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Jankowski, Eric Arthur

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The Super Combinatorics of Toric Supervarieties

by

Eric Jankowski

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

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in the

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of the

University of California, Berkeley

Committee in charge:

Professor Vera Serganova, Chair

Professor David Nadler

Professor Alexander Givental

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Eric Jankowski

Abstract

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Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor Vera Serganova, Chair

We develop and study the notion of a toric supervariety, a generalization of the classical notion of a toric variety. Toric supervarieties are defined as normal supervarieties admitting an action by an algebraic supertorus for which there is a dense open orbit. We classify toric supervarieties via decorated fans, which consist of the familiar polyhedral data from classical toric geometry together with additional fermionic structure. In special cases, this correspondence yields an equivalence of categories, and in general we show that a decorated fan encodes substantial geometric information about its associated toric supervariety. Toric supervarieties whose underlying toric varieties are projective are similarly described in terms of decorated polytopes.

We present several classes of examples, with particular emphasis on closures of supertorus orbits in the isomeric supergrassmannian. We show that this space admits an odd momentum map whose image coincides with the decorated polytope of the generic supertorus stratum. Moreover, the images of its toric subvarieties are shown to agree with their corresponding decorated polytopes. These decorated polytopes are then used to compute the Picard groups and Π -Picard sets of the isomeric supergrassmannian and its toric subvarieties.

We also develop foundational aspects of algebraic supergeometry needed for this theory, with particular attention to the notion of normality. To this end, we introduce and study integral, reduced, and normal superrings, establishing analogues of several classical results, including properties of superfields and versions of Serre's criteria. These foundations provide a framework for studying superschemes that behave like supermanifolds on a dense open subset while allowing controlled singularities.

“Possums”
by Sheila Black

A kind of thrill—to lie on a road
and flatten yourself,

white fur like a ball of winter,

like the March blossoms on the fruit trees,
each one folded in like

the fledgling that never made it
from the nest.

They do this when they feel threatened,
remain motionless

even when curious people come prod
them with sticks,

stiffening their pearly claws as a tree stiffens
its twigs for winter. What is it to be dead?

The possums know—that eternal watchfulness
by which the dead in their stately wisdom

watch us
who keep moving.

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Chapter 1

Introduction

1.1 For a general audience

Symmetry plays an essential role throughout mathematics. It provides a powerful lens through which we can understand geometric structure (for instance, the rotations and reflections of a circle), counting and probability (such as exchanging heads and tails in a coin flip), and many other quantifiable phenomena. To study symmetry systematically, mathematicians organize the symmetries of a given object into an algebraic structure called a *group*. The rich internal structure of groups has led to the development of group theory, one of the foundational subjects of modern mathematics.

Many symmetry groups are infinite. For example, there are infinitely many ways to rotate a circle: one may rotate by any angle between 0 and 2π . Moreover, rotating by 0 and by 2π produce the same effect, so the set of rotations naturally “wraps around.” In this sense, the group of rotations of a circle itself has the structure of a circle. This observation reflects a deeper principle: symmetry groups often carry geometric structure of their own.

The systematic study of such “continuous symmetries” was initiated by Sophus Lie, leading to the theory of Lie groups and their actions on geometric spaces. A guiding idea in modern geometry is that a space with many symmetries can often be understood in terms of the group acting on it. In particularly favorable situations, the symmetry group occupies a large portion of the space itself.

A *toric variety* is a striking example of this phenomenon. It is a geometric space that contains a dense subset that is itself a group called an *algebraic torus*. Concretely, this torus consists of tuples of nonzero complex numbers; for instance, the space $\mathbb{C}^2$ can be equipped with such symmetries by scaling each coordinate independently.

One of the most remarkable features of toric varieties is their deep connection to *convex geometry*. Foundational work of Demazure, Mumford, Danilov, Sumihiro, and others revealed that toric varieties can be described combinatorially in terms of convex polytopes and polyhedral fans. In the projective case, a polytope serves as a kind of blueprint: its faces encode how to assemble the variety from simpler pieces, while its combinatorial structure reflects geometric properties of the resulting space. This correspondence has made toric varieties an indispensable bridge between algebraic geometry, combinatorics, and even areas such as mirror symmetry.

In this thesis, we extend these ideas to the *supersymmetric* setting. Supersymmetry, originating in theoretical physics through the work of Wess, Zumino, and many others, posits a spacetime symmetry between two fundamental classes of particles, bosons and fermions. Capturing such symmetries mathematically requires a generalization of classical geometry: one replaces ordinary spaces and groups with *superspaces* and *supergroups*, in which both commuting (“even”) and anticommuting (“odd”) coordinates coexist. The mathematics of supersymmetry have been studied thoroughly by authors such as Witten, Manin, Penkov, Serganova, Vaintrob, and Deligne.

Our goal is to develop a theory of *toric supervarieties*, in which a suitably defined *supertorus* acts with a dense orbit, extending the classical theory of toric varieties into this richer setting. As in the classical case, we seek to understand how geometric and combinatorial structures interact, and to what extent familiar correspondences with convex geometry persist in the realm of supersymmetry.

1.2 Summary of main results

We begin with a brief discussion of toric varieties before describing supergeometric fundamentals, toric supervarieties, and the main results of this thesis.

Toric varieties

We work over $\mathbb{C}$, though the theory will be equivalent over any algebraically closed field of characteristic 0. Recall that a toric variety is a normal variety X equipped with a torus action for which there is an open orbit. If N is a lattice, a polyhedral cone $\sigma \in N_{\mathbb{R}}$ specifies a recipe for an affine toric variety $\text{Spec } \mathbb{C}[\sigma^{\vee} \cap N^*]$ with the torus whose coweight lattice is N . A fan is a collection of polyhedral cones closed under taking faces and such that the intersection of two cones is a face of each. The connection between toric varieties and fans is realized through the following theorem, due to Sumihiro [48].

Theorem 1.2.1. *Let the torus T act on a normal variety X . Then every point $x \in X$ has a T -invariant affine open neighborhood.*

Corollary 1.2.2. *The category of toric varieties is equivalent to the category of fans.*

The relationship between toric varieties and fans is extremely deep; it includes combinatorial criteria for geometric information such as smoothness and properness and Picard groups, as well as a correspondence between cones of the fan and torus orbits in the variety. We direct the reader to Chapter 2 for more details on this relationship, or to [9] for a full account.

Foundational supergeometry

The foundational question of the meaning of a normal supervariety must first be considered. We establish the correct super analogs of fields, integral domains and schemes, and reduced rings and schemes, as well as of Serre’s criteria (R_k) and (S_k) for schemes. We prove

several foundational results about superfields, integral superdomains, and reduced superrings, including the equivalence of several characterizations of these objects. Note that $J_{\mathcal{A}}$ refers to the ideal of $\mathcal{A}$ generated by the odd part of $K(\mathcal{A})$, the total superring of fractions of $\mathcal{A}$.

Theorem 1.2.3 (Lemma 3.4.6). *Let $\mathcal{A}$ be a superring.*

(a) *The following are equivalent:*

- (i) $\mathcal{A}$ is a superfield
- (ii) $\overline{\mathcal{A}}$ is a field
- (iii) $J_{\mathcal{A}} = (\mathcal{A}_{\overline{1}})$ is the set of non-units
- (iv) $(\mathcal{A}_{\overline{1}})$ is a maximal ideal

(b) *The following are equivalent:*

- (i) $\mathcal{A}$ is an integral superdomain
- (ii) $\overline{\mathcal{A}}$ is an integral domain and $\overline{\text{zdiv}(\mathcal{A})} = \text{zdiv}(\overline{\mathcal{A}})$
- (iii) $K(\mathcal{A})$ is a superfield
- (iv) $\mathcal{A}$ is reduced and $\text{Spec } \mathcal{A}$ is irreducible

(c) *The following are equivalent:*

- (i) $\mathcal{A}$ is a reduced superring
- (ii) $\overline{\mathcal{A}}$ is a reduced ring and $\overline{\text{zdiv}(\mathcal{A})} = \text{zdiv}(\overline{\mathcal{A}})$
- (iii) $J_{\mathcal{A}} = \text{Nil}(\mathcal{A})$ and $\text{zdiv}(\mathcal{A}) = \bigcup_{\mathfrak{p} \in \min \text{Spec } \mathcal{A}} \mathfrak{p}$
- (iv) $\overline{\mathcal{A}}$ is a reduced ring and $\text{zdiv}(\mathcal{A}) = \bigcup_{\mathfrak{p} \in \min \text{Spec } \mathcal{A}} \mathfrak{p}$
- (v) $S^{-1}\mathcal{A}$ is a reduced superring for any multiplicatively closed subset $S \subseteq \mathcal{A}_{\overline{0}}$
- (vi) $\mathcal{A}_{\mathfrak{m}}$ is a reduced superring for any maximal ideal $\mathfrak{m}$

An important additional condition is generic smoothness in the odd directions, i.e. generic fermionic regularity or GFR. The following result demonstrates the effectiveness of GFRR (GFR + reduced) superrings in generalizing the essential characteristics of reduced rings, and of $(R_1) + (S_2)$ as the best generalization of normality.

Theorem 1.2.4 (Proposition 3.4.16). *A Noetherian superring $\mathcal{A}$ is GFRR if and only if it satisfies (R_0) and (S_1) .*

These definitions and results are extended to supervarieties as local conditions.

Toric supervarieties

Let G be an algebraic supergroup with $G_{\bar{0}}$ reductive, where $G_{\bar{0}}$ is the underlying algebraic group of G . We call such supergroups quasi-reductive. Cartan subgroups T of G are centralizers of maximal tori $T_{\bar{0}}$ in $G_{\bar{0}}$; such a supergroup T will be called a supertorus. We would like to consider normal supervarieties X with an especially large amount of symmetry exhibited by a supertorus T , in the sense that there is an open orbit under a T -action. Such a supervariety will be called toric, generalizing the classical notion to the super setting.

It is interesting to ask to what extent Corollary 1.2.2 generalizes. It will be seen that classification of toric supervarieties in full generality is ultimately a wild problem (Proposition 5.5.2), but we obtain several partial results in various levels of generality. In particular, if we impose an enhancement of the usual (R_1) condition of Serre (Proposition/Definition 5.5.3, proven in Appendix A), then we can classify toric supervarieties by their decorated fans. A decorated fan is essentially an ordinary polyhedral fan Σ , where each ray $\rho \in \Sigma(1)$ is decorated by some additional odd data. (See Definition 5.6.4 for more details).

Theorem 1.2.5 (Theorem 5.6.5). *There is a bijective correspondence between (HR_1) toric supervarieties and decorated fans, up to isomorphism.*

If the underlying variety is projective, we also describe a notion of a decorated polytope, with a corresponding notion of the normal decorated fan of a decorated polytope.

We will see that morphisms of (HR_1) toric supervarieties are difficult to describe, so a categorical version of this theorem remains elusive. However, we offer a partial rectification in the form of the following theorem about “large-orbit” toric supervarieties, i.e. those whose orbits have odd codimension no greater than their even codimension. All large-orbit toric supervarieties admit the (HR_1) property, and most toric supervarieties that occur “in nature”, including all toric supervarieties with supertorus $Q(1)^n$, are large-orbit. Large-orbit decorated fans are particularly easy to describe; each ray ρ of the fan Σ need only be decorated by a single 0- or 1-dimensional subspace of $\mathfrak{t}_{\bar{1}}$.

Theorem 1.2.6 (Theorem 5.10.5). *The category of large-orbit toric supervarieties is equivalent to the category of large-orbit decorated fans.*

In the more general (HR_1) case, we obtain from the decorated fan a smoothness criterion, which naturally simplifies in the presence of large orbits since condition (c) is immaterial.

Theorem 1.2.7 (Theorem 5.11.2). *Let X be a (HR_1) toric supervariety with decorated fan $(N, \Sigma, \mathfrak{t}, \mathfrak{h}, \{V_{\rho,i}\})$. Then X is smooth if and only if for every cone $\sigma \in \Sigma$, the following three conditions hold:*

- (a) σ is a smooth cone.
- (b) There is a basis of $\mathfrak{t}_{\bar{1}}$ such that for all $\rho \in \sigma(1)$ and $i \geq 0$, $V_{\rho,i}$ is the span of a subset of that basis.
- (c) For each $\rho \in \sigma(1)$, if $\text{codim}(\mathfrak{h}, V_{\rho,0}) \geq 2$, then $[V_{\rho,0}, V_{\rho',0}] \subseteq \mathfrak{t}_{N_{\rho'}}$ for all $\rho' \in \sigma(1)$.

We summarize the correspondence between decorated cones and orbits as follows:

Theorem 1.2.8 (Section 3.4). *Let X be the (HR_1) toric supervariety of the decorated fan $(N, \Sigma, \mathfrak{t}, \mathfrak{h}, \{V_{\rho,i}\})$. Then:*

(a) *There is a bijective correspondence*

$$\begin{aligned} \{\sigma \in \Sigma\} &\longleftrightarrow \{T\text{-orbits in } X\} \\ \sigma &\longleftrightarrow \text{Orb}(\sigma) \cong T/(T_{N_\sigma}, \mathfrak{t}_{N_\sigma} \oplus V_{\sigma,0}) \end{aligned}$$

(b) *For each cone $\sigma \in \Sigma$, $\dim \text{Orb}(\sigma) = (\text{codim}(\sigma, N_{\mathbb{R}}) | \text{codim}(V_{\sigma,0}, \mathfrak{t}_{\mathbb{T}}))$*

A case of significant interest is the action of the isomeric supergroup $Q(n)$ on the homogeneous space $\text{QGr}(r, n)$, called the isomeric supergrassmannian. It is the super moduli space of Π -invariant $(r|r)$ -dimensional subspaces of a fixed $(n|n)$ -dimensional super vector space V equipped with an odd involution Π (i.e. an isomeric vector space in the sense of [35]). Cartan subgroups of $Q(n)$ are isomorphic to the $(n|n)$ -dimensional supertorus $T = Q(1)^n = Q(1) \times \dots \times Q(1)$. The T -orbit closures of closed points in $\text{QGr}(r, n)$ are all normal, and therefore can be described as toric supervarieties.

The complex isomeric supergrassmannian $\text{QGr}(r, n)$ is in fact also a homogeneous space for the isomeric unitary supergroup $UQ(n)$ over $\mathbb{R}$. Via this realization, we describe an “odd momentum map” $\text{QGr}(r, n) \rightarrow \Pi \mathfrak{uq}(n)^* \cong \mathfrak{uq}(n)$, whose image can be understood as a decorated polytope. The image of each T -orbit closure is likewise a subpolytope as in the classical case (see e.g. [1], [20], [15]).

Theorem 1.2.9 (Sections 6.3-6.4 and Proposition 6.4.2). *The supergrassmannian $\text{QGr}(r, n)$ is isomorphic to an odd coadjoint orbit of the real supergroup $UQ(n)$ acting on $\Pi \mathfrak{uq}(n)^*$ such that the corresponding “odd momentum map”*

$$\text{QGr}(r, n) \rightarrow \Pi \mathfrak{uq}(n)^* \rightarrow \Pi \mathfrak{t}_{\mathbb{R}}^* \rightarrow \mathfrak{t}_{\mathbb{R}}^*$$

has image on $\mathbb{R}[\xi]$ -points equal to the decorated polytope of the generic supertorus stratum. Moreover, the image of each toric subvariety is the corresponding decorated polytope.

We study two different kinds of locally free sheaves on $\text{QGr}(r, n)$ and its toric subvarieties: invertible sheaves and Π -invertible sheaves. An invertible sheaf is merely a $(1|0)$ -dimensional vector bundle; as usual, these assemble into the Picard group $\text{Pic}(X) \cong H^1(X, \mathcal{O}_{X, \bar{0}}^\times)$. A Π -invertible sheaf, originally studied by Manin and Skornyakov in is a $(1|1)$ -dimensional vector bundle equipped with an odd involution; these assemble into the Π -Picard set $\text{Pic}_\Pi(X) \cong H^1(X, \mathcal{O}_X^\times)$, which is not a group but a pointed set with distinguished point $\mathcal{O}_X \oplus \Pi \mathcal{O}_X$. While invertible sheaves characterize morphisms to projective superspace $\mathbb{P}^{m|n}$, Π -invertible sheaves characterize morphisms to Π -projective superspace $\mathbb{P}_\Pi^n \cong \text{QGr}(1, n+1)$. Such embeddings have been studied by various authors, in e.g. [29], [28], and [42].

Theorem 1.2.10 (Theorem 6.9.1). *Let X be $\mathrm{QGr}(r, n)$ or the closure of a $Q(1)^n$ orbit inside $\mathrm{QGr}(r, n)$. Then*

$$\mathrm{Pic}(X) \cong \prod_{\text{factors } \mathbb{P}_{\Pi}^1 \text{ of } X} \mathbb{Z} \quad \text{and}$$

$$\mathrm{Pic}_{\Pi}(X) \cong \left\{ (\ell_i, c_i) \in \prod_{\text{factors } \mathbb{P}_{\Pi}^{d_i} \text{ of } X} \mathbb{Z} \oplus \mathbb{C} \mid \text{at most one } c_i \neq 0, \text{ and if } d_i > 1, \text{ then } c_i^2 = \ell_i \right\}.$$

In particular,

$$\mathrm{Pic}(\mathrm{QGr}(r, n)) = \begin{cases} \mathbb{Z} & r = 1, n = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Pic}_{\Pi}(\mathrm{QGr}(r, n)) = \begin{cases} \mathbb{Z} \oplus \mathbb{C} & r = 1, n = 2 \\ \{(\ell, c) \in \mathbb{Z} \oplus \mathbb{C} \mid c^2 = \ell\} & r \in \{1, n-1\}, n > 2 \\ 0 & \text{otherwise} \end{cases}$$

The number of factors of Π -projective space $\mathbb{P}_{\Pi}^d \cong \mathrm{QGr}(1, d+1)$ can be detected easily by counting the number of d -simplex factors in the defining polytope. These calculations help us understand morphisms to projective superspace $\mathbb{P}^{m|n} \cong \mathrm{Gr}(1|0, m+1|n)$ and Π -projective superspace $\mathbb{P}_{\Pi}^n \cong \mathrm{QGr}(1, n+1)$, respectively. Since all such bundles can be interpreted in terms of symmetric powers of tautological bundles on $\mathbb{P}^{m|n}$ and $\mathbb{P}_{\Pi}^n$, any such morphisms turn out to be fairly trivial. This rigidity is common but not ubiquitous in supergeometry; for example, the Picard group of $\mathbb{P}^{1|m}$ has a very large continuous component [4].

Corollary 1.2.11 (Corollary 6.9.2). *Let X be the closure of a $Q(1)^n$ orbit inside $\mathrm{QGr}(r, n)$. Then:*

- (a) *An invertible sheaf on X is given by the tensor product of an invertible sheaf $\mathcal{O}(\ell)$, $\ell \in \mathbb{Z}$ on each factor of $\mathbb{P}_{\Pi}^1$ in X .*
- (b) *A Π -invertible sheaf on X is given by the tensor product of such an invertible sheaf with either:*
 - (i) *A Π -invertible sheaf $\mathcal{O}_{\Pi}(\ell)$, $\ell \in \mathbb{Z}$ on some factor $\mathbb{P}_{\Pi}^d$ for $d > 1$, or*
 - (ii) *A Π -invertible sheaf arising from $H^1(\mathbb{P}^1, \Omega^1) \cong \mathbb{C}$ on some factor of $\mathbb{P}_{\Pi}^1$.*

1.3 Structure of thesis

In chapter 2 we briefly discuss the theory of toric varieties over $\mathbb{C}$, explicitly spelling out the connection with convex geometry. Chapter 3 then introduces background on linear superalgebra and algebraic supergeometry, including some novel results on reduced, integral, and normal supervarieties. Chapter 4, the final background chapter, completes the

supergeometric exposition by expounding the theory of supergroups along with several key examples.

In chapter 5, we systematically build up the theory of toric supervarieties, and generalize several key theorems from the classical setting. In particular, we provide an equivalence of categories between certain toric supervarieties and *decorated fans*, which we show encode key geometric properties about the corresponding supervarieties.

We narrow our focus in chapter 6 to a supertorus action on $\mathrm{QGr}(r, n)$, the type Q supergrassmannian. We describe the image of an “odd momentum map” on $\mathrm{QGr}(r, n)$ as a *superpolytope*, showing that the supertorus orbit closures within likewise correspond with subpolytopes. Finally, we calculate the Picard groups and Π -Picard sets of $\mathrm{QGr}(r, n)$ and its toric subvarieties.

1.4 Conventions and notation

A sub-object of a supervariety, supergroup, etc. will be called a subvariety, subgroup, etc. (respectively) instead of a cumbersome term like subsupervariety. Commutative will mean supercommutative when referring to a superalgebra or superring.

Toric Varieties

- M and N the weight and coweight lattices of a toric variety
- $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{C}} = N \otimes_{\mathbb{Z}} \mathbb{C}$
- T_N the torus with coweight lattice N
- σ a strongly convex, rational, polyhedral cone
- ρ a ray, i.e. a 1-dimensional cone
- u_{ρ} the primitive generator of ρ
- $\sigma^{\vee}$ the dual cone of σ
- $\tau \leq \sigma$ a face of σ
- $\sigma(1)$ the set faces of σ which are rays
- H_m for $m \in M$ the hyperplane $\{u \in N_{\mathbb{R}} \mid \langle m, u \rangle = 0\}$
- $\sigma_m = H_m \cap \sigma$ for $m \in \sigma^{\vee} \cap M$, a face of σ
- Σ a polyhedral fan
- $\Sigma(1)$ the rays in Σ
- X_{Σ} and X_P the toric varieties corresponding to the fan Σ and the polytope P
- σ_F the normal cone of a face $F \leq P$

- Σ_P the normal fan of P
- $m' \leq_\sigma m$ the statement that $m - m' \in \sigma^\vee$

Superalgebra

- k a field of characteristic not equal to 2 or 3
- $\mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$
- $V = V_{\bar{0}} \oplus V_{\bar{1}}$ the parity decomposition of a super vector space
- ΠV the parity shift of V
- (V, Π_V) an isomeric vector space
- $T(V)$, $S(V)$, and $\Lambda(V)$ the tensor, symmetric, and alternating superalgebras of V
- $\mathfrak{g}$ a Lie superalgebra
- $U\mathfrak{g}$ the universal enveloping superalgebra of $\mathfrak{g}$
- $\mathfrak{h}$ or $\mathfrak{t}$ a Cartan subalgebra
- $\mathcal{A}$ an associative superalgebra or superring, usually also unital and commutative
- $K(\mathcal{A})$ the total superring of fractions of $\mathcal{A}$
- $J_{\mathcal{A}}$ the ideal of $\mathcal{A}$ generated by $K(\mathcal{A})_{\bar{1}}$
- $\bar{\mathcal{A}} = \mathcal{A}/J_{\mathcal{A}}$ the underlying ring of $\mathcal{A}$
- $\mathcal{A}^\times$ the group of invertible elements of $\mathcal{A}$
- $\text{zdiv}(\mathcal{A})$ and $\text{nonzdiv}(\mathcal{A})$ the zero divisors and non-zero divisors of $\mathcal{A}$
- $\text{Nil}(\mathcal{A})$ the nilradical of $\mathcal{A}$
- $\mathfrak{p}$ a prime ideal and $\mathfrak{m}$ a maximal ideal
- (R_k) and (S_k) the criteria of Serre

Supergeometry

- X a supervariety
- $(|X|, \mathcal{O}_X)$ the topological space and structure sheaf of X
- $\mathcal{F}$ a quasi-coherent sheaf
- $\mathcal{O}_{X,x}$ and $\mathcal{F}_x$ the stalks at $x \in |X|$
- $\mathcal{O}_{X,\bar{0}} \oplus \mathcal{O}_{X,\bar{1}}$ and $\mathcal{F}_{\bar{0}} \oplus \mathcal{F}_{\bar{1}}$ the parity decompositions of $\mathcal{O}_X$ and $\mathcal{F}$

- $\mathcal{K}_X$ the sheaf of rational functions on X
- $\mathcal{I}_X$ the ideal sheaf of $\mathcal{O}_X$ generated by $\mathcal{K}_{X,\bar{1}}$
- $X_{\bar{0}} = (|X|, \mathcal{O}_X/\mathcal{I}_X)$ the underlying variety of X
- $k[X] = \Gamma(X, \mathcal{O}_X)$ and $k(X) = \Gamma(X, \mathcal{K}_X)$
- $X(\mathcal{A})$ the $\mathcal{A}$ -points of X
- $\mathcal{T}_X$ the tangent sheaf of X
- $T_x X$ the tangent space of X at $x \in |X|$

Supergroups

- G an algebraic supergroup with underlying group $G_{\bar{0}}$ and Lie superalgebra $\mathfrak{g}$
- $(G_{\bar{0}}, \mathfrak{g})$ a super Harish-Chandra pair
- $G \cdot x$ the G -orbit of $x \in X(\mathbb{C})$ in X
- $\overline{G \cdot x}$ the closure of $G \cdot x$ in X
- $\text{Stab}_G(x)$ the stabilizer subgroup of G with respect to $x \in |X|$
- DS the Duflo-Serganova functor

Toric supervarieties

- T an algebraic supertorus with underlying torus $T_{\bar{0}}$ and Lie superalgebra $\mathfrak{t}$
- T/H the open orbit of a toric supervariety, where H is an odd abelian subgroup of T
- $L_{\mathcal{A}}(m)$ the m -weight space of the $T_{\bar{0}}$ -action on $\mathcal{A} \subseteq \mathbb{C}[T]$
- $\mathcal{A}_{\sigma}$ the superalgebra of regular functions on an affine toric supervariety with cone σ
- $\mathcal{A}_{\sigma,m}$ the subalgebra of $\mathcal{A}$ generated by $L_{\mathcal{A}_{\sigma}}(m')$ for all $m' \leq_{\sigma} m$
- $H_{\sigma,m} = \text{Stab}_T(1)$ in $\text{Spec } \mathcal{A}_{\sigma,m}$
- $V_{\rho,i} = \mathfrak{h}_{\rho,m}$ for m such that $\langle m, u_{\rho} \rangle = i$
- $(N, \Sigma, \mathfrak{t}, \mathfrak{h}, \{V_{\rho,i}\})$ a decorated fan
- (HR_1) homological regularity in codimension 1

Chapter 2

Toric Varieties

We first introduce the classical theory of toric varieties over $\mathbb{C}$, closely following [9].

2.1 Algebraic tori

Definition 2.1.1. An *algebraic torus* is an algebraic group T isomorphic to $(\mathbb{C}^\times)^n$ for some $n \in \mathbb{Z}_{\geq 0}$, where $\mathbb{C}^\times \cong \operatorname{Spec} \mathbb{C}[x^{\pm 1}]$ is the multiplicative group of the field of complex numbers.

A torus T has both a *character lattice* (or *weight lattice*) $M = \operatorname{Hom}(T, \mathbb{C}^\times)$ and a *cocharacter lattice* (or *coweight lattice*) $N = \operatorname{Hom}(\mathbb{C}^\times, T)$. These lattices are dual to each other, admitting a nondegenerate perfect pairing

$$\langle -, - \rangle : M \times N \rightarrow \mathbb{Z}$$

given by composing the morphisms of algebraic groups $\mathbb{C}^\times \rightarrow T \rightarrow \mathbb{C}^\times$. Given an arbitrary lattice N , one may define the torus $T_N := N \otimes_{\mathbb{Z}} \mathbb{C}^\times$ and observe that T_N has cocharacter lattice N and character lattice $M = N^*$. The character lattice M is naturally a basis for the regular functions on T_N , in the sense that the coordinate ring satisfies $\mathbb{C}[T_N] = \mathbb{C}[M]$. If we fix an identification $N \cong \mathbb{Z}^n$, then we likewise obtain $M \cong \mathbb{Z}^n$ and $T_N \cong (\mathbb{C}^\times)^n$, so that $\mathbb{C}[T_N] \cong \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$.

We collect a few results about algebraic tori here:

Lemma 2.1.2. (a) *Any connected closed subgroup of an algebraic torus is itself an algebraic torus.*

(b) *If $\phi : T \rightarrow T'$ is a morphism of algebraic groups where T and T' are both tori, then the image of ϕ is a closed subgroup (and therefore subtorus) of T' .*

2.2 Definition of a toric variety

Following [21], by algebraic variety we mean a reduced separated scheme of finite type over $\mathbb{C}$. A *normal* variety X is one that satisfies the R_1 (i.e. regular in codimension 1) and S_2 (i.e. regular functions on $X - Y$ extend to X if $\operatorname{codim}(Y, X) \geq 2$) criteria of Serre. Equivalently, a variety is normal if the local ring at every point is an integrally closed domain.

Definition 2.2.1. A *toric variety* is an irreducible normal variety X equipped with a left action by an algebraic torus T and a T -equivariant open immersion $T \rightarrow X$.

Remark 2.2.2. Notice that if X is any (irreducible normal) variety with an open orbit of a torus T , then the open orbit T/H itself is an algebraic torus and X is a toric variety for T/H . In this sense, any such variety with an open orbit of a torus can be modeled as a toric variety according to the above definition.

Definition 2.2.3. A *toric morphism* $X \rightarrow X'$ is a morphism of algebraic varieties that restricts to a group homomorphism on their underlying tori. Toric morphisms are T -equivariant.

2.3 Cones and affine toric varieties

If $N \cong \mathbb{Z}^n$ is a lattice, we write $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ for the real span of its points.

Definition 2.3.1. A (rational polyhedral) *cone* σ in $N_{\mathbb{R}}$ is the positive real span in $N_{\mathbb{R}}$ of finitely many lattice points in N . That is,

$$\sigma = \mathbb{R}_+ S = \left\{ \sum_{u \in S} \lambda_u u \mid \lambda_u \in \mathbb{R}_{\geq 0} \right\}$$

for some finite subset $S \subseteq N$. The *dimension* of σ is the dimension of the vector space $\mathbb{R}\sigma$.

Examples of cones include orthants (e.g. quadrants in $\mathbb{R}^2$, octants in $\mathbb{R}^3$, etc.), rays emanating from the origin, and the origin and $\mathbb{R}^n$ themselves. All of these examples, with the exception of $\mathbb{R}^n$, are strongly convex according to the following definition.

Definition 2.3.2. A cone is *strongly convex* if it contains no positive-dimensional subspaces. Henceforth in this document, all cones shall be assumed strongly convex. A *ray* is a 1-dimensional cone.

Let $M = N^*$ be the dual lattice of N , so $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ is the dual vector space of $N_{\mathbb{R}}$. The *dual cone* of σ is the cone $\sigma^{\vee} = \{m \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma\}$ in $M_{\mathbb{R}}$. If σ is a strongly convex rational polyhedral cone, then the lattice points $\sigma^{\vee} \cap M$ of $\sigma^{\vee}$ form a semigroup called S_{σ} . From S_{σ} we may form its semigroup algebra $\mathbb{C}[S_{\sigma}]$, which leads to the classic characterization of affine toric varieties.

Proposition 2.3.3. Let X be an affine variety. Then X is a toric variety for the torus T_N if and only if $X \cong X_{\sigma} = \text{Spec } \mathbb{C}[S_{\sigma}]$ for some cone $\sigma \subset N_{\mathbb{R}}$. In this case, the open immersion $T \rightarrow X$ is given by the dual of the natural inclusion $\mathbb{C}[S_{\sigma}] \rightarrow \mathbb{C}[M]$.

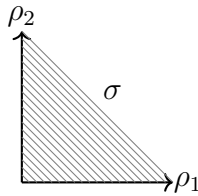
Definition 2.3.4. A *face* of the cone σ is a sub-cone $H_m \cap \sigma$ of σ , where $H_m := \ker m \cap \sigma = \{u \in \sigma \mid \langle m, u \rangle = 0\}$ for some $m \in \sigma^{\vee}$. A *facet* is a face of codimension 1.

Faces of σ correspond naturally to the open affine toric subvarieties of X_{σ} , as well as to the T -orbits in X_{σ} . We first expound the nature of faces before examining this correspondence in more depth.

Lemma 2.3.5. *Let σ be a cone. Then:*

- (a) *Every face of σ is a cone.*
- (b) *An intersection of two faces of σ is itself a face of σ .*
- (c) *A face of a face of σ is itself a face of σ .*

Example 2.3.6. Let $N = \mathbb{Z}^2$ and $\sigma = \mathbb{R}_+\{e_1, e_2\}$, shown in the figure below, be the first quadrant. Then σ has four faces: itself, the origin, and the two rays ρ_1, ρ_2 emanating from the origin.



Since $\mathbb{C}[S_\sigma] = \mathbb{C}[t_1, t_2]$, it follows that $X_\sigma \cong \mathbb{A}^2$.

Every ray ρ contains a unique *primitive generator*, the smallest nonzero lattice point along ρ . Likewise, every cone σ has a unique minimal *primitive generating set*, given by the primitive generators of its 1-dimensional faces.

Lemma 2.3.7. *Let σ be a cone which is not necessarily strongly convex. Then the following are equivalent:*

- (a) *σ is strongly convex*
- (b) *the origin is a face of σ*
- (c) $\dim \sigma^\vee = \dim N$

There is a partial ordering on the set of cones in $N_{\mathbb{R}}$ given by $\tau \leq \sigma$ if τ is a face of σ . The partially ordered set of the faces of σ is isomorphic to set of affine open toric subvarieties of X_σ , ordered by inclusion. At the level of coordinate algebras, the inclusion $X_\tau \subset X_\sigma$ is dual to the localization $\mathbb{C}[S_\sigma] \rightarrow \mathbb{C}[S_\tau]$ via inversion of a character m such that $\tau = H_m \cap \sigma$.

Likewise, the T -orbits of X_σ are enumerated by the faces of σ . In particular, if τ is a face, then the corresponding orbit is $\text{Orb}(\tau) := X_\tau - \bigcup_{\tau' < \tau} X_{\tau'}$, and the closure of $\text{Orb}(\tau)$ is $\bigcup_{\tau' \geq \tau} \text{Orb}(\tau')$. The dimension of τ is the codimension of $\text{Orb}(\tau)$ in X_σ , so X_σ has a fixed point if and only if σ is full-dimensional.

2.4 Fans and arbitrary toric varieties

The language of cones is a powerful one, as it allows us to translate results about toric varieties into results about convex geometry, and vice-versa. Moreover, it allows for a nice characterization of arbitrary normal toric varieties via polyhedral fans:

Definition 2.4.1. A *fan* Σ in $N_{\mathbb{R}}$ is a finite collection of cones σ such that Σ is closed under taking faces, and the intersection of any two cones in Σ is a face of each. A *morphism of fans* $(N, \Sigma) \rightarrow (N', \Sigma')$ is a lattice map $\bar{\phi} : N \rightarrow N'$ such that for every cone $\sigma \in \Sigma$, there is $\sigma' \in \Sigma'$ such that $\phi_{\mathbb{R}}(\sigma) \subseteq \sigma'$.

One may construct a toric variety X_{Σ} from a fan Σ by gluing together the open affines X_{σ} for $\sigma \in \Sigma$. In particular, we glue X_{σ} to $X'_{\sigma'}$ along $X_{\sigma \cap \sigma'}$. The following theorem, due to Sumihiro [48], says that every toric variety arises in this way.

Theorem 2.4.2. *The category of toric varieties is equivalent to the category of fans.*

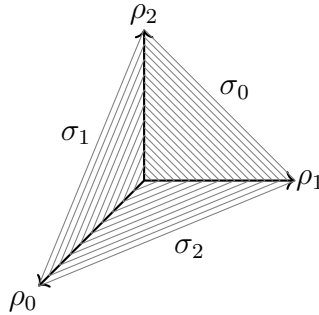
The correspondence between toric varieties and fans conveys deep geometric information. We provide only a taste of that here, directing the reader to [9] for more details.

Proposition/Definition 2.4.3. Let $\Sigma \subseteq N_{\mathbb{R}}$ be a fan, and $\sigma \in \Sigma$ a cone.

- (a) We say σ is *smooth* if its primitive generators form a basis for N . A fan is smooth if all its cones are.
- (b) We say a fan is *complete* if its support $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$ equals $N_{\mathbb{R}}$.

The toric variety X_{Σ} is smooth (respectively, complete) if and only if Σ is smooth (respectively, complete).

Example 2.4.4. Let ρ_1, ρ_2 be as before, and add a third ray $\rho_0 = \mathbb{R}_+(-x_1 - x_2)$. The complete fan Σ on this set of rays is depicted in the following figure.



We have

$$\begin{aligned} \mathbb{C}[S_{\sigma_0}] &= \mathbb{C}[t_1, t_2] \\ \mathbb{C}[S_{\sigma_1}] &= \mathbb{C}[t_1^{-1}, t_1^{-1}t_2] \\ \mathbb{C}[S_{\sigma_2}] &= \mathbb{C}[t_1t_2^{-1}, t_2^{-1}], \end{aligned}$$

from which we see that each $X_{\sigma_i} \cong \mathbb{A}^2$ is an affine chart on $X_{\Sigma} \cong \mathbb{P}^2$.

Remark 2.4.5. As in the prior section, cones of a fan Σ correspond bijectively with both the T -orbits in X_{Σ} , and the T -stable subvarieties of X_{Σ} .

The following proposition occurs in [12].

Proposition 2.4.6. *If Σ contains at least one cone of maximal dimension, then $\text{Pic}(X_\Sigma) \cong \mathbb{Z}^{r-n-\lambda}$ where r is the number of rays in Σ , n is its dimension, and λ is the dimension of the space of linear dependencies of rays of maximal cones.*

2.5 Polytopes and projective toric varieties

A particularly elegant refinement of the correspondence between convex geometry and toric varieties arises in the projective setting.

Definition 2.5.1. A *lattice polytope* $P \subset M$ is the convex hull in $M_{\mathbb{R}}$ of finitely many points in M .

To such a polytope, we may associate a projective toric variety by considering the semigroup of lattice points in the cone over P . More concretely, define the cone

$$\sigma_P = \{(u, t) \in M_{\mathbb{R}} \times \mathbb{R}_{\geq 0} \mid u \in tP\}.$$

The semigroup $S_P = \sigma_P \cap (M \times \mathbb{Z})$ determines a finitely generated graded algebra

$$\mathbb{C}[S_P] = \bigoplus_{k \geq 0} \left(\bigoplus_{u \in kP \cap M} \mathbb{C} \cdot \chi^u \right),$$

and the associated projective variety

$$X_P := \text{Proj } \mathbb{C}[S_P]$$

is a projective toric variety.

This construction yields a polarized toric variety $(X_P, \mathcal{O}_{X_P}(1))$, where the ample line bundle corresponds to the grading on $\mathbb{C}[S_P]$. The lattice points of P index a basis of global sections of this line bundle, giving a natural embedding

$$X_P \hookrightarrow \mathbb{P}^{|P \cap M| - 1}.$$

Conversely, every projective toric variety arises from a lattice polytope in this way. More precisely, if X is a projective toric variety together with an ample T -invariant line bundle $\mathcal{L}$, then the associated moment polytope

$$P_{\mathcal{L}} = \{u \in M_{\mathbb{R}} \mid \chi^u \text{ extends to a global section of } \mathcal{L}\}$$

is a lattice polytope, and $X \cong X_{P_{\mathcal{L}}}$.

Definition 2.5.2. For each face $F \subseteq P$, the associated *normal cone* is

$$\sigma_F = \{v \in N_{\mathbb{R}} \mid \langle u', v \rangle \leq \langle u, v \rangle \text{ for all } u' \in P \text{ and all } u \in F\},$$

where $N = \text{Hom}(M, \mathbb{Z})$ is the dual lattice. The collection of cones σ_F as F ranges over all faces of P forms a complete fan Σ_P in $N_{\mathbb{R}}$, called the *normal fan* of P .

From the perspective of fans, Σ_P recovers the underlying toric variety X_P (without polarization). In this sense, the polytope encodes strictly more data: it determines both the toric variety and a choice of ample line bundle. Two lattice polytopes give rise to isomorphic polarized toric varieties if and only if they differ by a lattice translation.

Chapter 3

Supergeometry

In this chapter we develop the requisite supergeometry to study toric supervarieties. Essentially, it is supergeometry in the category of super vector spaces. Throughout, $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ and k is a field of characteristic not equal to 2 or 3. Starting in section 3.6, we will focus primarily on the field $\mathbb{C}$ of complex numbers, though any algebraically closed field of characteristic 0 would work just as well.

3.1 Super vector spaces

Definition 3.1.1. The category $\mathbf{SVec}$ of super vector spaces over k has as its objects the $\mathbb{Z}_2$ -graded k -vector spaces $V = V_{\bar{0}} \oplus V_{\bar{1}}$, with morphisms $\mathrm{Hom}(V, W)$ the linear maps $V \rightarrow W$. If $v \in V_{\bar{i}}$ is a homogeneous element, we write $|v| = \bar{i}$ for its parity. Note that $\mathrm{Hom}(V, W)$ is itself a super vector space, with $\mathrm{Hom}(V, W)_{\bar{0}}$ the parity-preserving linear maps and $\mathrm{Hom}(V, W)_{\bar{1}}$ the parity-reversing linear maps. In this sense, $\mathbf{SVec}$ is enriched in itself.

Example 3.1.2. For $m, n \in \mathbb{Z}_{\geq 0}$, the super vector space $k^{m|n}$ has grading

$$(k^{m|n})_{\bar{0}} = k^m \quad \text{and} \quad (k^{m|n})_{\bar{1}} = k^n.$$

Definition 3.1.3. The tensor product of V and W has grading

$$\begin{aligned} (V \otimes W)_{\bar{0}} &= (V_{\bar{0}} \otimes W_{\bar{0}}) \oplus (V_{\bar{1}} \otimes W_{\bar{1}}) \\ (V \otimes W)_{\bar{1}} &= (V_{\bar{0}} \otimes W_{\bar{1}}) \oplus (V_{\bar{1}} \otimes W_{\bar{0}}). \end{aligned}$$

Perhaps the most significant difference between the category of super vector spaces and the usual category of vector spaces is the braiding in the symmetric monoidal structure. In the super case, the braiding isomorphism is

$$\begin{aligned} s_{V,W} : V \otimes W &\rightarrow W \otimes V \\ s_{V,W}(v \otimes w) &= (-1)^{|v||w|} w \otimes v \end{aligned}$$

on homogeneous elements $v \in V, w \in W$. That is, if the order of two odd elements is reversed, a -1 sign appears.

Writing $k := k^{1|0}$, the category $\mathbf{SVec}$ has duals given by $V^* = \text{Hom}(V, k) \cong V_0^* \oplus V_1^*$, and admits a natural identification $\text{Hom}(V, W) \cong W \otimes V^*$. We obtain a notion of *superdimension* $\text{sdim}(V)$ via the composition

$$k \xrightarrow{\text{coev}} V \otimes V^* \xrightarrow{s_{V, V^*}} V^* \otimes V \xrightarrow{\text{ev}} k,$$

yielding $\text{sdim}(V) = \dim(V_0) - \dim(V_1)$. More commonly, we will simply refer to the dimension of V , the tuple $\dim(V) = (\dim V_0 | \dim V_1)$.

Definition 3.1.4. If V is a super vector space, we define its parity shift to be $\Pi V = k^{0|1} \otimes V$; i.e. $(\Pi V)_0 = V_1$ and $(\Pi V)_1 = V_0$. The endofunctor Π on $\mathbf{SVec}$ is an involution in the sense that $\Pi^2 V$ is naturally isomorphic to V .

The object

$$V^{\otimes m|n} = \underbrace{V \otimes \cdots \otimes V}_{m \text{ times}} \otimes \underbrace{\Pi V \otimes \cdots \otimes \Pi V}_{n \text{ times}}$$

is well-defined via the usual universal property with respect to

$$V^{m|n} = \underbrace{V \times \cdots \times V}_{m \text{ times}} \times \underbrace{\Pi V \times \cdots \times \Pi V}_{n \text{ times}}.$$

We will write $V^{\otimes n} := V^{\otimes n|0}$.

Isomeric vector spaces

Definition 3.1.5. An *isomeric vector space* is a super vector space V equipped with an odd involution $\Pi_V : V \rightarrow V$. Likewise, an *isomeric subspace* of V is a subspace which is preserved by Π_V . Essentially, it is just a super vector space with a preferred isomorphism between the even and odd parts.¹

Let V and W be isomeric vector spaces and $f \in \text{Hom}(V, W)$. We say f is a *isomeric* if $f \circ \Pi_V = (-1)^{|f|} \Pi_W \circ f$, and we write $\text{Hom}_\Pi(V, W)$ for the isomeric morphisms $V \rightarrow W$.

3.2 Superalgebras

Definition 3.2.1. A *superalgebra* is an algebra object in the category of super vector spaces; that is, it is a super vector space $\mathcal{A}$ together with a multiplication morphism $\tau : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$.

- (a) It is *commutative* if $\tau \circ s_{\mathcal{A}, \mathcal{A}} = \tau$.
- (b) It is *associative* if $\tau \circ \tau \circ \text{id} = \tau \circ \text{id} \circ \tau$.
- (c) It is *unital* if there is $1 \in \mathcal{A}_0$ such that $\tau(1 \otimes a) = \tau(a \otimes 1) = a$ for all $a \in \mathcal{A}$.

¹We prefer the term *isomeric*, as suggested in [35], to the more dated term *queer*.

The tensor product $\mathcal{A} \otimes \mathcal{B}$ of two superalgebras $\mathcal{A}$ and $\mathcal{B}$ is a superalgebra with multiplication

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|}(ac \otimes bd)$$

on homogeneous elements. Note again that a -1 sign appears exactly when the order of two odd elements is reversed.

Example 3.2.2. Let V be a super vector space. The *tensor superalgebra* of V is

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$$

together with the the product operation coming from the standard bilinear maps $V^{\otimes r} \otimes V^{\otimes s} \rightarrow V^{\otimes(r+s)}$.

If $\sigma \in S_n$ is a permutation, we write

$$\sigma(v_1 \otimes \cdots \otimes v_n) = (-1)^{|\sigma|}(v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}),$$

where $|\sigma|$ is the sign of σ , for the natural action of S_n on $V^{\otimes n}$. We write $\mathfrak{a}_n$ and $\mathfrak{b}_n$, respectively, for the subspaces of $V^{\otimes n}$ spanned by

$$v_1 \otimes \cdots \otimes v_n - \sigma_n(v_1 \otimes \cdots \otimes v_n)$$

and

$$v_1 \otimes \cdots \otimes v_n - (-1)^{|\sigma|}\sigma_n(v_1 \otimes \cdots \otimes v_n),$$

respectively. The respective quotients form the *nth symmetric power*

$$S^n(V) = V^{\otimes n} / \mathfrak{a}_n$$

and the *nth exterior power*

$$\Lambda^n(V) = V^{\otimes n} / \mathfrak{b}_n.$$

The corresponding infinite direct sums $\mathfrak{a} = \bigoplus_{n \geq 0} \mathfrak{a}_n$ and $\mathfrak{b} = \bigoplus_{n \geq 0} \mathfrak{b}_n$ are ideals whose respective quotients yield the *symmetric superalgebra*

$$S(V) = T(V) / \mathfrak{a}$$

and the *exterior superalgebra*

$$\Lambda(V) = T(V) / \mathfrak{b}.$$

Both are unital associative algebras, and the symmetric superalgebra is moreover commutative.

3.3 Lie superalgebras

Definition 3.3.1. A *Lie superalgebra* is a Lie algebra object in $\mathbf{SVec}$. More concretely, it is a super vector space

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

equipped with a morphism $[-, -] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ called the *bracket*, such that

$$(a) \quad [x, y] + (-1)^{|x||y|}[y, x] = 0$$

$$(b) \quad [x, [y, z]] + (-1)^{|x||y|+|x||z|}[y, [z, x]] + (-1)^{|y||z|+|x||z|}[z, [x, y]] = 0$$

for homogeneous $x, y, z \in \mathfrak{g}$. A morphism of Lie superalgebras is a morphism of super vector spaces which preserves the bracket. Note that the two axioms are simply antisymmetry and the Jacobi identity, with the addition of the “super” sign rule arising from the braiding $s_{\mathfrak{g}, \mathfrak{g}}$.

Observe that $\mathfrak{g}_{\bar{0}}$ is an ordinary Lie algebra, and $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$ -module via the adjoint action. On the odd part, the bracket is a morphism $S^2 \mathfrak{g}_{\bar{1}} \rightarrow \mathfrak{g}_{\bar{0}}$.

Example 3.3.2. Any associative superalgebra $\mathcal{A}$ can be made into a Lie superalgebra by means of the commutator bracket

$$[a, b] = ab - (-1)^{|a||b|}ba$$

on homogeneous elements.

Example 3.3.3. Consider the associative superalgebra $\text{End}(V) = \text{Hom}(V, V)$. As in the prior example, it can be made into a Lie superalgebra with the commutator bracket

$$[X, Y] = XY - (-1)^{|X||Y|}YX,$$

and we refer to it as $\mathfrak{gl}(V)$.

We will often write $\mathfrak{gl}(m|n) := \mathfrak{gl}(k^{m|n})$. We have

$$\begin{aligned} \mathfrak{gl}(m|n)_{\bar{0}} &= \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \mid A \in \mathfrak{gl}(n), D \in \mathfrak{gl}(m) \right\} \quad \text{and} \\ \mathfrak{gl}(m|n)_{\bar{1}} &= \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \mid B \in \text{Mat}_{m \times n}, C \in \text{Mat}_{n \times m} \right\}. \end{aligned}$$

If $X \in \mathfrak{gl}(V)$, we may compute its *supertrace* $\text{str}(X)$ using the usual diagram

$$k \xrightarrow{\text{coev}} V \otimes V^* \xrightarrow{X \otimes \text{id}} V \otimes V^* \xrightarrow{s_{V, V^*}} V^* \otimes V \xrightarrow{\text{ev}} k,$$

obtaining $\text{str}(X) = \text{tr}(X|_{V_{\bar{0}}}) - \text{tr}(X|_{V_{\bar{1}}})$ as the aggregate composition. Thus, if

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{gl}(m|n),$$

then $\text{str}(X) = \text{tr}(A) - \text{tr}(D)$.

Example 3.3.4. The special linear Lie superalgebra on V is

$$\mathfrak{sl}(V) = \{X \in \mathfrak{gl}(V) \mid \text{str}(X) = 0\},$$

so

$$\mathfrak{sl}(m|n) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{gl}(m|n) \mid \text{tr}(A) - \text{tr}(D) = 0 \right\}.$$

Many other classical series exist; see [8] for a more complete list, including $\mathfrak{osp}(m|2n)$, $D(2|1, \alpha)$, $F(3|1)$, $G(3)$, and others of more exotic type. Because it is central to many of the important results of this dissertation, we spell out the type Q case here:

Example 3.3.5. Let (V, Π_V) be an isomeric vector space, and define

$$\mathfrak{q}(V) = \{X \in \mathfrak{gl}(V) \mid [X, \Pi_V] = 0\}$$

to be the subalgebra of endomorphisms that commute with Π_V . If we fix the “identity” isomorphism between $k^{n|0}$ and $k^{0|n}$ in $k^{n|n}$, then we may write

$$\mathfrak{q}(n) := \mathfrak{q}(k^{n|n}) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right\}.$$

Example 3.3.6. Let $\mathcal{A}$ be a superalgebra. We say a linear transformation $D \in \text{End}(\mathcal{A})$ is a derivation if $D(ab) = D(a)b + (-1)^{|a||D|}aD(b)$ for all $a, b \in \mathcal{A}$. The subspace $\text{Der}(\mathcal{A})$ of $\text{End}(\mathcal{A})$ is closed under the commutator bracket and is therefore a Lie superalgebra.

Universal enveloping superalgebra and representations

Definition 3.3.7. The *universal enveloping superalgebra* of a Lie superalgebra $\mathfrak{g}$ is $U(\mathfrak{g}) = T(\mathfrak{g})/I$, where I is the ideal generated by elements of the form $xy - (-1)^{|x||y|}yx - [x, y]$ for all $x, y \in \mathfrak{g}$. It admits a natural filtration by degree-at-most- k monomials $U^k(\mathfrak{g})$, where in particular $U^0(\mathfrak{g}) = k$ and $U^1(\mathfrak{g}) = k \oplus \mathfrak{g}$.

By the PBW theorem, the natural morphism $\mathfrak{g} \rightarrow U(\mathfrak{g})$ is injective, and the natural morphism $S\mathfrak{g} \rightarrow \text{gr } U(\mathfrak{g})$ is an isomorphism. The universal enveloping superalgebra admits the structure of a cocommutative Hopf superalgebra with comultiplication $x \mapsto x \otimes 1 + 1 \otimes x$, counit $x \mapsto 0$, and antipode $x \mapsto -x$ for all $x \in \mathfrak{g}$. There is a natural correspondence between Lie superalgebra morphisms $\mathfrak{g} \rightarrow \mathcal{A}$ and superalgebra morphisms $U\mathfrak{g} \rightarrow \mathcal{A}$ for $\mathcal{A}$ any associative superalgebra.

Definition 3.3.8. A *representation* of a Lie superalgebra $\mathfrak{g}$ on a super vector space V is a morphism of Lie superalgebras $\mathfrak{g} \rightarrow \mathfrak{gl}(V)$, or equivalently a morphism of associative algebras $U(\mathfrak{g}) \rightarrow \mathfrak{gl}(V)$. We will often refer to V itself as a $\mathfrak{g}$ -representation or a $\mathfrak{g}$ -module.

The representations of $\mathfrak{g}$ form an abelian, symmetric monoidal category. Let V, V_1, V_2 be representations of $\mathfrak{g}$. The action of $\mathfrak{g}$ on $V_1 \otimes V_2$ is given by $x \cdot (v_1 \otimes v_2) = (x \cdot v_1) \otimes v_2 + (-1)^{|x||v_1|}v_1 \otimes (x \cdot v_2)$, and the action of $\mathfrak{g}$ on V^* is given by $(x \cdot \varphi)(v) = -(-1)^{|x||\varphi|}\varphi(x \cdot v)$. The action of $\mathfrak{g}$ on ΠV is given by the action on $k^{0|1} \otimes V$, where $\mathfrak{g}$ acts by 0 on $k^{0|1}$.

Quasireductive Lie superalgebras

Definition 3.3.9. A finite-dimensional Lie superalgebra $\mathfrak{g}$ is *quasireductive* if $\mathfrak{g}_{\bar{0}}$ is reductive and the adjoint action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is semisimple.

