume a and the set of boundary data $\tilde{h}', \tilde{k}'$, then

$$\|\tilde{\mathcal{N}}_\epsilon(\phi, f, \lambda, \tilde{h}, \tilde{k}) - \tilde{\mathcal{N}}_\epsilon(\phi, f, \lambda, \tilde{h}', \tilde{k}')\| \leq c_\Lambda (R_\epsilon^{-1} \|(\tilde{h} - \tilde{h}', \tilde{k} - \tilde{k}')\| + R_\epsilon^{3-2n-\delta} |\nu - \nu'| + R_\epsilon^{4-4n-\delta} \|Y - Y'\|). \quad (3.51)$$

Proof. The proof follows mostly from [10] Proposition 6.2.1. Only note that near the exceptional divisor, we again use Lemma 3.6.2.1 and mean value theorem. $\square$

We use [10] Lemma 7.0.2 to match the metrics on the two pieces and obtain a metric ω_{sol} on $Bl_p X \setminus E$. To show the $\mathcal{D}^* \mathcal{D}(\psi_{f_\epsilon})$ term vanishes, we use the condition (3.3). To make sense of the condition, we first discuss the definition of the extremal vector field.

Definition 3.7.1. For a compact manifold X with smooth divisor E , we say a holomorphic vector field Z is parallel to E if their normal components vanish along E , i.e., if $E = \{z_1 = 0\}$, and locally we write $Z = \Re \left(\sum_{i=1}^m f_i \frac{\partial}{\partial z_i} \right)$, then $f_1|_E = 0$. We denote the space of holomorphic potentials parallel to E as $h_E^{//}$.

A key property of such vector fields is that they are L^2 with respect to Poincaré type Kähler metric on $X \setminus E$. For ω a Poincaré type Kähler metric in $[\omega_X]$, Auvray in [15] defined Poincaré type Futaki character of $Z \in h_E^{//}$ with respect to E ,

$$\mathcal{F}_{[\omega_X]}^E(Z) := \int_{X \setminus E} s(\omega) f_\omega^Z \frac{\omega^n}{n!},$$

here f_ω^Z is the holomorphy potential for Z with respect to ω . This quantity is independent of the choice of representative in the given Kähler class. For $Z_f, Z_g \in h_E^{//}$ normalized to $\int_{X \setminus E} f_\omega^Z \frac{\omega^n}{n!} = \int_{X \setminus E} g_\omega^Z \frac{\omega^n}{n!} = 0$, we define the following inner product

$$\langle Z_f, Z_g \rangle := \int_{X \setminus E} f_\omega^Z g_\omega^Z \frac{\omega^n}{n!}.$$

Take $T \subset K$ a compact torus whose Lie algebra is $\mathfrak{t}$. Then we see $\langle \cdot, \cdot \rangle$ is positive definite on $\mathfrak{t}$. The extremal vector field is then defined by the duality of the Futaki character under this inner product.

Note with similar arguments as in the cscK case, and we know that $\nabla^{1,0}s(\omega_{sol})$, when restricted to E , becomes the extremal vector field $\nabla^{1,0}s(\omega_{FS})$ on E , which equals to zero. On the other hand, from Equation 3.48, we decompose $\nabla^{1,0}s(\omega_{sol})$ into two parts. For the parts involving Y , which lives in the space of holomorphy potential, we know from the condition (3.3) that $\nabla^{1,0}(\epsilon^4 Y)|_E = 0$. For the $\mathcal{D}_{\omega_{HS}}^* \mathcal{D}_{\omega_{HS}}(\psi_{f_\epsilon})$ term, it gives nonzero terms when restricted to E . Hence it has to vanish, and the metric is indeed extremal.

Chapter 4

Toric Scalar-flat Kähler Surfaces

Let X be a non-compact toric symplectic 4-manifold. In [4], Abreu and Sena-Dias construct complete toric scalar-flat Kähler metrics on strictly unbounded toric symplectic 4-manifold X . We say X is strictly unbounded if the moment polytope of X is unbounded with the unbounded edges being non-parallel. This condition is equivalent to saying there exists a finite sequence of blow-downs of X from which we obtain a minimal resolution of $\mathbb{C}^2/\Gamma$ for some finite cyclic subgroup $\Gamma \subset U(2)$. The metrics constructed in [4] include the well-known examples of the LeBrun-Simanca metrics [46], the (multi-)Taub-NUT metrics [45], the gravitational instantons of Gibbons-Hawking [35] and Kronheimer [44], etc.

First, we focus on the case where X is strictly unbounded, or equivalently, as a complex surface, it arises as a finite sequence of blow-ups of minimal resolution of $\mathbb{C}^2/\Gamma$, as described in Definition-Proposition 4.1.1. The non-intersection condition on the divisors can be relaxed, as discussed in more detail in Remark 4.2.1. Along the proof, we will see the complex structures induced from these metrics are biholomorphic to that on X away from the divisor D . Here the *generalized Taub-NUT metrics* live on $\mathbb{C}^2$ and are scalar-flat Kähler generalizations of the Ricci-flat Kähler Taub-NUT metrics. These metrics are introduced by Donaldson in [27] and later explored by Abreu and Sena-Dias in [4] and Weber in [68] and [69]. They all have quadratic curvature decay and cubic volume growth, but except for the standard Taub-NUT metric, they are not asymptotically locally flat(ALF). The *exceptional Taub-NUT metrics* are also scalar-flat Kähler metrics living on $\mathbb{C}^2$. These metrics are studied by Weber in [69], where he showed these metrics have quadratic curvature decay and quartic volume growth but are not ALE.

A previously known example of toric scalar-flat Kähler metric of Poincaré type is first discussed by Fu-Yau-Zhou in [31], and later studied by the author in [29], which we refer to as the *Hwang-Singer metric* ω_{HS} . It lives on $\mathbb{C}^2 - \{0\}$. Near the origin, it has Poincaré type singularity, and it is asymptotically Euclidean on the other end. This S^1 -invariant metric is in fact toric and we will discuss it in detail in Example 4.2.1. Theorem 1.2 then gives us a two-parameter family deformation of ω_{HS} .

Intuitively, Poincaré type metrics can be viewed as the limit of a conical family of met-

rics when the cone angle approaches 0. Similarly, smooth metrics can be viewed as the limit when the cone angle approaches 2π . This degeneration phenomenon was established by Guenancia [37] in the Kähler-Einstein setting. In the toric case, the relation between boundary weights, conical angles, and Poincaré-type asymptotics already appears in [54], where positive weights yield conical angles and vanishing weights lead to cusp behavior. For the metrics constructed in Theorem 1.2, we can explicitly write down a conical family of toric Kähler metrics connecting the Poincaré type metrics with those in [4]. This family of conical metrics is *not* necessarily scalar-flat, though.

Earlier, Legendre [48] observed that in the toric setting, the cone angles of conical Kähler metrics correspond to the inward normals attached to the facets of the moment polytope, and described such metrics in terms of symplectic potentials via the Legendre transform. Weber [67] later constructed toric scalar-flat Kähler metrics with conical singularities along the divisor, where the cone angle along each edge is encoded by a boundary "label" describing the growth of a vanishing Killing field. In contrast, we adopt a boundary-value viewpoint, treating the cone angle as a boundary condition for the Kähler potential. More precisely, following the method of Abreu and Sena-Dias, we give an independent construction of conical toric scalar-flat Kähler metrics.

Theorem 4.1. *(Theorem 4.3.1) Consider the same setting as in Theorem 1.2. Fix $\theta_i \in (0, 1)$ for $i = 1, \dots, m$, on X , we have*

- *a conical toric scalar-flat Kähler metric with angle $2\pi\theta_i$ along D_i and is asymptotically locally Euclidean(ALE) on the remaining end;*
- *a two-parameter family of them with angle $2\pi\theta_i$ along D_i and are asymptotic to either the generalized Taub-NUT metrics or the exceptional Taub-NUT metrics on the remaining end.*

Besides the general case where X is strictly unbounded, we construct toric scalar-flat Kähler metrics of Poincaré type when the unbounded edges of the momentum polytope are parallel:

Theorem 4.2. *Consider the same setting as in Theorem 1.2 except that the unbounded edges of X are parallel. On $X \setminus D$, we have a one-parameter family of toric scalar-flat Kähler metrics with Poincaré type singularity along D and are asymptotic to the model product metric on $S^2 \times \mathbb{R}^2$ on the remaining end.*

The scalar-flat metrics we constructed belong to a particular class of Poincaré type metrics, characterized by the specific behavior of their potential functions along the divisor, which we denote as the $S_{A,B}$ type (Definition 4.1.6). We have the following *uniqueness* result:

Theorem 4.3. (Theorem 4.4.1) *Given the same setting as in Theorem 1.2. Assume g is a toric scalar-flat Kähler metric on $X \setminus D$, and its symplectic potential u is of $S_{A,B}$ type along D , then g can only be one of the metrics constructed in Theorem 1.2.*

Naturally, we would ask if we still have the uniqueness result without assuming u to be of $S_{A,B}$ type:

Question 4.0.1. (Strong uniqueness) *Without assuming the potential function is of $S_{A,B}$ type along the divisor in Theorem 4.3, can we still obtain the uniqueness result?*

A related question is to determine, locally, whether $S_{A,B}$ type represents the only Guillemin boundary behavior for scalar-flat Kähler metric of Poincaré type.

4.1 Preliminaries

In this section, we recall some basics of toric Kähler metrics on toric symplectic 4-manifolds and discuss the local ansatz for finding scalar-flat Kähler metrics. First, we recall the basic definition of a toric symplectic 4-manifold:

Definition 4.1.1. *A symplectic 4-manifold (X, ω) is said to be **toric** if it admits an effective Hamiltonian $\mathbb{T}^2$ -action τ of the standard torus to the diffeomorphism group of (X, ω) such that the corresponding moment map $\mu : X \rightarrow \mathbb{R}^2$ is proper onto its image P .*

Remark 4.1.1. *Our definition differs slightly from the standard one in [42], where properness of the moment map is not assumed, and instead aligns with the convention adopted in [8].*

Here the moment map of the $\mathbb{T}^2$ -action is a map $\mu : X \rightarrow \mathbb{R}^2$ such that $\iota_\xi \omega = -d\mu$ for each infinitesimal generator ξ of $\mathbb{T}^2$. For a compact symplectic 4-manifold, the moment image of μ is the convex hull of the image of fixed points of $\mathbb{T}^2$ in X , and the classical Atiyah-Bott-Guillemin-Sternberg theory tells us this image is a polytope. For a non-compact symplectic 4-manifold, we first introduce the definition of the moment polytope:

Definition 4.1.2. ([4], Definition 2.2) *We say a convex polytope $P \subset \mathbb{R}^2$ is a moment polytope if*

- (i) *for each edge, we can find a primitive vector of $\mathbb{Z}^2$, which is an interior normal to this edge;*
- (ii) *for each pair of intersecting edges, their chosen interior normals form a $\mathbb{Z}$ -basis of $\mathbb{Z}^2$.*

We say two moment polytopes are equivalent if there exists a translation in $\mathbb{R}^2$ and a $GL(2, \mathbb{Z})$ transformation mapping one to the other, and two symplectic toric manifolds are equivalent if there is an equivariant symplectomorphism mapping one to the other. Delzant's theorem [25] tells us in the compact setting, the moment polytope determines the symplectic toric 4-manifold up to equivariant symplectomorphism. It turns out that in the non-compact setting, we also have the following correspondence:

Proposition 4.1.1. ([42]) *There is a bijective correspondence between the equivalence class of symplectic toric 4-manifolds and the equivalence class of moment polytopes.*

We are particularly interested in the following class of non-compact symplectic toric 4-manifold:

Definition-Proposition 4.1.1. ([4] Definition 2.4, Proposition 2.8) *A symplectic toric 4-manifold is said to be **strictly unbounded** if the following equivalent conditions hold:*

- (i) *the image of the moment map, P , is an unbounded polytope with finitely many edges, with the unbounded edges being non-parallel;*
- (ii) *X as a complex surface, arises as a finite sequence of blow-ups of a minimal resolution of $\mathbb{C}^2/\Gamma$ for some finite cyclic group $\Gamma \subset U(2)$ such that $\mathbb{C}^2/\Gamma$ has an isolated singularity point at the origin.*

Let P be a moment polytope given by

$$P := \{x \in \mathbb{R}^2 : \ell_i(x) := \langle x, \nu_i \rangle + \lambda_i \geq 0, i = 1, \dots, d\}. \quad (4.1)$$

Here $\nu_i = (\alpha_i, \beta_i) \in \mathbb{Z}^2$ are the primitive interior normals to the edges. Let

$$L_1 := \ell_1^{-1}(0), \dots, L_i := \ell_i^{-1}(0), \dots, L_d := \ell_d^{-1}(0)$$

be the edges of the polytope. We order its edges so that L_1, L_d are the unbounded edges and for $i = 1, \dots, d-1$, $L_i \cap L_{i+1} \neq \emptyset$. From [4] Remark 2.5, we know we can further assume the Delzant condition for the polytope:

$$\det(\nu_i, \nu_{i+1}) = -1, \text{ for } i = 1, \dots, d-1.$$

Let (X_P, ω_P, τ_P) be the associated symplectic 4-manifold of P with moment map μ_P . It admits a canonical integrable, torus-invariant, compatible complex structure J_P . We denote the resulting Kähler surface by $(X_P, \omega_P, J_P, g_P)$. Let P° be the interior of P , and consider $X_P^\circ := \mu_P^{-1}(P^\circ)$, then

$$X_P^\circ \cong P^\circ \times \mathbb{T}^2 = \{(x, \theta) : x = (x_1, x_2) \in P^\circ, \theta = (\theta_1, \theta_2) \in \mathbb{R}^2/\mathbb{Z}^2\}.$$

Here (x, θ) are interpreted as the **action-angle coordinates** for ω_P , i.e.,

$$\omega_P = dx_1 \wedge d\theta_1 + dx_2 \wedge d\theta_2.$$

From [38], we know the symplectic potential $u_P \in C^\infty(P^\circ)$ is written as

$$u_P = \frac{1}{2} \sum_{i=1}^d \ell_i(x) \log \ell_i(x).$$

The metric g_P is given by

$$g_P = \sum_{i,j=1}^2 ((\text{Hess } u_P)_{ij} dx_i \otimes dx_j + (\text{Hess } u_P)^{ij} d\theta_i \otimes d\theta_j).$$

From [2] and [9], given any toric complex structure J which is ω_P -compatible, there exist action-angle coordinates (x, θ) on P° such that for some symmetric and positive-definite 2×2 matrix $U(x)$, we can write J in the following form

$$J = - \sum_{i,j=1}^2 (U(x)^{ij} \frac{\partial}{\partial x^i} \otimes d\theta_j + U(x)_{ij} \frac{\partial}{\partial \theta^i} \otimes dx_j).$$

Furthermore, the integrability of J is equivalent to the existence of $u \in C^\infty(P^\circ)$ such that $U(x) = \text{Hess}_x(u)$. Then u is the potential corresponding to J , and the Kähler metric is written as

$$g = \sum_{i,j=1}^2 ((\text{Hess } u)_{ij} dx_i \otimes dx_j + (\text{Hess } u)^{ij} dx_i \otimes dx_j). \quad (4.2)$$

For simplicity concern, we will use u_{ij}, u^{ij} to denote $(\text{Hess } u)_{ij}, (\text{Hess } u)^{ij}$ respectively. From [1], [3], we know when the Hessian of the symplectic potential u on P° is positive-definite and the boundary behavior of u is specified by the Guillemin's boundary condition, it determines a complex structure on X_P° which extends to (X_P, ω_P, τ_P) . We say u satisfies Guillemin's boundary condition if modulo a smooth function,

$$u(x) = \frac{1}{2} \sum_{i=1}^d \ell_i(x) \log \ell_i(x), \quad (4.3)$$

and its restriction to the interior of each face of P is strictly convex and smooth.

Definition 4.1.3. ([8], Definition 4.2) Given P , write $L := \{L_1(x), \dots, L_d(x)\}$. We say a symplectic potential $u : P^\circ \rightarrow \mathbb{R}$ belongs to the class $S(P, L)$ if it is smooth, strictly convex, and satisfies the Guillemin boundary condition.

We mention that as discussed in [9] and [8] Proposition 4.3, there is an equivalent characterization of $S(P, L)$, which we refer to as the first-order boundary conditions.

Definition 4.1.4 ([8]). The space $S(P, L)$ consists of all smooth functions u on P° such that $H^u := u^{ij}$ satisfies:

- H^u extends smoothly to P ;
- for any edge $L_j \subset \partial P$ with inward normal ν_j and any point $x \in \ell_j$,

$$H_x^u(\nu_j, \cdot) = 0, \quad (dH^u)_x(\nu_j, \nu_j) = 2\nu_j;$$

- H^u is positive definite on P° , as well as in the tangent directions along the interior of each edge.

