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Maximum Principles for High Order Accurate Finite Difference Discretizations of the
Laplace Operator

by

William Thacher

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Applied Science and Technology

in the

Graduate Division

of the

University of California, Berkeley

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Abstract

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Doctor of Philosophy in Applied Science and Technology

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Professor Phillip Colella, Co-Chair

Professor Per-Olof Persson, Co-Chair

In this thesis we prove maximum principles for high order accurate, lattice-based finite difference discretizations of the Laplace operator in one, two, and three dimensions. The primary finding of this work is that with the introduction of carefully constructed perturbations, we can recover “local” maximum principles for high order accurate operators which are similar to those that naturally hold for second order accurate operators. From these local maximum principles we prove global results which are discrete analogues of the classical weak maximum principle.

We first discuss precisely why the typical discrete maximum principle fails for high order accurate discrete second derivative operators in one dimension. We then introduce a quantity which is the maximum of several perturbations of a grid function, and we establish a local maximum principle using these perturbations. We show that the proof can be reduced to the satisfaction of a system of linear inequalities. Next, we show how this proof can be interpreted as a “smoothing” operation on certain basis functions.

To extend this machinery to higher dimensions, we obtain two generalizations of the one dimensional results. The first is showing the equivalence of the perturbed variable local maximum principle and a certain linear program. The second is a higher dimensional version of the basis function smoothing argument. We combine these results into an inductive procedure which simultaneously builds up perturbations and increasingly accurate Laplacian stencils which satisfy an associated local maximum principle. Using this procedure, we generate sequences of high order accurate Laplacians in two and three dimensions.

Lastly, we obtain global maximum principles from these local results, and prove stability estimates using comparison functions. We conclude with a discussion of future research.

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Chapter 1

Introduction

The finite difference method is one of the oldest and most widely used numerical techniques for solving second order elliptic boundary value problems. The prototypical example is the Dirichlet problem for the Poisson equation; that is, find a function u which satisfies

$$\Delta u = f \in \Omega \tag{1.1}$$

$$u = g \in \partial\Omega \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ is an open set and f, g are given functions. We consider u to be a classical solution if it satisfies these conditions pointwise in Ω , with derivatives defined in the usual sense. The essential idea of the finite difference method is to replace the domain with a discrete set of points, and to replace partial derivative operators with difference quotients. The hope is that if the finite difference operator is a good approximation of the elliptic operator, then the solution to the discretized boundary value problem will be a good approximation of the classical solution to the boundary value problem.

There are several other notions of what it means for u to “solve” this PDE which are based on properties that classical solutions satisfy. Generalized or weak solutions, strong solutions, and viscosity solutions are three such examples. The most basic question for a particular notion of a solution is whether such a u exists and whether it is unique, given certain assumptions on Ω, f, g . To answer this question, one needs a technique for deducing information about u given knowledge of Δu in Ω and u on $\partial\Omega$.

One such technique is to make use of a maximum principle. The basic statement is that if $\Delta u \geq 0 \in \Omega$, then $\sup_{\Omega} u = \sup_{\partial\Omega} u$; in words, a subharmonic function cannot achieve its maximum in the interior of the domain. There are different versions of the maximum principle for different notions of solution. From this maximum principle one can prove existence and uniqueness, as well as more refined results about elliptic equations [13], [9].

In a very natural way, standard second order accurate finite difference schemes satisfy a discrete version of the classical weak maximum principle [12]. This facilitates relatively simple proofs of existence and uniqueness of solutions to the difference equations, as well as the rate of convergence of the numerical solution to the classical solution. However, the proof of this discrete maximum principle for the second order case relies on a local property

which explicitly cannot hold for higher (greater than second) order accurate methods. It is for this reason that there has been very little work on proving discrete maximum principles for high order accurate schemes. In fact, there is very little stability analysis period on these schemes, as there are no alternative analytical tools that have been able to surmount this second order barrier.

In contrast, there is an extremely rich theory for Galerkin methods based on analytical tools for weak solutions which extends to higher order accuracy [7]. In many cases the analysis is not inherently bound to low order accurate schemes or a particular dimension; it is often quite possible to build function spaces using high order polynomials which satisfy the same essential properties that make low order accurate schemes stable. Thus there has been a great development of high order accurate, provably stable Galerkin schemes for solving second order elliptic equations.

Despite the lack of analytical tools available, there has also been substantial development of high order accurate finite difference schemes, with authors pushing the order of accuracy, geometric complexity, and incorporation into simulation codes. The thesis [11] contains a thorough literature search on “immersed” boundary schemes, in which the boundary can be arbitrarily oriented in a Cartesian mesh, as well as applications of these methods to a variety of elliptic equations. High order accurate finite volume methods on block structured grids [5] have a similar formulation to lattice-based finite difference methods. For unstructured grids, interpolation techniques such as moving least squares and radial basis functions have been utilized to construct high order accurate schemes [10]. Compact finite difference schemes for solving the Poisson equation were suggested as early as [6], and were studied using Fourier analysis in [15]. Numerical evidence and adoption into sophisticated physics codes demonstrate that it is possible to design arbitrarily accurate methods that do indeed “work” as expected, meaning high order convergence of the error is observed.

Empirically, these high order accurate schemes appear to share certain stability properties with second order accurate schemes, yet there is nothing explicit in their construction that enforces these properties. The motivation for this thesis is to understand this phenomenon. The primary obstacle to this analysis appears to be the order of accuracy, rather than the particulars of the grid or the elliptic equation of interest, and consequently we will narrow our focus.

The definition of a linear second order elliptic operator only demands that the coefficients of the second order partial derivatives satisfy a certain ellipticity condition. However, it is often the case that results which hold for the Laplace operator hold in some form for general elliptic operators. The same appears to be true to some extent for finite difference schemes, as past authors have been able to deal with variable coefficients as well as first and zeroth order terms. Thus in this thesis we deal solely with the Laplace operator, which we believe will not diminish the applicability of our tools to other elliptic equations.

There are also many different choices one can make when designing a finite difference scheme, including the layout of nodes, the method for deriving consistent finite difference stencils, the order of accuracy, and the enforcement of boundary conditions. In this thesis we consider the simplest of these options: lattice-based schemes, centered difference operators,

and some type of “immersed” or “embedded” boundary treatment. We now review prior work on maximum principle type results for this class of finite difference schemes, focusing on the Laplace operator.

1.1 Prior Work

We first discuss the second order accurate case.

1.1.1 Second Order Accurate Methods

NOTATION. Let $h > 0$ be the grid spacing parameter. Points $i \in \mathbb{Z}^n$ with components i_d will be the indices of points $x_i := ih \in \mathbb{R}^n$ with components $x_{i,d}$. We let B_r denote the closed ball on $\mathbb{Z}^n$ of radius r , or $\{i \in \mathbb{Z}^n : |i|_{\ell^2} \leq r\}$, and ∂B_r denote the set of all lattice points on the boundary of B_r . Let Q_r be the closed cube on $\mathbb{Z}^n$ of radius r , or $\{i \in \mathbb{Z}^n : |i|_{\ell^\infty} \leq r\}$. $B_r(i), Q_r(i)$ will indicate translations of these sets. If Ω is a bounded, connected, open set in $\mathbb{R}^n$, we let Ω^h indicate the set of all grid points x_i which are contained inside Ω . When not specified, assume $h = 1$. We will work with various notions of a discrete boundary $\partial\Omega^h$ throughout this work; for example, it may be the set of points formed by slightly enlarging the set Ω^h . $\bar{\Omega}^h$ will be defined as $\Omega^h \cup \partial\Omega^h$.

Let u_i denote the value of the grid function u^h at the grid point x_i . If L^h is an operator acting on grid functions, let $L^h u_i$ denote the value of L^h applied to u^h at the grid point x_i . Let T_k be the translation operator:

$$T_k u_i = u_{i+k} \quad (1.3)$$

and define the discrete second derivative operator:

$$\partial^{2,h} := \frac{1}{h^2} (T_{-1} - 2T_0 + T_1) \quad (1.4)$$

Let Δ^h denote the second order accurate discrete Laplace operator:

$$\Delta^h := \sum_{d=1}^n \partial_d^{2,h} \quad (1.5)$$

where $\partial_d^{2,h}$ indicates a discrete partial derivative in the d coordinate direction, and e_d indicates the unit vector in direction d . This stencil has an important property:

DEFINITION. A discrete operator is **positive type** if the center coefficient is negative and all other coefficients are nonnegative.

From this property one can prove a discrete maximum principle which is often first credited to [12]. Let Ω^h be a bounded subset of $\mathbb{Z}^n$. Let $\partial\Omega^h$ be the smallest set of points such that $Q_1(i)$ is contained in $\bar{\Omega}^h$ for all $i \in \Omega^h$. Set $h = 1$. Then we have the following analogue of the classical weak maximum principle:

THEOREM 1.1.1. *Suppose $\Delta^h u^h \geq 0 \in \Omega^h$. Then $\max_{\overline{\Omega^h}} u^h = \max_{\partial\Omega^h} u^h$.*

Proof: First suppose that $\Delta^h u^h > 0 \in \Omega^h$. Since Δ^h is positive type at each index i in Ω^h , we have that

$$\Delta^h u_i = \sum_{d=1}^n \partial^{2,h} u_i > 0 \quad (1.6)$$

which from (1.4) implies that

$$u_i < \frac{1}{2n} \sum_{d=1}^n (u_{i+e_d} + u_{i-e_d}) \leq \max_{j \in Q_1(i)} u_j \quad (1.7)$$

Therefore if $\Delta^h u > 0 \in \Omega^h$, the maximum of u^h cannot occur at any point in Ω^h , meaning we must have $\max_{\overline{\Omega^h}} u^h = \max_{\partial\Omega^h} u^h$.

Now suppose $\Delta^h u^h \geq 0 \in \Omega^h$. Let v^h be the restriction of the function x_1^2 to $\overline{\Omega^h}$. Then we have for any $\epsilon > 0$ that $\Delta^h(u + \epsilon v) > 0 \in \Omega^h$ which implies $\max_{\overline{\Omega^h}} u^h + \epsilon v^h = \max_{\partial\Omega^h} u^h + \epsilon v^h$, and we can send $\epsilon \rightarrow 0$ to obtain the desired result. $\square$

From this maximum principle one can use another quadratic comparison function to prove a stability result of the form: Let $u^h \in \ell^\infty(\overline{\Omega^h})$. Then there exists a constant C which depends only on $\text{diam}(\Omega)$ such that

$$|u^h|_{\ell^\infty(\Omega^h)} \leq C |\Delta^h u^h|_{\ell^\infty(\Omega^h)} + |u^h|_{\ell^\infty(\partial\Omega^h)} \quad (1.8)$$

This type of stability result can be employed to prove that particular finite difference schemes for the Poisson equation with Dirichlet boundary conditions produce convergent solutions; for example see [2].

1.1.2 Non-Positive Type Methods

The proofs for the second order accurate case relied on the fact that Δ^h is a positive type operator. For Laplace operators which are greater than third order accurate, one can show that the stencil cannot be of positive type; see Lemma 3.1.1. This has posed a major obstacle to proving maximum principle type results for higher order accurate schemes. We below report two strategies that have proved successful for fourth order schemes in two dimensions.

Monotone Matrices

The works of [16] and [1] deal with fourth order accurate variable coefficient elliptic operators in two dimensions. The fourth order accurate centered difference Laplace operator is defined

as:

$$\partial_4^{2,h} := \frac{1}{h^2} \left(-\frac{5}{2}T_0 + \frac{4}{3}(T_1 + T_{-1}) - \frac{1}{12}(T_2 + T_{-2}) \right) \quad (1.9)$$

$$\Delta_4^h := \sum_{d=1}^n \partial_{4,d}^{2,h} \quad (1.10)$$

which is clearly not of positive type, so the discrete maximum principle which worked for the second order stencil does not hold.

A matrix which has the property $Ax \geq 0 \implies x \geq 0$ is known as a monotone matrix. If we consider our finite difference operator as a matrix, we can see that the discrete maximum principle property which holds for the second order accurate operators from the previous section is equivalent to monotonicity. Letting L^h be the matrix which defines the finite difference Laplace operator with zero Dirichlet boundary conditions, if the discrete maximum principle holds then $-L^h u^h \geq 0 \implies u^h \geq 0$. The approach in [1] and [16] is to prove that the matrix $-L^h$ for a particular fourth-order accurate scheme is monotone, even though $-L^h$ is not itself positive type, by using a specific matrix decomposition. It is not clear that this technique has applicability beyond this specific case, so we refer to [16] for the details of this proof.

Brandt's Generalized Local Maximum Principle

In his paper [4], Brandt introduces the concept of a “generalized local maximum principle” for finite difference stencils which are not of positive type. If $L^h u_i > 0$, it is not necessarily true that u_i is less than a neighboring value. However, Brandt shows that it is possible to define a “perturbed” variable $\bar{u}$ such that $L^h u_i > 0$ implies that $\bar{u}_i$ is less than a neighboring value $\bar{u}_j$.

Again consider the fourth order finite difference stencil for the Laplacian in two dimensions. We simplify the proof of Proposition 5.1 in [4] for this case:

THEOREM 1.1.2. *Define*

$$\bar{u}_i := \max_{s \in \{0, \pm e_d\}} \left\{ \frac{8}{7}u_i - \frac{1}{7}u_{i+s} \right\} \quad (1.11)$$

Then the following holds:

$$\Delta_4^h u_0 > 0 \implies \exists j : \bar{u}_0 < \bar{u}_j \quad (1.12)$$

Proof: Suppose $\bar{u}_0 = u_0 > \max_{s \in \{\pm e_d\}} \left\{ \frac{8}{7}u_0 - \frac{1}{7}u_s \right\}$. Then we have for $s \in \{\pm e_d\}$

$$u_0 > \frac{8}{7}u_0 - \frac{1}{7}u_s \implies \quad (1.13)$$

$$\bar{u}_0 = u_0 < u_s \leq \bar{u}_s \quad (1.14)$$

which proves the claim for the case that $\bar{u}_0 = u_0 > \max_{s \in \{\pm e_d\}} \left\{ \frac{8}{7}u_0 - \frac{1}{7}u_s \right\}$. Now consider the case that $\bar{u} = \frac{8}{7}u_0 - \frac{1}{7}u_{e_1}$; the other three cases will follow from symmetry. We have

$$\Delta_4^h u_0 = -5u_0 + \frac{4}{3}(u_{\pm e_d}) - \frac{1}{12}(u_{\pm 2e_d}) \quad (1.15)$$

$$= -\frac{35}{8} \left(\frac{8}{7}u_0 - \frac{1}{7}u_{e_1} \right) + \frac{1}{24}u_{e_1} + \frac{2}{3}(u_{-e_1} + u_{\pm e_2}) + \frac{7}{12} \sum_{s \in \{s \in \pm e_d\}} \left(\frac{8}{7}u_s - \frac{1}{7}u_{2s} \right) \quad (1.16)$$

meaning if $\Delta_4^h u_0 > 0$, we have:

$$\left(\frac{8}{7}u_0 - \frac{1}{7}u_{e_1} \right) < \frac{8}{35} \left(\frac{1}{24}u_{e_1} + \frac{2}{3}(u_{-e_1} + u_{\pm e_2}) + \frac{7}{12} \sum_{s \in \{s \in \pm e_d\}} \left(\frac{8}{7}u_s - \frac{1}{7}u_{2s} \right) \right) \quad (1.17)$$

$$\bar{u}_0 < \max_{s \in \pm e_d} \bar{u}_s \quad (1.18)$$

which proves the claim for $\bar{u}_0 = \frac{8}{7}u_0 - \frac{1}{7}u_s$ for $s \in \{\pm e_d\}$. $\square$

As we can see from this proof, while the fourth order stencil is not of positive type, it can be rewritten as a positive type stencil after introducing variables which are derived from u^h . In this work, Brandt goes through several commonly used finite difference stencils for elliptic operators in two dimensions which are not of positive type, such as the nine point stencil for $\partial_x \partial_y$. For each case, a $\bar{u}$ is defined, and it is shown algebraically that the stencil is positive type in the perturbed variables. In some of these cases, $\bar{u}$ varies in space, as it may depend on the coefficients of the operator, and also must be modified near the boundary where the interior stencil cannot be used. If $\bar{u}$ is suitably defined so that at each point in Ω^h the local maximum principle holds, then one can obtain a “global” maximum principle:

$$L^h u^h > 0 \in \Omega^h \implies \max_{\Omega^h} \bar{u}^h \leq \max_{\partial \Omega^h} \bar{u}^h \quad (1.19)$$

If we can set $\bar{u} = u^h \in \partial \Omega^h$, then we will recover the discrete maximum principle:

$$L^h u^h > 0 \in \Omega^h \implies \max_{\Omega^h} u^h \leq \max_{\Omega^h} \bar{u}^h \leq \max_{\partial \Omega^h} \bar{u}^h = \max_{\partial \Omega^h} u^h \quad (1.20)$$

since by definition $\bar{u}_i \geq u_i$.

The strength of this approach is that it provides a flexible template by which one could conceivably prove discrete maximum principles for other high order accurate finite difference operators: if one can find a suitable perturbation and prove that the local maximum principle holds at each point in the domain, then a global maximum principle quickly follows.

The weakness of this approach is that these perturbations are apparently obtained by inspection of the stencil coefficients, and they are not justified by some property of the high order accurate operator. While the example above has a relatively simple definition for $\bar{u}$, other examples in [4] are considerably more complex, and it is not clear how one would adapt

these perturbations for even higher order, larger width stencils. The fact that the definition of $\bar{u}$ needs to be modified near the boundary is another potential difficulty.

However, it turns out that there is a great deal of additional structure hidden in Brandt's approach that allows us to extend it to stencils in one, two and three dimensions, for very high orders of accuracy. His ideas are the starting point for this thesis.

1.2 Outline of Thesis

Broadly, the goal of this thesis is to develop tools for the analysis of high order accurate finite difference methods for elliptic equations. More specifically, this work develops the machinery by which one can prove discrete maximum principles for high order accurate finite difference discretizations of the Laplace operator.

In Chapter 2 we focus on the one dimensional case. We examine the failure of the typical discrete maximum principle when applied to high order accurate operators, and propose a solution using the perturbed variable technique of [4]. In Chapter 3, we fully generalize the perturbed variable approach, and build a method for generating Laplacian stencils in higher dimensions that satisfy local maximum principles. In Chapter 4, we use these local maximum principles to prove global maximum principles and stability results. We conclude with a discussion of applications of these results as well as future research directions.

Chapter 2

Maximum Principles in One Dimension

In this chapter we will motivate and introduce the analytical framework used in this thesis by studying the one dimensional case. Our objective is to obtain a result similar to the classical weak maximum principle [13]:

THEOREM 2.0.1. *Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ for some bounded open set $\Omega \subset \mathbb{R}^n$. If $\Delta u > 0 \in \Omega$ then:*

$$\sup_{\overline{\Omega}} u = \sup_{\partial\Omega} u \quad (2.1)$$

Proof: If u achieves a local maximum at a point x in Ω , then we must have $D^2u(x) \preceq 0$, meaning $\Delta u(x) \leq 0$. Equivalently, if $\Delta u(x) > 0$, $u(x)$ cannot be a local maximum. Therefore, if $\Delta u > 0 \in \Omega$, the maximum must be achieved on the boundary. $\square$

In this chapter we will follow this same two part argument: The first is establishing a relationship between local maxima and second order partial derivatives; we will refer to this as a “local maximum principle.” The second is to establish a relationship between local maxima and global maxima, which we will refer to as a “global maximum principle.” For the second order case both of these steps are relatively straightforward, but as mentioned in the introduction, these same tools cannot be directly applied to higher order accurate stencils which are not of positive type. The majority of this chapter is concerned with building tools that allow us to prove local maximum principles for high order accurate second derivative stencils. We conclude by proving a global maximum principle and a stability result.

