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RADIANCE BOUNDS

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March 1959
Index Number NS 714-100

Bureau of Ships
Contract NObs-72092

SIO REFERENCE 59-20

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INTRODUCTION

If a truck travels along an open highway from city A to city B one hundred miles away, and if it is known that the truck cannot go faster than forty-five miles an hour and that the driver does not go slower than forty miles an hour in trucks on open highways, then it is easy to bracket the expected time of travel of this truck from city A to city B. Evidently an upper bound on the time of travel is 100/40 = 2 1/2 hours, and a lower bound on the time of travel is 100/45 = 2 2/9 hours. Hence the time of travel is predictable, in this case, to within 5/18 of an hour, or about seventeen minutes.

In many problems of mathematical physics, workers encounter problems similar to the simple example above, wherein the magnitude of a certain variable cannot, by the nature of things, be predicted exactly. Nevertheless as in the preceding example, there exists certain information about the variable which will allow a reasonable estimate of its range of variation in a particular context. A quantity which is smaller than all the quantities in this range is called a lower bound, and one which is greater than all the quantities in the range is called an upper bound.

In the study of the penetration of (either natural or artificial) light into the sea and the atmosphere, the most important variable is the radiance distribution $N(x, \bullet)$ which describes the angular distribution of the "brightness" of monochromatic light about each point x in the medium. While much is known about the theory of the radiance distribution, the equations governing it are extremely difficult to solve in detail. It would then be of great help in practical everyday problems to be able to obtain rough estimates--in a way similar to the case of the truck problem above--of upper and lower bounds of the radiance distribution about a given point in the medium.

To understand the basic mathematical procedure involved in a bounding-process, we consider once again the preceding illustration. In that problem, the variable of interest was the time of travel T from city A to city B. The distance D between them was presumed known, and the exact law governing the relation between D, T and the instantaneous speed V of the truck was known: $D = \int_0^T V(t)\,dt$. However, the speed was known only approximately, that is, it was known to lie between $\overline{V} = 45$ miles/hour and $\underline{V} = 40$ miles/hour. Hence the exact law could be used to obtain the following inequalities:

$$\int_0^T \underline{V}\,dt \leq \int_0^T V(t)\,dt \leq \int_0^T \overline{V}\,dt ,$$

or

$$\underline{V}T \leq D \leq \overline{V}T .$$

Since $\underline{V}$, $\bar{V}$ and D are now known, it follows from the left-hand inequality that $T \leq D/\underline{V}$, and the right-hand inequality that $T \geq D/\bar{V}$, hence T --the time of travel--must lie between the limits:

$$\frac{D}{\bar{V}} \leq T \leq \frac{D}{\underline{V}} .$$

Substituting the values of $\underline{V}$, $\bar{V}$, and D in these inequalities, we obtain the numbers arrived at above.

In this note, we turn to the exact laws governing the radiance distribution $N(x, \cdot)$ and try to bracket it between upper and lower bounds. The laws governing $N(x, \cdot)$ differ only in degree, not in kind, from the simple law governing D, V, and T in the truck problem. The pertinent variables used in this bracketing, or bounding, process are the inherent optical properties σ, α of the medium (the volume scattering and volume attenuation functions) and the inherent radiance N^{o} of the source of light feeding radiant energy to the medium.

We shall be able to bound the radiance distributions even when they are in the process of building up in time and space immediately after a source has been turned on. Hence the bounds so obtained (or bounds obtained by following the pattern set below) can be used to estimate the magnitudes of the time-varying diffusely scattered radiance returned to a given point after a short pulse of light has been shot into a medium. Of course, by letting time increase indefinitely, we can also obtain bounds for steady state radiance distributions.

Not only can bounds for the observable time-dependent radiance distributions be found, but also the bounds for the time-dependent n-ary scattered radiance (i.e., radiance composed of radiant flux which has been scattered n and only n times). Such information can be invaluable in finite-series approximations to the observable radiance distributions; for by knowing that, say, the upper bounds to fifth order scattered light and all higher orders fall sufficiently below the threshold of a certain radiance detector, we need not bother to calculate n-ary radiances for $n \geq 5$.

With this background in mind, we now turn to the details.