Quasireductive Lie superalgebras encompass virtually all examples that will arise in this thesis. An important feature is the existence of a *Cartan subalgebra* $\mathfrak{h}$: Since $\mathfrak{g}_{\bar{0}}$ is reductive it will have a Cartan subalgebra $\mathfrak{h}_{\bar{0}}$; the centralizer of $\mathfrak{h}_{\bar{0}}$ in $\mathfrak{g}$ is a Lie superalgebra $\mathfrak{h}$ whose even part is indeed $\mathfrak{h}_{\bar{0}}$. Such Cartan subalgebras are quasiabelian in the following sense:

Definition 3.3.10. A Lie superalgebra $\mathfrak{h}$ is *quasiabelian* if $\mathfrak{h}_{\bar{0}}$ lies in the center of $\mathfrak{h}$.

In many important cases, such as in $\mathfrak{sl}(m|n)$ and $\mathfrak{osp}(m|2n)$, a Cartan subalgebra is purely even. In $\mathfrak{q}(n)$, however, there is a Cartan subalgebra given by

$$\mathfrak{q}(1)^n \cong \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mid A, B \text{ diagonal} \right\}.$$

All Cartan subalgebras of a fixed quasireductive Lie superalgebra are isomorphic.

3.4 Commutative superalgebras and superrings

Henceforth all superalgebras will be commutative, associative, and unital unless stated otherwise. A finitely generated superalgebra is isomorphic to a quotient of the polynomial superalgebra

$$k[x_1, \dots, x_m, \xi_1, \dots, \xi_n]$$

in m even variables and n odd variables. We note that Latin (respectively, Greek) letters will typically be used for even (respectively, odd) elements of a superalgebra. Due to the commutativity rule $ab = (-1)^{|a||b|}ba$, odd elements ξ are always nilpotent: $\xi^2 = 0$.

Analogously, a superring is a $\mathbb{Z}_2$ -graded ring $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$. We likewise assert that all superrings will be commutative, associative, and unital.

Ideals and Modules

Many similarities exist between the theories of ideals in the classical and super cases.

Let $\mathcal{A}$ be a commutative superring. The notions of left, right, and two-sided ideals all coincide in $\mathcal{A}$, and must be compatibly $\mathbb{Z}_2$ -graded. A prime ideal $\mathfrak{p}$ is one that satisfies the usual condition: if $ab \in \mathfrak{p}$, then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. The ideal $\mathfrak{p} \subset \mathcal{A}$ is prime if and only if $\mathcal{A}/\mathfrak{p}$ is an integral domain. Likewise, a maximal ideal is one that is maximal with respect to inclusion; $\mathfrak{m} \subset \mathcal{A}$ is maximal if and only if $\mathcal{A}/\mathfrak{m}$ is a field.

Maximal (respectively, prime) ideals in particular correspond bijectively to maximal (respectively, prime) ideals of $\mathcal{A}_{\bar{0}}$, or equivalently $\mathcal{A}/(\mathcal{A}_{\bar{1}})$, since prime ideals contain all nilpotents. As above, if $S \subseteq \mathcal{A}$, we write (S) for the ideal generated by S .

A (left) $\mathcal{A}$ -module (for $\mathcal{A}$ a superalgebra) is a super vector space M equipped with a morphism $\mathcal{A} \otimes_k M \rightarrow M$ of super vector spaces satisfying the usual axioms. Right modules are defined analogously, and we observe that we can define a right module structure on a left module M (or vice versa) via the sign rule

$$m \cdot a = (-1)^{|a||m|}am.$$

Modules over superrings are defined similarly.

Localization

Localization behaves the same as in the ordinary situation, except that we insist the multiplicatively closed subset $S \subset \mathcal{A}$ that we invert must be contained in $\mathcal{A}_{\bar{0}}$. That is, if S is such a subset, $S^{-1}\mathcal{A} = S^{-1}\mathcal{A}_{\bar{0}} \oplus S^{-1}\mathcal{A}_{\bar{1}}$ where we treat $\mathcal{A}_{\bar{1}}$ as an $\mathcal{A}_{\bar{0}}$ -module. To avoid degenerate cases, we will further assume $0 \notin S$, so that localization preserves non-zero divisors.

We write $\text{zdiv}(\mathcal{A})$ for the set of zero divisors of $\mathcal{A}$, $\text{nonzdiv}(\mathcal{A})$ for the non-zero divisors, $\mathcal{A}^\times$ for the units, and $\text{Nil}(\mathcal{A})$ for the nilpotents.

Definition 3.4.1. The *total superring of fractions* of $\mathcal{A}$ is $K(\mathcal{A}) := \text{nonzdiv}(\mathcal{A})^{-1}\mathcal{A}$. We denote by $i : \mathcal{A} \rightarrow K(\mathcal{A})$ the natural inclusion, by $J_{\mathcal{A}}$ the ideal $i^{-1}((K(\mathcal{A})_{\bar{1}})) \subseteq \mathcal{A}$, and by $\overline{\mathcal{A}}$ the *underlying ring* $\mathcal{A}/J_{\mathcal{A}}$. If S is any subset of $\mathcal{A}$, we write $\overline{S}$ for its image in $\overline{\mathcal{A}}$.

Notice that our usage of the label $J_{\mathcal{A}}$ differs from the common convention, in which people use $J_{\mathcal{A}}$ to refer to the smaller ideal $(\mathcal{A}_{\bar{1}})$.

Example 3.4.2. Let $\mathcal{A} = k[x, \xi, \eta, y]/(\xi\eta - xy, y\xi, y\eta, y^2) \cong k[x, x\xi_1, \xi_2, \xi_1\xi_2]$. Then $(\mathcal{A}_{\bar{1}}) = (\xi, \eta)$ but $J_{\mathcal{A}} = (\xi, \eta, y)$. The difference is that $J_{\mathcal{A}}$ captures every element which generically arises from the odd part, while $(\mathcal{A}_{\bar{1}})$ captures only those which globally arise from the odd part.

We record the following super version of the prime avoidance lemma, whose proof is completely identical to that of the standard version.

Lemma 3.4.3. *Let I be an ideal of a superring $\mathcal{A}$ such that $I \subseteq \mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_n$ for some prime ideals $\mathfrak{p}_i$. Then $I \subseteq \mathfrak{p}_i$ for some i .*

Reduced superrings and integral superdomains

We introduce some terminology which has seen conflicting usage in the literature. We believe the following definitions are most in line with the classical situation. The common theme here is that badly-behaved elements should all arise from the odd part of the total superring of fractions.

Definition 3.4.4. Let $\mathcal{A}$ be a superring.

- (a) $\mathcal{A}$ is a *superfield* if $J_{\mathcal{A}} = \mathcal{A} \setminus \mathcal{A}^\times$
- (b) $\mathcal{A}$ is an *integral superdomain* if $J_{\mathcal{A}} = \text{zdiv}(\mathcal{A})$
- (c) $\mathcal{A}$ is *reduced* if $J_{\mathcal{A}} = \text{Nil}(\mathcal{A})$ and $\overline{\text{zdiv}(\mathcal{A})} = \text{zdiv}(\overline{\mathcal{A}})$

In Lemma 3.4.6 we will provide several equivalent characterizations of these types of superrings. We first prove a lemma that will allow us to conflate $\overline{\mathcal{A}_{\mathfrak{p}}}$ and $\overline{\mathcal{A}}_{\overline{\mathfrak{p}}}$ for prime ideals $\mathfrak{p}$ of a reduced superring $\mathcal{A}$.

Lemma 3.4.5. *Let $\mathcal{A}$ be a reduced superring and $S \subseteq \mathcal{A}_{\bar{0}}$ a multiplicatively closed subset.*

- (a) $J_{S^{-1}\mathcal{A}} = S^{-1}J_{\mathcal{A}}$

$$(b) \overline{S^{-1}\mathcal{A}} = \overline{S}^{-1}\overline{\mathcal{A}}$$

Proof. (a) Since $\text{Nil}(S^{-1}\mathcal{A}) = S^{-1}\text{Nil}(\mathcal{A}) = S^{-1}J_{\mathcal{A}}$, it suffices to show that $\text{Nil}(S^{-1}\mathcal{A}) = J_{S^{-1}\mathcal{A}}$. Let $\frac{a}{s} \in S^{-1}\mathcal{A}$ be nilpotent, so there is $t \in S$ such that $ta^n = 0$. Then $ta \in \text{Nil}(\mathcal{A}) = J_{\mathcal{A}}$, so $ta = \frac{\eta}{b}$ for $\eta \in (\mathcal{A}_{\overline{1}})$ and $b \in \text{nonzdiv}(\mathcal{A})$. Then $b \cdot \frac{a}{s} = \frac{\eta}{st}$ in $S^{-1}\mathcal{A}$ for $\frac{\eta}{st} \in (S^{-1}\mathcal{A}_{\overline{1}})$. Hence $\frac{a}{s} \in J_{S^{-1}\mathcal{A}}$, since localization preserves non-zero divisors. Therefore $\text{Nil}(S^{-1}\mathcal{A}) = J_{S^{-1}\mathcal{A}}$ and we are finished.

(b) Follows immediately from part (a). $\square$

Lemma 3.4.6. *Let $\mathcal{A}$ be a superring.*

(a) *The following are equivalent:*

- (i) $\mathcal{A}$ is a superfield
- (ii) $\overline{\mathcal{A}}$ is a field
- (iii) $J_{\mathcal{A}} = (\mathcal{A}_{\overline{1}})$ is the set of non-units
- (iv) $(\mathcal{A}_{\overline{1}})$ is a maximal ideal

(b) *The following are equivalent:*

- (i) $\mathcal{A}$ is an integral superdomain
- (ii) $\overline{\mathcal{A}}$ is an integral domain and $\overline{\text{zdiv}(\mathcal{A})} = \text{zdiv}(\overline{\mathcal{A}})$
- (iii) $K(\mathcal{A})$ is a superfield
- (iv) $\mathcal{A}$ is reduced and $\text{Spec } \mathcal{A}$ is irreducible

(c) *The following are equivalent:*

- (i) $\mathcal{A}$ is a reduced superring
- (ii) $\overline{\mathcal{A}}$ is a reduced ring and $\overline{\text{zdiv}(\mathcal{A})} = \text{zdiv}(\overline{\mathcal{A}})$
- (iii) $J_{\mathcal{A}} = \text{Nil}(\mathcal{A})$ and $\text{zdiv}(\mathcal{A}) = \bigcup_{\mathfrak{p} \in \min \text{Spec } \mathcal{A}} \mathfrak{p}$
- (iv) $\overline{\mathcal{A}}$ is a reduced ring and $\text{zdiv}(\mathcal{A}) = \bigcup_{\mathfrak{p} \in \min \text{Spec } \mathcal{A}} \mathfrak{p}$
- (v) $S^{-1}\mathcal{A}$ is a reduced superring for any multiplicatively closed subset $S \subseteq \mathcal{A}_{\overline{0}}$
- (vi) $\mathcal{A}_{\mathfrak{m}}$ is a reduced superring for any maximal ideal $\mathfrak{m}$

Proof. (a) The equivalence of (iii) and (iv) is immediate since every maximal ideal contains $(\mathcal{A}_{\overline{1}})$. These equivalent conditions imply $(\mathcal{A}_{\overline{1}}) = J_{\mathcal{A}}$, which then implies (i) and (ii), which are equivalent because every maximal ideal contains $J_{\mathcal{A}}$.

Let us now prove (i) $\implies$ (iv). It suffices to show that $J_{\mathcal{A}} = (\mathcal{A}_{\overline{1}})$. Let $\eta \in J_{\mathcal{A}}$, so we may write $\eta = \frac{\eta'}{a}$ for $\eta' \in (\mathcal{A}_{\overline{1}})$ and $a \in \text{nonzdiv}(\mathcal{A})$. But $a \notin J_{\mathcal{A}}$ implies $\mathcal{A}$ is a unit, so indeed $\eta \in (\mathcal{A}_{\overline{1}})$ and we are finished.

- (b) First assume (i), so that $K(\mathcal{A}) = \mathcal{A}_{J_{\mathcal{A}}}$ has maximal ideal $K(\mathcal{A})J_{\mathcal{A}} = (K(\mathcal{A})_{\overline{1}})$ and (iii) follows. For the converse, note that a zero divisor $a \in \mathcal{A}$ does not receive an inverse in $K(\mathcal{A})$, so $a \in J_{\mathcal{A}}$. Thus, we have established (i) $\iff$ (iii).

Now observe that (ii) just says $\overline{\text{zdiv}(\mathcal{A})} = 0$, so indeed $J_{\mathcal{A}} = \text{zdiv}(\mathcal{A})$, which is (i). Conversely, since nonzerodivisors form a multiplicatively closed subset, $\mathcal{A}$ integral implies $\overline{\mathcal{A}}$ integral and so we obtain (i) $\iff$ (ii). Finally, (ii) $\iff$ (iv) follows from part (c).

- (c) That (i) $\iff$ (ii) and (iii) $\iff$ (iv) is immediate from the definitions. To see the equivalence of (i) and (ii) with (iii) and (iv), it suffices to show that if $\overline{\mathcal{A}}$ is reduced, then $\overline{\text{zdiv}(\mathcal{A})} = \text{zdiv}(\overline{\mathcal{A}})$ if and only if $\text{zdiv}(\mathcal{A}) = \bigcup_{\mathfrak{p} \in \min \text{Spec } \mathcal{A}} \mathfrak{p}$.

For the forward implication, let $a \in \text{zdiv}(\mathcal{A})$, so there is $b \in \mathcal{A}$ such that $b \notin J_{\mathcal{A}}$ and $ab \in J_{\mathcal{A}}$. Then $ab \in \mathfrak{p}$ for all primes $\mathfrak{p}$, but $b \notin \mathfrak{p}'$ for some minimal prime $\mathfrak{p}'$ since $b \notin \text{Nil}(\mathcal{A})$. Hence $a \in \mathfrak{p}'$, so $\text{zdiv}(\mathcal{A}) \subseteq \bigcup_{\mathfrak{p} \in \min \text{Spec } \mathcal{A}} \mathfrak{p}$. That minimal primes consist of zero divisors is standard. Conversely, if $\text{zdiv}(\mathcal{A}) = \bigcup_{\mathfrak{p} \in \min \text{Spec } \mathcal{A}} \mathfrak{p}$, then $\overline{\text{zdiv}(\mathcal{A})} = \overline{\bigcup_{\mathfrak{p} \in \min \text{Spec } \mathcal{A}} \mathfrak{p}} = \bigcup_{\mathfrak{p} \in \min \text{Spec } \mathcal{A}} \overline{\mathfrak{p}} = \text{zdiv}(\overline{\mathcal{A}})$ since $\overline{\mathcal{A}}$ is reduced.

We now show (iii) $\implies$ (v). That $\text{Nil}(S^{-1}\mathcal{A}) = J_{S^{-1}\mathcal{A}}$ is proven in Lemma 3.4.5. For $\text{zdiv}(S^{-1}\mathcal{A}) = \bigcup_{\mathfrak{p} \in \min \text{Spec } S^{-1}\mathcal{A}} \mathfrak{p}$, first take $\frac{a}{s} \in \text{zdiv}(S^{-1}\mathcal{A})$, so $a \in \text{zdiv}(\mathcal{A})$. Then a belongs to a minimal prime $\mathfrak{p}$ of $\mathcal{A}$. If $\mathfrak{p} \cap S = \emptyset$, then $S^{-1}\mathfrak{p}$ is a minimal prime of $S^{-1}\mathcal{A}$ and we are finished. Otherwise we may assume that a belongs to no such minimal prime. Let $\mathfrak{q}$ be a prime that does contain a , so $\mathfrak{q}$ contains a minimal prime that meets S . Then $S^{-1}\mathfrak{q} = S^{-1}\mathcal{A}$, so $s^{-1}a$ is a unit in $S^{-1}\mathcal{A}$, a contradiction. Hence $\text{zdiv}(S^{-1}\mathcal{A}) = \bigcup_{\mathfrak{p} \in \min \text{Spec } S^{-1}\mathcal{A}} \mathfrak{p}$, so indeed (iii) $\implies$ (v).

Since (v) $\implies$ (vi) is immediate, it remains to show (vi) $\implies$ (i). First let $a \in \text{Nil}(\mathcal{A})$, define $I = \{d \in \mathcal{A} \mid ad \in J_{\mathcal{A}}\}$, and pick a maximal ideal $\mathfrak{m}$. Then $\frac{a}{1} \in \text{Nil}(\mathcal{A}_{\mathfrak{m}}) = J_{\mathcal{A}_{\mathfrak{m}}}$, so $\frac{b}{t} \cdot \frac{a}{1} = \frac{\alpha}{s}$ for $\frac{\alpha}{s} \in ((\mathcal{A}_{\mathfrak{m}})_{\overline{1}})$ and $\frac{b}{t} \in \text{nonzdiv}(\mathcal{A}_{\mathfrak{m}})$. Hence there is $u \notin \mathfrak{m}$ such that $u(abs - t\alpha) = 0$ in $\mathcal{A}$. Since $u, b, s \notin \mathfrak{m}$ and $ut\alpha \in J_{\mathcal{A}}$, it follows that $ubs \in I \setminus \mathfrak{m}$, so I is not contained in $\mathfrak{m}$. Therefore I is the unit ideal, so indeed $a \in J_{\mathcal{A}}$.

Now if $\mathcal{A}$ is not reduced, there is a nonzero $\xi \in J_{\mathcal{A}}$ such that $\overline{\text{Ann}_{\mathcal{A}}(\xi)}$ is not contained in $\text{zdiv}(\overline{\mathcal{A}})$. Let $a \in \text{Ann}_{\mathcal{A}}(\xi)$ be such a “bad” element, so that $\overline{a} \notin \text{zdiv}(\overline{\mathcal{A}})$. Moreover, let $\mathfrak{m}$ be a maximal ideal containing $\text{Ann}_{\mathcal{A}}(\xi)$. Then $\text{Ann}_{\mathcal{A}}(a) \subseteq J_{\mathcal{A}} \subseteq \mathfrak{m}$, so the relationship $a\xi = 0$ remains nontrivial in $\mathcal{A}_{\mathfrak{m}}$. This contradicts reducedness of $\mathcal{A}_{\mathfrak{m}}$, so indeed $\mathcal{A}$ must be reduced as well. $\square$

We now collect some results about reduced superrings in the general and Noetherian cases. First, we record the standard definitions of Noetherianity.

Definition 3.4.7. A superring $\mathcal{A}$ is *Noetherian* if it satisfies any of the following equivalent conditions.

- (a) $\mathcal{A}$ satisfies the ascending chain condition on ideals
- (b) Every ideal of $\mathcal{A}$ is finitely generated
- (c) Every nonempty set of ideals in $\mathcal{A}$ has an inclusion-maximal element

Lemma 3.4.8. *Let $\mathcal{A}$ be a reduced superring. Then $\mathcal{A}_{\mathfrak{p}}$ is a superfield for all $\mathfrak{p} \in \min \operatorname{Spec} \mathcal{A}$.*

Proof. If $\mathfrak{p}$ is a minimal prime, then $\mathcal{A}_{\mathfrak{p}}$ is a reduced local superring with a unique prime ideal $\mathfrak{p}\mathcal{A}_{\mathfrak{p}} = \operatorname{Nil}(\mathcal{A}_{\mathfrak{p}}) = J_{\mathcal{A}_{\mathfrak{p}}}$ and hence a superfield. $\square$

Lemma 3.4.9. *Let $\mathcal{A}$ be a Noetherian reduced superring and $S \subseteq \mathcal{A}_{\bar{0}}$ a multiplicative subset.*

$$(a) \ K(\mathcal{A}) \cong \prod_{\mathfrak{p} \in \min \operatorname{Spec} \mathcal{A}} \mathcal{A}_{\mathfrak{p}}$$

$$(b) \ S^{-1}K(\mathcal{A}) = K(S^{-1}\mathcal{A})$$

Proof. (a) By the universal property of localization, there are natural maps $K(\mathcal{A}) \rightarrow \mathcal{A}_{\mathfrak{p}}$, since minimal primes consist only of zero divisors. Hence we get a map $K(\mathcal{A}) \rightarrow \prod_{\mathfrak{p} \in \min \operatorname{Spec} \mathcal{A}} \mathcal{A}_{\mathfrak{p}}$. By the prime avoidance lemma 3.4.3, any nonminimal prime contains a non-zero divisor, so $\operatorname{Spec} K(\mathcal{A}) = \min \operatorname{Spec} \mathcal{A}$, a finite discrete set, as a subspace of $\operatorname{Spec} \mathcal{A}$. Passing to spectra, it follows that $K(\mathcal{A}) \rightarrow \prod_{\mathfrak{p} \in \min \operatorname{Spec} \mathcal{A}} \mathcal{A}_{\mathfrak{p}}$ is an isomorphism.

(b) We know that $S^{-1}\mathcal{A}$ is reduced by Lemma 3.4.6(c). Hence $K(S^{-1}\mathcal{A})$ is the product of $(S^{-1}\mathcal{A})_{S^{-1}\mathfrak{p}}$ over minimal primes $\mathfrak{p}$ of $\mathcal{A}$ such that $\mathfrak{p} \cap S = \emptyset$. If S meets $\mathfrak{p}$, then $S^{-1}(\mathcal{A}_{\mathfrak{p}}) = 0$ since $\mathfrak{p}\mathcal{A}_{\mathfrak{p}} = ((\mathcal{A}_{\mathfrak{p}})_{\bar{1}})$ and hence consists of only nilpotents. Therefore $S^{-1}K(\mathcal{A}) \cong \prod_{\mathfrak{p} \in \min \operatorname{Spec} S^{-1}\mathcal{A}} \mathcal{A}_{\mathfrak{p}} \cong K(S^{-1}\mathcal{A})$. $\square$

Lemma 3.4.10. *Let $\mathcal{A}$ be a Noetherian superring. The following are equivalent:*

(i) $\mathcal{A}$ is reduced

(ii) $\mathcal{A}_{\mathfrak{p}}$ is a superfield for all $\mathfrak{p} \in \min \operatorname{Spec} \mathcal{A}$, and $\mathcal{A} \rightarrow \prod_{\mathfrak{p} \in \min \operatorname{Spec} \mathcal{A}} \mathcal{A}_{\mathfrak{p}}$ is injective

Proof. We have (i) $\implies$ (ii) from Lemmas 3.4.8 and 3.4.9.

Now assume (ii), and let $s \in \mathcal{A} \setminus \bigcup_{\mathfrak{p} \in \min \operatorname{Spec} \mathcal{A}} \mathfrak{p}$. Then the image of s in $\prod_{\mathfrak{p} \in \min \operatorname{Spec} \mathcal{A}} \mathcal{A}_{\mathfrak{p}}$ is invertible and hence a non-zero divisor in $\mathcal{A}$, so indeed

$$\operatorname{zdiv}(\mathcal{A}) = \bigcup_{\mathfrak{p} \in \min \operatorname{Spec} \mathcal{A}} \mathfrak{p}.$$

We now show $\operatorname{Nil}(\mathcal{A}) = J_{\mathcal{A}}$. Let $a \in \operatorname{Nil}(\mathcal{A})$. Then $\frac{a}{1} \in J_{\mathcal{A}_{\mathfrak{p}}} \cong (J_{\mathcal{A}})_{\mathfrak{p}}$, i.e. there is $s \notin \mathfrak{p}$ such that $as \in J_{\mathcal{A}}$. Let $I = \{s \in \mathcal{A} \mid as \in J_{\mathcal{A}}\}$, so $I \not\subseteq \mathfrak{p}$ for any minimal prime $\mathfrak{p}$. Using the prime avoidance lemma 3.4.3, let $s \in I \setminus \bigcup_{\mathfrak{p} \in \min \operatorname{Spec} \mathcal{A}} \mathfrak{p}$. As before, $s \in \operatorname{nonzdiv}(\mathcal{A})$, so indeed $a \in J_{\mathcal{A}}$. $\square$

When we generalize the commutative superalgebra of this section to algebraic supergeometry in the next, we will often want our spaces to contain smooth points. Part (c) of the following definition provides a suitably general algebraic model for such spaces.

Definition 3.4.11. Let $\mathcal{A}$ be a superring.

- (a) $\mathcal{A}$ is *fermionically regular* (or *FR*) if it is of the form ΛM for some commutative ring R and R -module M . In this situation, we have $J_{\mathcal{A}} = (\mathcal{A}_{\bar{1}})$, and we may assume $R \cong \bar{\mathcal{A}}$ and $M \cong J_{\mathcal{A}}/J_{\mathcal{A}}^2$.

- (b) $\mathcal{A}$ is *generically fermionically regular* (or *GFR*) if $\mathcal{A}_{\mathfrak{p}}$ is a FR superring for all minimal prime ideals $\mathfrak{p}$.
- (c) $\mathcal{A}$ is *GFRR* if it is GFR and reduced (so that each $\mathcal{A}_{\mathfrak{p}}$ is a FR superfield for $\mathfrak{p}$ minimal).

Serre's criteria

Let $\mathcal{A}$ be a superring. Following [44], we say an element $t \in \mathcal{A}_{\bar{0}}$ is $\mathcal{A}$ -regular if the multiplication map by t is injective but not bijective. Likewise, we say $\xi \in \mathcal{A}_{\bar{1}}$ is $\mathcal{A}$ -regular if the cohomology of the multiplication map by ξ is trivial. An $\mathcal{A}$ -regular sequence is a sequence of homogeneous elements $r_1, \dots, r_k$ such that each r_i is regular in $\mathcal{A}/(r_1, \dots, r_{i-1})$.

Definition 3.4.12. Let $\mathcal{A}$ be a Noetherian local superring with maximal ideal $\mathfrak{m}$.

- (a) $\mathcal{A}$ is *regular* if $\mathfrak{m}$ is generated by an $\mathcal{A}$ -regular sequence
- (b) The *depth* of $\mathcal{A}$ is the minimum $i \geq 0$ such that $\text{Ext}^i(\mathcal{A}/\mathfrak{m}, \mathcal{A}) \neq 0$.

The usual theorem of Rees still holds in this setting by the same argument:

Lemma 3.4.13. *The depth of a Noetherian local superring $(\mathcal{A}, \mathfrak{m})$ is the length of any maximal $\mathcal{A}$ -regular sequence of even elements in $\mathfrak{m}$.*

Definition 3.4.14. Let $\mathcal{A}$ be a Noetherian superring. We say $\mathcal{A}$ satisfies

- (a) (R_k) if $\mathcal{A}_{\mathfrak{p}}$ is regular for all prime ideals $\mathfrak{p}$ of height $\leq k$
- (b) (S_k) if $\text{depth}(\mathcal{A}_{\mathfrak{p}}) \geq \min(k, \text{ht}(\mathfrak{p}))$ for all prime ideals $\mathfrak{p}$

Lemma 3.4.15. *A superfield is regular if and only if it is FR.*

The following proposition justifies our usage of GFRR superrings as super-versions of reduced rings, as well as our definitions of the conditions (R_k) and (S_k) .

Proposition 3.4.16. *A Noetherian superring $\mathcal{A}$ is GFRR if and only if it satisfies (R_0) and (S_1) .*

Proof. If $\mathcal{A}$ is GFRR, let $\mathfrak{p}$ be a minimal prime. Then $\mathcal{A}_{\mathfrak{p}}$ is a FR superfield, so (R_0) holds. Now let $\mathfrak{p}$ be a prime ideal with $\text{ht}(\mathfrak{p}) \geq 1$. For (S_1) , it suffices to show that there is a nonzerodivisor in $\mathfrak{p}\mathcal{A}_{\mathfrak{p}}$. We know by Lemma 3.4.6 that $\mathcal{A}_{\mathfrak{p}}$ is reduced, so we have $\text{zdiv}(\mathcal{A}_{\mathfrak{p}}) = \bigcup_{\mathfrak{q} \in \min\text{Spec}(\mathcal{A}_{\mathfrak{p}})} \mathfrak{q} = \bigcup_{\mathfrak{q} \in \min\text{Spec}(\mathcal{A}), \mathfrak{q} \subseteq \mathfrak{p}} \mathfrak{q}\mathcal{A}_{\mathfrak{p}}$. If $\mathfrak{p}\mathcal{A}_{\mathfrak{p}}$ is contained in this set, then $\mathfrak{p}$ is one of these minimal primes by the prime avoidance lemma 3.4.3. But $\text{ht}(\mathfrak{p}) \geq 1$, so indeed $\mathfrak{p}\mathcal{A}_{\mathfrak{p}}$ contains a nonzerodivisor.

Conversely, suppose $\mathcal{A}$ satisfies (R_0) and (S_1) . We will show that $\mathcal{A}_{\mathfrak{p}}$ is reduced for all primes $\mathfrak{p}$. If $\mathfrak{p}$ is a minimal prime, then by (R_0) we know $\mathcal{A}_{\mathfrak{p}}$ is regular. But $\mathfrak{p}\mathcal{A}_{\mathfrak{p}}$ is the unique prime ideal of $\mathcal{A}_{\mathfrak{p}}$, so $\mathfrak{p}\mathcal{A}_{\mathfrak{p}} = \text{Nil}(\mathcal{A}_{\mathfrak{p}})$. Regularity implies that $\text{Nil}(\mathcal{A}_{\mathfrak{p}}) = ((\mathcal{A}_{\mathfrak{p}})_{\bar{1}})$, so $\mathcal{A}_{\mathfrak{p}}$ is a superfield and therefore is reduced.

Now if $\text{ht}(\mathfrak{p}) \geq 1$, we know by (S_1) that $\mathfrak{p}\mathcal{A}_{\mathfrak{p}}$ contains a nonzerodivisor, i.e. an element a for which the localization map $\mathcal{A}_{\mathfrak{p}} \rightarrow \mathcal{A}_{\mathfrak{p}}[a^{-1}]$ is injective. Since any superring, particularly

$\mathcal{A}_{\mathfrak{p}}[a^{-1}]$, is contained in the product of its localizations at prime ideals, we know that $\mathcal{A}_{\mathfrak{p}}$ is a subring of a product of localizations of $\mathcal{A}$ at primes $\mathfrak{q} \subsetneq \mathfrak{p}$ with $a \notin \mathfrak{q}$. Since $\text{ht}(\mathfrak{q}) < \text{ht}(\mathfrak{p})$, we may induct on the height to find that $\mathcal{A}_{\mathfrak{p}}$ is a subring of a product of reduced superrings which are localizations of $\mathcal{A}_{\mathfrak{p}}$ at minimal primes. It follows by Lemma 3.4.10 that $\mathcal{A}_{\mathfrak{p}}$ (and hence also $\mathcal{A}$) is reduced. $\square$

Definition 3.4.17. A *normal superring* is a Noetherian superring satisfying (R_1) and (S_2) .

Lemma 3.4.18. If $\mathcal{A}$ is normal, then so is $\overline{\mathcal{A}}$.

Proof. Proven in more generality in Lemma 3.6.7. $\square$

3.5 Superschemes

In classical algebraic geometry, a vector space V can be viewed as the affine scheme $\text{Spec}(S(V^*))$, such that the coordinate ring of k^n is isomorphic to $k[x_1, \dots, x_n]$. The same framework applies in supergeometry: if $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a super vector space, it can be represented by the *affine superscheme* $\text{Spec}(S(V^*))$; recall that $S(V^*) = S(V_{\bar{0}}^*) \otimes \Lambda(V_{\bar{1}}^*)$. Thus, the coordinate ring of $k^{m|n}$ will be the polynomial superalgebra $k[x_1, \dots, x_m, \xi_1, \dots, \xi_n]$. Let us now introduce the notion of a superscheme in more detail.

Generalities on superspaces

Definition 3.5.1. A *superspace* $X = (|X|, \mathcal{O}_X)$ is a pair consisting of a topological space $|X|$ and a sheaf $\mathcal{O}_X$ of superrings such that each stalk $\mathcal{O}_{X,x}$ is a local superring.

A *morphism of superspaces* is a pair $\phi = (|\phi|, \phi^{\#}) : X \rightarrow Y$ for a continuous map $|\phi| : |X| \rightarrow |Y|$ and a morphism of $\phi^{\#} : \mathcal{O}_Y \rightarrow |\phi|_* \mathcal{O}_X$ of sheaves on $|Y|$ which becomes a local morphism of local superrings on each stalk.

We say that the superspace X is a *superscheme* if $(|X|, (\mathcal{O}_X)_{\bar{0}})$ is a scheme and the odd part $(\mathcal{O}_X)_{\bar{1}}$ of the structure sheaf is a quasi-coherent sheaf of $(\mathcal{O}_X)_{\bar{0}}$ -modules. An *open subscheme* of X is given by the restriction of $\mathcal{O}_X$ to an open set $U \subseteq |X|$, and a *closed subscheme* is an equivalence class of closed immersions in the usual sense.

Superschemes are locally modeled on *affine superschemes*

$$\text{Spec } \mathcal{A} = (|\text{Spec } \mathcal{A}_{\bar{0}}|, \tilde{\mathcal{A}})$$

where $\tilde{\mathcal{A}}$ is the sheaf on $|\text{Spec } \mathcal{A}_{\bar{0}}|$ given by $\Gamma(D(f), \tilde{\mathcal{A}}) = \mathcal{A}_f$ for any $f \in \mathcal{A}_{\bar{0}}$. The meaning of $D(f)$ in $\text{Spec } \mathcal{A}$ is the same as in $\text{Spec } \mathcal{A}_{\bar{0}}$, as their underlying topological spaces do not differ. If X is a superscheme and $\mathcal{A}$ is a superring, then we have a canonical isomorphism $\text{Hom}(X, \text{Spec } \mathcal{A}) \cong \text{Hom}(\mathcal{A}, \Gamma(X, \mathcal{O}_X))$. If X is over a field k , then $\Gamma(X, \mathcal{O}_X) \cong \text{Hom}(X, \mathbb{A}_k^{1|1})$.

Superschemes (and likewise supervarieties) form a full subcategory of superspaces. Before we introduce supervarieties, however, we first need the following definitions.

Functor of points

Given two superschemes X, Y we write $X(Y) := \text{Hom}(Y, X)$. If $\mathcal{A}$ is a superring, we write $X(\mathcal{A}) := X(\text{Spec } \mathcal{A})$; the functor $X(-)$ is completely determined by its values on superrings. If X is a superscheme over a field k , then the set $X(k)$ of k -points is the set of closed points of $|X|$.

If $X = \text{Spec } \mathcal{B}$ is also affine, then $X(\mathcal{A}) = \text{Hom}(\mathcal{B}, \mathcal{A})$. In particular, if $X = \text{Spec } S(V^*)$ is a super vector space V viewed as a supervariety, then in the category of super vector spaces we have $X(\mathcal{A}) = \text{Hom}(V^*, \mathcal{A})_{\bar{0}} \cong (\mathcal{A} \otimes V)_{\bar{0}}$. We therefore define the functor of points of a super vector space V (or a Lie superalgebra $\mathfrak{g}$) by $\mathcal{A} \mapsto (\mathcal{A} \otimes V)_{\bar{0}}$ (or $\mathcal{A} \mapsto (\mathcal{A} \otimes \mathfrak{g})_{\bar{0}}$). Notice that $(\mathcal{A} \otimes V)_{\bar{0}}$ lies in the category of vector spaces (and $(\mathcal{A} \otimes \mathfrak{g})_{\bar{0}}$ lies in the category of ordinary Lie algebras); the only difference from the functor of points of an ordinary vector space is that the domain is enlarged from the category of commutative algebras to the category of commutative superalgebras.

Reduced, integral, and GFR superschemes

Definition 3.5.2. Let X be a superscheme.

- (a) X is *irreducible* if $|X|$ is irreducible as a topological space.
- (b) X is *reduced* if $\mathcal{O}_X(U)$ is a reduced superring for each affine open U .
- (c) X is *integral* if $\mathcal{O}_X(U)$ is an integral superdomain for each affine open U .
- (d) X is *GFR* if there is a collection of points $\{\eta\}$ whose closure is X and for which each $\mathcal{O}_{X,\eta}$ is FR.
- (e) X is *GFRR* if it is GFR and reduced.
- (f) X is *Noetherian* if it admits a finite cover by open affine subschemes $\text{Spec } \mathcal{A}_i$ such that each $\mathcal{A}_i$ is a Noetherian superring.

Lemma 3.5.3. Let X be a superscheme.

- (a) X is integral if and only if it is irreducible and reduced.
- (b) X is reduced if and only if $\mathcal{O}_{X,x}$ is a reduced superring for all points $x \in |X|$.

Proof. (a) Follows from the affine case in Lemma 3.4.6(b) and by passing to the reduced induced closed subschemes for the purpose of using e.g. [Stacks, Tag 01OM].

- (b) Immediate from Lemma 3.4.6(c). □

The importance of considering only those opens U which are affine in the definition of reduced and integral superschemes is highlighted in the following example.

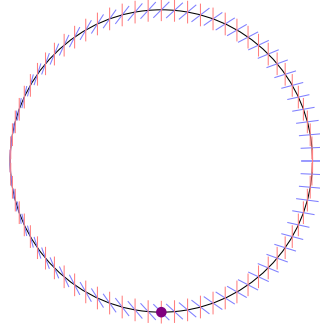


Figure 3.1: A real picture of X , with blue fuzz for the ξ_1 direction, red fuzz for the ξ_2 direction, and a purple “thick point” at $0 \in U_1$.

Example 3.5.4. Consider the non-affine superscheme X obtained from gluing the affines $U_1 = \operatorname{Spec} \mathbb{C}[x, x\xi_1, \xi_2, \xi_1\xi_2]$ and $U_2 = \operatorname{Spec} \mathbb{C}[x^{-1}, \xi_1, \xi_2]$ along their common localization $\operatorname{Spec} \mathbb{C}[x^{\pm 1}, \xi_1, \xi_2]$. Then the superring $\mathcal{O}_X(X) \cong \mathbb{C}[\xi_2, \xi_1\xi_2]$ of global functions is not reduced, but X certainly is. Geometrically, U_1 can be realized as a certain orbit closure of the natural action of $GL(1|2)$ on $S^2(\mathbb{C}^{1|2})^*$, appearing in [46].

This supervariety is depicted in Figure 3.1 as having a non-reduced thick point at $0 \in U_1$. Such behavior is common in the world of supervarieties, when two odd directions interact to produce this type of singularity. We propose that such a singularity be called a “pill,” in analogy with the small balls of lint that occur on surfaces of fabric. In both the mathematical and textile contexts, pills occur when fuzz interacts to produce a small bump.

Normal superschemes

Definition 3.5.5. Let X be a Noetherian superscheme. We say X is (R_k) , (S_k) , *regular*, or *normal* if its local rings $\mathcal{O}_{X,x}$ all admit the respective property.

Definition 3.5.6. Let X be a Noetherian superscheme and Z a closed subset of $|X|$ of codimension at least 2. We say X satisfies the *Hartogs property* at Z if the restriction map $\mathcal{O}_X(X) \rightarrow \mathcal{O}_X(X \setminus Z)$ is an isomorphism. Further, we say that X satisfies the Hartogs property (irrespective of any closed subset) if it satisfies the Hartogs property at all closed subsets of codimension at least 2.

We record the following mild generalization of a standard lemma.

Lemma 3.5.7. *Let X be a Noetherian superscheme. Then X satisfies (S_2) if and only if it satisfies (S_1) and the Hartogs property.*

Proof. Since all relevant properties are local, we may assume $X = \operatorname{Spec} \mathcal{A}$ is affine. Note that $\mathcal{O}_X$ is a coherent sheaf of $(\mathcal{O}_X)_{\bar{0}}$ -modules, and $(X, (\mathcal{O}_X)_{\bar{0}})$ is an affine scheme, so $H^1(X, \mathcal{O}_X) = 0$. Now let $\mathfrak{p} \in |\operatorname{Spec} \mathcal{A}|$ be a point and Z its closure. If $\operatorname{ht} \mathfrak{p} \leq 1$ then we are finished because (S_2) implies (S_1) . Now suppose $\operatorname{ht} \mathfrak{p} \geq 2$. The long exact sequence in cohomology with supports truncates as

$$0 \rightarrow H_Z^0(X, \mathcal{O}_X) \rightarrow H^0(X, \mathcal{O}_X) \rightarrow H^0(X \setminus Z, \mathcal{O}_X|_{X \setminus Z}) \rightarrow H_Z^1(X, \mathcal{O}_X) \rightarrow 0$$

so X satisfies the Hartogs property at Z if and only if $H_Z^i(X, \mathcal{O}_X) = 0$ for $i = 0, 1$. Using [21] Exercises III.3.3-4, this condition is equivalent to $\text{Ext}^i(\mathcal{A}_{\mathfrak{p}}/\mathfrak{p}, \mathcal{A}_{\mathfrak{p}}) = 0$ for $i = 0, 1$, which is (S_2) . $\square$

3.6 Supervarieties

The meanings of separated and finite type are the same as in the usual setting.

Definition 3.6.1. An *algebraic supervariety* is a GFRR separated superscheme of finite type over an algebraically closed field.

This definition is designed to mimic the commonly-used definition of a variety found in e.g. [21], although we allow for reducible supervarieties by saying “reduced” instead of “integral”. If we drop the “GFRR” condition and add “integral”, it becomes equivalent to the formulation given in [46], although our notion of integrality differs slightly. Here, we consider only those supervarieties which are generically fermionically regular in order to guarantee the existence of smooth points.

As usual, a closed subvariety is a closed subscheme which is itself a supervariety, and an open subvariety is an open subscheme of X , viewed as a supervariety. A morphism of supervarieties is a morphism of superschemes between supervarieties.

Lemma 3.6.2. *Let X be a supervariety. Then X is affine if and only if $X_{\bar{0}}$ is.*

Proof. See [49]. $\square$

Definition 3.6.3. The *dimension* of an irreducible supervariety X is the dimension of the tangent space $T_{\eta}X$ at the generic point η . Equivalently, its even dimension is the dimension of $X_{\bar{0}}$, and its odd dimension is the odd dimension of the super vector space $\text{Spec } \mathcal{O}_{X, \eta}$ over $\mathcal{O}_{X_{\bar{0}}, \eta}$.

Lemma 3.6.4. *Let $U \subseteq V \subseteq |X|$ be open subsets of an irreducible supervariety X . Then the restriction map $\mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U)$ is injective.*

Proof. Let η be the generic point. The map $\mathcal{O}_X(V) \rightarrow \mathcal{O}_{X, \eta}$ is injective, and factors through $\mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U)$. $\square$

Proposition 3.6.5. *The reduced induced subscheme structure on X is given locally by the underlying rings $\bar{\mathcal{A}} = \mathcal{A}/J_{\mathcal{A}}$.*

Proof. It suffices to show that $J_{\mathcal{A}} = \{a \in \mathcal{A} \mid a_{\mathfrak{p}} \in \mathfrak{p}\mathcal{A}_{\mathfrak{p}} \text{ for all } \mathfrak{p} \in \text{Spec } \mathcal{A}\}$, where $a_{\mathfrak{p}}$ denotes the image of a in $\mathcal{A}_{\mathfrak{p}}$. Note that $\supseteq$ holds by definition of $J_{\mathcal{A}}$ using the minimal prime ideal $\mathfrak{p} = J_{\mathcal{A}}$. On the other hand, if $a \in J_{\mathcal{A}}$, then a is nilpotent and so $a_{\mathfrak{p}}$ belongs to every prime ideal of $\mathcal{A}_{\mathfrak{p}}$. $\square$

We will refer to this closed subvariety of X as the *underlying variety* $X_{\bar{0}}$, and the defining ideal as $\mathcal{J}_X$. Using [49] and [21] Exercise III.3.1, we see that X is affine if and only if $X_{\bar{0}}$ is affine.

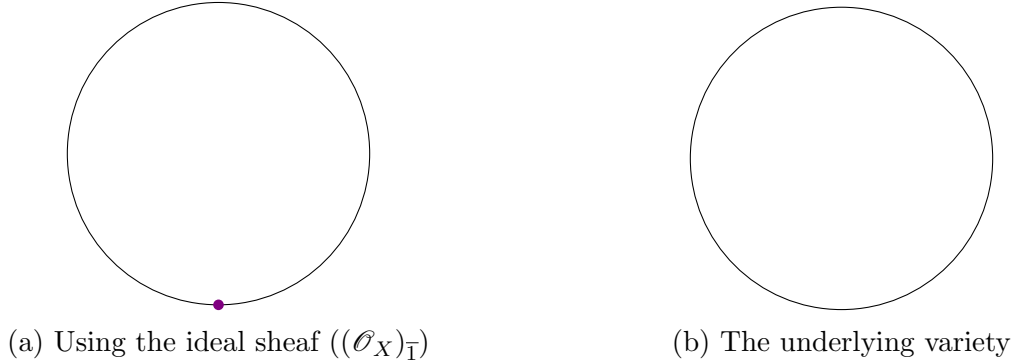


Figure 3.2: Two different closed subschemes of X as in Example 3.5.4

Example 3.6.6. Let X be as in Example 3.5.4, so $X_{\bar{0}} \cong \mathbb{P}^1$. If we instead consider the closed subscheme of X corresponding to the smaller ideal sheaf $((\mathcal{O}_X)_1) \subsetneq \mathcal{I}_X$, then we obtain $\mathbb{P}^1$ with a thick point at $0 \in U_1$, a nonreduced scheme depicted in Figure 3.2(A). It is therefore critical to use the larger ideal sheaf $\mathcal{I}_X$ if we want to genuinely obtain an underlying variety from a supervariety.

Lemma 3.6.7. *If X is a normal supervariety, then so is $X_{\bar{0}}$.*

Proof. The condition (R_1) is immediate, and implies (S_1) . For (S_2) , we note that the Hartogs property of Lemma 3.5.7 implies the Hartogs property for $X_{\bar{0}}$, since $\overline{\mathcal{O}_X(X)} \rightarrow \overline{\mathcal{O}_X(X \setminus Z)}$ is also an isomorphism. The claim then follows by Lemma 3.5.7. $\square$

3.7 Quasi-coherent sheaves and vector bundles

If $X = \operatorname{Spec} \mathcal{A}$ is an affine superscheme and M is an $\mathcal{A}$ -module, we write $\widetilde{M}$ for the sheaf of $\mathcal{O}_X$ -modules defined by $\Gamma(D(f), \widetilde{M}) = M_f$. Such a sheaf will be called quasi-coherent, and when $\mathcal{A}$ is finitely generated, it is moreover called coherent if M is finitely generated over $\mathcal{A}$.

Definition 3.7.1. Let X be a supervariety and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$ -modules. We say that $\mathcal{F}$ is quasi-coherent (respectively, coherent) if it is quasi-coherent (respectively, coherent) on each open affine.

With the usual collection of morphisms, the categories of quasi-coherent and coherent sheaves on a supervariety X are abelian. They admit the usual bifunctors $(-) \otimes_{\mathcal{O}_X} (-)$ and $\mathcal{H}om_{\mathcal{O}_X}(-, -)$, as well as pullback and pushforward functors along morphisms of supervarieties.

Let $\mathcal{F}$ be a coherent sheaf on a supervariety X . Then for $x \in |X|$, the fiber of $\mathcal{F}$ at x is $\mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x$. The parity swap of $\mathcal{F}$, denoted $\Pi \mathcal{F}$, is the sheaf given by $\Gamma(U, \Pi \mathcal{F}) = \Pi(\Gamma(U, \mathcal{F}))$ for any open $U \subseteq |X|$.

Definition 3.7.2. If a coherent sheaf $\mathcal{F}$ on a supervariety X is locally isomorphic to $\mathcal{O}_X^{m|n}$ as an $\mathcal{O}_X$ -module, we call $\mathcal{F}$ a vector bundle on X with generic fiber $\mathbb{C}^{m|n}$ and dimension $(m|n)$.

A line bundle, or invertible sheaf, is a $(1|0)$ -dimensional vector bundle. As usual, these are the objects which are invertible with respect to $\otimes$ and the trivial sheaf $\mathcal{O}_X$.

A Π -bundle, or Π -invertible sheaf, is a pair $(\mathcal{F}, \Pi_{\mathcal{F}})$ where $\mathcal{F}$ is a $(1|1)$ -dimensional vector bundle and $\Pi_{\mathcal{F}}$ is an odd isomorphism $\Pi_{\mathcal{F}} : \mathcal{F} \rightarrow \Pi\mathcal{F}$. Much like in the context of isomeric vector spaces, we can equivalently think of $\Pi_{\mathcal{F}}$ as an odd involution.

The study of Π -invertible sheaves originated with Manin and Skornyakov in [31], where it is expounded that $\text{Pic}(X) \cong H^1(X, \mathcal{O}_{X,0}^\times)$ and $\text{Pic}_\Pi(X) \cong H^1(X, \mathcal{O}_X^\times)$. In many ways the latter isomorphism makes the Π -Picard set a good analogue of the classical Picard group of a variety, but in other ways it is not so nice. For instance, although the tensor product of two Π -invertible sheaves decomposes into a sum of two rank- $(1|1)$ locally free sheaves (see e.g. [30] or [32]), the summands cannot in general be equipped with odd involutions to make them Π -invertible.

Nonetheless, the calculation of $\text{Pic}_\Pi(X)$ for various supervarieties X remains an interesting problem. Cacciatori and Noja calculate it for projective superspace $\mathbb{P}^{m|n}$ in [4, Corollaries 1 and 2], as well as for a family of supermanifolds over $\mathbb{P}^2$ with Re in [5, Theorem 7.8]. Manin and Skornyakov do so for Π -projective superspace $\mathbb{P}_\Pi^n$ in [32, Chapter 2, Section 8.11.f]. Manin, Voronov, and Penkov generalize this to $\text{QGr}(r, n)$ in [30, p. 2078], although the description is somewhat opaque and does not clearly indicate when the result is 0.

There have also been some efforts to find the Picard groups of various supervarieties. The result for the type-A supergrassmannian is due to Penkov and Skornyakov in [38], while Cacciatori and Noja treat the exceptional case of $\mathbb{P}^{1|n}$ in [4]. The computation of $\text{Pic}(\text{QGr}(r, n))$, which will be done in Proposition 6.5.5, is known to experts and often attributed to Penkov [39]. However, neither he nor the author could locate a written copy of his proof.

For more details on matters such as Picard superschemes, which captures some subtleties of superschemes over a base, see [3].

3.8 Projective and Π -projective supervarieties

One of the most distinctive features of supergeometry is the relative lack of projective supervarieties (see [7]). In this section we introduce projective supervarieties, as well as their “type Q analog”, Π -projective supervarieties.

Projective

Let $S = \bigoplus_{n \geq 0} S_n$ be a $\mathbb{Z}_{\geq 0}$ -graded commutative superalgebra, where each S_n is a super vector space, and write $S_+ = \bigoplus_{n \geq 1} S_n$. Though not strictly necessary, for simplicity we will assume that S is generated by S_1 over S_0 .

We may construct $\text{Proj } S$ in the standard way: The topological space $|\text{Proj } S|$ is the space of homogeneous prime ideals of S which do not contain S_+ , with a basis of open sets

given by $D_+(f)$ for each $f \in S_0$. The points in $D_+(f)$ are those prime ideals not containing either S_+ or f . The structure sheaf $\mathcal{O}_{\text{Proj } S}$ is given by $\Gamma(D_+(f), \mathcal{O}_{\text{Proj } S}) = S_{(f)}$, where $S_{(f)}$ is the set of degree-0 elements in S_f . (Note the distinction between degree 0 and degree $\bar{0}$.) In particular, we have $D_+(f) \cong \text{Spec } S_{(f)}$.

Definition 3.8.1. We say a supervariety is *projective* if it is isomorphic to some $\text{Proj } S$ where $S_0 = \mathbb{C}$ and S_1 is finite-dimensional.

The classic example of projective superspace $\mathbb{P}^{m|n}$ arises from $S_1 = k^{m+1|n}$; any projective supervariety is a closed subvariety of some $\mathbb{P}^{m|n}$. As we will see later when discussing supergrassmannians, it is the super moduli space of $(1|0)$ -dimensional subspaces of $k^{m+1|n}$. Morphisms $X \rightarrow \mathbb{P}^{m|n}$, for X a supervariety, correspond bijectively with (isomorphism classes) of a line bundle $\mathcal{F}$ together with a choice of homogeneous sections $s_0, \dots, s_m, \sigma_1, \dots, \sigma_n$ that globally generate $\mathcal{F}$ (see [7]).

Π -projective

If we further assume that $S_0 = \mathbb{C}$ and S_1 is a finite dimensional isomeric vector space (with odd involution Π_S), then Π_S extends to a derivation, which by abuse of notation we will also refer to as Π_S , on S . In this case we can construct $\text{Proj}_{\Pi}(S)$ as follows: The underlying topological space remains the same as in the case of $\text{Proj } S$. However, the structure sheaf $\mathcal{O}_{\text{Proj}_{\Pi} S}$ is given by $\Gamma(D_+(f), \mathcal{O}_{\text{Proj}_{\Pi} S}) = S_{(f + \Pi_S f)}$, where $S_{(f + \Pi_S f)}$ is the superalgebra of $\mathbb{Z}_2$ -homogeneous parts of elements of the intersection of $\ker \Pi_S$ and the (non-super, non-commutative) algebra generated by the Π_S -fixed points and $(f + \Pi_S f)^{-1}$.