This article focuses on finding scalar-flat Kähler metrics. Direct calculations show the scalar curvature of the metric has the following expression:

$$s = - \sum_{i,j} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j}.$$

Then the scalar-flat equation we aim at solving becomes $\sum_{i,j} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} = 0$. In [26], Donaldson gave a reformulation of Joyce's construction in [41], which allows us to write down explicit symplectic potentials of scalar-flat Kähler metrics on complex surfaces. The key is to use the ***axi-symmetric harmonic function*** as the local model. More precisely:

Theorem 4.1.1. ([26], local model) *Let ξ_1, ξ_2 be two solutions to*

$$\frac{\partial^2 \xi}{\partial H^2} + \frac{\partial^2 \xi}{\partial r^2} + \frac{1}{r} \frac{\partial \xi}{\partial r} = 0 \quad (4.4)$$

on

$$\mathbb{H} := \{(H, r) \in \mathbb{R}^2 : r > 0\}.$$

Then the 1-forms

$$\epsilon_1 = r \left(\frac{\partial \xi_2}{\partial r} dH - \frac{\partial \xi_2}{\partial H} dr \right), \quad \epsilon_2 = -r \left(\frac{\partial \xi_1}{\partial r} dH - \frac{\partial \xi_1}{\partial H} dr \right)$$

are closed. Let x_1, x_2 be their primitives, then the 1-form $\epsilon = \xi_1 dx_1 + \xi_2 dx_2$ is also closed. Let u be its primitive. Assume for $\xi = (\xi_1, \xi_2)$, we have

$$\det D\xi > 0. \quad (4.5)$$

Then u is a local symplectic potential for a scalar-flat Kähler toric metric on $\mathbb{R}^4$.

Some known solutions to (4.4) include

$$aH + b, \quad a \log r + b, \quad \frac{1}{2} \log(H + \sqrt{(H+a)^2 + r^2}) \quad (4.6)$$

where $a, b \in \mathbb{R}$. In [4], the authors used these solutions to construct scalar-flat Kähler metrics on unbounded symplectic toric 4-manifolds. These metrics belong to the class $S(P, L)$, and are precisely those whose complex structures are equivariantly biholomorphic to J_P .

The situation is different for toric scalar-flat Kähler metrics of Poincaré type. Let D be a torus-invariant divisor in X , and let L_F be the edge corresponding to D in the moment polytope, defined by $\ell_F^{-1}(0)$. Although the Guillemin boundary condition does not have a

straightforward extension to describe the behavior of u near ℓ_F , the first-order boundary condition does, as pointed out in [8] Definition 4.6.

Definition 4.1.5 ([8]). *Let $S(P, L, F)$ denote the space of smooth functions u on P° such that $H^u := u^{ij}$ satisfies:*

- H^u extends smoothly to P ;
- for any edge $L_j \subset F$ with inward normal ν_j and any $x \in L_j$,

$$H_x^u(\nu_j, \cdot) = 0, \quad (dH^u)_x(\nu_j, \nu_j) = 0,$$

and for any edge $L_j \subset \partial P \setminus F$ with inward normal ν_j and $x \in L_j$,

$$H_x^u(\nu_j, \cdot) = 0, \quad (dH^u)_x(\nu_j, \nu_j) = 2\nu_j;$$

- H^u is positive definite on P° , as well as in the tangent directions along the interior of each edge.

Within this class $S(P, L, F)$, [8] showed in Proposition 4.16 that there is a subclass, denoted by $S_{A,B}(P, L, F)$, which is characterized by the Guillemin boundary condition.

Definition 4.1.6. ([8], Definition 4.16) *Given $A \in \mathbb{R}^+$, $B \in \mathbb{R}$, we say a symplectic potential $u : P^\circ \rightarrow \mathbb{R}$ belongs to the class $S_{A,B}(P, L, F)$ if it is strictly convex and smooth on P° , its restriction to the interior of each edge of P is strictly convex and smooth, and*

$$u + (A + B\ell_F) \log \ell_F - \frac{1}{2} \sum_{j \neq F} \ell_j \log \ell_j \quad (4.7)$$

is smooth on P .

Furthermore, from [8] Theorem 4.18, we know for $u \in S_{A,B}(P, L, F)$, the induced Kähler metric (4.2) exhibits Poincaré type behavior along D . Let $d\lambda$ be the Lebesgue measure on $\mathbb{R}^2$, we define a measure $d\lambda_i$ on ℓ_i by

$$-d\ell_i \wedge d\lambda_i = -\nu_i \wedge d\lambda_i = d\lambda.$$

On ℓ_F , the induced measure vanishes for the class $S(P, L, F)$, which is obtained by sending the corresponding label ℓ_i to infinity. Note that $S_{A,B}(P, L, F)$ is a proper subset of $S(P, L, F)$, as can be seen by comparing [8] Definition 4.6 and Proposition 4.19.

We write

$$u_{P,F,A,B} = \frac{1}{2} \sum_{j \neq F} \ell_j \log \ell_j - (A + B\ell_F) \log \ell_F$$

as the potential of the model metric for the class $S_{A,B}(P, L, F)$. To construct scalar-flat Kähler metrics of Poincaré type in this class, we need other solutions to (4.4) besides the ones in (4.6). We consider

$$\xi = \frac{1}{2} \frac{1}{\sqrt{(H+a)^2 + r^2}}, \quad a \in \mathbb{R}.$$

This solution, together with the solutions to (4.4) mentioned above, serve as local models for our construction in the next section.

4.2 Construction of scalar-flat Kähler metrics of Poincaré type

Given X a strictly unbounded symplectic toric 4-manifold and let P be its moment polytope defined by (4.1). Write $\ell = \{\ell_1(x), \dots, \ell_d(x)\}$. Let $D = \sum_{i=1}^m D_i$ be a smooth divisor on X such that each D_j is fixed by the torus action. Let $L_{i_j} = \ell_{i_j}^{-1}(0)$ be its image on the moment polytope. Assume

$$i_1 > 1, i_2 > i_1 + 1, \dots, d > i_m > i_{m-1} + 1. \quad (4.8)$$

Let $I = \{i_1, \dots, i_m\} \subset \{1, 2, \dots, d\}$ be the index set and $L_I = \bigcup_{j=1}^m L_{i_j}$ be the union of the edges corresponding to D_i , then $P \setminus L_I$ is the moment polytope of $X \setminus D$.

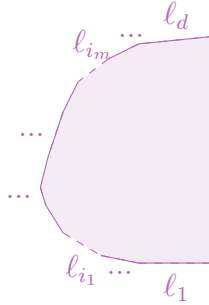


Figure 4.1: The moment polytope $P \setminus L_I$

We prove Theorem 1.2 by giving an explicit construction of the toric scalar-flat Kähler metrics of Poincaré type on $X \setminus D$. Consider $\nu = (\alpha, \beta)$ a vector in $\mathbb{R}^2$ such that

$$\det(\nu, \nu_1), \det(\nu, \nu_d) \geq 0. \quad (4.9)$$

As discussed in [4], since P is strictly unbounded, this set of vectors forms a cone bounded by $-\nu_1$ and ν_d .

Theorem 4.2.1. *For X, D, P, I, ν defined as above, for $(H, r) \in \mathbb{H}$, set*

$$\xi_1 := \alpha_1 \log r + \frac{1}{2} \sum_{i=1}^{d-1} (\alpha_{i+1} - \alpha_i) \log(H + a_i + \sqrt{(H + a_i)^2 + r^2}) - \frac{1}{2} \sum_{k \in I} \frac{\Lambda_k \alpha_k}{\sqrt{(H + a_{k-1})^2 + r^2}} + \alpha H,$$

$$\xi_2 := \beta_1 \log r + \frac{1}{2} \sum_{i=1}^{d-1} (\beta_{i+1} - \beta_i) \log(H + a_i + \sqrt{(H + a_i)^2 + r^2}) - \frac{1}{2} \sum_{k \in I} \frac{\Lambda_k \beta_k}{\sqrt{(H + a_{k-1})^2 + r^2}} + \beta H.$$

Here $a_1, \dots, a_{d-1}$ are real numbers determined by P satisfying

$$a_{j-1} > a_j \text{ if } j \notin I \text{ and } a_{j-1} = a_j \text{ if } j \in I. \quad (4.10)$$

Let x_1, x_2 be the primitives of

$$\epsilon_1 = r \left(\frac{\partial \xi_2}{\partial r} dH - \frac{\partial \xi_2}{\partial H} dr \right), \quad \epsilon_2 = -r \left(\frac{\partial \xi_1}{\partial r} dH - \frac{\partial \xi_1}{\partial H} dr \right).$$

Then there exist constants $\Lambda_{i_1}, \dots, \Lambda_{i_m} > 0$ determined by the polytope P , such that x_1, x_2 define the momentum action coordinates on P° of some toric scalar-flat Kähler metric of Poincaré type on $X \setminus D$ whose symplectic potential satisfies

$$du = \xi_1 dx_1 + \xi_2 dx_2.$$

Furthermore, for each $k \in I$,

$$u \in S_{\frac{\Lambda_k}{2}, \frac{1}{2} \det(\nu_{k-1}, \nu_{k+1})}(P, L, L_k). \quad (4.11)$$

Proof. The first step is to show the assumption (4.5) in Theorem 4.1.1 is satisfied. We compute $D\xi$:

$$D\xi = \begin{bmatrix} \alpha + \frac{1}{2} \sum_{i=1}^{d-1} \frac{\alpha_{i+1} - \alpha_i}{\rho_i} + \frac{1}{2} \sum_{k \in I} \frac{\Lambda_k \alpha_k H_{k-1}}{\rho_{k-1}^3} & \frac{\alpha_1}{r} + \frac{1}{2} \sum_{i=1}^{d-1} \frac{(\alpha_{i+1} - \alpha_i)r}{\rho_i(\rho_i + H_i)} + \frac{1}{2} \sum_{k \in I} \frac{\Lambda_k \alpha_k r}{\rho_{k-1}^3} \\ \beta + \frac{1}{2} \sum_{i=1}^{d-1} \frac{\beta_{i+1} - \beta_i}{\rho_i} + \frac{1}{2} \sum_{k \in I} \frac{\Lambda_k \beta_k H_{k-1}}{\rho_{k-1}^3} & \frac{\beta_1}{r} + \frac{1}{2} \sum_{i=1}^{d-1} \frac{(\beta_{i+1} - \beta_i)r}{\rho_i(\rho_i + H_i)} + \frac{1}{2} \sum_{k \in I} \frac{\Lambda_k \beta_k r}{\rho_{k-1}^3} \end{bmatrix}. \quad (4.12)$$

Here $H_i := H + a_i$, $\rho_i := \sqrt{H_i^2 + r^2}$. Set $a_0 := -\infty$, $a_d := \infty$ and set H_0, H_d, ρ_0, ρ_d accordingly, we rewrite Equation (4.12) as

$$D\xi = \begin{bmatrix} \alpha + \frac{1}{2} \sum_{i=1}^d \alpha_i \left(\frac{1}{\rho_{i-1}} - \frac{1}{\rho_i} \right) + \frac{1}{2} \sum_{k \in I} \frac{\Lambda_k \alpha_k H_{k-1}}{\rho_{k-1}^3} & \frac{1}{2r} \sum_{i=1}^d \alpha_i \left(\frac{H_{i-1}}{\rho_{i-1}} - \frac{H_i}{\rho_i} \right) + \frac{1}{2} \sum_{k \in I} \frac{\Lambda_k \alpha_k r}{\rho_{k-1}^3} \\ \beta + \frac{1}{2} \sum_{i=1}^d \beta_i \left(\frac{1}{\rho_{i-1}} - \frac{1}{\rho_i} \right) + \frac{1}{2} \sum_{k \in I} \frac{\Lambda_k \beta_k H_{k-1}}{\rho_{k-1}^3} & \frac{1}{2r} \sum_{i=1}^d \beta_i \left(\frac{H_{i-1}}{\rho_{i-1}} - \frac{H_i}{\rho_i} \right) + \frac{1}{2} \sum_{k \in I} \frac{\Lambda_k \beta_k r}{\rho_{k-1}^3} \end{bmatrix}. \quad (4.13)$$

Note from [4] Theorem 4.1, we know the determinant of the following matrix is positive:

$$\begin{bmatrix} \alpha + \frac{1}{2} \sum_{i=1}^d \alpha_i \left(\frac{1}{\rho_{i-1}} - \frac{1}{\rho_i} \right) & \frac{1}{2r} \sum_{i=1}^d \alpha_i \left(\frac{H_{i-1}}{\rho_{i-1}} - \frac{H_i}{\rho_i} \right) \\ \beta + \frac{1}{2} \sum_{i=1}^d \beta_i \left(\frac{1}{\rho_{i-1}} - \frac{1}{\rho_i} \right) & \frac{1}{2r} \sum_{i=1}^d \beta_i \left(\frac{H_{i-1}}{\rho_{i-1}} - \frac{H_i}{\rho_i} \right) \end{bmatrix}. \quad (4.14)$$

Comparing the expression of $\det(D\xi)$ with the above, it suffices to show their difference is still positive. A key step in showing positivity is the following lemma.

Lemma 4.2.1.1. *Given $k \in I$, $\forall i$, consider the terms involving i and k in the expression of $\det(D\xi)$:*

$$\frac{\Lambda_k}{4} (\alpha_i \beta_k - \alpha_k \beta_i) \left(\frac{1}{\rho_{i-1}} - \frac{1}{\rho_i} \right) \cdot \frac{r}{\rho_{k-1}^3} - \frac{\Lambda_k}{4} (\alpha_i \beta_k - \alpha_k \beta_i) \left(\frac{H_{i-1}}{\rho_{i-1}} - \frac{H_i}{\rho_i} \right) \cdot \frac{H_{k-1}}{\rho_{k-1}^3}.$$

Then it can be simplified to the following non-negative term

$$\Lambda_k \cdot (\beta_k \alpha_i - \alpha_k \beta_i) \frac{1}{4r \rho_{k-1}^3} \left(\frac{r^2 + H_{k-1} H_{i-1}}{\rho_{i-1}} - \frac{r^2 + H_{k-1} H_i}{\rho_i} \right). \quad (4.15)$$

Proof. We rewrite the simplified expression, Equation (4.15), as follows:

$$\Lambda_k (\beta_k \alpha_i - \alpha_k \beta_i) \frac{1}{4 \rho_{k-1}^3} \left(r \left(\frac{1}{\rho_{i-1}} - \frac{1}{\rho_i} \right) + H_{k-1} \left(\frac{H_{i-1}}{r \rho_{i-1}} - \frac{H_i}{r \rho_i} \right) \right).$$

Note for any $i, r, \rho_i > 0$, and from $\det(\nu_i, \nu_{i+1}) = -1$ we deduce that $\beta_k \alpha_i - \alpha_k \beta_i > 0 \iff i > k$. Thus, it suffices to show

$$\rho_i (r^2 + H_{k-1} H_{i-1}) - \rho_{i-1} (r^2 + H_{k-1} H_i) > 0 \text{ for } i > k \quad (4.16)$$

and

$$\rho_i (r^2 + H_{k-1} H_{i-1}) - \rho_{i-1} (r^2 + H_{k-1} H_i) < 0 \text{ for } i < k. \quad (4.17)$$

When $i > k$, we rewrite the expression as

$$\rho_i (r^2 + H_{k-1} H_{i-1}) - \rho_{i-1} (r^2 + H_{k-1} H_i) = \frac{(\rho_i - \rho_{i-1}) r^2}{\rho_i H_{i-1} + \rho_{i-1} H_i} (\rho_i (H_{i-1} - H_{k-1}) + \rho_{i-1} (H_i - H_{k-1})). \quad (4.18)$$

We claim that $\rho_{i-1} H_i - \rho_i H_{i-1} > 0$. It is because $f(x) = \frac{x}{\sqrt{x^2 + y^2}}$ is an increasing function given $y > 0$, then $\frac{H_i}{\sqrt{H_i^2 + r^2}} > \frac{H_{i-1}}{\sqrt{H_{i-1}^2 + r^2}}$. Hence, we know

$$\rho_i H_{i-1} + \rho_{i-1} H_i > 0 \iff (\rho_{i-1} H_i - \rho_i H_{i-1})(\rho_i H_{i-1} + \rho_{i-1} H_i) > 0 \iff r^2 (H_i^2 - H_{i-1}^2) > 0. \quad (4.19)$$

On the other hand, we have

$$\rho_i - \rho_{i-1} > 0 \iff (\rho_i + \rho_{i-1})(\rho_i - \rho_{i-1}) > 0 \iff H_i^2 - H_{i-1}^2 > 0. \quad (4.20)$$