2.1 Local Maximum Principles

The basic approach for obtaining a local maximum principle in [4] is to define a perturbed variable and show that the stencil of interest can be rewritten as a positive type stencil in that

perturbed variable. Algebraically, one can see why this is a reasonable approach by looking at the example of the fourth order operator in two dimensions given in the introduction: negative off center stencil values are accounted for by adding a perturbation to values at neighboring stencil points. In [4], the exact form of these perturbations changes for different stencils, and it is not clear how they are related to the stencil itself, or how one could come up with the correct perturbation for a given stencil.

In this section we build intuition on why the low order local maximum principle fails in certain cases for high order accurate second derivative operators, and introduce perturbations that deal with these failure modes. We will first step back and talk about how high order accurate second derivative stencils are constructed. Refer to Section 1.1.1 for notation.

NOTATION. We define the discrete derivative operators:

$$\partial^{1,h} := \frac{1}{2h} (T_1 - T_{-1}) \quad (2.2)$$

$$\partial^{2,h} := \frac{1}{h^2} (T_{-1} - 2T_0 + T_1) \quad (2.3)$$

$$\partial^{2q,h} := \partial^{2,h} \circ_{q-1} \partial^{2,h} \quad (2.4)$$

$$\partial^{2q+1,h} := \partial^{1,h} \circ \partial^{2q,h} \quad (2.5)$$

where $\circ$ indicates operator composition, and $\circ_m$ indicates m operator compositions. We will use $[\partial^{p,h}]$ to indicate a grid function corresponding to the stencil coefficients which is centered at the origin. $\partial^{p,kh}$ indicates a discrete derivative operator with grid spacing kh .

DEFINITION. We will say that a discrete derivative operator is an **order P accurate r^{th} derivative operator** if it correctly reproduces the r^{th} derivative of polynomials of degree $P + r - 1$. We will use $\partial_P^{r,h}$ to denote the discrete derivative operator of minimal width (the support of the stencil is contained in the smallest possible interval) that satisfies this definition. It can be shown inductively that all operators $\partial^{q,h}$ defined above are second order accurate.

We can derive high order accurate second derivative operators in a variety of ways. A straightforward method is to assume that $\partial_P^{2,h}$ is a weighted sum of translation operators:

$$\partial_P^{2,h} = \sum_{i=-P/2}^{P/2} \left[\partial_P^{2,h} \right]_i T_i \quad (2.6)$$

and obtain the coefficients by solving a Vandermonde system of equations. An equivalent formulation is to consider the “basis” functions δ_i which are 1 at i and zero elsewhere. We can fit an order P polynomial to this grid function on the interval $[-P/2, P/2]$, and then read off the stencil coefficients as:

$$\left[\partial_P^{2,h} \right]_i := \partial^2 \delta_i(0) \quad (2.7)$$

We can also represent this operator as a sum of high order discrete derivative operators:

$$\partial_P^{2,h} = \sum_{q=1}^{P/2} \mu_q \frac{\partial^{2q,h}}{(2q!)} \quad (2.8)$$

The coefficients μ_q can be obtained from the following polynomials $p_q(x)$, which are defined to satisfy:

$$\frac{\partial^{s,h}}{s!} p_q(0) = 1 \text{ if } s = q, 0 \text{ otherwise} \quad (2.9)$$

These are essentially centered versions of classical Newton polynomials, and can likewise be defined sequentially:

$$p_0(x) = 1 \quad (2.10)$$

$$p_1(x) = x \quad (2.11)$$

$$p_q(x) = p_{q-2}(x) \cdot (x^2 - (\lfloor (q-1)/2 \rfloor)^2) \quad (2.12)$$

They have the property that for even q they are even and nonnegative at all grid points, and for odd q they are odd. In addition, for even q we have: $p_q(i) = 0, |i| < q/2, p_q(\lfloor q/2 \rfloor) = 1, p_q(i) > 1, |i| > q/2$. We can represent a grid function u^h on $[-P/2, P/2]$ using the expansion:

$$u_i = \sum_{q=0}^P \frac{\partial^{q,h} u_0}{q!} p_q(i) \quad (2.13)$$

We can then read off the coefficients μ_q as

$$\mu_q = \partial^2 p_q(0) \quad (2.14)$$

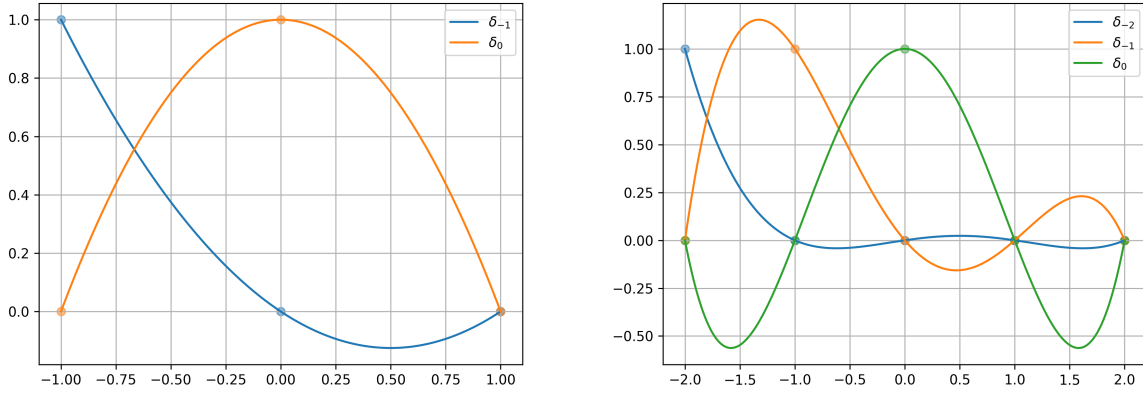
This particular representation of our operators shows in what sense high order accurate operators are “close” to each other, in that they can be obtained sequentially by adding another high order difference operator. We can calculate these μ_q sequentially by applying the product rule: $\mu_1 = 2, \mu_2 = -2, \mu_3 = \mu_2(-4) = 8, \mu_4 = \mu_3(-9) = -72$, with general formula: $\mu_{q+1} = \mu_q(-q^2)$.

2.1.1 The Low Order Local Maximum Principle

For the operator $\partial^{2,h}$, the discrete maximum principle states that if u_0 obtains a local maximum over the ball of radius 1, then $\partial^{2,h} u_0 \leq 0$. We will refer to this as the *low order local maximum principle*. Equivalently, if $\partial^{2,h} u_0 > 0$, then u_0 is strictly less than one of its neighboring values. This follows from the fact that $\partial^{2,h}$ is positive type:

$$u_{-1} - 2u_0 + u_1 > 0 \implies \quad (2.15)$$

$$u_0 < \frac{1}{2}(u_{-1} + u_1) \leq \max\{u_{-1}, u_1\} \quad (2.16)$$


 Figure 2.1: Lagrange interpolating polynomials for $P = 2$ and $P = 4$.

Why exactly is this stencil positive type? The stencil coefficient at -1 is given by taking the second derivative of the quadratic interpolant of δ_{-1} at the origin; see Figure 2.1. A quadratic function cannot have any changes in convexity, so the low order local maximum principle succeeds in this case because we are simply testing the convexity of a quadratic function which interpolates the three grid values.

High order accurate stencils, such as the fourth order accurate second derivative operator, have off center stencil coefficients which are negative. This immediately invalidates the low order local maximum principle: Consider the basis function $-\delta_2$. This function has a maximum at the origin over the ball of radius 2, but $\partial_4^{2,h}(-\delta_2) = \frac{1}{12} > 0$. This counterexample simply utilized the fact that this stencil is not of positive type. Why is this stencil not of positive type? The polynomial interpolant of δ_2 is concave at the origin; see Figure 2.1. The low order local maximum principle fails here because fourth order polynomials can have changes in convexity over intervals smaller than the grid spacing.

Another counterexample is given by the grid function v^h defined by its discrete derivatives on $[-2, 2]$. Let $\partial^{2,h}v_0 = -2$, $\partial^{4,h}v_0 = -28$, which gives lattice grid values $[-18, -1, 0, -1, -18]$, meaning a local maximum is achieved at the origin. The fourth order accurate discrete derivative operator is given in the difference quotient representation as:

$$\partial_4^{2,h} = \partial^{2,h} - \frac{1}{12}\partial^{4,h} \quad (2.17)$$

meaning $\partial_4^{2,h}v^h(0) = 1/3 > 0$. We visualize this example in Figure 2.2 by plotting the polynomial of degree four that interpolates these five values.

We also want to understand which operators are “compatible” with the low order local maximum principle, meaning if u^h has a local maximum over the ball of radius R , these operators are nonpositive at this point. These are exactly the positive type operators, meaning

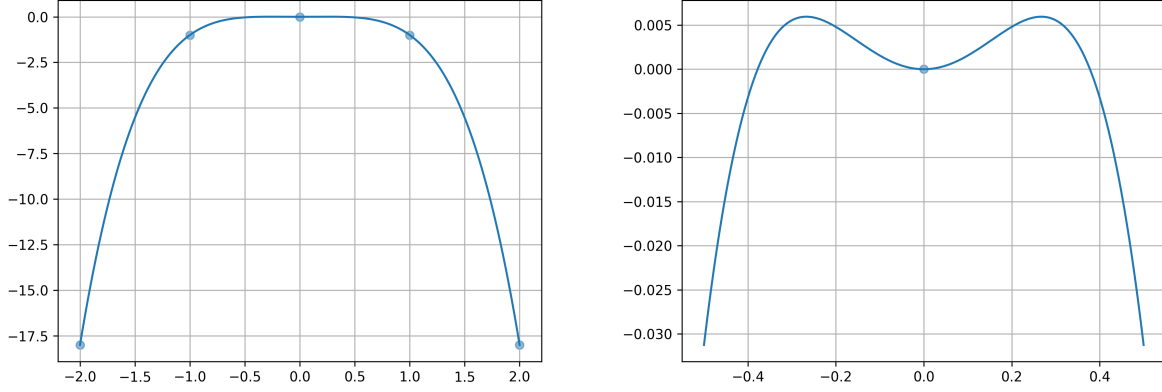


Figure 2.2: Counter example to the low order local maximum principle: the grid function achieves a discrete local maximum at the origin, but it is convex at that point. Right hand figure shows detail in a neighborhood of the origin.

a nonnegative combination of translation operators of the form:

$$-s_0 T_0 + \sum_{i=-R, i \neq 0}^R s_i T_i \quad (2.18)$$

for any $s_i \geq 0$. The set of these operators is a convex cone.

We can represent an operator in this set as a combination of discrete derivative operators. It will be helpful here to utilize the Newton polynomial basis. In this basis, we have the inequalities:

$$u_0 \geq \frac{1}{2} (u_1 + u_{-1}) \implies \quad (2.19)$$

$$u_0 \geq u_0 + \frac{\partial^{2,h} u_0}{2} p_2(1) \quad (2.20)$$

As we move out from the origin, we “see” higher order difference quotients, since $p_q(i) = 0$ for $i < q/2$ for even q , so we have:

$$0 \geq \sum_{q=1}^i \frac{\partial^{2q,h} u_0}{(2q)!} p_{2q}(i) \quad (2.21)$$

Thus if u^h achieves a local maximum at the origin over the interval $[-R, R]$, then we have for any $s_i \geq 0$:

$$0 \geq \sum_{i=1}^R s_i \sum_{q=1}^i \frac{\partial^{2q,h} u_0}{(2q)!} p_{2q}(i) = \sum_{q=1}^R \frac{\partial^{2q,h} u_0}{(2q)!} \sum_{i=q}^R s_i p_{2q}(i) \quad (2.22)$$

Thus a combination of even discrete derivative operators:

$$\sum_{q=1}^R \gamma_q \frac{\partial^{2q,h}}{(2q)!} \quad (2.23)$$

lies in this cone if $\gamma_q = \sum_{i=1}^R s_i p_{2q}(i)$ for some nonnegative s_i . Since the p_{2q} are nonnegative, this means at the very least that any operator with a negative coefficient on any (even) discrete derivative term will not be compatible with the low order local maximum principle.

The counterexamples reveal two issues with applying the low order local maximum principle to high order accurate second derivative operators:

1. Off center stencil coefficients may be negative due to concavity of basis functions at the origin
2. The existence of a local maximum of u^h does not allow us to infer needed information about particular difference quotients whose sign is negative in the operator

We see that the low order local maximum principle is not well suited to high order polynomials and high order accurate second derivative operators. If we would like to maintain the structure of the low order local maximum principle, we will need to modify either the operator under consideration, the quantity we are bounding, or both. The basic device to deal with this is to introduce a new quantity derived from u^h which is the maximum over several perturbations of u^h .

2.1.2 High Order Local Maximum Principles

NOTATION. Let $\{\alpha_0, \alpha_1 \dots \alpha_M\} =: \{\alpha_m\}$ be $M+1$ discrete operators supported in some neighborhood of the origin, where α_0 is the zero operator, and let K be the set of all convex combinations of these operators, notated as $\langle \alpha_m \rangle$. We will define

$$\psi(K, u)_i = \bar{\psi}(K, u)_i := \max_{\alpha_m} \{u_i + \alpha_m(u)_i\} \quad (2.24)$$

$$\underline{\psi}(K, u)_i := \min_{\alpha_m} \{u_i + \alpha_m(u)_i\} \quad (2.25)$$

Note that since K contains the zero map, we will always have $u_i \leq \psi(K, u)_i$.

For now we will define $\alpha_0 = 0, \alpha_1 = -ah^2\partial^{2,h}$ for some parameter $a > 0$, so that we have:

$$\psi(K, u)_i = \max \{u_i, u_i - ah^2\partial^{2,h}u_i\} \quad (2.26)$$

We will again set $h = 1$ for clarity. We would like to be able to prove a statement that has the same structure as the low order local maximum principle: if $\psi(K, u)$ has a maximum at the origin, then $\partial_4^{2,h}u_0 \leq 0$.

We can first check that this device is compatible with the counterexamples posed earlier: we have that $\psi(K, -\delta_2)_0 = 0$ and $\psi(K, -\delta_2)_1 = a$, which means that we no longer have a

local maximum of $\psi(K, -\delta_2)$ at the origin. For the second example, we have that $\psi(K, v)_1 = -1 - a(0 + 2 - 18) = -1 + 16a$ and $\psi(K, v)_0 = 0 - a(-1 - 2 * (0) - 1) = 2a$, so for large enough a we will no longer have a local maximum of $\psi(K, v)$ at the origin.

This device somehow detects that the fourth order difference quotient may be significantly negative relative to the second order difference coefficient. One can see this by looking at the difference quotient expansion of u^h . Firstly, if $\psi(K, u)_0$ is a local maximum, we must have $\psi(K, u)_0 = u_0 - a\partial^{2,h}u_0$, meaning $\partial^{2,h}u_0 \leq 0$. Otherwise we would have $\partial^{2,h}u_0 > 0$, which would mean $\psi(K, u)_0 = u_0 < \max\{u_{-1}, u_1\} \leq \max\{\psi(K, u)_{-1}, \psi(K, u)_1\}$.

From the fact that $\psi(K, u)_0$ is a local maximum, we have the inequality:

$$u_0 - a\partial^{2,h}u_0 \geq \frac{1}{2}(\psi(K, u)_1 + \psi(K, u)_{-1}) \quad (2.27)$$

$$\geq \frac{1}{2}(u_1 - a_1\partial^{2,h}u_1 + u_{-1} - a_1\partial^{2,h}u_{-1}) \quad (2.28)$$

where $0 \leq a_1 \leq a$, using the fact that by definition $\psi(K, u)_i \geq u_i - a_i\partial^{2,h}u_i$ for any $0 \leq a_i \leq a$. Expanding this expression using the definition of the p_q in terms of difference quotients, we have:

$$u_0 - a\partial^{2,h}u_0 \geq u_0 + \sum_{q=1}^2 \frac{\partial^{2q,h}u_0}{(2q!)} (p_{2q}(1) - a_1\partial^{2,h}p_{2q}(1)) \quad (2.29)$$

$$\left(a + \frac{p_2(1) - a_1\partial^{2,h}p_2(1)}{2}\right) |\partial^{2,h}u_0| \geq \frac{\partial^{4,h}u_0}{4!} (p_4(1) - a_1\partial^{2,h}p_4(1)) \quad (2.30)$$

where in the last line we used the fact that $\partial^{2,h}u_0 \leq 0$. Since p_4 is convex on the grid at 1, we may be able to choose a such that we can compare how negative $\partial^{4,h}u_0$ is relative to $\partial^{2,h}u_0$. Evaluating this expression gives us:

$$\left(a + \frac{1 - 2a_1}{2}\right) |\partial^{2,h}u_0| \geq \frac{\partial^{4,h}u_0}{4!} (-12a_1) \quad (2.31)$$

In particular, setting $a_1 = a$ gives the inequality:

$$|\partial^{2,h}u_0| \geq \partial^{4,h}u_0(-a) \quad (2.32)$$

We also have from the fact that $\psi(K, u)_0$ is a local maximum:

$$u_0 - a\partial^{2,h}u_0 \geq \frac{1}{2}(u_{-2} + u_2) \quad (2.33)$$

$$= u_0 + \frac{\partial^{2,h}u_0}{2!}p_2(2) + \frac{\partial^{4,h}u_0}{4!}p_4(2) \quad (2.34)$$

$$|\partial^{2,h}u_0| \geq \frac{1}{2(a+2)}\partial^{4,h}u_0 \quad (2.35)$$

Setting $a = 1$ for example, we can conclude that

$$|\partial^{4,h}u_0| \leq \max\{a, 2(a+2)\} |\partial^{2,h}u_0| = 6 |\partial^{4,h}u_0| \quad (2.36)$$

so we can see that the statement that $\psi(K, u)$ has a local maximum allows us to make a strong conclusion about the relative magnitudes of difference quotients of different orders. Because $\psi(K, u)$ is the maximum over several perturbations of u^h , when $\psi(K, u)$ has a maximum over some region, we can generate a huge number of inequalities that contain information about the difference quotients of u^h at the origin.

The above inequalities are in fact sufficient to prove our local maximum principle for $\psi(K, u)$ for the fourth order accurate second derivative operator: setting $a = 1/12$ in (2.32), we have

$$\partial^{2,h}u_0 - \frac{1}{12}\partial^{4,h}u_0 \leq 0 \quad (2.37)$$

since $\partial^{2,h}u_0 \leq 0$. Proving this required the manipulation of several inequalities, which will become more complicated for other cases. We now show how to reduce this proof to the solution of a small system of linear equations.