Example 3.8.2. Consider $S_0 = \mathbb{C}$ and $S_1 = \mathbb{C}^{n+1|n+1}$ equipped with the obvious odd involution Π_S . That is,

$$S = \mathbb{C}[x_0, \dots, x_n, x_0\xi_0, \dots, x_n\xi_n]$$

where Π_S interchanges each x_i with the corresponding $x_i\xi_i$, i.e.

$$\Pi_S = \sum_{i=1}^n x_i\xi_i \frac{\partial}{\partial x_i} + \frac{\partial}{\partial \xi_i}.$$

We choose to label each odd generator by $x_i\xi_i$ rather than ξ_i for later notational convenience. Then the (non-super, non-commutative) algebra generated by Π_S -fixed points is $\mathcal{A} = \mathbb{C}[x_0(1 + \xi_0), \dots, x_n(1 + \xi_n)]$. If we want to find $\Gamma(D_+(x_0), \mathcal{O}_{\text{Proj}_{\Pi} S})$, we need to invert $x_0(1 + \xi_0)$ in $\mathcal{A}$, and find the kernel of Π_S

$$\mathbb{C}[x_0^{-1}x_1(1 + \xi_1 - \xi_0 - \xi_0\xi_1), \dots, x_0^{-1}x_n(1 + \xi_n - \xi_0 - \xi_0\xi_n)]$$

within. Then, writing $z_i = x_0^{-1}x_i(1 - \xi_0\xi_i)$ and $\eta_i = \xi_i - \xi_0$, we obtain

$$\begin{aligned} \Gamma(D_+(x_0), \mathcal{O}_{\text{Proj}_{\Pi} S}) &= S_{(x_0(1+\xi_0))} \\ &= \mathbb{C}[z_1, \dots, z_n, z_1\eta_1, \dots, z_n\eta_n]. \end{aligned}$$

and

$$\begin{aligned} \Gamma(D_+(x_i), \mathcal{O}_{\text{Proj}_{\Pi} S}) \\ = \mathbb{C}[z_i^{-1}z_1(1 - \eta_i\eta_1), \dots, z_i^{-1}, \dots, z_i^{-1}z_n(1 - \eta_i\eta_n), z_i^{-1}z_1(\eta_1 - \eta_i), \dots, \eta_i, \dots, z_i^{-1}z_n(\eta_n - \eta_i)]. \end{aligned}$$

Definition 3.8.3. We say a supervariety is Π -projective if it is isomorphic to some $\text{Proj}_\Pi S$ where $S_0 = \mathbb{C}$ and S_1 is finite-dimensional and isomeric.

The classic example of Π -projective superspace $\mathbb{P}_\Pi^n$ arises from $S_1 = \mathbb{C}^{n+1|n+1}$ equipped with the obvious odd involution Π_S ; any Π -projective supervariety is a closed subvariety of some $\mathbb{P}_\Pi^n$. It is the super moduli space of Π_V -invariant $(1|1)$ -dimensional subspaces of $\mathbb{C}^{n+1|n+1}$, and morphisms $X \rightarrow \mathbb{P}_\Pi^n$ correspond bijectively with (isomorphism classes of) a Π -bundle $\mathcal{F}$ together with a choice of even sections $s_0, \dots, s_n$ (and their odd counterparts $\sigma_0 = \Pi_{\mathcal{F}} s_0, \dots, \sigma_n = \Pi_{\mathcal{F}} s_n$) that globally generate $\mathcal{F}$.

3.9 Tangents and smoothness of supervarieties

Let X be a supervariety.

Definition 3.9.1. The *tangent sheaf* $\mathcal{T}_X$ is the sheaf defined on affine opens $U = \text{Spec } \mathcal{A}$ by $\Gamma(U, \mathcal{T}_X) = \text{Der}(\mathcal{A})$, as defined in Example 3.3.6. The tangent sheaf is a vector bundle of Lie superalgebras on X , and $\Gamma(U, \mathcal{T}_X)$ acts by derivations on $\Gamma(U, \mathcal{O}_X)$. A *vector field* on X is an element of $\Gamma(X, \mathcal{T}_X)$; if $X = \text{Spec } \mathcal{A}$ is affine, then $\text{Der}(\mathcal{A})$ is the space of vector fields on X .

Definition 3.9.2. If $x \in X(\mathbb{C})$ is a closed point, the *tangent space* at x is the super vector space $T_x X = \text{Der}(\mathcal{O}_{X,x}, \mathbb{C})$ of point derivations $\delta : \mathcal{O}_{X,x} \rightarrow \mathbb{C}$, i.e. linear maps $\delta \in \mathcal{O}_{X,x}^*$ such that $\delta(fg) = \delta(f)g(x) + (-1)^{|\delta||f|} f(x)\delta(g)$. The sign $(-1)^{|\delta||f|}$ is not especially important, since if $|f| = \bar{1}$ then $f(x) = 0$.

As usual, there is a natural identification $T_x X \cong (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$, and $\mathcal{T}_{X,x} \cong \text{Der}(\mathcal{O}_{X,x})$. We have a natural map $\mathcal{T}_{X,x} \rightarrow T_x X$ given by composing a derivation $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{X,x}$ with the evaluation map $\mathcal{O}_{X,x} \rightarrow \mathbb{C}$ at x . That is, $D \mapsto (f \mapsto D(f)(x))$.

Definition 3.9.3. We say a supervariety X is *smooth* at the point $x \in X(\mathbb{C})$ if the natural map $\mathcal{T}_{X,x} \rightarrow T_x X$ is surjective.

Essentially, smoothness captures the notion of being locally isomorphic to a super vector space. Next we give an abridged list of equivalent conditions for smoothness that will be relevant to this thesis; see [46] for more.

Proposition 3.9.4. Suppose X is a supervariety, $x \in X(\mathbb{C})$ is a closed point, and $\mathcal{A} := \mathcal{O}_{X,x}$ with maximal ideal $\mathfrak{m} = \mathfrak{m}_x$. Let $t_1, \dots, t_m, \xi_1, \dots, \xi_n \in \mathfrak{m}$ project to a homogeneous basis of $\mathfrak{m}/\mathfrak{m}^2$, where $|t_i| = \bar{0}$ and $|\xi_i| = 1$. Then the following are equivalent:

- (a) The natural map $\mathcal{T}_{X,x} \rightarrow T_x X$ is surjective
- (b) $\bar{\mathcal{A}}$ is a regular local ring such that $\mathcal{A} \cong \bar{\mathcal{A}}[\xi_1, \dots, \xi_n]$
- (c) There is a neighborhood $U = \text{Spec } B$ of x such that $\bar{B}$ is regular and $B \cong \bar{B}[\xi_1, \dots, \xi_n]$
- (d) $\mathcal{A}/(\mathcal{A}_{\bar{1}})$ is a regular local ring such that $\mathcal{A} \cong (\mathcal{A}/(\mathcal{A}_{\bar{1}}))[\xi_1, \dots, \xi_n]$

(e) *There is a neighborhood $U = \operatorname{Spec} B$ of x such that $B/(B_{\bar{1}})$ is regular and $B \cong (B/(B_{\bar{1}}))[\xi_1, \dots, \xi_n]$*

(f) *$\mathcal{A}$ is a regular local superalgebra*

Proof. The equivalence (a) $\iff$ (d) $\iff$ (e) is proven in [46], and (d) $\iff$ (f) is proven in [44]. Meanwhile, the same argument for (d) $\iff$ (e) works for (b) $\iff$ (c) as well.

It remains to show (b) $\iff$ (d). If (b) holds, then since $\mathcal{A} \cong \overline{\mathcal{A}}[\xi_1, \dots, \xi_n]$, it follows that $(\mathcal{A}_{\bar{1}}) = (\xi_1, \dots, \xi_n) = \overline{\mathcal{A}}$, so (d) holds too. Conversely, if (d) is true, then regularity of $\mathcal{A}/(\mathcal{A}_{\bar{1}})$ implies that $\mathcal{A}$ has no pill singularity; that is, $\mathcal{A}/(\mathcal{A}_{\bar{1}})$ has no nilpotents, so $(\mathcal{A}_{\bar{1}})$ contains all the nilpotents. Then $(\mathcal{A}_{\bar{1}}) = J_{\mathcal{A}}$, so we are finished. $\square$

Chapter 4

Supergroups

In this chapter we briefly introduce algebraic supergroups, the global versions of Lie superalgebras. Particular emphasis will be placed on their actions, Cartan subgroups, and homogeneous spaces. We work over $\mathbb{C}$, though the theory is equivalent over any algebraically closed field of characteristic zero.

4.1 Equivalent definitions of supergroups

Definition 4.1.1. An *(algebraic) supergroup* is a group object in the category of affine supervarieties. That is, a supergroup is an affine supervariety G equipped with multiplication $m : G \times G \rightarrow G$, inversion $s : G \rightarrow G$, and identity $e : \text{Spec } \mathbb{C} \rightarrow G$ morphisms satisfying the usual properties

$$\begin{aligned} m \circ (m \times \text{id}) &= m \circ (\text{id} \times m) \\ m \circ (e \times \text{id}) &= m \circ (\text{id} \times e) = \text{id} \\ m \circ (\text{id} \times s) \circ \Delta &= m \circ (s \times \text{id}) \circ \Delta = e \circ c \end{aligned}$$

where $\text{id} : G \rightarrow G$ is the identity, $\Delta : G \rightarrow G \times G$ is the diagonal embedding, and $c : G \rightarrow \text{Spec } \mathbb{C}$ is the constant morphism. A morphism of supergroups $\phi : G \rightarrow H$ is a morphism of supervarieties that respects the structure morphisms m, s , and e on G and H .

Since supergroups are by necessity affine, the coordinate superalgebra $\mathcal{A}$ is in fact a Hopf superalgebra with comultiplication, antipode, and counit given by the duals of multiplication, inversion, and identity, respectively. Conversely, given a finitely generated commutative Hopf superalgebra which is an integral superdomain, its spectrum is a supergroup. In fact, this correspondence describes an anti-equivalence of categories.

Likewise, via the functor of points approach, a supergroup is an affine supervariety $G = \text{Spec } \mathcal{A}$ whose functor of points $G(\mathcal{B}) := \text{Hom}(\mathcal{A}, \mathcal{B})$ factors through the forgetful functor from the category of groups to the category of sets. The category of *supergroup functors* defined as such, with natural transformations of groups for morphisms, is of course equivalent to the category of supergroups. The only difference between a supergroup and an ordinary algebraic group is that the domain category of the functor of points is enlarged from the category of commutative algebras to the category of commutative superalgebras. That is, supersymmetry is merely symmetry in supercommutative coordinates.

Theorem 4.1.2 (incomplete version of theorem 4.3.2). *The following categories are equivalent:*

- (a) *The category of algebraic supergroups*
- (b) *The opposite category of finitely-generated commutative Hopf superalgebras which are integral superdomains*
- (c) *The category of supergroup functors*

We now provide some examples of supergroups.

Example 4.1.3. Let V be a finite-dimensional super vector space. We define $\mathrm{GL}(V)$ to be the supergroup with coordinate superring $\mathbb{C}[\mathrm{End}(V), \det_{\bar{0}}^{-1}, \det_{\bar{1}}^{-1}]$, where $\det_{\bar{i}}$ is the determinant polynomial on $\mathrm{End}(V_{\bar{i}}) \subseteq \mathrm{End}(V) = \mathrm{End}(V)_{\bar{0}}$. For a given superalgebra $\mathcal{A}$, $\mathrm{GL}(V)(\mathcal{A})$ is the group of $\mathcal{A}$ -linear automorphisms of the free $\mathcal{A}$ -module $\mathcal{A} \otimes_{\mathbb{C}} V$. The underlying algebraic group of $\mathrm{GL}(V)$ is $\mathrm{GL}(V_{\bar{0}}) \times \mathrm{GL}(V_{\bar{1}})$.

If $V = \mathbb{C}^{m|n}$, we write $\mathrm{GL}(m|n) = \mathrm{GL}(V)$. We have

$$\mathrm{GL}(m|n)(\mathcal{A}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\}$$

where W and Z are invertible $m \times m$ and $n \times n$ matrices, respectively, with entries in $\mathcal{A}_{\bar{0}}$, and where X and Y are any $m \times n$ and $n \times m$ matrices, respectively, with entries in $\mathcal{A}_{\bar{1}}$. The underlying group, $\mathrm{GL}(m) \times \mathrm{GL}(n)$, consists of those matrices where B and C are both 0.

Example 4.1.4. The *Berezinian*, or *superdeterminant*, is a specific surjective morphism of algebraic groups $\mathrm{Ber} : \mathrm{GL}(V) \rightarrow \mathbb{G}_m$. (See [7] for more details.) On $\mathrm{GL}(m|n)$, it is given at the level of $\mathcal{A}$ -points by

$$\mathrm{Ber} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BD^{-1}C) \det(D^{-1}).$$

We write $\mathrm{SL}(V)$ for the kernel of Ber , so $\mathbb{C}[\mathrm{SL}(V)] \cong \mathbb{C}[\mathrm{End}(V)]/(\mathrm{Ber} - 1)$ where in the quotient we have written Ber for the polynomial function on the coordinates. The underlying group is $\{(A, D) \in \mathrm{GL}(V_{\bar{0}}) \times \mathrm{GL}(V_{\bar{1}}) \mid \det(A) = \det(D)\}$.

Example 4.1.5. Let (V, Π_V) be an isomeric vector space. The *isomeric supergroup* $Q(V)$ is the intersection of $\mathrm{GL}(V)$ with the kernel of $[-, \Pi_V]$ in $\mathrm{End}(V)$. That is,

$$Q(V)(\mathcal{A}) = \{M \in \mathrm{GL}(V)(\mathcal{A}) \mid [M, \mathrm{id}_{\mathcal{A}} \otimes \Pi_V] = 0\}.$$

The coordinate superalgebra is $\mathbb{C}[\mathrm{End}(V), \det_{\bar{0}}^{-1}]/([- , \Pi_V])$, where we write $[-, \Pi_V]$ for the collection of polynomial functions that together assert $[f, \Pi_V] = 0$. The choice of Π_V is immaterial; two different choices lead to isomorphic supergroups.

If $V = \mathbb{C}^{n|n}$, we write $Q(n) = Q(V)$ and obtain

$$Q(n)(\mathcal{A}) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right\}$$

where A is an invertible $n \times n$ matrix with entries in $\mathcal{A}_{\bar{0}}$, and B is an $n \times n$ matrix with entries in $\mathcal{A}_{\bar{1}}$.

4.2 The Lie superalgebra of a supergroup

There are myriad ways to construct the Lie algebra of a supergroup. We will employ three different methods here.

The functor of points approach

Let G be a supergroup, viewed through the functor of points approach. Let $\mathcal{A}$ be a commutative superalgebra, and let $\mathcal{A}(\varepsilon) := \mathcal{A}[\varepsilon]/(\varepsilon^2)$, where we assert $|\varepsilon| = \bar{0}$. As a super vector space, we have a non-homogeneous splitting $\mathcal{A}(\varepsilon) = \mathcal{A} \oplus \varepsilon\mathcal{A}$, along with morphisms

$$i : \mathcal{A} \rightarrow \mathcal{A}(\varepsilon), \quad p : \mathcal{A}(\varepsilon) \rightarrow \mathcal{A}$$

where i is the natural inclusion, and p is the natural projection given by $p(\varepsilon) = 0$ and $p(a) = a$ for all a .

Definition 4.2.1. Write $G(p) : G(\mathcal{A}(\varepsilon)) \rightarrow G(\mathcal{A})$. The Lie superalgebra of G is the functor $\text{Lie}(G)(\mathcal{A}) = \ker(G(p))$.

As proven in [7], the functor $\text{Lie}(G)$ is always representable by some Lie superalgebra $\mathfrak{g}$. Since we can calculate the value of this functor at $\mathcal{A} = \mathbb{C}[\xi]$ and obtain $\text{Lie}(G)(\mathbb{C}[\xi]) = (\mathbb{C} \otimes \mathfrak{g}_{\bar{0}}) \oplus (\mathbb{C}\xi \otimes \mathfrak{g}_{\bar{1}})$, we can fully recover the Lie superalgebra from the functor. If $f : G \rightarrow H$ is a morphism of supergroups, it induces a morphism $\text{Lie}(f) : \text{Lie}(G) \rightarrow \text{Lie}(H)$ given on a superalgebra $\mathcal{A}$ by the restriction of $f(\mathcal{A}(\varepsilon)) : G(\mathcal{A}(\varepsilon)) \rightarrow H(\mathcal{A}(\varepsilon))$ to $\text{Lie}(G)(\mathcal{A})$.

Left-invariant derivations and point derivations

Recall that for a superalgebra $\mathcal{A}$, its super vector space of derivations $\text{Der}(\mathcal{A})$ is the subspace of $\text{End}(\mathcal{A})$, where $\mathcal{A}$ is viewed as a super vector space, of maps $D : \mathcal{A} \rightarrow \mathcal{A}$ such that $D(ab) = D(a)b + (-1)^{|D||a|}aD(b)$. Note that for a supergroup $G = \text{Spec } \mathbb{C}[G]$, we have $\text{Der}(\mathbb{C}[G]) = \Gamma(G, \mathcal{T}_G)$, the space of vector fields.

Definition 4.2.2. Let G be a supergroup, and write $\mathbb{C}[G]$ for its Hopf superalgebra of global sections. Denote by Δ, ε , and S the comultiplication, counit, and antipode of G , respectively. We say that a derivation $D \in \text{Der}(\mathbb{C}[G])$ is *left-invariant* if $\Delta \circ D = (id \otimes D) \circ \Delta$.

It can be shown (as in [7]) that the commutator bracket

$$[D_1, D_2] = D_1 D_2 - (-1)^{|D_1||D_2|} D_2 D_1$$

on $\text{Der}(\mathbb{C}[G])$ preserves the space of left-invariant vector fields, making it a Lie superalgebra. Denote this Lie superalgebra by $L(G)$.

Now consider the tangent space $T_e G = \text{Der}(\mathbb{C}[G], \mathbb{C})$, where e is the identity in G . We can extend any point derivation $\delta \in T_e G$ to a left-invariant derivation $D = (id \otimes \delta) \circ \Delta$, and the resulting bijective correspondence allows us to define a Lie superalgebra structure on $T_e G$.

Theorem 4.2.3. *Let G be a supergroup. Then the following Lie superalgebras are isomorphic:*

- (a) $\mathrm{Lie}(G)$, the kernel of the morphism $G(\mathcal{A}(\varepsilon)) \rightarrow G(\mathcal{A})$
- (b) $L(G)$, the space of left-invariant vector fields on G
- (c) $T_e G$, the tangent space at the identity of G

We will refer to these notions interchangeably as the definition of the Lie superalgebra of a supergroup.

Example 4.2.4. Recall the supergroups $\mathrm{GL}(V)$, $\mathrm{SL}(V)$, and $Q(V)$ from Examples 4.1.3, 4.1.4, and 4.1.5. Their respective Lie superalgebras are $\mathfrak{gl}(V)$, $\mathfrak{sl}(V)$, and $\mathfrak{q}(V)$ from Examples 3.3.3, 3.3.4, and 3.3.5.

4.3 Super Harish-Chandra pairs

Another important perspective on supergroups is via super Harish-Chandra pairs.

Definition 4.3.1. A *super Harish-Chandra pair*, or *SHCP*, is a tuple $(G_{\bar{0}}, \mathfrak{g})$ where $G_{\bar{0}}$ is an affine algebraic group, $\mathfrak{g}$ is a finite-dimensional Lie superalgebra such that $\mathrm{Lie} G_{\bar{0}} = \mathfrak{g}_{\bar{0}}$, and $\mathfrak{g}$ is equipped with a $G_{\bar{0}}$ -module structure such that the action on $\mathfrak{g}_{\bar{0}}$ is adjoint.

A morphism of SHCPs $(G_{\bar{0}}, \mathfrak{g}) \rightarrow (H_{\bar{0}}, \mathfrak{h})$ consists of a morphism $\phi_{\bar{0}} : G_{\bar{0}} \rightarrow H_{\bar{0}}$ of algebraic groups and a morphism $d\phi : \mathfrak{g} \rightarrow \mathfrak{h}$ of Lie superalgebras such that the even part of $d\phi$ is indeed $d\phi_{\bar{0}}$.

The purely even group $G_{\bar{0}}$ contains global “even” data, while the Lie superalgebra $\mathfrak{g} = \mathrm{Lie} G$ contains infinitesimal data of mixed parity. It turns out that these data suffice to characterize the entire supergroup. Intuitively, since odd coordinates are nilpotent, any odd aspect of the geometry will already exhibit infinitesimal behavior.

It was proven in [27] that the functor $G \rightarrow (G_{\bar{0}}, \mathrm{Lie} G)$ is an equivalence of categories from supergroups to SHCPs, with inverse given by $(G_{\bar{0}}, \mathrm{Lie} G) \mapsto \mathrm{Spec} \mathrm{Hom}_{\mathfrak{g}_{\bar{0}}}(U\mathfrak{g}, \mathbb{C}[G_{\bar{0}}])$. We do not explicitly spell out the Hopf superalgebra structure on $\mathbb{C}[G] = \mathrm{Hom}_{\mathfrak{g}_{\bar{0}}}(U\mathfrak{g}, \mathbb{C}[G_{\bar{0}}])$ here, but will we certainly leverage the equivalence of categories.

Theorem 4.3.2 (complete version of theorem 4.1.2). *The following categories are equivalent:*

- (a) *The category of algebraic supergroups*
- (b) *The opposite category of finitely-generated commutative Hopf superalgebras which are integral superdomains*
- (c) *The category of supergroup functors*
- (d) *The category of super Harish-Chandra pairs*

4.4 Actions of supergroups

Definition 4.4.1. A (left) action of a supergroup G on a supervariety X is a morphism $a_X : G \times X \rightarrow X$ of supervarieties satisfying all the usual properties. In particular, each $a_X(\mathcal{A}) : G(\mathcal{A}) \times X(\mathcal{A}) \rightarrow X(\mathcal{A})$ an action of the group $G(\mathcal{A})$ on the set $X(\mathcal{A})$, and this relationship is functorial in $\mathcal{A}$.

Definition 4.4.2. An *representation* of G (or G -representation or G -module) on a super vector space V is a morphism $G \rightarrow \mathrm{GL}(V)$ of supergroups.

In the case where a G -module V is infinite-dimensional, it can be expressed as the unite of finite-dimensional G -modules. Either way, we will write $a_v : G \times V \rightarrow V$ for the resulting morphism of supervarieties.

Definition 4.4.3. A $\mathbb{C}[G]$ -comodule is a morphism $a_V^* : V \rightarrow \mathbb{C}[G] \otimes V$ of super vector spaces such that $(1 \otimes a_V^*) \circ a_V^* = (m_G^* \otimes 1) \circ a_V^*$, where m_G^* is comultiplication.

With the analogous notion of morphisms in the usual setting, we can form the (symmetric monoidal) categories of G -modules and $\mathbb{C}[G]$ -modules. As usual (see [25]), these categories are equivalent.

Actions of Lie G and SHCPs

Definition 4.4.4. An action of a Lie superalgebra $\mathfrak{g}$ on a supervariety X is a morphism of Lie superalgebras $\mathfrak{g} \rightarrow \Gamma(X, \mathcal{T}_X)$.

Given an action a_X of G on X , we obtain an action of $\mathfrak{g} = \mathrm{Lie} G$ on X as follows. For an open set $U \subseteq X$, choose an open subset $U_G \subseteq G$ such that $a_X|_{U_G \times U}$ lands in U . Then for any $f \in \Gamma(U, \mathcal{O}_X)$ and $u \in \mathfrak{g} = \mathrm{Der}(\mathbb{C}[G], \mathbb{C})$, define $u \cdot f = -(u \otimes 1) \circ a_X^*(f)$, a vector field on X . If V is a G -module, it naturally inherits the structure of a $\mathrm{Lie}(G)$ -module in an identical way. That is, the action is given by $u \cdot v = -(u \otimes 1) \circ a_V^*(v)$ for $u \in \mathfrak{g}$ and $v \in V$.

Definition 4.4.5. An action of a SHCP $(G_{\bar{0}}, \mathfrak{g})$ on a supervariety is a pair of actions $a_{X, \bar{0}} : G_{\bar{0}} \times X \rightarrow X$ and $\rho : \mathfrak{g} \rightarrow \Gamma(X, \mathcal{T}_X)$, compatible in the sense that the action of $\mathrm{Lie}(G_{\bar{0}})$ coincides with that of $\mathfrak{g}_{\bar{0}}$, and ρ commutes with the adjoint action of $G_{\bar{0}}$ on $\mathfrak{g}$.

Naturally, a G -action determines a $(G_0, \mathrm{Lie}(G))$ -action. The converse is proven for supermanifolds in [2] and for supervarieties in [46], so the categories of such actions are indeed equivalent. The same is therefore true for G -modules and $(G_0, \mathrm{Lie}(G))$ -modules. From this, we obtain the following corollary:

Corollary 4.4.6. *Let G be a supergroup with an action on a supervariety X such that the open $U_{\bar{0}} \subseteq X_{\bar{0}}$ is $G_{\bar{0}}$ -stable. Then U is G -stable as well.*

4.5 Cartan subgroups of quasi-reductive supergroups

Definition 4.5.1. A supergroup G is *quasi-reductive* if $G_{\bar{0}}$ is reductive.

The Lie superalgebra of a quasi-reductive supergroup is quasi-reductive.

Remark 4.5.2. As in the classical situation, a Cartan subgroup of an algebraic group is the centralizer of a maximal torus. In the case of a reductive group, maximal tori are their own centralizers. A similar phenomenon holds in the case of quasi-reductive supergroups; the even part of the centralizer of any maximal torus is indeed itself. However, sometimes such a centralizer will have a nontrivial odd component.

We supply the following definition here and at the beginning of Chapter 5 due to its centralizing nature in both this thesis and quasi-reductive supergroups.

Definition 4.5.3. A *supertorus* is a supergroup which occurs as the centralizer of a maximal torus in a quasi-reductive supergroup.

Because the maximal torus lies in $G_{\bar{0}}$ and is its own even centralizer, the even part $T_{\bar{0}}$ of a supertorus T is just an ordinary algebraic torus. Meanwhile, the odd part can admit non-abelian structure, so there are multiple supertori in each dimension. We will often refer to a supertorus by its SHCP $(T_{\bar{0}}, \mathfrak{t})$ where $\mathfrak{t}$ is a *quasiabelian* (or *quasitoral*) Lie superalgebra in the sense that $\mathfrak{t}_{\bar{0}}$ is central in $\mathfrak{t}$.

$$\begin{array}{ccc} G_{\bar{0}} & \subset & G \\ \cup & & \cup \\ T_{\bar{0}} & \subset & T \end{array}$$

Example 4.5.4. A Cartan subgroup of $\mathrm{GL}(m|n)$ or $\mathrm{SL}(m|n)$ is purely even; in the respective cases it must be isomorphic to $(\mathbb{C}^\times)^{m+n}$ and $(\mathbb{C}^\times)^{m+n-1}$. However, a Cartan subgroup of $Q(n)$ is instead isomorphic to $Q(1)^n$, the n -fold direct product of $Q(1)$ with itself.

4.6 Homogeneous supervarieties and orbits

Let G be a supergroup and K a closed subgroup of G . We may construct a supervariety G/K representing the quotient of G by the right action of K . We will spell out only what we need in this thesis; see [33] and [34] for more details.

The underlying topological space of G/K will be $G_{\bar{0}}/K_{\bar{0}}$, and we write $\pi_{\bar{0}} : G_{\bar{0}} \rightarrow G_{\bar{0}}/K_{\bar{0}}$ for the natural projection. If $U \subseteq G_{\bar{0}}/K_{\bar{0}}$ is open, then $\pi_{\bar{0}}^{-1}(U) \subseteq G_{\bar{0}}$ is stable under the right action of $K_{\bar{0}}$. By Corollary 4.4.6, it follows that the corresponding open subscheme of G is stable under the right action of K . The structure sheaf of G/K will therefore be $\mathcal{O}_{G/K} = ((\pi_{\bar{0}})_* \mathcal{O}_G)^K$. More explicitly, we have

$$\Gamma(U, \mathcal{O}_{G/K}) = \Gamma(\pi_{\bar{0}}^{-1}(U), \mathcal{O}_G)^K,$$

the right K -invariants of $\Gamma(\pi_{\bar{0}}^{-1}(U), \mathcal{O}_G)$. Hence $(G/K)_{\bar{0}} = G_{\bar{0}}/K_{\bar{0}}$, and

$$\mathbb{C}[G/K] = \mathbb{C}[G]^K = \mathbb{C}[G]^{(K_{\bar{0}}, \mathfrak{k})}$$

where $K \sim (K_{\bar{0}}, \mathfrak{k})$ acts on $\mathbb{C}[G]$ via pullback of the right action on G .

The natural projection $\pi : G \rightarrow G/K$ is equivariant with respect to the natural G -action by left translation. This action differentiates to an action by $\mathfrak{g}$, which induces a surjection $\mathfrak{g} \rightarrow T_e(G)$ whose kernel is $\mathfrak{k}$. In particular, G/K is smooth by Proposition 3.9.4.

Orbits and stabilizers

Let $a_X : G \times X \rightarrow X$ be an action of a supergroup on a supervariety. If $x \in X(\mathbb{C})$ is a closed point, then there is an orbit map $a_x : G \rightarrow X$ given by $a_x \circ (id_G \times i_x)$, where $i_x : x \rightarrow X$ is the natural inclusion. The *stabilizer subgroup* $\text{Stab}_G(x)$ is a closed subgroup (see [13]) of G defined as the fiber $a_x^{-1}(x)$. The orbit map $a_x : G \rightarrow X$ factors through the natural surjection $G \rightarrow G/\text{Stab}_G(x)$, and $G/\text{Stab}_G(x) \rightarrow X$ is an immersion of supervarieties. If this immersion is open, we refer to it as an open orbit of G .

The orbit closure of x , denoted $\overline{G \cdot x}$, is the closed subscheme of X determined by the sheaf of ideals given by the kernel of the pullback map $\mathcal{O}_X \rightarrow (a_x)_* \mathcal{O}_G$. It is seen in [46] that $\overline{G \cdot x}$ is a supervariety with open orbit isomorphic to $G/\text{Stab}_G(x)$.

Chapter 5

Toric supervarieties

5.1 Algebraic supertori

Definition 5.1.1. An *algebraic supertorus* is an algebraic supergroup $T = (T_{\bar{0}}, \mathfrak{t})$ such that $T_{\bar{0}} \cong (\mathbb{C}^\times)^n$ is an ordinary algebraic torus lying in the center of T .

This definition is motivated by the theory of Cartan subgroups of quasireductive supergroups as in [45]. A *quasireductive* supergroup G is one whose even part $G_{\bar{0}}$ is reductive. In this setting, one chooses a Cartan subgroup (i.e. maximal torus) $T_{\bar{0}}$ of $G_{\bar{0}}$, and defines T as the centralizer of $T_{\bar{0}}$ in G . Thus, an algebraic supertorus is an algebraic supergroup which occurs as a Cartan subgroup of a quasireductive supergroup. The representation theory of supertori has been studied in [17] and [47].

From Theorem 4.3.2, one sees that a supertorus is determined by an ordinary algebraic torus $T_{\bar{0}}$ and a super vector space $\mathfrak{t} = \mathfrak{t}_{\bar{0}} \oplus \mathfrak{t}_{\bar{1}}$ (such that $\mathfrak{t}_{\bar{0}} = \text{Lie } T_{\bar{0}}$) equipped with a linear map $S^2 \mathfrak{t}_{\bar{1}} \rightarrow \mathfrak{t}_{\bar{0}}$.

Recall that we write M and N for the weight and coweight lattices, respectively, of $T_{\bar{0}}$. Conversely, if N is a lattice, we write T_N for the algebraic torus whose coweight lattice is N . The perfect pairing between M and N will be written as

$$\langle -, - \rangle : M \times N \rightarrow \mathbb{Z},$$

and we note that it extends $\mathbb{C}$ -linearly to $M \times \mathfrak{t}_{\bar{0}}$ since $\mathfrak{t}_{\bar{0}} \cong N_{\mathbb{C}} := N \otimes_{\mathbb{Z}} \mathbb{C}$.

Lemma 5.1.2. Let $\phi : T \rightarrow T'$ be a morphism of supergroups, where T and T' are both supertori. Then the image of ϕ is a supertorus which is closed in T' .

Proof. We may view the morphism as living in the category of super Harish-Chandra pairs, so that we have a morphism $\phi_{\bar{0}} : T_{\bar{0}} \rightarrow T'_{\bar{0}}$ of algebraic groups and a compatible morphism $d\phi : \mathfrak{t} \rightarrow \mathfrak{t}'$ of Lie superalgebras. By Lemma 2.1.2, $\phi_{\bar{0}}(T_{\bar{0}})$ is a closed subtorus of $T'_{\bar{0}}$. Moreover, $d\phi(\mathfrak{t})$ is a Lie superalgebra whose even part is the Lie algebra of $\phi_{\bar{0}}(T_{\bar{0}})$. Since $\mathfrak{t}'_{\bar{0}}$ lies in the center of $\mathfrak{t}'$, the same holds for $\text{Lie } \phi_{\bar{0}}(T_{\bar{0}}) \subseteq \mathfrak{t}'_{\bar{0}}$ in $d\phi(\mathfrak{t}) \subseteq \mathfrak{t}'$, so the lemma follows. $\square$

The algebra of regular functions on T_0 is isomorphic to the group algebra $\mathbb{C}[M]$, and likewise $\mathbb{C}[T] \cong \mathbb{C}[M] \otimes_{\mathbb{C}} \Lambda \mathfrak{t}_1^*$. In coordinates, we will often fix compatible isomorphisms

$$\begin{aligned} T_N &\cong (\mathbb{C}^\times)^p \\ \mathbb{C}[T_N] &\cong \mathbb{C}[t_1^{\pm 1}, \dots, t_p^{\pm 1}], \\ \mathbb{C}[T] &\cong \mathbb{C}[t_1^{\pm 1}, \dots, t_p^{\pm 1}, \xi_1, \dots, \xi_q], \\ \mathfrak{t} &\cong \mathbb{C}\{x_1, \dots, x_p\} \oplus \mathbb{C}\{\theta_1, \dots, \theta_q\}. \end{aligned}$$

where we view the x_i (respectively, θ_i) $\in \text{Der}_{\mathbb{C}[T]}(\mathbb{C}[T], \mathbb{C})$ as even (respectively, odd) point derivations at the identity such that $x_i(t_j) = \delta_{ij} = \theta_i(\xi_j)$.

We then choose $x_{ij} = (x_{ij})_1 x_1 + \dots + (x_{ij})_p x_p \in \mathfrak{t}_0$ for $i, j = 1, \dots, q$ such that $x_{ij} + x_{ji} = [\theta_i, \theta_j]$ for all such i and j . In some cases we will use $x_{ij} = \frac{1}{2}[\theta_i, \theta_j]$, although in some situations it will be more convenient to make different choices. We will express the Hopf superalgebra structure on $\mathbb{C}[T]$, the group law of T , and the action of $\mathfrak{t}$ on $\mathbb{C}[T]$ in terms of these choices, but we stress that any two choices are equivalent via automorphisms of T .

The Hopf superalgebra structure on $\mathbb{C}[T]$ is given by the rules

$$\begin{aligned} \Delta t^m &= t^m \otimes t^m \left(1 - \sum_{i,j=1}^q \langle m, x_{ij} \rangle \xi_i \otimes \xi_j \right) & St^m &= t^{-m} & \varepsilon t^m &= 1 \\ \Delta \xi_i &= \xi_i \otimes 1 + 1 \otimes \xi_i & S \xi_i &= -\xi_i & \varepsilon \xi_i &= 0 \end{aligned}$$

where $t^m \in \mathbb{C}[M]$ is the function corresponding to the character $m \in M$, and the group law on $\mathcal{A}$ -points is

$$\begin{aligned} &(a_1, \dots, a_p \mid \alpha_1, \dots, \alpha_q) \cdot (b_1, \dots, b_p \mid \beta_1, \dots, \beta_q) \\ &= \left(a_1 b_1 \left(1 - \sum_{i,j=1}^q (x_{ij})_1 \alpha_i \beta_j \right), \dots, a_p b_p \left(1 - \sum_{i,j=1}^q (x_{ij})_p \alpha_i \beta_j \right) \right. \\ &\quad \left. \mid \alpha_1 + \beta_1, \dots, \alpha_q + \beta_q \right) \end{aligned}$$

for $a_i, b_i \in (\mathcal{A}_0)^\times$ and $\alpha_i, \beta_i \in \mathcal{A}_1$.

The left action of T on itself induces the left regular representation of T on $\mathbb{C}[T]$. This action differentiates to one of $\mathfrak{t}$ on $\mathbb{C}[T]$, wherein

$$\begin{aligned} x_i &\mapsto -t_i \frac{\partial}{\partial t_i} \\ \theta_i &\mapsto \sum_{j=1}^q \xi_j \left\langle t \frac{\partial}{\partial t}, x_{ij} \right\rangle - \frac{\partial}{\partial \xi_i} \end{aligned} \tag{5.1}$$

and where we have written $\langle t \frac{\partial}{\partial t}, x_{ij} \rangle$ to signify $(x_{ij})_1 t_1 \frac{\partial}{\partial t_1} + \dots + (x_{ij})_p t_p \frac{\partial}{\partial t_p}$, by abuse of the notation $\langle -, - \rangle : M \times N \rightarrow \mathbb{Z}$. For $x \in \mathfrak{t}$, we write x for the corresponding derivation. In general when we refer to “the” action of T (or $\mathfrak{t}$) on $\mathbb{C}[T]$, we will mean the left regular

representation. However, we will also later need the right regular representation, for which we will write x^{right} , so we record this action of $\mathfrak{t}$ here:

$$\begin{aligned} x_i^{\text{right}} &\mapsto t_i \frac{\partial}{\partial t_i} \\ \theta_i^{\text{right}} &\mapsto \sum_{j=1}^q \xi_j \left\langle t \frac{\partial}{\partial t}, x_{ji} \right\rangle + \frac{\partial}{\partial \xi_i} \end{aligned}$$

When H is a subgroup of T , we will write $\mathbb{C}[T]^H$ (or $\mathcal{A}^H$) for the H -invariants of $\mathbb{C}[T]$ (or of $\mathcal{A} \subseteq \mathbb{C}[T]$) under the right regular representation, so that $\mathbb{C}[T]^H \cong \mathbb{C}[T/H]$. Likewise, if x is a vector field on $\mathcal{A}$, we write $\mathcal{A}^x$ for its kernel.

Example 5.1.3. A recurring example in this paper will be the supertorus $Q(1)^n$, for which $p = q = n$ and $\frac{1}{2}[\theta_i, \theta_j] = \delta_{ij}x_i$.

5.2 Definition of toric supervarieties

If we wish to generalize Definition 2.2.1 to the supergeometric setting in a way that respects Remark 2.2.2, we must contend with the realm of homogeneous spaces T/H for a supertorus T . Since $T_{\bar{0}}$ is central in T , any purely even subgroup $H_{\bar{0}} \subseteq T$ is a normal subgroup, and so $T/H_{\bar{0}}$ is a supertorus. The same is not true for an arbitrary supergroup H ; in fact, T/H need not be a supergroup at all!

Example 5.2.1. Consider the supergroup $T = Q(1)^2 := Q(1) \times Q(1)$, with subgroup $H = Q(1)$ embedded diagonally. It is routine to check that H is not a normal subgroup, and so T/H does not inherit the structure of a supergroup.

Consequently, in our definition, we make the compromise that the open orbit of a toric supervariety is isomorphic merely to a homogeneous space T/H for a purely odd (and therefore abelian) subgroup H .

Definition 5.2.2. A *toric supervariety* is an irreducible normal supervariety X equipped with a left action by a supertorus T , and a T -equivariant map $T \rightarrow X$ which factors through an open immersion of a homogeneous space T/H . We will assume $H_{\bar{0}} = 1$. If $H = 1$, then the toric supervariety will be called *faithful*.

Occasionally we will wish to drop the normality criterion, at which times it will be clarified that the supervariety is not necessarily normal.

Lemma 5.2.3. *Let X be a (not necessarily normal) toric supervariety with torus T . Then the underlying variety $X_{\bar{0}}$ is a (not necessarily normal) toric variety with torus $T_{\bar{0}}$.*

Proof. The action of $T_{\bar{0}}$ on X preserves the defining ideal $\mathcal{I}_X$ of $X_{\bar{0}}$. □

5.3 Structure of affine toric supervarieties

Lemma 5.3.1. *Let $X = \operatorname{Spec} \mathcal{A}$ be an affine supervariety. Then:*

- (a) *X is toric (but not necessarily normal) for a quotient of the supertorus T if and only if $\mathcal{A}$ is isomorphic to a finitely-generated subalgebra of $\mathbb{C}[T]$ which is also a $\mathfrak{t}$ -subrepresentation.*
- (b) *If X is toric (but not necessarily normal) for the supertorus T , then it is faithful if and only if there is a non-nilpotent $T_{\bar{0}}$ -weight vector $f \in \mathcal{A}$ such that $\mathcal{A}[f^{-1}] \cong \mathbb{C}[T]$.*

Proof. First assume $\mathcal{A} \subseteq \mathbb{C}[T]$ is such a subalgebra as in (a). The inclusion $\mathcal{A} \rightarrow \mathbb{C}[T]$ dualizes to map $T \rightarrow X$ which factors through an open immersion $U \rightarrow X$ with $T \rightarrow U$ surjective. Because $\mathcal{A}$ is a subrepresentation of $\mathbb{C}[T]$, all the above maps of supervarieties are T -equivariant. Hence $U \cong T/H$ for some subgroup $H \subseteq T$, so X is a toric supervariety for $T/H_{\bar{0}}$. If there is a suitable weight vector $f \in \mathcal{A}$ as in (b), then $U \cong T$ is the basic open $D(f)$ and so X is faithful.

Conversely, suppose X is toric with open orbit isomorphic to T/H . By Lemma 3.6.4, $\mathcal{A} \rightarrow \mathbb{C}[T/H] = \mathbb{C}[T]^H \rightarrow \mathbb{C}[T]$ is injective. Moreover, the action of T on X becomes an action of T on $\mathbb{C}[X]$, so the conditions of (a) are satisfied. Now if X is faithful, consider the underlying variety $X_{\bar{0}} \supseteq T_{\bar{0}}$ with reduced coordinate algebra $\bar{\mathcal{A}} \subseteq \mathbb{C}[T_{\bar{0}}]$. Here, there is a $T_{\bar{0}}$ -weight vector $t^m \in \bar{\mathcal{A}}$ such that $\bar{\mathcal{A}}[t^{-m}] = \mathbb{C}[T_{\bar{0}}]$; i.e. $T_{\bar{0}} = D(t^m)$. Let $f \in \mathcal{A}$ lie in the preimage of x^m under the projection $\mathcal{A} \rightarrow \bar{\mathcal{A}}$. Then the open subvariety $D(f)$ has underlying variety $T_{\bar{0}}$ and so $\mathcal{A}[f^{-1}] \cong \mathbb{C}[T]$. $\square$

Henceforth consider only those toric supervarieties which are normal. By Lemma 3.6.7, this means the underlying toric varieties are normal, so the affines are determined by strongly convex, rational, polyhedral cones $\sigma \subset N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. When we say “cone” we implicitly assume all the prior adjectives.

More generally, a toric variety X_{Σ} for the torus T_N induces a fan Σ in $N_{\mathbb{R}}$, i.e. a collection of cones which contains all the faces of each of its cones, and such that the intersection of any two cones in Σ is a face of each. If τ is a face of σ , we write $\tau \leq \sigma$, so that $\mathbb{C}[S_{\tau}] \supseteq \mathbb{C}[S_{\sigma}]$ at the level of semigroup algebras, where $S_{\sigma} = \sigma^{\vee} \cap M$ is the semigroup of lattice points in the dual cone $\sigma^{\vee}$. For $m, m' \in S_{\sigma}$, we write $m' \leq_{\sigma} m$ if $m - m' \in S_{\sigma}$. We refer to [9] for any missing exposition on the subject.

We will use the symbol $\bullet$ to refer to the 0-dimensional cone, and ρ to refer to a ray. The set of rays belonging to the fan Σ (respectively, the cone σ) will be denoted by $\Sigma(1)$ (respectively, $\sigma(1)$). For an arbitrary cone $\sigma \in \Sigma$, we write $\mathcal{A}_{\sigma}$ for the coordinate superalgebra of an affine toric supervariety such that $\bar{\mathcal{A}}_{\sigma} = \mathbb{C}[S_{\sigma}]$.

If $\operatorname{Spec} \mathcal{A}$ is an affine toric supervariety where $\mathcal{A} = \mathcal{A}_{\sigma}$ has underlying toric variety $\operatorname{Spec} \mathbb{C}[S_{\sigma}]$, we write $\mathcal{A}_{\tau}$ for the coordinate superalgebra of the affine toric open subvariety corresponding to the face $\tau \leq \sigma$. Moreover, if $m \in S_{\sigma}$, we write $L_{\mathcal{A}_{\sigma}}(m)$ for the $-m$ -weight space of $\mathcal{A}_{\sigma}$ as a $T_{\bar{0}}$ -module. (The apparent sign discrepancy occurs because $T_{\bar{0}}$ acts by the character $-m$ on the monomial t^m .) Sometimes we will suppress the subscript and write $L(m)$ if the meaning is clear from context.

The notation $L(m)$ is intended to be evocative of, but not confused with, the typical notation for the irreducible weight modules of a torus. However, we stress that $L_{\mathbb{C}[T]}(m)$ is not usually irreducible for a supertorus T .

Lemma 5.3.2. *Let $\mathcal{A}_\rho$ be the coordinate superalgebra of a (not necessarily normal) affine toric supervariety corresponding to a ray ρ . Then $\mathcal{A}_\rho$ may be written as*

$$\mathbb{C} [t^{m_1}(1 + \dots), (t^{m_2}(1 + \dots))^{\pm 1}, \dots, (t^{m_p}(1 + \dots))^{\pm 1}, t^{\ell_1 m_1}(\dots), \dots, t^{\ell_r m_1}(\dots)]$$

for m_i a basis of M and $\ell_i \in \mathbb{Z}_{\geq 0}$, where any ellipsis within parentheses, $(1 + \dots)$ or $(\dots)$, refers to nilpotent terms in $L(0)$.

Proof. We know $\overline{\mathcal{A}_\rho} \cong \mathbb{C}[t^{m_1}, t^{\pm m_2}, \dots, t^{\pm m_p}]$. If $f, g \in \mathcal{A}_\rho$ are lifts of inverse elements in $\overline{\mathcal{A}_\rho}$, then $fg = 1 + \eta$ for some $\eta \in J_{\mathcal{A}_\rho}$, so $f(1 - \eta - \eta^2 - \dots) = g^{-1}$ and hence we may assume that inverses lift to inverses. The rest follows since we may generate $\mathcal{A}_\rho$ by T_0 -weight vectors. $\square$

Proposition 5.3.3. *Let $X = \text{Spec } \mathcal{A}$ be a (not necessarily normal) affine toric supervariety. Then X is normal if and only if all of the following conditions hold:*

- (a) $\overline{\mathcal{A}} \cong \mathbb{C}[S_\sigma]$ for some cone $\sigma \subseteq N_{\mathbb{R}}$
- (b) $\mathcal{A}_\sigma = \bigcap_{\rho \in \sigma(1)} \mathcal{A}_\rho$
- (c) $\mathcal{A}_\rho$ is FR for all ρ

Proof. First assume X is normal, so $X_{\overline{0}}$ is too and hence (a) holds. We also have (b) by the Hartogs property of Lemma 3.5.7, so it remains to prove (c). Using the expression for $\mathcal{A}_\rho$ obtained in Lemma 5.3.2, consider the height-1 prime ideal generated by nilpotent elements and $t^{m_1}(1 + \dots)$. Localization at this prime preserves the nilpotent generators, so they must form a regular sequence. Then $\mathcal{A}_\rho$ is isomorphic to the exterior algebra of the span of this regular sequence over $\overline{\mathcal{A}_\rho}$, so (c) follows.

For the converse, we note that (a) and (c) together imply that

$$U = \bigcup_{\rho \in \sigma(1)} \text{Spec } \mathcal{A}_\rho$$

is regular and therefore normal. Then the singular locus of X has codimension at least 2, so (R_1) , and hence also (S_1) , holds. For (S_2) , it suffices by Lemma 3.5.7 to show that X admits the Hartogs property. Let Z be a closed subset of $|X|$ with codimension at least 2, so $Z \cap U$ is likewise a closed subset of U with codimension at least 2. Since U is smooth and hence (S_2) , we have

$$\begin{array}{ccc} \mathcal{O}_X(X) & \xrightarrow{\sim} & \mathcal{O}_X(U) \\ \downarrow & & \downarrow \sim \\ \mathcal{O}_X(X \setminus Z) & \hookrightarrow & \mathcal{O}_X(U \setminus Z) \end{array}$$

so indeed X has the Hartogs property at Z . $\square$

Thus, to determine an affine toric supervariety, one needs only a cone $\sigma \subset N_{\mathbb{R}}$ and a suitable FR superalgebra $\mathcal{A}_\rho$ for each of its rays $\rho \in \sigma(1)$. In the following section, we propose a method of extracting classifying data from the coordinate superalgebras.

5.4 Decorated fans

Let X be a toric supervariety (for the supertorus T) with open orbit T/H , and suppose $X_{\bar{0}} \cong X_{\Sigma}$. Using Lemma 5.3.1(a), we may assume that each $\mathcal{A}_{\sigma}$ (for $\sigma \in \Sigma$) is a subalgebra of $\mathbb{C}[T]$ in such a way that $\mathcal{A}_{\sigma}$ localizes to $\mathcal{A}_{\tau}$ whenever $\tau \leq \sigma$. For each cone $\sigma \in \Sigma$ and each character $m \in S_{\sigma}$, let $\mathcal{A}_{\sigma,m}$ be the subalgebra of $\mathcal{A}_{\sigma}$ generated by $L_{\mathcal{A}_{\sigma}}(m')$ for all $m' \leq_{\sigma} m$. Then $\mathcal{A}_{\sigma,m}$ also satisfies the condition of Lemma 5.3.1(a), so we may define $H_{\sigma,m}$ as the stabilizer of the image of 1, i.e. the subtorus of T for which the T -action on $\text{Spec } \mathcal{A}_{\sigma,m}$ has open orbit $T/H_{\sigma,m}$.

The following lemma says that the choice of the point 1 was largely irrelevant.

Lemma 5.4.1. *Let T be a supertorus, Y a homogeneous space for T , and $x, y \in Y$ two closed points. Then $\text{Stab}_T x = \text{Stab}_T y$.*

Proof. Let $g \in T_{\bar{0}}$ be such that $gx = y$. Then $\text{Stab}_T y$ can be obtained from $\text{Stab}_T x$ via conjugation by g , which is central. Therefore $\text{Stab}_T x = \text{Stab}_T y$. $\square$

We write $\mathfrak{h} := \text{Lie } H$ and $\mathfrak{h}_{\sigma,m} := \text{Lie } H_{\sigma,m}$. Moreover, for $m \in S_{\sigma}$, write $\sigma_m = \{x \in \sigma \mid \langle m, x \rangle = 0\}$ for the face of σ consisting of its intersection with the hyperplane $\ker m$. We also write N_{σ} for the sublattice of N generated by $\sigma \cap N$.

Lemma 5.4.2. *Using the above notation, we have:*

- (a) $\overline{\mathcal{A}_{\sigma,m}} \cong \mathbb{C}[\sigma^{\vee} \cap \sigma_m^{\perp} \cap M]$
- (b) $(H_{\sigma,m})_{\bar{0}} = T_{N_{\sigma_m}}$
- (c) $(\mathfrak{h}_{\sigma,m})_{\bar{1}} = \{\theta \in \mathfrak{t}_1 \mid \theta^{\text{right}} L_{\mathcal{A}_{\sigma}}(m) = 0\}$
- (d) $H_{\sigma,m} \supseteq H_{\sigma,m'}$ whenever $m \leq_{\sigma} m'$
- (e) $\mathfrak{h}_{\sigma,m} \supseteq \sum_{\rho \in \sigma(1)} \mathfrak{h}_{\rho,m}$

Proof. (a) This is because $\sigma^{\vee} \cap \sigma_m^{\perp} \leq \sigma^{\vee}$ is the dual face of $\sigma_m \leq \sigma$, and m belongs to the relative interior of $\sigma_m^{\perp}$.

(b) The open $T_{\bar{0}}$ -orbit of $\text{Spec } \overline{\mathcal{A}_{\sigma,m}}$ has coordinate algebra

$$\mathbb{C}[\sigma_m^{\perp} \cap M] \cong \mathbb{C}[(N/N_{\sigma_m})^*]$$

by part (a).

(c) Since $\mathbb{C}[T/H_{\sigma,m}] \cong \mathbb{C}[T_{\bar{0}}/T_{N_{\sigma}}] \otimes \Lambda(\mathfrak{t}_{\bar{1}}/(\mathfrak{h}_{\sigma,m})_{\bar{1}})^*$ is a localization of $\mathcal{A}_{\sigma,m}$, we have

$$\begin{aligned} (\mathfrak{h}_{\sigma,m})_{\bar{1}} &= \{\theta \in \mathfrak{t}_1 \mid \theta^{\text{right}} L_{\mathbb{C}[T/H_{\sigma,m}]}(0) = 0\} \\ &= \{\theta \in \mathfrak{t}_1 \mid \theta^{\text{right}} L_{\mathcal{A}_{\sigma}}(m) = 0\}. \end{aligned}$$

(d) Immediate from construction of $\mathcal{A}_{\sigma,m} \subseteq \mathcal{A}_{\sigma,m'}$.

- (e) We have $L_{\mathcal{A}_\sigma}(m) = \bigcap_{\rho \leq \sigma} L_{\mathcal{A}_\rho}(m)$ by Proposition 5.3.3(b), so the statement holds for the odd part by (c). Equality of the even parts follows from (b). $\square$

If $\rho \in \Sigma(1)$, we write u_ρ for the primitive generator of the ray ρ . That is, u_ρ is the first nonzero lattice point along ρ . The following lemma is an immediate consequence of Lemma 5.4.2(d).

Lemma 5.4.3. *Let $\rho \in \Sigma(1)$. Then $\mathfrak{h}_{\rho,m}$ depends only on the nonnegative integer $\langle m, u_\rho \rangle$.*

For $i \geq 0$, we write $V_{\rho,i}$ (or $V_{\rho,m}$) for any subspace $(\mathfrak{h}_{\rho,m})_{\bar{1}} \subseteq \mathfrak{t}_{\bar{1}}$ such that $\langle m, u_\rho \rangle = i$. The prior lemma asserts that the choice of m is immaterial. We collect some results about the subspaces $V_{\rho,i}$ here:

Lemma 5.4.4. *Let $V_{\rho,i}$ be as above. Then:*

- (a) $V_{\rho,0} \supseteq V_{\rho,1} \supseteq V_{\rho,2} \supseteq \dots$
- (b) *The chain $V_{\rho,0} \supseteq V_{\rho,1} \supseteq V_{\rho,2} \supseteq \dots$ stabilizes at $\mathfrak{h}$.*
- (c) *If $\rho, \rho' \in \sigma(1)$ and $m \in \sigma^\vee \cap M$, then $[V_{\rho,m}, V_{\rho',m}] \subseteq \mathfrak{t}_{N_{\sigma_m}}$. In particular, $[V_{\rho,0}, V_{\rho,0}] \subseteq \mathfrak{t}_{N_\rho}$ and $[V_{\rho,1}, V_{\rho,1}] = 0$.*
- (d) *Either $[V_{\rho,0}, V_{\rho,0}] = 0$ or $\text{codim}(\mathfrak{h}, V_{\rho,0}) = 1$.*

Proof. (a) Lemma 5.4.2(d).