Combining $H_{i-1} - H_{k-1} > 0, H_i - H_{k-1} > 0$ with (4.19) and (4.20), we obtain (4.16). Similarly, for $i < k$,

$$\rho_i(r^2 + H_{k-1}H_{i-1}) - \rho_{i-1}(r^2 + H_{k-1}H_i) = \frac{(\rho_i - \rho_{i-1})r^2}{\rho_i H_{i-1} + \rho_{i-1} H_i} (\rho_i(H_{i-1} - H_{k-1}) + \rho_{i-1}(H_i - H_{k-1})) < 0. \quad (4.21)$$

□

For $k, k' \in I$, the term involving k and k' in $\sum_{k \in I} \frac{\alpha_k H_{k-1}}{\rho_{k-1}^3} \sum_{k \in I} \frac{\beta_k r}{\rho_{k-1}^3} - \sum_{k \in I} \frac{\alpha_k r}{\rho_{k-1}^3} \sum_{k \in I} \frac{\beta_k H_{k-1}}{\rho_{k-1}^3}$ has the following expression and is non-negative:

$$\frac{r}{\rho_k^3 \rho_{k'}^3} (\alpha_k \beta_{k'} - \alpha_{k'} \beta_k) (H_{k-1} - H_{k'-1}). \quad (4.22)$$

Finally, for ν satisfying (4.9), given $k \in I$, $\det D\xi$ changes by adding

$$\begin{aligned} \det(\nu, \nu_1) \left(\frac{1}{r} - \frac{r}{2\rho_1(H_1 + \rho_1)} \right) + \frac{r}{2} \sum_{i=1}^{d-1} \det(\nu, \nu_i) \left(\frac{1}{\rho_{i-1}(H_{i-1} + \rho_{i-1})} - \frac{1}{\rho_i(H_i + \rho_i)} \right) \\ + \frac{r}{2} \det(\nu, \nu_d) \frac{1}{\rho_d(H_d + \rho_d)} + (\alpha\beta_k - \beta\alpha_k) \frac{r}{\rho_{k-1}^3}. \end{aligned} \quad (4.23)$$

With loss of generality, we assume one of the unbounded edges of P is the x_1 -axis, then $\nu_1 = (0, 1)$. Consider the interior normals ν_i satisfying $\det(\nu_{i-1}, \nu_i) = -1$, $i = 2, \dots, d$, we obtain $\alpha_i > 0$, $i = 2, \dots, d$ and $\frac{\beta_d}{\alpha_d} < \frac{\beta_{d-1}}{\alpha_{d-1}} < \dots < \frac{\beta_2}{\alpha_2}$. The condition (4.9) implies

$$\alpha\beta_k - \beta\alpha_k \geq 0 \text{ for } k \in I.$$

Thus by comparing with [4] Equation (7), we know the above additional term is non-negative. Combining (4.14), (4.15), (4.22), and (4.23), we conclude $\det(D\xi) > 0$.

Next, we prove that $x = (x_1, x_2)$ define global symplectic action coordinates on $P \setminus L_I$ and at the same time $u(x)$ has the desired boundary behavior on $\partial P \setminus L_I$. Note for

$$\xi = \frac{1}{2\sqrt{(H+a)^2 + r^2}},$$

the primitive of $\epsilon = r(\frac{\partial \xi}{\partial r} dH - \frac{\partial \xi}{\partial H} dr)$, up to constants, is given by

$$\frac{1}{2} \left(1 - \frac{H+a}{\sqrt{(H+a)^2 + r^2}} \right).$$

Then for $\nu = 0$, up to constants, we have

$$x_1 = \beta_1 H + \frac{1}{2} \sum_{i=1}^{d-1} (\beta_{i+1} - \beta_i)(H_i - \rho_i) - \frac{1}{2} \sum_{k \in I} \Lambda_k \beta_k (1 - \frac{H_{k-1}}{\rho_{k-1}}),$$

$$x_2 = -\alpha_1 H - \frac{1}{2} \sum_{i=1}^{d-1} (\alpha_{i+1} - \alpha_i)(H_i - \rho_i) + \frac{1}{2} \sum_{k \in I} \Lambda_k \alpha_k (1 - \frac{H_{k-1}}{\rho_{k-1}}).$$

Then we see x extends continuously to $r = 0$ except at the points $(H, r) = (-a_k, 0)$ for $k \in I$. The point $(H, r) = (-a_k, 0)$ corresponds to the cusp edge L_k for $k \in I$. Consider the behavior of x in the intervals on the H -axis:

- (i) For $H > -a_1$, then $x_1 = \beta_1 H$, and $x_2 = -\alpha_1 H$;
- (ii) for $1 \leq j < i_1 - 1$, $-a_{j+1} < H < -a_j$, then $x_1 = \beta_{j+1} H + \sum_{i=1}^j a_i (\beta_{i+1} - \beta_i)$, and $x_2 = -\alpha_{j+1} H - \sum_{i=1}^j a_i (\alpha_{i+1} - \alpha_i)$;
- (iii) for $j = i_1 - 1$, $-a_{i_1+1} < H < -a_{i_1-1} = -a_{i_1}$, then $x_1 = \beta_{i_1+1} H + \sum_{i=1}^{i_1} a_i (\beta_{i+1} - \beta_i) - \Lambda_{i_1} \beta_{i_1}$, and $x_2 = -\alpha_{i_1+1} H - \sum_{i=1}^{i_1} a_i (\alpha_{i+1} - \alpha_i) + \Lambda_{i_1} \alpha_{i_1}$;
- (iv) similar calculations show for $2 \leq k \leq m$, and $i_{k-1} + 1 \leq j < i_k - 1$, with $-a_{j+1} < H < -a_j$, we have $x_1 = \beta_{j+1} H + \sum_{i=1}^j a_i (\beta_{i+1} - \beta_i) - \sum_{\ell=1}^{k-1} \Lambda_{i_\ell} \beta_{i_\ell}$, and $x_2 = -\alpha_{j+1} H - \sum_{i=1}^j a_i (\alpha_{i+1} - \alpha_i) + \sum_{\ell=1}^{k-1} \Lambda_{i_\ell} \alpha_{i_\ell}$; for $j = i_k - 1$, with $-a_{i_k+1} < H < -a_{i_k-1} = -a_{i_k}$, we have $x_1 = \beta_{i_k+1} H + \sum_{i=1}^{i_k} a_i (\beta_{i+1} - \beta_i) - \sum_{\ell=1}^k \Lambda_{i_\ell} \beta_{i_\ell}$, and $x_2 = -\alpha_{i_k+1} H - \sum_{i=1}^{i_k} a_i (\alpha_{i+1} - \alpha_i) + \sum_{\ell=1}^k \Lambda_{i_\ell} \alpha_{i_\ell}$;
- (v) for $H < -a_{d-1}$, then $x_1 = \beta_d H + \sum_{i=1}^{d-1} a_i (\beta_{i+1} - \beta_i) - \sum_{\ell=1}^m \Lambda_{i_\ell} \beta_{i_\ell}$, and $x_2 = -\alpha_d H - \sum_{i=1}^{d-1} a_i (\alpha_{i+1} - \alpha_i) + \sum_{\ell=1}^m \Lambda_{i_\ell} \alpha_{i_\ell}$.

We want to show the following:

1. $x = (x_1, x_2)$ gives a proper homeomorphism

$$x(H, 0) : \partial \mathbb{H} \setminus \bigcup_{k \in I} (a_{k-1}, 0) \rightarrow \partial P \setminus L_I. \quad (4.24)$$

2. $u(x)$ modulo a smooth function is given by

$$\frac{1}{2} \sum_{i \notin I} \ell_i(x) \log \ell_i(x) + \frac{1}{2} \sum_{k \in I} (\det(\nu_{k+1}, \nu_{k-1}) \ell_k(x) - \Lambda_k) \log \ell_k(x). \quad (4.25)$$

First, we discuss the choice of a_j . Note from [4] Theorem 1.2, there exist real numbers $a'_1 < a'_2 < \dots < a'_{d-1}$ determined by P such that for $1 \leq j \leq d-1$,

$$\sum_{i=1}^j a'_i \det(\nu_{i+1} - \nu_i, \nu_{j+1}) = \lambda_{j+1}. \quad (4.26)$$

More precisely, from [57] Lemma 7.2, we know $a'_i - a'_{i-1} = \frac{|L_i|}{2\pi|\nu_i|^2}$. Here $|L_i|$ is the length of the i th edge of P . On the other hand, from the boundary behavior of the non-cusp edges in (4.25), with the same proof as [57] Lemma 7.2, we have

$$a_i - a_{i-1} = \frac{|L_i|}{2\pi|\nu_i|^2}.$$

For $k \in I$, we set $a_k = a_{k-1}$. Later we will see that this ensures the desired boundary behavior. Then we obtain the following relations between a_j and a'_j :

$$a_j = a'_j \text{ if } j < i_1; a_j = a'_j + \sum_{\ell=1}^k (a'_{i_\ell-1} - a'_{i_\ell}) \text{ if } i_k \leq j < i_{k+1}; a_j = a'_j + \sum_{\ell=1}^m (a'_{i_\ell-1} - a'_{i_\ell}) \text{ if } i_m \leq j < d. \quad (4.27)$$

First, we look at condition (4.24). For simplicity, we write $k = i_1$. Near the edge L_{k+1} , as $r = 0$ and $-a_{k+1} < H < -a_k$, we have

$$x_1 = \beta_{k+1}H + \sum_{i=1}^k a_i(\beta_{i+1} - \beta_i) - \Lambda_k \beta_k, \quad x_2 = -\alpha_{k+1}H - \sum_{i=1}^k a_i(\alpha_{i+1} - \alpha_i) + \Lambda_k \alpha_k.$$

Then condition (4.24) $\ell_{k+1}(x) = 0$ translates to

$$\sum_{i=1}^k a_i \det(\nu_{i+1} - \nu_i, \nu_{k+1}) + \Lambda_k = \lambda_{k+1}. \quad (4.28)$$

Using (4.26) and the relation (4.27), it simplifies to

$$\Lambda_k = a'_k - a'_{k-1}. \quad (4.29)$$

For edges ℓ_j with $i < k$, the argument is the same as the standard case as in [4] Theorem 4.1; for $i = k+j$, with $1 \leq j < i_2 - k$, the condition (4.24) $\ell_{k+j}(x) = 0$ on $-a_{k+j} < H < a_{k+j-1}$ becomes

$$\sum_{i=1}^{k+j-1} a_i \det(\nu_{i+1} - \nu_i, \nu_{k+j}) + \Lambda_k \det(\nu_{k+j}, \nu_k) = \lambda_{k+j}.$$

Using (4.26) it suffices to check $\det(\nu_{k+j}, \nu_k)(\Lambda_k + a'_{k-1} - a'_k) = 0$, which holds from (4.29). Thus we've shown condition (4.25) for $1 \leq j < i_2$ given the choice of Λ_k as in (4.29). It remains to apply the same procedure to indices $i_2, \dots, i_m$ respectively. For simplicity, we write $t = i_2$. For $-a_{t+1} < H < -a_t$, we have

$$x_1 = \beta_{t+1}H + \sum_{i=1}^t a_i(\beta_{i+1} - \beta_i) - \Lambda_t \beta_t - \Lambda_k \beta_k, \quad x_2 = -\alpha_{t+1}H - \sum_{i=1}^t a_i(\alpha_{i+1} - \alpha_i) + \Lambda_t \alpha_t + \Lambda_k \alpha_k.$$

Then condition (4.24) $\ell_{t+1}(x) = 0$ on $-a_{t+1} < H < -a_t$ translates to

$$\sum_{i=1}^t a_i \det(\nu_{i+1} - \nu_i, \nu_{t+1}) + \Lambda_t + \Lambda_k \det(\nu_{t+1}, \nu_k) = \lambda_{t+1}, \quad (4.30)$$

and it simplifies to

$$\Lambda_t = a'_t - a'_{t-1}. \quad (4.31)$$

For $i_2 < i < i_3$, condition (4.24) can be shown in exactly the same way. For $i_3, \dots, i_m$, it's now clear this procedure also works. We deduce that given the choices of $\Lambda_i = a'_i - a'_{i-1}$ for all $i \in I$, $x : (\mathbb{H}, \partial\mathbb{H} \setminus \bigcup_{k \in I} (a_{k-1}, 0)) \rightarrow (P, \partial P \setminus L_I)$ is a proper homeomorphism, and its restriction to $\mathbb{H}$ is a smooth proper diffeomorphism onto P° .

Next, we look at the condition (4.25). Near a non-cusp edge L_i , we have $\xi = \nu_i \log r + O(1)$, and the boundary condition is standard as desired. We focus on the situation near the cusp edge L_k . For this, we prove the following lemmas:

Lemma 4.2.1.2. *Assume for $k \in I$, $\Lambda_k = a'_k - a'_{k-1}$, then*

$$\ell_k(x) = \rho_{k-1} + O(r^2), \quad (4.32)$$

and there exist smooth positive function $\delta_{k-1,k,k+1}$ such that

$$\rho_{k-1} + H_{k-1} = \rho_{k-1} \cdot \frac{2\ell_{k+1}(x) + O(r^2)}{\delta_{k-1,k,k+1}(x)}, \quad \rho_{k-1} - H_{k-1} = \rho_{k-1} \cdot \frac{2\ell_{k-1}(x) + O(r^2)}{\delta_{k-1,k,k+1}(x)}. \quad (4.33)$$

Proof. We first consider $k = i_1$. When both H_{k-1} and r go to zero, the image approaches the cusp edge L_k on the polytope. Note that (H_{k-1}, r) can never reach $(0, 0)$, since L_k is excluded from the polytope. For $i > k$, we have $H_i - \sqrt{H_i^2 + r^2} = -\frac{r^2}{2a_i}$, and for $i < k-1$, we have $H_i - \sqrt{H_i^2 + r^2} = 2a_i$. Then for the behavior of x near $\ell_k(x) = 0$, we write

$$x_1 = \beta_{k-1}H + \frac{1}{2}(\beta_{k+1} - \beta_{k-1})(H_{k-1} - \rho_{k-1}) - \frac{1}{2}\Lambda_k \beta_k \left(1 + \frac{H_{k-1}}{\rho_{k-1}}\right) + \sum_{i=1}^{k-2} a_i(\beta_{i+1} - \beta_i) + O(r^2),$$

$$x_2 = -\alpha_{k-1}H - \frac{1}{2}(\alpha_{k+1} - \alpha_{k-1})(H_{k-1} - \rho_{k-1}) + \frac{1}{2}\Lambda_k\alpha_k(1 + \frac{H_{k-1}}{\rho_{k-1}}) - \sum_{i=1}^{k-2} a_i(\alpha_{i+1} - \alpha_i) + O(r^2).$$

Then

$$\nu_k \cdot x = \rho_{k-1} - a_{k-1} + \sum_{i=1}^{k-2} a_i \det(\nu_k, \nu_{i+1} - \nu_i) + O(r^2).$$

To prove Equation (4.32), it's equivalent to show

$$\sum_{i=1}^{k-2} a_i \det(\nu_k, \nu_{i+1} - \nu_i) - a_{k-1} + \lambda_k = 0. \quad (4.34)$$

From the relations between a_j and a'_j in Equation (4.27) and the expression of λ_j in Equation (4.26), direct computation shows this automatically holds.