THEOREM 2.1.1. *Let*

$$\psi(K, u)_i := \max\{u_i, u_i - a\partial^{2,h}u_i\} \quad (2.38)$$

Then for $a = 1$, $P \leq 8$, $\partial_P^{2,h}$ satisfies the following local maximum principle:

$$\psi(K, u)_0 \geq \max_{i \in [-P/2, P/2]} \{\psi(K, u)_i\} \implies \partial_P^{2,h}u_0 \leq 0 \quad (2.39)$$

Proof: First, suppose that $\psi(K, u)_0 = u_0 > u_0 - a\partial^{2,h}u_0$ is a local maximum of $\psi(K, u)$. Then we have

$$u_0 > u_0 - a\partial^{2,h}u_0 \implies \partial^{2,h}u_0 > 0 \implies \quad (2.40)$$

$$u_0 \leq \max\{u_1, u_{-1}\} \leq \max\{\psi(K, u)_1, \psi(K, u)_{-1}\} \quad (2.41)$$

which contradicts $\psi(K, u)_0$ being a local maximum. Thus if $\psi(K, u)_0$ is a local maximum, it must be the case that $\psi(K, u)_0 = u_0 - a\partial^{2,h}u_0$, meaning $\partial^{2,h}u_0 \leq 0$. This proves the statement for $P = 2$. By the definition of $\psi(K, u)$ and the fact that it achieves a local maximum at the origin, we have:

$$u_0 - a\partial^{2,h}u_0 \geq \psi(K, u)_i \geq u_i - a_i\partial^{2,h}u_i \quad (2.42)$$

for any $a_i \in [0, a]$. We can add inequalities of this form multiplied by coefficients $s_{P,i} \geq 0$ to obtain:

$$0 \geq \sum_{i=-P/2}^{P/2} s_{P,i} [-(u_0 - a\partial^{2,h}u_0) + (u_i - a_i\partial^{2,h}u_i)] \quad (2.43)$$

$$0 \geq -\left(\sum_{i=-P/2}^{P/2} s_{P,i}\right)(u_0 - a\partial^{2,h}u_0) + \sum_{i=-P/2}^{P/2} s_{P,i}(u_i - a_i\partial^{2,h}u_i) \quad (2.44)$$

We will set

$$s_{P,0} := - \left(\sum_{i=-P/2}^{P/2} s_{P,i} \right) \quad (2.45)$$

For each $P = 4, 6, 8$, we want to solve for sets of coefficients such that for any u^h , we have:

$$\partial_P^{2,h} u_0 = s_{P,0} (u_0 - a \partial^{2,h} u_0) + \sum_{i=-P/2}^{P/2} s_{P,i} (u_i - a_i \partial^{2,h} u_i) \quad (2.46)$$

where $a_i \in [0, a]$ and $a_0 = a$. Then by (2.44), we will have:

$$0 \geq s_{P,0} (u_0 - a \partial^{2,h} u_0) + \sum_{i=-P/2}^{P/2} s_{P,i} (u_i - a_i \partial^{2,h} u_i) \implies \quad (2.47)$$

$$0 \geq \partial_P^{2,h} u_0 \quad (2.48)$$

which will prove the claim. Note that if this statement holds for any $s_{P,i}$, the condition $s_{P,0} = - \left(\sum_{i=-P/2}^{P/2} s_{P,i} \right)$ will automatically be satisfied. Letting 1^h be the function which is constantly 1, we have

$$0 = \partial_P^{2,h} 1_0 = s_{P,0} (1_0 - a \partial^{2,h} 1_0) + \sum_{i=-P/2}^{P/2} s_{P,i} (1_i - a_i \partial^{2,h} 1_i) \quad (2.49)$$

$$= s_{P,0} + \sum_{i=-P/2}^{P/2} s_{P,i} \quad (2.50)$$

We can solve for these $s_{P,i}$ by matching the coefficients of the translation operator expansion of $\partial_P^{2,h}$; that is, we must satisfy:

$$\partial_P^{2,h} = \sum_{i=-P/2}^{P/2} \left[\partial_P^{2,h} \right]_i T_i = \sum_{i=-P/2}^{P/2} s_{P,i} T_i \circ (I - a_i \partial^{2,h}) + s_{P,0} (I - a \partial^{2,h}) \quad (2.51)$$

We can reduce the number of coefficients by assuming symmetry:

$$\partial_P^{2,h} = s_{P,0} (I - a \partial^{2,h}) + \sum_{k=1}^{P/2} s_{P,k} [(T_k + T_{-k}) \circ (I - a_k \partial^{2,h})] \quad (2.52)$$

We set $a_{P/2} = 0$. For $P = 4$ we have:

$$\begin{bmatrix} 1 + 2a_0 & -2a_1 & 0 \\ -a_0 & 1 + 2a_1 & 0 \\ 0 & -a_1 & 1 \end{bmatrix} \begin{bmatrix} s_{4,0} \\ s_{4,1} \\ s_{4,2} \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} \\ \frac{4}{3} \\ -\frac{1}{12} \end{bmatrix} \quad (2.53)$$

For $P = 6$ we have:

$$\begin{bmatrix} 1+2a_0 & -2a_1 & 0 & 0 \\ -a_0 & 1+2a_1 & -a_2 & 0 \\ 0 & -a_1 & 1+2a_2 & 0 \\ 0 & 0 & -a_2 & 1 \end{bmatrix} \begin{bmatrix} s_{6,0} \\ s_{6,1} \\ s_{6,2} \\ s_{6,3} \end{bmatrix} = \begin{bmatrix} -\frac{49}{18} \\ \frac{3}{2} \\ -\frac{3}{20} \\ \frac{1}{90} \end{bmatrix} \quad (2.54)$$

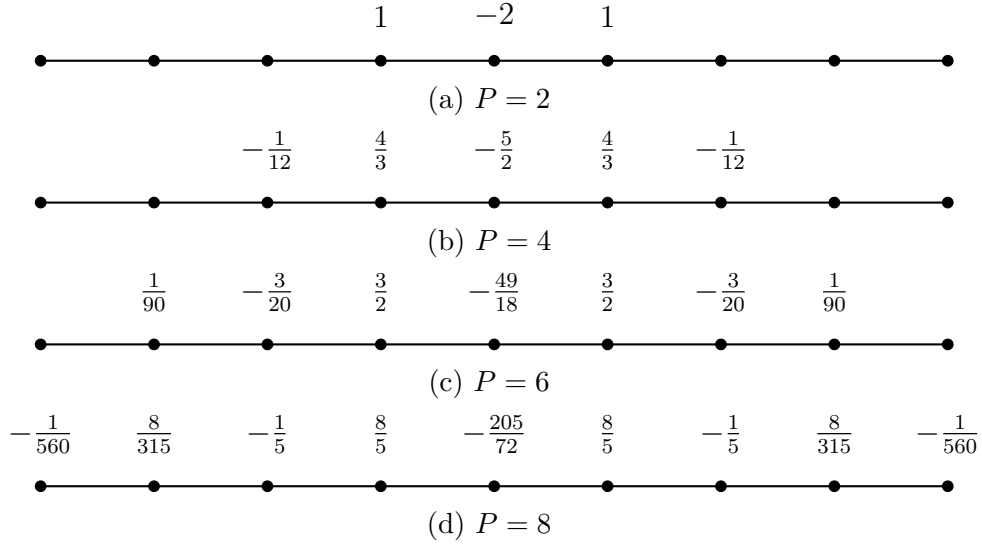
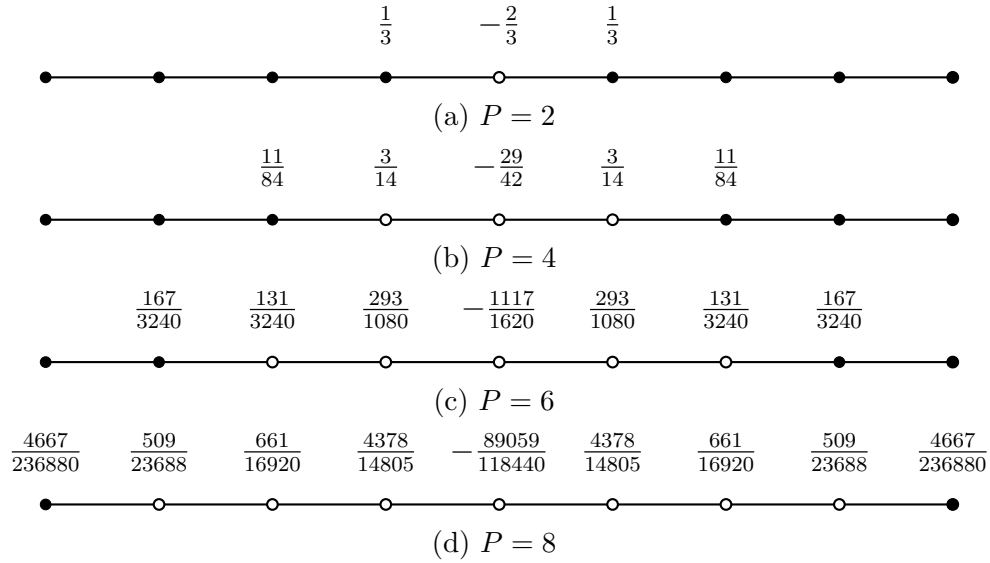
For $P = 8$ we have:

$$\begin{bmatrix} 1+2a_0 & -2a_1 & 0 & 0 & 0 \\ -a_0 & 1+2a_1 & -a_2 & 0 & 0 \\ 0 & -a_1 & 1+2a_2 & -a_3 & 0 \\ 0 & 0 & -a_2 & 1+2a_3 & 0 \\ 0 & 0 & 0 & -a_3 & 1 \end{bmatrix} \begin{bmatrix} s_{8,0} \\ s_{8,1} \\ s_{8,2} \\ s_{8,3} \\ s_{8,4} \end{bmatrix} = \begin{bmatrix} -\frac{205}{72} \\ \frac{8}{5} \\ -\frac{1}{5} \\ \frac{8}{315} \\ -\frac{1}{560} \end{bmatrix} \quad (2.55)$$

It can be verified by hand that if we set $a_k = 1$, this system is satisfied for $P \leq 8$ by $s_{P,0} < 0, s_{P,k} > 0$; these coefficients are shown in Figure 2.4. $\square$

In Figure 2.5 we show the algebraic decomposition of $\partial^{2,h}$; note that negative off-diagonal coefficients are contributed by perturbed coefficients at neighbors. It is in fact the case that $\partial_P^{2,h}$ satisfies a local maximum principle for this same $\psi(K, u)$ for much higher orders P with the same choice of a_k , which can be verified numerically by deriving stencils and solving a similar system of equations.

Recall that one of our issues with the low order local maximum principle was that it was not compatible with off center basis functions which are concave at the origin. In the above proof, we showed that we could “rewrite” the stencil as a positive type stencil in specific perturbed variables. We built some intuition in terms of difference quotients as to why this device allowed us to prove a local maximum principle for $\psi(K, u)$. We now turn to the question of interpreting these rewritten stencil coefficients, and in doing so provide another proof of this local maximum principle.

Figure 2.3: Order P centered second derivative stencils $\partial_P^{2,h}$ in one dimension.Figure 2.4: Positive type representations of $\partial_P^{2,h}$ in one dimension. Open circles indicate perturbed variables $u_i - \partial_P^{2,h} u_i$.

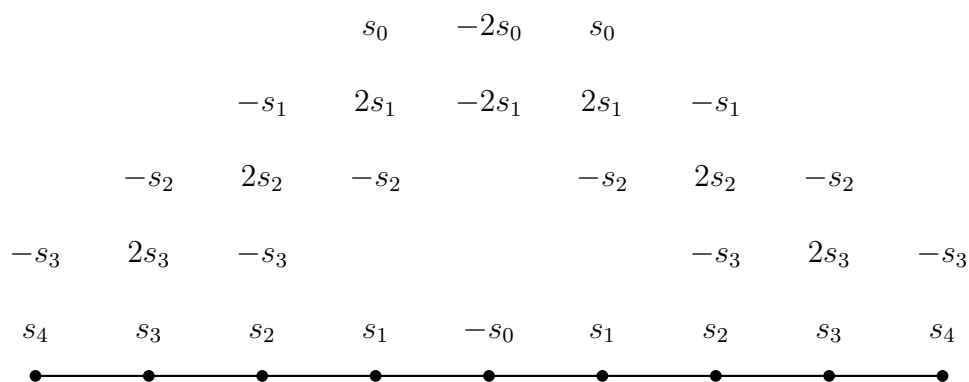


Figure 2.5: Algebraic decomposition of $\partial_8^{2,h}$ stencil with $a_k = 1, k \in [-3, 3], a_{\pm 4} = 0$.

2.1.3 Basis Function Smoothing

Given a grid function u^h , our perturbed variables are obtained by a linear mapping. For example, for the $P = 6$, seven point stencil we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a & 1+2a & -a & 0 & 0 & 0 & 0 \\ 0 & -a & 1+2a & -a & 0 & 0 & 0 \\ 0 & 0 & -a & 1+2a & -a & 0 & 0 \\ 0 & 0 & 0 & -a & 1+2a & -a & 0 \\ 0 & 0 & 0 & 0 & -a & 1+2a & -a \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{-3} \\ u_{-2} \\ u_{-1} \\ u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_{-3} \\ u_{-2} - a\partial^{2,h}u_{-2} \\ u_{-1} - a\partial^{2,h}u_{-1} \\ u_0 - a\partial^{2,h}u_0 \\ u_1 - a\partial^{2,h}u_1 \\ u_2 - a\partial^{2,h}u_2 \\ u_3 \end{bmatrix} \quad (2.56)$$

Set $R := P/2$. We can view this as a change of basis: locally, we have

$$u_j = \sum_{i=-R}^R (u_i - a_i \partial^{2,h} u_i) \widehat{\delta}_i(j) \quad (2.57)$$

where we denoted the “smoothed” basis functions as $\widehat{\delta}_i$, which correspond to solving (2.56) for a right hand side which is 1 at i and zero elsewhere, or finding a grid function which satisfies:

$$\widehat{\delta}_i(j) - a_i \partial^{2,h} \widehat{\delta}_i(j) = \delta_i(j) \quad (2.58)$$

From these we can read off the coefficients of our rewritten positive type representation as:

$$s_{P,i} = \partial_P^{2,h} \widehat{\delta}_i(0) \quad (2.59)$$

This is equivalent to solving the transpose of (2.56) with the stencil coefficients as a right hand side; for the example above we would have

$$\begin{bmatrix} 1 & -a & 0 & 0 & 0 & 0 & 0 \\ 0 & 1+2a & -a & 0 & 0 & 0 & 0 \\ 0 & -a & 1+2a & -a & 0 & 0 & 0 \\ 0 & 0 & -a & 1+2a & -a & 0 & 0 \\ 0 & 0 & 0 & -a & 1+2a & -a & 0 \\ 0 & 0 & 0 & 0 & -a & 1+2a & 0 \\ 0 & 0 & 0 & 0 & 0 & -a & 1 \end{bmatrix} \begin{bmatrix} s_{6,-3} \\ s_{6,-2} \\ s_{6,-1} \\ s_{6,0} \\ s_{6,1} \\ s_{6,2} \\ s_{6,3} \end{bmatrix} = \begin{bmatrix} \left[\partial_6^{2,h} \right]_{-3} \\ \left[\partial_6^{2,h} \right]_{-2} \\ \left[\partial_6^{2,h} \right]_{-1} \\ \left[\partial_6^{2,h} \right]_0 \\ \left[\partial_6^{2,h} \right]_1 \\ \left[\partial_6^{2,h} \right]_2 \\ \left[\partial_6^{2,h} \right]_3 \end{bmatrix} \quad (2.60)$$

which is the technique we used to prove Theorem 2.1.1.

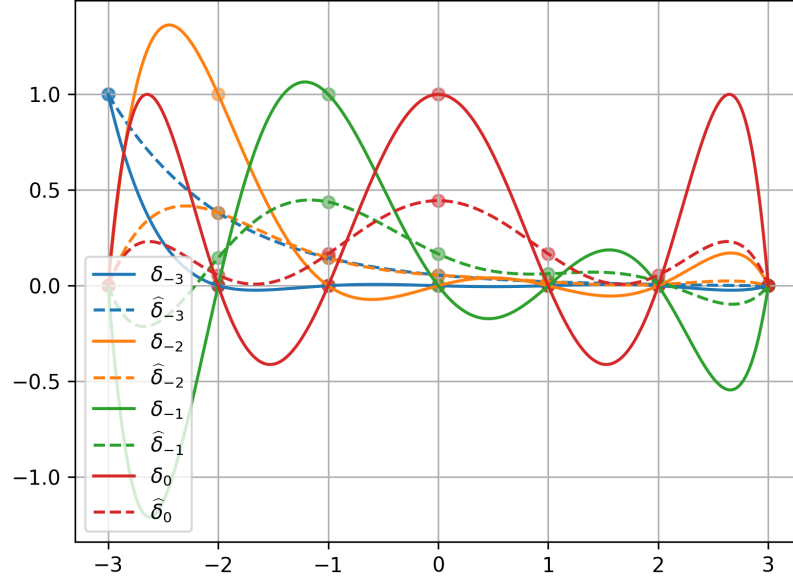


Figure 2.6: Smoothed basis functions for $P = 6$, $a = 1$. Only $\hat{\delta}_i$ for $i \leq 0$ are shown due to symmetry.

Now, since we set $a_R = 0$ and $a_i = a$, we can see that these $\hat{\delta}_i$ are in fact the discrete Green's functions associated with the discrete elliptic boundary value problem:

$$(I - a\partial^{2,h})\hat{\delta}_i(j) = \delta_i(j), j \in [-R+1, R-1] \quad (2.61)$$

$$\hat{\delta}_i(j) = \delta_i(j), j \in \{-R, R\} \quad (2.62)$$

Heuristically, we are “smoothing” out the basis function δ_i .

See Figure 2.6 for an example of smoothed basis functions. These functions have the following property:

LEMMA 2.1.1. *For $i \neq 0$, let $\hat{\delta}_i$ satisfy the discrete boundary value problem (2.61)-(2.62). Then*

$$\partial^{2,h}\hat{\delta}_i(0) \geq 0 \quad (2.63)$$

Proof: Consider the set of points j in $[-R+1, R-1]$ where $\hat{\delta}_i(j) < 0$. At those points, we have

$$a\partial^{2,h}\hat{\delta}_i(j) = \hat{\delta}_i(j) - \delta_i(j) < 0 \quad (2.64)$$

which means that $\widehat{\delta}_i(j)$ cannot achieve a local minimum at those points; therefore $\widehat{\delta}_i(j)$ cannot achieve a negative minimum in $[-R+1, R-1]$. Letting $(\cdot)^- := \min\{\cdot, 0\}$, this gives us:

$$\min_{[-R+1, R-1]} \widehat{\delta}_i \geq \min_{\{-R, R\}} \left(\widehat{\delta}_i \right)^- \geq 0 \quad (2.65)$$

which is a discrete minimum principle. Now from the discrete PDE we have:

$$\widehat{\delta}_i(0) - a\partial^{2,h}\widehat{\delta}_i(0) = 0 \implies \quad (2.66)$$

$$a\partial^{2,h}\widehat{\delta}_i(0) = \widehat{\delta}_i(0) \geq 0 \quad (2.67)$$

which proves the claim. $\square$

The higher order accurate discrete derivative operator can be written as

$$\partial_P^{2,h} = \partial^{2,h} + \sum_{q=2}^P \mu_q \frac{\partial^{2q,h}}{q!} = \partial^{2,h} + C_P^h \quad (2.68)$$

Intuitively, we may expect these high order difference quotients to be small relative to $\partial^{2,h}$ due to the “smoothness” of $\widehat{\delta}_i$, i.e. we can compute a c_P that satisfies:

$$\left| C_P^h \widehat{\delta}_i(0) \right| \leq c_P \left| \partial^{2,h} \widehat{\delta}_i(0) \right| \quad (2.69)$$

For example, we set $a = 1$ and can calculate this constant numerically as:

$$c_P := \min_c : \left| C_P^h \widehat{\delta}_i(0) \right| \leq c \left| \partial^{2,h} \widehat{\delta}_i(0) \right| \quad \forall i \in [-P/2, P/2] \setminus 0 \quad (2.70)$$

For $P = 4, 6, 8$ we obtain rounded values:

$$c_4 = .5 \quad (2.71)$$

$$c_6 = .62778 \quad (2.72)$$

$$c_8 = .73730 \quad (2.73)$$

so we have that $c_P \leq 3/4$ for $a = 1$ for $P = 4, 6, 8$. From this calculation we can prove:

LEMMA 2.1.2. *Let $\widehat{\delta}_i$ satisfy (2.61)-(2.62) for $i \neq 0$, for $R = 2, 3, 4$ with $a = 1$. If $i \neq 0$, then $\partial_{2R}^{2,h} \widehat{\delta}_i(0) \geq 0$, and $\partial_{2R}^{2,h} \widehat{\delta}_0 < 0$*

Proof: For any $i \neq 0$, we have

$$\partial_{2R}^{2,h} \widehat{\delta}_i(0) = \partial^{2,h} \widehat{\delta}_i(0) + C_P^h \widehat{\delta}_i(0) \quad (2.74)$$

$$\geq \partial^{2,h} \widehat{\delta}_i(0) - c_P \left| \partial^{2,h} \widehat{\delta}_i(0) \right| \quad (2.75)$$

$$\geq (1 - c_P) \partial^{2,h} \widehat{\delta}_i(0) \geq 0 \quad (2.76)$$

which follows from $c_P < 1$ and Lemma 2.1.1. Now, let u^h be a grid function on $[R, R]$ which is constantly 1. Set

$$s_{2R,i} := \partial_{2R}^{2,h} \widehat{\delta}_i(0) \quad (2.77)$$

Then we have that $u_i - a\partial_{2R}^{2,h}u_i = u_i$, meaning for $a(i) = a \in [-R+1, R-1]$, 0 else:

$$0 = \partial_{2R}^{2,h}u_0 = \sum_{i=-R}^R s_{2R,i} (u_i - a(i)\partial_{2R}^{2,h}u_i) \quad (2.78)$$

$$= \sum_{i=-R}^R s_{2R,i} \quad (2.79)$$

$$= \sum_{i=-R}^R \partial_{2R}^{2,h} \widehat{\delta}_i(0) \implies \quad (2.80)$$

$$\partial_{2R}^{2,h} \widehat{\delta}_0(0) = - \sum_{i=-R, i \neq 0}^R \partial_{2R}^{2,h} \widehat{\delta}_i(0) \leq 0 \quad (2.81)$$

where we are using the fact that both the original stencil coefficients and the rewritten stencil coefficients satisfy the zeroth moment condition. Now suppose u^h is the restriction of the quadratic function x^2 to the grid. Then

$$2 = \partial_{2R}^{2,h}u_0 = \sum_{i=-R}^R s_{2R,i} (u_i - a(i)\partial_{2R}^{2,h}u_i) \quad (2.82)$$

$$= \sum_{i=-R}^R s_{2R,i} (i^2 - a(i)2) \quad (2.83)$$

Since $(i^2 - a(i)2) \neq 0$ for some $i \in [-R, R]$, this means that not all of the $s_{2R,i} = 0$, meaning by the sign condition that $s_{2R,0} < 0$. $\square$

This lemma showed another technique by which we could prove that our high order accurate discrete derivative operator satisfied a local maximum principle for $\psi(K, u)$. While it was a less direct proof technique, this perspective showed how these perturbations induce a particular smoothing of the basis functions that modifies their curvature at the origin to get a positive type stencil.