- (b) By Lemma 5.3.2, $\mathcal{A}_\rho \rightarrow \mathcal{A}_\bullet = \mathbb{C}[T/H]$ is a localization via the inversion of a single even weight vector $f = t^{m_1}(1 + \dots) \in \mathcal{A}_\rho$. Hence

$$L_{\mathbb{C}[T/H]}(0) = \sum_{n \geq 0} L_{\mathcal{A}_\rho}(nm_1)[f^{-n}],$$

so there is $n \geq 0$ for which

$$L_{\mathbb{C}[T/H]}(0) = L_{\mathcal{A}_\rho}(nm_1)[f^{-n}].$$

Therefore $V_{\rho,n} = \mathfrak{h}$.

- (c) Lemma 5.4.2 parts (b) and (e).
 (d) Suppose $\text{codim}(\mathfrak{h}, V_{\rho,0}) \geq 2$ and $[V_{\rho,0}, V_{\rho,0}] \neq 0$, so we may write

$$\begin{aligned} \mathcal{A}_\rho &= \mathbb{C}[t_1(1 + \dots), (t_2(1 + \dots))^{\pm 1}, \dots, (t_p(1 + \dots))^{\pm 1}, \\ &\quad \xi_1, \dots, \xi_r, t_1^{\ell_{r+1}}(\xi_{r+1} + \dots), \dots, t_1^{\ell_q}(\xi_q + \dots)] \end{aligned}$$

and pick $\theta \in V_{\rho,0}$ such that $[\theta, \theta] \neq 0$. We may assume $\theta = \xi_q t_1 \frac{\partial}{\partial t_1} - \frac{\partial}{\partial \xi_q}$. Then for $i = r+1, \dots, q-1$, we have $\theta \cdot (t_1^{\ell_i}(\xi_i + \dots)) = t_1^{\ell_i}(\xi_q \xi_i + \xi_q(\dots) - \frac{\partial}{\partial \xi_q}(\dots))$. But $\frac{\partial}{\partial \xi_q}(\dots)$ has no terms that can cancel with $\xi_q \xi_i$, contradicting that $\xi_q \xi_i$ first appears in $L(\ell_i + \ell_q)$. $\square$

In other words, a toric supervariety induces a decorated fan in the following sense:

Definition 5.4.5. A *decorated fan* is a tuple

$$(N, \Sigma, \mathfrak{t}, \mathfrak{h}, \{V_{\rho,i}\}_{\rho \in \Sigma(1), i \geq 0})$$

consisting of a finite-dimensional lattice N , a fan Σ in $N_{\mathbb{R}}$, a quasiabelian Lie superalgebra $\mathfrak{t}$ with $\mathfrak{t}_{\bar{0}} = \mathfrak{t}_N$, an odd abelian subalgebra $\mathfrak{h} \subseteq \mathfrak{t}$, and a collection of descending chains $V_{\rho,0} \supseteq V_{\rho,1} \supseteq \dots$ of subspaces of $\mathfrak{t}_{\bar{1}}$, stabilizing at $\mathfrak{h}$.

Beginning in Definition 5.6.4, we will additionally impose compatibility conditions on the $V_{\rho,i}$.

Our objective now is threefold: to determine which decorated fans arise from toric supervarieties, to determine to what extent we can recover a toric supervariety from its decorated fan, and to extract geometric information about a toric supervariety from its decorated fan. A first step towards the latter goal occurs in the following proposition, wherein we write $V_{\sigma,0} = \sum_{\rho \in \sigma(1)} V_{\rho,0}$ for the odd part of the Lie superalgebra $\mathfrak{h}_{\sigma,0}$.

Proposition 5.4.6. *Let X be a toric supervariety, $(N, \Sigma, \mathfrak{t}, \mathfrak{h}, \{V_{\rho,i}\})$ its decorated fan, and $y \in X$ a closed point in the $T_{\bar{0}}$ -orbit corresponding to $\sigma \in \Sigma$. Then $\text{Stab}_T y = (T_{N_{\sigma}}, \mathfrak{t}_{N_{\sigma}} \oplus V_{\sigma,0})$.*

Proof. The even part of the orbit is determined entirely classically. Meanwhile, the odd part is $V_{\sigma,0}$ because any odd coordinates $t^m(\dots)$ in $\mathcal{A}_{\sigma}$ will be fixed at 0 in this orbit unless $m \in \sigma^{\perp}$. $\square$

5.5 Uniqueness of decorated fans

In many important situations, a decorated fan will correspond to a unique toric supervariety. However, for supertori with large odd abelian subalgebras, this is far from the case.

Example 5.5.1. Let T be the abelian supertorus of dimension $(1|4)$, so that its Lie superalgebra acts by the derivations $t \frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \xi_i}$ for $i = 1, \dots, 4$ on $\mathbb{C}[T] = \mathbb{C}[t^{\pm 1}, \xi_1, \xi_2, \xi_3, \xi_4]$. We consider the following two superalgebras:

$$\begin{aligned} \mathcal{A} &= \mathbb{C}[t, t\xi_1, t\xi_2, t\xi_3, t^2\xi_4] \\ \mathcal{A}' &= \mathbb{C}[t, t\xi_1, t\xi_2, t\xi_3, t^2(\xi_4 + \xi_1\xi_2\xi_3)] \end{aligned}$$

Both $\text{Spec } \mathcal{A}$ and $\text{Spec } \mathcal{A}'$ admit the same decorations $\mathfrak{t}_{\bar{1}} \supseteq \mathbb{C}\theta_4 \supseteq 0 \supseteq \dots$, and both are isomorphic to $\mathbb{A}^{1|4}$ as a supervariety. However, the actions of T are decidedly different.

In fact, for particularly large abelian supertori, classification is a wild problem.

Proposition 5.5.2. *A classification of toric supervarieties contains a classification of triples of skew-symmetric bilinear forms up to simultaneous congruence.*

Proof. Let T be the abelian supertorus of dimension $(3|n+4)$. Write

$$\mathbb{C}[T] = \mathbb{C}[t_A^{\pm 1}, t_B^{\pm 1}, t_C^{\pm 1}, \xi_1, \dots, \xi_n, \xi_A, \xi_B, \xi_C, \xi'],$$

and assume the odd part $\mathfrak{t}_{\bar{1}}$ of the Lie superalgebra $\mathfrak{t}$ acts by $\theta_i = -\frac{d}{d\xi_i}$.

Let σ be the positive orthant, corresponding to the affine toric variety with coordinate ring $\mathbb{C}[t_A, t_B, t_C]$. Write ρ_A, ρ_B, ρ_C for its rays, and let

$$\begin{aligned} V_{\rho_\ell, 0} &= \mathbb{C}\{\theta_1, \dots, \theta_n, \theta_\ell, \theta'\} \\ V_{\rho_\ell, 1} &= \mathbb{C}\{\theta_\ell, \theta'\} \\ V_{\rho_\ell, 2} &= \mathbb{C}\{\theta'\} \\ V_{\rho_\ell, 3} &= 0 \end{aligned}$$

for $\ell = A, B, C$.

We will attempt to classify toric supervarieties with the decorated fan $(\mathbb{Z}^3, \sigma, \mathfrak{t}, 0, \{V_{\rho, i}\})$ specified above. For each ℓ , suppose $\theta_i \theta_j \theta_k$ kills $L_{\mathcal{A}_{\rho_\ell}}(2)$ for any $i, j, k \in \{1, \dots, n\}$. Then the possible coordinate superalgebras of U_{ρ_ℓ} are

$$\begin{aligned} \mathcal{A}_{\rho_A} &= \mathbb{C} \left[t_A, t_B^{\pm 1}, t_C^{\pm 1}, t_A \xi_1, \dots, t_A \xi_n, t_A^2 \xi_A, \xi_B, \xi_C, t_A^3 \left(\xi' + \xi_A \left(\sum_{i < j}^n a_{ij} \xi_i \xi_j \right) \right) \right] \\ \mathcal{A}_{\rho_B} &= \mathbb{C} \left[t_A^{\pm 1}, t_B, t_C^{\pm 1}, t_B \xi_1, \dots, t_B \xi_n, \xi_A, t_B^2 \xi_B, \xi_C, t_B^3 \left(\xi' + \xi_B \left(\sum_{i < j}^n b_{ij} \xi_i \xi_j \right) \right) \right] \\ \mathcal{A}_{\rho_C} &= \mathbb{C} \left[t_A^{\pm 1}, t_B^{\pm 1}, t_C, t_C \xi_1, \dots, t_C \xi_n, \xi_A, \xi_B, t_C^2 \xi_C, t_C^3 \left(\xi' + \xi_C \left(\sum_{i < j}^n c_{ij} \xi_i \xi_j \right) \right) \right] \end{aligned}$$

and hence we may write $(a_{ij}), (b_{ij}), (c_{ij})$ as skew-symmetric matrices. But these matrices depend on a choice of basis of $\mathbb{C}\{\xi_1, \dots, \xi_n\}$, and such changes of basis are equivalent to simultaneous congruence transformations. $\square$

In summary, one decorated fan can describe many different toric supervarieties. We introduce the following geometric criterion to ensure that the correspondence becomes bijective.

Homological regularity

Proposition/Definition 5.5.3. Let X be a toric supervariety and $(N, \Sigma, \mathfrak{t}, \mathfrak{h}, \{V_{\rho, i}\})$ its decorated fan. We say X is *homologically regular in codimension 1*, or (HR_1) , if it satisfies any of the following equivalent conditions:

- (a) For every $\rho \in \Sigma(1)$, there is a parameterization as in (5.1) of the derivations by which $\mathfrak{t}$ acts on $\mathbb{C}[T]$ such that

$$\mathcal{A}_\rho = \mathbb{C}[t_1, t_2^{\pm 1}, \dots, t_p^{\pm 1}, \xi_1, \dots, \xi_r, t_1^{\ell_{r+1}} \xi_{r+1}, \dots, t_1^{\ell_q} \xi_q]$$

for some positive integers ℓ_i .

- (b) For every $\rho \in \Sigma(1)$, $\mathcal{A}_\rho$ is the subalgebra of $\mathbb{C}[T]$ generated by $L_{\mathbb{C}[T]W}(n + \rho^\perp)$ for all $n \geq 0$ and all codimension-at-most-1 subspaces $W \subseteq V_{\rho, 0}$ containing $V_{\rho, n}$.
- (c) For every $\rho \in \Sigma(1)$ and proper subspace $W \subset V_{\rho, 0}$ containing $\mathfrak{h}$, the quotient T/W of T/H embeds into the toric supervariety whose decorated fan is $(N, \rho, \mathfrak{t}, W, \{V_{\rho, i} + W\})$.

- (d) For every point $x \in |X|$ of codimension at most 1, the maximal ideal $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ is generated by a regular sequence $r_1, \dots, r_d$ such that whenever $\theta \in (\text{Stab}_{\mathfrak{t}} x)_{\overline{\mathbb{I}}}$ and $\overline{\theta \cdot r_i} = 0$, we have $\theta \cdot r_i \in (r_i)$.
- (e) For every point $x \in |X|$ of codimension at most 1 and for every $\theta \in (\text{Stab}_{\mathfrak{t}} x)_{\overline{\mathbb{I}}}$, $DS_{\theta}(\mathcal{O}_{X,x}/((\mathcal{O}_{X,x}^{\mathfrak{t}_{\overline{0}}})_{\overline{\mathbb{I}}}))$ is FR.

Proof. We defer this somewhat lengthy proof to Appendix A. In essence, we show that these conditions always hold when $[V_{\rho,0}, V_{\rho,0}] \neq 0$, while for $[V_{\rho,0}, V_{\rho,0}] = 0$ we mainly employ the fact that $\theta_j^{\text{right}} = \frac{\partial}{\partial \xi_j}$ for $\theta_j \in V_{\rho,0}$. $\square$

Remark 5.5.4. The various equivalent characterizations of (HR_1) can be summarized as follows:

Characterizations (a) and (b) mean it is relatively straightforward to construct X from its decorated fan. Meanwhile, (c) says that locally, away from a closed subset of codimension ≥ 2 , X is a categorical limit of large-orbit toric supervarieties. Large-orbit toric supervarieties are those whose orbits have odd codimension no greater than their even codimension, and will be studied more in Section 5.8.

Characterization (d) is part of the reason we baptize this property (HR_1) , in analogy with the (R_1) condition of Definition 3.4.14. In particular, we can think of (HR_1) as an enhancement of (R_1) in which we further demand that the odd vector fields behave nicely away from a closed subset of codimension ≥ 2 . (The “homologically” part should be interpreted as arising from characterization (e), however.)

Finally, (e) links this property with the Duflo-Serganova functor DS_{θ} of [11] and [18], whose definition we recall here: If θ is an odd vector field acting semisimply on an affine supervariety $X = \text{Spec } \mathcal{A}$, we write $DS_{\theta}\mathcal{A}$ for the cohomology of θ on the θ^2 -invariants of $\mathcal{A}$, i.e.

$$\begin{aligned} DS_{\theta}\mathcal{A} &= \ker(\theta : \mathcal{A}^{\theta^2}) / \text{Im}(\theta : \mathcal{A}^{\theta^2}) \\ &= \mathcal{A}^{\theta} / \theta(\mathcal{A}^{\theta^2}), \end{aligned}$$

which is naturally a superalgebra.

The quotient $\mathcal{O}_{X,x}/((\mathcal{O}_{X,x}^{\mathfrak{t}_{\overline{0}}})_{\overline{\mathbb{I}}}) = \mathcal{O}_{X,x}/((\mathcal{O}_{X,x}^{\text{Stab}_{\mathfrak{t}} x})_{\overline{\mathbb{I}}})$ is the local ring at the point x of a particular closed subvariety of $\text{Spec } \mathcal{A}_{\rho}$ which admits an action of the abelian supertorus $(T_{\overline{0}}, \mathfrak{t}_{\overline{0}} \oplus V_{\rho,0})$. Condition (e) then says that the cohomology of any odd vector field induced by this action is fermionically regular.

In Example 5.5.1, it is routine to check that $\text{Spec } \mathcal{A}$ satisfies (HR_1) , but $\text{Spec } \mathcal{A}'$ does not. In particular, consider the vector field θ_1 , and write $\mathfrak{p}, \mathfrak{p}'$ for the height-1 prime ideals of $\mathcal{A}, \mathcal{A}'$, respectively, generated by t and the nilpotents. Then

$$\begin{aligned} DS_{\theta_1}\mathcal{A}_{\mathfrak{p}} &= \mathbb{C}[t, t\xi_2, t\xi_3, t^2\xi_4]_{\mathfrak{p}} \cap \mathbb{C}[t, t\xi_2, t\xi_3, t^2\xi_4] / (t) \\ &\cong \mathbb{C}[t\xi_2, t\xi_3, t^2\xi_4] \end{aligned}$$

is a FR superalgebra, but

$$\begin{aligned} DS_{\theta_1}\mathcal{A}'_{\mathfrak{p}'} &= \mathbb{C}[t, t\xi_2, t\xi_3, t^3\xi_4]_{\mathfrak{p}'} \cap \mathbb{C}[t, t\xi_2, t\xi_3, t^3\xi_4] / (t, t^2\xi_2\xi_3, t^3\xi_4) \\ &\cong \mathbb{C}[t\xi_2, t\xi_3] / (t^2\xi_2\xi_3) \end{aligned}$$

is not.

5.6 Classification of toric supervarieties

Thanks to this means of constructing (HR_1) toric supervarieties, we can now determine the exact conditions under which a decorated fan describes a (HR_1) toric supervariety. We begin with the following lemma, wherein we write $V_{\sigma,0}$ for $\mathfrak{h}_{\sigma,0}$, or equivalently $\sum_{\rho \in \sigma(1)} V_{\rho,0}$.

Lemma 5.6.1. *Let $X = \text{Spec } \mathcal{A}_\sigma$ be a (HR_1) toric supervariety, and $W \subseteq V_{\sigma,0}$. Then $\mathcal{A}_\sigma \supseteq L_{\mathbb{C}[T]^W}(m) \neq 0$ if and only if W satisfies*

- (a) $\langle m, [W, W] \rangle = 0$
- (b) $V_{\rho,m} \subseteq W$ for all $\rho \in \sigma(1)$
- (c) $\sum_{i=0}^{\langle m, u_\rho \rangle - 1} (\dim V_{\rho,i} - \dim V_{\rho,i} \cap W) \leq \langle m, u_\rho \rangle$ for all $\rho \in \sigma(1)$

Proof. First observe that condition (a) is equivalent to $L_{\mathbb{C}[T]^W}(m) \neq 0$. It remains to show equivalence of (b) and (c) with $\mathcal{A}_\sigma \supseteq L_{\mathbb{C}[T]^W}(m)$.

We begin with the case of $\sigma = \rho$ a ray. Consider the expression

$$\mathcal{A}_\rho = \mathbb{C}[t_1, t_2^{\pm 1}, \dots, t_p^{\pm 1}, t_1^{\ell_1} \xi_1, \dots, t_1^{\ell_q} \xi_q]$$

obtained in Proposition/Definition 5.5.3. If $W \subseteq V_{\rho,0}$ satisfies the three properties, assume without loss of generality that $\xi_1, \dots, \xi_d$ is a basis for $(\mathfrak{t}_1/W)^*$ compatible with

$$V_{\rho,0} = V_{\rho,0} + W \supseteq V_{\rho,1} + W \supseteq \dots \supseteq V_{\rho,m} + W = W$$

in the sense that each $(\mathfrak{t}_1/(V_{\rho,j} + W))^*$ is spanned by a subset of this basis. Then ℓ_j , for $j = 1, \dots, d$, is the smallest nonnegative integer such that ξ_j does not vanish on $V_{\rho,j} + W$. Hence

$$\begin{aligned} \sum_{j=1}^d \ell_j &= \sum_{\ell=1}^{\langle m, u_\rho \rangle} \ell \cdot (\dim(V_{\rho,\ell-1} + W) - \dim(V_{\rho,\ell} + W)) \\ &= \sum_{i=0}^{\langle m, u_\rho \rangle - 1} (\dim(V_{\rho,i} + W) - \dim(V_{\rho,m} + W)) \\ &= \sum_{i=0}^{\langle m, u_\rho \rangle - 1} (\dim V_{\rho,i} + \dim W - \dim(V_{\rho,i} \cap W) - \dim W) \\ &= \sum_{i=0}^{\langle m, u_\rho \rangle - 1} (\dim V_{\rho,i} - \dim V_{\rho,i} \cap W) \\ &\leq \langle m, u_\rho \rangle \end{aligned}$$

by property (c), so indeed $\mathcal{A}_\rho$ contains all $s^m \xi_I$ for $I \subseteq \{1, \dots, d\}$. Therefore $\mathcal{A}_\sigma \supseteq L_{\mathbb{C}[T]^W}(m)$.

Conversely, if $\mathcal{A}_\rho \supseteq L_{\mathbb{C}[T]^W}(m)$, then the same chain of equalities above shows that $\sum_{i=0}^{\langle m, u_\rho \rangle - 1} (\dim V_{\rho, i} - \dim V_{\rho, i} \cap W) \leq \langle m, u_\rho \rangle$ and $V_{\rho, m} \subseteq W$.

We now observe that properties (b) and (c) for $W \subseteq V_{\sigma, 0}$ are equivalent to the same two properties for $W \cap V_{\rho, 0} \subseteq V_{\rho, 0}$ for each $\rho \in \sigma(1)$. Hence, if $W \subseteq V_{\sigma, 0}$ satisfies (b) and (c), then

$$\begin{aligned} L_{\mathbb{C}[T]^W}(m) &\subseteq L_{\mathbb{C}[T]^{W \cap V_{\rho, 0}}}(m) \\ &\subseteq \mathcal{A}_\rho \end{aligned}$$

for all ρ , so indeed $L_{\mathbb{C}[T]^W}(m) \subseteq \mathcal{A}_\sigma$. Conversely, suppose $L_{\mathbb{C}[T]^W}(m) \subseteq \mathcal{A}_\sigma$. Since $L_{\mathbb{C}[T]^{V_{\rho, 0}}}(0)$ is contained within $\mathcal{A}_\rho$, it follows that $L_{\mathbb{C}[T]^{W \cap V_{\rho, 0}}}(m) \subseteq \mathcal{A}_\rho$, so $W \cap V_{\rho, 0}$ satisfies (b) and (c). Therefore the lemma is proven. $\square$

In view of both the prior lemma and those forthcoming, we make the following definition.

Definition 5.6.2. We say $W \subseteq V_{\sigma, 0}$ satisfies the *dimension jumping property with respect to σ and m* , or $DJ(\sigma, m)$, if W satisfies the three properties of Lemma 5.6.1. If ρ is a ray, sometimes we will write $DJ(\rho, n)$ when $\langle m, u_\rho \rangle = n$.

The following lemma is essentially a stronger version of Lemma 5.4.4(c). Since $W \subseteq V_{\sigma, 0}$ means $[W, W] \subseteq \mathfrak{t}_{N_\sigma}$, we note that condition (a) of Lemma 5.6.1 is equivalent to $[W, W] \subseteq \mathfrak{t}_{N_{\sigma_m}}$.

Lemma 5.6.3. *Let X be a (HR_1) toric supervariety and $(N, \Sigma, \mathfrak{t}, \mathfrak{h}, \{V_{\rho, i}\})$ its decorated fan. Then for each $\sigma \in \Sigma$ and $m \in \sigma^\vee \cap M$, there is a subspace $W \subseteq V_{\sigma, 0}$ satisfying $DJ(\sigma, m)$.*

Proof. Let $\sigma \in \Sigma$ and $m \in \sigma^\vee \cap M$. Then $L_{\mathcal{A}_\sigma}(m) \neq 0$, so it contains a simple T -submodule $C(m)$. We have $C(m) = L_{\mathbb{C}[T/S]}(m)$ for some $S \subseteq T$ with $S_0 = T_{\ker m}$. Let $W = \mathfrak{s}_1 \cap V_{\sigma, 0}$, so that $L_{\mathbb{C}[T]^W} \subseteq L_{\mathcal{A}_\sigma}(m)$ and hence W satisfies $DJ(\sigma, m)$ by Lemma 5.6.1. $\square$

It turns out that the condition of Lemma 5.4.4(d) is sufficient to provide a converse to Lemma 5.6.3. Therefore we will hereafter refer exclusively to the following enhanced notion of a decorated fan.

Definition 5.6.4. A *decorated fan* is a tuple $(N, \Sigma, \mathfrak{t}, \mathfrak{h}, \{V_{\rho, i}\}_{\rho \in \Sigma(1), i \geq 0})$ consisting of a finite-dimensional lattice N , a fan Σ in $N_\mathbb{R}$, a quasiabelian Lie superalgebra $\mathfrak{t}$ with $\mathfrak{t}_0 = \mathfrak{t}_N$, an odd abelian subalgebra $\mathfrak{h} \subseteq \mathfrak{t}$, and a collection of descending chains $V_{\rho, 0} \supseteq V_{\rho, 1} \supseteq \dots$ of subspaces of $\mathfrak{t}_1$, subject to the following compatibility conditions:

- (a) For each $\rho \in \Sigma(1)$, the chain $V_{\rho, 0} \supseteq V_{\rho, 1} \supseteq \dots$ stabilizes at $\mathfrak{h}$.
- (b) For each $\sigma \in \Sigma$ and $m \in \sigma^\vee \cap M$, there is a subspace $W \subseteq V_{\sigma, 0}$ satisfying $DJ(\sigma, m)$.
- (c) For each $\rho \in \Sigma(1)$, either $[V_{\rho, 0}, V_{\rho, 0}] = 0$ or $\text{codim}(\mathfrak{h}, V_{\rho, 0}) = 1$.

Theorem 5.6.5. *Any decorated fan induces a unique (HR_1) toric supervariety, up to isomorphism.*

Proof. Proposition/Definition 5.5.3 gives us both uniqueness and a method to construct the candidate toric supervariety. This method depends on conditions (b), (c), and (d) of Lemma 5.4.4, which are guaranteed by parts (a), (b), and (c), respectively, of Definition 5.6.4. So, for $\rho \in \Sigma(1)$, let $\mathcal{A}_\rho$ be as in Proposition/Definition 5.5.3, and let $\mathcal{A}_\sigma = \bigcap_{\rho \in \sigma(1)} \mathcal{A}_\rho$.

We first verify that $\mathcal{A}_\sigma$ is finitely generated. First note that since each weight space is finite-dimensional, we may assume it contains only finitely many generators. Now choose $m \in \sigma^\vee \cap M$ such that $L_{\mathcal{A}_\sigma}(m) = L_{\mathbb{C}[T]^H}(m)$. Then we may assume every generator occurs in $L_{\mathcal{A}_\sigma}(m')$ for $m' \leq_\sigma m$, of which there are finitely many (up to $\sigma^\perp$). We conclude that $\mathcal{A}_\sigma$ is finitely generated, and hence that $\text{Spec } \mathcal{A}_\sigma$ is a (not necessarily normal) toric supervariety by Lemma 5.3.1.

For normality, we show that $\overline{\mathcal{A}_\sigma} = \mathbb{C}[S_\sigma]$. Since every nonzero submodule of $\mathbb{C}[T]$ contains a non-nilpotent element, it suffices to show that $L_{\mathcal{A}_\sigma}(m)$ is nonzero for every $m \in \sigma^\vee \cap M$. Using condition (b) of the definition, we obtain from Lemma 5.6.1 the desired result.

We now claim that $\mathcal{A}_\sigma$ localizes to $\mathcal{A}_\rho$. It suffices to show that for each $n \geq 0$, there is $m \in \sigma^\vee \cap M$ such that $\langle m, u_\rho \rangle = n$ and $L_{\mathcal{A}_\rho}(m) = L_{\mathcal{A}_\sigma}(m)$. This is merely a consequence of the convex geometry of cones. In particular, the intersection $(n + \rho^\perp) \cap \sigma^\vee \cap M$ is infinite, so its elements have arbitrarily large pairings with $u_{\rho'}$ for $\rho' \in \sigma(1) \setminus \{\rho\}$. Therefore there is $m \in (n + \rho^\perp) \cap \sigma^\vee \cap M$ such that $L_{\mathcal{A}_{\rho'}}(m) = L_{\mathbb{C}[T]^H}(m)$ for all such ρ' , so indeed $L_{\mathcal{A}_\rho}(m) = L_{\mathcal{A}_\sigma}(m)$. It then follows by Proposition 5.3.3 that $\text{Spec } \mathcal{A}_\sigma$ is normal, and by Proposition/Definition 5.5.3 that $\text{Spec } \mathcal{A}_\sigma$ is (HR_1) , completing the proof. $\square$

5.7 The abelian case

Suppose T is an abelian supertorus, and let X be a toric supervariety for T . Since any subgroup of T is normal, we may assume that the morphism $T \rightarrow X$ is an open immersion, i.e. $H = 1$.

In this situation, since the bracket is 0, every $V_{\sigma,0}$ satisfies $DJ(\sigma, m)$, and conditions (b) and (c) of Definition 5.6.4 are immaterial. It follows that a decorated fan can be expressed simply as a fan Σ such that each ray $\rho \in \Sigma(1)$ is decorated by a decreasing chain of subspaces $V_{\rho,0} \supseteq V_{\rho,1} \supseteq V_{\rho,2} \supseteq \dots$ of $\mathfrak{t}_T$ that stabilizes at 0.

Example 5.7.1. We may write projective superspace

$$\mathbb{P}^{p|q} = \text{Proj } \mathbb{C}[z_0, \dots, z_p, \zeta_1, \dots, \zeta_q]$$

as a toric supervariety for an abelian supertorus T of dimension $p|q$ as follows: As in the usual case, let T_0 act by $m \in \mathbb{Z}^p$ on the monomial $z_0^r z_1^{m_1} \dots z_p^{m_p}$ for any $r \in \mathbb{Z}$. Likewise, let $\mathfrak{t}_T$ act by translations on the odd part so that θ_i acts by $\frac{\partial}{\partial \zeta_i}$ on the coordinate superalgebras.

As usual, we obtain a complete fan Σ whose rays are $\rho_i = \mathbb{R}_+ e_i$ for $i = 1, \dots, n$ and $\rho_0 = \mathbb{R}_+(-e_1 - \dots - e_p)$. If σ is the positive orthant $\mathbb{R}_+(e_1, \dots, e_p)$, then $\mathcal{A}_\sigma = \mathbb{C}[\frac{z_1}{z_0}, \dots, \frac{z_p}{z_0}, \frac{\zeta_1}{z_0}, \dots, \frac{\zeta_q}{z_0}] \cong \mathbb{C}[t_1, \dots, t_p, \xi_1, \dots, \xi_q]$ and the other coordinate superalgebras of maximal cones are

$$\mathbb{C}[t_i^{-1} t_1, \dots, t_i^{-1} t_p, t_i^{-1} \xi_1, \dots, t_i^{-1} \xi_q]$$

for $i = 1, \dots, p$. Hence the decorations consist of $0 \supseteq 0 \supseteq \dots$ on $\rho_1, \dots, \rho_p$ and $\mathfrak{t}_T \supseteq 0 \supseteq \dots$ on ρ_0 .

Alternatively, by adjusting the characters by which $T_{\bar{0}}$ acts on the odd parts of the affine charts, one could arrange for any collection of subspaces $V_{\rho_i,0}$ as the decorations, so long as they are linearly independent and span $\mathfrak{t}_{\bar{1}}$.

Since the action of $\mathfrak{t}_{\bar{1}}$ on $\mathbb{C}[T]$ is by $\frac{\partial}{\partial \xi}$ for $\xi \in \mathfrak{t}_{\bar{1}}^*$, it follows that there exists a natural splitting $\bar{\mathcal{A}} \rightarrow \mathcal{A}$ of the projection $\mathcal{A} \rightarrow \bar{\mathcal{A}}$. These splittings assemble into a global splitting $X \rightarrow X_{\bar{0}}$, so that any (HR_1) toric supervariety for an abelian supertorus is *split* in this sense.

5.8 The case of large orbits

In this section we will assume that $\text{codim}(\mathfrak{h}, V_{\rho,0}) \leq 1$, so that the odd codimension of an orbit never exceeds the even codimension. Such a toric supervariety will be said to have *large orbits*. In view of Proposition/Definition 5.5.3, it is clear that toric supervarieties with large orbits are always (HR_1) . Moreover, $\sum_{\rho \in \sigma(1)} V_{\rho,m}$ always satisfies $DJ(\sigma, m)$, so condition (b) of Definition 5.6.4 can be weakened to the condition of Lemma 5.4.4(c). These observations are summarized in the following definition and proposition:

Definition 5.8.1. A *large-orbit decorated fan* is a tuple $(N, \Sigma, \mathfrak{t}, \mathfrak{h}, \{V_{\rho,i}\})$ consisting of a finite-dimensional lattice N , a fan Σ in $N_{\mathbb{R}}$, a quasiabelian Lie superalgebra $\mathfrak{t}$ with $\mathfrak{t}_{\bar{0}} = \mathfrak{t}_N$, an odd abelian subalgebra $\mathfrak{h} \subseteq \mathfrak{t}$, and a collection of descending chains $V_{\rho,0} \supseteq V_{\rho,1} \supseteq \dots$ of subspaces of $\mathfrak{t}_{\bar{1}}$, subject to the following compatibility conditions:

- (a) For each $\rho \in \Sigma(1)$, the chain $V_{\rho,0} \supseteq V_{\rho,1} \supseteq \dots$ stabilizes at $\mathfrak{h}$.
- (b) If $\rho, \rho' \in \sigma(1)$ and $m \in \sigma^\vee \cap M$, then $[V_{\rho,m}, V_{\rho',m}] \subseteq \mathfrak{t}_{N_{\sigma_m}}$.
- (c) For each $\rho \in \Sigma(1)$, $\text{codim}(\mathfrak{h}, V_{\rho,0}) \leq 1$.

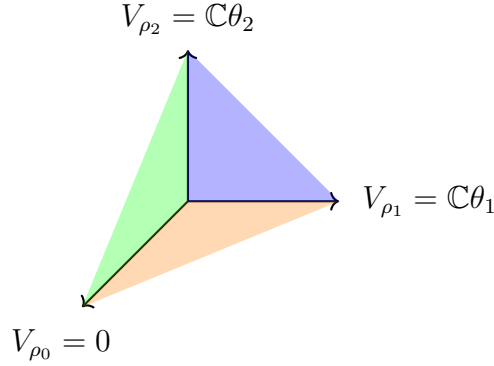
In particular, each chain $V_{\rho,0} \supseteq V_{\rho,1} \supseteq \dots$ is constant except possibly for a one-dimensional jump at a single index.

Proposition 5.8.2. *Up to T -equivariant isomorphism, there is a bijective correspondence between large-orbit decorated fans and toric supervarieties with large orbits.*

For supertori such as $T = Q(1)^n$ that have no nonzero square roots of 0 (or more generally, no purely odd subalgebras properly containing $\mathfrak{h}$), every toric supervariety has large orbits by Lemma 5.4.4(d). Moreover, since $[V_{\rho,1}, V_{\rho,1}] = 0$, it holds that $V_{\rho,1} = \mathfrak{h}$. Consequently, all the data of such a toric supervariety are concentrated in the fan decorations $V_{\rho,0}$ for $\rho \in \Sigma(1)$. The data of a decorated fan can then be simplified as in the following corollary.

Corollary 5.8.3. *Let T be a supertorus such as $Q(1)^n$ with no nonzero square roots of 0. Then a decorated fan with $\mathfrak{t} = \text{Lie } T$ can be reduced to the data of $(N, \Sigma, \mathfrak{t}, \{V_{\rho}\}_{\rho \in \Sigma(1)})$ satisfying the following compatibility conditions.*

- (a) If $\rho, \rho' \in \sigma(1)$, then $[V_{\rho}, V_{\rho'}] \subseteq \mathfrak{t}_{N_{\sigma}}$.
- (b) For each $\rho \in \Sigma(1)$, $\dim(V_{\rho}) \leq 1$.


 Figure 5.1: The decorated fan of an action of $Q(1)^2$ on $\mathbb{P}^{2|2}$

We will explore some examples of such toric supervarieties presently, as well as in Chapter 6.

Example 5.8.4. Let us use Proposition 5.8.2 and Corollary 5.8.3 to classify toric supervarieties with supertorus $Q(1)^n$ and underlying variety $\mathbb{P}^n \cong X_\Sigma$ for the complete fan whose rays are $\rho_i = \mathbb{R}_+ x_i$ for $i = 1, \dots, n$ and $\rho_0 = \mathbb{R}_+(-x_1 - \dots - x_n)$.

For $i = 1, \dots, n$, the ray ρ_i must be decorated by a subspace V_{ρ_i} such that $[V_{\rho_i}, V_{\rho_i}] \subseteq \mathbb{C}x_i$. Hence $V_{\rho_i} = \mathbb{C}\theta_i$ or 0. Likewise, unless $V_{\rho_0} = 0$, we have $V_{\rho_0} = \mathbb{C}(\theta_1 \pm \theta_2 \pm \dots \pm \theta_n)$ for some 2^{n-1} choices of $\pm$.

It is straightforward to verify that condition (b) of the definition of a large-orbit decorated fan holds regardless of which subspaces are chosen. We therefore obtain a collection of $2^n(1 + 2^{n-1})$ toric supervarieties which are not equivariantly isomorphic. Many, however, are isomorphic via toric morphisms (to be defined in the following section).

Notice that if all but one of these decorations is nonzero, then the decorated fan describes a supervariety isomorphic to projective superspace $\mathbb{P}^{n|n}$. For instance, if $V_{\rho_i} = \mathbb{C}\theta_i$ and $V_{\rho_0} = 0$, then the coordinate superalgebras of the affine charts can be written as

$$\mathbb{C}[t_1, \dots, t_n, t_1\xi_1, \dots, t_n\xi_n]$$

and

$$\mathbb{C}[t_i^{-1}t_1, \dots, t_i^{-1}, \dots, t_i^{-1}t_n\xi_1, \dots, \xi_i, \dots, t_i^{-1}t_n\xi_n].$$

Figure 5.1 depicts the corresponding decorated fan for $n = 2$,

If instead we change the decoration of ρ_0 to $\theta_1 + \dots + \theta_n$, then the resulting affine charts have coordinate superalgebras

$$\mathbb{C}[t_1, \dots, t_n, t_1\xi_1, \dots, t_n\xi_n]$$

and

$$\mathbb{C}[t_i^{-1}t_1(1 + \xi_i\xi_1), \dots, t_i^{-1}, \dots, t_i^{-1}t_n(1 + \xi_i\xi_n), t_i^{-1}t_1(\xi_i - \xi_1), \dots, t_i^{-1}\xi_i, \dots, t_i^{-1}t_n(\xi_i - \xi_n)]$$

so the supervariety is decidedly not isomorphic to projective superspace. Figure 5.2(B) depicts corresponding decorated fan for $n = 1$.

In general, when $T = Q(1)^n$, there are finitely many possible decorations for each ray. Namely, for $\rho = \mathbb{R}_+(a_1x_1 + \dots + a_nx_n)$, there are 2^{d-1} many possible “square root subspaces” $\mathbb{C}(\sqrt{a_1}\theta_1 \pm \dots \pm \sqrt{a_n}\theta_n)$, where d is the number of indices $i = 1, \dots, n$ for which $a_i \neq 0$.

The prior example admits no issues of compatibility between different rays of the same cone. This is not ordinarily the case; if $\rho_1 = \mathbb{R}_+(a_1x_1 + \dots + a_nx_n)$ and $\rho_2 = \mathbb{R}_+(b_1x_1 + \dots + b_nx_n)$, compatibility is less common if the linear matroid on the n vectors

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \dots, \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

has certain independence properties. For example, if two rays span a 2-dimensional cone and the corresponding matroid contains a 3-circuit, then one of the decorations V_ρ must be 0. While we do not explore this idea in depth in this paper, we provide the following example of this phenomenon.

Example 5.8.5. Let $n = 3$ and consider the cone whose rays are $\rho_1 = \mathbb{R}_+(x_1 + x_2)$ and $\rho_2 = \mathbb{R}_+(x_1 + x_3)$. If V_{ρ_1} and V_{ρ_2} are both nonzero, then $[V_{\rho_1}, V_{\rho_2}] = \mathbb{C}x_1$, which is not contained within the span of ρ_1 and ρ_2 . Hence the decorations are compatible if and only if at least one of them is 0.

5.9 Decorated polytopes

Recall from section 2.5 that a toric variety X_Σ is projective if and only if Σ is the normal fan Σ_P of a lattice polytope P in $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. In particular, a face F of dimension r induces a cone σ_F of codimension r , which in turn corresponds to a torus orbit of dimension r .

Suppose X is a toric supervariety whose underlying variety is X_{Σ_P} . Due to the bijection between faces of P and cones of Σ_P , we may assign the same data $V_{\sigma_F,0} \supseteq V_{\sigma_F,1} \supseteq \dots$ to each face F of codimension 1 in P . Since $M_{\mathbb{R}}$ is a real form of $\mathfrak{t}_0^*$ and $V_{\sigma_F} \subseteq \mathfrak{t}_1$, we will prefer the assignment $W_{F,i} := (\mathfrak{t}_1/V_{\sigma_F,i})^* \subseteq \mathfrak{t}_1^*$ when attempting to decorate such a polytope.

Of course, in order for a decorated polytope to define a toric supervariety, one requires compatibility conditions on the decorations $W_{F,i}$. We neglect to state these on the grounds that they are essentially the same as those in Definitions 5.6.4 and 5.8.1, and that they are easier to express in terms of the $V_{\rho,i}$.

Suppose the decorated polytope $(P, \{W_{F,i}\})$ corresponds to a toric supervariety with large orbits, and define $W_F = (\mathfrak{t}_1/V_{\sigma_F,0})^*$ for all faces $F \subseteq P$. We may then view the decorated polytope as the subset $\mathcal{P} = \bigcup_{F \in P} \text{int } F \times W_F$ of $M_{\mathbb{R}} \oplus \mathfrak{t}_1^*$, where $\text{int } F$ is the relative interior of F . This subset fails to be bounded or even a polyhedron, but it is at least convex and locally closed. The upshot of this viewpoint will be explored more in Chapter 6.

Example 5.9.1. Continuing Example 5.8.4 in the case of $n = 1$, we note that the decorations $V_{\rho_0} = V_{\rho_1} = \mathfrak{t}_1$ describe the “isomeric projective line,” or $\text{QGr}(1,2)$, to be explored more in Chapter 6. The corresponding decorated polytope and decorated fan are illustrated in Figure 5.2.



Figure 5.2: Two combinatorial descriptions of $\mathrm{QGr}(1, 2)$ as a toric supervariety for $\mathbb{Q}(1)$.

5.10 Morphisms and equivalence of categories

Towards a categorical version of our classification of toric supervarieties, we now consider toric morphisms. Let X and X' be toric supervarieties with open orbits T/H and T'/H' , respectively.

Definition 5.10.1. A *toric morphism* $X \rightarrow X'$ is a commutative diagram

$$\begin{array}{ccccc}
 T & \longrightarrow & T/H & \longrightarrow & X \\
 \downarrow \phi & & \downarrow \tilde{\phi} & & \downarrow \varphi \\
 T' & \longrightarrow & T'/H' & \longrightarrow & X'
 \end{array}$$

where ϕ is a morphism of supergroups such that $\phi(H) \subseteq H'$, $\tilde{\phi}$ is the induced map $T/H \rightarrow T'/H'$, and $\varphi : X \rightarrow X'$ extends $\tilde{\phi}$.

A toric morphism of toric supervarieties naturally drops to a toric morphism of the underlying toric varieties. So, we obtain from the pair (ϕ, φ) a morphism $\phi^\circ : N \rightarrow N'$ of lattices such that for each $\sigma \in \Sigma$, there is $\sigma' \in \Sigma'$ for which $\phi_{\mathbb{R}}^\circ(\sigma) \subseteq \sigma'$. Such a ϕ° is called a *morphism of fans*.

In addition, we obtain a morphism $d\phi : \mathfrak{t} \rightarrow \mathfrak{t}'$ of Lie superalgebras such that $\phi_{\mathbb{C}}^\circ = d\phi_{\mathbb{C}}$. It remains to describe the compatibility conditions with respect to the decorations. Suppose $\phi_{\mathbb{R}}^\circ(\sigma) \subseteq \sigma'$, so we may reduce to the affine case $\varphi : \mathrm{Spec} \mathcal{A}_\sigma \rightarrow \mathrm{Spec} \mathcal{A}_{\sigma'}$. The map φ dualizes to $\varphi^* : \mathcal{A}_{\sigma'} \rightarrow \mathcal{A}_\sigma$, a restriction of $\phi^* : \mathbb{C}[T'] \rightarrow \mathbb{C}[T]$. The underlying map of rings $\overline{\varphi}^* : \mathbb{C}[\sigma'^\vee \cap M'] \rightarrow \mathbb{C}[\sigma^\vee \cap M]$ sends $m' \in M' \cong \mathrm{Hom}(N', \mathbb{Z})$ to $m' \circ \phi^\circ \in M \cong \mathrm{Hom}(N, \mathbb{Z})$, so it must hold that $\phi^*(L_{\mathcal{A}_{\sigma'}}(m')) \subseteq L_{\mathcal{A}_\sigma}(m' \circ \phi^\circ)$.

Thus, a toric morphism is determined by a morphism of fans $\phi^\circ : (N, \Sigma) \rightarrow (N', \Sigma')$ together with a compatible morphism of Lie superalgebras $d\phi : \mathfrak{t} \rightarrow \mathfrak{t}'$ such that $\phi^*(L_{\mathcal{A}_{\sigma'}}(m')) \subseteq L_{\mathcal{A}_\sigma}(m' \circ \phi^\circ)$ whenever $\phi_{\mathbb{R}}^\circ(\sigma) \subseteq \sigma'$ and $m' \in \sigma'^\vee \cap M$. One would hope that a condition involving only the decorations $V_{\rho,i}$ and $V'_{\rho',i}$ would suffice. For large-orbit toric supervarieties, such a condition is indeed viable due to the following property.

Lemma 5.10.2. *Let X be a large-orbit toric supervariety and $(N, \Sigma, \mathfrak{t}, \mathfrak{h}, \{V_{\rho,i}\})$ its decorated fan. Then for $\sigma \in \Sigma$, it holds that $L_{\mathcal{A}_\sigma}(m) = L_{\mathbb{C}[T]^{\vee_{\sigma,m}}}(m)$ where $V_{\sigma,m} = \sum_{\rho \in \sigma(1)} V_{\rho,m}$.*

Proof. The claim is immediate in the case of $\dim \sigma \leq 1$ by construction of $\mathcal{A}_\sigma$ in Proposition/Definition 5.5.3. Otherwise we have

$$\begin{aligned} L_{\mathcal{A}_\sigma}(m) &= \bigcap_{\rho \in \sigma(1)} L_{\mathcal{A}_\rho}(m) \\ &= \bigcap_{\rho \in \sigma(1)} L_{\mathbb{C}[T]^{V_{\rho,m}}}(m) \\ &= L_{\mathbb{C}[T]^{V_{\sigma,m}}}(m) \end{aligned}$$

and we are finished. $\square$

Hence the condition that $\phi^*(L_{\mathcal{A}'_{\sigma'}}(m')) \subseteq L_{\mathcal{A}_\sigma}(m' \circ \phi^\circ)$ is equivalent to

$$\phi^*(L_{\mathbb{C}[T']^{V'_{\sigma',m'}}}(m')) \subseteq L_{\mathbb{C}[T]^{V_{\sigma,m' \circ \phi^\circ}}}(m' \circ \phi^\circ)$$

or simply $d\phi(V_{\sigma,m' \circ \phi^\circ}) \subseteq V'_{\sigma',m'}$.

Definition 5.10.3. Let $(N, \Sigma, \mathfrak{t}, \mathfrak{h}, \{V_{\rho,i}\})$ and $(N', \Sigma', \mathfrak{t}', \mathfrak{h}', \{V'_{\rho',i}\})$ be large-orbit decorated fans, with $\phi^\circ : (N, \Sigma) \rightarrow (N', \Sigma')$ a morphism of fans and $d\phi : \mathfrak{t} \rightarrow \mathfrak{t}'$ a compatible morphism of Lie superalgebras. We say $d\phi$ is *compatible with the decorations* if $d\phi(V_{\sigma,m' \circ \phi^\circ}) \subseteq V'_{\sigma',m'}$ for any $\sigma \in \Sigma$ and $\sigma' \in \Sigma'$ such that $\phi^\circ_\mathbb{R}(\sigma) \subseteq \sigma'$, and any $m' \in \sigma'^\vee \cap M'$. In this situation, we call the pair $(\phi^\circ, d\phi)$ a *morphism of large-orbit decorated fans*.

That is, we have the following proposition:

Proposition 5.10.4. *Let X and X' be toric supervarieties with large orbits. A morphism $\phi : T \rightarrow T'$ of supergroups drops to $T/H \rightarrow T'/H'$ and subsequently extends to a toric morphism $X \rightarrow X'$ if and only if $(\phi^\circ, d\phi)$ is a morphism of large-orbit decorated fans.*

Proof. As above, this follows from Lemma 5.10.2. $\square$

We may then define the category of large-orbit decorated fans via the objects and morphisms of Definitions 5.8.1 and 5.10.3. Combining Propositions 5.8.2 and 5.10.4, we immediately obtain the following theorem.

Theorem 5.10.5. *The category of large-orbit toric supervarieties is equivalent to the category of large-orbit decorated fans.*

Generalizing this equivalence to (HR_1) toric supervarieties turns out to be surprisingly difficult. While it is not difficult to show, in analogy with Lemma 5.10.2, that $\mathcal{A}_\rho$ is spanned by $L_{\mathbb{C}[T]^W}(m)$ for W satisfying $DJ(\rho, m)$, it is not the case that $\mathcal{A}_\sigma$ is spanned by $L_{\mathbb{C}[T]^W}(m)$ for W satisfying $DJ(\sigma, m)$. The following example demonstrates the strangeness of the (HR_1) situation.

Example 5.10.6. In this example we construct a (HR_1) toric supervariety for which some $L_{\mathcal{A}_\sigma}(m)$ is not spanned by induced representations. Consider the $(3|4)$ -dimensional abelian supertorus $T = (\mathbb{C}^\times)^3 \times \mathbb{A}^{0|4}$. As before, we write $\mathbb{C}[T] = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, \xi_1, \xi_2, \xi_3, \xi_4]$ and $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ a dual basis of $\mathfrak{t}_1$ to $\{\xi_1, \xi_2, \xi_3, \xi_4\}$, so that θ_i acts by $\frac{\partial}{\partial \xi_i}$ on $\mathbb{C}[T]$.

Let σ be the positive orthant in $\mathbb{R}^3$, with rays $\rho_i = \mathbb{R}_+ x_i$ for $i = 1, 2, 3$. We decorate the rays with

$$\begin{array}{ccccccc}
 V_{\rho_1,0} & \supseteq & V_{\rho_1,1} & \supseteq & V_{\rho_1,2} & \supseteq & V_{\rho_1,3} \\
 \parallel & & \parallel & & \parallel & & \parallel \\
 \mathbb{C}\{\theta_1, \theta_2, \theta_3, \theta_4\} & & \mathbb{C}\{\theta_2, \theta_3, \theta_4\} & & \mathbb{C}\{\theta_2, \theta_3\} & & \mathbb{C}\{\theta_2\}
 \end{array}$$

and

$$\begin{array}{ccccccc}
 V_{\rho_2,0} & \supseteq & V_{\rho_2,1} & \supseteq & V_{\rho_2,2} & \supseteq & V_{\rho_2,3} \\
 \parallel & & \parallel & & \parallel & & \parallel \\
 \mathbb{C}\{\theta_1, \theta_2, \theta_3, \theta_4\} & & \mathbb{C}\{2\theta_1 + \theta_3, \theta_2, \theta_4\} & & \mathbb{C}\{2\theta_1 + \theta_3, \theta_2 - 2\theta_4\} & & \mathbb{C}\{\theta_2 - 2\theta_4\}
 \end{array}$$

and

$$\begin{array}{ccccccc}
 V_{\rho_3,0} & \supseteq & V_{\rho_3,1} & \supseteq & V_{\rho_3,2} & \supseteq & V_{\rho_3,3} \\
 \parallel & & \parallel & & \parallel & & \parallel \\
 \mathbb{C}\{\theta_1, \theta_2, \theta_3, \theta_4\} & & \mathbb{C}\{\theta_1, \theta_2 - \theta_4, \theta_3\} & & \mathbb{C}\{\theta_1 + \theta_3, \theta_2 - \theta_4\} & & \mathbb{C}\{\theta_1 + \theta_3\}
 \end{array}$$

and all other decorations 0. Hence

$$\begin{aligned}
 \mathcal{A}_{\rho_1} &= \mathbb{C}[t_1, t_2^{\pm 1}, t_3^{\pm 1}, t_1\xi_1, t_1^2\xi_4, t_1^3\xi_3, t_1^4\xi_2] \\
 \mathcal{A}_{\rho_2} &= \mathbb{C}[t_1^{\pm 1}, t_2, t_3^{\pm 1}, t_2(\xi_1 - 2\xi_3), t_2^2(2\xi_2 + \xi_4), t_2^3(2\xi_1 - \xi_3), t_2^4(\xi_2 + 2\xi_4)] \\
 \mathcal{A}_{\rho_3} &= \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, t_3, t_3(\xi_2 + \xi_4), t_3^2(\xi_1 - \xi_3), t_3^3\xi_4, t_3^4\xi_1]
 \end{aligned}$$

and a straightforward calculation shows that

$$\begin{aligned}
 L_{\mathcal{A}_\sigma}(5, 5, 5) &= \bigcap_{i=1}^3 L_{\mathcal{A}_{\rho_i}}(5, 5, 5) \\
 &= \mathbb{C}t_1^5 t_2^5 t_3^5 \{1, \xi_1, \xi_2, \xi_3, \xi_4, \xi_1\xi_2 + \xi_3\xi_4\}.
 \end{aligned}$$

Since $\xi_1\xi_2 + \xi_3\xi_4$ is an indecomposable element of Λ_1^* , it follows that $L_{\mathcal{A}_\sigma}(5, 5, 5)$ is not the span of induced representations of the form $L_{\mathbb{C}[T]^w}(5, 5, 5)$.

5.11 Smoothness

In this section, we will provide a criterion for smoothness of a toric supervariety based on its decorated fan.

In the classical situation, a toric variety X is smooth if and only if each cone σ in its fan Σ is a *smooth cone* in the sense that the primitive generators of its rays form part of a basis for N . This has the effect that every smooth affine toric variety is isomorphic to $\mathbb{C}^n \times (\mathbb{C}^\times)^{p-n}$ for some $p \geq n \geq 0$, a fact which we will use in our proof of Theorem 5.11.2.

Definition 5.11.1. Let X be a supervariety and $x \in X(\mathbb{C})$ a closed point. We say X is *smooth* at x if $\mathcal{O}_{X,x}$ is a regular local superring. We say X is *smooth* if it is smooth at all closed points.

There exist many equivalent formulations for smoothness of a supervariety (or more broadly, smoothness of a morphism of superschemes), several of which can be found in Appendix B of [46].

Theorem 5.11.2. *Let X be a (HR_1) toric supervariety and $(N, \Sigma, \mathfrak{t}, \mathfrak{h}, \{V_{\rho,i}\})$ its decorated fan. Then X is smooth if and only if for every cone $\sigma \in \Sigma$, the following three conditions hold:*

- (a) σ is a smooth cone.
- (b) There is a basis of $\mathfrak{t}_{\bar{1}}$ such that for all $\rho \in \sigma(1)$ and $i \geq 0$, $V_{\rho,i}$ is the span of a subset of that basis.
- (c) For each $\rho \in \sigma(1)$, if $\text{codim}(\mathfrak{h}, V_{\rho,0}) \geq 2$, then $[V_{\rho,0}, V_{\rho',0}] \subseteq \mathfrak{t}_{N_{\rho'}}$ for all $\rho' \in \sigma(1)$.