Similarly, we have

$$\nu_{k-1} \cdot x = \frac{1}{2} \det(\nu_{k+1}, \nu_{k-1})(\rho_{k-1} - H_{k-1}) + \frac{\Lambda_k}{2} (1 - \frac{H_{k-1}}{\rho_{k-1}}) + \sum_{i=1}^{k-2} a_i \det(\nu_{k-1}, \nu_{i+1} - \nu_i) + O(r^2).$$

Under the normalization assumption $\det(\nu_j, \nu_{j+1}) = -1$ for all j , straightforward calculations give

$$\det(\nu_{j+1}, \nu_{j-1}) = \frac{\alpha_{j+1} + \alpha_{j-1}}{\alpha_j} = \frac{\beta_{j+1} + \beta_{j-1}}{\beta_j}. \quad (4.35)$$

Combining (4.35) and (4.32) with the above calculations, we obtain

$$H_{k-1} = \rho_{k-1} \cdot \frac{(\nu_{k+1} - \nu_{k-1}) \cdot x + \lambda_k \det(\nu_{k+1}, \nu_{k-1}) + 2 \sum_{i=1}^{k-2} a_i \det(\nu_{k-1}, \nu_{i+1} - \nu_i) + \Lambda_k + O(r^2)}{(\nu_{k+1} + \nu_{k-1}) \cdot x + \lambda_k \det(\nu_{k+1}, \nu_{k-1}) + \Lambda_k + O(r^2)}.$$

Thus to prove Equation (4.33), it suffices to show

$$\lambda_{k+1} = \Lambda_k + \lambda_k \det(\nu_{k+1}, \nu_{k-1}) + \sum_{i=1}^{k-2} a_i \det(\nu_{k-1}, \nu_{i+1} - \nu_i), \quad (4.36)$$

and

$$\lambda_{k-1} = - \sum_{i=1}^{k-2} a_i \det(\nu_{k-1}, \nu_{i+1} - \nu_i). \quad (4.37)$$

We claim that

$$\lambda_{k+1} - \lambda_k \det(\nu_{k+1}, \nu_{k-1}) + \lambda_{k-1} - \Lambda_k = 0. \quad (4.38)$$

Using (4.26), we have

$$\lambda_{k+1} + \lambda_{k-1} = a_k - a_{k-1} + a_{k-1} \det(\nu_{k+1}, \nu_{k-1}) + \sum_{i=1}^{k-2} a_i \det(\nu_{k-1} + \nu_{k+1}, \nu_{i+1} - \nu_i),$$

then by simplifying this equation with (4.27), we obtain (4.38). Then (4.36) is reduced to (4.37), which is straightforward from (4.26). Then we obtain Equation (4.33) with

$$\delta_{k-1,k,k+1} = \ell_{k+1}(x) + \ell_{k-1}(x) + O(r^2).$$

It remains to show Equations (4.32) and (4.33) for $i_2, \dots, i_m$. Write $t = i_2$, near $\ell_t(x) = 0$, we write

$$x_1 = \beta_{t-1}H + \frac{1}{2}(\beta_{t+1} - \beta_{t-1})(H_{t-1} - \rho_{t-1}) - \frac{1}{2}\Lambda_t\beta_t(1 + \frac{H_{t-1}}{\rho_{t-1}}) + \sum_{i=1}^{t-2} a_i(\beta_{i+1} - \beta_i) - \Lambda_k\beta_k + O(r^2),$$

$$x_2 = -\alpha_{t-1}H - \frac{1}{2}(\alpha_{t+1} - \alpha_{t-1})(H_{t-1} - \rho_{t-1}) + \frac{1}{2}\Lambda_t\alpha_t(1 + \frac{H_{t-1}}{\rho_{t-1}}) - \sum_{i=1}^{t-2} a_i(\alpha_{i+1} - \alpha_i) - \Lambda_k\alpha_k + O(r^2).$$

Then Equation (4.32) holds automatically under the assumption $\Lambda_t = a'_t - a'_{t-1}$, and the first equation in Equation (4.33) simplifies to

$$0 = \lambda_{t+1} - \lambda_t \det(\nu_{t+1}, \nu_{t-1}) + \lambda_{t-1} - \Lambda_t. \quad (4.39)$$

Direct computation as the previous situation shows this equation holds. For $i_3, \dots, i_m$, it's now clear this procedure also works. $\square$

Lemma 4.2.1.3. *There exists a smooth and strictly positive function δ on P such that*

$$\det(\text{Hess}_x(u)) = (\delta \prod_{i=1}^d \ell_i \prod_{k \in I} \ell_k)^{-1}. \quad (4.40)$$

Proof. From [27], we know $r = (\det \text{Hess}_x(u))^{-1/2}$. Then it suffices to show there exists a smooth and strictly positive function δ , such that

$$\prod_{i=1}^d \ell_i \prod_{k \in I} \ell_k = \frac{r^2}{\delta}. \quad (4.41)$$

From the above discussions, we know

$$\frac{\partial x_1}{\partial r} = -\frac{r}{2} \sum_{i=1}^{d-1} \frac{\beta_{i+1} - \beta_i}{\rho_i} - \frac{r}{2} \sum_{k \in I} \Lambda_k \beta_k \frac{H_{k-1}}{\rho_{k-1}^3}, \quad \frac{\partial x_2}{\partial r} = \frac{r}{2} \sum_{i=1}^{d-1} \frac{\alpha_{i+1} - \alpha_i}{\rho_i} + \frac{r}{2} \sum_{k \in I} \Lambda_k \alpha_k \frac{H_{k-1}}{\rho_{k-1}^3}.$$

Then

$$\frac{\partial \ell_j}{\partial r} = \frac{\partial x_1}{\partial r} \alpha_j + \frac{\partial x_2}{\partial r} \beta_j \quad (4.42)$$

$$= \frac{r}{2} \sum_{i=1}^{d-1} \frac{\det(\nu_{i+1} - \nu_i, \nu_j)}{\rho_i} + \frac{r}{2} \sum_{k \in I} \frac{\Lambda_k H_{k-1}}{\rho_{k-1}^3} (\alpha_k \beta_j - \alpha_j \beta_k) \quad (4.43)$$

$$= \frac{r}{2} \left(-\frac{\det(\nu_1, \nu_j)}{\rho_1} + \sum_{i=2}^{d-1} \det(\nu_i, \nu_j) \left(\frac{1}{\rho_{i-1}} - \frac{1}{\rho_i} \right) + \frac{\det(\nu_d, \nu_j)}{\rho_{d-1}} \right) + \frac{r}{2} \sum_{k \in I} \frac{\Lambda_k H_{k-1}}{\rho_{k-1}^3} \det(\nu_k, \nu_j). \quad (4.44)$$

We want to show for $j \notin I$, as we approach each edge E_j of P , $\frac{\partial \ell_j}{\partial r} = r \delta_j$ for some smooth and positive function δ_j ; for $j \in I$, as H approaches $-a_j$ and r approaches 0, $\ell_j^2 \ell_{j-1} \ell_{j+1} = r^2 \delta_j$ for some smooth and positive function δ_j . To show these, we have the following discussions:

- (i) When $j = 1$, for $r = 0$, $-a_1 < H$, it's immediate to see each term of (4.44) is positive. This implies $\frac{\partial \ell_j}{\partial r} = r \delta_j$ for some function δ_j smooth and strictly positive.
- (ii) When $j > 1$ and $j, j+1 \notin I$. For $r = 0$, $-a_j < H < -a_{j-1}$, given $1 \leq i < j$, we have $\det(\nu_i, \nu_j) < 0$, $\frac{1}{\rho_{i-1}} - \frac{1}{\rho_i} < 0$; given $j+1 \leq i < d$, we have $\det(\nu_i, \nu_j) > 0$, $\frac{1}{\rho_{i-1}} - \frac{1}{\rho_i} > 0$; given $k \in I, k < j$, we have $\det(\nu_k, \nu_j) < 0$, $H_{k-1} < 0$; given $k \in I, k > j$, we have $\det(\nu_k, \nu_j) > 0$, $H_{k-1} > 0$. Hence, each term of (4.44) is again positive.
- (iii) When $j = d$, for $r = 0$, $H < -a_d$, similar arguments show each term is positive.
- (iv) When $j+1 \in I$, for $r = 0$, $-a_j = -a_{j+1} < H < -a_{j-1}$, the only difference is when $i = j+1 \in I$, we have $\rho_i = \rho_{i-1}$ and $\det(\nu_i, \nu_j) > 0$, $H_j > 0$, thus the involved terms are still positive. All other terms are still positive from the discussion in the second case.
- (v) When $j \in I$. For $H = -a_j = -a_{j-1}$, r doesn't take the value 0, we let $r \rightarrow 0$, then $\rho_j, \rho_{j-1} \rightarrow 0$. In this case, $\frac{1}{\rho_{j-1}}$ goes to infinity, and we need a different argument. Using Equation (4.33), we obtain

$$r^2 = (\rho_{j-1} + H_{j-1})(\rho_{j-1} - H_{j-1}) = \ell_j^2 \cdot \frac{(2\ell_{j-1} + O(r^2))(2\ell_{j+1} + O(r^2))}{\delta_{j-1,j,j+1}^2},$$

here $\delta_{j-1,j,j+1}$ is a smooth and positive function. Thus $\ell_j^2 \ell_{j-1} \ell_{j+1} = r^2 \delta_j$ for some smooth and positive function δ_j .

□

With the preparation of these lemmas, we are ready to show the boundary condition (4.25) holds for the cusp edges of P . Consider $k \in I$, near $\ell_k(x) = 0$,

$$\begin{aligned}\xi_1 &= \alpha_{k-1} \log r + \frac{1}{2}(\alpha_{k+1} - \alpha_{k-1}) \log(H_{k-1} + \rho_{k-1}) - \frac{\Lambda_k}{2} \frac{\alpha_k}{\rho_{k-1}} + O(1), \\ \xi_2 &= \beta_{k-1} \log r + \frac{1}{2}(\beta_{k+1} - \beta_{k-1}) \log(H_{k-1} + \rho_{k-1}) - \frac{\Lambda_k}{2} \frac{\beta_k}{\rho_{k-1}} + O(1).\end{aligned}$$

From Lemma 4.2.1.3, we know $r^2 = \delta' \ell_{k-1} \ell_k^2 \ell_{k+1}$ for some smooth and strictly positive function δ' near ℓ_k . Note from Equations (4.32) and (4.33), we obtain from (4.35) that

$$\begin{aligned}du &= \frac{1}{2}(\log(\ell_{k-1})\alpha_{k-1} + \log(\ell_{k+1})\alpha_{k+1} + (\det(\nu_{k+1}, \nu_{k-1}) \log(\ell_k) - \frac{\Lambda_k}{\ell_k})\alpha_k)dx_1 \\ &+ \frac{1}{2}(\log(\ell_{k-1})\beta_{k-1} + \log(\ell_{k+1})\beta_{k+1} + (\det(\nu_{k+1}, \nu_{k-1}) \log(\ell_k) - \frac{\Lambda_k}{\ell_k})\beta_k)dx_2 + O(1).\end{aligned}\quad (4.45)$$

This is the desired boundary behavior. Hence proving (4.25). This completes the construction of the metric. The corresponding potential $u \in S_{\frac{\Lambda_k}{2}, \frac{1}{2} \det(\nu_{k-1}, \nu_{k+1})}(P, L, L_k)$ for $k \in I$ is immediate from the construction.

□

Remark 4.2.1. We remark here that we can modify the assumption (4.8) on the index set to $1 < i_1 < i_2 < \dots < i_d < d$. If $k, k+1 \in I$ and $k-1, k+2 \notin I$, which means there are two adjacent cusp edges, then we have $a_{k-1} = a_k = a_{k+1}$. Now define for $i < k$, $\alpha'_i := \alpha_i, \beta'_i := \beta_i$; for $i = k$, $\alpha'_k := \alpha_k + \alpha_{k+1}, \beta'_k := \beta_k + \beta_{k+1}$; for $i > k$, $\alpha'_i := \alpha_{i-1}, \beta'_i := \beta_{i-1}$. Then, the situation is reduced to that discussed in the theorem. Similarly, we can extend the arguments to the case for any number of adjacent cusp edges with this argument. From this, we can relax the assumption of $D_i \cap D_j = \emptyset$ for all $i \neq j$ to allow some non-empty intersection of them.

We are interested in studying the asymptotic behavior of these Poincaré type scalar-flat Kähler metrics away from the divisor. For this, we compare them with the scalar-flat Kähler metrics on $\mathbb{C}^2$ discussed in [27] and [69]. From [69], we know the symplectic coordinates

$$x_1 = \frac{1}{2}(-H + \sqrt{H^2 + r^2}) + \alpha H^2, \quad x_2 = \frac{1}{2}(H + \sqrt{H^2 + r^2}) + \beta H^2, \quad \text{for } \alpha, \beta \geq 0$$

induce the Taub-NUT metric when $\alpha = \beta > 0$, the generalized Taub-NUT metrics when $\alpha > 0, \beta > 0$, and the exceptional Taub-NUT metrics when $\alpha = 0, \beta > 0$ or $\beta = 0, \alpha > 0$.

Theorem 4.2.2. *Consider the metrics constructed in Theorem 4.2.1. The metric is (1) ALE if $\alpha = \beta = 0$; (2) asymptotic to a generalized Taub-NUT metric if $\alpha, \beta > 0$; (3) asymptotic to an exceptional Taub-NUT metric if either $\alpha = 0, \beta > 0$ or $\alpha > 0, \beta = 0$.*

Proof. From [26], Section 3, given angular coordinates θ_1, θ_2 for x_1, x_2 , the metrics constructed in Theorem 4.2.1 have the following form

$$\sum u_{ij} dx_i \otimes dx_j + \sum u^{ij} d\theta_i \otimes d\theta_j.$$

Equivalently $\sum u_{ij} dx_i \otimes dx_j$ can be written as

$$r \cdot \det D\xi(dH^2 + dr^2).$$

For $H \geq 0$, and $\rho := \sqrt{H^2 + r^2}$, as $\rho \rightarrow \infty$, we have

$$\frac{1}{\rho_i + H_i} - \frac{1}{\rho + H} = O\left(\frac{1}{\rho^2}\right), \quad \frac{1}{\rho_i} - \frac{1}{\rho} = O\left(\frac{1}{\rho^2}\right),$$

then as $\rho \rightarrow \infty$, we rewrite $D\xi$ as

$$D\xi = \begin{bmatrix} \alpha & 0 \\ \beta & 0 \end{bmatrix} + \frac{1}{r} \begin{bmatrix} \frac{r \sum_{i=1}^{d-1} (\alpha_{i+1} - \alpha_i)}{2\rho} & \alpha_1 + \frac{\sum_{i=1}^{d-1} (\alpha_{i+1} - \alpha_i) r^2}{2\rho(\rho+H)} \\ \frac{r \sum_{i=1}^{d-1} (\beta_{i+1} - \beta_i)}{2\rho} & \beta_1 + \frac{\sum_{i=1}^{d-1} (\beta_{i+1} - \beta_i) r^2}{2\rho(\rho+H)} \end{bmatrix} + O\left(\frac{1}{\rho^2}\right). \quad (4.46)$$

We see that compared to the Abreu and Sena-Dias case, the additional terms of the form $\frac{\alpha_k}{\rho_k}$ in ξ only create terms of the form $O\left(\frac{1}{\rho^2}\right)$. Then under similar computations as in [4] Proposition 5.1, we know

$$r \det D\xi = \frac{\det(\nu_d, \nu_1)}{2\rho} + O\left(\frac{1}{\rho^2}\right), \quad \text{for } \alpha = \beta = 0;$$

$$r \det D\xi = \det(\nu, \nu_1) \left(1 - \frac{r^2}{2\rho(H + \rho)}\right) + \det(\nu, \nu_d) \frac{r^2}{2\rho(H + \rho)} + \frac{\det(\nu_d, \nu_1)}{2\rho} + O\left(\frac{1}{\rho^2}\right), \quad \text{for } (\alpha, \beta) \neq (0, 0).$$

For $H \leq 0$, we argue the same way by considering $(H, r) \mapsto (-H, r), (\nu_1, \dots, \nu_d) \mapsto (\nu_d, \dots, \nu_1), (a_1, \dots, a_d) \mapsto (-a_d, \dots, -a_1)$. Hence, we deduce the desired asymptotic behavior for the first two cases. The completeness of these metrics is immediate with the above calculations, for details, see [4] Proposition 5.1.

For the last case where either $\alpha > 0, \beta = 0$ or $\alpha > 0, \beta = 0$, we compare $\det D\xi$ with that for the model exceptional Taub-NUT metrics on $\mathbb{C}^2$, introduced in [69]. Let ξ_{AS} be the Legendre transform of its momentum coordinate for the toric scalar-flat metric constructed in [4] for the given polytope P , and ξ_{exc} be the one for the exceptional Taub-NUT metric, then we know from [68] Section 5 that $\det D\xi_{AS} = \det D\xi_{exc} + O(\rho^{-2})$. For our case, from

the above expression (4.46), we have $\det D\xi = \det D\xi_{AS} + O(\rho^{-2})$, and thus the metrics are asymptotic to the exceptional Taub-NUT metrics. $\square$

Example 4.2.1. (*Hwang-Singer metric, [29], [31]*) Consider the polytope with three edges whose normal vectors are $\nu_1 = (0, 1), \nu_2 = (1, 1), \nu_3 = (1, 0)$ and $I = \{2\}$. Let

$$L = \{x_1, x_2, x_1 + x_2 - 1\}.$$

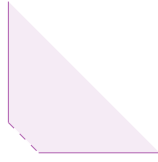


Figure 4.2: The moment polytope of the Hwang-Singer metric

Then, from the construction, we obtain

$$\xi_1 = \frac{1}{2} \log(H + \sqrt{H^2 + r^2}) - \frac{1}{2} \frac{1}{\sqrt{H^2 + r^2}}, \quad \xi_2 = \frac{1}{2} \log(-H + \sqrt{H^2 + r^2}) - \frac{1}{2} \frac{1}{\sqrt{H^2 + r^2}};$$

$$x_1 = \frac{1}{2}(H + \sqrt{H^2 + r^2}) + \frac{1}{2} \frac{H}{\sqrt{H^2 + r^2}}, \quad x_2 = \frac{1}{2}(-H + \sqrt{H^2 + r^2}) - \frac{1}{2} \frac{H}{\sqrt{H^2 + r^2}}.$$

Then, we obtain the symplectic potential u , which is

$$u = \frac{1}{2} x_1 \log x_1 + \frac{1}{2} x_2 \log x_2 + \frac{1}{2} (x_1 + x_2 - 1) \log(x_1 + x_2) + h$$

for some smooth function h . Let ω be the corresponding Kähler form. It lives on the complement of the zero section E of the total space of the line bundle $\mathcal{O}(-1)$ over $\mathbb{CP}^1$, which we denote by Y . Then $u \in S_{\frac{1}{2}, -\frac{1}{2}}(Y, L, E)$. Consider the momentum coordinate

$$\tau := 2(x_1 + x_2) = 2\sqrt{H^2 + r^2},$$

let X be the generator of the S^1 -action satisfying $i_X \omega = -d\tau$, then from [67] Section 2.3, we can compute the norm $\|X\|^2$ as follows:

$$\|X\|^2 = r \cdot \frac{\left(\frac{\partial \tau}{\partial H}\right)^2 + \left(\frac{\partial \tau}{\partial r}\right)^2}{\frac{\partial \tau}{\partial H} \frac{\partial x_1}{\partial r} - \frac{\partial \tau}{\partial r} \frac{\partial x_1}{\partial H}} = \frac{2\tau^2}{2 + \tau}.$$

We see it is exactly the momentum profile of the Hwang-Singer metric discussed in [29] Section 2 and Section 3. If we vary the length of the edge corresponding to the divisor, then we obtain a one-parameter family of toric scalar-flat Kähler metrics of Poincaré type introduced in [31]. More precisely, consider

$$\xi_1 = \frac{1}{2} \log(H + \sqrt{H^2 + r^2}) - \frac{a}{2} \frac{1}{\sqrt{H^2 + r^2}}, \quad \xi_2 = \frac{1}{2} \log(-H + \sqrt{H^2 + r^2}) - \frac{a}{2} \frac{1}{\sqrt{H^2 + r^2}},$$

the corresponding symplectic potential is given by

$$u = \frac{1}{2} x_1 \log x_1 + \frac{1}{2} x_2 \log x_2 + \frac{1}{2} (x_1 + x_2 - a) \log(x_1 + x_2) + h_a$$

for some smooth function h_a . Then $u \in S_{\frac{a}{2}, \frac{1}{2}}(Y, L, E)$.