We can also interpret this smoothing in terms of its effect on on a grid function u^h . Define:

$$\varphi^h := (I - a\partial_{2R}^{2,h}) u^h \in Q_{P/2-1} \quad (2.84)$$

$$\varphi^h = u^h \in \partial Q_{P/2} \quad (2.85)$$

so that u^h satisfies the boundary value problem:

$$(I - a\partial^{2,h}) u^h = \varphi^h \in Q_{P/2-1} \quad (2.86)$$

$$u^h = \varphi^h \in \partial Q_{P/2} \quad (2.87)$$

We will denote the solution operator for this boundary value problem as $(I - a\partial^{2,h})^{-1}$. Then we have:

THEOREM 2.1.2. *If φ^h achieves a maximum over $[-P/2, P/2]$ at the origin, then $\partial_P^{2,h} u_0 \leq 0$ for $a = 1$, $P \leq 8$.*

Note that we have:

$$\partial_P^{2,h} u_0 = \partial_P^{2,h} \circ (I - a\partial^{2,h})^{-1} \varphi_0 \quad (2.88)$$

Intuitively, the mapping $(I - a\partial^{2,h})^{-1}$ smooths φ^h , so we expect that for $(I - a\partial^{2,h})^{-1} \varphi^h$ the high order difference quotients are small relative to its second order difference quotient. As in Lemma 2.1.1, we can show that if φ^h has a maximum at the origin, then $\partial^{2,h} u_0 = \partial^{2,h} \circ (I - a\partial^{2,h})^{-1} \varphi_0 \leq 0$: Without loss of generality, assume $\varphi_0 = 0$. Then:

$$\varphi^h = (I - a\partial^{2,h}) u^h \leq \varphi_0 = 0 \in Q_{P/2-1} \quad (2.89)$$

$$\varphi^h = u^h \leq \varphi_0 = 0 \in \partial Q_{P/2} \quad (2.90)$$

By the low order local maximum principle, we have that

$$(I - a\partial^{2,h}) u^h \leq 0 \in Q_{P/2-1} \implies \quad (2.91)$$

$$\max_{Q_{P/2-1}} u^h \leq \max_{\partial Q_{P/2}} (u^h)^+ \leq 0 \quad (2.92)$$

Then we have

$$\varphi_0 = (I - a\partial^{2,h}) u_0 = 0 \implies \quad (2.93)$$

$$a\partial^{2,h} u_0 = u_0 \leq 0 \quad (2.94)$$

as desired.

Proof of 2.1.2: Let $\widehat{\delta}_i$ satisfy (2.61)-(2.62) for $R = P/2$. By our definition of φ^h , we have:

$$\varphi^h = \sum_{i=-P/2}^{P/2} \varphi_i \delta_i \implies \quad (2.95)$$

$$(I - a\partial^{2,h})^{-1} \varphi^h = \sum_{i=-P/2}^{P/2} \varphi_i (I - a\partial^{2,h})^{-1} \delta_i \quad (2.96)$$

$$= \sum_{i=-P/2}^{P/2} \varphi_i \widehat{\delta}_i \quad (2.97)$$

Then we have that

$$\partial_P^{2,h} u_0 = \partial_P^{2,h} \circ (I - a\partial^{2,h})^{-1} \varphi_0 \quad (2.98)$$

$$= \sum_{i=-P/2}^{P/2} \varphi_i \partial_P^{2,h} \widehat{\delta}_i(0) = \sum_{i=-P/2}^{P/2} \varphi_i s_{P,i} \quad (2.99)$$

where we have again defined:

$$s_{P,i} := \partial_P^{2,h} \widehat{\delta}_i(0) \quad (2.100)$$

From Lemma 2.1.2, we have that

$$s_{P,i} \geq 0, i \neq 0 \quad (2.101)$$

$$s_{P,0} < 0 \quad (2.102)$$

$$\sum_{i=-P/2}^{P/2} s_{P,i} = 1, \text{ or } \sum_{i=-P/2, i \neq 0}^{P/2} s_{P,i} = -s_{P,0} \quad (2.103)$$

Now assume that φ is maximal at the origin. Then for all $i \neq 0$:

$$\varphi_0 \geq \varphi_i \implies \quad (2.104)$$

$$s_{P,i} \varphi_0 \geq s_{P,i} \varphi_i \implies \quad (2.105)$$

$$\sum_{i=-P/2, i \neq 0}^{P/2} s_{P,i} \varphi_0 \geq \sum_{i=-P/2, i \neq 0}^{P/2} s_{P,i} \varphi_i \implies \quad (2.106)$$

$$-s_{P,0} \varphi_0 \geq \sum_{i=-P/2, i \neq 0}^{P/2} s_{P,i} \varphi_i \implies \quad (2.107)$$

$$0 \geq s_{P,0} \varphi_0 + \sum_{i=-P/2, i \neq 0}^{P/2} s_{P,i} \varphi_i = \partial_P^{2,h} u_0 \quad (2.108)$$

where on the last line we have used (2.99). □

2.1.4 Expansion of Set of Local Maximum Principle Compatible Operators

As pointed out earlier, a local maximum principle generates a convex cone of discrete operators which are compatible with that local maximum principle. For the low order local maximum principle, we showed that these must be of the form given in (2.18) in terms of translation operators and (2.23) in terms of even order difference quotients. This set of operators is restrictive and does not include high order accurate second derivative operators.

In the same way we can characterize the set of operators which are compatible with a local maximum principle for $\psi(K, u)$.

If $\psi(K, u)$ has a maximum at the origin over the interval $[-R, R]$, then we showed that it must be equal to $u_0 - a\partial^{2,h}u_0$. This gives us the inequality:

$$u_0 - a\partial^{2,h}u_0 \geq u_i - a_i\partial^{2,h}u_i \quad (2.109)$$

for $a_i \in [0, a]$ and $i \in [-R, R]$. Expanding u^h in terms of its difference quotients we have for any $s_i \geq 0$, $\sum_i s_i = 1$:

$$u_0 - a\partial^{2,h}u_0 \geq \sum_{q=0}^{2R} \frac{\partial^{q,h}u_0}{q!} (p_q(i) - a(i)\partial^{2,h}p_q(i)) \implies \quad (2.110)$$

$$0 \geq a\partial^{2,h}u_0 + \sum_{i=1}^R s_i \sum_{q=1}^R \frac{\partial^{2q,h}u_0}{(2q)!} (p_{2q}(i) - a(i)\partial^{2,h}p_{2q}(i)) \quad (2.111)$$

$$\geq a\partial^{2,h}u_0 + \sum_{q=1}^R \frac{\partial^{2q,h}u_0}{(2q)!} \sum_{i=1}^R s_i (p_{2q}(i) - a(i)\partial^{2,h}p_{2q}(i)) \quad (2.112)$$

where in the second line we only consider even difference quotients by symmetry. For any fixed choice of $a(x)$, we can define $\hat{\gamma}(a(x))_q := \sum_{i=1}^R s_i (p_{2q}(i) - a(i)\partial^{2,h}p_{2q}(i))$, and we have that any operator of the form:

$$a\partial^{2,h} + \sum_{q=1}^R \frac{\partial^{2q,h}}{(2q)!} \hat{\gamma}(a(x))_q \quad (2.113)$$

is compatible with the local maximum principle for $\psi(K, u)$. Since the quantity $\partial^{2,h}p_{2q}(i)$ is positive at certain points i , it is possible to make some of the $\hat{\gamma}(a(x))_q$ negative. Thus the device $\psi(K, u)$ greatly expands the set of discrete operators which are compatible with the associated local maximum principle. The geometry of this set is not trivial, as for each choice of $a(x)$ we have a convex set. We proved explicitly that this set contains high order accurate discrete second derivative operators; however it may be the case that it contains other high order accurate elliptic operators.

2.2 Global Maximum Principles

Now that we have established local maximum principles, we will show how to extend these results to global maximum principles, following the course of the classical weak maximum principle. We will then use the typical comparison function technique to prove stability results.

Suppose we have properly selected $\psi(K, u)$ so that the local maximum principle holds, or:

$$\partial_P^{2,h} u_i > 0 \implies \exists j \in Q_R(i) : \psi(K, u)_i < \psi(K, u)_j \quad (2.114)$$

Now let Ω^h be some subset of the lattice $\mathbb{Z}$. We will define the set $G^h = G_R^h(\Omega^h)$ as the smallest set of lattice points outside of Ω^h such that for each point $i \in \Omega^h$, $Q_R(i) \subset \Omega^h \cup G_R^h(\Omega^h)$. For this case of an interval, $G_R^h([0, N_x]) = [-R, N_x + R] \setminus [0, N_x]$. We then have the following result:

LEMMA 2.2.1. *Let $u^h \in \ell^\infty(\mathbb{Z})$. Let $\Omega^h := [0, N_x]$. Suppose that $\partial_P^{2,h}$ satisfies a local maximum principle for $\psi(K, u)$. If $\partial_P^{2,h} u > 0 \in \Omega^h$, then*

$$\max_{\Omega^h} u^h \leq \max_{\Omega^h} \psi(K, u) \leq \max_{G_R^h} \psi(K, u) \quad (2.115)$$

with $R = P/2$.

Proof: Since $\partial_P^{2,h} u > 0 \in \Omega^h$, by the local maximum principle, the maximum of $\psi(K, u)$ over $\Omega^h \cup G^h$ cannot occur at any point in Ω^h , meaning it must occur in G^h . By definition, $u^h \leq \psi(K, u)$. $\square$

This result follows simply from the local maximum principle for $\psi(K, u)$, which leads to a very important remark: $\psi(K, u)$ will typically not vary in space, except possibly near the boundaries. At a single point, it would be trivial to propose a perturbed variables such that if $\partial_P^{2,h} u_0 > 0$, u_0 would be bounded above by these perturbed variables. For example:

$$\partial_4^{2,h} u_0 > 0 \implies u_0 < \max \left\{ \frac{16}{15} u_1 - \frac{1}{15} u_2, \frac{16}{15} u_{-1} - \frac{1}{15} u_{-2} \right\} \quad (2.116)$$

However, we have no obvious way to control either quantity on the right hand side using the information that $\partial_4^{2,h} u_{\pm 1} > 0$. This breaks the structure of the global maximum principle: a function which is subharmonic at a point must be smaller there than at a neighboring point; if the function is subharmonic at that neighboring point then there must be another point at which the function is greater, and so on until we have reached the boundary. Thus when we generate auxiliary variables in pursuit of a local maximum principle at one point, we must be able to bound these auxiliary variables using a local maximum principle at a neighboring point. If we keep the definition of $\psi(K, u)$ constant throughout the interior of the domain, then when we prove the local maximum principle for a single point, we will have proved it for all interior points, maintaining the structure of the global maximum principle.

From Lemma 2.2.1 we can prove a stability result.

THEOREM 2.2.1. *Suppose the discrete Laplace operator L^h satisfies a local maximum principle for $\psi(K, u)$, where*

$$\psi(K, u)_i = \max \{ u_i, u_i - ah^2 \partial^{2,h} u_i \} \quad (2.117)$$

Let $G^h \cup \Omega^h$ be contained in the strip $0 \leq x \leq D$. Then we have

$$\max_{\Omega^h} |u^h| \leq (D^2) \max_{\Omega^h} |L^h u^h| + \max_{G^h} (|u^h| + ah^2 |\partial^{2,h} u^h|) \quad (2.118)$$

Proof: Define $M := \max_{\Omega^h} |L^h u^h|$ and

$$N := \max \left\{ \max_{G^h} |\psi(K, u^h)|, \max_{G^h} |\psi(K, u^h)| \right\} \quad (2.119)$$

$$\leq \max_{G^h} (|u^h| + ah^2 |\partial^{2,h} u^h|) \quad (2.120)$$

Define the comparison function $v^h = -\frac{M+\epsilon}{2}(x^2 - D^2) + N$ for $\epsilon > 0$. Then we have

$$L^h(u^h - v^h) \geq -M + (M + \epsilon) > 0 \quad (2.121)$$

We have the following bounds on v^h in $\Omega^h \cup G^h$:

$$N \leq v^h \leq \frac{M + \epsilon}{2} D^2 + N \quad (2.122)$$

Then we have the bound:

$$\max_{G^h} \psi(K, u^h - v^h) \leq \max_{G^h} \psi(K, u^h) + \psi(K, -v^h) \quad (2.123)$$

$$\leq N + \max_{G^h} \psi(K, -v^h) \quad (2.124)$$

$$= N + \max_{i \in G^h} \max \{-v_i, -v_i + ah^2 \partial^{2,h} v_i\} \quad (2.125)$$

$$\leq N + \max_{i \in G^h} -v_i \leq 0 \quad (2.126)$$

where we use the fact that $\partial^{2,h} v \leq 0 \in G^h$. We can apply the previous lemma to obtain:

$$u^h - v^h \leq \max_{\Omega^h} \psi(K, u^h - v^h) \leq 0 \implies \quad (2.127)$$

$$u^h \leq \frac{M + \epsilon}{2} D^2 + N \quad (2.128)$$

Applying the same logic to $(u^h + v^h)$, we get

$$\max_{\Omega^h} |u^h| \leq \frac{M + \epsilon}{2} D^2 + N \quad (2.129)$$

and sending $\epsilon \rightarrow 0$ we obtain the desired result. $\square$

In the case of an interval with $[0, N_x]$, we can set $D = N_x + R$.

We can apply this theorem to show existence and uniqueness for a particular discrete boundary value problem. Suppose P is such that for all $2r \leq P$, $\partial_{2r}^{2,h}$ satisfies a local maximum principle in $\psi(K, u)$ for

$$\psi(K, u)_i = \max \{u_i, u_i - ah^2 \partial^{2,h} u_i\} \quad (2.130)$$

Then we have:

THEOREM 2.2.2. *Let $\Omega^h = [1, N_x - 1]$ and $\partial\Omega^h := \{0, N_x\}$. Let the operator L^h be given by*

$$L^h = \sum_{r=1}^{P/2} a_r(x) \partial_{2r}^{2,h} \quad (2.131)$$

where $a_r(x) \geq 0$, and

$$\text{supp}(a_r(x)) \subseteq [r, N_x - r] \quad (2.132)$$

$$\forall x \in \Omega^h, \sum_{r=1}^{P/2} a_r = 1 \quad (2.133)$$

Then we have

$$\max_{\Omega^h} |u^h| \leq \max_{\partial\Omega^h} |u^h| + (N_x^2) \max_{\Omega^h} |L^h u^h| \quad (2.134)$$

Note that this result quickly implies uniqueness of solutions to the boundary value problem:

$$L^h u^h = f^h \in \Omega^h \quad (2.135)$$

$$u^h = g^h \in \partial\Omega^h \quad (2.136)$$

which we can see by setting f^h, g^h to zero, which implies $\max_{\Omega^h} |u^h| = 0$. In finite dimensions this uniqueness implies existence.

Proof of 2.2.2: We first claim that L^h satisfies a local maximum principle for $\psi(K, u)$. Suppose $\psi(K, u)$ achieves a maximum at some index $i \in \Omega^h$. Let R_i be defined as

$$R_i := \min \{ \text{dist}(i, \partial\Omega^h), P/2 \} \quad (2.137)$$

Then for all $\partial_{2r}^{2,h}$ for which $r \leq R_i$, we have by the local maximum principle:

$$\partial_{2r}^{2,h} u_i \leq 0 \quad (2.138)$$

By assumption that at least one of the a_r is positive at any point in the domain, this implies that

$$L^h u_i \leq 0 \quad (2.139)$$

which proves the claim. We will then set $G^h = G_1(\Omega^h) = \partial\Omega^h$, and $\psi(K, u) = u^h \in \partial\Omega^h$. By the assumption on the support of a_r , this means that we have the global maximum principle:

$$L^h u_i > 0 \implies \exists j \in Q_{R_i}(i) : \psi(K, u)_i < \psi(K, u)_j \quad (2.140)$$

where we are using the fact that at the points $\{1, N_x - 1\}$, $L^h = a_1(x)\partial^{2,h}$, so the local maximum principle holds with $\psi(K, u) = u^h \in \partial\Omega^h$. We now apply Theorem 4.1.1, with $a = 0 \in \partial\Omega^h$ in the definition of $\psi(K, u)$ to get

$$\max_{\Omega^h} |u^h| \leq (N_x^2) \max_{\Omega^h} |L^h u^h| + \max_{\partial\Omega^h} |u^h| \quad (2.141)$$

□

We have demonstrated here basic results on how local maximum principles can be used to prove global results. We continue this effort in Chapter 4; however for now we address local maximum principles in two and three dimensions.

Chapter 3

Local Maximum Principles in Higher Dimensions

In the previous chapter, we showed that it was possible to prove local maximum principles for certain high order accurate second derivative operators in one dimension by considering local maxima of a perturbation of u^h . The proof of the local maximum principle was reduced to solving a small system of linear equations and verifying that the output satisfied a certain sign condition. In other words, given the form of the perturbation and given a stencil, we showed that proving that this stencil was compatible with the local maximum principle defined by these perturbations could be cast as a system of linear inequalities.

Next, we provided an alternative perspective on this local maximum principle by showing that the perturbations induced a “smoothing” of basis functions which strongly suppressed high order behavior. The logical extension of this technique to higher dimensions would be to define:

$$\psi(K, u)_i = \max \{u_i, u_i - a\Delta^h u_i\} \quad (3.1)$$

and try to prove a similar local maximum principle for dimension split Laplacian stencils. If this fails, we must modify $\psi(K, u)$ or the dimension split stencil itself.

In this chapter, we generalize the results from the one dimensional case in two ways. Firstly, we show that the process of proving that a stencil satisfies a local maximum principle in a perturbed variable is a linear program; this formalizes the technique first used in [4]. From this result, we demonstrate that the problem of constructing a stencil of a given order of accuracy that is compatible with a given local maximum principle can also be cast as a linear program. Secondly, we generalize and refine the idea of basis function smoothing to higher dimensions. Combining these two generalizations, we build an inductive procedure which produces sequences of high order accurate, local maximum principle compatible Laplacians.

We begin this chapter by considering how to derive consistent stencils in higher dimensions.