Proof. We may assume X is affine, and that $X_{\bar{0}} \cong \mathbb{C}^n \times (\mathbb{C}^\times)^{p-n}$, so X corresponds to the cone σ whose rays are $\rho_i = \mathbb{R}_+ e_i$ for $i = 1, \dots, n$, where e_i is the i th unit vector in $\mathbb{Z}^p$. If X is smooth, then (a) is immediate and we may write

$$\mathcal{A}_\sigma = \mathbb{C}[t_1, \dots, t_n, t_{n+1}^{\pm 1}, \dots, t_p^{\pm 1}, t^{m_1}(\xi_1 + \dots), \dots, t^{m_q}(\xi_q + \dots)]$$

for some non-nilpotent t_i , linearly independent $\xi_i \in \mathfrak{t}_{\bar{1}}^*$, and products of nilpotent terms represented by ellipses. Localizing to some $\mathcal{A}_{\rho_i}$, we see that we may assume each $\xi_j + \dots$ simply equals ξ_j . We therefore have

$$\mathcal{A}_\sigma = \mathbb{C}[t_1, \dots, t_n, t_{n+1}^{\pm 1}, \dots, t_p^{\pm 1}, t^{m_1} \xi_1, \dots, t^{m_q} \xi_q]$$

and in particular

$$\mathcal{A}_{\rho_i} = \mathbb{C}[t_1^{\pm 1}, \dots, t_i, \dots, t_p^{\pm 1}, t_i^{\langle m_1, e_i \rangle} \xi_1, \dots, t_i^{\langle m_q, e_i \rangle} \xi_q]$$

for $i = 1, \dots, n$. Choosing a basis of $\mathfrak{t}_{\bar{1}}^*$ including the ξ_i yields a dual basis $\theta_1, \dots, \theta_q, \theta_{q+1}, \dots, \theta_r$ of $\mathfrak{t}_{\bar{1}}$ for which $V_{\rho_i, m} = \mathbb{C}\{\theta_j \mid \langle m, e_i \rangle < \langle m_j, e_i \rangle\} \oplus \mathbb{C}\{\theta_{q+1}, \dots, \theta_r\}$. Hence (b) holds.

Now for (c), note that if $\text{codim}(\mathfrak{h}, V_{\rho_i,0}) \geq 2$, then by construction of $\mathcal{A}_\rho$ in Proposition/Definition 5.5.3, we have $t_i \in \mathbb{C}[T]^{V_{\rho_i,0}}$. Since $t_i \in \mathbb{C}[T]^{V_{\rho_\ell,0}}$ for all $\ell \neq i$ as well, it follows that $t_i \in \mathbb{C}[T]^{[V_{\rho_i,0}, V_{\rho_\ell,0}]}$ and therefore $\langle \varepsilon_i, [V_{\rho_i,0}, V_{\rho_\ell,0}] \rangle = 0$, where $\{\varepsilon_i\}$ in M is dual to $\{e_i\}$ in N . Hence $[V_{\rho_i,0}, V_{\rho_\ell,0}] \subseteq \mathfrak{t}_{N_{\rho_\ell}}$ and we are finished.

Conversely, suppose properties (a), (b), and (c) hold. As above, property (c) allows us to find $t_i \in \mathbb{C}[T]^{\sum_{\ell=1}^n V_{\rho_\ell,0}}$ if $\text{codim}(\mathfrak{h}, V_{\rho_i,0}) \geq 2$, or otherwise in $\mathbb{C}[T]^{V_{\rho_i,1} + \sum_{\ell \neq i} V_{\rho_\ell,0}}$. Now pick such a basis $\theta_1, \dots, \theta_r$ as in (b), where $\mathfrak{h} = \mathbb{C}\{\theta_{q+1}, \dots, \theta_r\}$, and let $\xi_1, \dots, \xi_r$ be the dual basis of $\mathfrak{t}_{\bar{1}}^*$. Then each

$$\mathcal{A}_{\rho_i} = \mathbb{C}[t_1^{\pm 1}, \dots, t_i, \dots, t_p^{\pm 1}, t_i^{n_{i1}} \xi_1, \dots, t_i^{n_{iq}} \xi_q]$$

where $n_{ij} \geq 0$ is minimal such that $\theta_j \notin V_{\rho_i, n_{ij}}$. Then let $m_j = n_{1j} \varepsilon_1 + \dots + n_{qj} \varepsilon_q$, so

$$\mathcal{A}_\sigma = \mathbb{C}[t_1, \dots, t_n, t_{n+1}^{\pm 1}, \dots, t_p^{\pm 1}, t^{m_1} \xi_1, \dots, t^{m_q} \xi_q]$$

and indeed X is smooth. □

This theorem simplifies in the case of a toric supervariety with large orbits. In particular, condition (c) is immaterial, so smoothness in the odd directions depends entirely on finding such a basis as in (b).

This condition on basis elements is equivalent to Klyachko's condition [26] that a descending filtration $E = \dots \supseteq E^\rho(-1) \supseteq E^\rho(0) \supseteq E^\rho(1) \supseteq \dots = 0$ for each $\rho \in \Sigma(1)$ specifies a toric vector bundle with fiber E over the identity. To obtain a Klyachko filtration from our $V_{\rho,0} \supseteq V_{\rho,1} \supseteq \dots$, one must take $E^\rho(i) = 0$ for $i > 0$ and $E^\rho(i) = (\mathfrak{t}_1/V_{\rho,-i})^*$ for $i \leq 0$. These Klyachko data yield the vector bundle $\mathcal{J}_X/\mathcal{J}_X^2$, the conormal bundle of $X_{\bar{0}}$ in X .

Note the sign discrepancy in $E^\rho(i) = (\mathfrak{t}_1/V_{\rho,-i})^*$; this occurs because $E^\rho(i)$ is the im_1 -eigenspace of $J_{\mathcal{A}_\rho}/J_{\mathcal{A}_\rho}^2$ for the left regular representation, wherein the even torus acts by the character $-m$ on the monomial t^m .

Using this characterization of smoothness, it is not difficult to construct a toric resolution of singularities for a toric supervariety.

Proposition 5.11.3. *Let X be a toric supervariety for the supertorus T . Then there exists a smooth toric supervariety X' for T and a proper, birational, T -equivariant toric morphism $X' \rightarrow X$.*

Proof. Let $(N, \Sigma, \mathfrak{t}, \mathfrak{h}, \{V_{\rho,i}\})$ be the decorated fan of X . Construct a refinement Σ' of Σ (with the same support) by introducing new rays until each maximal cone is smooth and contains at most one ray from the original fan Σ . Decorate each new ray ρ' by $V'_{\rho',i} = \mathfrak{h}$ for all $i \geq 0$. Since each cone contains at most one of the original rays, the resulting decorated fan $(N, \Sigma', \mathfrak{t}, \mathfrak{h}, \{V'_{\rho',i}\})$ satisfies the conditions of Theorem 5.11.2.

We now verify that the identity $T \rightarrow T$ extends to $X' \rightarrow X$. It suffices to consider a ray $\rho' \in \Sigma'$ and a cone $\sigma \in \Sigma$ such that $\rho' \subseteq \sigma$. If $\rho' \notin \Sigma$, then ρ' has trivial decorations and so each $L_{\mathcal{A}_{\rho'}}(m) = L_{\mathbb{C}[T]^H}(m) \supseteq L_{\mathcal{A}_\sigma}(m)$. Otherwise $\rho' \in \Sigma$, in which case $\mathcal{A}_{\rho'} = \mathcal{A}_\rho$. Therefore we obtain a T -equivariant morphism $X' \rightarrow X$. Birationality is immediate since the open orbits are both isomorphic to T/H . Finally, properness of the underlying morphism of varieties follows since the fans have the same support; this is sufficient since properness depends only on the underlying varieties (see e.g. [3]). $\square$

Example 5.11.4. The toric supervariety

$$X = \text{Spec } \mathbb{C}[t_1, t_2, t_3, t_1 t_2 \xi_1, t_1 t_3 \xi_2, t_2 t_3 (\xi_1 + \xi_2), t_1 t_2 t_3 \xi_1 \xi_2]$$

has smooth underlying variety, but is not itself smooth. Its failure to be smooth can be detected by checking that the subspaces $\mathbb{C}\theta_2, \mathbb{C}\theta_1$, and $\mathbb{C}(\theta_1 - \theta_2)$ occur among the decorations. The decorated fan for X , as well as that of a (non-affine) resolution of singularities X' , can be found in Figure 5.3. Note that the resolution depicted is not the one constructed in Proposition 5.11.3, since the fan is not sufficiently refined.

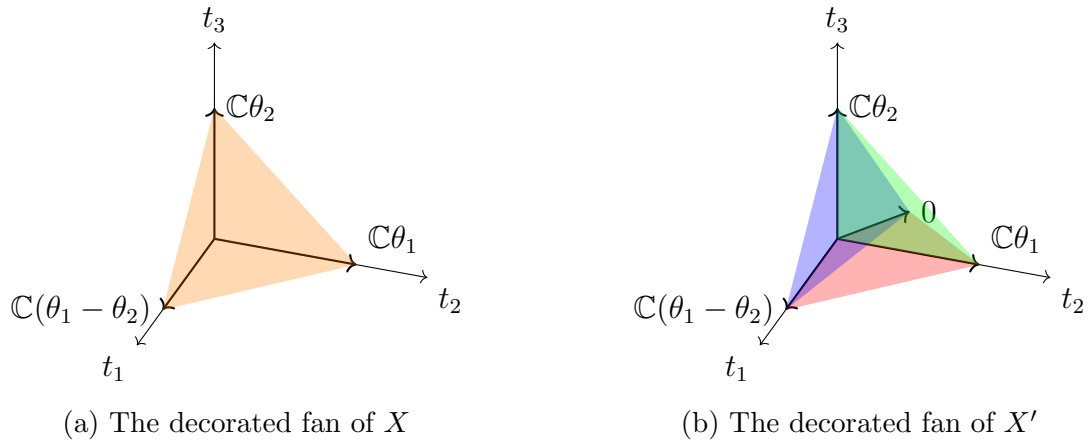


Figure 5.3: Decorated fans for a resolution of singularities $X' \rightarrow X$ as in Example 5.11.4. The additional ray is generated by $(1, 1, 1)$ and decorated by the 0 subspace of $\mathfrak{t}_{\bar{1}}$.

Chapter 6

Toric subvarieties of the isomeric grassmannian

In this chapter, based on [23] and [22], we examine a natural collection of examples of toric supervarieties for quotients of the supertorus $Q(1)^n$. We also illustrate a relationship between their decorated polytopes and some naturally-obtained subsets of $\mathfrak{t}^*$.

6.1 The classical story

We begin with an introduction to the classical analog in type A , following [1], [20], [14], [16] and [15].

Let $T = (\mathbb{C}^\times)^n$ act on $\mathbb{C}^n$ by coordinate-wise scaling, and hence on the Grassmannian $\text{Gr}(r, n)$ via its natural action on r -dimensional subspaces. This action has infinitely many orbits but finitely many orbit types, and the closures of these orbits admit a combinatorial description.

Recall that $\text{Gr}(r, n)$ admits the Plücker embedding

$$\text{Gr}(r, n) \hookrightarrow \mathbb{P}(\Lambda^r \mathbb{C}^n),$$

with homogeneous coordinates indexed by r -element subsets $I \subset \{1, \dots, n\}$. The torus T acts on $\Lambda^r \mathbb{C}^n$ with weight decomposition

$$\Lambda^r \mathbb{C}^n = \bigoplus_{|I|=r} \mathbb{C} \cdot e_I,$$

where $e_I = e_{i_1} \wedge \dots \wedge e_{i_r}$ has weight equal to the sum of the standard basis vectors indexed by I .

Given a point $x \in \text{Gr}(r, n)$ with Plücker coordinates $(p_I(x))$, the associated *weight polytope* is defined as

$$P_x := \text{conv}\{e_I \in \mathbb{R}^n \mid p_I(x) \neq 0\}.$$

This is a lattice polytope whose vertices and edges are a subset of those in the hypersimplex

$$\Delta_{r,n} = \left\{ (u_1, \dots, u_n) \in \mathbb{R}^n \mid 0 \leq u_i \leq 1, \sum u_i = r \right\},$$

which is the convex hull of the vectors in $\{0, 1\}^n$ with exactly r instances of 1.

Proposition 6.1.1. *Let x be a closed point in $\mathrm{Gr}(r, n)$. Then $\overline{T \cdot x}$ is a normal projective toric variety whose associated polytope coincides with P_x as described above.*

From a symplectic perspective, this polytope also arises as the image of a momentum map. Identifying $\mathrm{Gr}(r, n)$ with a coadjoint orbit of $\mathrm{U}(n)$, the compact torus $(S^1)^n$ acts in a Hamiltonian fashion, and the corresponding moment map

$$\mu : \mathrm{Gr}(r, n) \rightarrow \mathbb{R}^n$$

has image equal to the hypersimplex $\Delta(r, n)$. Here, the target space $\mathbb{R}^n$ is really the dual space of $t_{\mathbb{R}}$, a real form of the Lie algebra of T . The image of the orbit closure $\overline{T \cdot x}$ under μ is precisely the polytope P_x .

6.2 The isomeric supergrassmannian

Let (V, Π_V) be an isomeric vector space as in Definition 3.1.5

Write $Q(V, \Pi_V)$ for the supergroup of (even) automorphisms of V that commute with Π_V , so $Q(V, \Pi_V) \cong Q(n)$ for $n = \dim V_{\bar{0}} = \dim V_{\bar{1}}$.

Definition 6.2.1. Let (V, Π_V) be an isomeric vector space. The *isomeric supergrassmannian* is the super moduli space $\mathrm{QGr}(r, V)$ of $(r|r)$ -dimensional isomeric subspaces of V . Oftentimes we will assume $V = \mathbb{C}^{n|n}$ with $\Pi_V : \mathbb{C}^{n|0} \rightarrow \mathbb{C}^{0|n}$ the “identity” on $\mathbb{C}^n$, in which case we will write $\mathrm{QGr}(r, n)$.

For $\mathcal{A}$ a superalgebra, the $\mathcal{A}$ -points of $\mathrm{QGr}(r, n)$ may be written as the space of full-rank matrices of the form

$$M = \left(\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1r} & \alpha_{11} & \cdots & \alpha_{1r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nr} & \alpha_{n1} & \cdots & \alpha_{nr} \\ \hline \alpha_{11} & \cdots & \alpha_{1r} & a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nr} & a_{n1} & \cdots & a_{nr} \end{array} \right),$$

for $a_{ij} \in \mathcal{A}_{\bar{0}}$ and $\alpha_{ij} \in \mathcal{A}_{\bar{1}}$, modulo the right action of $Q(r)(\mathcal{A})$. As seen in [36], it is covered by $\binom{n}{r}$ affine charts (one for each size- $(n-r)$ subset I of $\{1, \dots, n\}$) of the form

$$M' = \left(\begin{array}{c|c} a_I & \alpha_I \\ \hline 1_{r \times r} & 0_{r \times r} \\ \hline \alpha_I & a_I \\ \hline 0_{r \times r} & 1_{r \times r} \end{array} \right),$$

where $(a_I \mid \alpha_I)$ is the submatrix of M consisting of the rows corresponding the indices in I , and where the rows of M' have been rearranged accordingly for notational convenience. It is clear from this affine cover that $\mathrm{QGr}(r, n)$ is a smooth supervariety with underlying variety $\mathrm{Gr}(r, n)$.

The isomeric supergrassmannian as a homogeneous space

By way of the transitive action of $Q(n)$ on $\text{QGr}(r, n)$, we see that $\text{QGr}(r, n) \cong Q(n)/P$ for P a maximal parabolic subgroup. The same holds if we replace $Q(n)$ by its subgroup $SQ(n)$ consisting of those linear transformations with Q -determinant 1, or even its quotient $PSQ(n)$ by scalars. It is from this construction that we view $\text{QGr}(r, n)$ as the isomeric analog of the ordinary Grassmannian.

However, for the purposes of this paper, we are more interested in the action of the isomeric unitary supergroup $UQ(n)$ consisting of automorphisms which commute with Π_V and preserve a super-Hermitian form h in the sense of [6], i.e. one that satisfies $h(u, v) = (-i)^{|u||v|} \bar{h}(u, v)$ for a usual Hermitian form $\bar{h}$. From h we obtain the notion of super-Hermitian adjoint

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}^* = \begin{pmatrix} A^\dagger & -iB^\dagger \\ -iB^\dagger & A^\dagger \end{pmatrix}$$

where † represents the usual Hermitian adjoint with respect to $\bar{h}$.

We may therefore write the real Lie supergroup $UQ(n)$ as the one whose $\mathcal{A}$ -points (for $\mathcal{A}$ a *real* commutative superalgebra) are

$$UQ(n)(\mathcal{A}) = \{X \in Q(n)(\mathbb{C} \otimes_{\mathbb{R}} \mathcal{A}) \mid X^{-1} = X^*\}$$

and its Lie superalgebra

$$\begin{aligned} \mathfrak{uq}(n) &= \{X \in \mathfrak{q}(n) \mid X + X^* = 0\} \\ &= \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mid A = -A^\dagger, B = iB^\dagger \right\} \\ &= \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mid A \in iH(n), B \in (1+i)H(n) \right\} \\ &= iHQ(n) \end{aligned}$$

where $H(n)$ is the space of ordinary $n \times n$ Hermitian matrices, and

$$\begin{aligned} HQ(n) &= \{X \in \mathfrak{q}(n) \mid X^* = X\} \\ &= \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mid A = A^\dagger, B = -iB^\dagger \right\} \\ &= \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mid A \in H(n), B \in (1-i)H(n) \right\} \end{aligned}$$

is the space of isomeric super Hermitian endomorphisms. We may realize $\Pi HQ(n)$ (the parity shift of $HQ(n)$) as the dual space to $\mathfrak{uq}(n)$ via the following odd pairing:

$$\begin{aligned} \langle -, - \rangle : HQ(n) \times iHQ(n) &\rightarrow \mathbb{R} \\ \left\langle \begin{pmatrix} A & B \\ B & A \end{pmatrix}, \begin{pmatrix} A' & B' \\ B' & A' \end{pmatrix} \right\rangle &= \frac{1}{1+i} \text{tr}(AB' + A'B) \end{aligned}$$

As in the ordinary situation, $UQ(n)$ acts transitively on $\text{QGr}(r, n)$. The stabilizer of a fixed isomeric subspace is $UQ(r) \times UQ(n-r)$, so the following lemma holds:

Lemma 6.2.2. *If $0 \leq r \leq n$, then*

$$\mathrm{QGr}(r, n) \cong UQ(n)/UQ(r) \times UQ(n-r).$$

The isomeric supergrassmannian as an adjoint orbit

Consider the adjoint action of $UQ(n)$ on $\mathfrak{uq}(n) \cong iHQ(n)$:

$$\begin{aligned} UQ(n)(\mathcal{A}) \times \mathfrak{uq}(n)(\mathcal{A}) &\rightarrow \mathfrak{uq}(n)(\mathcal{A}) \\ g \cdot X &\mapsto gXg^{-1} \end{aligned}$$

Here, $\mathcal{A}$ is a real commutative superalgebra. Let us fix the closed point

$$M_r = i \begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & \mathbb{1}_r \end{pmatrix}$$

of $\mathfrak{uq}(n)$ where $\mathbb{1}_r$ is the diagonal matrix whose first r diagonal entries are 1, with the rest 0. Since the stabilizer of M_r is exactly $UQ(r) \times UQ(n-r)$, it follows by Lemma 6.2.2 that the adjoint orbit $UQ(n) \cdot M_r$ can be equipped with a complex structure so that it is isomorphic to $\mathrm{QGr}(r, n)$. We will equivalently think of $\mathrm{QGr}(r, n)$ as an “odd coadjoint orbit,” i.e. the orbit of $-iM_r$ as a closed point in the parity shift $\Pi HQ(n)$.

We emphasize that there is no way to view $UQ(n) \cdot M_r$ as an even coadjoint orbit, since the adjoint and coadjoint representations of $UQ(n)$ are only isomorphic up to parity shift. In particular, the stabilizer of the closed point

$$(1-i) \begin{pmatrix} 0 & \mathbb{1}_r \\ \mathbb{1}_r & 0 \end{pmatrix}$$

is not $UQ(r) \times UQ(n-r)$, but $U(r) \times UQ(n-r)$.

6.3 Realization as a superpolytope

Denote by $(\mathfrak{t}_{\mathbb{R}})_{\bar{0}}$ the Cartan subalgebra of $\mathfrak{uq}_{\bar{0}}(n)$ consisting of diagonal matrices. Let $\mathfrak{t}_{\mathbb{R}}$ be its centralizer in $\mathfrak{uq}(n)$, consisting of block matrices for which both A and B are diagonal. Observe that $\mathfrak{t}_{\mathbb{R}}$ is the Lie superalgebra of a compact real form of the supertorus $Q(1)^n \subset Q(n)$, and that $\mathfrak{t}_{\mathbb{R}}^*$ also consists of matrices for which both A and B are diagonal.

Consider the natural inclusion of supermanifolds $\mathrm{QGr}(r, n) \rightarrow \Pi \mathfrak{uq}(n)^*$, resembling an odd version of a momentum map. Composing it with the natural projection $\Pi \mathfrak{uq}(n)^* \rightarrow \Pi \mathfrak{t}_{\mathbb{R}}^*$ yields a new map $\mu : \mathrm{QGr}(r, n) \rightarrow \Pi \mathfrak{t}_{\mathbb{R}}^*$. We now calculate the image of μ on $\mathbb{R}[\xi]$ -points so that we may understand the image of μ as a subset of the super vector space $\Pi \mathfrak{t}_{\mathbb{R}}^*$.

Write $A + B$ for $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ in order to compress notation. Let $A + B \in UQ(n)(\mathcal{A})$, so $(A + B)^{-1} = A^\dagger - iB^\dagger$. Write $A = (a_{jk} + ia'_{jk})$ and $B = (\beta_{jk} + i\beta'_{jk})$. Then

$$\begin{aligned} (A + B) \cdot (-iM_r) &= (A + B)(-iM_r)(A^\dagger - iB^\dagger) \\ &= (a_{jk} + ia'_{jk} + \beta_{jk} + i\beta'_{jk})_{k \leq r} (a_{kj} - ia'_{kj} - \beta'_{kj} - i\beta_{kj}), \end{aligned}$$

a matrix whose j th diagonal entry is

$$\sum_{k=1}^r \left([a_{jk}^2 + (a'_{jk})^2] + (1-i)[-a_{jk}\beta'_{jk} + a'_{jk}\beta_{jk} + a_{jk}\beta_{jk} + a'_{jk}\beta'_{jk}] \right).$$

Since $(A+B)(A^\dagger - iB^\dagger) = (A^\dagger - iB^\dagger)(A+B) = 1$, one may verify that

$$\sum_{k=1}^n \left([a_{jk}^2 + (a'_{jk})^2] + (1-i)[-a_{jk}\beta'_{jk} + a'_{jk}\beta_{jk} + a_{jk}\beta_{jk} + a'_{jk}\beta'_{jk}] \right) = 1$$

for all j , and similarly

$$\sum_{j=1}^n \left([a_{jk}^2 + (a'_{jk})^2] + (1-i)[-a_{jk}\beta'_{jk} + a'_{jk}\beta_{jk} + a_{jk}\beta_{jk} + a'_{jk}\beta'_{jk}] \right) = 1$$

for all k . Thus the sum of the diagonal entries of $(A+B) \cdot (-iM_r)$ is equal to r .

Write $d_1, \dots, d_n$ for these diagonal entries. We find from the above equations $\sum_{i=1}^n d_i = r$ and $d_i \in [0, 1] + (1-i)\mathbb{R}\xi$. Moreover, if the even part of such an entry is 0, then $a_{jk} = a'_{jk} = 0$ for all $k = 1, \dots, r$, so the odd part is 0 as well. Likewise, if the even part is 1, then the odd part is the odd part of the trace, which is also 0. On the other hand, if the even part belongs to the open interval $(0, 1)$, then the odd part can take on any value in $\mathbb{R}\xi$.

In this sense, we may view the “image” of $\mu : M \rightarrow \Pi \mathfrak{t}_{\mathbb{R}}^*$ as a decorated polytope $\mathcal{P}$ whose even part is the (r, n) -hypersimplex $\Delta(r, n)$. If F is a face of $\Delta(r, n)$, let $I_F \subseteq \{1, \dots, n\}$ be the collection of indices i for which the i th coordinate of F is 0 or 1. Then to the face F we attach the subspace $W_F \subseteq \Pi(\mathfrak{t}_{\mathbb{R}})_0^* \cong \mathbb{R}^n$ consisting of vectors whose entries sum to 0, and whose i th entry is 0 for any $i \in I$. Therefore

$$\begin{aligned} \mathcal{P} &= \bigcup_{F \leq \Delta(r, n)} \text{int } F \times W_F \\ &\cong \left\{ (b_1, \dots, b_n \mid \beta_1, \dots, \beta_n) \in \mathbb{R}[\xi]^n \right. \\ &\quad \left. \left| (b_i, \beta_i) \in \{0, 1\} \times \{0\} \cup (0, 1) \times \mathbb{R}[\xi] \text{ and } \sum_{i=1}^n (b_i, \beta_i) = (r, 0) \right. \right\} \end{aligned}$$

6.4 Supertorus orbits

By restricting the action of $Q(n)$ to the Cartan subgroup $Q(1)^n$ of diagonal matrices, we obtain a supertorus action on $\text{QGr}(r, n)$. Let us consider its action (by matrix multiplication) on the $\mathcal{A}$ -points:

$$\left(\begin{array}{ccc|ccc} b_1 & & & b_1\beta_1 & & \\ & \ddots & & & \ddots & \\ & & b_n & & & b_n\beta_n \\ \hline b_1\beta_1 & & & b_1 & & \\ & \ddots & & & \ddots & \\ & & b_n\beta_n & & & b_n \end{array} \right) \cdot \left(\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1r} & \alpha_{11} & \cdots & \alpha_{1r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nr} & \alpha_{n1} & \cdots & \alpha_{nr} \\ \hline \alpha_{11} & \cdots & \alpha_{1r} & a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nr} & a_{n1} & \cdots & a_{nr} \end{array} \right),$$

or equivalently

$$\begin{pmatrix} b_1(1+\beta_1) & & \\ & \ddots & \\ & & b_n(1+\beta_n) \end{pmatrix} \cdot \begin{pmatrix} a_{11} + \alpha_{11} & \cdots & a_{1r} + \alpha_{1r} \\ \vdots & \ddots & \vdots \\ a_{n1} + \alpha_{n1} & \cdots & a_{nr} + \alpha_{nr} \end{pmatrix}$$

Although this action does not in general have an open orbit, we may still consider the closures of orbits of closed points as toric supervarieties for a quotient of $T = Q(1)^n$.

As before, we will write $\mathfrak{t} = \mathfrak{q}(1)^n = \mathbb{C}\{x_1, \dots, x_n\} \oplus \mathbb{C}\{\theta_1, \dots, \theta_n\}$ where $\frac{1}{2}[\theta_i, \theta_j] = \delta_{ij}x_i$. Abusing the language of Harish-Chandra pairs, we will often conflate a supertorus with its Lie superalgebra when writing stabilizer subgroups.

Lemma 6.4.1. *Let y be a closed point of $\text{QGr}(r, n)$.*

- (a) $\text{Stab}_{T_{\bar{0}}}(y)$ is a span of independent elements of the form $x_{i_1} + \dots + x_{i_\ell}$, such that each x_i occurs in exactly one such element.
- (b) $\text{Stab}_T(y)_{\bar{0}} = \text{Stab}_{T_{\bar{0}}}(y)$
- (c) $\text{Stab}_T(y)_{\bar{1}} = \mathbb{C}\{\theta_{i_1} + \dots + \theta_{i_\ell} \mid x_{i_1} + \dots + x_{i_\ell} \in \text{Stab}_{T_{\bar{0}}}(y)\}$

Proof. (a) Let y be the closed point

$$\begin{pmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nr} \end{pmatrix} (\mathbb{C}^\times)^r$$

for $a_{ij} \in \mathbb{C}$. Assume without loss of generality that $a_{ij} = \delta_{ij}$ for $i \leq r$, so that

$$y = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ a_{(r+1)1} & \cdots & a_{(r+1)r} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nr} \end{pmatrix} (\mathbb{C}^\times)^r$$

and hence $(t_1, \dots, t_n) \in T_{\bar{0}}$ stabilizes y if and only if $t_i = t_j$ whenever $a_{ij} \neq 0$.

- (b) Immediate from the definitions.
- (c) The same argument as in part (a) applies, using the compressed expression for the group action as matrix multiplication. $\square$

Write $\Pi_{\mathfrak{t}} : \mathfrak{t} \rightarrow \mathfrak{t}$ for the odd involution $x_i \mapsto \theta_i$ induced by Π_V , so that $\text{Stab}_T(y)$ is $\Pi_{\mathfrak{t}}$ -invariant. In particular, if y belongs to the T_0 -orbit corresponding to σ , then $\text{Stab}_T(y) = \mathfrak{t}_{N_\sigma} \oplus \Pi \mathfrak{t}_{N_\sigma}$.

Proposition 6.4.2. *Let y be a closed point of $\text{QGr}(r, n)$. The closure of the orbit $T \cdot y$ is a large-orbit toric supervariety corresponding to the decorated polytope $\mathcal{P}$ consisting of the usual even polytope P of $\overline{T_0 \cdot y}$ and odd subspaces $W_F = (\mathfrak{t}_{\overline{1}}/\Pi \mathfrak{t}_{N_{\sigma_F}})^*$, where σ_F the normal cone to the face F of P .*

Proof. Let U be an open affine chart containing y ; without loss of generality

$$U = \left\{ \begin{pmatrix} 1_{r \times r} \\ a_I \end{pmatrix} \right\}$$

at the level of $\mathbb{C}$ -points. We have

$$\begin{aligned} & \begin{pmatrix} b_1(1 + \beta_1) & & \\ & \ddots & \\ & & b_n(1 + \beta_n) \end{pmatrix} \cdot \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ a_{(r+1)1} & \cdots & a_{(r+1)r} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nr} \end{pmatrix} \\ &= \begin{pmatrix} b_1(1 + \beta_1) & & \\ & \ddots & \\ & & b_r(1 + \beta_r) \\ b_{r+1}(1 + \beta_{r+1})a_{(r+1)1} & \cdots & b_{r+1}(1 + \beta_{r+1})a_{(r+1)r} \\ \vdots & \ddots & \vdots \\ b_n(1 + \beta_n)a_{n1} & \cdots & b_n(1 + \beta_n)a_{nr} \end{pmatrix} \\ &= \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ b_i^{-1}b_j(1 - \beta_i + \beta_j + \beta_i\beta_j)a_{ji} \end{pmatrix} \end{aligned}$$

so that $\mathbb{C}[U \cap \overline{T \cdot y}]$ is generated by $t_i^{-1}t_j(1 + \xi_i\xi_j)$ and $t_i^{-1}t_j(-\xi_i + \xi_j)$ for $1 \leq i \leq r < j \leq n$. Hence $U \cap \overline{T \cdot y}$ is the large-orbit affine toric supervariety with $V_{\rho,0} = \Pi \mathfrak{t}_{N_\rho}$ and $V_{\rho,1} = 0$, so we are finished. $\square$

Generically, up to complexification and composition with $\Pi_{\mathfrak{t}}^* : \Pi \mathfrak{t}^* \rightarrow \mathfrak{t}^*$, this is the same decorated polytope obtained in section 6.2! Other orbits correspond to decorated subpolytopes in a predictable manner. In effect, this means that the theory of torus strata developed in [15] generalizes easily to the situation of $Q(1)^n$ acting on $\text{QGr}(r, n)$.

6.5 Picard group of the isomeric supergrassmannian

Recall from section 3.7 that if X is a supervariety, its Picard group $\text{Pic}(X)$ of invertible $(1|0)$ -dimensional sheaves is isomorphic to $H^1(X, \mathcal{O}_{X,\bar{0}}^\times)$.

Let $X = \text{QGr}(r, n)$ for some integers $n \geq r \geq 0$.

Lemma 6.5.1. *If $r = 1$ and $n = 2$, so that $X = \mathbb{P}_\Pi^1$, then $\text{Pic}(X) = \mathbb{Z}$.*

Proof. Since the odd dimension of $\mathbb{P}_\Pi^1$ is 1, we have $\mathcal{O}_{\mathbb{P}_\Pi^1, \bar{0}} = \mathcal{O}_{\mathbb{P}^1}$ and hence $\text{Pic}(\mathbb{P}_\Pi^1) = \text{Pic}(\mathbb{P}^1) = \mathbb{Z}$. $\square$

Recall that $X_{\bar{0}} \cong \text{Gr}(r, n)$ is the underlying variety of $X \cong \text{QGr}(r, n)$, given by quotienting the structure sheaf by the ideal $\mathcal{I}_X = i_X^{-1}((\mathcal{K}_{X, \bar{1}}))$.

Lemma 6.5.2. *The restriction map $\text{Pic}(X) \rightarrow \text{Pic}(X_{\bar{0}})$ is injective.*

Proof. Consider the sequence of successive quotients

$$\mathcal{O}_{X, \bar{0}} \rightarrow \mathcal{O}_{X, \bar{0}} / (\mathcal{I}_X^{2k})_{\bar{0}} \rightarrow \mathcal{O}_{X, \bar{0}} / (\mathcal{I}_X^{2(k-1)})_{\bar{0}} \rightarrow \cdots \rightarrow \mathcal{O}_{X, \bar{0}} / (\mathcal{I}_X^4)_{\bar{0}} \rightarrow \mathcal{O}_{X, \bar{0}} / (\mathcal{I}_X^2)_{\bar{0}}$$

where $k = \lfloor \frac{n}{2} \rfloor$, and note that $\mathcal{I}_X^{2(k+1)} = 0$. Write X_i for the scheme $(|X|, \mathcal{O}_{X, \bar{0}} / (\mathcal{I}_X^{2i})_{\bar{0}})$. For $i = 1, \dots, k$, we have an exact sequence of sheaves of abelian groups

$$0 \rightarrow (\mathcal{I}_X^{2i})_{\bar{0}} / (\mathcal{I}_X^{2(i+1)})_{\bar{0}} \rightarrow \left(\mathcal{O}_{X, \bar{0}} / (\mathcal{I}_X^{2(i+1)})_{\bar{0}} \right)^\times \rightarrow \left(\mathcal{O}_{X, \bar{0}} / (\mathcal{I}_X^{2i})_{\bar{0}} \right)^\times \rightarrow 0$$

in which the first nonzero map is given by $f \mapsto 1 + f$. This yields the long exact sequence in cohomology

$$\cdots \rightarrow H^1(X, (\mathcal{I}_X^{2i})_{\bar{0}} / (\mathcal{I}_X^{2(i+1)})_{\bar{0}}) \rightarrow \text{Pic}(X_{i+1}) \rightarrow \text{Pic}(X_i) \rightarrow \cdots$$

where we have used the isomorphism $\text{Pic}(X) \cong H^1(X, \mathcal{O}_{X, \bar{0}}^\times)$.

We claim that $\text{Pic}(X_{i+1}) \rightarrow \text{Pic}(X_i)$ is injective, for which it suffices to show that $H^1(X, (\mathcal{I}_X^{2i})_{\bar{0}} / (\mathcal{I}_X^{2(i+1)})_{\bar{0}}) = 0$. We first observe that $(\mathcal{I}_X^{2i})_{\bar{0}} / (\mathcal{I}_X^{2(i+1)})_{\bar{0}} \cong \mathcal{I}_X^{2i} / \mathcal{I}_X^{2i+1}$. It is seen in [36] that $\mathcal{I}_X / \mathcal{I}_X^2 \cong \Omega_{X_{\bar{0}}^1}$, and indeed $\mathcal{I}_X^{2i} / \mathcal{I}_X^{2i+1} \cong \Omega_{X_{\bar{0}}^{2i}}$. But the Hodge numbers $h^q(X_{\bar{0}}, \Omega_{X_{\bar{0}}^p}^p)$ are nonzero only if $p = q$ [43], so $H^1(X, (\mathcal{I}_X^{2i})_{\bar{0}} / (\mathcal{I}_X^{2(i+1)})_{\bar{0}}) = 0$. Therefore

$$\text{Pic}(X) \rightarrow \text{Pic}(X_k) \rightarrow \cdots \rightarrow \text{Pic}(X_1) \rightarrow \text{Pic}(X_{\bar{0}})$$

is injective, and we are finished. $\square$

We now introduce an embedding of Grassmannians (see e.g. [40]) that will be a useful tool in Lemmas 6.5.4 and 6.7.5.

Definition 6.5.3. A *standard extension* of Grassmannians is an embedding of the form

$$\begin{aligned} \text{Gr}(r, n) &\rightarrow \text{Gr}(r', n') \\ \{V \subseteq \mathbb{C}^n\} &\mapsto \{V \oplus W \subseteq \mathbb{C}^n \oplus \mathbb{C}^{n'-n}\} \end{aligned}$$

where W is a fixed $(r' - r)$ -dimensional subspace of $\mathbb{C}^{n'-n}$.

Standard extensions generalize easily to the super setting. In the type-Q case, the map is given by taking the direct sum with a fixed isomeric $(r' - r|r' - r)$ -dimensional subspace W of $\mathbb{C}^{n'-n|n'-n}$. Functorially, the map on A -points can be written as

$$M \mapsto \begin{pmatrix} M & 0_{n \times (r'-r)} \\ 0_{(r'-r) \times r} & 1_{(r'-r) \times (r'-r)} \\ 0_{(n'-n-r'+r) \times r} & 0_{(n'-n-r'+r) \times (r'-r)} \end{pmatrix}.$$

Lemma 6.5.4. *If $0 < r < n - 1$, then $\text{Pic}(X) = 0$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \mathbb{P}^2 & \longrightarrow & \text{Gr}(r, n) \\ \downarrow & & \downarrow \\ \mathbb{P}_{\Pi}^2 & \longrightarrow & \text{QGr}(r, n) \end{array}$$

where the vertical arrows are the natural closed immersions and the horizontal arrows are standard extensions. Applying the Picard group functor, we obtain

$$\begin{array}{ccc} \mathbb{Z} & \xleftarrow{\sim} & \mathbb{Z} \\ \uparrow & & \uparrow \\ \text{Pic}(\mathbb{P}_{\Pi}^2) & \xleftarrow{\quad} & \text{Pic}(\text{QGr}(r, n)), \end{array}$$

so it suffices to show that $\text{Pic}(\mathbb{P}_{\Pi}^2) = 0$.

Using the description of $X = \mathbb{P}_{\Pi}^2$ as a toric supervariety with supertorus $Q(1)^2$, it is straightforward to compute the even coordinates on the affine charts:

$$\begin{aligned} \mathbb{C}[U_0]_{\bar{0}} &= \mathbb{C}[t_1, t_2, t_1 t_2 \xi_1 \xi_2] \\ \mathbb{C}[U_1]_{\bar{0}} &= \mathbb{C}[t_1^{-1}, t_1^{-1} t_2 (1 + \xi_1 \xi_2), t_1^{-2} t_2 \xi_1 \xi_2] \\ \mathbb{C}[U_2]_{\bar{0}} &= \mathbb{C}[t_2^{-1}, t_1 t_2^{-1} (1 - \xi_1 \xi_2), t_1 t_2^{-2} \xi_1 \xi_2] \\ \mathbb{C}[U_{01}]_{\bar{0}} &= \mathbb{C}[t_1^{\pm 1}, t_2, t_2 \xi_1 \xi_2] \\ \mathbb{C}[U_{02}]_{\bar{0}} &= \mathbb{C}[t_1, t_2^{\pm 1}, t_1 \xi_1 \xi_2] \\ \mathbb{C}[U_{12}]_{\bar{0}} &= \mathbb{C}[t_1^{-1}, (t_1^{-1} t_2 (1 + \xi_1 \xi_2))^{\pm 1}, t_1^{-1} \xi_1 \xi_2] \end{aligned}$$

Now from the exact sequence

$$0 \rightarrow \mathcal{O}_X^{\times} \rightarrow \mathcal{K}_X^{\times} \rightarrow \mathcal{K}_X^{\times} / \mathcal{O}_X^{\times} \rightarrow 0$$

we obtain

$$H^0(\mathcal{K}_X^{\times}) \rightarrow H^0(\mathcal{K}_X^{\times} / \mathcal{O}_X^{\times}) \rightarrow \text{Pic}(X) \rightarrow 0$$

since $\mathcal{K}_X^{\times}$ is a constant sheaf. Using the trivialization over the open cover $\{U_0, U_1, U_2\}$, we note that

$$H^0(\mathcal{K}_X^{\times} / \mathcal{O}_X^{\times}) = \{(f_0, f_1, f_2) \mid f_i \in \mathbb{C}(U_i)^{\times} / \mathbb{C}[U_i]^{\times}, f_i / f_j \in \mathbb{C}[U_{ij}]^{\times}\}.$$

Hence $\text{Pic } X$ consists of the data

$$\begin{aligned}\frac{f_1}{f_0} &= t_1^\ell \cdot \left(1 + \sum_{m_2 > 0} b_m t^m \xi_1 \xi_2 \right) \\ \frac{f_2}{f_0} &= t_2^\ell \cdot \left(1 + \sum_{m_1 > 0} c_m t^m \xi_1 \xi_2 \right)\end{aligned}$$

for $\ell \in \mathbb{Z}$ and $b_m, c_m \in \mathbb{C}$, such that

$$\frac{f_2}{f_1} = (t_1^{-1} t_2 (1 + \xi_1 \xi_2))^\ell \cdot (1 + \dots).$$

An explicit computation reveals that this is not possible unless $\ell = 0$, meaning the image of $\text{Pic}(\mathbb{P}_{\Pi}^2) \rightarrow \text{Pic}(\mathbb{P}^2) \cong \mathbb{Z}$ is 0, as desired. $\square$

Accumulating the results of the Lemmas 6.5.1 and 6.5.4 together with the cases of X a point, we obtain:

Proposition 6.5.5. *Let $0 \leq r \leq n$. Then*

$$\text{Pic}(\text{QGr}(r, n)) = \begin{cases} \mathbb{Z} & r = 1, n = 2 \\ 0 & \text{otherwise} \end{cases}$$

6.6 Picard groups of toric subvarieties

In this section we use essentially the same techniques as in the prior section to compute the Picard group of toric supervarieties in $\text{QGr}(r, n)$.

Recall that we write $\mathfrak{q}(1)^n = \text{Span}_{\mathbb{C}}\{x_1, \dots, x_n, \theta_1, \dots, \theta_n\}$, where $[\theta_i, \theta_j] = \delta_{ij} x_i$, for the Lie superalgebra of the supertorus $Q(1)^n$.

Let $y \in |\text{QGr}(r, n)|$ be a closed point. As in [23], its stabilizer is

$$\text{Stab}_{q(1)^n} y = \text{Span}_{\mathbb{C}} \left\{ \sum_{i \in I_j} x_i \mid j = 1, \dots, s \right\} + \text{Span}_{\mathbb{C}} \left\{ \sum_{i \in I_j} \theta_i \mid j = 1, \dots, s \right\}$$

for some partition $I_1 \sqcup \dots \sqcup I_s = \{1, \dots, n\}$. We may then let $T \subseteq Q(1)^n$ be the subgroup

$$T = \prod_{j=1}^s \prod_{i \in I_j \setminus \max(I_j)} Q(1)$$

so that the orbit closure $X = \overline{Q(1)^n \cdot y}$ is a faithful toric supervariety for T . Its Lie superalgebra is

$$\mathfrak{t} = \text{Span}_{\mathbb{C}}\{x_i, \theta_i \mid i \neq \max(I_j) \text{ for any } j = 1, \dots, s\},$$



Figure 6.1: One parallel component in a triangular prism; three in a cube

and we write $\mathfrak{t}_{\mathbb{R}}$ for the $\mathbb{R}$ -linear span of the same basis vectors. Then, since stabilizers of points are sums of unit vectors, it follows that the decorated fan will be a complete fan on rays of the form $\pm\mathbb{R}_+(x_{i_1} + \cdots + x_{i_k})$ decorated by $\mathbb{C}(\theta_{i_1} + \cdots + \theta_{i_k})$.

In particular, the fermionic sheaf $\mathcal{F}_X \cong \Omega_{X_0}^1$ as in the case of $X = \text{QGr}(r, n)$. This can also be seen at the level of affine toric charts of X , each of which is generated by a collection of even weight vectors that reduce to the ordinary even generators, as well as one odd generator in each of the same weight spaces. We then have:

Lemma 6.6.1. *Let X be the closure of a $Q(1)^n$ orbit in $\text{QGr}(r, n)$. Then the restriction map $\text{Pic}(X) \rightarrow \text{Pic}(X_0)$ is injective.*

Proof. A result of Danilov ([10, Corollary 12.7]) yields $H^1(X, \Omega_X^q) = 0$ for $q > 1$, so the argument of Lemma 6.5.2 applies. $\square$

Let P be the polytope corresponding to X_0 , and note that such a polytope induces a unique toric supervariety $X_P \subseteq \text{QGr}(r, n)$, up to isomorphism. In particular, X_P may be described by a decorated polytope which is uniquely determined by P , and equals the image of X_P under an “odd momentum map” $\text{QGr}(r, n) \rightarrow \Pi\mathfrak{uq}(n)^*$ (see [23, Section 5]). Considering P as a subset of $(\mathfrak{t}_{\mathbb{R}})_0^*$, its edges are parallel to either x_i^* or $x_i^* - x_j^*$ for some $i, j \in \{1, \dots, n\}$. The fan Σ_P dual to P consists of rays of the form $\pm\mathbb{R}_+(x_{i_1} + \cdots + x_{i_k})$ as above.

Definition 6.6.2. The *parallel component number* $pc(P)$ of a polytope $P \subseteq \mathbb{R}^n$ is any of the following equivalent notions:

- (a) The number of lines $L \in \mathbb{P}(\mathbb{R}^n)$ such that the 1-skeleton of P becomes disconnected upon removal of the edges parallel to L .
- (b) The number of 2-colorings of the vertices of P such that all the edges between differently colored vertices are parallel.
- (c) The number of codimension-1 hyperplanes $H \subseteq (\mathbb{R}^n)^*$ such that $\{\sigma \cap H \mid \sigma \in \Sigma + P\}$ is a subfan of Σ_P .
- (d) The number of surjective toric morphisms $X_P \rightarrow \mathbb{P}_{\Pi}^1$, up to automorphism of $\mathbb{P}_{\Pi}^1$.

Example 6.6.3. The supervarieties $\mathbb{P}_{\Pi}^1 \times \mathbb{P}_{\Pi}^2$ and $\mathbb{P}_{\Pi}^1 \times \mathbb{P}_{\Pi}^1 \times \mathbb{P}_{\Pi}^1$ are described by a triangular prism and a cube, respectively. These polytopes have parallel component numbers 1 and 3, respectively, as seen in Figure 6.1. In view of the forthcoming Proposition 6.6.4, their respective Picard groups are $\mathbb{Z}$ and $\mathbb{Z}^3$.

Proposition 6.6.4. $\text{Pic}(X_P) \cong \mathbb{Z}^{pc(P)}$, a saturated subgroup of $\text{Pic}((X_P)_{\bar{0}})$

Proof. An element of $\text{Pic}(X) \cong H^1(\mathcal{O}_{X,\bar{0}}^\times)$ contains the data of a transition function from each maximal affine toric open to each other with which it shares a maximal proper subset that is an affine toric open, i.e. a codimension-1 cone in the language of fans. This can be encoded by a transition function from each vertex of P to each adjacent vertex, along the incident edge. Since the 1-skeleton of P is connected, the data of these transition functions conversely determine a unique element of $\text{Pic}(X)$.

A transition function f_{vw} from U_v to U_w is a unit in $\mathcal{O}_{X,\bar{0}}(U_v \cap U_w)$, and they are subject to cocycle conditions $f_{v_1 v_2} f_{v_2 v_3} \cdots f_{v_{k-1} v_k} f_{v_k v_1} = 1$. By Lemma 6.6.1, we may write the transition functions in the form t_i^ℓ or $(t_i t_j^{-1} (1 - \xi_i \xi_j))^\ell$ for some $\ell \in \mathbb{Z}$, depending on whether the relevant edge is parallel to x_i^* or $x_i^* - x_j^*$.

Applying the cocycle condition, we see that the only way to cancel a $(1 - \xi_i \xi_j)^\ell$ term is with an opposite $(1 - \xi_i \xi_j)^{-\ell}$ term. Hence $\ell = 0$ unless the cycle contains another edge parallel to $x_i^* - x_j^*$. It then follows that t_i^ℓ can be canceled out only if the cycle contains another edge parallel to x_i^* . In particular, a transition function along an edge e can be nonzero only if every cycle containing e contains another edge parallel to e . The number of such directions is counted by part (a) of Definition 6.6.2

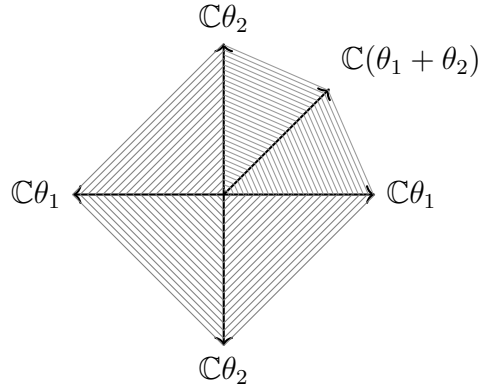
Conversely, for each direction counted by $pc(P)$, the choice of $\ell \in \mathbb{Z}$ is free and independent of other such choices. These choices yield the Picard group $\text{Pic}(X_P)$, as desired. $\square$

The same conclusion applies to the $Q(1)^n$ toric supervariety of any complete fan whose rays are of the form $\pm \mathbb{R}_+(x_{i_1} + \cdots + x_{i_k})$, decorated by $\theta_{i_1} + \cdots + \theta_{i_k}$. However, we caution that not every fan of this form can admit these decorations. For example, the rays $\mathbb{R}_+(x_1 + x_2)$ and $\mathbb{R}_+(x_1 + x_3)$ cannot together form a 2-cone because $[\theta_1 + \theta_2, \theta_1 + \theta_3] \notin \text{Span}_{\mathbb{C}}\{x_1 + x_2, x_1 + x_3\}$. In terms of the polytope, the edges must be parallel to x_i^* or $x_i^* - x_j^*$.

Example 6.6.5. Let X be the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at a torus-fixed point, so the corresponding polytope P is a pentagon as in Figure 6.2. If the rays of Σ_P are appropriately decorated by θ_1, θ_2 , and $\theta_1 + \theta_2$ as in Figure 6.3, then the Picard group of the resulting toric supervariety X_P is isomorphic to $\mathbb{Z}^2$. The edge in P (equivalently, the ray in Σ_P) added by blowing up does not affect the Picard group because no other edge is parallel to it (equivalently, the opposite ray is not present). That is, any transition function across this edge is trivial, and the morphism $X_P \rightarrow \mathbb{P}_{\Pi}^1 \times \mathbb{P}_{\Pi}^1$ induces an isomorphism at the level of Picard groups.

Note that X_P is not a blowup of $\mathbb{P}_{\Pi}^1 \times \mathbb{P}_{\Pi}^1$; any new rays introduced by a blowup operation on a toric supervariety must be decorated by 0.

Remark 6.6.6. In the case that X_P does arise as a supertorus orbit closure in $\text{QGr}(r, n)$, then the argument of the forthcoming Lemma 6.8.5 shows that each contribution to $pc(P)$ arises from a factor of $\mathbb{P}_{\Pi}^1$ in X_P . Due to the isomorphism of $\text{Pic}(\mathbb{P}_{\Pi}^1) \cong H^1(\mathcal{O}_{X,\bar{0}}^\times) \cong \text{Pic}(\mathbb{P}^1)$, it follows that any invertible sheaf on X_P can be understood as a tensor product of a choice of invertible sheaf $\mathcal{O}(\ell_i)$ on each factor of $\mathbb{P}_{\Pi}^1$. That is, if $X_P \cong (\mathbb{P}_{\Pi}^1)^d \times Y$ for some superscheme Y with d maximal, then an invertible sheaf on X_P is determined entirely by its restriction to any closed subvariety $(\mathbb{P}^1)^d \times pt$.


 Figure 6.2: Two parallel components in the pentagon P of Example 6.6.5

 Figure 6.3: The decorated fan Σ_P of Example 6.6.5

6.7 Π -Picard set of the isomeric supergrassmannian

If X is a supervariety, write $X/\mathcal{J}^2$ for its closed subvariety $(|X|, \mathcal{O}_X/\mathcal{J}_X^2)$. By removing the grading, since $\mathcal{O}_X/\mathcal{J}_X^2$ is a sheaf of *commutative* rings, we may view $X/\mathcal{J}^2$ as an honest (non-super) scheme, which we call X_{thick} . Observe that $\text{Pic}_{\Pi}(X/\mathcal{J}^2) = H^1(X, (\mathcal{O}_X/\mathcal{J}_X^2)^{\times}) = \text{Pic}(X_{\text{thick}})$ and $\text{Pic}_{\Pi}(X_{\bar{0}}) = H^1(X, (\mathcal{O}_X/\mathcal{J}_X)^{\times}) = \text{Pic}(X_{\bar{0}})$.

Once again, let $X = \text{QGr}(r, n)$ for some integers $n \geq r \geq 0$.

Lemma 6.7.1. *The restriction map $\text{Pic}_{\Pi}(X) \rightarrow \text{Pic}_{\Pi}(X_{\bar{0}})$ factors as an injection $\text{Pic}_{\Pi}(X) \rightarrow \text{Pic}_{\Pi}(X/\mathcal{J}^2)$ followed by a surjection $\text{Pic}_{\Pi}(X/\mathcal{J}^2) \rightarrow \text{Pic}_{\Pi}(X_{\bar{0}})$.*

Proof. Injectivity of $\text{Pic}_{\Pi}(X) \rightarrow \text{Pic}_{\Pi}(X/\mathcal{J}^2)$ is the same as in Lemma 6.5.2. Likewise, surjectivity of $\text{Pic}_{\Pi}(X/\mathcal{J}^2) \rightarrow \text{Pic}_{\Pi}(X_{\bar{0}})$ follows from $H^2(\text{Gr}(r, n), \Omega^1) = 0$ [43]. $\square$

This lemma has two important consequences. For one, we now have a frame of reference in which to describe the cohomology set $\text{Pic}_{\Pi}(X)$; that is, we can describe it as a subset of the group $\text{Pic}_{\Pi}(X/\mathcal{J}^2) = \text{Pic}(X_{\text{thick}})$. Secondly, it yields the following lemma.