ACTIVITY WITHIN THE CHARACTERISTIC ELLIPSOID

Our first task in the bounding problem is to find the region within which the activity of a time-dependent multiple scattering process takes place. This task is simplified by some previous work on the time-dependent problem in which it was found that the characteristic ellipsoid is precisely the region within which such activity takes place for any given source-receiver pair of points x_0, x and epoch time T.[1] Specifically, let x_0 be the source point of known inherent radiance distribution $N^0(x_0, \cdot)$. Let x be the receiver point a distance d away from x_0. Then at epoch time T, the point pair (x_0, x) defines a characteristic ellipsoid $\mathcal{E}(x_0, x\,; T)$ whose analytical representation is:

$$r(d, \psi, T) = \frac{D^2 - d^2}{2(D - d\cos\psi)}\,, \quad D = vT \geq d\,, \tag{1}$$

where v is the speed of light in the medium.

The geometric significance of (1) is easily described: x_0 and x are at the foci of $\mathcal{E}(x_0, x\,; T)$ which has a major diameter $D = vT$. The radius vector $r(d, \psi, T)$ is measured from x as an origin, and ψ is the angle between the major axis of $\mathcal{E}(x_0, x\,; T)$ and the radius vector ($\cos\psi = \xi \cdot \xi_0$, where ξ_0 is the unit vector at x pointing at x_0). For any given d, ψ , and T, $r(d, \psi, T)$ is the distance from x to the boundary of $\mathcal{E}$ in the direction ψ at epoch time T .

The physical significance of (1) is also easily described: The characteristic ellipsoid $\mathcal{E}(x_0, x\,; T)$ is the subregion of the optical medium which at epoch time T is actively contributing scattered radiant flux from x_0 to point x . The region outside of $\mathcal{E}(x_0, x\,; T)$ does not contribute flux to x at epoch time T .

THE RADIANCE OPERATOR

We now introduce the notion of the radiance operator $\mathcal{R}$. Despite its formidable appearance, the operator $\mathcal{R}$ is essentially a mathematical version of Jacob's ladder: it allows an easy step-by-step climb up into the hierarchy of functions which describe the amounts of multiply scattered radiant flux in an optical medium. The radiance operator which we shall need in the present study is a close kin of the integral operator used in everyday steady-state computations leading to the path radiance N_r^*:

$$N_r^*(x, \xi) = \int_0^r T_{r'}(x, \xi)\, N_*(x', \xi)\, dr'. \tag{1}$$

Here N_* is the path function, $T_{r'}$ the beam transmittance of a path of length r. The integral is taken over the path defined by the triple (x, ξ, r) where x is the vector which locates the observation point, ξ the unit vector which defines the direction of observation, and r is the length of the path of sight (measured from x). The path function N_* is defined by:

$$N_*(x, \xi) = \int_{\Xi} \sigma(x; \xi; \xi')\, N(x, \xi')\, d\Omega(\xi'), \tag{2}$$

where Ξ is the collection of all directions about any point x in the medium X.

Combining (1) and (2) we have:

$$N_r^*(x, \xi) = \int_0^r \int_{\Xi} T_{r'}(x, \xi)\, \sigma(x'; \xi; \xi')\, N(x', \xi')\, d\Omega(\xi')\, dr'. \tag{3}$$

An examination of (3) shows that $N_r^*(x, \xi)$ is simply the weighted integral of the radiance distributions $N(x', \cdot)$ over the path (x, ξ, r). The weighting functions are T_r and σ. If we let r increase in the direction ξ until the endpoint $x + r\xi$ of the path touches the boundary of X, then $N_r^*(x, \xi)$ represents the "space light" generated along the extent of (x, ξ, r). If the inherent radiance of the boundary were zero, then $N_r^*(x, \xi)$ would be the observable radiance at x in the direction ξ.

Suppose now that the point source at x_o is turned on at epoch time $T = 0$, and that we await the scattered radiation at the observation point x a distance d from x_o. At epoch times $T \geqslant d/v$ the field radiance produced by the source will generally be observed. To find $N_r^*(x, \xi, T)$ the path radiance associated with (x, ξ, r) at these epoch times, we merely use Equation (3) in which $r(d, \psi, T)$ of Equation (1) has been used in the upper limit of the integration along (x, ξ, r):

$$N_r^*(x, \xi) = \int_0^{\min\{r(d,\psi,T),\, r\}} \int_{\Xi} T_{r'}(x, \xi)\, \sigma(x'; \xi; \xi')\, N(x'; \xi')\, d\Omega(\xi')\, dr' . \tag{4}$$

The reason for the rather odd looking upper limit should be clear after reflecting on the definition of the characteristic spheroid.