With a similar approach, we can prove Theorem 4.2, again by explicitly constructing the toric metrics:

Proof of Theorem 4.2. Compared with the case in Theorem 4.2.1, all arguments of the construction work for the parallel edges case except for the choice of $\nu = (\alpha, \beta)$. Since $\nu_1 = -\nu_d$, ν needs to satisfy

$$\det(\nu, \nu_1) = 0; \quad \det(\nu, \nu_k) \geq 0, \forall k \in I,$$

then with the same arguments, we see ξ_1, ξ_2 give a one-parameter family of toric scalar-flat Kähler metrics. To understand the asymptotic behavior of these metrics, we compare them with that of the product metric g_{prod} on $S^2 \times \mathbb{R}^2$. On S^2 , we take the round metric, and on $\mathbb{R}^2$, we take the hyperbolic metric. Then $S^2 \times \mathbb{R}^2$ is biholomorphic to $\mathbb{CP}^1 \times D$. The symplectic coordinates of this product metric, as discussed in [67], can be written as

$$dx_1 = \frac{1}{2} (-1 + \sqrt{H^2 + r^2} + \sqrt{(H-1)^2 + r^2}), \quad dx_2 = \frac{1}{2} (1 - \sqrt{H^2 + r^2} + \sqrt{(H-1)^2 + r^2}).$$

The toric scalar-flat metrics whose moment polytope has parallel unbounded edges are discussed by Weber in [67] and [69]. These metrics satisfy the Guillemin boundary condition. Let ξ_{Web} be the Legendre coordinate of the momentum coordinate of the metric for the given polytope, and ξ_{prod} be the one for the model metric. Then we know from [68] Section 5 that $\det D\xi_{Web} = \det D\xi_{prod} + O(\rho^{-2})$. For our case, from the formula in Equation (4.46), we have $\det D\xi = \det D\xi_{Web} + O(\rho^{-2})$, thus conclude the asymptotic behavior of the constructed metrics. $\square$

At the end of this section, we remark that for a toric scalar-flat Kähler metric whose symplectic potential lives in $S_{A,B}(P, F)$, the A, B are **uniquely determined** by the polytope. We recall the following theorem in [8]:

Theorem 4.2.3. ([8], Proposition 4.20) Consider $u \in S_{A,B}(P, L, F)$ satisfies

$$-\sum_{i,j} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} = s_{(P,L,F)}$$

for some extremal affine linear function $s_{(P,L,F)}$ of (P, L, F) . Then A, B are determined by the data (P, L, F) .

Following the proof of this theorem, an explicit expression of A and B are obtained in Equations (46) and (48). For our case where $s_{(P,L,F)} = 0$, we see A, B are expressed in terms of functions on F , thus for non-compact P , they are still determined by the data (P, L, F) . Hence, for a fixed polytope P , if the symplectic potential of the toric scalar-flat Kähler metric lives in $S_{A,B}$, then the choices of A, B must agree with those in Theorem 4.2.1. In the Appendix 4.4, we will discuss a uniqueness result under this prescribed class of symplectic potential.

4.3 A conical family of toric metrics

Given a cone angle $2\pi\theta_0$ for $\theta_0 \in (0, 1)$, motivated by the conical family in [34] Remark 1.2, we consider the following boundary behavior of potential u of a toric metric which has conical singularity along the divisor corresponding to the edge $L(x) = 0$ on its moment polytope:

$$u(x) = \frac{1}{2\theta_0} \cdot \ell(x) \log \ell(x) + h_0(x) \quad (4.47)$$

for some smooth function h_0 . Consider the ALE scalar-flat Kähler metric of Poincaré type constructed in Theorem 4.2.1 (i.e., $\nu = 0$), denote its moment polytope by $P \setminus L_I$. Let u be its symplectic potential, we write

$$u = \frac{1}{2} \sum_{i \notin I} \ell_i \log \ell_i + \frac{1}{2} \sum_{i \in I} (\alpha_i + \beta_i \ell_i) \log \ell_i + h$$

for some $h \in C^\infty(P)$. Here $\alpha_i = -\frac{1}{2}\Lambda_i, \beta_i = \frac{1}{2}\det(\nu_{i+1}, \nu_{i-1})$ for $i \in I$. Consider the smooth scalar-flat Kähler metric on P constructed by Abreu and Sena-Dias in [4], let u_{AS} be the symplectic potential, we can write

$$u_{AS} = \frac{1}{2} \sum_{i=1}^{d-1} \ell_i \log \ell_i + h_{AS}$$

for some $h_{AS} \in C^\infty(P)$. For any $\theta = (\theta_{i_1}, \dots, \theta_{i_m})$, consider

$$u^{(\theta)} := \frac{1}{2} \sum_{i \notin I} \ell_i \log \ell_i + \frac{1}{2} \sum_{i \in I} \frac{1}{\theta_i} \cdot \ell_i \log \ell_i + v_\theta + \prod_{i \in I} (1 - \theta_i) \cdot h + \prod_{i \in I} \theta_i \cdot h_{AS}. \quad (4.48)$$

Here $v_\theta = 0$ for $\theta = 1$ and for $\theta \in (0, 1)$,

$$v_\theta := \sum_{i \in I} [(\beta_i - \frac{1}{2\theta_i})\ell_i \log(\ell_i + \frac{\theta_i}{1 - \theta_i}) + \alpha_i \log(\ell_i + \frac{\theta_i}{1 - \theta_i}) + (\beta_i - \frac{1}{2})\ell_i \log \frac{1}{1 - \theta_i} - \alpha_i \log \frac{1}{1 - \theta_i}].$$

Then $u^{(\theta)} \rightarrow u$ as $\theta_i \rightarrow 0$ for all i and $u^{(\theta)} \rightarrow u_{AS}$ as $\theta_i \rightarrow 1$ for all i . This family of conical metrics, however, are not necessarily scalar-flat.

In the remaining part of this section, we will follow the framework of Abreu and Sena-Dias to explicitly construct toric conical scalar-flat Kähler metrics:

Theorem 4.3.1. *Given X satisfying the same conditions as in Theorem 4.2.1 and let P be the moment polytope for X . Consider an index set $J = \{j_1, \dots, j_m\} \subset \{1, 2, \dots, d\}$ with $1 \leq j_1 < \dots < j_m \leq d$. For each $j_i \in J$, fix a cone angle $2\pi\theta_{j_i}$, we write $\theta = (\theta_{j_1}, \dots, \theta_{j_m})$. For the normal vector $\nu_i = (\alpha_i, \beta_i)$, consider*

$$\alpha'_i := \alpha_i \text{ if } i \notin J, \alpha'_i := \frac{\alpha_i}{\theta_i} \text{ if } i \in J; \quad \beta'_i := \beta_i \text{ if } i \notin J, \beta'_i := \frac{\beta_i}{\theta_i} \text{ if } i \in J. \quad (4.49)$$

Let $\ell_J = \bigcup_{i=1}^m \ell_{j_i}$ be the union of edges ℓ_{j_i} indexed by the elements in J , here ℓ_{j_i} corresponds to $D_i, \forall i = 1, \dots, m$. Consider $\nu = (\alpha, \beta)$ a vector in $\mathbb{R}^2$ satisfying (4.9). Set

$$\begin{aligned} \xi_1 &:= \alpha'_1 \log r + \frac{1}{2} \sum_{i=1}^{d-1} (\alpha'_{i+1} - \alpha'_i) \log(H + a_i^{(\theta)} + \sqrt{(H + a_i^{(\theta)})^2 + r^2}) + \alpha H, \\ \xi_2 &:= \beta'_1 \log r + \frac{1}{2} \sum_{i=1}^{d-1} (\beta'_{i+1} - \beta'_i) \log(H + a_i^{(\theta)} + \sqrt{(H + a_i^{(\theta)})^2 + r^2}) + \beta H, \end{aligned}$$

where $a_1^{(\theta)}, \dots, a_{d-1}^{(\theta)}$ are real numbers determined by P and θ . Let x_1, x_2 be the primitives of

$$\epsilon_1 = r\left(\frac{\partial \xi_2}{\partial r} dH - \frac{\partial \xi_2}{\partial H} dr\right), \quad \epsilon_2 = -r\left(\frac{\partial \xi_1}{\partial r} dH - \frac{\partial \xi_1}{\partial H} dr\right).$$

Then they define the momentum action coordinates on P° of some conical toric scalar-flat Kähler metric on X whose cone angle along D_i is $2\pi\theta_{j_i}$. Its symplectic potential satisfies

$$du = \xi_1 dx_1 + \xi_2 dx_2.$$

Proof. Firstly, the positivity of $\det D\xi$ can be proved with the same arguments as in Theorem 4.2.1, it is because (α'_i, β'_i) either equals to (α_i, β_i) , or is rescaled by a positive constant $\frac{1}{\theta_i}$ from (α_i, β_i) along edges corresponding to conical divisors. Next, to check the boundary behavior of u , we note for $-a_j^{(\theta)} < H < -a_{j-1}^{(\theta)}$, we have

$$\xi_1 = \alpha'_1 \log r + \sum_{i=1}^{j-2} (\alpha'_{i+1} - \alpha'_i) \log r = \alpha'_j \log r + O(1), \quad \xi_2 = \beta'_j \log r + O(1),$$

which gives

$$du = \log r (\alpha'_j dx_1 + \beta'_j dx_2) + O(1).$$

We claim that we still have $r = (\delta \prod_{i=1}^d \ell_i)^{1/2}$ for some smooth and positive function δ . The reason is that with a similar analysis as in Lemma 4.2.1.3, we obtain

$$\frac{\partial x}{\partial r} = -\frac{r}{2} \sum_{i=1}^{d-1} \frac{\beta'_{i+1} - \beta'_i}{\rho_i^{(\theta)}}, \quad \frac{\partial \ell_j}{\partial r} = \frac{r}{2} \sum_{i=1}^{d-1} \frac{\det(\nu'_{i+1} - \nu'_i, \nu_j)}{\rho_i^{(\theta)}},$$

here $\nu'_i := (\alpha'_i, \beta'_i)$, $H_i^{(\theta)} := H + a_i^{(\theta)}$ and $\rho_i^{(\theta)} := \sqrt{(H_i^{(\theta)})^2 + r^2}$. Again, since α'_i, β'_i are rescaled from α_i, β_i , we still have

$$\frac{r}{2} \left(-\frac{\det(\nu'_1, \nu_j)}{\rho_1^{(\theta)}} + \sum_{i=2}^{d-1} \det(\nu'_i, \nu_j) \left(\frac{1}{\rho_{i-1}^{(\theta)}} - \frac{1}{\rho_i^{(\theta)}} \right) + \frac{\det(\nu'_d, \nu_j)}{\rho_{d-1}^{(\theta)}} \right) > 0,$$

and this proves the claim. Thus, $du = \log \ell_j (\alpha'_j dx_1 + \beta'_j dx_2) + O(1)$, giving the desired boundary behavior.

Lemma 4.3.1.1. *Given $j \in J$,*

$$a_j^{(\theta)} - a_{j-1}^{(\theta)} = \theta_j \cdot \frac{|L_j|}{2\pi |\nu_j|^2}. \quad (4.50)$$

Proof. Note

$$\omega|_{L_j} = r \frac{\partial \xi_2}{\partial r} dH \wedge d\theta_1 - r \frac{\partial \xi_1}{\partial r} dH \wedge d\theta_2,$$

then from $\xi = \frac{1}{\theta_j} \nu_j \log r + O(1)$ on L_j , we get $\omega|_{L_j} = \frac{|\nu_j|^2}{\theta_j} dH \wedge dt$. Then

$$L_j = \int_{L_j} \omega = 2\pi (a_j^{(\theta)} - a_{j-1}^{(\theta)}) \cdot \frac{|\nu_j|^2}{\theta_j},$$

concluding (4.50). □

Also we know for $j \notin J$, $a_j^{(\theta)} - a_{j-1}^{(\theta)} = a'_j - a'_{j-1}$. Then, we obtain the relation between $a_j^{(\theta)}$ and a'_j :

$$\begin{aligned}
a_j^{(\theta)} &= a'_j \text{ if } j < j_1; \quad a_j^{(\theta)} = a'_j + \sum_{\ell=1}^k (1 - \theta_{j_\ell})(a'_{j_{\ell-1}} - a'_{j_\ell}) \text{ if } j_k \leq j < j_{k+1}; \\
&\text{and } a_j^{(\theta)} = a'_j + \sum_{\ell=1}^m (1 - \theta_{j_\ell})(a'_{j_{\ell-1}} - a'_{j_\ell}) \text{ if } j_m \leq j \leq d. \quad (4.51)
\end{aligned}$$

From the expression of ξ_1, ξ_2 we deduce the expression of x_1, x_2 as follows:

$$x_1 = \beta'_1 H + \frac{1}{2} \sum_{i=1}^{d-1} (\beta'_{i+1} - \beta'_i)(H_i^{(\theta)} - \rho_i^{(\theta)}), \quad x_2 = -\alpha'_1 H - \frac{1}{2} \sum_{i=1}^{d-1} (\alpha'_{i+1} - \alpha'_i)(H_i^{(\theta)} - \rho_i^{(\theta)}).$$

When $r = 0$, we have

$$x_1 = \beta'_1 H + \frac{1}{2} \sum_{i=1}^{d-1} (\beta'_{i+1} - \beta'_i)(H_i^{(\theta)} - |H_i^{(\theta)}|), \quad x_2 = -\alpha'_1 H - \frac{1}{2} \sum_{i=1}^{d-1} (\alpha'_{i+1} - \alpha'_i)(H_i^{(\theta)} - |H_i^{(\theta)}|).$$

Equivalently, we know

- (i) If $-a_1^{(\theta)} < H$, then $x_1 = \beta'_1 H$, $x_2 = -\alpha'_1 H$;
- (ii) if $-a_{j+1}^{(\theta)} < H < -a_j^{(\theta)}$, then $x_1 = \beta'_{j+1} H + \sum_{i=1}^j a_i^{(\theta)}(\beta'_{i+1} - \beta'_i)$, $x_2 = -\alpha'_{i+1} H - \sum_{i=1}^j a_i^{(\theta)}(\alpha'_{i+1} - \alpha'_i)$;
- (iii) if $H < -a_{d-1}^{(\theta)}$, then $x_1 = \beta'_d H - \sum_{i=1}^{d-1} a_i^{(\theta)}(\beta'_{i+1} - \beta'_i)$, $x_2 = -\alpha'_d H - \sum_{i=1}^{d-1} a_i^{(\theta)}(\alpha'_{i+1} - \alpha'_i)$.