Lemma 6.7.2. *If $0 < r < n$, then $\text{Pic}_{\Pi}(X/\mathcal{J}^2) \cong \mathbb{Z} \oplus \mathbb{C}$. In particular, $\text{Pic}_{\Pi}(\mathbb{P}_{\Pi}^1) = \mathbb{Z} \oplus \mathbb{C}$.*

Proof. From the exact sequence

$$0 \rightarrow \Omega^1 \rightarrow (\mathcal{O}_X/\mathcal{J}_X^2)^{\times} \rightarrow (\mathcal{O}_X/\mathcal{J}_X)^{\times} \rightarrow 0$$

and from surjectivity of $H^0((\mathcal{O}_X/\mathcal{J}_X^2)^\times) \rightarrow H^0((\mathcal{O}_X/\mathcal{J}_X)^\times)$, we obtain the exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(\Omega^1) & \longrightarrow & \mathrm{Pic}(X_{\mathrm{thick}}) & \longrightarrow & \mathrm{Pic}(X_{\bar{0}}) \longrightarrow 0 \\ & & \parallel & & & & \parallel \\ & & \mathbb{C} & & & & \mathbb{Z} \end{array}$$

and hence $\mathrm{Pic}_\Pi(X/\mathcal{J}^2) = \mathrm{Pic}(X_{\mathrm{thick}}) \cong \mathbb{Z} \oplus \mathbb{C}$. $\square$

It is then straightforward to see that for $X = \mathbb{P}_\Pi^n$, the transition functions for a Π -invertible sheaf on $X/\mathcal{J}^2$ take the form

$$f_i f_0^{-1} = t_i^\ell (1 + c \xi_i)$$

for some $\ell \in \mathbb{Z}$ and $c \in \mathbb{C}$. To see which transition functions also work for X itself, we calculate

$$\begin{aligned} f_i f_j^{-1} &= (f_i f_0^{-1})(f_j f_0^{-1})^{-1} \\ &= (t_i t_j^{-1})^\ell (1 + c(\xi_i - \xi_j) - c^2 \xi_i \xi_j) \end{aligned}$$

so that the only requirement is $c^2 = \ell$. Therefore

Lemma 6.7.3. *If $n > 1$, then $\mathrm{Pic}_\Pi(\mathbb{P}_\Pi^n) = \{(\ell, c) \in \mathbb{Z} \oplus \mathbb{C} \mid c^2 = \ell\}$.*

The only difference between the elements $(\ell, +\sqrt{\ell})$ and $(\ell, -\sqrt{\ell})$ is that the odd involution is negated; the sheaves themselves are isomorphic.

Remark 6.7.4. We write $\mathcal{O}_\Pi(\ell)$ for the sheaf represented by $(\ell, \pm\sqrt{\ell})$. Naturally, $\mathcal{O}_\Pi(1)$ is the tautological sheaf whose fiber at the point $W \in \mathbb{P}_\Pi^n$ is W^* , and $\mathcal{O}_\Pi(-1)$ is the dual sheaf with fiber W . We observe that all $\mathcal{O}_\Pi(\ell)$ are symmetric powers of these. For $\ell > 0$,

$$\begin{aligned} \mathcal{O}_\Pi(\ell) &\cong S^\ell \mathcal{O}_\Pi(1) \\ \mathcal{O}_\Pi(-\ell) &\cong S^\ell \mathcal{O}_\Pi(-1) \end{aligned}$$

and so their fibers at W are $S^\ell W^*$ and $S^\ell W$, respectively. Note however that the odd involution must be rescaled appropriately following this operation.

If an isomeric basis of $\mathbb{C}^{n+1|n+1}$ is chosen, as must be done when we fix a supertorus $Q(1)^{n+1} \subseteq Q(n+1)$, then the $\mathcal{O}_\Pi(\ell)$ can also be defined in terms of the standard representation $V = \mathbb{C}^{n+1|n+1}$ of $Q(n+1)$. Fix a closed point $W \in \mathbb{P}_\Pi^n$, and let $W^\perp$ be its orthogonal complement with respect to the given basis. Then $Q(W) \times Q(W^\perp) \cong Q(1) \times Q(n)$ is a block-diagonal subgroup of $Q(V) \cong Q(n+1)$, and we can ask for the highest weight spaces of $S^\ell W^*$ and $S^\ell W$ for the action of $Q(W)$. For $\ell > 0$, these highest weight spaces assemble into the sheaves $\mathcal{O}_\Pi(\ell)$ and $\mathcal{O}_\Pi(-\ell)$, respectively.

Lemma 6.7.5. *If $1 < r < n-1$, then $\mathrm{Pic}_\Pi(X) = 0$.*

Proof. We employ the same strategy that was used in Lemma 6.5.4. It is straightforward to see that a standard extension $\mathrm{QGr}(2, 4) \rightarrow \mathrm{QGr}(r, n)$ induces an isomorphism on Π -Picard sets at the level of $X/\mathcal{J}^2$. That is, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{Z} \oplus \mathbb{C} & \xleftarrow{\sim} & \mathbb{Z} \oplus \mathbb{C} \\ \uparrow & & \uparrow \\ \mathrm{Pic}_{\Pi}(\mathrm{QGr}(2, 4)) & \xleftarrow{\quad} & \mathrm{Pic}_{\Pi}(\mathrm{QGr}(r, n)), \end{array}$$

so it suffices to show that $\mathrm{Pic}_{\Pi}(\mathrm{QGr}(2, 4)) = 0$.

Choose two embeddings $\mathbb{P}_{\Pi}^2 \rightarrow \mathrm{QGr}(2, 4)$ whose momentum map images are adjacent triangles in the hypersimplex $\Delta_{2,4}$. As above, these embeddings induce isomorphisms

$$\mathrm{Pic}_{\Pi}((\mathrm{QGr}(2, 4))_{\mathrm{thick}}) \rightarrow \mathrm{Pic}_{\Pi}((\mathbb{P}_{\Pi}^2)_{\mathrm{thick}}).$$

Since they occur as adjacent triangles, the clockwise directions within the two triangles disagree on their shared edge. (Later, in the proof of Lemma 6.8.4, we will illustrate this phenomenon more explicitly.) Thus we may assume without loss of generality that these two maps $\mathbb{Z} \oplus \mathbb{C} \rightarrow \mathbb{Z} \oplus \mathbb{C}$ are given by ± 1 . It follows that an element $(\ell, c) \in \mathrm{Pic}_{\Pi}(\mathrm{QGr}(2, 4)) \subseteq \mathbb{Z} \oplus \mathbb{C}$ must satisfy $c^2 = \ell$ and $(-c)^2 = -\ell$, so indeed $\mathrm{Pic}_{\Pi}(\mathrm{QGr}(2, 4)) = 0$. $\square$

We now collect the results of Lemmas 6.7.2, 6.7.3, and 6.7.5 in the following proposition. Note in particular that the only nonzero cases are $\mathbb{P}_{\Pi}^n$.

Proposition 6.7.6. *Let $0 \leq r \leq n$. Then*

$$\mathrm{Pic}_{\Pi}(\mathrm{QGr}(r, n)) = \begin{cases} \mathbb{Z} \oplus \mathbb{C} & r = 1, n = 2 \\ \{(\ell, c) \in \mathbb{Z} \oplus \mathbb{C} \mid c^2 = \ell\} & r \in \{1, n-1\}, n > 2 \\ 0 & \text{otherwise} \end{cases}$$

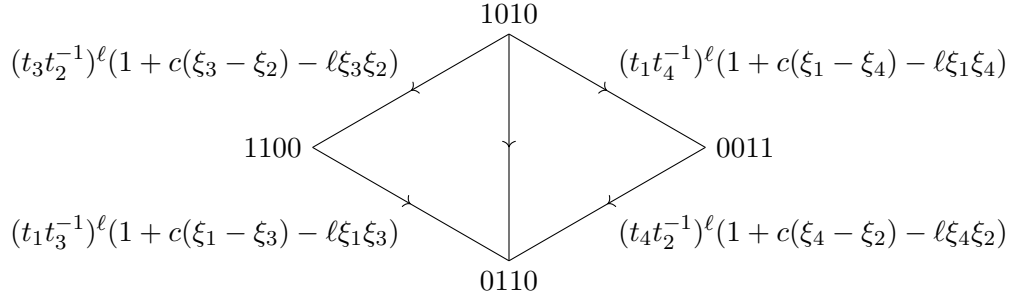
6.8 Π -Picard sets of toric subvarieties

Now let X be the closure of a $Q(1)^n$ orbit in $\mathrm{QGr}(r, n)$, and let P be the image of its underlying variety under the momentum map, a subpolytope of $\Delta_{r,n}$ whose vertices and edges are also vertices and edges of $\Delta_{r,n}$.

Lemma 6.8.1. *The restriction map $\mathrm{Pic}_{\Pi}(X) \rightarrow \mathrm{Pic}_{\Pi}(X_{\bar{0}})$ factors as an injection $\mathrm{Pic}_{\Pi}(X) \rightarrow \mathrm{Pic}_{\Pi}(X/\mathcal{J}^2)$ followed by a surjection $\mathrm{Pic}_{\Pi}(X/\mathcal{J}^2) \rightarrow \mathrm{Pic}_{\Pi}(X_{\bar{0}})$.*

Proof. As in Lemma 6.7.1, injectivity of $\mathrm{Pic}_{\Pi}(X) \rightarrow \mathrm{Pic}_{\Pi}(X/\mathcal{J}^2)$ is the same as in Lemma 6.6.1. However, unlike in Lemma 6.7.1, it is not necessarily the case that $H^2(X, \Omega^1) = 0$. Instead, we employ a slightly more manual argument to show that $\mathrm{Pic}_{\Pi}(X/\mathcal{J}^2) \rightarrow \mathrm{Pic}_{\Pi}(X_{\bar{0}})$ is surjective.

Since the even part of $\mathcal{O}_X/\mathcal{J}_X^2$ is exactly $\mathcal{O}_{X_{\bar{0}}}$, there is a splitting $X/\mathcal{J}^2 \rightarrow X_{\bar{0}}$ of the closed immersion $X_{\bar{0}} \rightarrow X/\mathcal{J}^2$. Since the composition of these maps is the identity on $X_{\bar{0}}$, the same is true for the corresponding maps on $\mathrm{Pic}_{\Pi}(X_{\bar{0}})$. $\square$


 Figure 6.4: Incompatible transition functions (unless $\ell = c = 0$)

As before, this implies that $\text{Pic}_{\Pi}(X/\mathcal{J}^2) \cong \text{Pic}(X_{\bar{0}}) \oplus H^1(\Omega^1)$. Of course, each transition function along an edge of P consists of an integer and a complex number. These two pieces of data are intimately linked; indeed, one should expect that the continuous part of $\text{Pic}_{\Pi}(X/\mathcal{J}^2)$ has the same rank as $\text{Pic}(X_{\bar{0}})$. It is therefore no coincidence that $\text{rk Pic}(X_{\bar{0}}) = \dim H^1(\Omega^1)$. See e.g. [9] for the case of P a simple polytope.

Lemma 6.8.2. *It holds that $\text{Pic}_{\Pi}(X/\mathcal{J}^2) \cong (\mathbb{Z} \oplus \mathbb{C})^{\text{rk Pic}(X_{\bar{0}})}$.*

We record the following fact about 2-faces of P .

Lemma 6.8.3. *A 2-face of P is either a triangle or a square.*

Proof. It is a straightforward exercise to verify that any set of more than three coplanar vertices in $\Delta_{r,n}$ are the vertices of a square. $\square$

Lemma 6.8.4. *If two triangular faces in P meet at an edge and are not part of a larger simplicial face, then any Π -invertible sheaf trivializes on the union of the corresponding cells of X .*

Proof. The same proof as in Lemma 6.7.5 works here as well. Since the transition functions are nicer on toric supervarieties than on $\text{QGr}(r, n)$, we elect to include some more explicit computations in Figure 6.4.

Without loss of generality, two such triangles occur at the vertices 1100, 1010, 0110, and 0011, and transition functions between vertices are as pictured in Figure 6.4, for some $n \in \mathbb{Z}$ and $a \in \mathbb{C}$. In the simplest situation, these transition functions arise by considering the closure of a $Q(1)^4$ orbit of a generic point in $\text{QGr}(2, 4)$.

The vertical downward arrow must be $(t_1 t_2^{-1})^{\ell}(1 + c(\xi_1 - \xi_2) - \ell \xi_1 \xi_2)$. If this equals the product of the two transition functions on the left, then $c^2 = \ell$. However, if it equals the product of those on the right, then $c^2 = -\ell$. Therefore $\ell = c = 0$, so the Π -invertible sheaf trivializes over these cells. $\square$

Lemma 6.8.5. *If P is not the product of a smaller polytope with a simplex, then $\text{Pic}_{\Pi}(X) = 0$.*

Proof. We prove the contrapositive. Let $S \subseteq P$ be a maximal simplicial face, and assume that there is a Π -invertible sheaf that does not trivialize over S . By Lemmas 6.8.3 and 6.8.4, all other edges incident to the vertices of S must be parts of squares. In particular, P consists of parallel translates of S , connected via parallel edges. That is, P is the product of S with some other polytope. $\square$

From this lemma, it is clear that any nonzero contribution to $\text{Pic}_\Pi(X)$ will occur as a result of simplex factors in P , or equivalently $\mathbb{P}_\Pi^n$ factors in X . Let I be the collection of these factors, and for $i \in I$ write d_i for the dimension of the factor $\mathbb{P}_\Pi^{d_i}$. Then

$$\text{Pic}_\Pi(X) \subseteq (\mathbb{Z} \oplus \mathbb{C})^{|I|} \subseteq (\mathbb{Z} \oplus \mathbb{C})^{\text{rk Pic}(X_{\bar{0}})},$$

a fact that we will use in the following proposition.

Proposition 6.8.6. *It holds that*

$$\text{Pic}_\Pi(X) = \{(\ell_i, c_i) \in (\mathbb{Z} \oplus \mathbb{C})^{|I|} \mid \text{at most one } c_i \neq 0, \text{ and if } d_i > 1, \text{ then } c_i^2 = \ell_i\}.$$

Proof. The $c_i^2 = \ell_i$ conditions come from Lemma 6.7.3. Now for any two distinct simplex factors, say $1, 2 \in I$, choose a transition function f_1, f_2 along an edge of each. Then $f_1 f_2 f_1^{-1} f_2^{-1} = 1$, so f_1 and f_2 commute. It follows that at least one of them must be purely even, so either $c_1 = 0$ or $c_2 = 0$. $\square$

Remark 6.8.7. Observe that only one factor of $\mathbb{P}_\Pi^n$ for $n > 1$ can contribute nonzero transition functions to any particular Π -invertible sheaf. That is, the Π -Picard set of a product of such $\mathbb{P}_\Pi^n$ is the coproduct of their Π -Picard sets, in the category of pointed sets.

Consequently, if $X \cong \prod_{i=1}^\infty (\mathbb{P}_\Pi^{d_i}) \times Y$ for some superscheme Y with d_i maximal, then a Π -invertible sheaf on X is given by the tensor product of a Π -invertible sheaf on a single factor of $\mathbb{P}_\Pi^{d_i}$ as determined in Remark 6.7.4 with an invertible sheaf on $(\mathbb{P}_\Pi^1)^{d_1}$ as determined in Remark 6.6.6.

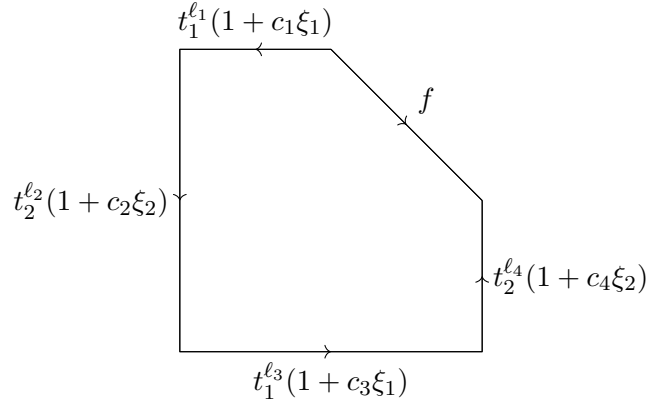
Previously, we indicated that the result of Proposition 6.6.4 can be generalized to any toric supervariety whose polytope's edges are parallel to x_i^* or $x_i^* - x_j^*$, such as in Example 6.6.5 and the corresponding Figure 6.2. However, analogously generalizing the result on Π -invertible sheaves is not quite so straightforward.

Example 6.8.8. Let X be the toric supervariety of Example 6.6.5. The transition functions of a Π -invertible sheaf on X are specified in Figure 6.5 such that

$$\begin{aligned} f &= t_2^{\ell_4} (1 + c_4 \xi_2) t_1^{\ell_3} (1 + c_3 \xi_1) t_2^{\ell_2} (1 + c_2 \xi_2) t_1^{\ell_1} (1 + c_1 \xi_1) \\ &= t_1^{\ell_1 + \ell_3} t_2^{\ell_2 + \ell_4} (1 + (c_1 + c_3) \xi_1 + (c_2 + c_4) \xi_2 - (c_1 c_2 + c_1 c_4 - c_2 c_3 + c_3 c_4) \xi_1 \xi_2). \end{aligned}$$

This result must satisfy the following compatibility criteria:

$$\begin{aligned} \ell_1 + \ell_2 + \ell_3 + \ell_4 &= 0 \\ c_1 + c_2 + c_3 + c_4 &= 0 \\ c_1 c_2 + c_1 c_4 - c_2 c_3 + c_3 c_4 &= \ell_2 + \ell_4 \end{aligned}$$


 Figure 6.5: Transition functions of a Π -invertible sheaf along a pentagon

For instance, consider the collection of Π -invertible sheaves whose underlying invertible sheaves on $X_{\bar{0}}$ (obtained by killing nilpotents in the transition functions) lie in the image of the pullback induced by the blowup $X_{\bar{0}} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$. In terms of the parameters $\ell_i \in \mathbb{Z}$ and $c_i \in \mathbb{C}$, these Π -invertible sheaves are the ones satisfying $\ell_1 + \ell_3 = \ell_2 + \ell_4 = 0$, so up to an element of $\text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$ they form the elliptic cone $(c_1 + c_3)^2 + 2c_2c_3 = 0$.

Remark 6.8.9. The compatibility conditions describing $\text{Pic}_{\Pi}(X)$ for a toric supervariety X can become rather unwieldy. In future work, it would be interesting to understand the connected components of $\text{Pic}_{\Pi}(X)$ as varieties in terms of the combinatorics of the decorated fan of X .

Additionally, it would be useful to determine combinatorial criteria for ampleness of a Π -invertible sheaf on a toric supervariety, as well as to concretely realize the resulting embeddings into $\mathbb{P}_{\Pi}^n$.

6.9 Summary of Picard results

The following theorem is the amalgamation of Propositions 6.5.5, 6.6.4, 6.7.6, and 6.8.6.

Theorem 6.9.1. *Let X be $\text{QGr}(r, n)$ or the closure of a $Q(1)^n$ orbit inside $\text{QGr}(r, n)$. Then*

$$\begin{aligned} \text{Pic}(X) &\cong \prod_{\text{factors } \mathbb{P}_{\Pi}^1 \text{ of } X} \mathbb{Z} \quad \text{and} \\ \text{Pic}_{\Pi}(X) &\cong \left\{ (\ell_i, c_i) \in \prod_{\text{factors } \mathbb{P}_{\Pi}^{d_i} \text{ of } X} \mathbb{Z} \oplus \mathbb{C} \mid \text{at most one } c_i \neq 0, \text{ and if } d_i > 1, \text{ then } c_i^2 = \ell_i \right\} \end{aligned}$$

The number of factors of Π -projective space $\mathbb{P}_{\Pi}^d \cong \text{QGr}(1, d+1)$ can be detected easily by counting the number of d -simplex factors in the defining polytope. We emphasize that

for $X = \text{QGr}(r, n)$, we have

$$\begin{aligned} \text{Pic}(\text{QGr}(r, n)) &= \begin{cases} \mathbb{Z} & r = 1, n = 2 \\ 0 & \text{otherwise} \end{cases} \\ \text{Pic}_{\Pi}(\text{QGr}(r, n)) &= \begin{cases} \mathbb{Z} \oplus \mathbb{C} & r = 1, n = 2 \\ \{(\ell, c) \in \mathbb{Z} \oplus \mathbb{C} \mid c^2 = \ell\} & r \in \{1, n-1\}, n > 2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

so in particular $\text{Pic}(\text{QGr}(r, n)) = \text{Pic}_{\Pi}(\text{QGr}(r, n)) = 0$ unless $\text{QGr}(r, n) \cong \mathbb{P}_{\Pi}^{n-1}$.

The setting $\mathbb{Z} \oplus \mathbb{C}$ of $\text{Pic}(X)$ (for $X = \mathbb{P}_{\Pi}^n$) arises as the Picard group of the (non-super) scheme with the same topological space and the structure sheaf $\mathcal{O}_X/\mathcal{I}_X^2$, where $\mathcal{I}_X$ is the odd ideal. Note that the $\mathbb{Z}/2\mathbb{Z}$ -grading must be forgotten in order to make it into a non-super scheme, though it is equivalent to consider the Π -Picard set of the corresponding superscheme. Equivalently, $\mathbb{Z}$ is the Picard group of the underlying variety (given by $\mathcal{O}_X/\mathcal{I}_X$), and $\mathbb{C} \cong H^1(X, \Omega^1)$.

These calculations help us understand morphisms from X to projective superspace $\mathbb{P}^{m|n} \cong \text{Gr}(1|0, m+1|n)$ and Π -projective superspace $\mathbb{P}_{\Pi}^n \cong \text{QGr}(1, n+1)$, respectively. Since all such bundles can be interpreted in terms of symmetric powers of tautological bundles on $\mathbb{P}^{m|n}$ and $\mathbb{P}_{\Pi}^n$, any such morphisms turn out to be fairly trivial. This rigidity is common but not ubiquitous in supergeometry; for example, the Picard group of $\mathbb{P}^{1|m}$ has a very large continuous component [4].

Corollary 6.9.2. *Let X be the closure of a $Q(1)^n$ orbit inside $\text{QGr}(r, n)$. Then:*

- (a) *An invertible sheaf on X is given by the tensor product of an invertible sheaf $\mathcal{O}(\ell)$, $\ell \in \mathbb{Z}$ on each factor of $\mathbb{P}_{\Pi}^1$ in X .*
- (b) *A Π -invertible sheaf on X is given by the tensor product of such an invertible sheaf with either:*
 - (i) *A Π -invertible sheaf $\mathcal{O}_{\Pi}(\ell)$, $\ell \in \mathbb{Z}$ on some factor $\mathbb{P}_{\Pi}^d$ for $d > 1$, or*
 - (ii) *A Π -invertible sheaf arising from $H^1(\mathbb{P}^1, \Omega^1) \cong \mathbb{C}$ on some factor of $\mathbb{P}_{\Pi}^1$.*

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Appendix A

Proof of Proposition 5.5.3

In this appendix we prove equivalence of the following five conditions for a toric supervariety X with decorated fan $(N, \Sigma, \mathfrak{t}, \mathfrak{h}, \{V_{\rho,i}\})$.

- (a) For every $\rho \in \Sigma(1)$, there is a parameterization as in (5.1) of the derivations by which $\mathfrak{t}$ acts on $\mathbb{C}[T]$ such that

$$A_\rho = \mathbb{C}[t_1, t_2^{\pm 1}, \dots, t_p^{\pm 1}, \xi_1, \dots, \xi_r, t_1^{\ell_{r+1}} \xi_{r+1}, \dots, t_1^{\ell_q} \xi_q]$$

for some positive integers ℓ_i .

- (b) For every $\rho \in \Sigma(1)$, A_ρ is the subalgebra of $\mathbb{C}[T]$ generated by $L_{\mathbb{C}[T]}^w(n + \rho^\perp)$ for all $n \geq 0$ and all codimension-at-most-1 subspaces $W \subseteq V_{\rho,0}$ containing $V_{\rho,n}$.
- (c) For every $\rho \in \Sigma(1)$ and proper subspace $W \subset V_{\rho,0}$ containing $\mathfrak{h}$, the quotient T/W of T/H embeds into a toric supervariety whose decorated fan is $(N, \rho, \mathfrak{t}, W, \{V_{\rho,i} + W\})$.
- (d) For every point $x \in |X|$ of codimension at most 1, the maximal ideal $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ is generated by a regular sequence $r_1, \dots, r_d$ such that whenever $\theta \in (\text{Stab}_{\mathfrak{t}} x)_{\overline{\mathbb{I}}}$ and $\overline{\theta \cdot r_i} = 0$, we have $\theta \cdot r_i \in (r_i)$.
- (e) For every point $x \in |X|$ of codimension at most 1 and for every $\theta \in (\text{Stab}_{\mathfrak{t}} x)_{\overline{\mathbb{I}}}$, $DS_\theta(\mathcal{O}_{X,x}/((\mathcal{O}_{X,x}^{\mathfrak{t}_0})_{\overline{\mathbb{I}}}))$ is FR.

First, we need the following lemma. In essence, it says that locally in codimension 1, the badness of a non- (HR_1) toric supervariety is contained in the badness of the case of an abelian supertorus. That is, each $\text{Spec } A_\rho$ contains a closed subvariety which is a toric supervariety for the abelian supertorus $(T_{\overline{0}}, \mathfrak{t}_{\overline{0}} \oplus V_{\rho,0})$. This closed toric subvariety is the subject of condition (e) of Proposition/Definition 5.5.3.

Lemma A.0.1. *Let $\text{Spec } A_\rho$ be a toric supervariety with $[V_{\rho,0}, V_{\rho,0}] = 0$, and parameterize the derivations so that $\theta_j^{\text{right}} = \frac{\partial}{\partial \xi_j}$ for $\theta_j \in V_{\rho,0}$. Then we may write*

$$A_\rho = \mathbb{C}[t_1, t_2^{\pm 1}, \dots, t_p^{\pm 1}, \xi_1, \dots, \xi_r, t_1^{\ell_{r+1}}(\xi_{r+1} + \dots), \dots, t_1^{\ell_q}(\xi_q + \dots)]$$

such that each $(\xi_j + \dots)$ for $j = r+1, \dots, q$ lies in $\Lambda\{\xi_{r+1}, \dots, \xi_q\}$.

Proof. By default, we know

$$A_\rho = \mathbb{C}[t_1(1 + \dots), t_2^{\pm 1}, \dots, t_p^{\pm 1}, \xi_1, \dots, \xi_r, t_1^{\ell_{r+1}}(\xi_{r+1} + \dots), \dots, t_1^{\ell_q}(\xi_q + \dots)]$$

without any assumptions on the ellipses. Our assumption on θ_j^{right} is equivalent to specifying $x_{ij} = 0$ for $j = r + 1, \dots, q$ in the language of 5.1. It follows that for $j = 1, \dots, q$, we have

$$\theta_j \cdot t^m(\dots) \in \mathbb{C}t^m(1 + \dots)\Lambda\{\xi_1, \dots, \xi_r\} - \frac{\partial}{\partial \xi_j} t^m(\dots),$$

of which the first summand is contained in A_ρ . Hence the second summand is too, so in particular A_ρ is closed under the action by $\frac{\partial}{\partial \xi_j}$.

By repeated application of these derivations, we conclude that $t_1 \in A_\rho$. Likewise, if some $t_1^{\ell_j}(\xi_j + \dots)$ contains a summand with a ξ_i term for $i = 1, \dots, r$, then we may subtract $\xi_i \frac{\partial}{\partial \xi_i} t_1^{\ell_j}(\xi_j + \dots)$ from it so that the result does not. $\square$

We are now ready to prove Proposition/Definition 5.5.3.

Proof. Case 1, (a) $\iff$ (b) $\iff$ (c): First suppose $[V_{\rho,0}, V_{\rho,0}] \neq 0$, so $\text{codim}(\mathfrak{h}, V_{\rho,0}) = 1$ and $V_{\rho,1} = \mathfrak{h}$. We claim that conditions (a)-(e) always hold in this case. First note that we may pick a basis $\theta_1, \dots, \theta_s$ of $\mathfrak{t}_1$ with dual basis $\xi_1, \dots, \xi_s$ so that $\mathfrak{h} = \mathbb{C}\{\theta_{q+1}, \dots, \theta_s\}$ and $V_{\rho,0} = \mathbb{C}\{\theta_q, \dots, \theta_s\}$. We may then parameterize $\mathbb{C}[T]$ so that

$$\begin{aligned} \theta_q^{\text{right}} &= \sum_{i=q}^s a_i \xi_i t_1 \frac{\partial}{\partial t_1} + \frac{\partial}{\partial \xi_q} \\ \theta_j^{\text{right}} &= \frac{\partial}{\partial \xi_j} \end{aligned} \quad j = q + 1, \dots, s$$

for some $a_i \in \mathbb{C}$. Therefore $L_{A_\rho}(\rho^\perp) = \mathbb{C}[T/H_{\rho,0}] = \mathbb{C}[t_2^{\pm 1}, \dots, t_p^{\pm 1}, \xi_1, \dots, \xi_{q-1}]$, and there is a new odd generator in $L(1)$. Since $L(1)$ must be 2^q -dimensional, it follows that $L_{A_\rho}(1 + \rho^\perp) = L_{\mathbb{C}[T/H]}(1 + \rho^\perp)$ and hence both (a) and (b) follow. In particular, we have

$$A_\rho = \mathbb{C}[t_1, t_2^{\pm 1}, \dots, t_p^{\pm 1}, \xi_1, \dots, \xi_{q-1}, t_1 \xi_q].$$

Note that (c) is trivial since the only possible W is $\mathfrak{h}$ itself.

A brief aside: We show that for the purposes of checking parts (d) and (e), even in the case of $[V_{\rho,0}, V_{\rho,0}] = 0$, it suffices to consider $T_{\bar{0}}$ -invariant points $x \in |X|$ of codimension 1. Such a point which is not $T_{\bar{0}}$ -invariant can be viewed as a prime ideal (of some $\overline{A}_\sigma$) generated by a single (regular) element which is not a monomial. As a result, the local ring $\mathcal{O}_{X,x}$ will contain all $\xi_1, \dots, \xi_q$, so $(\text{Stab}_t x)_{\bar{1}} = \mathfrak{h}$. This conclusion also holds for the codimension-0 point.

Since $\theta_j^{\text{right}} = \frac{\partial}{\partial \xi_j}$ kills $\mathcal{O}_{X,x}$ for $\theta_j \in \mathfrak{h}$, we note that θ_j simply takes each t^m to some multiple of it by a linear combination of $\xi_1, \dots, \xi_q$. Hence (d) will always hold for this choice of x . Meanwhile, (e) is trivial since $\mathcal{O}_{X,x}^{t_{\bar{0}}}$ contains all $\xi_1, \dots, \xi_q$.

On the other hand, a $T_{\bar{0}}$ -invariant point of codimension 1 can be viewed as the ideal generated by t_1 and the odd elements in some A_ρ . Since localization at this prime will not

affect future calculations, we will often suppress it from the notation and simply calculate with A_ρ .

Case 1, (a) $\iff$ (d) $\iff$ (e): Let us now prove (d) in this case. We have $(\text{Stab}_{\mathfrak{t}} x)_{\overline{1}} = \mathbb{C}\theta_q + \mathfrak{h}$. By the same argument as in the aside, any contribution of $\mathfrak{h}$ here is insignificant. Hence we check that the condition holds for $\theta = \theta_q$ and $r_1, \dots, r_d = t_1, \xi_1, \dots, \xi_{q-1}, t_1 \xi_q$. We have $\theta_q \cdot t_1 = \sum_{i=1}^q b_i \xi_i t_1$ for some $b_i \in \mathbb{C}$ and $\theta_q \cdot \xi_i = 0$ for all $i = 1, \dots, q-1$, as well as $\theta_q \cdot t_1 \xi_q = t_1 + \dots$, so (d) holds. For (e), we note that the ideal $((\mathcal{O}_{X,x}^{\mathfrak{t}_0})_{\overline{1}})$ is generated by $\xi_1, \dots, \xi_{q-1}$, and $A_\rho^{\theta_q^2} = \mathbb{C}[t_2^{\pm 1}, \dots, t_p^{\pm 1}, \xi_1, \dots, \xi_{q-1}]$. Thus the quotient is purely even, so we are finished.

Case 2, (a) $\iff$ (b) $\iff$ (c): We now assume $[V_{\rho,0}, V_{\rho,0}] = 0$. Suppose (a) holds, so we may write $\theta_j^{\text{right}} = \frac{\partial}{\partial \xi_j}$ for $j = r+1, \dots, s$, where $\mathfrak{h} = \mathbb{C}\{\theta_{q+1}, \dots, \theta_s\}$. By default, we have $L_{A_\rho}(\rho^\perp) = \mathbb{C}[T/H_{\rho,0}] = \mathbb{C}[t_2^{\pm 1}, \dots, t_p^{\pm 1}, \xi_1, \dots, \xi_r]$. Now for $W = \mathbb{C}\{\theta_{r+1}, \dots, \hat{\theta}_j, \dots, \theta_q\}$, we obtain $L_{\mathbb{C}[T]^W} = \mathbb{C}[t_1^{\pm 1}, \dots, t_p^{\pm 1}, \xi_1, \dots, \xi_r, \xi_j]$ and so (b) follows.

Conversely, if (b) holds, choose a basis $\xi_1, \dots, \xi_s$ of $\mathfrak{t}^*$ compatible with the chain $V_{\rho,0} \supseteq V_{\rho,1} \supseteq \dots$ in the sense that each $V_{\rho,i}$ is spanned by a subset of the dual basis. In particular, assume $\xi_1, \dots, \xi_r$ vanish on $V_{\rho,0}$, and $\xi_1, \dots, \xi_q$ vanish on $\mathfrak{h}$. Likewise, choose even coordinates $t_1, t_2^{\pm 1}, \dots, t_q^{\pm 1} \in \mathbb{C}[T]^{V_{\rho,0}}$, so that $\theta_i^{\text{right}} = \frac{\partial}{\partial \xi_i}$ for $i = r+1, \dots, q$ in terms of these elements. If $W \supseteq V_{\rho,n}$ is a codimension-1 subspace of $V_{\rho,0}$, then $\mathbb{C}[T]^W = \mathbb{C}[s_1^{\pm 1}, \dots, s_p^{\pm 1}, \xi_1, \dots, \xi_r, \xi_k]$ for ξ_k a functional vanishing on W but not $V_{\rho,0}$. It follows that $L_{\mathbb{C}[T]^W}(n + \rho^\perp)$ introduces the generator $s^n \xi_k$, so (a) is satisfied.

The equivalence of (b) and (c) is immediate.

Case 2, (a) $\iff$ (d) $\iff$ (e): First assume (a), so

$$A_\rho = \mathbb{C}[t_1, t_2^{\pm 1}, \dots, t_p^{\pm 1}, \xi_1, \dots, \xi_r, t_1^{\ell_{r+1}} \xi_{r+1}, \dots, t_1^{\ell_q} \xi_q]$$

and

$$\begin{aligned} \theta_j^{\text{right}} &= \frac{\partial}{\partial \xi_j} \\ \theta_j \cdot t^m &= \left(\sum_{i=1}^r \langle m, [\theta_i, \theta_j] \rangle \xi_i \right) t^m \\ \theta_j \cdot \xi_i &= -\delta_{ij} \end{aligned}$$

for $j = r+1, \dots, s$. Now consider the regular sequence

$$t_1, \xi_1, \dots, \xi_r, t_1^{\ell_{r+1}} \xi_{r+1}, \dots, t_1^{\ell_q} \xi_q.$$

We have

$$\begin{aligned} \theta_j \cdot t_1 &\in (t_1) \\ \theta_j \cdot \xi_i &= 0 & i = 1, \dots, r \\ \theta_j \cdot t_1^{\ell_i} \xi_i &\in (t_1^{\ell_i} \xi_i) & j \neq i > r \\ \overline{\theta_j \cdot t_1^{\ell_j} \xi_j} &\neq 0 \end{aligned}$$

for $j > r$, so indeed (d) holds. Likewise, we have $((\mathcal{O}_{X,x}^{\mathbf{t}_0})_{\overline{1}}) = (\xi_1, \dots, \xi_r)$, so checking (e) reduces to calculating the cohomology of $\frac{\partial}{\partial \xi_j}$ on

$$\mathbb{C}[t_1, t_2^{\pm 1}, \dots, t_p^{\pm 1}, t_1^{\ell_{r+1}} \xi_{r+1}, \dots, t_1^{\ell_q} \xi_q].$$

This cohomology is exactly

$$\mathbb{C}[t_1, t_2^{\pm 1}, \dots, t_p^{\pm 1}, t_1^{\ell_{r+1}} \xi_{r+1}, \dots, \widehat{t_1^{\ell_j} \xi_j}, \dots, t_1^{\ell_q} \xi_q] / (t_1^{\ell_j}),$$

which is certainly FR.

On the other hand, suppose (a) is not true, and write

$$A_\rho = \mathbb{C}[t_1, t_2^{\pm 1}, \dots, t_p^{\pm 1}, \xi_1, \dots, \xi_r, t_1^{\ell_{r+1}}(\xi_{r+1} + \dots), \dots, t_1^{\ell_q}(\xi_q + \dots)]$$

as in Lemma A.0.1. Assume without loss of generality that modulo $((A_\rho)_{\overline{1}}^2)$, the regular sequence $r_1, \dots, r_d$ is a permutation of the generators of A_ρ above. Now choose ℓ_k minimal such that the corresponding generator $t_1^{\ell_k}(\xi_k + \dots)$ contains a summand of the form ξ_J for $\sum_{j \in J} \ell_j > \ell_k$. Then for $j \in J \subseteq \{r+1, \dots, q\} \setminus \{k\}$, we have

$$\theta_j \cdot t_1^{\ell_k}(\xi_k + \dots) = \left(\sum_{i=1}^r \langle \ell_k, [\theta_i, \theta_j] \rangle \xi_i \right) t_1^{\ell_k}(\xi_k + \dots) - t_1^{\ell_k} \frac{\partial}{\partial \xi_j}(\dots)$$

wherein the first summand belongs to $(t_1^{\ell_j}(\xi_j + \dots))$ but the second does not. Hence (d) fails in this situation.

Finally, to see that (e) does not hold, we may reduce to the case in which $r = 0$ and $\theta_j = \frac{\partial}{\partial \xi_j}$. As above, let ℓ_k, J , and j be as above. Then the image of $\frac{\partial}{\partial \xi_j}$ contains $t_1^{\ell_k}(\xi_{J \setminus j} + \dots)$, meaning there is a relation in $DS_\theta \mathcal{O}_{X,x}$ among prior generators $t_1^{\ell_i} \xi_i$ for $i \in J \setminus \{j\}$. Hence the result is not FR, so we are finished. $\square$

Appendix B

Quasi-normality in one odd dimension

In this appendix we include the precursor [24] to the more robust theory of normal toric varieties above. By relaxing the normality criterion on toric supervarieties to quasi-normality, we obtain a wider zoo of toric supervarieties with one odd dimension. Henceforth we do not require toric varieties (or toric supervarieties) to be normal.

B.1 Alternate constructions of toric varieties

Recall that we write T_N for the algebraic torus with cocharacter lattice N and character lattice $M := N^*$.

Affine toric varieties

Given a finite set $\mathcal{A} = \{m_1, \dots, m_r\}$ in a lattice M' , one can define $N' = (M')^*$ and subsequently a morphism of varieties

$$\Phi_{\mathcal{A}} : T_{N'} \rightarrow \mathbb{C}^r$$

given by $\Phi_{\mathcal{A}}(t) = (t^{m_1}, \dots, t^{m_r})$. The Zariski closure of the image $\Phi_{\mathcal{A}}(T_{N'}) \subseteq \mathbb{C}^r$ becomes an affine toric variety $Y_{\mathcal{A}}$ with torus $\Phi_{\mathcal{A}}(T_{N'}) \cong T_{(\mathbb{Z}\mathcal{A})^*}$.

Such subvarieties of $\mathbb{C}^r$ may instead be characterized by their *toric ideals*, which are prime ideals of the form

$$(x^{\ell_+} - x^{\ell_-} \mid \ell_+, \ell_- \in \mathbb{N}^s \text{ and } \ell_+ - \ell_- \in L)$$

for some sublattice $L \subseteq \mathbb{Z}^r$. Equivalence with the prior construction may be established by viewing $\Phi_{\mathcal{A}}$ as a map $T_{N'} \rightarrow (\mathbb{C}^\times)^r$ and defining L as the kernel of the induced map $\bar{\Phi}_{\mathcal{A}}^* : \mathbb{Z}^r \rightarrow M$ on character lattices.

Equivalently, one may begin with an *affine semigroup*, i.e. a finitely generated unital semigroup S that can be embedded in a lattice. The semigroup algebra $\mathbb{C}[S]$ then becomes the coordinate ring of the toric variety $\text{Spec } \mathbb{C}[S]$ whose torus is $T_{(\mathbb{Z}S)^*}$. If $S = \mathbb{N}\mathcal{A}$, then $\text{Spec } \mathbb{C}[S] \cong Y_{\mathcal{A}}$.

Normal affine toric varieties

In order to obtain a normal affine toric variety $\text{Spec } \mathbb{C}[S]$, one requires that the semigroup S be *saturated*, meaning that if $km \in S$ for $m \in \mathbb{Z}S$ and $k \geq 0$, then $m \in S$.

Such semigroups can also be characterized in terms of cones. If σ is a cone, then the lattice points $\sigma^\vee \cap M$ of $\sigma^\vee$ form a saturated affine semigroup called S_σ . This construction leads to the classic characterization of normal affine toric varieties.

Proposition B.1.1. *Let X be an affine toric variety with torus T_N . Then the following are equivalent:*

- (a) X is normal.
- (b) $X \cong \text{Spec } \mathbb{C}[S]$ for a saturated affine semigroup $S \subseteq M = N^*$ such that $\mathbb{Z}S = M$.
- (c) $X \cong X_\sigma := \text{Spec } \mathbb{C}[S_\sigma]$ for a strongly convex rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$.

B.2 Supertori with one odd dimension

Supergroup structure

If T is a supertorus with $\dim T = (n|1)$, then $\mathfrak{t}$ is an $(n|1)$ -dimensional Lie superalgebra with $\mathfrak{t}_{\bar{0}} \subset Z(\mathfrak{t})$ and an odd generator θ such that $\theta^2 := \frac{1}{2}[\theta, \theta] \in \mathfrak{t}_{\bar{0}}$. An $(n|1)$ -dimensional supertorus is therefore parameterized by a point $c \in \mathfrak{t}_{\bar{0}} \cong N_{\mathbb{C}} := N \otimes_{\mathbb{Z}} \mathbb{C}$, where N is the cocharacter lattice of $T_{\bar{0}}$.

Definition B.2.1. Denote by T_N the ordinary algebraic torus with character lattice N , and $T_{N,c} = (T_N, \mathfrak{t})$ the supertorus with one odd dimension corresponding to the parameter $c = \theta^2 \in N_{\mathbb{C}}$.

Let $M = N^*$ be the character lattice of T_N . The supertorus $T_{N,c}$ will have coordinate ring $\mathbb{C}[T_{N,c}] = \mathbb{C}[M, \xi]$, with a Hopf superalgebra structure

$$\begin{aligned} \Delta x^m &= x^m \otimes x^m (1 - \langle m, c \rangle \xi \otimes \xi) & Sx^m &= x^{-m} & \varepsilon x^m &= 1 \\ \Delta \xi &= \xi \otimes 1 + 1 \otimes \xi & S\xi &= -\xi & \varepsilon \xi &= 0 \end{aligned}$$

where $\langle -, - \rangle$ is the perfect pairing $M \otimes N \rightarrow \mathbb{Z}$, extended $\mathbb{C}$ -linearly to $M \otimes N_{\mathbb{C}}$.

Fixing an identification $N \cong \mathbb{Z}^n$ so that $c = (c_1, \dots, c_n) \in \mathbb{C}^n$, write

$$T_{N,c}(\mathcal{A}) = \text{Hom}(\text{Spec } \mathcal{A}, T_{N,c})$$

for the functor of points of T evaluated on a superalgebra $\mathcal{A}$. Then the group law of $T_{N,c}(\mathcal{A})$ is given by

$$(t_1, \dots, t_n \mid \xi) \cdot (u_1, \dots, u_n \mid \eta) = (t_1 u_1 (1 - c_1 \xi \eta), \dots, t_n u_n (1 - c_n \xi \eta) \mid \xi + \eta)$$

for $t_i, u_i \in (\mathcal{A}_{\bar{0}})^\times$ and $\xi, \eta \in \mathcal{A}_1$.

Representation theory

The left action of $T_{N,c}$ on its coordinate ring $\mathbb{C}[T_{N,c}]$ differentiates to a left action by $\mathfrak{t}$, wherein the even part acts as usual (killing ξ) and the odd generator acts by $x^m \xi \mapsto -x^m$ and $x^m \mapsto \langle m, c \rangle x^m \xi$. Using the above identification $T_{\bar{0}} \cong \mathbb{Z}^n$, we may write $\mathfrak{t}$ as the subalgebra

$$\text{Span}_{\mathbb{C}} \left(-x_1 \frac{\partial}{\partial x_1}, \dots, -x_n \frac{\partial}{\partial x_n} \right) \oplus \text{Span}_{\mathbb{C}} \left(\xi \left(c_1 x_1 \frac{\partial}{\partial x_1} + \dots + c_n x_n \frac{\partial}{\partial x_n} \right) - \frac{\partial}{\partial \xi} \right)$$

of the Lie superalgebra of derivations on $\mathbb{C}[T_{N,c}] \cong \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, \xi]$. Thus, as a representation of $T_{N,c}$, $\mathbb{C}[T_{N,c}]$ splits into a direct sum of $(1|1)$ -dimensional representations $L(m) = \text{Span}_{\mathbb{C}}(x^m, x^m \xi)$ for $m \in M$.

Lemma B.2.2. *If $T = T_{N,c}$, then $L(m)$ is irreducible if and only if $\langle m, c \rangle \neq 0$. If $\langle m, c \rangle = 0$, then $L(m)$ is indecomposable but has a $(1|0)$ -dimensional subrepresentation $\langle x^m \rangle$.*

Proof. The even part preserves the span of each monomial, and the odd part acts as above. $\square$

Decomposability

Under an identification $T_N \cong (\mathbb{C}^\times)^n$ as above, we obtain $c = (c_1, \dots, c_n) \in \mathbb{C}^n$. Suppose there is a nontrivial $\mathbb{Q}$ -linearly dependency among the indices, say $q_1 c_1 + \dots + q_n c_n = 0$. We may assume $q_i \in \mathbb{Z}$, and that $\gcd(q_1, \dots, q_n) = 1$. Since $\text{GL}(T_{\mathbb{Z}^n}) \cong \text{GL}(n, \mathbb{Z})$, and since any n -tuple of integers with gcd 1 can be a row of an invertible integer matrix, we can choose another identification $T_N \cong (\mathbb{C}^\times)^n$ so that $c_n = 0$. Continuing this process inductively, it follows that we may assume the nonzero indices c_i are $\mathbb{Q}$ -linearly independent. That is,

$$T_{N,c} \cong T_{\mathbb{Z}^r, (c_1, \dots, c_r)} \times (\mathbb{C}^\times)^{n-r}$$

for rationally independent $c_1, \dots, c_r \in \mathbb{C}$.

Definition B.2.3. We say a supertorus T (of any dimension) is *indecomposable* if it cannot be written as a product $T' \times T''$ of two smaller supertori.

We have seen that if the entries of $c = (c_1, \dots, c_n) \in \mathbb{C}^n$ admit a $\mathbb{Q}$ -linear dependency, then $T_{\mathbb{Z}^n, c}$ is decomposable. The following proposition shows that this condition exactly determines indecomposability of a supertorus with one odd dimension.

Proposition B.2.4. *Let $T = T_{\mathbb{Z}^n, c}$ be a supertorus such that $c_1, \dots, c_n \in \mathbb{C}$ are $\mathbb{Q}$ -linearly independent. Then T is indecomposable.*

Proof. Suppose $T = T' \times T''$. Since T has only one odd dimension, we may assume without loss of generality that T'' is a purely even torus of nonzero dimension. Hence $T' = T_{\mathbb{Z}^r, c'}$ for some $c' \in \mathbb{C}^r$, and $T'' = (\mathbb{C}^\times)^{n-r}$. Then $c = (c'_1, \dots, c'_r, 0, \dots, 0)$ is a contradiction which proves the result. $\square$

Morphisms

We will need the following lemmas on morphisms between supertori.

The *image* of a morphism $\phi : G \rightarrow G'$ of supergroups is the supergroup whose $\mathcal{A}$ -points are $\phi(\mathcal{A})(G(\mathcal{A})) \subseteq G'(\mathcal{A})$. At the level of super Harish-Chandra pairs, the image of $(\phi, d\phi) : (G_{\bar{0}}, \mathfrak{g}) \rightarrow (G'_{\bar{0}}, \mathfrak{g}')$ is $(\phi(G_{\bar{0}}), d\phi(\mathfrak{g}))$.

Lemma B.2.5. *Let $\phi : T_{N,c} \rightarrow T_{N',c'}$ be a morphism of supervarieties. Then ϕ is a morphism of supergroups if and only if all of the following conditions hold.*

- (i) *There is a lattice map $\bar{\phi} : N \rightarrow N'$ such that $\phi|_{T_N} = \bar{\phi} \otimes 1_{\mathbb{C}^\times}$.*
- (ii) *The dual map $\phi^* : \mathbb{C}[M', \xi'] \rightarrow \mathbb{C}[M, \xi]$ satisfies $\phi^*(\xi') = a\xi$ for some $a \in \mathbb{C}$.*
- (iii) *$\bar{\phi}_{\mathbb{C}} = \bar{\phi} \otimes 1_{\mathbb{C}} : N_{\mathbb{C}} \rightarrow N'_{\mathbb{C}}$ satisfies $\bar{\phi}(c) = a^2 c'$.*

Proof. Condition (i) is required as in Lemma 2.1.2. In order for ϕ to be a morphism of supergroups, we require ϕ^* to be a morphism of Hopf superalgebras. In particular, we require $(\phi^* \otimes \phi^*) \circ \Delta' = \Delta \circ \phi^*$. Since each of these maps is a morphism of superalgebras, it suffices to check the identity on the generators $x^{m'}$ and ξ' . First, observe that

$$\begin{aligned}
 (\phi^* \otimes \phi^*) \circ \Delta'(\xi') &= (\phi^* \otimes \phi^*)(\xi' \otimes 1 + 1 \otimes \xi') \\
 &= \phi^*(\xi') \otimes 1 + 1 \otimes \phi^*(\xi') \\
 &= \left(\sum_{i=1}^{\ell} a_i x^{k_i} \xi \right) \otimes 1 + 1 \otimes \left(\sum_{i=1}^{\ell} a_i x^{k_i} \xi \right) \\
 \Delta \circ \phi^*(\xi') &= \Delta(\phi^*(\xi')) \\
 &= \sum_{i=1}^{\ell} a_i \Delta(x^{k_i} \xi) \\
 &= \sum_{i=1}^{\ell} a_i x^{k_i} \otimes x^{k_i} (\xi \otimes 1 + 1 \otimes \xi)
 \end{aligned}$$

where we write $\phi^*(\xi') = \sum_{i=1}^{\ell} a_i x^{k_i} \xi$ for some $a_i \in \mathbb{C}$ and $k_i \in M$. Since the two sides are equal, it follows that all $k_i = 0$ and hence $\phi^*(\xi') = a\xi$ for some $a \in \mathbb{C}$. Moreover, for $m' \in M$, we have

$$\begin{aligned}
 (\phi^* \otimes \phi^*) \circ \Delta'(x^{m'}) &= (\phi^* \otimes \phi^*)(x^{m'} \otimes x^{m'}(1 - \langle m', c' \rangle) \xi' \otimes \xi') \\
 &= x^{\bar{\phi}^*(m')} \otimes x^{\bar{\phi}^*(m')}(1 - \langle m', c' \rangle \phi^*(\xi') \otimes \phi^*(\xi')) \\
 \Delta \circ \phi^*(x^{m'}) &= \Delta(x^{\bar{\phi}^*(m')}) \\
 &= x^{\bar{\phi}^*(m')} \otimes x^{\bar{\phi}^*(m')}(1 - \langle \bar{\phi}^*(m'), c \rangle \xi \otimes \xi)
 \end{aligned}$$

so that $a^2 \langle m', c' \rangle = \langle \bar{\phi}^*(m'), c \rangle$ for all $m' \in M'$. Since $\langle -, - \rangle$ is a nondegenerate perfect pairing, we know $\langle \bar{\phi}^*(m'), c \rangle = \langle m', \bar{\phi}_{\mathbb{C}}(c) \rangle$ and hence $\bar{\phi}_{\mathbb{C}}(c) = a^2 c'$. The other axioms of a Hopf algebra morphism are routine to check, and yield no additional requirements. $\square$

The moduli space

One can verify that isomorphism classes of $(n|1)$ -dimensional supertori for $c \neq 0$ depend only on the line through $c \in N_{\mathbb{C}}$, up to automorphisms $\text{GL}(T_N)$ of the torus T_N . That is, the moduli space of $(n|1)$ -dimensional supertori is the disjoint union of $\mathbb{P}(N_{\mathbb{C}})/\text{GL}(T_N) \cong \mathbb{P}^{n-1}/\text{GL}(n, \mathbb{Z})$ and a point (representing the $c = 0$ case).

B.3 Affine toric supervarieties

We will begin by assuming X is affine, which by Lemma 3.6.2 is equivalent to the assumption that $X_{\bar{0}}$ is affine.