NOTATION. We define a discrete partial derivative operator, where q is a multi-index:

$$D^{q,h} := \partial_1^{q_1,h} \circ \partial_2^{q_2,h} \dots \partial_n^{q_n,h} \quad (3.2)$$

where $\partial_d^{q,h}$ indicates a difference operator in the d^{th} coordinate direction. We set $q! := \prod_d (q_d!)$. Letting $p_q(x) = \prod_d p_{q_d}(x_d)$, we have

$$D^{s,h} p_q(0) = q! \text{ if } s = q \quad (3.3)$$

$$= 0 \text{ otherwise} \quad (3.4)$$

where two multi-indices are equal if all components are equal, and we set $|q| := |q|_{\ell^1}$.

We then have that a grid function u on Q_R can be expressed as:

$$u_i = \sum_{|q|_{\ell^\infty} \leq R} \frac{D^{q,h} u_0}{q!} p_q(i) \quad (3.5)$$

The dimension split stencils are obtained by applying the Δ operator to each of the p_q at the origin. We will denote:

$$\Delta_P^h := \sum_{d=1}^n \partial_{P,d}^{2,h} \quad (3.6)$$

where $\partial_{P,d}^{2,h}$ again indicates a discrete partial derivative in the d^{th} coordinate direction. These stencils are not the only order $2R$ accurate Laplace operators. Since the discrete derivative operator $D^{q,h}$ is second order accurate, if u is a degree $2R$ polynomial restricted to the lattice, we have that $D^{q,h} u(0) = 0$ for all $|q| > 2R$. Therefore we can add any difference quotient of order strictly greater than $2R$ to the dimension split stencil without affecting the order of accuracy of the operator.

Given our initial guess (3.1) for $\psi(K, u)$, we will first display a counterexample that indicates that it will be necessary to modify the dimension split stencil, $\psi(K, u)$, or both. Consider the dimension split stencil Δ_6^h in two dimensions. As in Section 2.1.3, we will rewrite this stencil by transforming the basis functions δ_i , and then apply Δ_6^h to these smoothed basis functions at the origin. Generalizing the one dimensional construction to a cube Q_3 , we have that the smoothed basis functions satisfy:

$$(I - a\Delta^h) \widehat{\delta}_i = \delta_i \in Q_2 \quad (3.7)$$

$$\widehat{\delta}_i = \delta_i \in \partial Q_3 \quad (3.8)$$

and obtain the transformed stencil coefficients as:

$$s_{6,i} = \Delta_6^h \widehat{\delta}_i(0) \quad (3.9)$$

Setting $a = 1$, we can calculate these numerically and find that the transformed stencil is not positive type; see Figure 3.1. We find that $s_{6,(2,0)}$ is negative, which invalidates the local maximum principle for this choice $a = 1$; we find that the same failure holds for other nonnegative choices for a . This counterexample indicates that we will have to modify $\psi(K, u)$, the operator Δ_6^h , or both. We now step back and formalize the perturbed variable technique for proving local maximum principles.

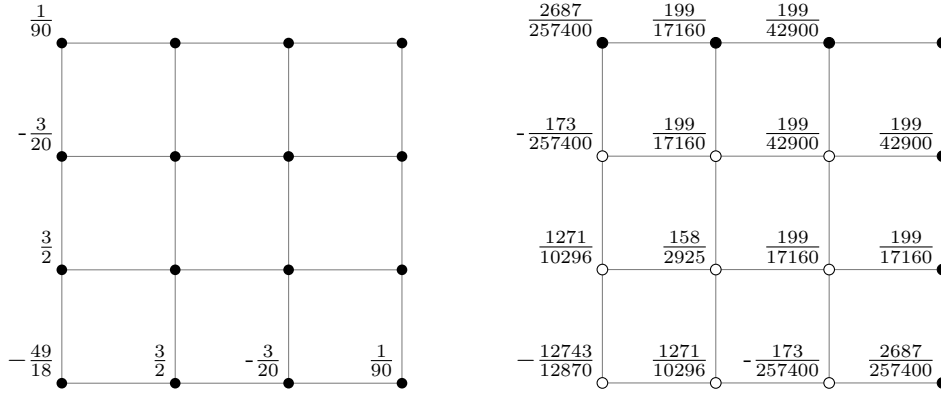


Figure 3.1: Left: Dimension split stencil Δ_6^h ; only top right quadrant of Q_3 is displayed due to symmetry. Right: Rewritten stencil coefficients $s_{6,i}$ for $a = 1$. Open circles indicate perturbed variables $u_i - a\Delta^h u_i$.

3.1 Local Maximum Principle Framework

NOTATION. Let $L^h : \ell^\infty(\mathbb{Z}^n) \rightarrow \ell^\infty(\mathbb{Z}^n)$ denote a discrete Laplace operator. In this section we will assume that L^h is translation invariant, and we will let L_i denote the coefficients of the stencil that define L^h , which are nonzero in some cube of radius R :

$$L^h := L_0 T_0 + \sum_{i \in Q_R} L_i T_i \quad (3.10)$$

We will set $[L^h]$ to be the vector representation of the operator.

DEFINITION. We say that L^h satisfies a **low order local maximum principle** if it has the property

$$L^h u_0 > 0 \implies \exists i \in Q_R : u_0 < u_i \quad (3.11)$$

or equivalently:

$$u_0 = \max_{i \in Q_R} u_i \implies L^h u_0 \leq 0 \quad (3.12)$$

Repeating a definition from the introduction, L^h is **positive type** if

$$L_0 < 0, L_i \geq 0 \quad (3.13)$$

Under mild assumptions, this property is equivalent to a low order local maximum principle.

THEOREM 3.1.1. *Suppose L^h satisfies $L_0 \neq 0$ and $L^h 1^h = 0$, where 1^h is a grid function which is unit everywhere. Then L^h satisfies a low order local maximum principle if and only if it is positive type.*

Proof of 3.1.1:

Suppose L^h satisfies (3.13). Then we have:

$$\sum_{i \in Q_R} L_i u_i > 0 \implies -L_0 u_0 < \sum_{i \in Q_R} L_i u_i \implies \quad (3.14)$$

$$u_0 < \frac{1}{|L_0|} \sum_{i \in Q_R} |L_i| u_i \leq \frac{1}{|L_0|} \sum_{i \in Q_R} |L_0| \max_{i \in Q_R} \{u_i\} \leq \max_{i \in Q_R} \{u_i\} \quad (3.15)$$

where we use the fact that $\sum_{j \in \mathcal{N}_i, j \neq i} |L_j| = |L_i|$.

Now suppose (3.11) holds. If $L_i < 0$ for some $i \neq 0$, then the grid function which is -1 at x_i and zero elsewhere contradicts (3.11), because $L^h u_0 > 0$ but there is no point in Q_R which is strictly greater than u_0 . If $L_0 > 0$, then the grid function which is 1 at x_0 and zero elsewhere contradicts (3.11). $\square$

High order discrete Laplace operators L^h will in general not be positive type due to high order consistency conditions:

LEMMA 3.1.1. *If L^h is a third-order accurate discrete Laplace operator, it cannot be of positive type.*

Proof: If L^h is third-order accurate, it is exact on polynomials of degree four. Letting Δ be the Laplace operator in the variable $x \in \mathbb{R}^n$ with components x_d , we have:

$$L_0 * 0 + \sum_{i \in Q_R} L_i x_{i,1}^2 = \Delta x_1^2(0) = 2 \quad (3.16)$$

$$L_i * 0 + \sum_{i \in Q_R} L_i x_{i,1}^4 = \Delta x_1^4(0) = 0 \quad (3.17)$$

Since $x_{i,1}^2 > 0$ for $i \neq 0$, the first condition implies that $L_i > 0$ for at least one i . Since $x_{i,1}^4 > 0$ for $i \neq 0$, in order to satisfy the second condition we must have $L_j < 0$ for at least one j . $\square$

Lemma 3.1.1 implies that it is useless to try to prove that operators which are not positive type satisfy a low order local maximum principle. As in one dimension, to maintain the structure of a discrete maximum principle we will use perturbations of u^h .

NOTATION. For clarity we will set

$$\phi(\alpha, u)_i = u_i + \alpha(u)_i \quad (3.18)$$

where α is a discrete operator. We can also in place of u have a discrete operator, and interpret this statement as a composition of operators.

DEFINITION. Let $K = \langle \alpha_0, \alpha_1 \dots \alpha_M \rangle$ with $\alpha_0 = 0$. We say that L^h satisfies a **local maximum principle for $\psi(K, u)$** if there exists $R \geq 0$ such that

$$L^h u_0 > 0 \implies \exists i \in Q_R : \bar{\psi}(K, u)_0 < \bar{\psi}(K, u)_i \quad (3.19)$$

or equivalently

$$\psi(K, u)_0 = \max_{i \in Q_R} \psi(K, u)_i \implies L^h u_0 \leq 0 \quad (3.20)$$

We say that L^h is **positive type in** $\psi(K, u)$ if for any $\alpha(0) \in \{\alpha_0 \dots \alpha_M\}$, there exist $\alpha(i) \in K, s_0, s_i \geq 0$ such that

$$L^h u_0 = -s_0 \phi(\alpha(0), u)_0 + \sum_{i \in Q_R} s_i \phi(\alpha(i), u)_i \quad \forall \quad u^h \quad (3.21)$$

In other words, the operator has a positive type representation using perturbed variables. Again, these two definitions are equivalent:

THEOREM 3.1.2. *Suppose L^h satisfies the same assumptions as 3.1.1. Then L^h satisfies a local maximum principle for $\psi(K, u)$ if and only if it is positive type in $\psi(K, u)$.*

Proof of 3.1.2: If L^h is positive type in $\psi(K, u)$, then for any $\alpha(0) \in \{\alpha_0 \dots \alpha_M\}$, we know there will exist coefficients $s_0, s_i \geq 0$ and $\alpha(i) \in K$ such that for any u^h

$$L^h u_0 = -s_0 \phi(\alpha(0), u)_0 + \sum_{i \in Q_R} s_i \phi(\alpha(i), u)_i \quad (3.22)$$

$$= -s_0 \phi(\alpha(0), u)_0 + \sum_{i \in Q_R} s_i \phi \left(\sum_{m=0}^M w_{i,m} \alpha_m, u \right)_i \quad (3.23)$$

$$= -s_0 \phi(\alpha(0), u)_0 + \sum_{i \in Q_R} s_i \sum_{m=0}^M w_{i,m} \phi(\alpha_m, u)_i \quad (3.24)$$

where $\sum_m w_{i,m} = 1 \quad \forall \quad i$. Define $s_{i,m} := s_i w_{i,m}$. Letting e_k be the grid function which is 1 at index k and zero elsewhere, we must have

$$L^h e_k = -s_0 \phi(\alpha(0), e_k)_0 + \sum_{i \in Q_R} \sum_{m=0}^M s_{i,m} \phi(\alpha_m, e_k)_i \quad \forall \quad k \in Q_R \quad (3.25)$$

We can back out $s_i = \sum_m s_{i,m}, w_{i,m} = s_{i,m}/s_i$. This is a linear system of equations with an inequality constraint, which we denote in matrix form as

$$A(\alpha(0)) s_{\alpha(0)} = [L^h] \quad , s_{\alpha(0)} \geq 0 \quad (3.26)$$

where $[L^h]$ is a vector. Since L^h is positive type in $\psi(K, u)$, for each $\alpha(0) \in \{\alpha_m\}$, there exists a vector $s_{\alpha(0)}$ which satisfies 3.26. Applying *Farkas' lemma* [8], a separation theorem from convex analysis, we have that

$$\left\{ \exists s_{\alpha(0)} \geq 0 : A(\alpha(0)) s_{\alpha(0)} = [L^h] \right\} \iff \left\{ [L^h]^T u > 0 \implies \exists i, m : (A(\alpha(0))^T u)_{i,m} > 0 \right\} \quad (3.27)$$

where we notate the grid function u^h as u , a vector with the same dimension as $[L^h]$, and the notation $(A(\alpha(0))^T u)_{i,m} > 0$ means that for all i , $\phi(\alpha_m, u)_i > 0$. The left hand statement is exactly the definition of L^h being positive type in $\psi(K, u)$. We claim that the right hand statement is exactly the definition of L^h satisfying a local maximum principle in $\psi(K, u)$: By our definition of $\phi(\alpha, u)$, we have that $\phi(\alpha, u + 1^h)_0 = \phi(\alpha, u) + 1$. Since $L^h 1_0^h = 0$, we can always shift u by a constant so that $\phi(\alpha(0), u)_i = 0$. Then the right hand statement reads

$$L^h u_0 > 0 \implies \phi(\alpha_m, u)_i > 0 \quad (3.28)$$

$$\psi(K, u)_i \geq \phi(\alpha_m, u)_i > 0 = \phi(\alpha(0), u)_i \quad (3.29)$$

which is exactly the definition 3.19. $\square$

In the previous chapter, we proved that one dimensional second derivative stencils satisfied a particular local maximum principle by using the “if” statement in this theorem; that is, we found a positive type representation of these operators. This theorem says that if we can prove that a stencil satisfies a local maximum principle for $\psi(K, u)$ by some other means, it must also be positive type in $\psi(K, u)$. This latter condition is verifiable numerically:

1. Guess $K = \langle 0, \alpha_1 \dots \alpha_M \rangle$
2. Verify numerically that for all $\alpha(0) \in \{\alpha_m\}$, there exists a solution to the linear equality and inequality constraints (3.26)

The second step involves the solution of a linear program. Although it is possible to pose a nonlinear optimization problem in which the α are parameters, this is a rather large search space, given that each α_m is itself a grid function. We have rather guessed the form of a simply parameterized α_m ; for example a single parameter a times some difference operator. The difficulty in this approach is precisely that which is mentioned in [4]; that is one must guess what these α_m are.

Given K , one can attempt to *construct* a consistent stencil which satisfies a local maximum principle for $\psi(K, u)$ by constraining L^h to satisfy (3.26) as well as consistency requirements. As mentioned previously, any consistent Laplacian stencil of order P will be the dimension split stencil plus an arbitrary linear combination of discrete derivative operators of order greater than P . We will only consider even order correction terms, meaning all indices in q are even, and assume symmetry over permutations of the multi-index of the difference quotient operator.

NOTATION. Let Δ_P^h denote the dimension split stencil of order P . Define

$$\tilde{D}^{q,h} := \sum_{q' \in \sigma(q)} D^{q',h} \quad (3.30)$$

where $\sigma(q)$ denotes the set of permutations of this multi-index. We will identify $\tilde{D}^{q,h}$ with the q which is arranged in non-decreasing order. Let $\mathcal{S}_P^n$ denote the set of length n multi-indices q which are even, arranged in non-decreasing order, and of order strictly greater than P .

Our approach in this chapter is to propose a form for $\psi(K, u)$, and then demonstrate that it is possible to solve for high order accurate stencils which satisfy a local maximum principle in this $\psi(K, u)$. We will continue using the notation from Theorem 3.1.2, and summarize the consistency and local maximum principle compatible requirements in this lemma:

LEMMA 3.1.2. *Let μ_q denote the coefficient of the discrete derivative operator $\tilde{D}^{q,h}$ in the operator*

$$L^h := \Delta_P^h + \sum_{q \in S_P^n} \mu_q \frac{\tilde{D}^{q,h}}{q!} \quad (3.31)$$

and set

$$D_P \mu := \left[\sum_{q \in S_P^n} \mu_q \frac{\tilde{D}^{q,h}}{q!} \right] \quad (3.32)$$

to be the matrix-vector product representation of this operator. Let $K = \langle 0, \alpha_1 \dots \alpha_M \rangle$. If the following inequalities are satisfied, then L^h satisfies a local maximum principle for $\psi(K, u)$:

$$\exists \mu : \forall m(0) \in \{0 \dots M\}, \exists s_{\alpha(0)} \geq 0 : \quad (3.33)$$

$$A(\alpha(0))s_{\alpha(0)} - D_P \mu = [\Delta_P^h] \quad (3.34)$$

Proof: Application of Theorem 3.1.2.

Supposing that for some K these inequalities are feasible, the set of order P accurate operators defined by coefficient vectors μ which satisfy a local maximum principle in $\psi(K, u)$ will form a convex set. We can choose the operator among these which minimizes an objective function $F(\mu)$. Define

$$F(\mu) := \sum_{q \in S_P^n} |\mu_q| e^{|q|} \quad (3.35)$$

with the purpose of penalizing a large stencil footprint. Minimization of an ℓ^1 norm subject to linear inequality constraints can be cast as a linear program by introducing auxiliary variables z_q which satisfy:

$$\mu_q \leq z_q, -\mu_q \leq z_q \quad (3.36)$$

and then we can minimize the auxiliary objective:

$$\sum_q z_q e^{|q|} \quad (3.37)$$

We remark here that this may be a rather large linear program. Letting $R = P/2$, we have that each $A(\alpha(0))$ is a $(2R + 1)^n \times (M + 1)(2R + 1)^n$ matrix, and there are $M + 1$

of these constraint matrices. We will typically limit the number of difference quotients by restricting $|q|$, so the system size is dominated by the inequalities involving $A(\alpha(0))$.

This framework shows that the perturbed variable technique of [4] is a linear program, and is a result that may be useful for operators other than the Laplacian. We will now be guided by our intuition from the one dimensional case for choosing a suitable K in higher dimensions.

3.2 Local Maximum Principle Compatible Laplace Operators

We here propose and motivate a procedure which has proved successful for building high order accurate Laplace operators.

3.2.1 Inductive Construction of $\psi(K, u)$

In one dimension, we showed that rewriting the stencil was equivalent to taking the second derivative at the origin of certain “smoothed” basis functions, which were the Green’s functions of the discrete elliptic operator $I - a\partial^{2,h}$. By applying the low order local maximum principle, we found that these functions satisfied $\partial^{2,h}\hat{\delta}_i(0) \geq 0$ for $i \neq 0$, and numerically, we showed that these functions had relatively small high order behavior. Therefore we will initially define K_1 as an extension of the device that worked in one dimension, that is:

$$\alpha_1 := -a_1\Delta^h \tag{3.38}$$

$$K_1 := \langle 0, -a_1\Delta^h \rangle \tag{3.39}$$

$$\psi(K_1, u)_i := \max \{u_i, u_i - a_1h^2\Delta^hu^h\} \tag{3.40}$$

for some parameter $a > 0$, and we will again set $h = 1$. To find a fourth order stencil which satisfies a local maximum principle for this $\psi(K_1, u)$, we will allow the addition of difference quotients of order 6 and greater to the dimension split stencil. We will limit the number of difference quotients by requiring in addition that $|q|_{\ell^\infty} \leq 4$. Letting μ_q denote the coefficients of these difference quotients, our goal is to find these such that the operator

$$\Delta_4^h + \sum_{q \in \mathcal{S}_4^n, |q|_{\ell^\infty} \leq 4} \mu_q \frac{\tilde{D}^{q,h}}{q!} \tag{3.41}$$

is positive type in $\psi(K_1, u)$. As in one dimension, we have that if $\psi(K_1, u)$ is maximal at the origin, it must be equal to $u_0 - a_1\Delta^hu_0$; otherwise we have a contradiction $u_0 < \max_{i \in \partial B_1(0)} \{u_i\} \leq \max_{i \in \partial B_1(0)} \{\psi(K_1, u)_i\}$. Therefore we only need to find a positive type representation for the case that $\psi(K_1, u)_0 = u_0 - a_1\Delta^hu_0$; that is in the notation of Theorem

3.1.2, we will solve the linear program:

$$\min_{\mu} F(\mu) \text{ such that} \quad (3.42)$$

$$A(\alpha_1)s_{\alpha_1} - D_4\mu = [\Delta_4^h] \quad (3.43)$$

$$s_{\alpha_1} \geq 0 \quad (3.44)$$

where D_4 is limited so that $|q|_{\ell^\infty} \leq 4$. Again using the notation of Theorem 3.1.2, we can select a perturbation $\alpha(i), i \neq 0$ to reduce $A(\alpha_1)$ to be a square matrix. Equivalently, we will construct a small boundary value problem to transform the basis functions δ_i . Setting

$$\alpha(i) = -a_1\Delta^h, \quad i \in Q_1 \quad (3.45)$$

$$\alpha(i) = 0, \quad i \in \partial Q_2 \quad (3.46)$$

is equivalent to setting up the boundary value problem:

$$(I - a_1\Delta^h)\hat{\delta}_i = \delta_i \in Q_1 \quad (3.47)$$

$$\hat{\delta}_i = \delta_i \in \partial Q_2 \quad (3.48)$$

We will define the matrix $M_4^h(a_1)$ as the matrix representation of this operator and boundary conditions; the vector representations of the grid functions $\hat{\delta}_i$ are exactly the columns of $M_4^h(a_1)^{-1}$. With this reduction, our linear program becomes:

$$\min_{\mu} F(\mu) \text{ such that} \quad (3.49)$$

$$M_4^h(a_1)s_4 - D_4\mu = [\Delta_4^h] \quad (3.50)$$

$$s_{4,i} \geq 0 \quad \forall i \neq 0 \quad (3.51)$$

where the components of s_4 are:

$$s_{4,i} = \left(\Delta_4^h + \sum_{q \in S_4^n, |q|_{\ell^\infty} \leq 4} \mu_q \frac{\tilde{D}^{q,h}}{q!} \right) \hat{\delta}_i(0) \quad (3.52)$$

As in one dimension, one can show that these $\hat{\delta}_i, i \neq 0$ have the property that

$$\Delta_i^h \hat{\delta}_i(0) \geq 0 \quad (3.53)$$

Thus, due to the smoothness of $\hat{\delta}_i$, it is reasonable to think that we might be able to generate a stencil L_4^h which maintains this sign property.