We want to show $x' = (x'_1, x'_2)$, when restricted to $(H, 0)$, defines a global proper homeomorphism. Again, we first focus on its behavior along L_j for $j_1 \leq j < j_2$. For simplicity, we write $k = j_1$, then

$$\begin{aligned}
x_1 &= \beta'_{k-1} H + \frac{1}{2}(\beta'_k - \beta'_{k-1})(H_{k-1}^{(\theta)} - \rho_{k-1}^{(\theta)}) + \sum_{i=1}^{k-2} a_i^{(\theta)}(\beta'_{i+1} - \beta'_i) + O(r^2), \\
x_2 &= -\alpha'_{k-1} H - \frac{1}{2}(\alpha'_k - \alpha'_{k-1})(H_{k-1}^{(\theta)} - \rho_{k-1}^{(\theta)}) - \sum_{i=1}^{k-2} a_i^{(\theta)}(\alpha'_{i+1} - \alpha'_i) + O(r^2).
\end{aligned}$$

From (4.51), we know $\ell_k(x) = 0$ holds. For $k < j < j_2$, note $\ell_j(x) = 0$ is equivalent to

$$\sum_{i=2}^{j-1} \det(\nu_i, \nu_j)(a_{i-1}^{(\theta)} - a_i^{(\theta)}) - a_1^{(\theta)} \det(\nu_1, \nu_j) = \sum_{i=2}^{j-1} \det(\nu'_i, \nu_j)(a'_{i-1} - a'_i) - a'_1 \det(\nu_1, \nu_j).$$

From (4.51), direct computation shows the above equation holds. With essentially the same arguments applied to $j_i \leq j < j_{i+1}$ for $i \geq 2$, we see $\ell_j(x) = 0$ holds. Hence x defines a global proper homeomorphism as desired. $\square$

For the asymptotic behavior of the conical metrics, note as $\rho \rightarrow \infty$,

$$r \det D\xi = \frac{\det(\nu'_d, \nu'_1)}{2\rho} + O\left(\frac{1}{\rho^2}\right), \text{ for } \alpha = \beta = 0;$$

$$r \det D\xi = \det(\nu, \nu'_1)\left(1 - \frac{r^2}{2\rho(H + \rho)}\right) + \det(\nu, \nu'_d)\frac{r^2}{2\rho(H + \rho)} + \frac{\det(\nu'_d, \nu'_1)}{2\rho} + O\left(\frac{1}{\rho^2}\right), \text{ for } (\alpha, \beta) \neq (0, 0).$$

The arguments in Theorem 4.2.2 still work here, and hence the asymptotic behavior of conical metrics coincide with those for the cuspidal metrics with a given choice of (α, β) .

4.4 Appendix: Uniqueness of toric metrics under given boundary conditions

Theorem 4.4.1. *Consider the same setting as in Theorem 4.2.1. Assume g is a toric scalar-flat Kähler metric of Poincaré type and its symplectic potential u satisfies the prescribed boundary behavior given in (4.11), then g can only be one of the metrics constructed in Theorem 4.2.1.*

Proof. The proof closely follows the arguments of Sena-Dias in [57], with the essential differences in Claim 4.4.1.3, Lemma 4.4.1.2 and Lemma 4.4.1.3.

Starting from a scalar-flat Kähler metric on a symplectic 4-manifold, Donaldson shows in [26] Theorem 1 that each solution of the scalar-flat Kähler equation locally arises from axisymmetric harmonic functions ξ_1, ξ_2 in the way described in Theorem 4.1.1. Here ξ_1, ξ_2 are unique up to translation in the H variable and addition of constants. More precisely, given $u(x)$ the symplectic potential of a scalar-flat Kähler metric, we consider (H, r) satisfying

$$r = (\det \text{Hess } u)^{-1/2}, \quad \frac{\partial H}{\partial x_1} = -\frac{u^{2j}}{r} \frac{\partial r}{\partial x_j}, \quad \frac{\partial H}{\partial x_2} = \frac{u^{1j}}{r} \frac{\partial r}{\partial x_j}. \quad (4.52)$$

Then for the moment P° endowed with the metric $g_{poly} = u_{ij} \sum_{i,j=1}^2 dx_i \otimes dx_j$, r is harmonic and H is its harmonic conjugate. The metric g_{poly} induces a complex structure J_{poly} via its Hodge star. We obtain a J_{poly} -holomorphic local coordinate on P° , written as

$$z := H + ir.$$

The coordinates (H, r) , as functions of x_1, x_2 , are known as the isothermal coordinates. Note the boundary behavior of u is determined on ∂P . Then r extends continuously to $\partial P \setminus L_I$, as does H . Thus, z extends to $\partial P \setminus L_I$ as a continuous function, denoted by $\tilde{z}$. Note $r = 0$ on $\partial P \setminus L_I$, then $\tilde{z}(\partial P \setminus L_I) \subset \partial \mathbb{H} \setminus \bigcup_{k \in I} (-a_k, 0)$. From the boundary behavior of the metric we know $\tilde{z}$ is a bijection from $\partial P \setminus L$ to $\partial \mathbb{H} \setminus \bigcup_{k \in I} (-a_k, 0)$.

Lemma 4.4.1.1. *The map $z : P^\circ \rightarrow \mathbb{H}$ is a bijection.*

Proof. The proof relies on the real sub-manifold associated with the symplectic 4-manifold X . From the discussions in [42] Theorem 6.7 and [57], we know since the moment map is proper, X is symplectomorphic to the quotient of some complex plane $\mathbb{C}^d$ by a sub-torus of the standard torus, with d being the number of edges of the moment map of X . Note complex conjugation descends to a function on X and D . We denote its fixed point set (which are real submanifolds) of X and D by $X_{\mathbb{R}}$ and $D_{\mathbb{R}}$, respectively. The moment map

$$\phi : X \setminus D \rightarrow \mathbb{R}^2 \setminus \bigcup_{k \in I} (-a_k, 0),$$

when restricted to $X_{\mathbb{R}} \setminus D_{\mathbb{R}}$ is denoted by $\phi_{\mathbb{R}}$. It is a 4 to 1 branched cover with the branched set being $\phi^{-1}(\partial P \setminus L_I)$ and write $\phi_{\mathbb{R}}^{-1}(P^\circ) = \bigcup_{j=0}^3 P_j$ as a disjoint union of the open sets P_j . Let $g_{\mathbb{R}}$ be the induced metric on $X_{\mathbb{R}} \setminus D_{\mathbb{R}}$ and P_0 , then $g_{poly} := \phi_{\mathbb{R}*}(g_{\mathbb{R}}) = u_{ij} dx_i \otimes dx_j$. Let

$$(X_{\mathbb{R}} \setminus D_{\mathbb{R}})^\circ \rightarrow X_{\mathbb{R}} \setminus D_{\mathbb{R}}$$

be the orientable double cover of $X_{\mathbb{R}} \setminus D_{\mathbb{R}}$ and $\phi_{\mathbb{R}}^\circ$ be the lifting of $\phi_{\mathbb{R}}$ to $(X_{\mathbb{R}} \setminus D_{\mathbb{R}})^\circ$. For each P_j , $k = 0, 1, 2$, we write P_j^0 and P_j^1 as the pre-image under the double cover. Via the Hodge star operator, we obtain from the metric induced by $g_{\mathbb{R}}$ on $(X_{\mathbb{R}} \setminus D_{\mathbb{R}})^\circ$ a complex structure $J_{\mathbb{R}}$, whose pushforward under $\phi_{\mathbb{R}}^\circ$ defines a complex structure J_{poly} on $P \setminus L_I$.

Claim 4.4.1.1. *Given $w \in \partial P \setminus L_I$, and p an element in $(\phi_{\mathbb{R}}^\circ)^{-1}(w) \subset (X_{\mathbb{R}} \setminus D_{\mathbb{R}})^\circ$, then there is a neighbourhood V_p of p in $(X_{\mathbb{R}} \setminus D_{\mathbb{R}})^\circ$ such that $z \circ \phi_{\mathbb{R}}^\circ$ extends to V_p as a holomorphic function for $J_{\mathbb{R}}$.*

The proof is a lifting and extension argument for the harmonic function r and the harmonic conjugate H on $X \setminus D$, and this argument is essentially the same as in [57] Lemma 6.2.

Now, we show the injectivity. For the holomorphic map z , we consider its degree; it suffices to show the degree is 1. If we consider a point $w_0 \in \partial \mathbb{H} \setminus \bigcup_{k \in I} (-a_k, 0)$ and let $w \in \partial P \setminus L_I$ be its pre-image of $\tilde{z}$. Fix p in the closure of $P_0^0 \cap P_1^0$. From Claim 4.4.1.1, there exists an extension of z to V_p . Assume V_p is small enough to admit a complex chart $z : V_p \rightarrow \mathbb{C}$. Then following the arguments of the proof of injectivity in [57] Section 6.1, we

know for any $\epsilon > 0$ such that $B_\epsilon \cap P \subset \phi_{\mathbb{R}}^\circ(V_p)$, there exists δ such that $z^{-1}(B_\delta(w_0)) \subset B_\epsilon(w)$; furthermore, given a point $w'_0 \in B_\delta(w_0) \cap \mathbb{H}$, we can enlarge the loop γ enclosing all pre-images of z so that it also encloses w , then as w'_0 tends to w_0 . The number of pre-images of w'_0 given by the integral

$$\int_{\gamma} \frac{\frac{dz}{ds}(s)ds}{z(s) - w'_0}$$

equals to that of w_0 , which is 1.

Then we show the surjectivity. The different boundary behavior for the symplectic potential in our case compared to that in [57] doesn't cause any essential difference to the arguments of the proof. Note P is non-compact and admits non-trivial harmonic functions, from the uniformization theorem, we know there exists a holomorphic map $\kappa : P^\circ \rightarrow \mathbb{H}$.

Claim 4.4.1.2. *κ extends as a homeomorphism to $P \setminus L_I \rightarrow \overline{\mathbb{H}} \setminus \bigcup_{k \in I} (-a_k, 0)$, and it's a bijection.*

Proof. First, we show the map is extendable. The different boundary behavior for the symplectic potential in our case compared to that in [57] doesn't cause any essential difference to the arguments of the proof, for details we refer the readers to Lemma 5.4.

To see the extension is bijective, we argue by contradiction. Assume it's not injective, let $w, v \in \partial P \setminus L_I$ such that $\tilde{\kappa}(w) = \tilde{\kappa}(v)$. Take $o \in P^\circ$ and consider the Jordan curve going through $\kappa(o)$ and $\tilde{\kappa}(w) = \tilde{\kappa}(v)$. Let $\mathcal{C}$ be the interior of this Jordan curve, then the segment $\mathcal{S}$ joining w and v satisfies $\tilde{\kappa}(\mathcal{S}) \subset \partial A \cap \partial \mathbb{H}$, and thus $\tilde{\kappa}$ is constant on $\mathcal{S}$. In particular, we see $\mathcal{S}$ doesn't contain ℓ_k for any $k \in I$. Then, the same arguments as in [57] give a contradiction. Similarly, we can prove that the inverse is also injective. $\square$

We use this extension map as an auxiliary to show the surjectivity of z . Let $U := z(P^\circ)$, write

$$\partial U = (\partial U \cap \partial \mathbb{H}) \cup (\partial U \cap \mathbb{H}),$$

then the surjectivity of z is equivalent to $\partial U \cap \mathbb{H} = \emptyset$. Consider $z_\kappa := z \circ \kappa^{-1} : \mathbb{H} \rightarrow \mathbb{H}$, it is a holomorphic, injective map which can be extended bijectively to $\partial \mathbb{H} \setminus \bigcup_{k \in I} (-a_k, 0) \rightarrow \partial \mathbb{H} \setminus \bigcup_{k \in I} (-a_k, 0)$. Consider

$$f(w) := \frac{1}{z_\kappa(-\frac{1}{w})} : \mathbb{H} \rightarrow \mathbb{H},$$

as in [57], the same arguments show that it is holomorphic and can be extended to a holomorphic function on $\mathbb{C}^*$ with 0 being an isolated singularity using the Schwarz reflection principle. Rewrite $U = z_\kappa(\mathbb{H})$, we claim that

Claim 4.4.1.3. $\partial U \cap \mathbb{H} = \{\lim z_\kappa(w_{n_k}), (w_k) \text{ unbounded with } (w_{n_k}) \text{ being a subsequence}\}.$

Proof. For any point z_∞ in the above set, we take a sequence $w_k \in \mathbb{H}$ such that $z_\kappa(w_k) \rightarrow z_\infty$. If the sequence is bounded, there is a convergent subsequence w_{n_k} in $\overline{\mathbb{H}}$ converging to $w \in \overline{\mathbb{H}}$. We have the following possibilities:

- (i) If $w \in \mathbb{H}$, then $\tilde{z}_\kappa(w) = z_\kappa(w) \in U$ but since U is open, $z_\infty \notin \partial U$;
- (ii) if $w \in \partial\mathbb{H} \setminus \bigcup_{k \in I} (-a_k, 0)$, then $\tilde{z}_\kappa \in \partial\mathbb{H} \setminus \bigcup_{k \in I} (-a_k, 0)$, then $z_\infty \notin \mathbb{H}$;
- (iii) if $w = (-a_k, 0)$ for some $k \in I$, consider $\kappa^{-1}(w_{n_k}) := p_{n_k}$, we have $z(p_{n_k}) \rightarrow z_\infty \in \mathbb{H}$. From $z : P \setminus L_I \rightarrow \overline{\mathbb{H}} \setminus \bigcup_{k \in I} (-a_k, 0)$ is bijective we know $p_{n_k} \rightarrow z^{-1}(z_\infty) \in P \setminus L_I$ and from $\kappa : P \setminus L_I \rightarrow \overline{\mathbb{H}} \setminus \bigcup_{k \in I} (-a_k, 0)$ is bijective we know $\kappa(p_{n_k}) \rightarrow z_\kappa^{-1}(z_\infty) \in \overline{\mathbb{H}} \setminus \bigcup_{k \in I} (-a_k, 0)$. This implies w_{n_k} tends to an element in $\overline{\mathbb{H}} \setminus \bigcup_{k \in I} (-a_k, 0)$, a contradiction.

□

Equivalently we know

$$\partial U \cap \mathbb{H} = \{\lim f(w_{n_k}), w_k \in \mathbb{H}, w_k \rightarrow 0, (w_{n_k}) \text{ a subsequence}\} := f(0).$$

Now, this set contains either ∞ or a single point, which is a pole. If the latter holds, we know z_κ^{-1} is a holomorphic function with an isolated pole, but the image can not lie in $\mathbb{H}$. Hence we conclude that $\partial U \cap \mathbb{H} = \emptyset$. This concludes the proof. □

Now we know z is a bijection, define $\mu := z^{-1}$, and let $\mu_{ALE} := z_{ALE}^{-1}$ be the corresponding map for the ALE metric g_{ALE} (i.e., the case where $\nu = 0$) constructed in Theorem 4.2.1.

Lemma 4.4.1.2. *Consider $\mu_0 := \mu - \mu_{ALE}$, we have $\mu_0 = r^2 f$ for some $f \in C^\infty(\overline{\mathbb{H}} \setminus \bigcup_{k \in I} (-a_k, 0))$, and f satisfies*

$$f_{HH} + f_{rr} + \frac{3f_r}{r} = 0.$$

Proof. Consider

$$\eta := (u_{x_1}, u_{x_2}), \quad \eta_{ALE} := (u_{ALE, x_1}, u_{ALE, x_2}),$$

we write

$$\xi(H, r) = \eta \circ \mu(H, r), \quad \xi_0(H, r) = \eta \circ \mu_0(H, r).$$

To show μ_0 extends as an analytic function to $\overline{\mathbb{H}} \setminus \bigcup_{k \in I} (-a_k, 0)$, it's equivalent to show ξ_0 extends as an analytic function on $\overline{\mathbb{H}} \setminus \bigcup_{k \in I} (-a_k, 0)$. Since ξ_0 is an axi-symmetric harmonic function on $\mathbb{H}$, from the mean value theorem, it is sufficient to show ξ_0 is bounded in a neighborhood of each point on $\partial\mathbb{H} \setminus \bigcup_{k \in I} (-a_k, 0)$. Write

$$\xi_0 = \eta_{ALE} \circ \mu_{ALE} - \eta \circ \mu = (\eta_{ALE} \circ \mu_{ALE} - \eta_{ALE} \circ \mu) + (\eta_{ALE} \circ \mu - \eta \circ \mu).$$

Note $\eta_{ALE} - \eta \in C^\infty$, and μ extends as a continuous function on $\overline{\mathbb{H}} \setminus \bigcup_{k \in I} (-a_k, 0)$, we know the second term is bounded. For the first term, rewrite it as $\xi_{ALE} - \xi_{ALE} \circ (\mu_{ALE}^{-1} \circ \mu)$. Near $\partial\mathbb{H} \setminus \bigcup_{k \in I} (-a_k, 0)$, there is a singularity of ξ_{ALE} with

$$\xi_{ALE} = \nu_i \log r + O(1) \text{ on each interval } -a_{i+1} < H < -a_i \text{ and } r = 0 \text{ given } i \notin I.$$

It's equivalent to show as $r \rightarrow 0$, $\log \frac{r}{r(\mu_{ALE}^{-1} \circ \mu)}$ is bounded. Composing with μ^{-1} it suffices to show

- (i) For the conjugate harmonic coordinate H' for z , for $i+1 \notin I$, $-a_{i+1} < H < -a_i \iff -a_{i+1} < H < -a_i$, and for $i+1 \in I$, $H = -a_{i+1} \iff H' = -a_{i+1}$;
- (ii) $\log \frac{r \circ \mu^{-1}}{r \circ \mu_{ALE}^{-1}}$ is bounded as r approaches 0.