From a finite set in a monoid

Given a supertorus $T' = T_{N', c'}$, let $M' = (N')^*$ be the character lattice of $T'_0 = T_{N'}$ and $\mathcal{A} = \{m_1, \dots, m_r\}$ a finite set in M' . We wish to construct a super-analog of the variety $Y_{\mathcal{A}}$ described in section B.1. Define

$$\begin{aligned}\mathcal{A}' &= \mathcal{B}' = \{m_i \in \mathcal{A} \mid \langle m_i, c' \rangle \neq 0\} = \{m_1, \dots, m_q\} \\ \mathcal{A}'' &= \{m_i \in \mathcal{A} \mid \langle m_i, c' \rangle = 0\} = \{m_{q+1}, \dots, m_r\}\end{aligned}$$

so that $\mathcal{A} = \mathcal{A}' \sqcup \mathcal{A}''$. Moreover, let $\mathcal{B}'' = \{n_{q+1}, \dots, n_s\}$ be a finite set in the monoid $\mathbb{N}^{\mathcal{A}''}$, and set $\mathcal{B} = \mathcal{B}' \sqcup \mathcal{B}''$. (We assume henceforth that $\mathcal{B}$ is nonempty so that we do not degenerate to the classical, purely-even case.) We will write $n_j = m_j$ for $j = 1, \dots, q$ so that $\mathcal{B} = \{n_1, \dots, n_s\}$, and $n_j = \sum_{i=q+1}^r p_{i,j} m_i$ for $j = q+1, \dots, s$ for some $p_{i,j} \in \mathbb{N}$. We can do the same for $j = 1, \dots, q$, in which case we have $p_{i,j} = 1$ if $i = j$ and $p_{i,j} = 0$ otherwise.

Now define a morphism of supervarieties

$$\Phi_{\mathcal{A}, \mathcal{B}} : T' \rightarrow \mathbb{C}^{r|s}$$

at the level of $\mathcal{A}$ -points (for $\mathcal{A}$ a superalgebra) by

$$\Phi_{\mathcal{A}, \mathcal{B}}(t \mid \xi) = (t^{m_1}, \dots, t^{m_r} \mid t^{n_1} \xi, \dots, t^{n_s} \xi) \in \mathbb{C}^{r|s}(\mathcal{A})$$

for $t \in ((\mathcal{A}_{\bar{0}})^\times)^{\dim N'}$ and $\xi \in \mathcal{A}_{\bar{1}}$. Denote by $J \subseteq \mathbb{C}[x_1, \dots, x_r, \xi_1, \dots, \xi_s]$ the ideal of functions vanishing on the image of $\Phi_{\mathcal{A}, \mathcal{B}}$. Then the Zariski closure of the image is defined as the closed subvariety $V(J) := \text{Spec } \mathbb{C}[x_1, \dots, x_r, \xi_1, \dots, \xi_s]/J$ of $\mathbb{C}^{r|s}$, henceforth referred to as $Y_{\mathcal{A}, \mathcal{B}}$.

Proposition B.3.1. *Let $\mathcal{A}, \mathcal{B} \subseteq M$ be as above. Then*

- (a) *The image of $\Phi_{\mathcal{A}, \mathcal{B}}$ (on $\mathcal{A}$ -points) is contained in a supertorus in such a way that $\Phi_{\mathcal{A}, \mathcal{B}}$ becomes a morphism of supergroups.*
- (b) *$Y_{\mathcal{A}, \mathcal{B}}$ is an affine toric supervariety whose supertorus is $\Phi_{\mathcal{A}, \mathcal{B}}(T') = T_{(\mathbb{Z}_{\mathcal{A}})^*, (\bar{\Phi}_{\mathcal{A}, \mathcal{B}})_{\mathbb{C}}(c')}$, where $(\bar{\Phi}_{\mathcal{A}, \mathcal{B}})_{\mathbb{C}}$ is obtained from $\Phi_{\mathcal{A}, \mathcal{B}}$ as in Lemma B.2.5(iii)*

Proof. Let $T'' = (\mathbb{C}^\times)^r \times \mathbb{C}^{0|s} \subset \mathbb{C}^{r|s}$ as a supervariety without any assumption of a supergroup structure. We can view $\Phi_{\mathcal{A}, \mathcal{B}}$ as a map of supergroups $T' \rightarrow T''$ in the following way: The map $\Phi_{\mathcal{A}, \mathcal{B}}(\mathcal{A})$ to $\mathbb{C}^{r|s}(\mathcal{A})$ factors through $T''(\mathcal{A})$, so we may abuse notation and write $\Phi_{\mathcal{A}, \mathcal{B}} : T' \rightarrow T''$. Now it remains to impose a supertorus structure on T'' , so let $\mathcal{A}$ be a superalgebra and let $t, u \in ((\mathcal{A}_{\bar{0}})^\times)^{\dim N'}$ and $\xi, \eta \in \mathcal{A}_{\bar{1}}$. If $\Phi_{\mathcal{A}, \mathcal{B}}$ is to be a group

homomorphism, we must have

$$\begin{aligned}
& (t^{m_1}, \dots, t^{m_r} \mid t^{n_1}\xi, \dots, t^{n_s}\xi) \cdot (u^{m_1}, \dots, u^{m_r} \mid u^{n_1}\eta, \dots, u^{n_s}\eta) \\
&= \Phi_{\mathcal{A}, \mathcal{B}}(t \mid \xi) \cdot \Phi_{\mathcal{A}, \mathcal{B}}(u \mid \eta) \\
&= \Phi_{\mathcal{A}, \mathcal{B}}((t \mid \xi) \cdot (u \mid \eta)) \\
&= \Phi_{\mathcal{A}, \mathcal{B}}(t_1 u_1 (1 - c'_1 \xi \eta), \dots, t_n u_n (1 - c'_n \xi \eta) \mid \xi + \eta) \\
&= (t^{m_1} u^{m_1} (1 - \langle m_1, c' \rangle \xi \eta), \dots, t^{m_r} u^{m_r} (1 - \langle m_r, c' \rangle \xi \eta) \mid t^{n_1} u^{n_1} (\xi + \eta), \dots, t^{n_s} u^{n_s} (\xi + \eta))
\end{aligned}$$

where we identify T'_0 with $(\mathbb{C}^\times)^n$ and M' with $\mathbb{Z}^n$. Thus, a potential group law on $T''(\mathcal{A})$ is

$$(a_1, \dots, a_r \mid \alpha_1, \dots, \alpha_s) \cdot (b_1, \dots, b_r \mid \beta_1, \dots, \beta_s) = (d_1, \dots, d_r \mid \delta_1, \dots, \delta_s)$$

where

$$\begin{aligned}
d_i &= a_i b_i - \langle m_i, c' \rangle \alpha_i \beta_i && \text{for } i = 1, \dots, q \\
d_i &= a_i b_i && \text{for } i = q + 1, \dots, r \\
\delta_j &= b^{p \cdot j} \alpha_j + a^{p \cdot j} \beta_j && \text{for } j = 1, \dots, s.
\end{aligned}$$

and where we write $b^{p \cdot j}$ and $a^{p \cdot j}$ for $b_1^{p_{1,j}} \dots b_r^{p_{r,j}}$ and $a_1^{p_{1,j}} \dots a_r^{p_{r,j}}$, respectively. For $j = 1, \dots, q$, the latter equation simplifies to

$$\delta_j = b_j \alpha_j + a_j \beta_j.$$

It is routine to check that this operation makes $T''(\mathcal{A})$ into a group with identity element $(1, \dots, 1 \mid 0, \dots, 0)$, and moreover T'' into an $(r|s)$ -dimensional supertorus which decomposes as $Q(1)^q \times T'''$ where T''' is the $(r - q|s - q)$ -dimensional abelian supertorus.

It follows by Lemma 5.1.2 that $\Phi_{\mathcal{A}, \mathcal{B}}(T')$ is a supertorus closed in T'' , so $Y_{\mathcal{A}, \mathcal{B}} \cap T'' = \Phi_{\mathcal{A}, \mathcal{B}}(T')$ because $Y_{\mathcal{A}, \mathcal{B}}$ is the closure of $\Phi_{\mathcal{A}, \mathcal{B}}(T')$. As in the classical setting, $\Phi_{\mathcal{A}, \mathcal{B}}(T')$ is open in $Y_{\mathcal{A}, \mathcal{B}}$ and irreducible, so $Y_{\mathcal{A}, \mathcal{B}}$ is also irreducible.

We now verify that $\Phi_{\mathcal{A}, \mathcal{B}}(T')$ acts on $Y_{\mathcal{A}, \mathcal{B}}$ via the functor of points. Since $\Phi_{\mathcal{A}, \mathcal{B}}(T') \subset T''$, the action is by homeomorphisms and so takes supervarieties to supervarieties. For a superalgebra $\mathcal{A}$, it follows that $\Phi_{\mathcal{A}, \mathcal{B}}(T')(\mathcal{A}) = (t, \xi) \cdot \Phi_{\mathcal{A}, \mathcal{B}}(T')(\mathcal{A}) \subseteq (t, \xi) \cdot Y_{\mathcal{A}, \mathcal{B}}(\mathcal{A})$ for $t \in (\mathcal{A}_0^*)^{\dim N'}$, $\xi \in \mathcal{A}_1$. Hence $(t, \xi) \cdot Y_{\mathcal{A}, \mathcal{B}}(\mathcal{A})$ is the functor of points of a supervariety containing $\Phi_{\mathcal{A}, \mathcal{B}}(T')$, meaning $Y_{\mathcal{A}, \mathcal{B}}(\mathcal{A}) \subseteq (t, \xi) \cdot Y_{\mathcal{A}, \mathcal{B}}(\mathcal{A})$ by definition of the Zariski closure. Replacing (t, ξ) with its inverse $(t^{-1}, -\xi)$ yields the reverse inclusion, so in fact $Y_{\mathcal{A}, \mathcal{B}}(\mathcal{A}) = (t, \xi) Y_{\mathcal{A}, \mathcal{B}}(\mathcal{A})$. Therefore $Y_{\mathcal{A}, \mathcal{B}}$ is an affine toric supervariety with torus $\Phi_{\mathcal{A}, \mathcal{B}}(T')$.

As in the usual setting, the even part $\Phi_{\mathcal{A}, \mathcal{B}}(T')_{\bar{0}}$ of the torus will have character lattice $\mathbb{Z}\mathcal{A}$. Since we assume $s > 0$, we also have one dimension of odd functions (generated by ξ). Viewing $\Phi_{\mathcal{A}, \mathcal{B}}$ as a morphism $T' \rightarrow \Phi_{\mathcal{A}, \mathcal{B}}(T')$, we see that $\Phi_{\mathcal{A}, \mathcal{B}}^*(\xi) = \xi$ and hence the additional parameter c is $\bar{\Phi}_{\mathcal{A}, \mathcal{B}}(c')$ by Lemma B.2.5. $\square$

The ideal of an affine toric supervariety

As in the usual case, we are interested in describing the ideal $J(Y_{\mathcal{A}, \mathcal{B}})$ of functions that vanish on $Y_{\mathcal{A}, \mathcal{B}}$. The map $\Phi_{\mathcal{A}, \mathcal{B}}$ can be viewed as a map to an $(r|s)$ -dimensional torus, inducing a morphism of lattices

$$\hat{\Phi}_{\mathcal{A}, \mathcal{B}} : \mathbb{Z}^{r|s} \rightarrow M' \oplus \mathbb{Z}^{0|1}$$

mapping the coordinate vectors of the domain as

$$\begin{aligned} e_i &\mapsto (m_i \mid 0) & i = 1, \dots, r \\ e'_j &\mapsto (n_j \mid 1) & j = 1, \dots, s. \end{aligned}$$

The kernel L of this morphism records the relations among the m_i and n_j , so if $(\ell \mid \ell') \in L$, then $\sum_{i=1}^r \ell_i (m_i \mid 0) + \sum_{j=1}^s \ell'_j (n_j \mid 1) = 0$. Notice that $\sum_{j=1}^s \ell'_j = 0$ due to the “odd” coordinate.

For $\ell = (\ell_1, \dots, \ell_r \mid \ell'_1, \dots, \ell'_s) \in L$, set

$$\ell_+ = \sum_{\ell_i > 0} \ell_i e_i \quad \ell_- = - \sum_{\ell_i < 0} \ell_i e_i \quad \ell'_+ = \sum_{\ell'_j > 0} \ell'_j e'_j \quad \ell'_- = - \sum_{\ell'_j < 0} \ell'_j e'_j$$

so that $\ell = (\ell_+ - \ell_-) + (\ell'_+ - \ell'_-)$ and $\ell_{\pm} \in \mathbb{N}^r, \ell'_{\pm} \in \mathbb{N}^s$. It follows immediately that the binomial

$$x^{\ell_+} \xi^{\ell'_+} - x^{\ell_-} \xi^{\ell'_-}$$

vanishes on $\Phi_{\mathcal{A}, \mathcal{B}}(T')$ and hence also on its closure $Y_{\mathcal{A}, \mathcal{B}}$. Since each ξ_i is sent to a multiple of ξ , the monomials $\xi_i \xi_j$ also vanish on $Y_{\mathcal{A}, \mathcal{B}}$. Thus, it is sufficient to consider only those binomials which contain either no ξ_j or exactly one in each term.

Proposition B.3.2. *The ideal of the affine toric supervariety $Y_{\mathcal{A}, \mathcal{B}} \subseteq \mathbb{C}^{r|s}$ is*

$$\begin{aligned} J(Y_{\mathcal{A}, \mathcal{B}}) &= (x^{\ell_+} \xi^{\ell'_+} - x^{\ell_-} \xi^{\ell'_-} \mid \ell \in L) + (\xi_i \xi_j) \\ &= (x^{k_+} \xi^{k'_+} - x^{k_-} \xi^{k'_-} \mid k_{\pm} \in \mathbb{N}^r, k'_{\pm} \in \mathbb{N}^s \text{ and } k_+ - k_- + k'_+ - k'_- \in L) + (\xi_i \xi_j) \end{aligned}$$

Proof. The proof is largely the same as in the classical case: Firstly, the two ideals on the right are equal by an identical proof, and we let $J_L \subseteq J(Y_{\mathcal{A}, \mathcal{B}})$ denote this ideal. Fixing identifications $T'_0 \cong (\mathbb{C}^\times)^n$ and $M' \cong \mathbb{Z}^n$, the map $\Phi_{\mathcal{A}, \mathcal{B}} : T' \rightarrow \mathbb{C}^{r|s}$ is given by monomials t^{m_i} for $i = 1, \dots, r$ and $t^{n_j} \xi$ for $j = 1, \dots, s$ in the variables $t_1, \dots, t_n, \xi$.

Now pick a monomial order $\geq$ on $\mathbb{C}[x_1, \dots, x_r, \xi_1, \dots, \xi_s]$, say lexicographic order with $x_1 > \dots > x_r > \xi_1 > \dots > \xi_s$. While this is not precisely a monomial order in the traditional sense due to the presence of odd variables, we will not invoke any properties that do not hold in this situation. If $J_L \neq J(Y_{\mathcal{A}, \mathcal{B}})$, we can pick $f \in J(Y_{\mathcal{A}, \mathcal{B}}) \setminus J_L$ with minimal leading monomial $x^a = \prod_{i=1}^r x_i^{a_i}$ or $x^a \xi_j = (\prod_{i=1}^r x_i^{a_i}) \xi_j$. We may assume without loss of generality that f has leading coefficient 1, and that f contains no summands in $(\xi_i \xi_j)$.

Since $f(t^{m_1}, \dots, t^{m_r}, t^{n_1} \xi, \dots, t^{n_s} \xi) \equiv 0$ as a polynomial in $t_1, \dots, t_n, \xi$, and since f contains no summands in $(\xi_i \xi_j)$, there must be cancellation involving the leading term. Hence $f(x_1, \dots, x_r, \xi_1, \dots, \xi_s)$ contains a monomial $x^b < x^a$ or $x^b \xi_k < x^a \xi_j$, respectively, such that

$$\prod_{i=1}^r (t^{m_i})^{a_i} = \prod_{i=1}^r (t^{m_i})^{b_i} \quad \text{or} \quad \left(\prod_{i=1}^r (t^{m_i})^{a_i} \right) t^{n_j} \xi = \left(\prod_{i=1}^r (t^{m_i})^{b_i} \right) t^{n_k} \xi,$$

implying

$$\sum_{i=1}^r a_i m_i = \sum_{i=1}^r b_i m_i \quad \text{or} \quad \sum_{i=1}^r a_i m_i + n_j = \sum_{i=1}^r b_i m_i + n_k.$$

Therefore $a - b \in L$ or $(a + e'_j) - (b + e'_k) \in L$, so by the second description of J_L it follows that $f - x^a + x^b$ or $f - x^a \xi_j + x^b \xi_k$ also lies in $J(Y_{\mathcal{A}, \mathcal{B}}) \setminus J_L$ but has strictly smaller leading term. With this contradiction, we are finished. $\square$

Remark B.3.3. In the classical setting, there is a nice classification of affine toric varieties via toric ideals (i.e. prime ideals generated by binomials arising from a lattice as above). A super-analog of this result is possible but rather less interesting, since it fails to describe the torus action on the corresponding supervariety, and in particular the parameter $c \in N_{\mathbb{C}}$.

From an affine semigroup

Recall from section B.1 that an affine toric variety is isomorphic to $\text{Spec } \mathbb{C}[S]$ for some affine semigroup S , and every $\text{Spec } \mathbb{C}[S]$ has the structure of an affine toric variety. The generalization to the super case with one odd dimension requires the same data with one minor addition.

Let T_N be the ordinary torus with character lattice $N^* = M = \mathbb{Z}S$. For $c \in N_{\mathbb{C}}$, let J_c be the T_N -invariant ideal $(S - \ker c) = (x^m \in \mathbb{C}[S] \mid \langle m, c \rangle \neq 0)$ of $\mathbb{C}[S]$, where $S - \ker c$ denotes the set difference. Note that $J_c \neq 0$ unless $c = 0$, and that J_c is the ideal of the vanishing subscheme of c , viewed as a vector field on the toric variety $\text{Spec } \mathbb{C}[S]$.

Lemma B.3.4. (a) *Let S be an affine semigroup, and let $J \subseteq \mathbb{C}[S]$ be a nonzero T_N -invariant ideal that contains J_c . Then $\text{Spec}(\mathbb{C}[S] \oplus \xi J)$ is an affine toric supervariety whose torus is $T_{N,c}$.*

(b) *The ideal J has a finite generating set consisting of monic monomials. Moreover, there is a minimal generating set of this type, unique up to multiplication by invertible monomials in $\mathbb{C}[S]$. In particular, if S is a pointed semigroup (i.e. the only unit is 1), then there is a unique minimal generating set.*

(c) *Suppose $S = \mathbb{N}\mathcal{A}$ for a finite set $\mathcal{A}$ in a lattice M' ($\supset M = \mathbb{Z}S$) such that $\mathcal{A} = \mathcal{A}' \sqcup \mathcal{A}''$ as above where $\mathcal{A}'' = \{m \in \mathcal{A} \mid \langle m, c \rangle_{M \times N_{\mathbb{C}}} = 0\}$. Let $\mathcal{B}''$ be a finite set in $\mathbb{N}\mathcal{A}''$ such that $J = (x^{n_j} \mid n_j \in \mathcal{A}' \sqcup \mathcal{B}'')$. Then $\text{Spec}(\mathbb{C}[S] \oplus \xi J) \cong Y_{\mathcal{A}, \mathcal{B}}$.*

Proof. (a) Follows from (c) and Proposition B.3.1.

(b) Since J is T_N -invariant, it has a generating set by monic monomials. For finiteness, notice that $\mathbb{C}[S]$ is a quotient of some $\mathbb{C}[y_1, \dots, y_k]$, which is Noetherian. It therefore remains to prove uniqueness.

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two generating sets by monic monomials, and let $x^{m_1} \in \mathcal{S}_1 \setminus \mathcal{S}_2$. Since $x^{m_1} \in J$, we must have $x^{m_1} = ax^{m_2}$ for some monomial $a \in \mathbb{C}[S]$ and $m_2 \in \mathcal{S}_2$. Since $x^{m_2} \in J$, we must have $x^{m_2} = bx^{m_3}$ for some monomial $b \in \mathbb{C}[S]$ and $m_3 \in \mathcal{S}_1$. Hence $x^{m_1} = abx^{m_3}$ and so either $m_1 = m_3$ or x^{m_1} is redundant in $\mathcal{S}_1$. In the event that $m_1 = m_3$, it holds that $a = b^{-1}$, so $x^{m_1} \in \mathcal{S}_1$ and $x^{m_2} \in \mathcal{S}_2$ differ only by an invertible monomial.

- (c) We first observe that the defining condition $\langle m, c \rangle_{M \times N_{\mathbb{C}}} = 0$ of $\mathcal{A}''$ is equivalent to $\langle m, c' \rangle_{M' \times N'_{\mathbb{C}}} = 0$ because $M = \mathbb{Z}\mathcal{A}$ is a sublattice of M' via $\bar{\phi}^* : M \rightarrow M'$ and likewise N is a quotient of N' via $\bar{\phi} : N' \rightarrow N$.

The morphism $\Phi_{\mathcal{A}, \mathcal{B}} : T_{(M')^*, c'} \rightarrow \mathbb{C}^{r|s}$ corresponds to a map

$$\pi : \mathbb{C}[x_1, \dots, x_r, \xi_1, \dots, \xi_s] \rightarrow \mathbb{C}[M', \xi]$$

given by $x_i \mapsto x^{m_i}$ and $\xi_j \mapsto x^{n_j} \xi$. The image of π is exactly $\mathbb{C}[S] \oplus \xi J$ because J is the ideal generated by the x^{n_j} . Hence, if we can show that $\ker \pi$ is the ideal $J(Y_{\mathcal{A}, \mathcal{B}})$, then

$$\begin{aligned} \mathbb{C}[S] \oplus \xi J &= \text{Im } \pi \\ &\cong \mathbb{C}[x_1, \dots, x_r, \xi_1, \dots, \xi_s] / \ker \pi \\ &= \mathbb{C}[x_1, \dots, x_r, \xi_1, \dots, \xi_s] / J(Y_{\mathcal{A}, \mathcal{B}}) \\ &= \mathbb{C}[Y_{\mathcal{A}, \mathcal{B}}] \end{aligned}$$

and so (c) will hold. But this follows by Proposition B.3.2. $\square$

Just as J_c is the ideal of the vanishing subscheme of c , the torus-invariant ideals J that contain J_c correspond bijectively to the toric subschemes of $\text{Spec } \mathbb{C}[S]$ which are fixed by c . Equivalently, they admit the trivial action by the smallest subtorus of T_N whose Lie algebra contains c ; this is the even part of the nontrivial supertorus in the decomposition of section B.2.

Example B.3.5. If $S = \mathbb{N}$ and $c = 1$, then we obtain the supertorus $T_{\mathbb{Z}, 1} \cong Q(1)$ and the ordinary toric variety $\text{Spec } \mathbb{C}[\mathbb{N}] \cong \text{Spec } \mathbb{C}[x] \cong \mathbb{A}^1$. We observe that $J_c = (x)$, so $\text{Spec } \mathbb{C}[x, x\xi]$ and $\text{Spec } \mathbb{C}[x, \xi]$ are the two toric supervarieties with torus $Q(1)$, corresponding to the toric subschemes $\{0\} = \text{Spec } \mathbb{C}[x]/(x)$ and $\emptyset = \text{Spec } \mathbb{C}[x]/(1)$ of $\mathbb{A}^1$. While these two spaces are isomorphic as supervarieties, they admit different torus actions and are in fact not $Q(1)$ -equivariantly isomorphic.

The datum of the ideal J is equivalent to the datum of its minimal generating set by monomials, which can be characterized as follows.

Definition B.3.6. Let S be an affine semigroup and $T_{(\mathbb{Z}S)^*, c}$ a supertorus. For $m, m' \in S$, we write $m' \leq_S m$ if there exists $m'' \in S$ such that $m' + m'' = m$. A nonempty finite set $\mathcal{B} \subset S$ is *c-admissible* if:

- (i) For all $m \in S - \ker c$, there is $b \in \mathcal{B}$ such that $b \leq_S m$.

It is moreover *minimal* if:

- (ii) If $a, b \in \mathcal{B}$ satisfy $a \leq_S b$, then $a = b$.

Lemma B.3.7. A nonempty finite set $\mathcal{B} \subset S$ is (minimal) *c-admissible* if and only if it is a (minimal) generating set for a nonzero ideal $J \subseteq \mathbb{C}[S]$ containing J_c .

Proof. Condition (1) is equivalent to the ideal generated containing J_c , and condition (2) is equivalent to minimality. $\square$

Proposition B.3.8. *Let X be an affine supervariety. Then the following are equivalent:*

- (a) X is an affine toric supervariety whose torus is $T = T_{N,c}$.
- (b) $X = Y_{\mathcal{A},\mathcal{B}}$ for a finite set $\mathcal{A} = \mathcal{A}' \sqcup \mathcal{A}''$ in a lattice and $\mathcal{B} = \mathcal{A}' \sqcup \mathcal{B}''$ nonempty where $\mathcal{B}''$ is a finite set in $\mathbb{N}\mathcal{A}''$ and $\mathcal{A}'' = \{m \in \mathcal{A} \mid \langle m, c \rangle = 0\}$.
- (c) $X = \text{Spec}(\mathbb{C}[S] \oplus \xi J)$ for an affine semigroup S and a nonzero T_N -invariant ideal $J \subseteq \mathbb{C}[S]$ that contains J_c .
- (d) $X = \text{Spec}(\mathbb{C}[S] \oplus \xi J)$ for an affine semigroup S and the ideal J of a toric subscheme of $\text{Spec } \mathbb{C}[S]$ which is fixed by c .
- (e) $X = \text{Spec } \mathbb{C}[S, \xi \mathcal{B}]$ for an affine semigroup S and an c -admissible set $\mathcal{B} \subset S$.

Proof. We have already seen that (e) $\iff$ (d) $\iff$ (c) $\iff$ (b) $\implies$ (a). Now suppose (a) holds, so X is an affine toric supervariety containing the supertorus T . The inclusion $T \hookrightarrow X$ induces a map of coordinate rings

$$\mathbb{C}[X] \rightarrow \mathbb{C}[T]$$

which is injective by Lemma 3.6.4. Thus we may view $\mathbb{C}[X]$ as a subalgebra of $\mathbb{C}[T]$ and in particular $\mathbb{C}[X_{\bar{0}}]$ as a subalgebra of $\mathbb{C}[T_{\bar{0}}]$. From our knowledge of the usual case, this implies $\mathbb{C}[X_{\bar{0}}] = \mathbb{C}[X]_{\bar{0}} = \mathbb{C}[S]$ for an affine semigroup S .

Since T acts on X , it also acts on $\mathbb{C}[X]$, so that $\mathbb{C}[X]$ is a subrepresentation of $\mathbb{C}[T]$. For each monomial $x^m \xi \in \mathbb{C}[X]$, it holds that $x^m \in \mathbb{C}[X]$ as well, so $\mathbb{C}[X]_{\bar{1}} = \xi J$ for some T_N -invariant ideal $J \subset \mathbb{C}[S]$. If $L(m)$ is irreducible (i.e. if $\langle m, c \rangle \neq 0$), then $x^m \in \mathbb{C}[X]$ implies $x^m \xi \in \mathbb{C}[X]$, meaning J must contain J_c . Therefore (c) follows from (a). $\square$

Remark B.3.9. The c -admissible set $\mathcal{B}$ can be taken as the finite set $\mathcal{B}$ occurring in $Y_{\mathcal{A},\mathcal{B}}$, but it is not in general minimal: Consider the supertorus $Q(1) \cong T_{\mathbb{Z},1} \cong \text{Spec } \mathbb{C}[x^{\pm 1}, \xi]$ as a toric supervariety containing itself. Here, we may take $S = \mathbb{Z}$, so that $X \cong Y_{\mathcal{A},\mathcal{B}}$ for $\mathcal{A} = \mathcal{B} = \{\pm 1\}$ because $\mathcal{B}$ must contain $\mathcal{A} = \mathcal{A} - \ker c$. However, such $\mathcal{B}$ cannot be minimal, since 1 and -1 generate the same ideal of $\mathbb{C}[x^{\pm 1}]$.

Example B.3.10. For every affine toric variety $X_{\bar{0}} = \text{Spec } \mathbb{C}[S] \supset T_N$ and supertorus $T_{N,c} \supset T_N$, there exists a *trivial* toric supervariety structure given by $J = \mathbb{C}[S]$.

Example B.3.11. Suppose $T = T_{N,c}$ is an indecomposable supertorus. Then for a given affine toric variety $X_{\bar{0}} = \text{Spec } \mathbb{C}[S]$ with torus T_N , J_c is generated by all non-constant monomials in $\mathbb{C}[S]$. If S is not pointed (i.e. it contains two distinct elements which are inverses), then $J_c = (1)$ and the only toric supervariety whose underlying variety is $X_{\bar{0}}$ and whose torus is T , is the trivial one.

If S is pointed, then J_c is a maximal ideal, so there is a unique nontrivial toric supervariety structure, in which $J = J_c$.

Some further computations

We have seen from Proposition B.3.8 that an affine toric supervariety $X = \operatorname{Spec} \mathcal{A} \supset T_{N,c}$ with one odd dimension consists of the data of an ordinary affine toric variety $X_{\bar{0}} = \operatorname{Spec} \mathcal{A}_{\bar{0}} \supset T_N$ together with the additional data of a nonzero T_N -invariant ideal $J \subset \mathcal{A}_{\bar{0}}$ that contains J_c . In this case, it holds that $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ where $\mathcal{A}_{\bar{1}} = \xi J$. In this section, we provide some further information about the *fermionic ideal* J .

Corollary B.3.12. *Let $X_{\bar{0}} = \operatorname{Spec} \mathcal{A}_{\bar{0}}$ be an affine toric variety with torus $T_{\bar{0}} = T_N$. Affine toric supervarieties with underlying variety $X_{\bar{0}}$ correspond to nonzero $\mathcal{A}_{\bar{0}} \times T_{\bar{0}}$ -bimodules J such that $J_c \subseteq J \subseteq \mathcal{A}_{\bar{0}}$. Equivalently if $c \neq 0$, they correspond to T_N -invariant submodules (or quotients) of $\mathcal{A}_{\bar{0}}/J_c$.*

Example B.3.13. Let $T = T_{\mathbb{Z}^2, (1,0)} \cong Q(1) \times \mathbb{C}^\times$ and $X_{\bar{0}} = \operatorname{Spec} \mathcal{A}_{\bar{0}}$ for $\mathcal{A}_{\bar{0}} = \mathbb{C}[x_1^{\pm 1}x_2, x_2]$. Here, we have $J_{(1,0)} = (x_1^{\pm 1}x_2)$, so $\mathcal{A}_{\bar{0}}/J_{(1,0)} = \mathbb{C}[x_2]/(x_2^2)$. This module has three torus-invariant ideals, namely (0) , (x_2) , and itself. These ideals yield toric supervarieties whose coordinate rings are $\mathbb{C}[x_1^{\pm 1}x_2, x_2, x_1^{\pm 1}x_2\xi]$, $\mathbb{C}[x_1^{\pm 1}x_2, x_2, x_1^{\pm 1}x_2\xi, x_2\xi]$, and $\mathbb{C}[x_1^{\pm 1}x_2, x_2, \xi]$, respectively.

Let us consider the toric supervariety $X = \operatorname{Spec} \mathcal{A}$ for $\mathcal{A} = \mathbb{C}[x_1^{\pm 1}x_2, x_2, x_1^{\pm 1}x_2\xi]$ in more detail. To witness it as a subvariety of some $\mathbb{C}^{|s|}$, we take the three generators of $\mathcal{A}_{\bar{0}}$ and the two generators of $\mathcal{A}_{\bar{1}}$, arranging them into finite sets as follows:

$$\begin{aligned} \mathcal{A}' = \mathcal{B}' &= \{(1, 1), (-1, 1)\} \\ \mathcal{A}'' &= \{(0, 1)\} \\ \mathcal{B}'' &= \emptyset \end{aligned}$$

Then we obtain

$$\begin{aligned} \Phi_{\mathcal{A}, \mathcal{B}} : T_{\mathbb{Z}^2, (1,0)} &\rightarrow \mathbb{C}^{3|2} \\ \Phi_{\mathcal{A}, \mathcal{B}}(t_1, t_2, \xi) &= (t_1t_2, t_1^{-1}t_2, t_2 \mid t_1t_2\xi, t_1^{-1}t_2\xi) \end{aligned}$$

and indeed $X \cong Y_{\mathcal{A}, \mathcal{B}}$.

To illustrate the naturality of the data of the fermionic ideal, we may verify that the Duflo-Serganova functor

$$DS_Q : \mathbb{C}[X] \mapsto \frac{\ker Q}{Q(\ker Q^2)}$$

of [11] outputs the data of J when $Q = \xi \left(c_1x_1 \frac{\partial}{\partial x_1} + \dots + c_nx_n \frac{\partial}{\partial x_n} \right) - \frac{\partial}{\partial \xi}$ is the odd vector field arising from the action of $\mathfrak{t}_1$ on $\mathbb{C}[X]$.

Proposition B.3.14. *With notation as above, $DS_Q \mathbb{C}[X] \cong \mathcal{A}_{\bar{0}}/J$.*

Proof. We may assume that $\mathbb{C}[X] \subseteq \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, \xi]$. Since $Q^2 = -c_1x_1 \frac{\partial}{\partial x_1} - \dots - c_nx_n \frac{\partial}{\partial x_n}$, it holds that $Q^2(f(x) + g(x)\xi) = Q^2(f(x)) + Q^2(g(x))\xi$. For a monomial x^m , we obtain $Q^2(x^m) = -\langle m, c \rangle x^m$ and hence

$$\ker Q^2 = (H \cap \mathbb{C}[X]_{\bar{0}}) \oplus (\xi H \cap \mathbb{C}[X]_{\bar{1}})$$

where

$$H = \bigoplus_{m \in \ker c} \mathbb{C}x^m$$

is the “hyperplane” orthogonal to c .

We observe that $Q(H) = 0$ and $Q(g(x)\xi) = g(x)$, so

$$\begin{aligned} Q(\ker Q^2) &= Q(H \cap \mathbb{C}[X]_{\bar{0}}) + Q(\xi H \cap \xi J) \\ &= H \cap J \end{aligned}$$

and $\ker Q = H \cap \mathbb{C}[X]_{\bar{0}}$. Hence

$$\begin{aligned} DS_Q \mathbb{C}[X] &= \frac{H \cap \mathbb{C}[X]_{\bar{0}}}{H \cap J} \\ &\cong \mathbb{C}[X]_{\bar{0}}/J, \end{aligned}$$

where the final isomorphism is due to the fact that $J \supseteq J_c$ and $H + J_c = \mathbb{C}[X]_{\bar{0}}$. $\square$

Morphisms

Let $X = \text{Spec}(\mathbb{C}[S] \oplus \xi J)$ and $X' = \text{Spec}(\mathbb{C}[S'] \oplus \xi' J')$ be affine toric supervarieties with supertori $T = T_{N,c}$ and $T' = T_{N',c'}$, respectively. Recall that a morphism of supergroups $T \rightarrow T'$ consists of a morphism $\bar{\phi} : N \rightarrow N'$ of lattices and a number $a \in \mathbb{C}$ such that $\bar{\phi}_{\mathbb{C}}(c) = a^2 c'$.

Before we characterize toric morphisms, we will establish some notation. The corresponding maps on coordinate superalgebras are

$$\begin{aligned} \phi^* : \mathbb{C}[S'] \oplus \xi' J' &\rightarrow \mathbb{C}[S] \oplus \xi J \\ \phi|_T^* : \mathbb{C}[M', \xi'] &\rightarrow \mathbb{C}[M, \xi] \end{aligned}$$

where $M = \mathbb{Z}S$ and $M' = \mathbb{Z}S'$. Since the conditions in the above definition imply that $\phi_{\bar{0}} : X_{\bar{0}} \rightarrow X'_{\bar{0}}$ is a toric morphism, it follows that $\phi_{\bar{0}}^* : \mathbb{C}[S'] \rightarrow \mathbb{C}[S]$ arises from a morphism $\hat{\phi}^* : S' \rightarrow S$ of semigroups, which extends to a morphism $\bar{\phi}^* : M' \rightarrow M$ of lattices. We write $\bar{\phi} : N \rightarrow N'$ for the dual map of $\bar{\phi}^*$.

Proposition B.3.15. *Let T and T' be the supertori of the affine toric supervarieties X and X' as above.*

- (a) *A morphism $X \rightarrow X'$ of supervarieties is toric if and only if $\phi_{\bar{0}}^*$ arises from a semigroup morphism $\hat{\phi}^* : S' \rightarrow S$, and $\bar{\phi}_{\mathbb{C}}(c) = a^2 c'$ where $a \in \mathbb{C}$ is the ξ -coefficient of $\phi^*(\xi')$.*
- (b) *A morphism $\phi = (\bar{\phi}, a) : T \rightarrow T'$ of supergroups extends to a morphism $X \rightarrow X'$ of toric supervarieties if and only if $\bar{\phi}^*(S') \subseteq S$, and either $a = 0$ or $\phi^*(J') \subseteq J$.*
- (c) *A toric morphism is equivariant, meaning for any superalgebra $\mathcal{A}$, it holds that $\phi(t \cdot x) = \phi(t) \cdot \phi(x)$ for all $t \in T(\mathcal{A})$ and $x \in X(\mathcal{A})$.*

Proof. The proof is the same as in the usual case ([9], Proposition 1.3.14), except that we use Lemma B.2.5 to determine when $\phi|_T$ is a morphism of supergroups. Notice that the condition $a = 0$ or $\phi^*(J') \subseteq J$ is equivalent to the condition $\phi^*(\xi' J') \subseteq \xi J$, since $\phi^*(\xi') = a\xi$. $\square$

B.4 Quasnormal toric supervarieties

We call a supervariety *quasnormal* if its underlying variety is normal. In the case of an integral supervariety (e.g. a supervariety with one odd dimension), this is equivalent to the condition that the local ring at every point is an integral superdomain $\mathcal{A}$ such that $\mathcal{A}/(\mathcal{A}_1)$ is integrally closed.

The affine case

Suppose henceforth that $X_\sigma \supset T_N$ is the normal affine toric variety corresponding to the cone $\sigma \subset N_{\mathbb{R}}$, so $X_\sigma = \text{Spec } \mathbb{C}[S_\sigma]$ where S_σ is the semigroup of the dual cone $\sigma^\vee$. If τ is a face of σ , then X_σ contains $X_\tau = \text{Spec } \mathbb{C}[S_\tau]$ as an open subvariety, and $\mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma]_{x^{m_{\sigma,\tau}}}$ where $m_{\sigma,\tau} \in S_\sigma$ is such that $\tau = \{u \in \sigma \mid \langle m_{\sigma,\tau}, u \rangle = 0\}$.

Proposition B.4.1. *Using the above notation, suppose $X_{\sigma,J^\sigma} = \text{Spec}(\mathbb{C}[S_\sigma] \oplus \xi J^\sigma)$ is the affine toric supervariety with underlying variety X_σ and fermionic ideal J^σ .*

(a) *The localization*

$$(\mathbb{C}[S_\sigma] \oplus \xi J^\sigma)_{x^{m_{\sigma,\tau}}} = \mathbb{C}[S_\tau] \oplus \xi J^\tau$$

is the coordinate ring of the open subvariety X_{τ,J^τ} obtained by restricting to the open affine X_τ . In particular, it holds that $J_{x^{m_{\sigma,\tau}}}^\sigma = J^\tau$.

(b) *Let $J_c^\sigma = (x^m \in \mathbb{C}[S_\sigma] \mid \langle m, c \rangle \neq 0)$, and likewise for τ . It holds that $(J_c^\sigma)_{x^{m_{\sigma,\tau}}} = J_c^\tau$.*

(c) *As defined in part (a), X_{τ,J^τ} is an affine toric supervariety whose torus is $T_{N,c}$.*

Proof. (a) The fact that $\mathbb{C}[S_\sigma]_{x^{m_{\sigma,\tau}}} = \mathbb{C}[S_\tau]$ is standard, so it remains to prove $J_{x^{m_{\sigma,\tau}}}^\sigma = J^\tau$. This follows immediately by computing sections over the basic open set defined by $x^{m_{\sigma,\tau}}$.

(b) Let x^m be a generator of J_c^σ and $k \in \mathbb{N}$, so $\langle m, c \rangle \neq 0$ and $x^{m-km_{\sigma,\tau}} \in (J_c^\sigma)_{x^{m_{\sigma,\tau}}}$. Now consider the quantity $\langle m - km_{\sigma,\tau}, c \rangle$. If $\langle m_{\sigma,\tau}, c \rangle = 0$, then indeed $x^{m-km_{\sigma,\tau}} \in J_c^\tau$. Otherwise $\langle m_{\sigma,\tau}, c \rangle \neq 0$, so $x^{\pm m_{\sigma,\tau}} \in J_c^\tau$ and hence $J_c^\tau = (1)$.

To see that $(J_c^\sigma)_{x^{m_{\sigma,\tau}}} \supseteq J_c^\tau$, let x^m be a generator of J_c^τ , so $x^m \in \mathbb{C}[S_\tau]$ satisfies $\langle m, c \rangle \neq 0$. By part (a), $m = m' - km_{\sigma,\tau}$ for some $x^{m'} \in S_\sigma$ and $k \in \mathbb{N}$. If $\langle m_{\sigma,\tau}, c \rangle = 0$, then $\langle m', c \rangle \neq 0$ and hence $x^{m'} \in J_c^\sigma$, so $x^m \in (J_c^\sigma)_{x^{m_{\sigma,\tau}}}$. Otherwise $\langle m_{\sigma,\tau}, c \rangle \neq 0$, so $x^{\pm m_{\sigma,\tau}} \in (J_c^\sigma)_{x^{m_{\sigma,\tau}}}$ and therefore $(J_c^\sigma)_{x^{m_{\sigma,\tau}}} = (1)$.

(c) This follows from part (b), as $(J_c^\sigma)_{x^{m_{\sigma,\tau}}} = J_c^\tau$. $\square$

That is, in order to determine a quasnormal affine toric supervariety over the torus $T_{N,c}$, one requires the data of a strongly convex rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$ decorated by a nonzero T_N -invariant ideal $J^\sigma \supset J_c$ of $\mathbb{C}[S_\sigma]$. When restricting to faces of σ , one finds that we may decorate each face with a certain localization J^τ of J^σ . These decorations will play an important role later, when we describe the non-affine case.

First, we translate Proposition B.3.15 to the language of cones.

Proposition B.4.2. *Suppose we have strongly convex rational polyhedral cones $\sigma \subset N_{\mathbb{R}}$ and $\sigma' \subset N'_{\mathbb{R}}$, a morphism $\phi = (\bar{\phi}, a) : T_{N,c} \rightarrow T_{N',c'}$ of supergroups, and nonzero T_N -, $T_{N'}$ -invariant ideals $J \supseteq J_c^\sigma$ and $J' \supseteq J_{c'}^{\sigma'}$ of $\mathbb{C}[S_\sigma]$ and $\mathbb{C}[S_{\sigma'}]$, respectively. Then ϕ extends to a map of affine toric supervarieties $\phi : X_{\sigma,J} \rightarrow X_{\sigma',J'}$ if and only if $\bar{\phi}_{\mathbb{R}}(\sigma) \subseteq \sigma'$, and either $a = 0$ or $\phi^*(J') \subseteq J$.*

The following lemma will be useful in section B.8, but is also interesting in its own right.

Lemma B.4.3. *Let $(\bar{\phi}, a) : X_{\sigma,J} \rightarrow X_{\sigma',J'}$ be a map of toric supervarieties, and let*

$$\phi^* : \mathbb{C}[S_{\sigma'}] \oplus \xi' J' \rightarrow \mathbb{C}[S_\sigma] \oplus \xi J$$

be the dual map. Then $\phi^(\xi' J_{c'}) \subseteq \xi J_c$.*

Proof. Let $m' \in S_{\sigma'} - \ker c'$. Then

$$\begin{aligned} \langle \bar{\phi}^*(m'), c \rangle &= \langle m', \bar{\phi}_{\mathbb{C}}(c) \rangle \\ &= a^2 \langle m', c' \rangle, \end{aligned}$$

which is nonzero if $a \neq 0$. In this case, we have $\phi^*(J_{c'}) \subseteq J_c$. If $a = 0$, then $\phi^*(\xi') = 0$ and the lemma still holds. $\square$

The general case

Let X be a quasinormal toric supervariety whose torus is $T = T_{N,c}$. Then $X_{\bar{0}} = X_\Sigma$ for a fan Σ in $N_{\mathbb{R}}$. By the results of the previous section, each cone $\sigma \in \Sigma$ is decorated by a suitable ideal $J^\sigma \supset J_c^\sigma$ of $\mathbb{C}[S_\sigma]$ such that if τ is a face of σ , then J^τ is a particular localization of J^σ . We show that these data give a complete classification of quasinormal toric supervarieties for the given torus T .

Proposition B.4.4. *Let Σ be a fan in $N_{\mathbb{R}}$, $c \in N_{\mathbb{C}}$, and $\{J^\sigma \subseteq \mathbb{C}[S_\sigma] \mid \sigma \in \Sigma\}$ a collection of nonzero ideals such that each $J^\sigma \supseteq J_c^\sigma$. If $(J^\sigma)_{x^{m_{\sigma,\tau}}} = J^\tau$ whenever τ is a face of σ in Σ , then the affine toric supervarieties $U_\sigma = \text{Spec}(\mathbb{C}[S_\sigma] \oplus \xi J^\sigma)$ can be glued together along their intersections to produce a toric supervariety whose underlying variety is X_Σ and whose torus is $T_{N,c}$.*

Proof. This follows from Proposition B.4.1(a) and the classical version of this statement. $\square$

Hence, quasinormal toric supervarieties can be described by a fan Σ decorated by suitable ideals. These ideals arrange nicely into a coherent ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_{X_\Sigma}$, called the *fermionic sheaf*, obtained by gluing together the coherent sheaves $\tilde{J}^\sigma$ (the sheaf associated to the module J^σ in the sense of [21]) on each open affine described by a cone $\sigma \in \Sigma$. The fermionic sheaf of a quasinormal toric supervariety admits the following expected properties:

Proposition B.4.5. *Let $\mathcal{J}$ be the fermionic sheaf of a quasinormal toric supervariety with underlying variety X_Σ and torus $T = T_{N,c}$.*

- (a) $\mathcal{J}$ contains the minimal fermionic sheaf $\mathcal{J}_c$, obtained by gluing the sheaves $\tilde{J}_c^\sigma$ for all $\sigma \in \Sigma$.

(b) $\mathcal{J}$ is T_N -equivariant.

(c) $\mathcal{J}_c$ is the ideal sheaf of the vanishing subscheme $Z_c \subset X_\Sigma$ of c , and $\mathcal{J}$ is the ideal sheaf of a toric subscheme $Z \subset Z_c$.

Proof. (a) Note that $\mathcal{J}_c$ is well-defined by Proposition B.4.1(b). Since each $J^\sigma \supseteq J_c^\sigma$, the claim follows.

(b) We know the claim holds for $U = U_\sigma$. The rest is standard; see e.g. [41].

(c) This is just a global version of the observations following Lemma B.3.4. $\square$

We have thus proven:

Proposition B.4.6. *Let X be a quasinormal supervariety and $T = T_{N,c}$. Then X is a toric supervariety with supertorus T if and only if $X \cong X_{\Sigma, \mathcal{J}} := (X_\Sigma, \mathcal{O}_{X_\Sigma} \oplus \xi \mathcal{J})$ for a fan Σ in $N_\mathbb{R}$ and a nonzero T_N -equivariant ideal sheaf $\mathcal{J} \supseteq \mathcal{J}_c$ in $\mathcal{O}_{X_\Sigma}$.*

A combinatorial approach

The classification data $(\Sigma, \mathcal{J})$ above are only partially combinatorial in the sense that $\mathcal{J}$ is an algebraic object. In this section, we represent $\mathcal{J}$ with more combinatorial data; namely, we replace $\mathcal{J}$ with a collection $\mathcal{B}$ of local generators of $\mathcal{J}$.

Definition B.4.7. As in Definition B.3.6, for $m, m' \in S_\sigma$ we write $m' \leq_\sigma m$ if there is $m'' \in S_\sigma$ such that $m = m' + m''$.

Definition B.4.8. Let $T_{N,c}$ be a supertorus and Σ a fan in $N_\mathbb{R}$. For each $\sigma \in \Sigma$, let $\mathcal{B}_\sigma$ be a nonempty finite set in S_σ . We say that $\mathcal{B} = \{\mathcal{B}_\sigma\}_{\sigma \in \Sigma}$ is a set of *c-admissible decorations* for the fan Σ if for all cones $\sigma, \sigma' \in \Sigma$:

- (i) For all $m \in S_\sigma - \ker c$, there is $b \in \mathcal{B}_\sigma$ such that $b \leq_\sigma m$.
- (ii) For all $b \in \mathcal{B}_\sigma$, there is $b' \in \mathcal{B}_{\sigma'}$ such that $b' \leq_{\sigma \cap \sigma'} b$.
- (iii) If $a, b \in \mathcal{B}_\sigma$ satisfy $a \leq_\sigma b$, then $a = b$.

Notice that unlike in Definition B.3.6, here we require minimality (iii) as part of admissibility. This requirement is not strictly necessary for the classification, but it simplifies some matters in Section B.6.

Lemma B.4.9. *Condition (ii) of Definition B.4.8 holds for all $\sigma, \sigma' \in \Sigma$ if and only if it holds for all σ, σ' such that one is a face of the other.*

Proof. Suppose it holds in all cases of faces, and let $\tau = \sigma \cap \sigma'$. Now let $b \in \mathcal{B}_\sigma$. Then there is $a \in \mathcal{B}_\tau$ such that $a \leq_\tau b$, and likewise $b' \in \mathcal{B}_{\sigma'}$ such that $b' \leq_\tau a$ since τ is a face of both σ and σ' . But then $b' \leq_\tau a \leq_\tau b$, so we are finished. $\square$

That is, we may replace condition (ii) with

- (ii') Let τ be a face of σ . Then

- For all $b \in \mathcal{B}_\sigma$, there is $a \in \mathcal{B}_\tau$ such that $b \leq_\tau a$.
- For all $a \in \mathcal{B}_\tau$, there is $b \in \mathcal{B}_\sigma$ such that $a \leq_\tau b$.

The following lemma shows that $\mathcal{B}$ consists of local generating sets of the sheaf $\mathcal{J}$.

Lemma B.4.10. *For each $\sigma \in \Sigma$, let J^σ be a monomial ideal in $\mathbb{C}[S_\sigma]$ and $\mathcal{B}_\sigma \subset S_\sigma$ a finite generating set of J^σ . Then*

- (a) $J^\sigma \supseteq J_c^\sigma$ if and only if condition (i) of Definition B.4.8 holds.
- (b) $J_{x^{m_{\sigma,\tau}}}^\sigma = J^\tau$ if and only if condition (ii) $\approx$ (ii') of Definition B.4.8 holds.
- (c) $\mathcal{B}_\sigma$ is minimal if and only if condition (iii) of Definition B.4.8 holds.

Proof. First observe that for $m \in S_\sigma$, we have $x^m \in J^\sigma$ if and only if there is a generator $b \in \mathcal{B}_\sigma$ and an element $f \in \mathbb{C}[S]_\sigma$ such that $fx^b = x^m$. In particular, f will always be a monomial, so this is equivalent to $b \leq_\sigma m$.

- (a) The ideal J_c^σ is generated by $S_\sigma - \ker c$, so condition (i) exactly says that all generators of J_c^σ belong to J^σ .
- (b) Notice that J^τ and $J_{x^{m_{\sigma,\tau}}}^\sigma$ are both ideals of $\mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma]_{x^{m_{\sigma,\tau}}}$, generated by $\mathcal{B}_\tau$ and $\mathcal{B}_\sigma$, respectively. Let $a \in \mathcal{B}_\tau$. By (ii'), there is $b \in \mathcal{B}_\sigma$ such that $b \leq_\tau a$, i.e. that $x^a \in J_{x^{m_{\sigma,\tau}}}^\sigma$. Likewise, let $b \in \mathcal{B}_\sigma$, so there is $a \in \mathcal{B}_\tau$ such that $a \leq_\tau b$, i.e. that $b \in J^\tau$. Therefore $J_{x^{m_{\sigma,\tau}}}^\sigma = J^\tau$. The converse follows by tracing backwards through the same reasoning.
- (c) If $a \leq_\sigma b$ and $a \neq b$, then b remains in the ideal generated by $\mathcal{B}_\sigma - \{b\}$, so $\mathcal{B}_\sigma$ is not minimal. Conversely, if $\mathcal{B}_\sigma$ is not minimal, then we can find $a, b \in \mathcal{B}_\sigma$ satisfying $a \leq_\sigma b$. $\square$

We write $X_{\Sigma, \mathcal{B}}$ for the toric supervariety defined by gluing the affine toric supervarieties $X_{\sigma, \mathcal{B}_\sigma}$ for $\sigma \in \Sigma$ along their intersections as before. Notice by Lemma B.3.4(b) that each $\mathcal{B}_\sigma$ is unique up to invertible elements of S_σ .

Corollary B.4.11. *Let X be a quasinormal supervariety and $T = T_{N,c}$. Then X is a toric supervariety with supertorus T if and only if $X \cong X_{\Sigma, \mathcal{B}}$ for a fan Σ with c -admissible decorations $\mathcal{B}$.*

Example B.4.12. Consider $(n|1)$ -dimensional projective superspace as a supervariety; i.e. $\mathbb{P}^{n|1} = (\mathbb{P}^n, \Lambda^\bullet \mathcal{O}(-1)) = \text{Proj } \mathbb{C}[x_0, \dots, x_n, \xi]$. Its underlying variety $\mathbb{P}^n$ can be viewed as a toric variety X_Σ where Σ is the fan whose cones are the positive spans of the proper subsets of $\{e_1, \dots, e_n, -(e_1 + \dots + e_n)\}$.

Using the Proj construction, the affine charts of $\mathbb{P}^{n|1}$ have coordinate rings $\mathbb{C}[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}, \frac{\xi}{x_i}]$ for $i = 0, \dots, n$. Letting $z_i = \frac{x_i}{x_0}$ for $i = 1, \dots, n$ and $\eta = \frac{\xi}{x_0}$, we can write these rings as $\mathbb{C}[z_1, \dots, z_n, \eta]$ and $\mathbb{C}[z_i^{-1}, z_i^{-1}z_1, \dots, z_i^{-1}z_n, z_i^{-1}\eta]$ for $i = 1, \dots, n$. Notably, $\mathbb{C}[z_1, \dots, z_n]$ is the semigroup algebra of the dual cone to the $\mathbb{R}_+$ -span of $\{e_1, \dots, e_n\}$, and $\mathbb{C}[z_i^{-1}, z_i^{-1}z_1, \dots, z_i^{-1}z_n]$ is that of $\{e_1, \dots, \hat{e}_i, \dots, e_n, -(e_1 + \dots + e_n)\}$ where $\hat{e}_i$ denotes the exclusion of e_i . Denote these cones by σ_0 and σ_i for $i = 1, \dots, n$ respectively.