Suppose for now that this is indeed the case, meaning (3.49)-(3.51) is feasible, and is solved by some μ^4 . Then set

$$L_4^h := \Delta_4^h + \sum_{q \in S_4^n, |q|_{\ell^\infty} \leq 4} \mu_q^4 \frac{\tilde{D}^{q,h}}{q!} \quad (3.54)$$

We will now repeat this procedure to obtain a sixth order accurate stencil.

We will first set up the boundary value problem that defines the $\widehat{\delta}_i$:

$$(I - a_1(x)\Delta^h - a_2(x)L_4^h)\widehat{\delta}_i = \delta_i \in Q_3 \setminus \partial Q_3 \quad (3.55)$$

$$\widehat{\delta}_i = \delta_i \in \partial Q_3 \quad (3.56)$$

$$a_1(x) = a_1 \in Q_2, \text{ else } 0 \quad (3.57)$$

$$a_2(x) = a_2 \in Q_1, \text{ else } 0 \quad (3.58)$$

and let $M_6^h(\{a_1, a_2\})$ be the associated matrix operator.

Suppose that for this equation we have the following discrete minimum principle: (this is proved later in Theorem 4.1.3)

$$(I - a_1(x)\Delta^h - a_2(x)L_4^h)u^h \geq 0 \in Q_3 \setminus \partial Q_3 \implies \quad (3.59)$$

$$\min_{Q_3 \setminus \partial Q_3} \psi(K_1, u) \geq \min_{\partial Q_3} (u^h)^- \quad (3.60)$$

Applying this to a Green's function for a point $i \neq 0$, we have;

$$\min_{Q_3 \setminus \partial Q_3} \psi(K_1, \widehat{\delta}_i) \geq \min_{\partial Q_3} (\widehat{\delta}_i)^- = 0 \implies \quad (3.61)$$

$$(I - a_1(x)\Delta^h)\widehat{\delta}_i(0) \geq 0 \quad (3.62)$$

From the discrete PDE we have:

$$(I - a_1(x)\Delta^h - a_2(x)L_4^h)\widehat{\delta}_i(0) = \delta_i(0) = 0 \quad (3.63)$$

$$a_2(0)L_4^h\widehat{\delta}_i(0) = (I - a_1(0)\Delta^h)\widehat{\delta}_i(0) \geq 0 \quad (3.64)$$

which shows that we have

$$L_4^h\widehat{\delta}_i(0) \geq 0, i \neq 0 \quad (3.65)$$

Again, we expect that the high order behavior of these smoothed basis functions should be small relative to the size of the fourth order accurate Laplace operator, so it is reasonable that we may be able to build a sixth-order accurate stencil using the perturbed variables suggested by this local boundary value problem. Therefore, to construct a sixth order accurate stencil we can modify K_2 to be

$$\alpha_2 := -a_1\Delta^h - a_2L_4^h \quad (3.66)$$

$$K_2 := \langle 0, \alpha_1, \alpha_2 \rangle = \langle K_1, -a_1\Delta^h - a_2L_4^h \rangle \quad (3.67)$$

for some $a_2 \geq 0$. If $\psi(K_2, u)$ achieves a local maximum at the origin, we can, by similar arguments as for K_1 , prove that $\psi(K_2, u)_0 = u_0 - a_1\Delta^h u_0 - a_2L_4^h u_0$. This means that

again we only need to find a single positive type representation of our new operator using $M_6^h(\{a_1, a_2\})$. To obtain a sixth order accurate stencil, we can solve the linear program:

$$\min_{\mu} F(\mu) \text{ such that} \quad (3.68)$$

$$M_6^h(\{a_1, a_2\})s_6 - D_6\mu = [\Delta_6^h] \quad (3.69)$$

$$s_{6,i} \geq 0 \quad \forall i \neq 0 \quad (3.70)$$

where D_6 is limited so that $|q|_{\ell^\infty} \leq 6$. If this problem is feasible and solved by μ^6 , we will set:

$$L_6^h := \Delta_6^h + \sum_{q \in S_6^n, |q|_{\ell^\infty} \leq 6} \mu_q^6 \frac{\tilde{D}^{q,h}}{q!} \quad (3.71)$$

we can continue inductively constructing K_3 as

$$K_3 = \langle K_2, -a_1\Delta^h - a_2L_4^h - a_3L_6^h \rangle \quad (3.72)$$

and repeat the construction of a boundary value problem to define the smoothed basis functions.

We formalize this process in the following theorem:

THEOREM 3.2.1. *Let $\{L_{2r}^h\}$ be a collection of order $2r$ accurate Laplace operators, $1 \leq r \leq P/2$, with $L_2^h := \Delta^h$. Suppose P is such that for all $2r \leq P$, L_{2r}^h satisfies a local maximum principle in $\psi(K_r, u)$ for*

$$K_1 = \langle 0 \rangle \quad (3.73)$$

$$K_r = \left\langle K_{r-1}, -\sum_{s=1}^{r-1} a_s L_{2s}^h \right\rangle \quad (3.74)$$

with $a_s \geq 0, a_1 > 0$. Furthermore, suppose that L_{2r}^h satisfies a local maximum principle:

$$L_{2r}^h u_0 > 0 \implies \quad (3.75)$$

$$\exists j \in Q_r \setminus \{0\} : \psi(K_r, u)_0 < \psi(K_{1+r-|j|_{\ell^\infty}}, u)_j \quad (3.76)$$

Setting $R = (P+2)/2$, define:

$$a_r(i) = a_r \in Q_{1+R-r} \text{ else } 0 \quad (3.77)$$

for some $a_R \geq 0$, and define the Green's functions $\hat{\delta}_i$ of the discrete boundary value problem to satisfy:

$$\left(I - \sum_{r=1}^{R-1} a_r(x) L_{2r}^h \right) \hat{\delta}_i = \delta_i \in Q_R \setminus \partial Q_R^h \quad (3.78)$$

$$\hat{\delta}_i = \delta_i \in \partial Q_R^h \quad (3.79)$$

Let L_{P+2}^h be an operator of the form

$$\Delta_{P+2}^h + \sum_{q \in \mathcal{S}_{P+2}^n, |q|_{\ell^\infty} \leq P+2} \mu_q^{P+2} \frac{\tilde{D}^{q,h}}{q!} \quad (3.80)$$

Then if the following condition is satisfied:

$$\min_{i \in Q_R, i \neq 0} L_{P+2}^h \hat{\delta}_i(0) \geq 0 \quad (3.81)$$

We will have:

I. L_{P+2}^h satisfies a local maximum principle in $\psi(K_R, u)$ for

$$K_R = \left\langle K_{R-1}, -\sum_{r=1}^{R-1} a_r L_{2r}^h \right\rangle \quad (3.82)$$

II: L_{P+2}^h satisfies the local maximum principle:

$$L_{P+2}^h u_0 > 0 \implies \quad (3.83)$$

$$\exists j \in Q_R \setminus \{0\} : \psi(K_R, u)_0 < \psi(K_{1+R-|j|_{\ell^\infty}}, u)_j \quad (3.84)$$

Proof: Later in this chapter, in Theorem 4.1.3 we will prove that this discrete boundary value problem (3.78) - (3.79) is well posed. For now, we take this as an assumption. Now assume that $\psi(K_R, u)$ is maximal at the origin over Q_R . Then we must have that:

$$\psi(K_R, u)_0 = u_0 - \sum_{r=1}^{R-1} a_r L_{2r}^h u_0 \quad (3.85)$$

otherwise, suppose that for some $1 \leq m < R-1$ we have

$$\psi(K_R, u)_0 = u_0 - \sum_{r=1}^{R-1-m} a_r L_{2r}^h u_0 = \psi(K_{R-m}, u)_i \quad (3.86)$$

which implies

$$u_0 - ah^2 \sum_{r=1}^{R-1-m} L_{2r}^h u_0 > u_0 - \sum_{r=1}^{R-m} a_r L_{2r}^h u_0 \implies \quad (3.87)$$

$$L_{2(R-m)}^h u_0 > 0 \implies \quad (3.88)$$

$$\exists i \in Q_{R-m} : \psi(K_{R-m}, u)_0 < \psi(K_{R-m}, u)_i \leq \psi(K_R, u)_i \quad (3.89)$$

by assumption of conditions on L_{2r}^h , which contradicts the claim that $\psi(K_R, u)$ is maximal. Therefore to prove that L_{P+2}^h satisfies a local maximum principle in $\psi(K_r, u)$ we must show

that there exists a positive type representation of this operator for the case that $\psi(K_R, u)_0 = \phi(\alpha(0), u)_0 = u_0 - \sum_{r=1}^{R-1} a_r L_{2r}^h u_0$. At each point $i \neq 0 \in Q_R$, we will select a perturbation for this representation as follows:

$$\alpha(i) = - \sum_{r=1}^{R-|i|_{\ell^\infty}} a_r L_{2r}^h \quad (3.90)$$

Therefore to prove that L_{P+2}^h satisfies a local maximum principle for $\psi(K_r, u)$ we must show that there exist coefficients $s_{P+2,i} \geq 0, s_{P+2,0} < 0$ such that

$$L_{P+2}^h = \sum_{i \in Q_r} s_{P+2,i} \phi(\alpha(i), I)_i \quad (3.91)$$

which is equivalent to solving the inequalities 3.33 - 3.34, which by Lemma 3.1.2 verify that L_{P+2}^h satisfies the desired local maximum principle. This choice of perturbations can be equivalently written as the operator:

$$\phi(\alpha(i), u)_i = \left(I - \sum_{r=1}^{R-1} a_r(x) L_{2r}^h \right) u_i \quad (3.92)$$

Therefore, as in Section 2.1.3, the stencil coefficients $s_{P+2,i}$ can be obtained by:

$$s_{P+2,i} = L_{P+2}^h \widehat{\delta}_i \quad (3.93)$$

where $\widehat{\delta}_i$ satisfy the boundary value problem (3.78)-(3.79). Then by assumption for $i \neq 0$, these satisfy $s_{P+2,i} \geq 0$, so we have by the zeroth moment condition that $s_{P+2,0} \leq 0$. This proves claim I. Lastly, if this condition holds, then we have

$$L_{P+2}^h u_i > 0 \implies \quad (3.94)$$

$$\sum_{i \in Q_r} s_{P+2,i} \phi(\alpha(i), u)_i > 0 \implies \quad (3.95)$$

$$\phi(\alpha(0), u)_0 = u_0 - \sum_{r=1}^{R-1} a_r L_{2r}^h u_0 < \max_{i \in Q_R} \phi(\alpha(i), u)_i < \max_{i \in Q_R} \psi(K_{1+R-|i|_{\ell^\infty}}, u)_i \quad (3.96)$$

Thus if $\psi(K_R, u)_0 = \phi(\alpha(0), u)_0$, this proves claim II. If that is not the case, then we have for some $1 \leq m \leq R-1$, $\psi(K_R, u)_0 = \psi(K_{R-m}, u)_0$, meaning:

$$u_0 - \sum_{r=1}^{R-m} a_r L_{2r}^h u_0 < u_0 - \sum_{r=1}^{R-1-m} a_r L_{2r}^h u_0 \implies \quad (3.97)$$

$$L_{2(R-m)}^h u_0 > 0 \implies \quad (3.98)$$

$$\exists i \in Q_{R-m} : \psi(K_{R-m}, u)_0 < \psi(K_{1+R-m-|i|_{\ell^\infty}}, u)_i \leq \max_{i \in Q_R} \psi(K_{1+R-|i|_{\ell^\infty}}, u)_i \quad (3.99)$$

which proves II. $\square$

A quick corollary of this theorem is that if $\{L_{2r}^h\}, r \leq P/2$ and L_{P+2}^h satisfy the assumptions of the theorem, then the set $\{L_{2r}^h, L_{P+2}^h\}$ also satisfies the assumptions of the theorem. Since the conditions are trivially satisfied for $P = 2$, to prove that a sequence of high order accurate Laplace operators satisfy the constructed local maximum principle for a sequence $\{a_r\}$, we only need numerically verify the positivity condition (3.81) for each P holds for some μ^P . To do so we will report the coefficients s_i of the stencil rewritten in the perturbed variables. It turns out that for our choice of variables these coefficients obey a certain symmetry relation, and so we only need to report a small number of them.

DEFINITION. We define the set C_R^n as $\{i \in \mathbb{Z}_+^n : i_1 \leq R, i_{k+1} \leq i_k \forall k \in [1, n-1]\}$. We will use the indices of this set to compactly represent grid functions which are supported in Q_R and satisfy certain symmetry relations. We will define the set S_i as the set of all unique permutations of $\{\pm i_1, \pm i_2 \dots \pm i_n\}$. For example,

$$S_{(1,2)} = \{(\pm 1, \pm 2), (\pm 2, \pm 1)\} = \{(1, 2), (-1, 2), (1, -2), (-1, -2), (2, 1), (-2, 1), (2, -1), (-2, -1)\}$$

If a grid function supported in Q_R has the symmetry relation $v_j = v_i, j \in S_i$ then it can be fully defined by its values in C_R^n . Under certain assumptions, our positive type representations will satisfy these symmetry relations.

We next claim that the Green's functions which satisfy (3.78)-(3.79) have the property that $\widehat{\delta}_i(j) = \widehat{\delta}_{W(i)}(W(j))$ if W is an index mapping which may permute indices and flip sign of indices, for example:

$$W : (i_x, i_y) \rightarrow (+i_y, -i_x) \quad (3.100)$$

so that $W(j) \in S_i$. This follows from the fact that the operator $I - \sum_{r=1}^{R-1} a_r(x) L_{2r}^h$ is invariant under this coordinate mapping: it is made up of even order operators $\tilde{D}^{q,h}$ and $\delta_i(j) = \delta_{W(i)}(W(j))$, and $a_r(x)$ is invariant under this mapping as well. Lastly, if $j = W(i) \in S_i$, then:

$$s_i = L_P^h \widehat{\delta}_i(0) = \sum_{j \in Q_{P/2}} [L^h]_j \widehat{\delta}_i(j) \quad (3.101)$$

$$= \sum_{j \in Q_{P/2}} [L^h]_{W(j)} \widehat{\delta}_i(j) \quad (3.102)$$

$$= \sum_{j \in Q_{P/2}} [L^h]_{W(j)} \widehat{\delta}_{W(i)}(W(j)) \quad (3.103)$$

$$= L^h \widehat{\delta}_{W(i)}(0) = s_j \quad (3.104)$$

where we have used the fact that L_P^h as a grid function satisfies this same symmetry. Therefore we only need to report the coefficients $s_i, i \in C_R^n$ for a stencil which is supported on Q_R .

We summarize the computational procedure in Algorithm 1. Note that this procedure and the proofs that accompany it are independent of dimension. By setting $a_1 = 1$, and $a_r = 0$ otherwise we can recover the one dimensional stencils. We now verify in two and three dimensions that this machinery can be used to produce sequences of high order accurate Laplace operators.

We perform all computations, including solving the linear program, using exact rational arithmetic, so that the nonnegativity condition (3.81) can be verified exactly. We report rounded decimals for readability. The size of all computations is greatly reduced by taking advantage of symmetries, and the process is implemented using the SageMath package [17].

Algorithm 1 Inductive Stencil Construction Procedure

Input: $P_{max}, \{a_1, a_2 \dots a_{P_{max}/2-1}\}$

$P \leftarrow 2$

$K = \langle 0 \rangle$

while $P \leq P_{max}$ **do**

Solve:

$\min_{\mu} F(\mu)$ such that

$$M_P^h(\{a_1 \dots a_{P/2-1}\})s_P - D_P\mu = [\Delta_P^h]$$

$$s_{P,i} \geq 0 \quad \forall i \neq 0$$

$$L_P^h = \Delta_P^h + D_P\mu^P$$

$$\alpha_{P/2} = -\sum_{s=1}^{P/2} a_s L_{2s}^h$$

$$K \leftarrow \langle K, -\alpha_{P/2} \rangle$$

$$P \leftarrow P + 2$$

end while

Return: $\{\mu^2 \dots \mu^{P_{max}}\}, \{L_2^h \dots L_{P_{max}}^h\}$

3.2.2 Two Dimensions

To apply Theorem 3.2.1, we only need to report a sequence of coefficients $\{a_r\}$, a sequence of coefficients $\{\mu^{2r}\}$, and the coefficients $s_{2r,i}$ that verify condition (3.81). We will always set the maximum P to 8.

Firstly, we set $a_1 = 1/2$ and all other a_r to zero. This means we are solving for operators which satisfy a local maximum principle in

$$\psi(K, u)_i = \max \left\{ u_i, u_i - \frac{1}{2} \Delta^h u_i \right\} \quad (3.105)$$

as we did in one dimension.

THEOREM 3.2.2. *Let $P \leq 8$ and $a = 1/2$. There exist coefficients $\mu_{(2,2r)}^{2r}, r = 2, 3, 4$ such that the operator*

$$\Delta_P^h + \mu_{(2,P)}^P \tilde{D}^{(2,P),h} \quad (3.106)$$

is positive type in $\psi(K, u)$ for

$$K = \langle 0, -a\Delta^h \rangle \quad (3.107)$$

Proof: Restricting our linear program such that we can only add a difference operator $\tilde{D}^{(2,2r),h}$, we solve our linear program and find:

$$\mu_{(2,4)}^4 = 0 \quad (3.108)$$

$$\mu_{(2,6)}^6 = 1.14638 \cdot 10^{-3} \quad (3.109)$$

$$\mu_{(2,8)}^8 = -1.36167 \cdot 10^{-3} \quad (3.110)$$

The coefficients $s_{2r,i}$ satisfy (3.81):

r	2	3	4
$s_{2r,(0,0)}$	$-1.52381 \cdot 10^0$	$-1.62226 \cdot 10^0$	$-1.62221 \cdot 10^0$
$s_{2r,(1,0)}$	$2.14286 \cdot 10^{-1}$	$2.42975 \cdot 10^{-1}$	$2.23274 \cdot 10^{-1}$
$s_{2r,(1,1)}$	$7.14286 \cdot 10^{-2}$	$9.73741 \cdot 10^{-2}$	$1.28750 \cdot 10^{-1}$
$s_{2r,(2,0)}$	$2.38095 \cdot 10^{-2}$	0	0
$s_{2r,(2,1)}$	$3.57143 \cdot 10^{-2}$	$1.47560 \cdot 10^{-2}$	$1.04700 \cdot 10^{-2}$
$s_{2r,(2,2)}$	0	$4.91868 \cdot 10^{-3}$	$4.09727 \cdot 10^{-3}$
$s_{2r,(3,0)}$	0	$8.81834 \cdot 10^{-3}$	$3.27875 \cdot 10^{-3}$
$s_{2r,(3,1)}$	0	$8.52440 \cdot 10^{-3}$	$6.22623 \cdot 10^{-3}$
$s_{2r,(3,2)}$	0	$2.45934 \cdot 10^{-3}$	$1.82179 \cdot 10^{-3}$
$s_{2r,(3,3)}$	0	0	$6.07265 \cdot 10^{-4}$
$s_{2r,(4,0)}$	0	0	$2.57701 \cdot 10^{-3}$
$s_{2r,(4,1)}$	0	0	$1.75144 \cdot 10^{-3}$
$s_{2r,(4,2)}$	0	0	$9.10897 \cdot 10^{-4}$
$s_{2r,(4,3)}$	0	0	$3.03632 \cdot 10^{-4}$
$s_{2r,(4,4)}$	0	0	0

(3.111)

which completes the proof. $\square$

Next we will set all $a_{P/2}$ to be $1/2$. It turns out that this choice recovers the dimension split stencils exactly; that is we have $\mu^P = 0$.