The first claim follows from [57] Theorem 6.2 and our choice of coefficients (4.27). For the second claim, recall $r \circ \mu^{-1} = \det(\text{Hess } u \circ \mu^{-1})^{-1/2}$, then it's equivalent to show that

$$\frac{(\det \text{Hess } u \circ \mu^{-1})^{-1/2}}{(\det \text{Hess } u_{ALE} \circ \mu^{-1})^{-1/2}}$$

is bounded. This follows from the assumed boundary behavior of u and u_{ALE} . Direct calculation gives $f_{HH} + f_{rr} + \frac{3f_r}{r} = 0$, for details see [57] Lemma 6.3. $\square$

Lemma 4.4.1.3. *For the normal vector $\nu_1 = (1, 0)$, consider $f_1 := f \cdot \nu_1$, then f_1 is a constant.*

Proof. Since f_1 is harmonic, it suffices to show it is bounded. From $f = \frac{\mu_0}{r^2}$ and $\nu \cdot \nu_1 \geq 0$ we deduce that $f_1 \leq \frac{\mu_{ALE} \cdot \nu_1}{r^2}$. From the expression of x_1, x_2 in Theorem 4.2.1, we know

$$|\mu_{ALE}(H, r)| \leq C\sqrt{H^2 + r^2 + 1}.$$

As in [57], we view f as a harmonic function in $\mathbb{R}^5$, which only depends on the coordinate H and the distance to the H -axis, r . Then $\forall w \in \mathbb{R}^5$,

$$f_1(w) \leq \frac{C(|w| + 1)}{r^2}.$$

Let $\mathcal{A}$ denote the subset of $\mathbb{R}^5$ whose H coordinate does not take values $-a_k$ for any $k \in I$ when the distance r vanishes. For a fixed $z \in \mathcal{A}$, there exists $R_z > 0$ large enough such that $\partial B(z, R_z) \subset \mathcal{A}$, and with the mean value theorem, we get $f_1(z) \lesssim \frac{1}{R^3} \int_{\partial B(z, R)} \frac{dw}{r^2}$. Then from [57] Lemma 6.4, for R large enough, direct calculations show $\int_{\partial B(z, R)} \frac{dw}{r^2} \lesssim R^2$, thus f_1 is bounded from above. $\square$

Similarly, for the normal vector ν_2 , we can define $f_2 := f \cdot \nu_1$, then f_2 is a constant. Thus f is a constant, which implies $\xi_0 = H \cdot \nu$ for some constant vector. Hence when $\nu = 0$, g is the ALE metric constructed in Theorem 4.2.1 and otherwise g is the generalized or exceptional Taub-NUT ones constructed there.

□

Chapter 5

Construction of the neck region

The degeneration of complex structures arises in the study of compactification of moduli space of special metrics. Given $\pi : \mathcal{X} \rightarrow \mathbb{D}$ a flat family of algebraic varieties such that $X_t = \pi^{-1}(t)$ is a smooth manifold for $t \neq 0$ and $X_0 = \pi^{-1}(0)$ is singular. We are interested in the limiting geometric behavior under Gromov–Hausdorff convergence, in particular in the collapsing case.

The degeneration of polarized Calabi-Yau manifolds is studied in [59] and the degeneration of negative Kähler-Einstein metrics is studied in [50]. One essential step in the study of the limiting collapsing geometry is to directly construct the degeneration region by gluing an appropriate neck region with carefully chosen ends to obtain an approximate solution.

We make progress on the construction of the neck region in this chapter. By imposing an S^1 -invariant structure on the neck region, we use LeBrun ansatz 2.2 for scalar-flat Kähler metrics to approach the construction. We allow the S^1 -action to have fixed points, and thus are led to solve a distributional scalar-flat equation. More concretely, we add delta functions supported at isolated points to obtain the inhomogeneous equation [Section 5.1]. The degrees of the restrictions of the S^1 -bundle over D at the two ends are then expected to differ by the number of chosen points. For a detailed analysis of the geometries at the singular region and end region, see Section 5.2. We further extend the metric over the isolated singular points by constructing a compactification where the metric extend with some regularity [Section 5.3].

5.1 Distributional equation on $S^2 \times I$

Let $D = S^2$ and g_D be the round metric. We construct the neck region starting from the pair (u_0, w_0) corresponding to the cylinder $(D \times I, g_D \oplus dz^2)$. Explicitly, consider

$$u_0 := \ln \frac{z^2 + T^{-2}}{\left(1 + \frac{x^2 + y^2}{4}\right)^2},$$

$$w_0 := \frac{1}{z^2 + T^{-2}},$$

satisfying Equation 2.2 and Equation 2.1 respectively. Then with this fixed $u := u_0$, the equation involving w becomes

$$(1 + \frac{x^2 + y^2}{4})^2 (\partial_x^2 w + \partial_y^2 w) + \partial_z^2 ((z^2 + T^{-2})w) = 0.$$

We consider around a singular point $p \in D \times I$ the distributional equation

$$(1 + \frac{x^2 + y^2}{4})^2 (\partial_x^2 w + \partial_y^2 w) + \partial_z^2 ((z^2 + T^{-2})w) = -2\pi\delta_p. \quad (5.1)$$

A solution to this equation gives a scalar-flat metric on complex surface with an S^1 -action having a fixed point at p . Now it remains to solve this equation.

Solving the distributional equation

Using separation of variable, we expand w in the eigenfunctions of the Laplace-Beltrami operator $\Delta_{\mathbb{CP}^1}$. Note $\Delta_{\mathbb{CP}^1}$ has the following expression

$$\Delta_{\mathbb{CP}^1} = (1 + \frac{x^2 + y^2}{4})^2 (\partial_x^2 + \partial_y^2).$$

Then Equation 5.1 becomes

$$(z^2 + T^{-2})\partial_z^2 w + 4z\partial_z w + 2w + \Delta_{\mathbb{CP}^1} w = -2\pi\delta_p$$

Recall the Laplace-Beltrami operator on a compact Riemann surface has discrete spectrum, we take eigenfunctions ψ_λ of $\Delta_{\mathbb{CP}^1}$ with eigenvalue $-\lambda^2$, we normalization so that $\psi(p_D) \geq 0$ and $\int_D |\psi_\lambda|^2 \omega_D = 1$. We then have the expansion

$$w = \sum_{\lambda \geq 0} f_\lambda \psi_\lambda, \quad \delta_p = \sum_{\lambda \geq 0} \psi_\lambda(p_D) \psi_\lambda \delta_p.$$

For each given eigenvalue λ , we have

$$(z^2 + T^{-2})f_\lambda''(z) + 4zf_\lambda'(z) + (2 - \lambda^2)f_\lambda(z) = -2\pi\psi_\lambda(p)\delta_p(z). \quad (5.2)$$

Let $x = \frac{1+iTz}{2}$, we obtain the following ODE:

$$x(x-1)f_\lambda''(x) + 2(2x-1)f_\lambda'(x) + (2 - \lambda^2)f_\lambda(x) = -\pi T\psi_\lambda(p)\delta_p(x). \quad (5.3)$$

From the theory of hypergeometric functions we know

$$v_1^\lambda := {}_2F_1(\alpha, \beta, \gamma, x), \quad v_2^\lambda = {}_2F_1(\alpha, \beta, \gamma, \bar{x})$$

are solutions to the homogeneous equation, with

$$\alpha = \frac{3 + \sqrt{1 + 4\lambda^2}}{2}, \quad \beta = \frac{3 - \sqrt{1 + 4\lambda^2}}{2}, \quad \gamma = 2.$$

Here ${}_2F_1(\alpha, \beta, \gamma, x)$ is called the hypergeometric series. It is a convergent series defined for $\alpha, \beta \in \mathbb{C}$, $\gamma \in \mathbb{C} \setminus \mathbb{Z}_-$ and $|x| < 1$ as follows

$${}_2F_1(\alpha, \beta, \gamma, x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{k! (\gamma)_k} x^k.$$

Let

$$\Sigma := \{\lambda > 0 : \lambda \text{ is an eigenvalue of } \Delta_{\mathbb{CP}^1} \text{ and } \sqrt{1 + 4\lambda^2} \in \mathbb{Z}\}.$$

We see if $\lambda \notin \Sigma$, then v_1^λ, v_2^λ give independent solutions to the equation. Note Σ is discrete. To obtain a solution to our non-homogeneous equation, we use the method of variation of parameters. Let

$$b(z) := T\pi\psi_\lambda(p_D)\delta_0(z).$$

We use v^λ to denote the non-homogeneous solution to the ODE, we write

$$v^\lambda = \sum_{i=1}^2 c_i v_i^\lambda.$$

Let $W[v_1^\lambda, v_2^\lambda](z)$ be the Wronskian determinant of the basis v_1^λ, v_2^λ . Then

$$c_i = \int \frac{W_i(s)}{W(s)} ds,$$

where W_i is the Wronskian determinant of the basis with the i -th column replaced by $(0, b(z))$. Explicitly, we get

$$c_1(z) = \int_{-\infty}^z \frac{2T\pi\psi_\lambda\delta_0}{s^2 + 1} \frac{v_2^\lambda(s)}{W[v_1^\lambda, v_2^\lambda](s)} ds = T\pi i\psi_\lambda(p)\sigma_0(z) \frac{\gamma}{\alpha(\lambda)\beta(\lambda)} \frac{\Gamma(\frac{2+\alpha(\lambda)}{2})\Gamma(\frac{2+\beta(\lambda)}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{3+\alpha(\lambda)+\beta(\lambda)}{2})},$$

$$c_2(z) = - \int_{-\infty}^z \frac{2T\pi\psi_\lambda\delta_0}{s^2 + 1} \frac{v_1^\lambda(s)}{W[v_1^\lambda, v_2^\lambda](s)} ds = -T\pi i\psi_\lambda(p)\sigma_0(z) \frac{\gamma}{\alpha(\lambda)\beta(\lambda)} \frac{\Gamma(\frac{2+\alpha(\lambda)}{2})\Gamma(\frac{2+\beta(\lambda)}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{3+\alpha(\lambda)+\beta(\lambda)}{2})} = -c_1(z).$$

Hence

$$v^\lambda(z) = C_1\sigma_0(z)\text{Im}(v_1(z)),$$

where C_1 is a constant satisfying $C_1\sigma_0(z) = 2ic_1(z)$. We want to modify v^λ by adding a linear combination of homogeneous solutions so that f_λ has the desired growth behavior. Recall the following property of hypergeometric series from [6] Theorem 2.3.2:

$$\begin{aligned}
 F(\alpha, \beta, \gamma, x) &= \frac{\Gamma(\beta - \alpha)\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \alpha)}(-x)^{-\alpha}F(\alpha, \alpha + 1 - \gamma, \alpha + 1 - \beta, x^{-1}) \\
 &\quad + \frac{\Gamma(\alpha - \beta)\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)}(-x)^{-\beta}F(\beta, \beta + 1 - \gamma, \beta + 1 - \alpha, x^{-1}).
 \end{aligned}$$

Using this, we obtain the following estimate

Lemma 5.1.0.1. *For $\rho := -\tan \frac{\pi\beta}{2}$, we have*

$$\left(\sigma_0(z) - \frac{1}{2}\right) \operatorname{Im}(v_1^\lambda)(z) + \frac{\rho_1}{2} \operatorname{Re}(v_1^\lambda)(z) = \mathcal{O}((Tz)^{-\alpha}).$$

Define

$$f_\lambda(z) = \frac{C_1}{2} \left(-\operatorname{Im}(v_1^\lambda)(z) + \frac{\rho}{2} \operatorname{Re}(v_1^\lambda)(z) \right) + v^\lambda(z).$$

From the lemma we see

$$f_\lambda(z) = \mathcal{O}((Tz)^{-\alpha})$$

is a decaying solution to the inhomogeneous problem.

5.2 Analysis of the geometries

Decaying behavior near the end region

We first study the behavior of $f_\lambda(z)$, and then deduce the behavior of w near the end region.

First we look at $f_\lambda(z)$. We want to analyze its behavior around 0. First we note that v_1^λ, v_2^λ are solutions to Equation 5.3 away from 0.

Lemma 5.2.0.1. *Let f be a homogeneous solution to Equation 5.3, then f does not have negative local minimum nor positive local maximum on $(-\infty, 0) \cup (0, \infty)$.*

Proof. If f has a negative local minimum at z , then $f'(z) = 0, f(z) < 0, f''(z) \geq 0$, but it contradicts the equation since the coefficient before f is negative. For same reason f cannot admit positive local maximum. $\square$

Lemma 5.2.0.2.

$$f_\lambda \leq 0$$

More precisely, there exists $N > 0$ independent of T and λ such that as $\lambda \rightarrow \infty$,

$$|f_\lambda(0)| = T\mathcal{O}(\lambda^N).$$

Then we show away from the singularity, the behavior is controlled by the solution corresponding to $\lambda = 0$. More precisely,

Theorem 5.2.1. *Let $w = \sum_{\lambda \geq 0} f_\lambda^T \psi_0$ then there exists $\rho > 0$ such that $w = f_0^T \psi_0 + T\mathcal{O}((Tz)^{-2})$ on $|z| \geq \rho T^{-1}$ as $T \rightarrow \infty$.*

Proof. Given $\lambda > 0$, we have $\alpha(\lambda) > 2$, more precisely, given a fixed $\epsilon \in (0, 1)$,

$$\alpha(\lambda) > 2 + \epsilon\lambda, \quad \lambda \gg 0. \quad (5.4)$$

Define a new function

$$g_\lambda(z) := z^{\epsilon\lambda+2} \max_D |\psi_\lambda| f_\lambda(z),$$

under direct computation, we see Equation 5.2 for f_λ gives us the following equation for g_λ :

$$(1+z^2)z^2 g_\lambda''(z) - (2\epsilon\lambda z^3 + 2(\epsilon+2)z)g_\lambda'(z) + (\lambda^2(\epsilon^2 + z^2(\epsilon^2 - 1)) + \lambda(5\epsilon + z^2\epsilon) + 6)g_\lambda(z) = -2\pi\psi_\lambda\delta_p(z).$$

Assuming ϵ being sufficiently close to 1, we observe the coefficients for g_λ and g_λ'' in the above equation are positive. Assuming $\lambda \gg 0$, Lemma 5.2.0.2 tells us

$$g_\lambda(z) < 0.$$

Further, combining with 5.4, the decaying behavior of f_λ tells us $\lim_{z \rightarrow \pm\infty} |g_\lambda(z)| = 0$.

Note that the positivity of coefficients of g_λ and g_λ'' shows there does not exist a negative local maximum on $(0, \infty)$, which implies the existence of a constant $c_\lambda > 0$ such that on (c_λ, ∞) , $g_\lambda(z)$ is non-decreasing.

□

Now we generate the explicit expansion formula of w by giving the formula of f_0^λ . The solutions to the homogeneous equation are

$$v_1^0(x) = \frac{1}{x(1-x)}, \quad v_2^0(x) = \frac{1}{1-x}.$$

The Wronskian of this basis is $\frac{1}{x^2(1-x)^2}$ and $W_1(x) = \frac{1}{1-x}$, $W_2(x) = \frac{1}{x(1-x)}$. We switch to z coordinate, then a general solution to the homogeneous equation can be expressed as

$$\frac{T^2}{1 + (Tz)^2} (\mu_1 + \mu_2 z)$$

for $\mu_1, \mu_2 \in \mathbb{R}$. Apply variation of parameters, we obtain an inhomogeneous solution

$$v^0(z) = -2\pi\sigma_0(z)T^2\psi_0(p_D)\frac{z}{1 + (Tz)^2}.$$

From Theorem 5.2.1, we see for $|z| \geq \frac{\rho}{T}$,

$$w = T^2 \left(-2\pi\sigma_0\psi_0(p_D) \cdot \frac{z}{1 + (Tz)^2} + \mathcal{O}(T^{-3}z^{-2}) \right).$$

This w is a solution to the distributional Equation 5.1, and we can modify this solution by adding a constant multiple of w_0 . Recall our goal is to construct a SFK metric on the neck region which resembles Hwang-Singer model at the ends, we now show by adding an appropriate multiple of w_0 to w , we obtain a good candidate. Let

$$w_T := w_0 + w.$$

Here we introduce the subscript T to indicate the dependence of the solution w on T . Then for $|z| \geq \frac{\rho}{T}$,

$$w_T = T^2 \left(\frac{1}{1 + (Tz)^2} - 2\pi\sigma_0\psi_0(p_D) \cdot \frac{z}{1 + (Tz)^2} + \mathcal{O}(T^{-3}z^{-2}) \right).$$

We see that, for T sufficiently large, $w_T > 0$.