Instead of repeatedly writing out the unwieldy integration in (4), let us agree to abbreviate it as follows:

$$N_r^*(x, \xi, T) = Q_r\left[\, N(x', \xi', T')\,\right] . \tag{5}$$

The symbol $\mathcal{Q}_r$ is the general radiance operator for path radiance associated with the path (x, ξ, r). In what follows we will be interested only in path lengths r describing the distances from x to the boundary of X. With this continually in mind we can drop reference to r and simply write the radiance operator as $\mathcal{Q}$. Thus $\mathcal{Q}$ operates on the radiance function N to give the path radiance N_r^* associated with (x, ξ, r).

But $\mathcal{Q}$ can do more than just transform N into N_r^*: it can assign to N^n (the n-ary radiance along (x, ξ, r)) the function N^{n+1}. (For detailed derivation of this property, see, e.g., reference 2.) Thus:

$$N^{n+1}(x, \xi, T) = \mathcal{Q}\left[N^n(x', \xi', T')\right] \tag{6}$$
$$n = 0, 1, 2, \ldots .$$

We shall now consider the problem of obtaining bounds on the n-ary radiance functions.

n-ARY RADIANCE BOUNDS

In the interests of simplicity, we will assume from now on that X has an isotropic point source, is infinite and optically homogeneous, but that σ may have arbitrary angular structure. The resulting bounds derived from these assumptions then take on particularly simple forms which invite rather than discourage applications. However, the methods developed below can serve as a pattern for more general bound-estimates when needed.

Let $\overline{N}^n$ and $\underline{N}^n$ denote the <u>least upper</u> and <u>greatest lower bounds</u>, respectively, of the n-ary radiance function N^n, $n = 0, 1, 2, \ldots$, in $\mathcal{E}(x_0, x; T)$. These bounds generally exist and are well defined because of the continuity of the function $N^n(\cdot, \cdot, \cdot)$ on the closed bounded set $\mathcal{E} \times \Xi \times T$ (T is a finite time interval). The quantities $\overline{N}^n$ and $\underline{N}^n$ are very much like the $\overline{V}$ and $\underline{V}$ used in the simple example of the Introduction. In fact, we are now about to perform a bracketing computation essentially like that involving the integral $D = \int_0^T V(t)\,dt$ of the Introduction. Using (6) and replacing* ξ by ψ we have:

$$Q[\underline{N}^n] \leq N^{n+1}(x, \psi, T) \leq Q[\overline{N}^n] \,. \tag{7}$$

Now under the present assumptions,

$$T_r(x, \psi) = e^{-\alpha r}$$

* This replacement is permissible because the radiation field now has <u>axial</u> symmetry about the major axis of $\mathcal{E}(x_0, x; T)$.

for all (x, ψ, r) . Furthermore, the upper limit of the integral in (4) is clearly $r(d, \psi, T)$. Therefore, (7) becomes:

$$\bar{N}^n \bar{\omega}_0 \left[1 - exp\left\{ -\alpha r(d, \psi, T) \right\} \right]$$
$$\geqslant$$
$$N^{n+1}(x, \psi, T) \qquad (8)$$
$$\geqslant$$

where $$\underline{N}^n \omega_0 \left[1 - exp\left\{ -\alpha r(d, \psi, T) \right\} \right]$$

$$\omega_0 = s/\alpha, \quad s = \int_{\Xi} \sigma \, d\Omega \; .$$

These are the two fundamental bounds for time-dependent n-ary radiance.

We can obtain further bounds of a rather simple nature as follows. Let $\bar{r}(T) = (D+d)/2$, and $\underline{r}(T) = (D-d)/2$. These are respectively the maximum and minimum radii vectors from the observation point x to the boundary of $\mathcal{E}(x_0, x; T)$. It follows that

$$1 - e^{-\alpha \underline{r}(T)} \leqslant 1 - e^{-\alpha r(d, \psi, T)} \leqslant 1 - e^{-\alpha \bar{r}(T)} \, . \qquad (9)$$

Observe that these two bounds do not depend on ψ ; they depend only on the epoch time T and on d . Now, returning to (6), we find that by repeated application of $\mathcal{Q}$:

$$N^{n+j}(x,\psi,T) = \mathcal{Q}^{j}\left[N^{n}(x',\psi',T')\right] \tag{10}$$

$$j = 1, 2, \ldots .$$

That is, by (6),

$$N^{n+2}(x,\psi,T) = \mathcal{Q}\left[N^{n+1}(x',\psi',T')\right]$$

$$= \mathcal{Q}\left[\mathcal{Q}\left[N^{n}(x'',\psi'',T'')\right]\right] = \mathcal{Q}^{2}\left[N^{n}(x'',\psi'',T'')\right],$$

and so on.