THEOREM 3.2.3. *Let $P \leq 8$. The operator Δ_P^h satisfies a local maximum principle for $\psi(K_{P/2}, u)$, where*

$$\alpha_0 = 0 \quad (3.112)$$

$$\alpha_r = -\frac{1}{2} \sum_{s=1}^r \Delta_{2r}^h \quad (3.113)$$

$$K_1 = \langle 0 \rangle, K_r = \langle K_{r-1}, \alpha_r \rangle \quad (3.114)$$

Proof: We again apply Theorem 3.2.1 to reduce this proof to computation of $s_{2r,i}$, which are seen to satisfy (3.81):

r	2	3	4
$s_{2r,(0,0)}$	$-1.52381 \cdot 10^0$	$-9.05322 \cdot 10^{-1}$	$-6.36841 \cdot 10^{-1}$
$s_{2r,(1,0)}$	$2.14286 \cdot 10^{-1}$	$9.96795 \cdot 10^{-2}$	$5.98449 \cdot 10^{-2}$
$s_{2r,(1,1)}$	$7.14286 \cdot 10^{-2}$	$4.49449 \cdot 10^{-2}$	$3.03501 \cdot 10^{-2}$
$s_{2r,(2,0)}$	$2.38095 \cdot 10^{-2}$	$7.45709 \cdot 10^{-3}$	$3.66959 \cdot 10^{-3}$
$s_{2r,(2,1)}$	$3.57143 \cdot 10^{-2}$	$1.83568 \cdot 10^{-2}$	$1.16664 \cdot 10^{-2}$
$s_{2r,(2,2)}$	0	$6.11893 \cdot 10^{-3}$	$5.44094 \cdot 10^{-3}$
$s_{2r,(3,0)}$	0	$1.06863 \cdot 10^{-2}$	$8.16252 \cdot 10^{-3}$
$s_{2r,(3,1)}$	0	$7.30569 \cdot 10^{-3}$	$5.32946 \cdot 10^{-3}$
$s_{2r,(3,2)}$	0	$3.05947 \cdot 10^{-3}$	$3.00931 \cdot 10^{-3}$
$s_{2r,(3,3)}$	0	0	$1.00310 \cdot 10^{-3}$
$s_{2r,(4,0)}$	0	0	$2.47512 \cdot 10^{-3}$
$s_{2r,(4,1)}$	0	0	$2.34724 \cdot 10^{-3}$
$s_{2r,(4,2)}$	0	0	$1.27795 \cdot 10^{-3}$
$s_{2r,(4,3)}$	0	0	$5.01552 \cdot 10^{-4}$
$s_{2r,(4,4)}$	0	0	0

$\square$

3.2.3 Three Dimensions

We will first set $a_1 = 1/3$ and all other coefficients to zero. Then we have the following:

THEOREM 3.2.4. *There exist coefficients μ^{2r} such that the operator*

$$\Delta_{2r}^h + \sum_{q \in S_{2r}^3, |q|_{\ell^\infty} \leq 2r} \mu_q^{2r} \tilde{D}^{q,h} \quad (3.116)$$

is positive type in $\psi(K, u)$ for

$$K = \langle 0, -a\Delta^h \rangle \quad (3.117)$$

Proof: The nonzero components of μ^{2r} are computed as:

$$\mu_{0,2,4}^4 = -8.58333 \cdot 10^{-3} \quad (3.118)$$

$$\mu_{(0,2,6)}^6 = 2.77778 \cdot 10^{-3}, \mu_{(0,4,4)}^6 = 3.01690 \cdot 10^{-3} \quad (3.119)$$

$$\mu_{(0,2,8)}^8 = -1.19380 \cdot 10^{-3}, \mu_{(0,4,6)}^8 = -1.94528 \cdot 10^{-3}, \mu_{(0,4,8)}^8 = -2.35533 \cdot 10^{-4}, \quad (3.120)$$

$$\mu_{(0,6,6)}^8 = -1.32592 \cdot 10^{-4}, \mu_{(2,4,6)}^8 = -2.43664 \cdot 10^{-4} \quad (3.121)$$

and the coefficients $s_{2r,i}$ satisfy (3.81):

r	2	3	4
$s_{2r,(0,0,0)}$	$-2.19600 \cdot 10^0$	$-2.29810 \cdot 10^0$	$-2.32006 \cdot 10^0$
$s_{2r,(1,0,0)}$	$1.47000 \cdot 10^{-1}$	$1.39936 \cdot 10^{-1}$	$1.40817 \cdot 10^{-1}$
$s_{2r,(1,1,0)}$	$6.00000 \cdot 10^{-2}$	$8.13237 \cdot 10^{-2}$	$7.33802 \cdot 10^{-2}$
$s_{2r,(1,1,1)}$	$2.00000 \cdot 10^{-2}$	$2.82509 \cdot 10^{-2}$	$5.36998 \cdot 10^{-2}$
$s_{2r,(2,0,0)}$	0	0	0
$s_{2r,(2,1,0)}$	$1.14167 \cdot 10^{-2}$	$3.63798 \cdot 10^{-4}$	0
$s_{2r,(2,1,1)}$	$6.66667 \cdot 10^{-3}$	$3.42889 \cdot 10^{-3}$	0
$s_{2r,(2,2,0)}$	0	$1.29554 \cdot 10^{-3}$	$2.46874 \cdot 10^{-4}$
$s_{2r,(2,2,1)}$	0	$9.40767 \cdot 10^{-4}$	$1.10583 \cdot 10^{-3}$
$s_{2r,(2,2,2)}$	0	$3.13589 \cdot 10^{-4}$	$4.05756 \cdot 10^{-4}$
$s_{2r,(3,0,0)}$	0	0	0
$s_{2r,(3,1,0)}$	0	$2.89904 \cdot 10^{-3}$	$1.78906 \cdot 10^{-3}$
$s_{2r,(3,1,1)}$	0	$1.14296 \cdot 10^{-3}$	$1.10600 \cdot 10^{-3}$
$s_{2r,(3,2,0)}$	0	$4.31846 \cdot 10^{-4}$	$7.06598 \cdot 10^{-4}$
$s_{2r,(3,2,1)}$	0	$3.13589 \cdot 10^{-4}$	$2.63980 \cdot 10^{-4}$
$s_{2r,(3,2,2)}$	0	$1.04530 \cdot 10^{-4}$	$1.11433 \cdot 10^{-4}$
$s_{2r,(3,3,0)}$	0	0	$1.29923 \cdot 10^{-4}$
$s_{2r,(3,3,1)}$	0	0	$7.69417 \cdot 10^{-5}$
$s_{2r,(3,3,2)}$	0	0	$3.45933 \cdot 10^{-5}$
$s_{2r,(3,3,3)}$	0	0	$1.15311 \cdot 10^{-5}$
$s_{2r,(4,0,0)}$	0	0	$1.63091 \cdot 10^{-4}$
$s_{2r,(4,1,0)}$	0	0	$3.44684 \cdot 10^{-4}$
$s_{2r,(4,1,1)}$	0	0	$3.68667 \cdot 10^{-4}$
$s_{2r,(4,2,0)}$	0	0	0
$s_{2r,(4,2,1)}$	0	0	$8.79932 \cdot 10^{-5}$
$s_{2r,(4,2,2)}$	0	0	$3.71445 \cdot 10^{-5}$
$s_{2r,(4,3,0)}$	0	0	$4.33076 \cdot 10^{-5}$
$s_{2r,(4,3,1)}$	0	0	$2.56472 \cdot 10^{-5}$
$s_{2r,(4,3,2)}$	0	0	$1.15311 \cdot 10^{-5}$
$s_{2r,(4,3,3)}$	0	0	$3.84370 \cdot 10^{-6}$
$s_{2r,(4,4,0)}$	0	0	0
$s_{2r,(4,4,1)}$	0	0	0
$s_{2r,(4,4,2)}$	0	0	0
$s_{2r,(4,4,3)}$	0	0	0
$s_{2r,(4,4,4)}$	0	0	0

(3.122)

We now set all $a_r = 1/3$.

THEOREM 3.2.5. *Let $P \leq 8$. There exist μ^{2r} such that the operator $L_{2r}^h := \Delta_{2r}^h + \mu^{2r}$*

satisfies a local maximum principle for $\psi(K_r, u)$, where

$$\alpha_0 = 0 \tag{3.123}$$

$$\alpha_r = -\frac{1}{3} \sum_{s=1}^r L_{2r}^h \tag{3.124}$$

$$K_1 = \langle 0 \rangle, K_r = \langle K_{r-1}, \alpha_r \rangle \tag{3.125}$$

Proof: The nonzero components of μ^{2r} are computed as:

$$\mu_{0,2,4}^4 = -8.58333 \cdot 10^{-3} \tag{3.126}$$

$$\mu_{(0,2,6)}^6 = 1.51866 \cdot 10^{-3}, \mu_{(0,4,4)}^6 = -2.87498 \cdot 10^{-4} \tag{3.127}$$

$$\mu_{(0,2,8)}^8 = -1.13473 \cdot 10^{-4}, \mu_{(0,4,6)}^8 = -8.31166 \cdot 10^{-4} \tag{3.128}$$

and the coefficients $s_{2r,i}$ satisfy (3.81):

r	2	3	4
$s_{2r,(0,0,0)}$	$-2.19600 \cdot 10^0$	$-1.37459 \cdot 10^0$	$-9.67443 \cdot 10^{-1}$
$s_{2r,(1,0,0)}$	$1.47000 \cdot 10^{-1}$	$7.38142 \cdot 10^{-2}$	$3.93794 \cdot 10^{-2}$
$s_{2r,(1,1,0)}$	$6.00000 \cdot 10^{-2}$	$3.32779 \cdot 10^{-2}$	$2.30829 \cdot 10^{-2}$
$s_{2r,(1,1,1)}$	$2.00000 \cdot 10^{-2}$	$1.47720 \cdot 10^{-2}$	$1.11546 \cdot 10^{-2}$
$s_{2r,(2,0,0)}$	0	0	0
$s_{2r,(2,1,0)}$	$1.14167 \cdot 10^{-2}$	$3.90073 \cdot 10^{-3}$	$1.35856 \cdot 10^{-3}$
$s_{2r,(2,1,1)}$	$6.66667 \cdot 10^{-3}$	$4.77903 \cdot 10^{-3}$	$3.24199 \cdot 10^{-3}$
$s_{2r,(2,2,0)}$	0	$2.44916 \cdot 10^{-3}$	$2.60715 \cdot 10^{-3}$
$s_{2r,(2,2,1)}$	0	$1.43657 \cdot 10^{-3}$	$1.38718 \cdot 10^{-3}$
$s_{2r,(2,2,2)}$	0	$4.78857 \cdot 10^{-4}$	$6.16936 \cdot 10^{-4}$
$s_{2r,(3,0,0)}$	0	0	$3.73323 \cdot 10^{-3}$
$s_{2r,(3,1,0)}$	0	$3.12964 \cdot 10^{-3}$	$1.71429 \cdot 10^{-3}$
$s_{2r,(3,1,1)}$	0	$1.16131 \cdot 10^{-3}$	$1.01465 \cdot 10^{-3}$
$s_{2r,(3,2,0)}$	0	$4.33678 \cdot 10^{-4}$	$6.68079 \cdot 10^{-4}$
$s_{2r,(3,2,1)}$	0	$4.36593 \cdot 10^{-4}$	$5.65035 \cdot 10^{-4}$
$s_{2r,(3,2,2)}$	0	$1.59619 \cdot 10^{-4}$	$2.91151 \cdot 10^{-4}$
$s_{2r,(3,3,0)}$	0	0	$2.08941 \cdot 10^{-4}$
$s_{2r,(3,3,1)}$	0	0	$1.65809 \cdot 10^{-4}$
$s_{2r,(3,3,2)}$	0	0	$8.84610 \cdot 10^{-5}$
$s_{2r,(3,3,3)}$	0	0	$2.94870 \cdot 10^{-5}$
$s_{2r,(4,0,0)}$	0	0	0
$s_{2r,(4,1,0)}$	0	0	$4.74625 \cdot 10^{-4}$
$s_{2r,(4,1,1)}$	0	0	$3.14379 \cdot 10^{-4}$
$s_{2r,(4,2,0)}$	0	0	$1.85045 \cdot 10^{-4}$
$s_{2r,(4,2,1)}$	0	0	$1.54898 \cdot 10^{-4}$
$s_{2r,(4,2,2)}$	0	0	$7.90361 \cdot 10^{-5}$
$s_{2r,(4,3,0)}$	0	0	$5.99757 \cdot 10^{-5}$
$s_{2r,(4,3,1)}$	0	0	$5.02318 \cdot 10^{-5}$
$s_{2r,(4,3,2)}$	0	0	$2.77219 \cdot 10^{-5}$
$s_{2r,(4,3,3)}$	0	0	$9.82899 \cdot 10^{-6}$
$s_{2r,(4,4,0)}$	0	0	0
$s_{2r,(4,4,1)}$	0	0	0
$s_{2r,(4,4,2)}$	0	0	0
$s_{2r,(4,4,3)}$	0	0	0
$s_{2r,(4,4,4)}$	0	0	0

(3.129)

□

3.2.4 Basis Function Smoothing

In this section we will study the small discrete boundary value problem and the corresponding “smoothed” basis functions that we utilized in the proof of Theorem 3.2.1. Recall that the coefficients of the rewritten stencil are equivalent to the operator applied to these basis functions.

Let $\{L_{2r}^h\}$ be a collection of order $2r$ accurate Laplace operators, $1 \leq r \leq P/2$, with $L_2^h := \Delta^h$. Suppose P is such that for all $r \leq P/2$, L_{2r}^h satisfies a local maximum principle in $\psi(K_r, u)$ for

$$K_1 = \langle 0 \rangle \quad (3.130)$$

$$K_r = \left\langle K_{r-1}, \sum_{s=1}^{r-1} a_s L_{2s}^h \right\rangle \quad (3.131)$$

for $a_s \geq 0$, and additionally satisfies:

$$L_{2r}^h u_0 > 0 \implies \quad (3.132)$$

$$\exists j \in Q_r \setminus \{0\} : \psi(K_r, u)_0 < \psi(K_{1+r-|j|_{\ell^\infty}}, u)_j \quad (3.133)$$

Set $R = (P + 2)/2$ and define:

$$R_i := \min \{1 + R - |i|_{\ell^\infty}, R\} \quad (3.134)$$

THEOREM 3.2.6. *Let L^h be given by*

$$L^h = \sum_{r=1}^{P/2} a_r(x) L_{2r}^h \quad (3.135)$$

where $a_r \geq 0$, and

$$a_r(x) = a_r \in Q_{1+R-r}, \text{ else } 0 \quad (3.136)$$

Then we have

$$(I - L^h) u^h \geq 0 \in Q_{R-1} \implies \quad (3.137)$$

$$\min_{Q_R} \underline{\psi}(K, u) \geq \min_{\partial Q_R} (u^h)^- \quad (3.138)$$

where

$$K = K(i) = K_{R_i} \quad (3.139)$$

Proof: We first claim that L^h satisfies a local maximum principle for $\psi(K, u)$. Suppose $\psi(K, u)$ achieves a maximum at some index $i \in Q_R$. Then we must have that

$$\psi(K, u)_i = u_i - \sum_{r=1}^{R_i-1} a_r L_{2r}^h u_i \quad (3.140)$$

otherwise, suppose that for some $1 \leq m < R_i - 1$ we have

$$\psi(K, u)_i = u_i - \sum_{r=1}^{R_i-1-m} a_r L_{2r}^h u_i = \psi(K_{R_i-m}, u)_i \quad (3.141)$$

which implies

$$u_i - ah^2 \sum_{r=1}^{R_i-1-m} L_{2r}^h u_i > u_i - \sum_{r=1}^{R_i-m} a_r L_{2r}^h u_i \implies \quad (3.142)$$

$$L_{2(R_i-m)}^h u_i > 0 \implies \quad (3.143)$$

$$\exists j \in Q_{R_i-m} : \psi(K_{R_i-m}, u)_i < \psi(K_{R_i-m}, u)_j \leq \psi(K, u)_j \quad (3.144)$$

which contradicts the claim that $\psi(K, u)$ is maximal. Now by the local maximum principle for $L_{2R_i}^h$, we have that

$$L_{2R_i}^h u_i \leq 0 \quad (3.145)$$

Next we have:

$$\psi(K, u)_i = u_i - \sum_{r=1}^{R_i-1} a_r L_{2r}^h u_i \implies \quad (3.146)$$

$$u_i - \sum_{r=1}^{R_i-1} a_r L_{2r}^h u_i \geq u_i \implies \quad (3.147)$$

$$\sum_{r=1}^{R_i-1} a_r L_{2r}^h u_i \leq 0 \quad (3.148)$$

which means that

$$L^h u_i = \sum_{r=1}^{R_i} a_r(x) L_{2r}^h u_i \leq 0 \quad (3.149)$$

verifying the local maximum principle:

$$L^h u^h > 0 \in Q_{R-1} \implies \quad (3.150)$$

$$\max_{Q_R} \psi(K, u) \leq \max_{\partial Q_R} u^h \quad (3.151)$$

Now suppose that $(I - L^h) u^h \geq 0 \in Q_{R-1}$. Consider the set of points i where u^h is negative. At those points, we have

$$(I - L^h) u_i \geq 0 \implies \quad (3.152)$$

$$L^h u_i < 0 \quad (3.153)$$

meaning $\bar{\psi}(K, u)$ cannot achieve a negative minimum in Q_{R-1} , or

$$(I - L^h) u^h \geq 0 \in Q_{R-1} \implies \quad (3.154)$$

$$\min_{Q_R} \underline{\psi}(K, u) \geq \min_{\partial Q_R} (u^h)^- \quad (3.155)$$

which proves the claim. $\square$

It follows that this discrete boundary value problem is well-posed: If $(I - L^h) u^h = 0 \in Q_{R-1}$, $u^h = 0 \in \partial Q_R$, then by this minimum principle we must have $u^h = 0$. Thus the discrete Green's functions exist, and have the following property:

LEMMA 3.2.1. *Let $\widehat{\delta}_i$ satisfy*

$$(I - L^h) \widehat{\delta}_i = \delta_i \in Q_{P/2-1} \quad (3.156)$$

$$\widehat{\delta}_i = \delta_i \in \partial Q_{P/2} \quad (3.157)$$

Then for $i \neq 0$, we have:

$$L_P^h \widehat{\delta}_i(0) \geq 0 \quad (3.158)$$

Proof: By Theorem 4.1.3, we have that

$$(I - L^h) \widehat{\delta}_i \geq 0 \in Q_{R-1} \implies \quad (3.159)$$

$$\min_{Q_R} \underline{\psi}(K, \widehat{\delta}_i) \geq \min_{\partial Q_R} (\widehat{\delta}_i)^- \geq 0 \quad (3.160)$$

Using the fact that

$$(I - L^h) \widehat{\delta}_i(0) = 0 \implies \quad (3.161)$$

$$\widehat{\delta}_i(0) = \sum_{r=1}^{P/2} a_r L_{2r}^h \implies \quad (3.162)$$

$$\left(I - \sum_{r=1}^{(P-1)/2} a_r L_{2r}^h \right) \widehat{\delta}_i(0) = a_{P/2} L_P^h \widehat{\delta}_i(0) \quad (3.163)$$

By the definition of $\psi(K, u)$, we have that

$$\underline{\psi}(K, \widehat{\delta}_i) \leq \left(I - \sum_{r=1}^{(P-1)/2} a_r L_{2r}^h \right) \widehat{\delta}_i(0) \quad (3.164)$$

Combining (3.160) and (3.163), we have:

$$a_{P/2} L_P^h \widehat{\delta}_i(0) = \left(I - \sum_{r=1}^{(P-1)/2} a_r L_{2r}^h \right) \widehat{\delta}_i(0) \geq \underline{\psi}(K, \widehat{\delta}_i) \geq 0 \quad (3.165)$$

which proves the claim. $\square$

We again remark that since the $\widehat{\delta}_i$ should be smooth, it is reasonable to expect that we could construct an order $P + 2$ Laplace operator that retains this same sign property.