For $\psi_0 = \psi_0(z)dx \wedge dy$, $\Delta_D\psi_0 = 0$, $\int_D |\psi_0(z)|^2 \omega_D = 1$, from

$$\psi_0(p_D) = \int_{p_D} \star_D \psi_0 = \int_D \frac{1}{2\pi} \omega_D \wedge \star_D \psi_0 = \frac{1}{2\pi} \langle \omega_D, \psi_0 \rangle_{L^2(D)},$$

we see $2\pi\psi_0(p_D)\psi_0 = \omega_D$.

Fix an integer $k_- > 0$, we add a homogeneous solution $2\pi\psi_0(p_D)\frac{k_-z}{z^2+T^{-2}}$ to $v^0(z)$ and get an inhomogeneous solution

$$f_0(z) = -2\pi(\sigma_0(z) - k_-)\psi_0(p_D)\frac{z}{z^2 + T^{-2}}.$$

which implies the following

$$f_0(z)\psi_0 = \begin{cases} k_- \frac{z}{z^2+T^{-2}} \omega_D, & z < 0 \\ (k_- - 1) \frac{z}{z^2+T^{-2}} \omega_D, & z > 0 \end{cases} \quad (5.5)$$

$$w_T(z) = \begin{cases} \frac{1}{z^2+T^{-2}} + 2\pi k_- \frac{\psi_0(p_D)z}{z^2+T^{-2}} \psi_0(z) + \mathcal{O}(T^{-1}z^{-2}), & z < 0 \\ \frac{1}{z^2+T^{-2}} + 2\pi(k_- - 1) \frac{\psi_0(p_D)z}{z^2+T^{-2}} + \mathcal{O}(T^{-1}z^{-2}), & z > 0 \end{cases} \quad (5.6)$$

$$\tilde{\omega}_T = e^{u_0} w_T dx \wedge dy = \begin{cases} (1 + k_- z) \omega_D, & z < 0 \\ (1 + (k_- - 1)z) \omega_D, & z > 0 \end{cases} \quad (5.7)$$

Hence we see given T sufficiently large, when $z > \frac{C}{T}$, the metric is a small perturbation of the Hwang-Singer model on $\mathcal{O}(1 - k_-)$.

Remark 5.2.1. *It no longer makes sense to speak of the notion of momentum profile for our newly constructed metric: since w_T depends on x, y , the metric obtained from LeBrun ansatz no longer arises from the Calabi ansatz.*

Behavior near the singularity

In order to construct the neck region over the entire cylinder, we need to study the metric on S^1 -fibration around the singularity which corresponds to the fixed point of the S^1 -action.

Denote

$$\mathcal{D} := \left(1 + \frac{x^2 + y^2}{4}\right)^2 (\partial_x^2 + \partial_y^2) + \partial_z^2(z^2 + T^{-2}),$$

then Equation 2.1 becomes $\mathcal{D}w = 2\pi\delta_p$. We can decompose this operator as

$$\mathcal{D} = \Delta_{\mathbb{CP}^1} + T^{-2}\partial_z^2 + R$$

with $R = z^2\partial_z^2 + 4z\partial_z + 2$ being the remaining part.

Consider a change of variable $\ell = Tz$, then the operator $\mathcal{D}$ and R becomes

$$\mathcal{D}_\ell := \Delta_{\mathbb{CP}^1} + \partial_\ell^2 + \ell^2\partial_\ell^2 + 4\ell\partial_\ell + 2, \quad R_\ell := \ell^2\partial_\ell^2 + 4\ell\partial_\ell + 2,$$

and

$$\mathcal{D}_\ell w = 2T\pi\delta_p.$$

Now for the distance function $r_\ell = \sqrt{x^2 + y^2 + \ell^2}$, and the Laplacian on $\mathbb{R}^3$ $\Delta_{\mathbb{R}^3} = \partial_x^2 + \partial_y^2 + \partial_\ell^2$, we know

$$\Delta_{\mathbb{R}^3} \left(\frac{1}{2r_\ell} \right) = -2\pi\delta, \quad \Delta_{\mathbb{CP}^1} - \Delta_{\mathbb{R}^3} = \mathcal{O}(r_\ell^2)P,$$

with $P = P(x, y, \ell)$ being a second-order differential operator. Also note that

$$\mathcal{D}_\ell \left(\frac{T}{2r_\ell} \right) = T\pi\delta_p + \mathcal{O}(r_\ell^2)P \left(\frac{T}{2r_\ell} \right) + R_\ell \left(\frac{T}{2r_\ell} \right)$$

More explicitly, we apply the following lemma from [50]:

Lemma 5.2.1.1. ([50]) *Consider $f(x, y, \ell)$ a homogeneous polynomial of degree k and $\mathcal{L}'$ a differential operator such that $\mathcal{L}'(\mathcal{W}^1) \subset \mathcal{W}^1$, then there exists $u, v' \in \mathcal{W}^1$ satisfying*

$$(\Delta_{\mathbb{R}^3} + \mathcal{L})u = \frac{f}{r_\ell^{k+1}} + v'.$$

Note the operator R satisfies $R(\mathcal{W}^1) \subset \mathcal{W}^1$, thus applying the lemma to our situation, we obtain

$$R_\ell \left(\frac{T}{2r_\ell} \right) = \frac{p^4(x, y, \ell)}{r_\ell^5} + w^1, \quad P \left(\frac{T}{2r_\ell} \right) = g^{0,\alpha}.$$

with p^4 being a homogeneous degree 4 polynomial with respect to variables x, y, ℓ . Schauder regularity theorem for second order PDE gives that

$$w = \frac{T}{2r_\ell} + g^{2,\alpha} + w^1 \in C^\alpha. \quad (5.8)$$

5.3 Compactification

Now with the information of w near the singularity, we want to construct a compactification where the metric g extends with some regularity.

With the expression of Equation 5.8, we can compare the expression of the metric g with the metric given by Gibbons-Hawking ansatz, then we see the correct model to use to compactify the metric space near the singularity is the Taub-NUT model space. Here we recall the definition of Taub-NUT model arising from the Gibbons-Hawking ansatz:

Gibbons Hawking ansatz Let $U \subset \mathbb{R}^3$ be a contractible open subset and V is a positive harmonic function on U . Use x_1, x_2, x_3 to denote the coordinates on $\mathbb{R}^3$. Consider $\pi_{GH} : M \rightarrow \mathbb{R}^3 - \{0\}$ the S^1 -bundle with connection form Θ . We say an S^1 -invariant metric on $U \times S^1$ arises from the Gibbons-Hawking ansatz if it is of the following form

$$g_{GH} := V\pi^*(dx_1^2 + dx_2^2 + dx_3^2) + V^{-1}\Theta^2. \quad (5.9)$$

Let t be the fiber coordinate and we can write $\Theta = dt + \theta$ with θ being a 1-form on $\mathbb{R}^3$ satisfying $*_{g_0}dV = d\theta$ with g_0 being the Euclidean metric on $\mathbb{R}^3$. The complex structure J_{GH} is given by

$$dx_3 \mapsto V^{-1}(dt + \theta), dx_1 \mapsto dx_2$$

and the Kähler form is given by

$$\omega_{GH} := Vdx_2 \wedge dx_3 + dx_1 \wedge \Theta_{GH}.$$

Taub-NUT metric Taub-NUT metric is a special family of metrics arising from the Gibbons-Hawking ansatz. In Equation 5.9 $U = \mathbb{R}^3 - \{0\}$ and $V = \frac{1}{2r} + a$ with r being the distance function from the origin in $\mathbb{R}^3$ and $a > 0$ being the parameter for this family. In this case the total space of the S^1 -bundle is $\mathbb{C}^2 - \{0\}$ (see [45]), and $\pi_{TN} : \mathbb{C}^2 - \{0\} \rightarrow \mathbb{R}^3 - \{0\}$ is the Hopf fibration $(z_1, z_2) \mapsto (\text{Re}(y), \text{Im}(y), z)$, where

$$y = z_1 z_2, z = \frac{1}{2}(|z_1|^2 - |z_2|^2).$$

We rewrite the metric and the Kähler form in this special case

$$g_{TN} := \left(\frac{1}{2r} + a\right) \pi_{TN}^* g_0 + \left(\frac{1}{2r} + a\right)^{-1} \Theta_{TN}^2, \quad (5.10)$$

$$\omega_{TN} := \left(\frac{1}{2r} + a \right) \pi_{TN}^* \omega_{\mathbb{C}} + dz \wedge \Theta_{TN}. \quad (5.11)$$

Remark 5.3.1. *As pointed out in [46], Gibbons-Hawking ansatz is a special case of LeBrun ansatz when we take $u = 0$. The similarity of the form of the scalar-flat metric we intend to construct and the Taub-NUT model may suggest another reason why it is a natural candidate for doing the compactification.*

Upon closer look, for ω_{TN} to be Kähler and for J to be integrable, we need the following conditions to hold:

$$\begin{aligned} \partial_z^2 \left(\frac{1}{2r} + a \right) \pi_{TN}^* \omega_{\mathbb{C}} + d_D d_D^c \left(\frac{1}{2r} + a \right) &= 0, \\ \partial_z \left(\frac{1}{2r} + a \right) \pi_{TN}^* \omega_{\mathbb{C}} - dz \wedge d_D^c \left(\frac{1}{2r} + a \right) &= d\Theta_{TN} =: \Upsilon_{TN}. \end{aligned}$$

Now back to our complex surface with S^1 -symmetry. Knowing Υ is closed and thus defines $[\Upsilon] \in H^2(S^2 \times I - \{p\}; \mathbb{R})$, we have

Lemma 5.3.0.1. *$\frac{1}{2\pi}[\Upsilon]$ is integral.*

Hence there exists a principal S^1 -bundle

$$\pi : \mathcal{N} \rightarrow S^2 \times I - \{p\}$$

with curvature form $-i\Upsilon$. In order to do the compactification around singularity p , we want to clarify the geometry of $\mathcal{N}$ over a small neighborhood U around p . Consider

$$\pi' : L \rightarrow U - \{p\}$$

the Hopf fibration. Recall the fact that the topological data for the principal S^1 -bundle is determined by its first Chern class, note that both π and π' have the same degree, we conclude the existence of an isomorphism between $\mathcal{N}$ and L . This allows us to glue $\mathcal{N}$ and L to get a compactification $\bar{\mathcal{N}}$. The next task is to study the regularity of the metric and complex structure. Let $y_D, \bar{y}_D, \ell$ be the coordinates on $U^3 := U^2 \times I - \{p\}$ with $U \subset \mathbb{R}_{x,y}^2$ being a small neighborhood around p . Note here $y_D, \bar{y}_D$ is the holomorphic coordinate on D . Let $U = \pi^{-1}(U^3)$, we naturally obtain a Hopf fibration mapping

$$\pi_{Hopf} : U \rightarrow U^3, \quad (z_1, z_2) \mapsto (\Re(z_1 z_2), \Im(z_1 z_2), \frac{1}{2}(|z_1|^2 - |z_2|^2)).$$

Comparing the degree of π_{Hopf} with that of π , we conclude that they actually coincide on U , and thus obtain

$$2\pi^* r_\ell = |z_1|^2 + |z_2|^2 := s^2.$$

Recall $\Upsilon = \partial_z \tilde{\omega} - dz \wedge d^c h$, and $w = \frac{T}{2r_\ell} + g^{3,\alpha} + w^1$, then when replacing w by w_T , we have $\Upsilon_T := \partial_z \tilde{\omega}_T - dz \wedge d^c h$, and $w_T = w_0 + \frac{T}{2r_\ell} + g^{3,\alpha} + w^1$,

$$\begin{aligned} \Upsilon_T - \Upsilon_{TN} &= \partial_\ell (T\tilde{\omega}_T - \omega_{TN}) + d\ell \wedge d_D^c \left(\frac{1}{T} w_T - \frac{1}{2r_\ell} - 1 \right) \\ &= \partial_\ell (g^{3,\alpha} + w^1) \frac{i}{2} dy_D \wedge d\bar{y}_D + d\ell \wedge d_D^c (g^{3,\alpha} + w^1) \\ &= (g_1^{2,\alpha} + w^0) dy_D \wedge d\bar{y}_D + (g_2^{3,\alpha} + w^1) d\ell \wedge dy_D + (g_3^{3,\alpha} + w^1) d\ell \wedge d\bar{y}_D. \end{aligned}$$

Here we used

$$\begin{aligned} \tilde{\omega}_T &= e^{u_0} w_T dx \wedge dy = \frac{z^2 + T^{-2}}{(1 + \frac{x^2 + y^2}{4})} w_T dx \wedge dy \\ &= \frac{T^{-2}(\ell^2 + 1)}{(1 + \frac{x^2 + y^2}{4})^2} w_T dx \wedge dy \\ &= T^{-2}(\ell^2 + 1) w_T \omega_D \\ &= \left(1 + \frac{1}{2Tr_\ell} + \frac{g^{3,\alpha}}{T} + \frac{w^1}{T} \right) \omega_D, \end{aligned}$$

and

$$\pi^*(\omega_D - \omega_{\mathbb{C}}) = O(r_\ell^2) \pi^*(dy \wedge d\bar{y}).$$

Under the Hopf fibration, $\pi^* dy, \pi^* dz, \pi^* d\bar{y} \in sO'(1)$ (for notation, see remark below). The lowest regularity term above is the first one, which gives

$$(g^{2,\alpha} + w^0) O'(s^2) \in C^{1,\alpha}.$$

Notation As $r \rightarrow 0$, we say $f = \mathcal{O}'(r^k)$ if for each $\epsilon > 0$,

$$|\partial^m f| = \begin{cases} O(1), & \text{if } m \leq k - 1 \\ O(r^{k-m-\epsilon}) & \text{if } m \geq k \end{cases}$$

Comparing the metric on $\mathcal{N}$ over a neighborhood around the singularity p , we see their curvature still differ by a 2-form with $C^{1,\alpha}$ singularity. Fortunately, we have one more degree of freedom to fix the difference, i.e., we can perform a suitable gauge transformation.

Lemma 5.3.0.2. *There exists an S^1 -invariant 1-form with $C^{2,\alpha}$ regularity in a small neighborhood around p and smooth away from the singularity satisfying $d\theta = \Upsilon - \Upsilon_{TN}$ and $\theta(\partial_t) = 0$.*

Proof. The strategy is to impose the gauge-fixing condition $d\theta = \Upsilon - \Upsilon_{TN}$ and solve the elliptic system under the Neumann boundary condition, which is $\theta(nu) = 0$ on the boundary of the small neighborhood. The desired regularity follows from the standard elliptic regularity theorem and S^1 -invariance is easily obtained. Cartan's formula gives the vanishing of $d(\iota_{\partial_t}\theta)$, implying $\theta(\partial_t) = 0$. $\square$

From this we deduce that the metric g extends to a $C^{2,\alpha}$ metric with the given connection $\Theta_{\bar{N}} := \Theta + \theta$, giving us a $C^{2,\alpha}$ compactification. In fact, we can further raise its regularity:

Theorem 5.3.1. *The metric extends smoothly across the singularity, resulting in a smooth compactification.*

Proof. We recall a useful interpretation of the cscK equation

$$Scal(\omega_\varphi) = c,$$

where we view it as a second order system

$$\omega_\varphi^n = e^F \omega_0^n, \tag{5.12}$$

$$\Delta_\varphi F = tr_\varphi \theta - c. \tag{5.13}$$

Here θ is a given $(1,1)$ -form in $c_1(X)$ (explicitly, it can be written as $Ric(\omega_0)$) and φ is the Kähler potential. This interpretation basically comes from the fact that $Scal(\omega_\varphi) = -\Delta_\varphi \log \omega_\varphi^n$.

The benefit of viewing the cscK equation in this way is that both equations are second order elliptic equations, and we can apply standard elliptic bootstrap to improve the regularity. More precisely, since we started from scalar-flat metric on the cylinder $D \times I$, and we are looking for scalar-flat metric in the end, we have $c = 0$, $\theta = 0$, thus the RHS of Equation 5.13 vanishes. Starting from the $C^{2,\alpha}$ regularity of g , we can choose $\varphi \in C^{4,\alpha}$, then Δ_φ is a linear second order elliptic operator whose coefficients have regularity $C^{2,\alpha}$. Apply elliptic bootstrap to Equation 5.13, we get $C^{4,\alpha}$ regularity of F . Then we go back to 5.12, now with both F and φ having regularity $C^{4,\alpha}$, we know from standard Monge-Ampere equation bootstrap that $\varphi \in C^{6,\alpha}$. In this way we increased the regularity of φ . Iterating the above arguments, we obtain the smoothness of g . $\square$

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