The torus $T_{\mathbb{Z}^n}$ of this toric variety acts on the coordinate rings by

$$(t_1, \dots, t_n) \cdot x_0^{m_0} x_1^{m_1} \cdots x_n^{m_n} = x_0^{m_0} (t_1^{-1} x_1)^{m_1} \cdots (t_n^{-1} x_n)^{m_n}.$$

It follows that $\mathbb{P}^{n|1}$ can be expressed as a decorated fan with σ_0 decorated by $\{1\}$ and σ_i decorated by $\{x_i^{-1}\}$ (the opposites of the characters by which $T_{\mathbb{Z}^n}$ acts on the respective odd generators in the coordinate rings). Then if $\mathbb{P}^{n|1}$ is a toric supervariety, the torus must be $T_{\mathbb{Z}^n, c}$ where $\langle m, c \rangle = 0$ for all characters x^m of the form $z_i^{-1} z_j$ for $i, j > 0$. But $z_i^{-1} z_j = x_i^{-1} x_j$, so in particular we must have $c = 0$ or $c = (1, \dots, 1)$, up to scale.

In this sense, $\mathbb{P}^{n|1}$ is a toric supervariety with supertorus $T_{\mathbb{Z}^n, (1, \dots, 1)} \cong Q(1) \times T_{\mathbb{Z}^{n-1}}$.

Morphisms

The category of quasinormal toric supervarieties with one odd dimension can then be described as having the above objects $X_{\Sigma, \mathcal{B}}$ with T allowed to vary, and morphisms as in Definition 5.10.1, which we characterize below. First, recall that a lattice map $\bar{\phi} : N \rightarrow N'$ is called *compatible with the fans* Σ, Σ' in $N_{\mathbb{R}}, N'_{\mathbb{R}}$ if for each $\sigma \in \Sigma$, there is $\sigma' \in \Sigma'$ such that $\bar{\phi}_{\mathbb{R}}(\sigma) \subseteq \sigma'$.

Definition B.4.13. Let $\phi_{\bar{\phi}} : X_{\Sigma} \rightarrow X'_{\Sigma'}$ be a morphism of toric varieties induced by a lattice map $\bar{\phi} : N \rightarrow N'$ compatible with the fans Σ in $N_{\mathbb{R}}$ and $\Sigma'_{\mathbb{R}}$ in $N'_{\mathbb{R}}$. Let $\mathcal{B}$ and $\mathcal{B}'$ be c - and c' -admissible decorations for their respective fans. We say that $\bar{\phi}$ is *compatible with the decorations* if whenever $\bar{\phi}_{\mathbb{R}}(\sigma) \subseteq \sigma'$ and $b' \in \mathcal{B}'_{\sigma'}$, there is $b \in \mathcal{B}_{\sigma}$ such that $b \leq_{\sigma} \bar{\phi}^*(b')$.

Proposition B.4.14. Using the prior notation, let $\phi^{\#} : \mathcal{O}_{X_{\Sigma'}} \rightarrow \phi_* \mathcal{O}_{X_{\Sigma}}$ be the corresponding morphism of sheaves on $X_{\Sigma'}$, and let $Z \subset X_{\Sigma}$ and $Z' \subset X_{\Sigma'}$ be the closed subschemes corresponding to the ideal sheaves $\mathcal{J}$ and $\mathcal{J}'$ obtained from the decorations $\mathcal{B}$ and $\mathcal{B}'$, respectively. Then the following are equivalent:

- (a) $\bar{\phi}$ is compatible with the decorations.
- (b) For each $\sigma \in \Sigma$ and $\sigma' \in \Sigma'$ such that $\bar{\phi}_{\mathbb{R}}(\sigma) \subseteq \sigma'$, it holds that $\phi_{\bar{\phi}}^*|_{U_{\sigma}}(J^{\sigma'}) \subseteq J^{\sigma}$.
- (c) $\phi^{\#}|_{\mathcal{J}'}$ takes $\mathcal{J}'$ to $\phi_* \mathcal{J}$.
- (d) $\phi|_Z$ is a morphism of schemes $Z \rightarrow Z'$.

Proof. For (a) $\iff$ (b), observe that $\bar{\phi}^*(\mathcal{B}'_{\sigma'})$ generates $\phi_{\bar{\phi}}^*|_{U_{\sigma}}(J^{\sigma'})$. Hence $\phi_{\bar{\phi}}^*|_{U_{\sigma}}(J^{\sigma'}) \subseteq J^{\sigma}$ is equivalent to the condition that for all $b' \in \mathcal{B}'_{\sigma'}$, there is $b \in \mathcal{B}_{\sigma}$ such that $b \leq_{\sigma} \bar{\phi}^*(b')$.

The map $\phi_{\bar{\phi}}^*|_{U_{\sigma}} : \mathbb{C}[S_{\sigma'}] \rightarrow \mathbb{C}[S_{\sigma}]$ is the same as

$$\text{Res}_{U_{\sigma}}^{\phi^{-1}(U_{\sigma'})} \circ \phi^{\#}(U_{\sigma'}) : \mathcal{O}_{X_{\Sigma'}}(U_{\sigma'}) \rightarrow \mathcal{O}_{X_{\Sigma}}(U_{\sigma}),$$

so if we restrict further to $J^{\sigma'} \subseteq \mathbb{C}[S_{\sigma'}]$ and $\mathcal{J}'(U_{\sigma'}) \subseteq \mathcal{O}_{X_{\Sigma'}}(U_{\sigma'})$, we see immediately that (b) $\iff$ (c). The fact that (c) $\iff$ (d) is standard. $\square$

Hence if $X_{\sigma, J}$ and $X_{\sigma', J'}$ are affine, compatibility with decorations is exactly the condition that $\phi^*(J') \subseteq J$ as in Proposition B.4.2. The following condition is equivalent to the condition that $\phi^*(\xi' J') \subseteq \xi J$.

Definition B.4.15. Let $X_{\Sigma, \mathcal{B}}$ and $X_{\Sigma', \mathcal{B}'}$ be toric supervarieties with supertori $T_{N, c}$ and $T_{N', c'}$, respectively. If $(\bar{\phi}, a) : T_{N, c} \rightarrow T_{N', c'}$ is a morphism of supergroups, we say that $(\bar{\phi}, a)$ is *compatible with the decorated fans* if $\bar{\phi} : N \rightarrow N'$ is compatible with the fans, and either $a = 0$, or $\bar{\phi}$ is compatible with the decorations.

Proposition B.4.16. Let N, N' be lattices and Σ, Σ' fans in $N_{\mathbb{R}}, N'_{\mathbb{R}}$ with c, c' -admissible decorations $\mathcal{B}, \mathcal{B}'$, respectively. Suppose $\phi = (\bar{\phi}, a) : T_{N, c} \rightarrow T_{N', c'}$ is a morphism of supergroups. Then ϕ extends to a morphism of toric supervarieties $X_{\Sigma, \mathcal{B}} \rightarrow X_{\Sigma', \mathcal{B}'}$ if and only if $(\bar{\phi}, a)$ is compatible with the decorated fans.

Proof. First suppose that ϕ extends. By the analogous result in the purely even setting, $\bar{\phi}$ is compatible with the fans. Let $\sigma \in \Sigma$, so the extension ϕ restricts to a map

$$\phi|_{U_{\sigma, J\sigma}} : U_{\sigma, J\sigma} \rightarrow U_{\sigma', J\sigma'}$$

for any $\sigma' \in \Sigma'$ such that $\bar{\phi}_{\mathbb{R}}(\sigma) \subseteq \sigma'$. By Proposition B.4.2, we see that $a = 0$ or $\phi_0|_{U_{\sigma}}^*(J^{\sigma'}) \subseteq J^{\sigma}$, which is sufficient by Proposition B.4.14.

Conversely, suppose the compatibility conditions hold. By Proposition B.4.2, these conditions guarantee that for each open affine toric subvariety $U_{\sigma, \mathcal{B}_{\sigma}} \subseteq X_{\Sigma, \mathcal{B}}$, there is an open affine toric subvariety $U_{\sigma', \mathcal{B}'_{\sigma'}} \subseteq X_{\Sigma', \mathcal{B}'}$ such that ϕ extends to a toric morphism $U_{\sigma, \mathcal{B}_{\sigma}} \rightarrow U_{\sigma', \mathcal{B}'_{\sigma'}}$. Moreover, these partial extensions are compatible with each other on intersections because intersections are also open affine toric subvarieties. Therefore these partial extensions may be glued together into a global extension, finishing our proof. $\square$

Definition B.4.17. We define the category of *decorated fans* as follows: The objects are quadruples $(N, c, \Sigma, \mathcal{B})$ where N is a lattice, $c \in N_{\mathbb{C}}$, Σ is a fan in $N_{\mathbb{R}}$, and $\mathcal{B}$ is a c -admissible family of decorations for Σ . The morphisms are tuples $(\bar{\phi}, a) : (N, c, \Sigma, \mathcal{B}) \rightarrow (N', c', \Sigma', \mathcal{B}')$ where $\bar{\phi} : N \rightarrow N'$ is a lattice map, $a \in \mathbb{C}$ is such that $\bar{\phi}_{\mathbb{C}}(c) = a^2 c'$, and $(\bar{\phi}, a)$ is compatible with the decorated fans. Composition of morphisms is given by $(\bar{\phi}', a') \circ (\bar{\phi}, a) = (\bar{\phi}' \circ \bar{\phi}, aa')$.

Synthesizing the constructions of this section with Corollary B.4.11 and Proposition B.4.16, we have proven the following theorem:

Theorem B.4.18. *The category of quasinormal toric supervarieties with one odd dimension is equivalent to the category of decorated fans.*

B.5 Smoothness

Now that we have described the category of quasinormal toric supervarieties of one odd dimension, we seek to understand their geometry.

Smoothness in the odd direction

Roughly speaking, a supervariety is smooth if and only if it is smooth in both the even and odd directions. We first characterize the latter.

Definition B.5.1. A supervariety X is said to be *smooth in the odd directions* if the odd part of its cotangent bundle is locally trivial.

In one odd dimension, we may therefore take local triviality of $\mathcal{J}$ as our definition of smoothness in the odd directions.

Proposition B.5.2. *Let $X = X_{\Sigma, \mathcal{B}} = X_{\Sigma, \mathcal{J}} = X_{\Sigma, Z}$ be a quasinormal toric supervariety with torus $T_{N, c}$. The following are equivalent:*

- (a) X is smooth in the odd directions.
- (b) Each $\mathcal{B}_\sigma$ is a singleton.
- (c) Each J^σ is a principal ideal.
- (d) $\mathcal{J}$ is a line bundle on X_Σ .
- (e) Z is an effective Cartier divisor.

Proof. We have (a) $\iff$ (d) $\iff$ (e) and (b) $\iff$ (c) from the definitions. Now if (d) holds, then $\mathcal{J}|_{U_\sigma}$ is a line bundle on the affine toric variety U_σ . A result of Gubeladze [19] shows that an arbitrary vector bundle on an affine normal toric variety is trivial, so $\mathcal{J}|_{U_\sigma} = \tilde{J}^\sigma$ where $J^\sigma \cong \mathbb{C}[S_\sigma]$ as a $\mathbb{C}[S^\sigma]$ -module. That is, J^σ is principal for each $\sigma \in \Sigma$.

Conversely, if each ideal is principal, then each $\tilde{J}^\sigma$ is a trivial line bundle, and so $\mathcal{J}$ trivializes over the cover of X_Σ by the U_σ . $\square$

Definition B.5.3. We shall henceforth refer to a family $\mathcal{B}$ of decorations as *smooth* if each $\mathcal{B}_\sigma$ is a singleton.

Smooth toric supervarieties

We seek to establish a characterization of smooth toric supervarieties. Let us begin in the affine case.

Proposition B.5.4. *Let X be an affine toric supervariety with torus $T_{N, c}$. Then X is smooth if and only if $X \cong X_{\sigma, \mathcal{B}_\sigma}$ for a smooth cone σ and a singleton $\mathcal{B}_\sigma$.*

Proof. If X is smooth, then so is $X_{\bar{0}}$. But a smooth variety is normal, and so $X = X_{\sigma, \mathcal{B}_\sigma}$ for a smooth cone σ and a c -admissible finite set $\mathcal{B}_\sigma = \{m_1, \dots, m_s\} \subset S_\sigma$. Then the underlying variety is $X_{\bar{0}} \cong \text{Spec } \mathbb{C}[x_1, \dots, x_r, x_{r+1}^{\pm 1}, \dots, x_n^{\pm 1}] = \mathbb{C}^r \times (\mathbb{C}^\times)^{n-r}$ with torus $T_{\bar{0}} \cong (\mathbb{C}^\times)^n$. Localizing at $\mathfrak{p} = (0^r, 1^{n-r})$, we set $\mathcal{B} = \mathbb{C}[S]_{\mathfrak{p}}$ and find $\mathcal{B} \cong \bar{\mathcal{B}}[x^{m_1}\xi, \dots, x^{m_s}\xi]$ wherein $x^{m_i}\xi \notin \bar{\mathcal{B}}[x^{m_j}\xi]$ for $i \neq j$. By definition of smoothness, it follows that $s = 1$.

On the other hand, if σ is a smooth cone and $\mathcal{B}_\sigma$ is a singleton, then by the above reasoning we have $\mathcal{B} \cong \bar{\mathcal{B}}[x^m\xi] \cong \bar{\mathcal{B}}[\xi]$ with $\bar{\mathcal{B}}$ a regular local ring. Therefore $X_{\sigma, \mathcal{B}_\sigma}$ is smooth. $\square$

The general case follows immediately from Propositions B.5.2 and B.5.4.

Theorem B.5.5. *Let X be a toric supervariety of one odd dimension with supertorus $T_{N, c}$. Then X is smooth if and only if $X \cong X_{\Sigma, \mathcal{B}}$ for a smooth fan Σ and smooth decorations $\mathcal{B}$.*

Example B.5.6. In this example, we show that every $(1|1)$ -dimensional toric supervariety is smooth. Note that the three 1-dimensional ordinary toric varieties (up to isomorphism) are the torus $\mathbb{C}^\times$ and the affine and projective lines $\mathbb{A}^1$ and $\mathbb{P}^1$, all of which are smooth. Also, the two $(1|1)$ -dimensional supertori (up to isomorphism) are $T_{\mathbb{Z},0}$ and $T_{\mathbb{Z},1} \cong Q(1)$.

Observe that the only nonzero monomial ideal in $\mathbb{C}[\mathbb{C}^\times] \cong \mathbb{C}[x^{\pm 1}]$ is the whole ring (1) . However, in $\mathbb{C}[\mathbb{A}^1] \cong \mathbb{C}[x]$, we have (x^n) for n any nonnegative integer. Since all these ideals are principal and smoothness is affine-local, it indeed follows that every $(1|1)$ -dimensional toric supervariety is smooth.

We now consider the possibilities over $\mathbb{P}^1$. Let $m, n \geq 0$. If we glue together the affine toric supervarieties $\text{Spec } \mathbb{C}[x, x^m \xi]$ and $\text{Spec } \mathbb{C}[x^{-1}, x^{-n} \xi]$ along their intersection $\text{Spec } \mathbb{C}[x^{\pm 1}, \xi]$, then we obtain a toric supervariety whose underlying variety is $\mathbb{P}^1$ and whose fermionic sheaf is isomorphic to the line bundle $\mathcal{O}(-m-n)$. In particular, if the supertorus is $Q(1)$, then $m, n \leq 1$ and so there are three possibilities for the fermionic sheaf; namely $\mathcal{O}(0)$, $\mathcal{O}(-1)$, and $\mathcal{O}(-2)$.

Example B.5.7. Consider the Π -projective space $\mathbb{P}_\Pi^1$, also known as the Π -symmetric super-Grassmannian $\text{Gr}(1|1, 2|2)^\Pi$ of $(1|1)$ -dimensional subspaces of $\mathbb{C}^{2|2}$ that are invariant under parity switch Π . For $\mathcal{A}$ a superalgebra, its $\mathcal{A}$ -points consist of the full-rank matrices

$$\begin{pmatrix} x_1 & \xi_1 \\ x_2 & \xi_2 \\ \xi_1 & x_1 \\ \xi_2 & x_2 \end{pmatrix}$$

modulo the right action by $Q(1)$, where such a matrix is said to be full-rank if it has an invertible $(1|1) \times (1|1)$ submatrix (i.e. either x_1 or x_2 is invertible in $\mathcal{A}_0$).

It is seen in e.g. [37], Section 6 that $\text{Gr}(1|1, 2|2)^\Pi$ is covered by two charts with coordinates

$$\begin{pmatrix} y & \eta \\ 1 & 0 \\ \eta & y \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ z & \zeta \\ 0 & 1 \\ \zeta & z \end{pmatrix}$$

respectively, and with transition functions given by $z = y^{-1}$ and $\eta = -\zeta/z^2$. Using the prior example, we find that $\text{Gr}(1|1, 2|2)^\Pi$ has underlying variety $\mathbb{P}^1$ and fermionic sheaf $\mathcal{O}(-2)$.

This space admits a natural left action by the subgroup of the general linear supergroup $\text{GL}(2|2, \mathbb{C})$ that preserves Π ; the $\mathcal{A}$ -points of this subgroup are

$$Q(2)(\mathcal{A}) = \left\{ \begin{pmatrix} B & C \\ C & B \end{pmatrix} \right\}$$

where B and C are 2×2 matrices consisting of even and odd elements of $\mathcal{A}$, respectively. We observe that $Q(2)$ contains a copy of $Q(1)$ embedded as

$$\begin{pmatrix} a & 0 & \tau & 0 \\ 0 & 1 & 0 & 0 \\ \tau & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

so that the action on the first coordinate chart

$$\begin{pmatrix} a & 0 & \tau & 0 \\ 0 & 1 & 0 & 0 \\ \tau & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y & \eta \\ 1 & 0 \\ \eta & y \\ 0 & 1 \end{pmatrix}$$

corresponds to multiplication

$$\begin{pmatrix} a & \tau \\ \tau & a \end{pmatrix} \begin{pmatrix} y & \eta \\ \eta & y \end{pmatrix}$$

in $Q(1)$. Therefore $\text{Gr}(1|1,2|2)^\Pi$ is naturally a smooth toric supervariety with supertorus $Q(1)$.

B.6 Fibers of the fermionic sheaf

One way to glean information from the fermionic sheaf $\mathcal{J}$ is by computing its fibers, which will turn out to extract the fan decorations $\mathcal{B}$.

In the classical setting of a normal toric variety X_Σ with torus T_N , the torus orbits correspond bijectively to cones $\sigma \in \Sigma$ ([9], Theorem 3.2.6). We denote by $\text{Orb}(\sigma)$ the T_N -orbit of X_Σ corresponding to the cone $\sigma \in \Sigma$.

Lemma B.6.1. *Let $\mathfrak{p}$ be a closed point in $\text{Orb}(\sigma)$. Then $\mathcal{B}_\sigma$ is unique when viewed as a subset of the character lattice of $\text{Stab}_{T_N}(\mathfrak{p})$ via pullback along $\text{Stab}_{T_N}(\mathfrak{p}) \rightarrow T_N$.*

Proof. By Lemma B.3.4(b), $\mathcal{B}_\sigma$ is unique up to multiplication by monomials which are invertible in $\mathbb{C}[S_\sigma]$. The invertible monomials in $\mathbb{C}[S_\sigma]$ are exactly x^m for $m \in \sigma^\perp \cap M$, since $S_\sigma = \sigma^\vee \cap M$. Observe that $\sigma^\perp \cap M$ is the character lattice of the orbit $T_N \mathfrak{p}$, viewed as an algebraic torus.

Let $b \in \mathcal{B}_\sigma$, and let $m \in \sigma^\perp \cap M$. Both x^b and x^{b+m} can be viewed as characters $T_N \rightarrow \mathbb{C}^\times$. It suffices to show that they become the same character of $\text{Stab}_{T_N}(\mathfrak{p})$ when we compose them with the map $\text{Stab}_{T_N}(\mathfrak{p}) \rightarrow T_N$. We have $T_N \mathfrak{p} \cong T_N / \text{Stab}_{T_N}(\mathfrak{p})$ by the orbit-stabilizer theorem, so the statement that x^m belongs to the character lattice of $T_N \mathfrak{p}$ is exactly the statement that x^m is trivial on $\text{Stab}_{T_N}(\mathfrak{p})$, as desired. $\square$

Proposition B.6.2. *Let $X_{\Sigma, \mathcal{B}}$ be a quasinormal toric supervariety with supertorus $T_{N, c}$, and let $\sigma \in \Sigma$. Let $\mathfrak{p}$ be a closed point in $\text{Orb}(\sigma)$. Then the fiber $\mathcal{J}(\mathfrak{p})$, as a representation of $\text{Stab}_{T_N}(\mathfrak{p})$, is a direct sum of the characters in $\mathcal{B}_\sigma$.*

Proof. Localizing at $\mathfrak{p}$ means we invert monomials not in $\mathfrak{p}$. In the event that $\mathfrak{p}$ belongs to the T_N -orbit corresponding to σ , we invert the monomials in $\sigma^\perp$. Hence, when we quotient $J_\mathfrak{p}^\sigma$ by $\mathfrak{p}J_\mathfrak{p}^\sigma$, we are left with a direct sum of the desired characters. $\square$

Example B.6.3. Consider the torus $T = T_{\mathbb{Z}^3, (1,0,0)} \cong Q(1) \times (\mathbb{C}^\times)^2$ and the toric supervariety over T with underlying variety $X_{\bar{0}} = \mathbb{A}^3$ (from the cone σ which is the positive orthant in $\mathbb{R}^3$) and coordinate ring

$$\mathbb{C}[X] = \mathbb{C}[x_1, x_2, x_3, x_1\xi, x_2^{j_1}x_3^{k_1}\xi, \dots, x_2^{j_\ell}x_3^{k_\ell}\xi]$$

for some fixed $\ell \in \mathbb{N}$, $0 \leq j_1 < \dots < j_\ell$, and $k_1 > \dots > k_\ell \geq 0$ where $(\ell, j_1, k_1) \neq (1, 0, 0)$. Observe that $\mathcal{B}_\sigma = \{x_1, x_2^{j_i} x_3^{k_i}\}_{i=1, \dots, \ell}$. By Corollary B.3.12, every nontrivial toric supervariety with supertorus T and underlying variety $X_{\bar{0}}$ is of this form, since $J_{(1,0,0)} = (x_1)$.

Denote by τ_{ij} the two-dimensional cone generated by e_i and e_j , and by ρ_i the one-dimensional cone generated by e_i . One can verify the following computations, in which we conflate maximal ideals with classical points in $\mathbb{A}^3$:

- For $\mathfrak{p} \in \text{Orb}(\sigma) = \{(0, 0, 0)\}$, the fiber is

$$J(\mathfrak{p}) \cong \mathbb{C}x_1 \oplus \bigoplus_{i=1}^{\ell} \mathbb{C}x_2^{j_i} x_3^{k_i}$$

- For $\mathfrak{p} \in \text{Orb}(\tau_{12}) = 0 \times 0 \times \mathbb{C}^\times$, the fiber is

$$J(\mathfrak{p}) \cong \begin{cases} \mathbb{C}x_1 \oplus \mathbb{C}x_2^{j_1} & \text{if } \ell > 0 \text{ and } j_1 > 0 \\ \mathbb{C}1 & \text{if } \ell > 0 \text{ and } j_1 = 0 \\ \mathbb{C}x_1 & \text{if } \ell = 0 \end{cases}$$

- For $\mathfrak{p} \in \text{Orb}(\tau_{13}) = 0 \times \mathbb{C}^\times \times 0$, the fiber is

$$J(\mathfrak{p}) \cong \begin{cases} \mathbb{C}x_1 \oplus \mathbb{C}x_3^{k_\ell} & \text{if } \ell > 0 \text{ and } k_\ell > 0 \\ \mathbb{C}1 & \text{if } \ell > 0 \text{ and } k_\ell = 0 \\ \mathbb{C}x_1 & \text{if } \ell = 0 \end{cases}$$

- For $\mathfrak{p} \in \text{Orb}(\rho_1) = 0 \times \mathbb{C}^\times \times \mathbb{C}^\times$, the fiber is

$$J(\mathfrak{p}) \cong \begin{cases} \mathbb{C}1 & \text{if } \ell > 0 \\ \mathbb{C}x_1 & \text{if } \ell = 0 \end{cases}$$

- The fiber at any other closed point is $\mathbb{C}1$, since x_1^{-1} is present in the local ring and $x_1 \xi \in \mathbb{C}[X]$.

B.7 Orbits and stabilizers

In the ordinary setting, both the orbit and the stabilizer of a closed point in a toric variety can be viewed as algebraic tori. In particular, if the torus is T_N and $\mathfrak{p} \in \text{Orb}(\sigma)$, let N_σ be the sublattice of N spanned by the points in $\sigma \cap N$, and $N(\sigma) = N/N_\sigma$. Then $\text{Stab}_{T_N}(\mathfrak{p}) \cong T_{N_\sigma}$ and $T_N \cdot \mathfrak{p} = \text{Orb}(\sigma) \cong T_{N(\sigma)}$. Additionally, the Zariski closure of the orbit $\text{Orb}(\sigma)$ is a toric variety with torus $T_{N(\sigma)}$. We will prove analogous results in the super case.

We recall from [13] the notion of the stabilizer supergroup.

Definition B.7.1. Let G be an affine algebraic supergroup acting on a supervariety X . Given a closed point $\mathfrak{p} \in |X|$, the *stabilizer supergroup functor* is the group-valued functor $\text{Stab}_G(\mathfrak{p})$ on the category of superalgebras, given by

$$\text{Stab}_G(\mathfrak{p})(\mathcal{A}) = \{g \in G(\mathcal{A}) \mid g \cdot \mathfrak{p}_{\mathcal{A}} = \mathfrak{p}_{\mathcal{A}}\}$$

where $\mathfrak{p}_{\mathcal{A}}$ is $\mathfrak{p}$, viewed as an $\mathcal{A}$ -point of X . By [13], $\text{Stab}_G(\mathfrak{p})$ is naturally an affine algebraic supergroup.

We may similarly define the *orbit functor* $G \cdot \mathfrak{p}$ as

$$(G \cdot \mathfrak{p})(\mathcal{A}) = G(\mathcal{A}) \cdot \mathfrak{p}_{\mathcal{A}}$$

so that the orbit-stabilizer theorem holds, in the sense that

$$(G \cdot \mathfrak{p})(\mathcal{A}) \cong G(\mathcal{A}) / \text{Stab}_G(\mathfrak{p})(\mathcal{A})$$

at the level of sets.

Let $\mathfrak{p} \in |X|$ be a closed point in a toric supervariety X with one odd dimension. To compute its orbit and stabilizer, we may assume without loss of generality that $X = Y_{\mathcal{A}, \mathcal{B}}$ is affine, so that $X = \text{Spec } \mathbb{C}[x_1, \dots, x_r, \xi_1, \dots, \xi_s] / J(Y_{\mathcal{A}, \mathcal{B}})$ by the proof of Lemma B.3.4(c). Then $\mathfrak{p} = (a_1, \dots, a_r \mid 0, \dots, 0)$ for some $a_i \in \mathbb{C}$ on which all the ordinary polynomials in $J(Y_{\mathcal{A}, \mathcal{B}})$ vanish. Suppose that $\mathbb{C}[x_1, \dots, x_r, \xi_1, \dots, \xi_s] / J(Y_{\mathcal{A}, \mathcal{B}}) \cong \mathbb{C}[x^{m_1}, \dots, x^{m_r}, x^{n_1}\xi, \dots, x^{n_s}\xi]$, and recall that $n_j = \sum_{i=1}^r p_{i,j} m_i$ for some $p_{i,j} \in \mathbb{N}$, and $n_j = m_j$ for $j = 1, \dots, q$.

Proposition B.7.2. *Using the above notation, it holds that*

$$\text{Stab}_{T_{N,c}}(\mathfrak{p}) \cong \begin{cases} T_{N_{\sigma},c} & \text{if } a^{p \cdot j} = 0 \text{ for all } i = 1, \dots, s \\ T_{N_{\sigma}} & \text{otherwise} \end{cases}$$

where c lies in the subspace $(N_{\sigma})_{\mathbb{C}} \subset N_{\mathbb{C}}$ and

$$T_{N,c} \cdot \mathfrak{p} \cong \begin{cases} T_{N(\sigma)} & \text{if } a^{p \cdot j} = 0 \text{ for all } i = 1, \dots, s \\ T_{N(\sigma), \bar{c}} & \text{otherwise} \end{cases}$$

where $\bar{c}$ is the image of c under $N \rightarrow N(\sigma)$.

Proof. For $(t \mid \xi) \in T_{N,c}(\mathcal{A})$ and $(a_1, \dots, a_r \mid \eta_1, \dots, \eta_s) \in X(\mathcal{A})$, we use the group law from Proposition B.3.1 to calculate

$$\begin{aligned} & (t \mid \xi) \cdot (a_1, \dots, a_r \mid \eta_1, \dots, \eta_s) \\ &= (t^{m_1} a_1 - t^{m_1} \langle m_1, c \rangle \xi \eta_1, \dots, t^{m_q} a_1 - t^{m_q} \langle m_q, c \rangle \xi \eta_q, \\ & \quad t^{m_{q+1}} a_{q+1}, \dots, t^{m_r} a_r \mid \\ & \quad t^{n_1} a_1 \xi + t^{n_1} \eta_1, \dots, t^{n_q} a_q \xi + t^{n_q} \eta_q, \\ & \quad t^{n_{q+1}} a^{p \cdot q+1} \xi + t^{n_{q+1}} \eta_{q+1}, \dots, t^{n_s} a^{p \cdot s} \xi + t^{n_s} \eta_s) \end{aligned}$$

and so in particular,

$$\begin{aligned}
(t \mid \xi) \cdot \mathfrak{p}_{\mathcal{A}} &= (t \mid \xi) \cdot (a_1, \dots, a_r \mid 0, \dots, 0) \\
&= (t^{m_1} a_1, \dots, t^{m_r} a_r \mid t^{n_1} a^{p,1} \xi, \dots, t^{n_s} a^{p,s} \xi) \\
&= (t^{m_1} a_1, \dots, t^{m_r} a_r \mid t^{m_1} a_1 \xi, \dots, t^{m_q} a_q \xi, t^{n_{q+1}} a^{p,q+1} \xi, \dots, t^{n_s} a^{p,s} \xi).
\end{aligned}$$

It follows that $(t \mid \xi)$ stabilizes $\mathfrak{p}_{\mathcal{A}}$ if and only if $t \in \text{Stab}_{T_N}(\mathfrak{p}) \cong T_{N_\sigma}$ and either $a^{p,j} = 0$ for all $j = 1, \dots, s$, or $\xi = 0$.

In the former case, this implies that $a_j = 0$ for $j = 1, \dots, q$. Under the correspondence between closed points of $\text{Spec } \mathbb{C}[S_\sigma]$ and semigroup homomorphisms $S_\sigma \rightarrow \mathbb{C}$ ([9], Proposition 1.3.1), the homomorphism corresponding to $\mathfrak{p}$ sends $m_1, \dots, m_q$ to 0. Therefore these m_j are not invertible in S_σ , so in particular $m_j \notin \sigma^\perp$. In order to see that $c \in (N_\sigma)_{\mathbb{C}}$, we must show that $\langle \sigma^\perp, c \rangle = 0$. But $\sigma^\perp$ must be contained in the span of $\{m_{q+1}, \dots, m_r\}$ because $m_1, \dots, m_q \notin \sigma^\perp$. Since $\{m_{q+1}, \dots, m_r\} \subseteq \ker c$, it indeed follows that $c \in (N_\sigma)_{\mathbb{C}}$.

Now observe that in either case, $\text{Stab}_{T_{N,c}}(\mathfrak{p})(\mathcal{A})$ is a normal subgroup of $X(\mathcal{A})$, so the orbit identity also follows by taking the respective quotients. $\square$

Example B.7.3. Let $T = T_{\mathbb{Z}^2, (1,0)}$. We consider the family of toric supervarieties $X^n := \text{Spec } \mathbb{C}[x_1, x_2, x_1 \xi, x_2^n \xi]$ for $n \geq 0$ and $X^\infty = \text{Spec } \mathbb{C}[x_1, x_2, x_1 \xi]$. Let $\mathfrak{p} \in |X^n| = \mathbb{A}^2$ be the closed point (a, b) . We calculate the stabilizers for such points in each of these toric supervarieties:

$\text{Stab}_{T_{\mathbb{Z}^2, (1,0)}}(a, b)$			
	X^0	X^n for $n > 0$	X^∞
$a = b = 0$	$T_{\mathbb{Z}^2}$	$T_{\mathbb{Z}^2, (1,0)}$	$T_{\mathbb{Z}^2, (1,0)}$
$a = 0, b \neq 0$	$T_{\mathbb{Z} \times 0}$	$T_{\mathbb{Z} \times 0}$	$T_{\mathbb{Z} \times 0, 1}$
$a \neq 0, b = 0$	$T_{0 \times \mathbb{Z}}$	$T_{0 \times \mathbb{Z}}$	$T_{0 \times \mathbb{Z}}$
$a, b \neq 0$	T_{0^2}	T_{0^2}	T_{0^2}

The orbits can be calculated from these data using the orbit-stabilizer theorem.

If $X_{\Sigma, \mathcal{B}}$ is a quasinormal toric supervariety with torus $T_{N,c}$ and $\sigma \in \Sigma$, we denote by $\text{Orb}_c(\sigma) = T_{N,c} \cdot \mathfrak{p}$ the $T_{N,c}$ -orbit of a closed point $\mathfrak{p} \in \text{Orb}(\sigma)$. Recall that the (Zariski) closure of $\text{Orb}_c(\sigma)$ in $X_{\Sigma, \mathcal{B}}$ is the smallest closed subvariety of $X_{\Sigma, \mathcal{B}}$ containing $\text{Orb}_c(\sigma)$. We denote this closure by $X_{\Sigma, \mathcal{B}}(\sigma)$.

Proposition B.7.4. *Using the above notation, $X_{\Sigma, \mathcal{B}}(\sigma)$ is a toric supervariety with torus $\text{Orb}_c(\sigma)$. We have two cases:*

- (a) *If $\text{Orb}_c(\sigma) \cong T_{N(\sigma)}$ as in the first case of the previous proposition, then $X_{\Sigma, \mathcal{B}}(\sigma)$ is the ordinary toric variety with fan $\bar{\Sigma} = \{\bar{\tau} \mid \sigma \leq \tau\}$ where $\bar{\tau}$ denotes the image of τ in $N(\sigma)_{\mathbb{R}} = N_{\mathbb{R}} / (N_\sigma)_{\mathbb{R}}$.*
- (b) *If $\text{Orb}_c(\sigma) \cong T_{N(\sigma), \bar{c}}$ as in the second case of the previous proposition, then $X_{\Sigma, \mathcal{B}}(\sigma)$ is the toric supervariety with fan $\bar{\Sigma}$ as above, and decorations $\mathcal{B}_{\bar{\tau}} = \mathcal{B}_\tau \cap \sigma^\perp$ for $\tau \geq \sigma$.*

Proof. Part (a) follows immediately from the classical situation ([9], Proposition 3.2.7). We obtain the fan in part (b) likewise. The fact that $X_{\Sigma, \mathcal{B}}(\sigma)$ is a toric supervariety follows immediately from the definitions. It therefore remains to find the decorations of part (b).

Since $X_{\Sigma, \mathcal{B}}(\sigma)$ is a closed subvariety of $X_{\Sigma, \mathcal{B}}$, it is affine-locally a quotient at the level of rings. That is, if $U_{\bar{\tau}} \subseteq X_{\Sigma, \mathcal{B}}(\sigma)$ is the affine open corresponding to $\bar{\tau}$ for $\tau \geq \sigma$, then $\mathbb{C}[U_{\bar{\tau}, \mathcal{B}_{\bar{\tau}}}]$ is a quotient of $\mathbb{C}[U_{\tau, \mathcal{B}_{\tau}}]$. In particular, $\mathbb{C}[U_{\tau, \mathcal{B}_{\tau}}] \rightarrow \mathbb{C}[U_{\bar{\tau}, \mathcal{B}_{\bar{\tau}}}]$ is the surjection $\mathbb{C}[S_{\tau}, \xi \mathcal{B}_{\tau}] \rightarrow \mathbb{C}[S_{\tau} \cap \sigma^{\perp}, \xi(\mathcal{B}_{\tau} \cap \sigma^{\perp})]$ given by mapping any x^m with $m \notin \sigma^{\perp}$ to 0. Therefore $\mathcal{B}_{\bar{\tau}}$ can be viewed as $\mathcal{B}_{\tau} \cap \sigma^{\perp}$, so we are finished. $\square$

In both cases of the above proposition, we observe that $X_{\Sigma, \mathcal{B}}(\sigma)$ is quasinormal. Moreover, if the original supervariety $X_{\Sigma, \mathcal{B}}$ is smooth, then so is $X_{\Sigma, \mathcal{B}}(\sigma)$: the underlying variety remains smooth since the new fan $\bar{\Sigma}$ consists of smooth cones as in the classical setting. For smoothness in the odd directions, we note in case (b) that the decorations $\mathcal{B}_{\bar{\tau}}$ do not grow in size, so they remain singletons.

B.8 Fiber products

In this section, we determine when fiber products (i.e. pullbacks) exist in the category of quasinormal toric supervarieties of one odd dimension. Consider the diagram

$$\begin{array}{ccc} & (N', c', \Sigma', \mathcal{B}') & \\ & \downarrow (\bar{\phi}', a') & \\ (N'', c'', \Sigma'', \mathcal{B}'') & \xrightarrow{(\bar{\phi}'', a'')} & (Q, d, \Lambda, \mathcal{C}) \end{array}$$

in the category of decorated fans, and suppose a' and a'' are not both 0. We construct the fiber product as follows:

Define a lattice

$$N := N' \times_Q N'' = \{(n', n'') \in N' \times N'' \mid \bar{\phi}'(n') = \bar{\phi}''(n'')\}$$

and a fan

$$\Sigma := \Sigma' \times_{\Lambda} \Sigma'' = \{(\sigma', \sigma'') \in \Sigma' \times \Sigma'' \mid \text{there is } \lambda \in \Lambda \text{ such that } \bar{\phi}'_{\mathbb{R}}(\sigma'), \bar{\phi}''_{\mathbb{R}}(\sigma'') \subseteq \lambda\}$$

so that (N, Σ) is the fiber product of

$$\begin{array}{ccc} & (N', \Sigma') & \\ & \downarrow \bar{\phi}' & \\ (N'', \Sigma'') & \xrightarrow{\bar{\phi}''} & (Q, \Lambda) \end{array}$$

in the usual category of fans (or equivalently, toric varieties). Define

$$c = c' \times_d c'' := ((a'')^2 c', (a')^2 c'')$$

and observe that

$$\begin{aligned}\bar{\phi}'((a'')^2 c') &= (a' a'')^2 d \\ &= \bar{\phi}''((a')^2 c''),\end{aligned}$$

so $c \in N_{\mathbb{C}}$.

It remains to describe the decorations. Towards this end, we first describe the character lattice $M = N^*$. We also write $M' = (N')^*$, $M'' = (N'')^*$, and $P = Q^*$.

Definition B.8.1. Let S be a semigroup contained in a lattice L . The *saturation* of S in L is given by $\text{Sat}(S) = \{\ell \in L \mid k\ell \in S \text{ for some } k \in \mathbb{N}\}$.

We collect some elementary results about saturations of semigroups:

Lemma B.8.2. (a) If $S \subseteq L$ is a semigroup, then $\text{Sat}(S)$ is the smallest saturated semigroup in L that contains S .

(b) Let L' be a sublattice of a lattice L . Then L/L' is also a lattice if and only if L' is saturated in L .

(c) The saturation of an affine semigroup is affine.

We may now apply this lemma to describe M .

Lemma B.8.3. (a) The natural surjection $\pi : M' \times M'' \rightarrow M$ has kernel $\text{Sat}((\bar{\phi}')^* \times (-\bar{\phi}'')^*(P))$, so that $M \cong (M' \times M'') / \text{Sat}((\bar{\phi}')^* \times (-\bar{\phi}'')^*(P))$.

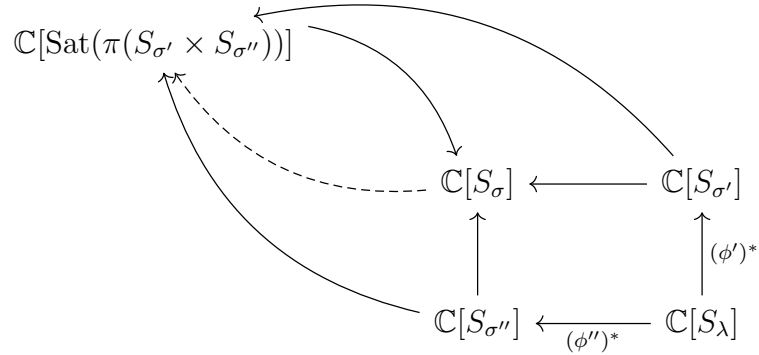
(b) If $\sigma = (\sigma', \sigma'') \in \Sigma$, then S_σ is the saturation of $\pi(S_{\sigma'} \times S_{\sigma''})$.

Proof. (a) We have a surjection $\pi : M' \times M'' \rightarrow M$ given by $\langle \pi(m', m''), (n', n'') \rangle = \langle m', n' \rangle + \langle m'', n'' \rangle$ for $(m', m'') \in M' \times M''$ and $(n', n'') \in N$. Notice that

$$\begin{aligned}\langle \pi(m' + (\bar{\phi}')^*(p), m'' - (\bar{\phi}'')^*(p)), (n', n'') \rangle \\ &= \langle m', n' \rangle + \langle m'', n'' \rangle + \langle (\bar{\phi}')^*(p), n' \rangle - \langle (\bar{\phi}'')^*(p), n' \rangle \\ &= \langle m', n' \rangle + \langle m'', n'' \rangle + \langle p, \bar{\phi}'(n') \rangle - \langle p, \bar{\phi}''(n'') \rangle \\ &= \langle m', n' \rangle + \langle m'', n'' \rangle\end{aligned}$$

since $\phi'(n') = \phi''(n'')$, so $\ker \pi$ contains $(\bar{\phi}')^* \times (-\bar{\phi}'')^*(P)$. But M is a lattice, so by Lemma B.8.2(b) we know $\ker \pi$ is saturated and hence by (a), $\ker \pi \supseteq \text{Sat}((\bar{\phi}')^* \times (-\bar{\phi}'')^*(P))$. Equality follows by counting dimensions.

(b) We first note that $S_\sigma \supseteq \pi(S_{\sigma'} \times S_{\sigma''})$ because if $(m', m'') \in S_{\sigma'} \times S_{\sigma''}$, then both summands of $\langle \pi(m', m''), (n', n'') \rangle = \langle m', n' \rangle + \langle m'', n'' \rangle$ are nonnegative for all $(n', n'') \in \sigma$. Then $\text{Sat}(\pi(S_{\sigma'} \times S_{\sigma''})) \subseteq S_\sigma$ is a saturated semigroup which is affine by Lemma B.8.2(c). Hence by the universal property of the fiber product, the following diagram commutes:



In this diagram, the dashed arrow arises from the universal property of the fiber product in the (opposite) category of toric varieties, and the unlabelled solid arrows arise from the natural morphisms of semigroups. The composition of the dashed arrow with the arrow in the reverse direction is the unique endomorphism of $\mathbb{C}[S_{\sigma}]$ satisfying the universal property. Since $\text{Sat}(\pi(S_{\sigma'} \times S_{\sigma''})) \subseteq S_{\sigma}$, these arrows are isomorphisms. Hence $S_{\sigma} = \text{Sat}(\pi(S_{\sigma'} \times S_{\sigma''}))$. $\square$

We now describe the decorations. For each $\sigma = (\sigma', \sigma'') \in \Sigma$, let

$$J^{\sigma} = \begin{cases} J_c^{\sigma} + ((\mathcal{B}'_{\sigma'} \times \{0\}) \cup (\{0\} \times \mathcal{B}''_{\sigma''})) & \text{if } a', a'' \neq 0 \\ J_c^{\sigma} + (\mathcal{B}'_{\sigma'} \times \{0\}) & \text{if } a' = 0 \\ J_c^{\sigma} + (\{0\} \times \mathcal{B}''_{\sigma''}) & \text{if } a'' = 0 \end{cases}$$

and define $\mathcal{B}_{\sigma} = \mathcal{B}'_{\sigma'} \times_{\mathcal{C}} \mathcal{B}''_{\sigma''}$ as the minimal generating set of J^{σ} . We write $\mathcal{B}' \times_{\mathcal{C}} \mathcal{B}''$ for the decorations $\{\mathcal{B}'_{\sigma'} \times_{\mathcal{C}} \mathcal{B}''_{\sigma''}\}_{(\sigma', \sigma'') \in \Sigma}$.

Lemma B.8.4. *The collection $\mathcal{B} = \mathcal{B}' \times_{\mathcal{C}} \mathcal{B}''$ defined above is c -admissible.*

Proof. It suffices to show that condition (ii') holds. Let $\tau = (\tau', \tau'')$ be a face of $\sigma = (\sigma', \sigma'')$. Let $m \in J^{\sigma}$, so either $m \in J_c^{\sigma}$ or m belongs to the ideal generated by $\mathcal{B}'_{\sigma'} \times \{0\}$ or $\{0\} \times \mathcal{B}''_{\sigma''}$. In the former case, condition (ii') holds because $(J_c^{\sigma})_{x^{m_{\sigma, \tau}}} = J_c^{\tau}$. In the latter case, we apply condition (ii') of $\mathcal{B}'$ or $\mathcal{B}''$ to arrive at the result. The other half of the argument follows by the same reasoning. $\square$

Lemma B.8.5. *Let $\bar{\psi}' : N \rightarrow N'$ and $\bar{\psi}'' : N \rightarrow N''$ be the natural maps. Then $(\bar{\psi}', a'')$ and $(\bar{\psi}'', a')$ are compatible with the decorated fans.*

Proof. Let $\sigma = (\sigma', \sigma'') \in \Sigma$. It suffices to show that $(\psi')^*(\xi' J^{\sigma'}) \subseteq \xi J^{\sigma}$ and $(\psi'')^*(\xi'' J^{\sigma''}) \subseteq \xi J^{\sigma}$. But the images of $J^{\sigma'}$ and $J^{\sigma''}$ are exactly the ideals generated by $\mathcal{B}'_{\sigma'} \times \{0\}$ and $\{0\} \times \mathcal{B}''_{\sigma''}$, respectively. These generators are present in J^{σ} exactly when $a' \neq 0$ and $a'' \neq 0$, respectively, so the lemma follows. $\square$

The above lemmas provide the data sufficient to describe the fiber product.

Proposition B.8.6. *If a' and a'' are not both 0, the fiber product is given by*

$$\begin{array}{ccc}
(N, c, \Sigma, \mathcal{B}) & \xrightarrow{(\bar{\psi}', a'')} & (N', c', \Sigma', \mathcal{B}') \\
(\bar{\psi}'', a') \downarrow & & \downarrow (\bar{\phi}', a') \\
(N'', c'', \Sigma'', \mathcal{B}'') & \xrightarrow{(\bar{\phi}'', a'')} & (Q, d, \Lambda, \mathcal{C})
\end{array}$$

where

$$\begin{aligned}
N &= N' \times_Q N'' \\
c &= c' \times_d c'' \\
\Sigma &= \Sigma' \times_\Lambda \Sigma'' \\
\mathcal{B} &= \mathcal{B}' \times_{\mathcal{C}} \mathcal{B}'' \\
\bar{\psi}' : N &\rightarrow N' \text{ and } \bar{\psi}'' : N \rightarrow N'' \text{ are the natural maps.}
\end{aligned}$$

Proof. We have already seen that the diagram commutes in the category of decorated fans. Suppose we have a commutative diagram

$$\begin{array}{ccc}
& & \xrightarrow{(\tilde{\psi}', \tilde{a}'')} \\
(\tilde{N}, \tilde{c}, \tilde{\Sigma}, \tilde{\mathcal{B}}) & & \searrow \\
& \xrightarrow{(\tilde{\psi}'', \tilde{a}')} & (N, c, \Sigma, \mathcal{B}) \xrightarrow{(\bar{\psi}', a'')} (N', c', \Sigma', \mathcal{B}') \\
& & \downarrow (\bar{\psi}'', a') \quad \downarrow (\bar{\phi}', a') \\
& & (N'', c'', \Sigma'', \mathcal{B}'') \xrightarrow{(\bar{\phi}'', a'')} (Q, d, \Lambda, \mathcal{C})
\end{array}$$

so there is a unique map $\tilde{\varphi} : \tilde{N} \rightarrow N$ compatible with the fans $\tilde{\Sigma}$ and Σ .

If $a' \neq 0$, then define $\tilde{a} = \frac{\tilde{a}'}{a'}$. Likewise if $a'' \neq 0$, we define $\tilde{a} = \frac{\tilde{a}''}{a''}$. If both are nonzero, we observe that $a'\tilde{a}'' = \tilde{a}'a''$ and hence $\tilde{a}$ is well-defined, and unique with respect to the property that $\tilde{a}a' = \tilde{a}'$ and $\tilde{a}a'' = \tilde{a}''$. Thus if

$$(\tilde{\varphi}, \tilde{a}) : (\tilde{N}, \tilde{c}, \tilde{\Sigma}, \tilde{\mathcal{B}}) \rightarrow (N, c, \Sigma, \mathcal{B})$$

is a morphism of decorated fans, it satisfies the universal property of the fiber product in the above diagram. We have

$$\begin{aligned}
\tilde{\varphi}_{\mathbb{C}}(\tilde{c}) &= ((\tilde{a}'')^2 c', (\tilde{a}')^2, c'') \\
&= \tilde{a}^2 ((a'')^2 c', (a')^2, c'') \\
&= \tilde{a}^2 c,
\end{aligned}$$

so it remains to show that if $\tilde{a} \neq 0$, then $\tilde{\varphi}$ is compatible with the decorations.

Suppose $\tilde{\varphi}_{\mathbb{R}}(\tilde{\sigma}) \subseteq \sigma$. Looking at the affine patches defined by $\tilde{\sigma}$ and $\sigma = (\sigma', \sigma'')$, we obtain the diagram

$$\begin{array}{ccccc}
& & & & (\tilde{\psi}')^* \\
& & & \swarrow & \\
\mathbb{C}[S_{\tilde{\sigma}}, \tilde{\xi} \tilde{\mathcal{B}}_{\tilde{\sigma}}] & & & & \\
& \nwarrow & & & \\
& & & & (\tilde{\psi}')^* \\
& & & & \\
& & \mathbb{C}[S_{\sigma}, \xi \mathcal{B}_{\sigma}] & \xleftarrow{(\psi')^*} & \mathbb{C}[S_{\sigma'}, \xi' \mathcal{B}'_{\sigma'}] \\
& & \uparrow (\psi'')^* & & \uparrow (\phi')^* \\
& & \mathbb{C}[S_{\sigma''}, \xi'' \mathcal{B}''_{\sigma''}] & \xleftarrow{(\phi'')^*} & \mathbb{C}[S_{\lambda}, \eta \mathcal{C}_{\lambda}]
\end{array}$$

Let $m \in J^{\sigma}$, where $\tilde{\varphi}_{\mathbb{R}}(\tilde{\sigma}) \subseteq \sigma$. Then either $m \in J_c^{\sigma}$, or $m \in (\mathcal{B}'_{\sigma'} \times \{0\})$ and $a'' \neq 0$, or $m \in (\{0\} \times \mathcal{B}''_{\sigma''})$ and $a' \neq 0$. If $m \in J_c^{\sigma}$, then $\varphi^*(m) \in J_c^{\tilde{\sigma}}$ by Lemma B.4.3. Otherwise we may assume without loss of generality that $m \in (\mathcal{B}'_{\sigma'} \times \{0\})$ and $a'' \neq 0$. Then there is $b \in \mathcal{B}'_{\sigma'}$ such that $b \leq_{\sigma} m$. It follows that $(\tilde{\psi}')^*(x^b \xi')$ is nonzero in $\mathbb{C}[S_{\tilde{\sigma}}, \tilde{\xi} \tilde{\mathcal{B}}_{\tilde{\sigma}}]$ since $a'' \neq 0$. Hence $(\tilde{\psi}')^*(b) \leq_{\tilde{\sigma}} \tilde{\varphi}^*(m)$ and $(\tilde{\psi}')^*(b) \in J^{\tilde{\sigma}}$, so indeed $\tilde{\varphi}^*(m) \in J^{\tilde{\sigma}}$, as desired. Therefore $\tilde{\varphi}$ is compatible with the decorations, and we are finished. $\square$

Remark B.8.7. If $a' = a'' = 0$, then the fiber product does not exist in general. In particular, commutativity of the diagram

$$\begin{array}{ccccc}
& & & & (\tilde{\psi}', \tilde{b}') \\
& & & \searrow & \\
(\tilde{N}, \tilde{c}, \tilde{\Sigma}, \tilde{\mathcal{B}}) & & & & \\
& \searrow & & & \\
& & & & (\tilde{\psi}'', \tilde{b}'') \\
& & & & \\
& & (N, c, \Sigma, \mathcal{B}) & \xrightarrow{(\bar{\psi}', b')} & (N', c', \Sigma', \mathcal{B}') \\
& & \downarrow (\bar{\psi}'', b'') & & \downarrow (\bar{\phi}', 0) \\
& & (N'', c'', \Sigma'', \mathcal{B}'') & \xrightarrow{(\bar{\phi}'', 0)} & (Q, d, \Lambda, \mathcal{C})
\end{array}$$

does not guarantee the existence of $\tilde{b}$ such that $\tilde{b}' = \tilde{b}b'$ and $\tilde{b}'' = \tilde{b}b''$.