Next, we will obtain a similar statement to Theorem 2.1.2, that is, if a particular transformation of u^h achieves a discrete maximum, then the high order Laplace operator applied to u^h is nonpositive: Define the operator L^h as in the previous theorem. Define

$$\varphi^h := (I - L^h) u^h \in Q_{P/2} \quad (3.166)$$

$$\varphi^h := u^h \in \partial Q_{P/2+1} \quad (3.167)$$

so that u^h satisfies the boundary value problem:

$$(I - L^h) u^h = \varphi^h \in Q_{P/2} \quad (3.168)$$

$$u^h = \varphi^h \in \partial Q_{P/2+1} \quad (3.169)$$

Let $(I - L^h)^{-1}$ denote the solution operator for this boundary value problem. Then we have the following:

LEMMA 3.2.2. *Let L_{P+2}^h be an order $P + 2$ accurate Laplace operator which satisfies:*

$$L_{P+2}^h \widehat{\delta}_i(0) \geq 0 \quad \forall i \neq 0 \quad (3.170)$$

$$L_{P+2}^h \widehat{\delta}_0(0) < 0 \quad (3.171)$$

where $\widehat{\delta}_i$ are the discrete Green's functions of (3.168) - (3.169). If φ^h achieves a maximum at the origin over $Q_{P/2+1}$, then $L_{P+2}^h u_0 \leq 0$.

Proof: Assume that φ^h is maximal over $Q_{P/2+1}$ at the origin. Then we have that

$$L_{P+2}^h u_0 = L_P^h \circ (I - L^h)^{-1} \varphi_0 \quad (3.172)$$

$$= \sum_{i \in Q_{P/2+1}} \varphi_i L_{P+2}^h \widehat{\delta}_i(0) \quad (3.173)$$

By assumption $L_{P+2}^h \circ (L^h)^{-1}$ is positive type in φ^h , the claim follows. $\square$

Note that our stencil construction procedure explicitly enforces this condition in (3.81).

3.2.5 Remarks

We remark here on several choices in this chapter that appear to be ad-hoc, and deserve attention in future studies.

Firstly, numerical experiments have shown that there is flexibility in choosing the sequence a_r ; for the sequences we consider ($a_1 = a, a_r = 0$ otherwise and $a_r = a$) there appears to be a range of acceptable values which shrinks as we move from dimension two to three. Other progressions, such as geometric progressions, appear to work as well. Ideally, we would establish a relationship between the a_r and the coefficients of the difference operators in the stencil.

Secondly, our results thus far have been given for $P \leq 8$; however, we have verified numerically that our procedure continues to work for much higher orders P with the same choice of coefficients a_r , so long as we allow a sufficient number of difference quotients in the correction term. For now it is an open question whether this process succeeds for arbitrary P ; it may be possible to prove such a statement if one has a tool for acquiring sharp difference quotient estimates on the smoothed basis functions. However, for now we are satisfied with the technique of solving a relatively small linear program.

Chapter 4

Global Maximum Principles in Higher Dimensions

In this chapter we prove a variety of global results which can be derived from local maximum principles. We are specifically interested in discretizations for which the tools developed in the preceding chapters can be directly applied. We will lower the order of accuracy, and thus the stencil footprint, as we approach the boundary to avoid additional complexity associated with extension procedures which implicitly create “ghost” or “fictitious” values. This assumption simplifies the work of proving a global maximum principle, and this case is not too far removed from discretizations used in practice; it is common to drop the order of accuracy near the boundary by two orders [1], [14], [11].

4.1 Global Maximum Principles

Suppose we have properly selected $\psi(K, u)$ so that the local maximum principle holds, or:

$$L^h u_i > 0 \implies \exists j \in Q_R(i) : \psi(K, u)_i < \psi(K, u)_j \quad (4.1)$$

Now let Ω^h be some connected subset of the lattice $\mathbb{Z}^n$.

DEFINITION. We will define the set $G^h = G_R^h(\Omega^h)$ as the smallest set of lattice points such that for each point $i \in \Omega^h$, $Q_R(i) \subset \Omega^h \cup G_R^h(\Omega^h)$. For a connected subset $\mathcal{O}^h \subset \mathbb{Z}^n$, we will denote $\mathcal{O}_r^h$ as the set of all points i contained in $\mathcal{O}^h$ such that $Q_r(i)$ is fully contained in $\mathcal{O}^h$.

We then have the following result:

LEMMA 4.1.1. *Let $u^h \in \ell^\infty(\mathbb{Z}^n)$. Let Ω^h be a connected subset of $\mathbb{Z}^n$. Suppose that L^h satisfies a local maximum principle for $\psi(K, u)$. If $L^h u > 0 \in \Omega^h$, then*

$$\max_{\Omega^h} u^h \leq \max_{\Omega^h} \psi(K, u) \leq \max_{G^h} \psi(K, u) \quad (4.2)$$

Proof: Since $L^h u > 0 \in \Omega^h$, by the local maximum principle, the maximum of $\psi(K, u)$ over $\Omega^h \cup G^h$ cannot occur at any point in Ω^h , meaning it must occur in G^h . By definition, $u^h \leq \psi(K, u)$. $\square$

We can extend this to particular variable coefficients and an identity term:

LEMMA 4.1.2. *Let L^h satisfy the assumptions of the previous theorem, and assume $a(x) > 0, c(x) \geq 0, \exists i \in \Omega^h : c(i) \neq 0$. Then we have $a(x)L^h - c(x)I > 0 \in \Omega^h$ implies:*

$$\max_{\Omega^h} \psi(K, u) \leq \max_{G^h} \psi(K, u)^+ \quad (4.3)$$

Proof: Consider the set of points i in Ω^h where $u^h > 0$. At those points we have $a(x)L^h u_i > c(x)u_i > 0$, meaning $\psi(K, u)$ cannot achieve a maximum at those points. Therefore u^h cannot achieve a positive maximum in Ω^h , meaning

$$\max_{\Omega^h} \psi(K, u) \leq \max \left\{ 0, \max_{G^h} \psi(K, u) \right\} = \max_{G^h} \psi(K, u)^+ \quad (4.4)$$

$\square$

Using comparison functions, we can derive a general stability result.

THEOREM 4.1.1. *Suppose the discrete Laplace operator L^h satisfies a local maximum principle for $\psi(K, u)$, where each $\alpha \in K$ has the property that if v^h is the restriction of a quadratic function to the grid, then*

$$\alpha(v^h) = -c_\alpha \Delta v^h \quad (4.5)$$

for some nonnegative constant $c_\alpha \leq c$. Let $G^h \cup \Omega^h$ be contained in the strip $0 \leq x_1 \leq D$. Then we have

$$\max_{\Omega^h} |u^h| \leq (D^2) \max_{\Omega^h} |L^h u^h| + \max_{G^h} \left(|u^h| + \sum_{\alpha \in K} |\alpha(u^h)| \right) \quad (4.6)$$

Proof: Define $M := \max_{\Omega^h} |L^h u^h|$ and

$$N := \max \left\{ \max_{G^h} |\psi(K, u^h)|, \max_{G^h} |\underline{\psi}(K, u^h)| \right\} \quad (4.7)$$

$$\leq \max_{G^h} \left(|u^h| + \sum_{\alpha \in K} |\alpha(u^h)| \right) \quad (4.8)$$

Define the comparison function $v^h = -\frac{M+\epsilon}{2}(x_1^2 - D^2) + N$ for $\epsilon > 0$. Then we have

$$L^h(u^h - v^h) \geq -M + (M + \epsilon) > 0 \quad (4.9)$$

We have the following bounds on v^h in $G^h \cup \Omega^h$:

$$N \leq v^h \leq \frac{M + \epsilon}{2} D^2 + N \quad (4.10)$$

Then we have the bound:

$$\max_{G^h} \psi(K, u^h - v^h) \leq \max_{G^h} \psi(K, u^h) + \psi(K, -v^h) \quad (4.11)$$

$$\leq N + \max_{G^h} \psi(K, -v^h) \quad (4.12)$$

$$= N + \max_{i \in G^h} \max_{\alpha \in K} (-v_i + \alpha(-v)_i) \quad (4.13)$$

$$\leq N + \max_{i \in G^h} -v_i \leq 0 \quad (4.14)$$

where we use the fact that $\Delta v \leq 0 \in G^h$. We can apply the previous lemma to obtain:

$$u^h - v^h \leq \max_{\Omega^h} \psi(K, u^h - v^h) \leq 0 \implies \quad (4.15)$$

$$u^h \leq \frac{M + \epsilon}{2} D^2 + N \quad (4.16)$$

Applying the same logic to $(u^h + v^h)$, we get

$$\max_{\Omega^h} |u^h| \leq \frac{M + \epsilon}{2} (D^2) + N \quad (4.17)$$

and sending $\epsilon \rightarrow 0$ we obtain the desired result. $\square$

We note that the K defined for the two and three dimensional Laplacian stencils satisfy this property because each α is a sum of second order accurate or higher Laplace operators times negative constants scaled by h^2 .

Let $\{L_{2r}^h\}$ be a collection of order $2r$ accurate Laplace operators, $1 \leq r \leq P/2$, with $L_2^h := \Delta^h$. Suppose P is such that for all $2r \leq P$, L_{2r}^h satisfies a local maximum principle in $\psi(K, u)$ for

$$\psi(K, u)_i = \max \{u_i, u_i - ah^2 \Delta^h u_i\} \quad (4.18)$$

Then we have:

THEOREM 4.1.2. *Let Ω^h be a connected subset of $\mathbb{Z}^n$ with radius $D/2$ and $\partial\Omega^h := G_1^h(\Omega^h)$, with $\overline{\Omega^h} := \Omega^h \cup \partial\Omega^h$. Let the operator L^h be given by*

$$L^h = \sum_{r=1}^{P/2} a_r(x) L_{2r}^h \quad (4.19)$$

where $a_r(x) \geq 0$, and

$$\text{supp}(a_r(x)) \subseteq \overline{\Omega^h_r} \quad (4.20)$$

$$\forall x \in \Omega^h, \sum_{r=1}^{P/2} a_r = 1 \quad (4.21)$$

Then we have

$$\max_{\Omega^h} |u^h| \leq \max_{\partial\Omega^h} |u^h| + (D^2) \max_{\Omega^h} |L^h u^h| \quad (4.22)$$

Note that this result quickly implies uniqueness of solutions to the boundary value problem:

$$L^h u^h = f^h \in \Omega^h \quad (4.23)$$

$$u^h = g^h \in \partial\Omega^h \quad (4.24)$$

which we can see by setting f^h, g^h to zero, which implies $\max_{\Omega^h} |u^h| = 0$. In finite dimensions this uniqueness implies existence.

Proof of 4.1.2: We first claim that L^h satisfies a local maximum principle for $\psi(K, u)$. Suppose $\psi(K, u)$ achieves a maximum at some index $i \in \Omega^h$. Let R_i be defined as

$$R_i := \max_R \{1 \leq R \leq P/2\} : i \in \overline{\Omega^h_R} \quad (4.25)$$

Then for all L^h_{2r} for which $r \leq R_i$, we have by the local maximum principle:

$$L^h_{2r} u_i \leq 0 \quad (4.26)$$

By assumption that at least one of the a_r is positive at any point in the domain, this implies that

$$L^h u_i \leq 0 \quad (4.27)$$

which proves the claim. We will then set $G^h = G_1(\Omega^h) = \partial\Omega^h$, and $\psi(K, u) = u^h \in \partial\Omega^h$. By the assumption on the support of a_r , this means that we have the local maximum principle:

$$L^h u_i > 0 \implies \exists j \in Q_{R_i}(i) : \psi(K, u)_i < \psi(K, u)_j \quad (4.28)$$

where we are using the fact that at the points $\overline{\Omega^h_1} \setminus \overline{\Omega^h_2}$, $L^h = a_1(x)\Delta^h$, so the local maximum principle holds with $\psi(K, u) = u^h \in \partial\Omega^h$. We now apply Theorem 4.1.1, with $a = 0 \in \partial\Omega^h$ in the definition of $\psi(K, u)$, and the constant c set to ah^2 to get

$$\max_{\Omega^h} |u^h| \leq (D^2) \max_{\Omega^h} |L^h u^h| + \max_{\partial\Omega^h} |u^h| \quad (4.29)$$

□

We will now consider the situation in which $\psi(K, u)$ is defined by the inductive procedure of the previous chapter. Let $\{L_{2r}^h\}$ be a collection of order $2r$ accurate Laplace operators, $1 \leq r \leq P/2$, with $L_2^h := \Delta^h$. Suppose P is such that for all $2r \leq P$, L_{2r}^h satisfies a local maximum principle in $\psi(K_r, u)$ for

$$K_1 = \langle 0 \rangle \quad (4.30)$$

$$K_r = \left\langle K_{r-1}, -ah^2 \sum_{s=1}^{r-1} L_{2s}^h \right\rangle \quad (4.31)$$

of the form:

$$L_{2r}^h u_0 > 0 \implies \quad (4.32)$$

$$\exists j \in Q_r : \psi(K_r, u)_0 < \psi(K_{1+r-|j|_{\ell^\infty}}, u)_j \quad (4.33)$$

THEOREM 4.1.3. *Let L^h be given by*

$$L^h = \sum_{r=1}^{P/2} a_r(x) L_{2r}^h \quad (4.34)$$

where $a_r(x) \geq 0$, and

$$a_{P/2}(x) = 1 \in \overline{\Omega^h}_{P/2}, \text{ else } 0 \quad (4.35)$$

$$a_r(x) = 1 \in \overline{\Omega^h}_r \setminus \overline{\Omega^h}_{r+1}, \text{ else } 0 \text{ for } 1 \leq r < P/2 \quad (4.36)$$

Then we have

$$\max_{\Omega^h} |u^h| \leq \max_{\partial\Omega^h} |u^h| + (D^2) \max_{\Omega^h} |L^h u^h| \quad (4.37)$$

Proof: Again let R_i be defined as

$$R_i := \max_R \{1 \leq R \leq P/2\} : i \in \overline{\Omega^h}_R \quad (4.38)$$

which means that

$$L^h u_i = L_{2R_i}^h u_i \quad (4.39)$$

We will define the spatially varying set $K = K(i)$ as

$$K(i) = K_{R_i} \quad (4.40)$$

We first claim that L^h satisfies a local maximum principle for $\psi(K, u)$: suppose $L^h u_i = L_{2R_i}^h u_i > 0$ at some index $i \in \Omega^h$. Then we have that on the cube $Q_{R_i}(i)$,

$$L_{2R_i}^h u_i > 0 \implies \quad (4.41)$$

$$\exists j \in Q_{R_i}(i) : \psi(K, u)_i = \psi(K_{R_i}, u)_i < \psi(K_{1+R_i-|j-i|_{\ell^\infty}}, u)_j \leq \psi(K, u)_j \quad (4.42)$$

We will then set $G^h = G_1(\Omega^h) = \partial\Omega^h$, and $\psi(K, u) = u^h \in \partial\Omega^h$. By the assumption on the support of a_r , this means that we have the local maximum principle for every $i \in \Omega^h$

$$L^h u_i > 0 \implies \exists j \in Q_{R_i}(i) : \psi(K, u)_i < \psi(K, u)_j \quad (4.43)$$

where we are using the fact that at the points $\overline{\Omega^h_1} \setminus \overline{\Omega^h_2}$, $L^h = \Delta^h$, so the local maximum principle holds with $\psi(K, u) = u^h \in \partial\Omega^h$. We now apply Theorem 4.1.1, with $a = 0 \in \partial\Omega^h$ in the definition of $\psi(K, u)$, and the constant c set to $a(P/2 - 1)h^2$ to get

$$\max_{\Omega^h} |u^h| \leq (D^2) \max_{\Omega^h} |L^h u^h| + \max_{\partial\Omega^h} |u^h| \quad (4.44)$$

□

Solving the Poisson equation using a discretization conforming to Theorem 4.1.3 corresponds to using maximal accuracy at each point in Ω^h , such that the cube bounding the stencil footprint is entirely contained in Ω^h , and using successively lower order accurate operators as the boundary is approached. In practice, to construct high order stencils at boundaries one typically uses an extension procedure to implicitly create “ghost” or “fictitious” points so that the interior stencil can be defined near the boundary; see [14] or [11] for examples. In this case to obtain a stability result it will be necessary to somehow disentangle the interior values from the boundary values used in the extension, which we expect will require additional machinery.

Chapter 5

Conclusion

In this thesis we proved maximum principles for high order accurate finite difference discretizations of the Laplacian in one, two and three dimensions. This is a significant advance in the stability analysis of finite difference methods for elliptic equations, which has, with very few exceptions, not progressed past second order schemes. After discussing precisely why the typical discrete maximum principle fails for high order accurate operators, we suggested a solution inspired by [4]. The key analytical device was the introduction of the perturbed variable $\psi(K, u)$. In one dimension, $\psi(K, u)$ took the form:

$$\psi(K, u)_i = \max \{u_i, u_i - a\partial^{2,h}u_i\} \quad (5.1)$$

and we showed that proving that a high order accurate second derivative operator $\partial_P^{2,h}$ satisfies a local maximum principle in $\psi(K, u)$ reduces to showing that $\partial_P^{2,h}$ can be rewritten as a positive type stencil using variables derived from $\psi(K, u)$. We next showed how this process induces a “smoothing” of basis functions by solving a small discrete elliptic boundary value problem with the operator $I - a\partial^{2,h}$. Equivalently, we showed that if $(I - a\partial^{2,h})u^h$ achieves a local maximum, then $\partial_P^{2,h}u_0 \leq 0$. In other words, if the transformation of u^h under this elliptic operator achieves a local maximum, then the high order accurate second derivative operator applied to u^h is nonpositive.

Next, we fully generalized the approach of [4] by casting it as a linear program. Based on our intuition from the one dimensional case, we proposed an inductive procedure which simultaneously built up $\psi(K, u)$ and order P accurate Laplacian stencils which were compatible with the local maximum principle generated by $\psi(K, u)$. Using this procedure, we generated sequences of high order accurate Laplacians.

In the final chapter, we proved global maximum principles that could be derived from the local results. While we did not fully treat boundary conditions as they are used in practice for high order accurate methods, this is a first step toward proving stability results for such methods. We expect that some additional tools will be required to deal with high order accurate stencils near boundaries.

There are many potential future research directions based on this work. The first will be to more fully address boundary conditions, starting with high order accurate extension

techniques. Next, we could apply this machinery to general elliptic operators with cross derivative terms and first order terms. The linear programming approach for constructing stencils may be generalizable to other elliptic operators, which we expect heuristically to follow the same local smoothing logic. However, we will likely need to prove more refined results about our inductive procedure for stencil construction.

This thesis represents a first effort toward gaining a more refined understanding of high order accurate finite difference methods for solving elliptic boundary value problems. In the study of classical solutions, the maximum principle is a powerful and flexible tool that can be used to prove regularity results, the Harnack inequality, and results about spectral properties of elliptic operators [3], [9], [13]. In addition, one can deal with variable coefficients and first and zeroth order terms. With a discrete maximum principle in hand, one can try to prove analogues of these results for finite difference schemes.

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