From (10), it follows that:

$$N^{n+j}(x,\psi,T) \leq \mathcal{Q}^{j}\left[\bar{N}^{n}\right] \tag{11}$$

But

$$\mathcal{Q}^{j}\left[\bar{N}^{n}\right] = \mathcal{Q}^{j-1}\left[\mathcal{Q}\left[\bar{N}^{n}\right]\right]$$

$$= \mathcal{Q}^{j-1}\left[\bar{N}^{n}\omega_{0}\left[1 - \exp\{-\alpha r(d,\psi,T)\}\right]\right]$$

$$\leq \mathcal{Q}^{j-1}\left[\bar{N}^{n}\omega_{0}\left(1 - e^{-\alpha \bar{F}(T)}\right)\right]. \tag{12}$$

Observe that the function on which $\mathcal{Q}$ operates in the last line of (12) is a constant function on $X \times \Xi$. It is easy to verify that if, in general, C is a constant function on $X \times \Xi$,

$$\mathcal{Q}[C] = C\,\omega_0\left[1 - \exp\{-\alpha F(d,\psi,T)\}\right].$$

Hence, in particular,

$$\mathcal{Q}[C] \leq C\,\omega_0\left[1 - e^{-\alpha \bar{F}(T)}\right]$$

Furthermore,

$$\mathcal{Q}^2[C] \leq C\left[\omega_0(1 - e^{-\alpha \bar{F}(T)})\right]^2,$$

and in general

$$\mathcal{Q}^k[C] \leq C\left[\omega_0(1 - e^{-\alpha \bar{F}(T)})\right]^k \tag{13}$$

$k = 1, 2, \ldots$.

Returning to (12), and applying (13), we have:

$$\mathcal{Q}^j[\bar{N}^n] \leq \bar{N}^n\left[\omega_0(1 - e^{-\alpha \bar{F}(T)})\right]^j,$$

And then returning to (11), we have, finally:

$$N^{n+j}(x,\psi,T) \leq \overline{N}^{n}\left[\omega_0\left(1-e^{-\alpha \overline{F}(T)}\right)\right]^{j} \tag{14}$$

In a similar way we can show that

$$\underline{N}^{n}\left[\omega_0\left(1-e^{-\alpha \underline{F}(T)}\right)\right]^{j} \leq N^{n+j}(x,\psi,T). \tag{15}$$

The expressions (14) and (15) constitute another useful set of bounds on the n-ary radiance functions. Perhaps the most useful ones of this variety are obtained by setting $n=0$, for then $\overline{N}^{0}$ is readily determinable from the inherent radiance data, which is usually known. These special bounds are summarized below:

$$\boxed{\underline{N}^{0}\left[\omega_0\left(1-e^{-\alpha \underline{F}(T)}\right)\right]^{j} \leq N^{j}(x,\psi,T) \leq \overline{N}^{0}\left[\omega_0\left(1-e^{-\alpha \overline{F}(T)}\right)\right]^{j}} \tag{16}$$

In order to use (16) or any formula derived therefrom, it must be re-emphasized that $\overline{N}^{0}$ and $\underline{N}^{0}$ are the <u>least upper</u> and <u>greatest lower</u> bounds of N^{0} in $\mathcal{E}(x_0, x; T)$.

RADIANCE BOUNDS

The observable radiance $N(x,\psi,T)$ at x in the direction ψ at time T may be represented as an infinite series of n-ary radiances:

$$N(x,\psi,T) = \sum_{n=0}^{\infty} N^n(x,\psi,T) \, . \tag{17}$$

Since each of these n-ary radiances has known upper and lower bounds, given for example by (16), it is then a simple matter to obtain a set of upper and lower bounds for $N(x,\psi,T)$. Thus,

$$\sum_{n=0}^{\infty} \underline{N}^n \leq N(x,\psi,T) \leq \sum_{n=0}^{\infty} \bar{N}^n , \tag{18}$$

so that from (16), we obtain:

$$\boxed{\frac{\underline{N}^0}{1-\omega_0\left(1-e^{-\alpha \underline{r}(T)}\right)} \leq N(x,\psi,T) \leq \frac{\bar{N}^0}{1-\omega_0\left(1-e^{-\alpha \bar{r}(T)}\right)}} \, . \tag{19}$$

REFERENCE

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26 February 1959